Two projects on equations over generalizations of hyperbolic groups

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INTRODUCTION

1.1 Equations over groups

Let G be a group and $\langle x \rangle$ be the infinite cyclic group generated by x. For each $g \in G$, let φ_g denote the homomorphism $G * \langle x \rangle \to G$ induced by taking the identity map on G and sending $x \mapsto g$. Given an element $w(x)$ of the free product $G * \langle x \rangle$, there is a corresponding function $G \to G$ whose evaluation at $g \in G$, which we will hereafter denote by $w(g)$, is

$$w(g) := \varphi_g(w(x)).$$

Building on this, we write $w(x) = 1$ to represent a *group equation* in the single variable x with coefficients in G whose *solution set* is

$$\{g \in G \,|\, w(g) = 1\}.$$

The solution set above is called the *primitive solution set corresponding to* $w(x)$. As with our familiar conception of equations, one may consider a system of equations corresponding to elements $w_1(x), w_2(x), \ldots \in G * \langle x \rangle$ whose solution set is the intersection of the corresponding primitive solution sets. Similarly, one may also consider equations in multiple variables $x_1, \ldots, x_n$ by replacing $\langle x \rangle$ with the free group $F_n = F(x_1, \ldots, x_n)$ with basis $\{x_1, \ldots, x_n\}$. (Then each n-tuple of elements in G defines a homomorphism $G * F_n \to G$ which fixes G, and so on.)

The purpose of this thesis is to present the results of two separate projects, both related to equations over groups and each concerning a particular generalization of hyperbolic groups. The remainder of this introduction is divided accordingly.

1.2 Project 1: Algebraic subgroups of acylindrically hyperbolic groups

Given a group G, the *elementary theory* of G, denoted $Th(G)$, is the set of all first order sentences in the language of group theory which are true in G. In [20], Kharlampovich and Myasnikov answered two long-standing conjectures of Tarski in the affirmative; they showed firstly that the elementary theory of all finitely generated non-abelian free groups coincide, i.e. that $Th(F_m) = Th(F_n)$ for $m, n \geq 2$, and secondly that the elementary theory of a free group is decidable. In [32]-[38], Sela independently proved this first result. Though the techniques of Kharlampovich and Myasnikov are primarily algebraic while those of Sela are largely geometric, both approaches to the conjecture rely heavily on an understanding of equations in free groups; in particular, they utilize variations on techniques developed by Makanin in [23] that were used by Razborov in [30] to describe the solution sets to arbitrary systems of equations in a free group in terms of what is now known as Makanin-Razborov diagrams.

In recent years, there has been an effort to generalize the results and equation-related groundwork in understanding the elementary theory of free groups to hyperbolic and relatively hyperbolic analogues. This includes the construction of Makanin-Razborov diagrams for hyperbolic and toral relatively hyperbolic groups ([31] and [14] respectively) as well as the several decidability results including

- Given torsion-free hyperbolic groups Γ_1, Γ_2, it is decidable whether $Th(\Gamma_1) = Th(\Gamma_2)$ [39, Thm. 7.11].
- The elementary theory of a torsion-free hyperbolic group is decidable [18, Thm. 3].

This push to generalize serves as motivation to achieve some groundwork understanding of equations over an even broader class of hyperbolic-like groups. The first project presented in this thesis produces results to this end for the class of acylindrically hyperbolic groups defined below.

1.2.1 Acylindrically hyperbolic groups

Let G be group acting on a metric space (S,d). Note that in this paper, all actions of groups on metric spaces are assumed to be isometric.

Definition 1.2.1. The action of G on S is called *acylindrical* if for each $\varepsilon > 0$ there exist $R,N > 0$ such that for every pair of points $x,y \in S$ with $d(x,y) \geq R$, there are at most N elements $g \in G$ such that

$$d(x,gx) \leq \varepsilon \quad \text{and} \quad d(y,gy) \leq \varepsilon$$

For example, every geometric (i.e. proper and cobounded) action is acylindrical. However, acylindricity is a much weaker condition. E.g., every action on a bounded space is acylindrical.

Definition 1.2.2. If S is a hyperbolic space, then the action of G on S is called *elementary* if the limit set of G on the Gromov boundary ∂S contains at most 2 points.

A group G is called *acylindrically hyperbolic* if it admits a non-elementary acylindrical action on a hyperbolic space. (Further preliminaries concerning groups acting acylindrically on hyperbolic spaces, including other equivalent definitions of acylindrically hyperbolic groups, can be found in Subsection 2.1.3. For a comprehensive discussion of acylindrically hyperbolic groups, see [27].) The class of acylindrically hyperbolic groups generalizes not only non-elementary hyperbolic groups (where the usual action of the group on its own Cayley graph corresponding to some finite generating set constitutes a non-elementary acylindrical action on a hyperbolic space), but also non-elementary relatively hyperbolic groups. Other important examples of acylindrically hyperbolic groups include infinite mapping class groups of punctured closed surfaces, $Out(F_n)$ for $n \geq 2$, directly indecomposable non-cyclic right angled Artin groups, most 3-manifold groups (see [27, Appendix] for details), and groups of deficiency ≥ 2 [28].

1.2.2 Additional definitions

In order to formally state the results of Project 1, we need the following definitions.

Definition 1.2.3. Given an acylindrically hyperbolic group G, a subgroup $H \leq G$ is called *non-elementary* if for some acylindrical action of G on a hyperbolic space S, the action of H on S is non-elementary.

Example 1.2.4. Let $G = H * \mathbb{Z}$, where $\mathbb{Z} = \langle z \rangle$ and $H = F(x,y)$ is a free group of rank 2. Then the action of G on the Bass-Serre tree associated to the free product structure is acylindrical and non-elementary, but the induced action of H is elementary. However, H is a non-elementary subgroup of G since there exists another acylindrical action of G on a hyperbolic space (namely the action of G on its Cayley graph with respect to the generating set $\{x,y,z\}$) such that H acts non-elementarily.

Definition 1.2.5. The *Zariski topology* (or *verbal topology* as in [5]) on G is defined by taking the collection of primitive solution sets to be a sub-basis for the closed sets of the topology. That is, each Zariski-closed set of G is of the form

$$\cap_{i \in I} S_i$$

for some indexing set I, where for each $i \in I$, the set S_i is a finite union of primitive solution sets corresponding to (single-variable) group equations with coefficients in G. Note that in general, the Zariski topology is not a group topology.

The Zariski topology (and its more general form in [2]) is useful in bringing notions from algebraic geometry to the group theoretical setting in order to aid in the study of equations over groups. The topology is also useful in the study of topological groups: Zariski-closed sets are closed in every T_0 group topology, and, in the case of countable groups, the Zariski-closed sets are the only such sets [24]. For some recent applications to topological groups, see [21] and references therein.

The following definition is motivated by the standard notion of an algebraic subgroup in algebraic geometry.

Definition 1.2.6. A Zariski-closed subgroup (or more generally, a subset) of G is called *algebraic*.

Example 1.2.7. For any subgroup $H \leq G$, the centralizer $C_G(H)$ is algebraic, as

$$C_G(H) = \cap_{h \in H} \{x \in G \,|\, [x,h] = 1\}.$$

1.2.3 Results of Project 1

In [19], Kharlampovich and Myasnikov show that given a torsion-free non-elementary hyperbolic group G, any proper, first-order definable subgroup $H \leq G$ is cyclic. In the language of [2], G is a domain, so any Zariski-closed subset of G may be expressed as an intersection of primitive solution sets. Furthermore, since G is equationally Noetherian ([39, Thm. 1.22]), this intersection may be taken to be finite, so algebraic subgroups of G are first-order definable, and in particular, the structural result from [19] for definable subgroups of G holds for algebraic subgroups. In Project 1, we obtain a structural result for algebraic subgroups in the more general case when G is an acylindrically hyperbolic group, specifically

Theorem 1.2.8. *Suppose that G is an acylindrically hyperbolic group and that $H \leq G$ is non-elementary. Then H is algebraic if and only if there exists a finite subgroup K of G such that $C_G(K) \leq H \leq N_G(K)$.*

It should be noted that we actually prove a stronger version of the forward implication of Theorem 1.2.8 (see Theorem 2.2.11) which is a result about the Zariski closure of non-elementary subgroups of acylindrically hyperbolic groups. In the course of proving Theorem 1.2.8, we obtain a technical result (Proposition 2.2.1) which seems to be of independent interest.

If H is an algebraic subgroup of a torsion-free non-elementary hyperbolic group G, then either the action of H on the Cayley graph of G is elementary (in which case H is either $\{1\}$ or infinite cyclic) or the action is non-elementary and $K = \{1\}$ so that $H = G$. Thus Theorem 1.2.8 recovers the structural result for algebraic subgroups implied by the result of Kharlampovich and Myasnikov in [19].

Theorem 1.2.8 also yields the following corollaries, which we prove in Section 2.3:

Corollary 1.2.9. *Let A and B be non-trivial groups, and let H be an algebraic subgroup of $A * B$. Then at least one of the following holds:*

(a) *H is either infinite cyclic or isomorphic to D_∞, the infinite dihedral group.*

(b) *H is conjugate to a subgroup of either A or B.*

(c) *$H = A * B$.*

In the case of torsion-free relatively hyperbolic groups, we obtain the following.

Corollary 1.2.10. *Let G be a torsion-free relatively hyperbolic group with peripheral subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$, and let $H \leq G$ be an algebraic subgroup. Then at least one of the following holds:*

(a) *$H = G$.*

(b) *H is cyclic.*

(c) *H is conjugate to a subgroup of some H_λ.*

Furthermore, if (c) holds for an abelian H_λ, then either $H = \{1\}$ or H is conjugate to H_λ.

Important examples of torsion-free relatively hyperbolic groups with abelian peripheral subgroups (the so-called toral relatively hyperbolic groups) include limit groups in the sense of Sela and fundamental groups of hyperbolic knot compliments. Limit groups are hyperbolic relative to their maximal abelian non-cyclic subgroups (see [7, Thm. 0.3]), while fundamental groups of hyperbolic knot compliments are hyperbolic relative to free abelian subgroups of rank 2 (cusp subgroups; see [10]).

Corollary 1.2.11. *Let G be an acylindrically hyperbolic group and let $H_1 \leq H_2 \leq H_3 \leq \dots$ be an ascending chain of algebraic subgroups of G. Then either*

(a) for each acylindrical action of G on a hyperbolic space S, the subgroup $\cup_{i \in \mathbb{N}} H_i$ acts on S with bounded orbits (in particular, each H_i is elliptic), or

(b) the chain stabilizes.

Note that in general, it is actually possible for an acylindrically hyperbolic group to have an ascending chain of (elliptic) algebraic subgroups which does not stabilize. Example 2.3.4 provides a construction of such a group.

1.3 Project 2: A mixed identity-free elementary amenable group

Let $F = F(x, y, \dots)$ denote the free group with basis $\{x, y, \dots\}$. A group G is said to satisfy the *mixed identity* $w = 1$ for some $w \in G * F$ if for every homomorphism $\varepsilon : G * F \to G$ which is identical on G, the image of w under ε is trivial. The mixed identity $w = 1$ is called *nontrivial* if w is a nontrivial element of $G * F$. In the framing of equations over groups, a mixed identity $w = 1$ for some $w \in G * F$ may be written as an equation

$$w(x_1, \dots, x_n) = 1$$

with coefficients in G and finitely many variables $x_1, \dots, x_n$ which are elements of some basis for F such that every n-tuple of elements in G is a solution.

Mixed identities may be viewed as generalizations of the usual group identites. For example, while any abelian group satisfies the nontrivial identity $[x, y] = 1$, any group G with nontrivial center $Z(G)$ satisfies the nontrivial mixed identity $[x, g] = 1$ for any element $g \in Z(G) \setminus \{1\}$. Importantly, a group can satisfy a nontrivial mixed identity while failing to satisfy any nontrivial identity, as shown below.

Example 1.3.1. Let A and B be nontrivial groups. Then direct product $G = A \times B$ satisfies the mixed identity $[[x,a],b] = 1$ which is nontrivial for any choice of $a \in A \backslash \{1\}$ and $b \in B \backslash \{1\}$. If, furthermore, A does not satisfy any nontrivial identity, then G also fails to satisfy any nontrivial identity, since if $w = 1$ holds in G for some $w \in F$, it must also hold in any subgroup of G.

G is called *mixed identity-free* (hereafter abbreviated *MIF*) if it does not satisfy any nontrivial mixed identity. Groups that are MIF are subject to strict structural restrictions. For example, MIF groups do not decompose as nontrivial direct products (as demonstrated above) and have infinite conjugacy class property. For finitely generated groups, the property of being MIF implies infinite girth. MIF groups also resemble free products from the model theoretic point of view, i.e. a countable group G is MIF if and only if G and $G * F_n$ are universally equivalent as G-groups for all $n \in \mathbb{N}$. (For proofs, see [17, Prop. 5.3, 5.4].)

The action of a group G on a set S is called *k-transitive* if $|S| \geq k$ and for any two k-tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ of distinct elements of S, there exists $g \in G$ such that $ga_i = b_i$. G is called *highly transitive* if there exists a set S such that G admits an action on S that is faithful and k-transitive for each $k \in \mathbb{N}$. As part of [17], Hull and Osin further demonstrate structural restrictions imposed by the MIF condition by proving a dichotomy for highly transitive groups, namely

Theorem 1.3.2. *([17, Thm. 1.6]) Let G be a highly transitive countable group. Then exactly one of the following holds. Either*

(a) G contains a normal subgroup isomorphic to $Alt(\mathbb{N})$*, or*

(b) G is MIF.

At the conclusion of their paper, Hull and Osin ask a number of follow-up questions to their results, including a call for an example of a finitely generated MIF amenable group. They also suggest that the elementary amenable lacunary hyperbolic groups constructed in [26] are reasonable candidates for such a group. In Project 2, we fulfill their request by making an appropriate modification to this construction. Before formally stating the results of Project 2, we provide a brief discussion of lacunary hyperbolic groups below.

1.3.1 Lacunary hyperbolic groups

A well-known result of Gromov [13] states that a finitely generated group is hyperbolic if and only if all of its asymptotic cones are $\mathbb{R}$-trees. In a generalization of this asymptotic cone characterization of hyperbolic groups, Ol'shanskii, Osin, and Sapir define a *lacunary hyperbolic group* to be a finitely generated group with at least one asymptotic cone that is an $\mathbb{R}$-tree [26]. (It should be noted that this generalization is indeed strict. See [40] for an example of a finitely generated group such that at least one but not all asymptotic cones are $\mathbb{R}$-trees.) Equivalently (via [26, Thm. 3.3]), lacunary hyperbolic groups may also be defined as follows.

Definition 1.3.3. Let G and H be groups with a homomorphism $\varepsilon : G \to H$, and let S a generating set for G. Provided ε is not injective, the *injectivity radius* of ε with respect to S is the maximal radius of a ball in the Cayley graph $\Gamma(G,S)$ about 1_G on which ε is injective. If ε is injective, then the injectivity radius of ε is infinity.

Definition 1.3.4. A group G is called *lacunary hyperbolic* if there exists of groups G_i with corresponding finite generating sets S_i and epimorphisms $\varepsilon_i : G_i \twoheadrightarrow G_{i+1}$ such that the following conditions hold.

(i) G is the direct limit of the sequence

$$G_0 \overset{\varepsilon_0}{\twoheadrightarrow} G_1 \overset{\varepsilon_1}{\twoheadrightarrow} G_2 \overset{\varepsilon_2}{\twoheadrightarrow} \dots$$

(ii) $\varepsilon(S_i) = S_{i+1}$.

(iii) Each G_i is δ_i-hyperbolic with respect to the generating set S_i.

(iv) Let $r(\varepsilon_i)$ denote the injectivity radius of ε_i with respect to the generating set S_i. Then $\delta_i = o(r(\varepsilon_i))$.

This direct limit characterization of lacunary hyperbolicity provides a clear road map for the construction of interesting example groups— attempt to pass a particular property to G via the choice of G_i's and check to ensure that conditions (ii)-(iv) are satisfied. This is the basic proof arc for the construction of the elementary amenable lacunary hyperbolic groups in [26, Section 3.5]. In Project 2, we follow this same arc, making appropriate adjustments to produce the following results.

1.3.2 Results of Project 2

Theorem 1.3.5. *There exists a* 2*-generated elementary amenable group which is MIF.*

Moreover, as a byproduct of the construction, we obtain the following:

Theorem 1.3.6. *For each prime p, there exists a locally finite p-group which is MIF.*

We conclude Project 2 with two brief discussion sections. The first section examines two other reasonable candidates for examples of finitely generated MIF amenable groups, the Grigorchuk group and the identity-free amenable groups of infinite girth constructed by Akhmedov in [1], and shows that in each case, the group in question satisfies a nontrivial mixed identity. The second section contextualizes the result of Project 2 in terms of a larger question asked by Hull and Osin in [17], namely "Does there exist a countable amenable highly transitive subgroup $A \leq Sym(\mathbb{N})$ such that $A \cap Alt(\mathbb{N}) = \{1\}$?"

Chapter 2

ALGEBRAIC SUBGROUPS OF ACYLINDRICALLY HYPERBOLIC GROUPS

2.1 Project 1 Preliminaries

In this section, we provide the remaining preliminaries necessary to prove Theorem 1.2.8.

2.1.1 Algebraic subgroups

Let G be a group, and let $\langle x\rangle$ be the infinite cyclic group generated by x. Notice that for two distinct elements $u(x), v(x) \in G * \langle x\rangle$, the equations $u(x) = 1$ and $v(x) = 1$ may have the same solution set. (For example, let $v(x) = w(x)^{-1}u(x)w(x)$ for any $w(x) \in G * \langle x\rangle$.) In this case we will regard the equations $u(x) = 1$ and $v(x) = 1$ as equivalent, as they both define the same subbasic closed set of the Zariski topology.

It is helpful, given a (not necessarily finite) generating set X of G, to view an element $w(x) \in G * \langle x\rangle$ as a word in the elements of $X \cup \{x\}$ and their inverses. In particular, note that for any cyclic reduction $w'(x)$ of the word $w(x)$, $w(x) = 1$ and $w'(x) = 1$ are the same equation and that therefore, we may always assume that the word on the left-hand side of an equation is cyclically reduced.

Remark 2.1.1. Viewing each $w(x) \in G * \langle x\rangle$ as a word in $(X \cup \{x\})^{\pm 1}$ allows us to observe that for any $g \in G$, the map $G \to G$ given by left (similarly, right) multiplication by g is a homeomorphism with respect to the Zariski topology. In particular, for each $g \in G$ and each word $w(x) \in G * \langle x\rangle$, let $w(g^{-1}x)$ be the word obtained from $w(x)$ by replacing each instance of the letter x with $g^{-1}x$ and each instance of the letter x^{-1} with $(g^{-1}x)^{-1} = x^{-1}g$, and note that if S is the solution set of the equation $w(x) = 1$, then gS is the solution set of the equation $w(g^{-1}x) = 1$.

Corollary 2.1.2. *If a subgroup $H \leq G$ contains a finite index algebraic subgroup A, then H is also algebraic.*

Proof. H is a finite union of cosets $h_1A, \ldots, h_nA$ ($h_1, \ldots, h_n \in H$), each of which is algebraic by Remark 2.1.1. $\square$

2.1.2 Hyperbolically embedded subgroups

Let G be a group, and let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a collection of subgroups of G. Given a subset $X \subseteq G$ such that G is generated by $X \cup (\cup_{\lambda \in \Lambda} H_\lambda)$, let $\Gamma(G, X \sqcup \mathscr{H})$ denote the Cayley graph of G whose edges are labeled by letters from the alphabet $X \sqcup \mathscr{H}$, where

$$\mathscr{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda.$$

It is important to note the use of disjoint unions here. In particular, if $g \in G$ appears multiple times in $X \sqcup \mathscr{H}$, then we include edges in $\Gamma(G, X \sqcup \mathscr{H})$ corresponding to *each* of the distinct copies of g. We think of the Cayley graphs $\Gamma(H_\lambda, H_\lambda)$ as complete subgraphs of $\Gamma(G, X \sqcup \mathscr{H})$.

Definition 2.1.3. ([8, Defn. 4.2]) For each $\lambda \in \Lambda$, define the relative metric $\widehat{d_\lambda} : H_\lambda \times H_\lambda \to [0, \infty]$ as follows. A (combinatorial) path p in $\Gamma(G, X \sqcup \mathscr{H})$ is called λ*-admissible* if it contains no edges of $\Gamma(H_\lambda, H_\lambda)$. Importantly, p may still contain vertices in $\Gamma(H_\lambda, H_\lambda)$ as well as edges labeled by letters from H_λ (provided those edges are not in $\Gamma(H_\lambda, H_\lambda)$). For each $h, k \in H_\lambda$, define $\widehat{d_\lambda}(h, k)$ to be the length of a shortest λ-admissible path in $\Gamma(G, X \sqcup \mathscr{H})$ connecting h to k if such a path exists. Otherwise, let $\widehat{d_\lambda}(h, k) = \infty$. It is easy to see that $\widehat{d_\lambda}$ satisfies the triangle inequality. The relative metric $\widehat{d_\lambda}$ is extended to G by setting

$$\widehat{d_\lambda}(f, g) := \begin{cases} \widehat{d_\lambda}(f^{-1}g, 1), & \text{if } f^{-1}g \in H_\lambda; \\ \infty, & \text{otherwise.} \end{cases}$$

Remark 2.1.4. The extension of $\widehat{d_\lambda}$ to G is a matter of notational convenience for the statement and use of Lemma 2.1.8.

Definition 2.1.5. ([8, Defn. 4.25]) Let G be a group and $X \subseteq G$. A collection of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of G is said to be *hyperbolically embedded in G with respect to X* (notated $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$) if the following conditions hold:

(a) G is generated by $X \cup (\cup_{\lambda\in\Lambda} H_\lambda)$

(b) The Cayley graph $\Gamma(G, X \sqcup \mathscr{H})$ is hyperbolic.

(c) For each $\lambda \in \Lambda$ the metric space $(H_\lambda, \widehat{d_\lambda})$ is proper, i.e., any ball of finite radius contains only finitely many elements.

Furthermore, $\{H_\lambda\}_{\lambda\in\Lambda}$ is said to be *hyperbolically embedded* in G (notated $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h G$) if there exists $X \subseteq G$ such that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$.

Lemma 2.1.6. *([8, Cor. 4.27]) Let G be a group $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, $X_1, X_2 \subseteq G$ such that*

$$G = \langle X_1 \cup (\cup_{\lambda\in\Lambda} H_\lambda)\rangle = \langle X_2 \cup (\cup_{\lambda\in\Lambda} H_\lambda)\rangle$$

and $|X_1 \triangle X_2| < \infty$. Then $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X_1)$ if and only if $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X_2)$.

Observe in particular that if $\{H_\lambda\}_{\lambda\in\Lambda}$ is hyperbolically embedded in G with respect to some X, then $\{H_\lambda\}_{\lambda\in\Lambda}$ is also hyperbolically embedded in G with respect to any set X' obtained from X by adding finitely many elements of G.

Definition 2.1.7. ([29, Defns. 2.19, 2.20]) Let $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$ and let q be a path in $\Gamma(G, X \sqcup \mathscr{H})$. A non-trivial subpath p of q is called an H_λ*-subpath* if the label of p is a word in the alphabet H_λ. An H_λ-subpath p of q is called an H_λ*-component* if it is not contained in a longer H_λ-subpath of q, and, in the case that q is loop, p is not contained in a longer H_λ-subpath of any cyclic shift of q.

Two H_λ-components p_1, p_2 of q are called *connected* if there exists a path c in $\Gamma(G, X \sqcup \mathscr{H})$ that connects some vertex of p_1 to some vertext of p_2 and the label of c is a word consisting only of letters from H_λ. Algebraically, this means that all vertices of

p_1 and p_2 belong to the same left coset of H_λ. An H_λ-component p of q is called *isolated* in q if it is not connected to any other H_λ-component of q.

For a path p in $\Gamma(G, X \sqcup \mathcal{H})$, let p_- and p_+ denote the initial and terminal vertices of p respectively. The following lemma is a simplification of [8, Prop. 4.14] and is integral to the technique of the main proof of Project 1:

Lemma 2.1.8. *Let $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$. Then there exists a constant $C > 0$ such that for any n-gon p with geodesic sides in $\Gamma(G, X \sqcup \mathcal{H})$, any $\lambda \in \Lambda$, and any isolated H_λ-component a of p, we have $\widehat{d}_\lambda(a_-, a_+) \leq Cn$.*

2.1.3 Groups acting on hyperbolic spaces

Definition 2.1.9. Given a group G acting on a hyperbolic space S, an element $g \in G$ is called *loxodromic* if the map $\mathbb{Z} \to S$ given by $n \mapsto g^n s$ is a quasi-isometric embedding for some (equivalently, any) $s \in S$. Each loxodromic element $g \in G$ has exactly two limit points $g^{\pm\infty}$ on the Gromov boundary ∂S. A pair of loxodromic elements $g, h \in G$ are called *independent* if the sets $\{g^{\pm\infty}\}$ and $\{h^{\pm\infty}\}$ are disjoint.

The following definition is due to Bestvina and Fujiwara in [3]:

Definition 2.1.10. For a group G acting on a metric space S, an element $h \in G$ satisfies the *weak proper discontinuity* condition (or h is a *WPD element*) if for each $\varepsilon > 0$ and each $x \in S$ there exists $M \in \mathbb{N}$ such that

$$|\{g \in G \,|\, d(x, gx) < \varepsilon, d(h^M x, gh^M x) < \varepsilon\}| < \infty$$

Let $\mathscr{L}_{WPD}(G,S)$ denote the set of elements $g \in G$ which are loxodromic WPD with respect to the action of G on S. It is easy to see that if G acts acylindrically (as in Definition 1.2.1) and S is a hyperbolic space, then every loxodromic element is a WPD element.

Lemma 2.1.11. *([8, Lem. 6.5, Cor. 6.6]) Let G be a group acting on a hyperbolic space S. Each element $h \in \mathscr{L}_{WPD}(G,S)$ is contained in a unique maximal virtually cyclic subgroup of G which is denoted by $E_G(h)$. Moreover, for every $g \in G$ the following are equivalent:*

(i) $g \in E_G(h)$.

(ii) $gh^n g^{-1} = h^{\pm n}$ *for some* $n \in \mathbb{N}$.

(iii) $gh^k g^{-1} = h^l$ *for some* $k,l \in \mathbb{Z}\backslash\{0\}$.

Corollary 2.1.12. *For each $h \in \mathscr{L}_{WPD}(G,S)$, the subgroup $E_G(h)$ is algebraic.*

Proof. It is not hard to see that $\langle h \rangle$ is a finite index subgroup of $E_G(h)$ so that $E_G(h)$ is a finite union of cosets $k_1\langle h \rangle, \ldots, k_n\langle h \rangle$ for some $k_1, \ldots, k_n \in E_G(h)$. Noting additionally that $C_G(h) \subseteq E_G(h)$ by (ii) of Lemma 2.1.11, we have

$$E_G(h) = \bigcup_{i=1}^{n} k_i \cdot C_G(h) = \bigcup_{i=1}^{n} k_i \cdot \{x \in G \,|\, [x,h] = 1\} = \bigcup_{i=1}^{n} \{x \in G \,|\, [k_i^{-1}x, h] = 1\}.$$

□

Theorem 2.1.13. *([27, Thm. 1.1]) Let G be a group acting acylindrically on a hyperbolic space. Then G satisfies exactly one of the following three conditions:*

(a) G has bounded orbits.

(b) G is virtually cyclic and contains a loxodromic element.

(c) G contains infinitely many independent loxodromic elements.

If G has bounded orbits, G is called *elliptic*.

Remark 2.1.14. In light of Definitions 1.2.2 and 2.1.9, Theorem 2.1.13 tells us that if a group G acts acylindrically on a hyperbolic space S, then the action is non-elementary if and only if G is not virtually cyclic and contains at least one loxodromic element.

Theorem 2.1.15. *([27, Thm. 1.2]) For any group G, the following conditions are equivalent:*

(i) *G admits a non-elementary acylindrical action on a hyperbolic space.*

(ii) *There exists a generating set X of G such that the corresponding Cayley graph* $\Gamma(G,X)$ *is hyperbolic,* $|\partial\Gamma(G,X)| > 2$*, and the natural action of G on* $\Gamma(G,X)$ *is acylindrical.*

(iii) *G is not virtually cyclic and admits an action on a hyperbolic space such that at least one element of G is loxodromic and satisfies the WPD condition.*

(iv) *G contains a proper, infinite, hyperbolically embedded subgroup.*

Definition 2.1.16. A group G is called *acylindrically hyperbolic* if it satisfies one of the four equivalent conditions above.

2.2 Proof of Theorem 1.2.8

2.2.1 Constructing a cobounded action

The proof of Theorem 1.2.8 relies on the existence of arbitrarily large finite collections of loxodromic elements in the non-elementary subgroup H with certain useful properties. Lemma 2.2.9 guarantees these elements, provided we can find a generating set of G satisfying the requisite assumptions. To find such a generating set, we prove Proposition 2.2.1, which also seems to be of independent interest.

Proposition 2.2.1. *Suppose that a group G acts on a hyperbolic space S and that H is a subgroup of G such that H is not virtually cyclic and* $H \cap \mathscr{L}_{WPD}(G,S) \neq \emptyset$*. Then there exists a generating set X of G such that the Cayley graph* $\Gamma(G,X)$ *is hyperbolic, the action of G on* $\Gamma(G,X)$ *is acylindrical, and the action of H on* $\Gamma(G,X)$ *is non-elementary.*

To prove Proposition 2.2.1, we need three other results.

Lemma 2.2.2. *([8, Cor. 2.9]) Let G be a group acting on a hyperbolic space. For each loxodromic WPD element $h \in G$, we have $E_G(h) \hookrightarrow_h G$.*

Lemma 2.2.3. *([27, Thm. 5.4]) Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a finite collection of subgroups of G, $X \subseteq G$, $\mathscr{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$. Then there exists $Y \subseteq G$ such that $X \subseteq Y$ and the following conditions hold:*

(a) $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,Y)$. In particular, the Cayley graph $\Gamma(G, Y \sqcup \mathscr{H})$ is hyperbolic.

(b) The action of G on $\Gamma(G, Y \sqcup \mathscr{H})$ is acylindrical.

Lemma 2.2.4. *([8, Cor. 6.12]) Let G be a group, $X \subseteq G$, $H \hookrightarrow_h (G,X)$ a proper, infinite subgroup. Then for every $a \in G \backslash H$, there exists $k \in H$ such that ak is loxodromic and satisfies WPD with respect to the action of G on $\Gamma(G, X \sqcup H)$.*

If, in addition, H is finitely generated and contains an element h of infinite order, then we can choose k to be a power of h.

We are now ready to prove Proposition 2.2.1.

Proof of Proposition 2.2.1. Let $h \in H \cap \mathscr{L}_{WPD}(G,S)$. By Lemma 2.2.2 there exists $Z \subseteq G$ such that $E_G(h) \hookrightarrow_h (G,Z)$. By Lemma 2.2.3 there exists $Y \subseteq G$ such that $Z \subseteq Y$ and

(a) $E_G(h) \hookrightarrow_h (G,Y)$. In particular, the Cayley graph $\Gamma(G, Y \sqcup E_G(h))$ is hyperbolic.

(b) The action of G on $\Gamma(G, Y \sqcup E_G(h))$ is acylindrical.

Now $E_G(h)$ is clearly infinite (since h is of infinite order), and $H \backslash E_G(h) \neq \emptyset$ (since H is not virtually cyclic). Furthermore, since $E_G(h)$ is virtually cyclic by definition, it is finitely generated. Therefore, we may apply Lemma 2.2.4 to $E_G(h) \hookrightarrow_h (G,Y)$ and any element $a \in H \backslash E_G(h)$ to conclude that there exists $n \in \mathbb{Z}$ such that ah^n is loxodromic with respect to the action of G on $\Gamma(G, Y \sqcup E_G(h))$. Then since $a, h \in H$, we have that $ah^n \in H$, so H contains an element which is loxodromic with respect to the action of G on $\Gamma(G, Y \sqcup E_G(h))$. By Remark 2.1.14, the fact that H is not virtually cyclic and contains a loxodromic element with respect to the action of G on $\Gamma(G, Y \sqcup E_G(h))$ implies that the action of H on $\Gamma(G, Y \sqcup E_G(h))$ is non-elementary.

Set $X = Y \cup E_G(h)$ and observe that the hyperbolicity of the Cayley graph $\Gamma(G, X)$, the acylindricity of the the action of G on $\Gamma(G, X)$, and the fact that the action of H on $\Gamma(G, X)$ is non-elementary all follow from the G-equivariant quasi-isometry $\Gamma(G, Y \sqcup E_G(h)) \to \Gamma(G, X)$ given by fixing vertices and unifying duplicate copies of edges wherever necessary. $\square$

The following is the first part of [16, Lem. 5.5] which, together with Proposition 2.2.1, yields a useful lemma:

Lemma 2.2.5. *Let Let G be a group and $X \subseteq G$ be a generating set of G such that the Cayley graph $\Gamma(G, X)$ is hyperbolic and the action of G on $\Gamma(G, X)$ is acylindrical. If $H \leq G$ acts non-elementarily on $\Gamma(G, X)$, then there exists a unique maximal finite subgroup of G which is normalized by H, denoted $K_G(H)$.*

Combining this result with Proposition 2.2.1, we can obtain $K_G(H)$ for any non-elementary subgroup H.

Lemma 2.2.6. *Suppose that G is an acylindrically hyperbolic group and H is a non-elementary subgroup of G. Then there exists a unique maximal finite subgroup of G which is normalized by H, denoted $K_G(H)$.*

Proof. Since G acts acylindrically on a hyperbolic space S such that the action of H on S is non-elementary, we have by Remark 2.1.14 that H is not virtually cyclic and $H \cap \mathscr{L}_{WPD}(G, S) \neq \emptyset$. So by Proposition 2.2.1, there exists generating set X of G such that the Cayley graph $\Gamma(G, X)$ is hyperbolic, the action of G on $\Gamma(G, X)$ is acylindrical, and the action of H on $\Gamma(G, X)$ is non-elementary. Then, by Lemma 2.2.5, there exists a unique maximal finite subgroup $K_G(H)$ of G which is normalized by H. $\square$

2.2.2 Proof of Theorem 1.2.8

We begin by proving a technical lemma:

Lemma 2.2.7. *Let G be a group, $\{H_\lambda\}_{\lambda\in\Lambda}$ a collection of subgroups of G, $\mathscr{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$, and $X \subseteq G$. Suppose that $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_h (G,X)$, $n \geq 2$, $g_1, g_2, \ldots, g_n \in X$, and $h_1, h_2, \ldots, h_n \in H_\lambda$ for some $\lambda \in \Lambda$ such that $g_1h_1g_2h_2\ldots g_nh_n = 1$. If for each i we have $\widehat{d_\lambda}(h_i, 1) > 2Cn$, where $\widehat{d_\lambda}$ is defined as in Definition 2.1.3 and C is the constant given by Lemma 2.1.8, then there exists $j \in \{1, \ldots, n\}$ such that $g_j \in H_\lambda$.*

Proof. Since $g_1h_1g_2h_2\ldots g_nh_n = 1$, there exists a geodesic $2n$-gon

$$p = a_1b_1a_2b_2\ldots a_nb_n$$

in the Cayley graph $\Gamma(G, X \sqcup \mathscr{H})$, where each side a_i is an edge labeled by $g_i \in X$ and each side b_i is an edge labeled by $h_i \in H_\lambda$ so that the closed loop p corresponds to the relation $g_1h_1g_2h_2\ldots g_nh_n = 1$. Then for each i, the side b_i is an H_λ-component of p.

It suffices to show that there exists $j \in \{1, \ldots, n\}$ such that the two H_λ-components adjacent to a_j are connected. If this is the case, then there exists an edge of $\Gamma(G, X \sqcup \mathscr{H})$ with label in H_λ which connects a_{j-} and a_{j+} so that $g_j \in H_\lambda$, finishing the proof.

Now for each i, we have $\widehat{d_\lambda}(h_i, 1) > 2Cn$, so Lemma 2.1.8 implies that none of the components $b_1, \ldots, b_n$ are isolated in p. Let b_r and b_s be connected H_λ-components such that the number, N, of sides of p between b_{r+} and b_{s-} is minimal. If $N = 1$, then choose a_j to be a_s, where $b_{r+} = a_{s-}$ and $b_{s-} = a_{s+}$.

Otherwise, there exists b_t which is an edge on the subpath of p (or cyclic shift of p) between b_{r+} and b_{s-}. Since b_r and b_s are connected, there exists an edge e of $\Gamma(G, X \sqcup \mathscr{H})$ with label in H_λ which connects b_{r+} and b_{s-}. Let p' be the geodesic $(N+1)$-gon formed by e and the sides of p between b_{r+} and b_{s-}. Then $N + 1 < 2n$, so

$$\widehat{d_\lambda}(h_t, 1) > 2Cn > C(N+1)$$

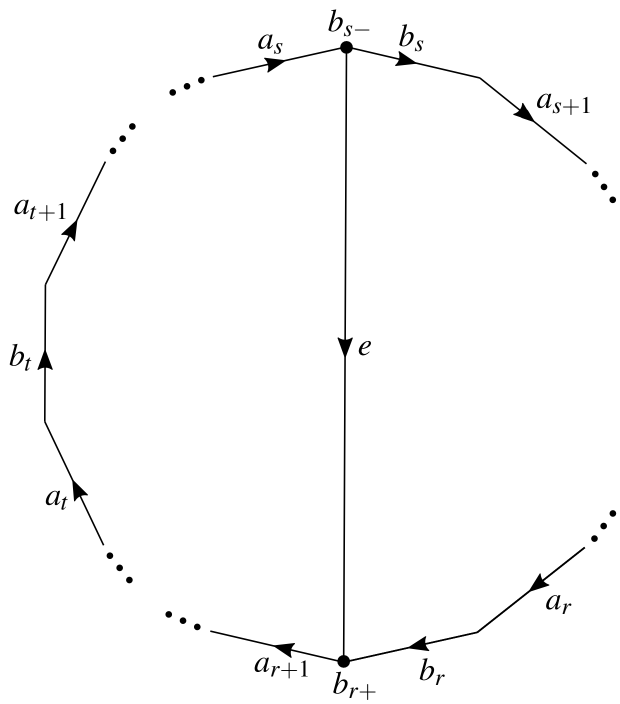

Figure 2.1: The $2n$-gon p with additional edge e

so that by Lemma 2.1.8, b_t is not an isolated H_λ-component of p'.

If b_t is connected to any other H_λ-component of p', then clearly N was not minimal, and if b_t is connected to e, then it is connected to both b_r and b_s, so again N was not minimal. Hence it must be that $N = 1$, finishing the proof.

□

We will also need two more results for the main proof of Project 1. Recall that two elements g, h of a group G are called *commensurable* if there exist $x \in G$ and $k, l \in \mathbb{Z} \backslash \{0\}$ such that $g^k = xh^l x^{-1}$ and called *non-commensurable* otherwise.

Lemma 2.2.8. *([8, Thm. 6.8]) Let G be a group acting on a hyperbolic space S. If $h_1, \dots, h_k$ are pairwise non-commensurable loxodromic WPD elements with respect to the action of G on S then there exists a set $X \subseteq G$ such that $\{E_G(h_1), \dots, E_G(h_k)\} \hookrightarrow_h (G, X)$.*

Lemma 2.2.9. *([16, Lem. 5.6]) Let Let G be a group and $X \subseteq G$ be a generating set of G such that the Cayley graph $\Gamma(G, X)$ is hyperbolic and the action of G on $\Gamma(G, X)$ is acylindrical. If $H \leq G$ acts non-elementarily on $\Gamma(G, X)$, then for each $k \in \mathbb{N}$, there exist pairwise* *non-commensurable loxodromic elements $a_1, \ldots, a_k \in H$ such that $E_G(a_i) = \langle a_i \rangle \times K_G(H)$.*

It will be helpful in the next proof (and later in the proof of Lemma 2.3.3) to have the following definition:

Definition 2.2.10. Let G be a group and $S \subseteq G$. We say that two equations $p(x) = 1$ and $q(x) = 1$ are *S-equivalent* if for each $s \in S$ we have that $p(s) = 1$ if and only if $q(s) = 1$.

We can now prove the following result:

Theorem 2.2.11. *Suppose that G is an acylindrically hyperbolic group and H is a non-elementary subgroup of G. Then the Zariski closure of H contains $C_G(K_G(H))$.*

Proof. Let V be the Zariski closure of H. Then

$$V = \cap_{\lambda \in \Lambda} V_\lambda$$

where Λ is some indexing set and each V_λ is a union of finitely many primitive solution sets. We want to show that $C_G(K_G(H)) \subseteq V_\lambda$ for each $\lambda \in \Lambda$. So let $p_1(x), \ldots, p_l(x)$ be elements of $G * \langle x \rangle$ such that

$$H \subseteq V_\lambda = \bigcup_{i=1}^{l} \{g \in G \,|\, p_i(g) = 1\}.$$

Without loss of generality, we may assume that for each i, $p_i(x)$ is cyclically reduced. Furthermore, we may assume (by taking a cyclic permutation of $p(x)$ if necessary) that $p_i(x)$ is of the form

$$p_i(x) = g_{1,i} x^{\alpha_{1,i}} g_{2,i} x^{\alpha_{2,i}} \ldots g_{m_i,i} x^{\alpha_{m_i,i}}$$

where $g_{1,i}, \ldots, g_{m_i,i} \in G$ and $\alpha_{1,i}, \ldots, \alpha_{m_i,i} \in \mathbb{Z}$, and if $m_i \geq 2$, then $g_{1,i}, \ldots, g_{m_i,i} \neq 1$ and $\alpha_{1,i}, \ldots, \alpha_{m_i,i} \neq 0$.

Set

$$k = l \cdot (\max\{m_1, \ldots, m_l\} + 1). \tag{2.1}$$

By assumption, G acts acylindrically on a hyperbolic space S such that the action of H on S is non-elementary. By Remark 2.1.14, H is not virtually cyclic and $H \cap \mathscr{L}_{WPD}(G,S) \neq \emptyset$, so by Proposition 2.2.1, there exists generating set W of G such that the Cayley graph $\Gamma(G,W)$ is hyperbolic, the action of G on $\Gamma(G,W)$ is acylindrical, and the action of H on $\Gamma(G,W)$ is non-elementary. Then, by Lemma 2.2.9, there exist pairwise non-commensurable elements $a_1, \ldots, a_k \in H$ which are loxodromic with respect to the action of G on $\Gamma(G,W)$ such that $E_G(a_i) = \langle a_i \rangle \times K_G(H)$. In particular, we have that

$$a_1, \ldots, a_k \in C_G(K_G(H))$$

and, since the elements $a_1, \ldots, a_k$ are pairwise non-commensurable,

$$E_G(a_i) \cap E_G(a_j) = K_G(H) \tag{2.2}$$

whenever $i \neq j$.

By Lemma 2.2.8, there exists a generating set X of G such that

$$\{E_G(a_1), \ldots, E_G(a_k)\} \hookrightarrow_h (G,X).$$

For each $i \in \{1, \ldots, l\}$ and each $n \leq m_i$, let $Z_i = \{g_{1,i}, \ldots, g_{m_i,i}\}$ and let

$$Y_{i,n} = \{g \in \langle Z_i \rangle \mid |g|_{Z_i} \leq 2^{m_i - n}\}.$$

I.e., $Y_{i,n}$ is the set of all elements of G which can be expressed as products of 2^{m_i-n} or fewer (not necessarily distinct) elements of $\{g_{1,i},\ldots,g_{m_i,i}\}$ and their inverses. Let

$$Y = \cup_{i=1}^{l} Y_{i,1}.$$

Since Y is finite, we may assume by Lemma 2.1.6 that X contains Y.

For each $i \in \{1,\ldots,k\}$, let $\widehat{d_i}$ denote the relative metric in the Cayley graph

$$\Gamma(G, X \sqcup (E_G(a_1) \sqcup \ldots \sqcup E_G(a_k))$$

which is induced by the hyperbolic embedding $\{E_G(a_1),\ldots,E_G(a_k)\} \hookrightarrow_h (G,X)$ and corresponds to $E_G(a_i)$ (see Definition 2.1.3). Since for each $i \in \{1,\ldots,k\}$, the element a_i has infinite order and the metric space $(E_G(a_i),\widehat{d_i})$ is proper, we may choose $\gamma \in \mathbb{N}$ large enough such that for any $\beta \in \mathbb{Z}\backslash\{0\}$ and any $i \in \{1,\ldots,k\}$,

$$\widehat{d_i}((a_i^{\gamma})^{\beta}, 1) > 2C \max\{m_1,\ldots,m_l\}$$

where C is the constant given by Lemma 2.1.8. By (2.1), there exists $j \in \{1,\ldots,l\}$ such that at least $m_j + 1$ elements of the set $\{a_1^{\gamma},\ldots,a_k^{\gamma}\}$ satisfy the equation $p_j(x) = 1$. Without loss of generality, the elements $a_1^{\gamma},\ldots,a_{m_j+1}^{\gamma}$ satisfy $p_j(x) = 1$.

Let Q be the subset of all $q(x) \in G * \langle x \rangle$ of the form

$$q(x) = h_1 x^{\beta_1} h_2 x^{\beta_2} \ldots h_N x^{\beta_N}$$

for some $N \le m_j$ and $\beta_1,\ldots,\beta_N \in \mathbb{Z}$ such that

(a) $h_1, h_2, \ldots, h_N \in Y_{j,N}$.

(b) $q(x) = 1$ is $C_G(K_G(H))$-equivalent to $p_j(x) = 1$ (see Definition 2.2.10).

Certainly Q is non-empty since $p_j(x)$ satisfies conditions (a), and (b). Let $q(x)$ be an element of Q whose corresponding value of N is minimal. We claim that $N = 1$ for $q(x)$.

Assuming that this is not the case, let $q(x) = h_1 x^{\beta_1} h_2 x^{\beta_2} \dots h_N x^{\beta_N}$ for some $N \geq 2$ where $h_1, \dots, h_N \in Y_{j,N}$ and $\beta_1, \dots, \beta_N \in \mathbb{Z}$. It follows that since $N \geq 2$ is minimal, we have that $h_1, \dots, h_N \neq 1$ and $\beta_1, \dots, \beta_N \neq 0$, since otherwise, there is a cyclic reduction of $q(x)$ which is strictly shorter, violating the minimality of N. For each $i \in \{1, \dots, m_j + 1\}$, we have that $a_i^{\gamma} \in C_G(K_G(H)) \cap H$ is a solution to the equation $p_j(x) = 1$ and hence by (c) to the equation $q(x) = 1$. Since

$$q(a_i^{\gamma}) = h_1 (a_i^{\gamma})^{\beta_1} h_2 (a_i^{\gamma})^{\beta_2} \dots h_N (a_i^{\gamma})^{\beta_N} = 1,$$

we have by Lemma 2.2.7 that there exists $r_i \in \{1, \dots, N\}$ for which $h_{r_i} \in E_G(a_i)$. Since this is true for each $i \in \{1, \dots, m_j + 1\}$ and since $N \leq m_j$, there exist $s, t \in \{1, \dots, m_j + 1\}$ with $s \neq t$ such that $r_s = r_t$ so that $h_{r_s} \in E_G(a_s) \cap E_G(a_t) = K_G(H)$ (see (2.2)).

Without loss of generality, we may assume that $r_s = 2$ (otherwise, take a cyclic permutation of $q(x)$). Then, since $h_2 \in K_G(H)$, the equation $q(x) = 1$ (and hence $p_j(x) = 1$) is $C_G(K_G(H))$-equivalent to $q'(x) = 1$ where

$$q'(x) = (h_1 h_2) x^{\beta_1 + \beta_2} \dots h_N x^{\beta_N}.$$

Furthermore, since $h_1, h_2, h_3, \dots, h_N \in Y_{j,N}$, we have that $(h_1 h_2), h_3, \dots, h_N \in Y_{j,N-1}$. Thus we have an element $q'(x) \in Q$ which contradicts our choice of $q(x)$ (namely the minimality of N). Therefore, $N = 1$ and so $q(x) = h_1 x^{\beta_1}$ for some $h_1 \in G$ and $\beta_1 \in \mathbb{Z}$.

It must also be that $\beta_1 = 0$, since otherwise $q(a_1^{\gamma}) = q(a_2^{\gamma}) = 1$ implies that there exists a non-zero integer β_1 such that $a_1^{\gamma\beta_1} = a_2^{\gamma\beta_1}$, showing that a_1 and a_2 are commensurable. Therefore $q(x) = h_1 \in G$ and hence it must be that $h_1 = 1$. The equation $q(x) = 1$ is trivially satisfied by all elements of $C_G(K_G(H))$, and therefore each $c \in C_G(K_G(H))$ also satisfies the equation $p_j(x) = 1$ by (c). Thus we obtain that

$$C_G(K_G(H)) \subseteq V_{\lambda} = \bigcup_{i=1}^{l} \{g \in G \,|\, p_i(g) = 1\}$$

as desired. □

We can now prove Theorem 1.2.8:

Proof of Theorem 1.2.8. To show the forward implication, observe that $C_G(K_G(H)) \leq H$ by Theorem 2.2.11 and that $H \leq N_G(K_G(H))$ by the definition of $K_G(H)$.

To show the reverse implication, observe that $C_G(K)$ is algebraic by Example 1.2.7 and that since $C_G(K) \leq H \leq N_G(K)$ and K is finite, it follows that $C_G(K)$ is of finite index in H. Then, by Corollary 2.1.2, H is algebraic. □

2.3 Applications

2.3.1 Free products

Proof of Corollary 1.2.9. Let T be the Bass-Serre tree corresponding to the graph of groups with vertex groups A and B and trivial edge group. It is easy to see that the natural action of $A*B$ on T is acylindrical. (This is a particular case of [**?** , Lem. 5.2].) By Theorem 1.2.8, either the action of H on T is elementary, or H acts non-elementarily and there exists a finite subgroup of $F \leq A*B$ such that $C_{A*B}(F) \leq H \leq N_{A*B}(F)$.

Before addressing each case, it will be helpful to recall the Kurosh Subgroup Theorem [22], which states that any subgroup K of $A*B$ decomposes as a free product

$$K = F(X) * (*_{i\in I} g_i^{-1} A_i g_i) * (*_{j\in J} h_j^{-1} B_j h_j) \tag{2.3}$$

where $X \subseteq G$ freely generates a subgroup of G, for all $i \in I$ we have $g_i \in A*B$ and $A_i \leq A$, and for all $j \in J$ we have $h_j \in A*B$ and $B_j \leq B$.

Assume first that H acts elementarily on T. Then by Theorem 2.1.13, either H is virtually cyclic and contains a loxodromic element or H acts on T with bounded orbits. If H is virtually cyclic, then since H decomposes as a free product as in (2.3), it must be that either

$$H \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong D_\infty$$

or H is equal to just a single factor of the free product in (2.3). In the latter case, either $H = F(X)$ (in which case H is cyclic), or H is conjugate to a subgroup of either A or B (which is not possible if H contains a loxodromic element since conjugates of A and B are vertex stabilizers of T and hence act with bounded orbits).

If H acts with bounded orbits, then by [4, Ch. 2, Cor. 2.8], H fixes a point in T and hence a vertex. Since vertex stabilizers of T are conjugates of A and B, H is conjugate to a subgroup of either A or B.

Now assume that H acts non-elementarily on T and there exists a finite subgroup of $F \leq A * B$ such that $C_{A*B}(F) \leq H \leq N_{A*B}(F)$. If F is trivial, then $H = A * B$. If F is non-trivial, then F decomposes as in (2.3), and in particular, F must be equal to some non-trivial $g_i^{-1}A_i g_i$ or $h_j^{-1}B_j h_j$. Without loss of generality, assume $F = g_i^{-1}A_i g_i \neq \{1\}$ for some $A_i \leq A$. We claim that in this case, $H \leq g_i^{-1}Ag_i$.

Let $f \in F \backslash \{1\}$. Since $H \leq N_{A*B}(F)$ and F fixes the vertex $g_i^{-1}A$ of T, we have for each $h \in H$ that

$$h^{-1}fh(g_i^{-1}A) = g_i^{-1}A$$

and thus

$$f(hg_i^{-1}A) = hg_i^{-1}A$$

so that f fixes the vertex $hg_i^{-1}A$ of T.

Now observe that each non-trivial element $g \in G \backslash \{1\}$ fixes at most one vertex of T. Indeed, if $g \in G$ fixes two vertices, then the unique path between those vertices is also fixed by g, so g stabilizes an edge and is therefore trivial. Since f is non-trivial, we must therefore have that $hg_i^{-1}A = g_i^{-1}A$ so that h fixes the vertex $g_i^{-1}A$ and is thus an element of $g_i^{-1}Ag_i$. So $H \leq g_i^{-1}Ag_i$ as desired. □

2.3.2 Torsion-free relatively hyperbolic groups

In this subsection, we prove Corollary 1.2.10, but to do so, we will need the two lemmas below:

Lemma 2.3.1. *Let A be an abelian group and let S be an algebraic subset of A. Suppose that there exists an infinite-order element $h \in A$ such that $|\langle h\rangle \cap S| = \infty$. Then $S = A$.*

Proof. It suffices to prove the statement in the case where S is the union of finitely many primitive solution sets, since if

$$S = \cap_{i\in I} S_i$$

where I is some indexing set and each S_i is a union of finitely many primitive solution sets, then we will have shown for each $i \in I$ that $S_i = A$, so indeed $S = A$. So let $p_1(x), \ldots, p_n(x)$ be elements of $A * \langle x\rangle$ such that

$$S = \bigcup_{i=1}^{n} \{a \in A \mid p_i(a) = 1\}.$$

Furthermore, since A is abelian, we may assume that for each $i \in \{1, \ldots, n\}$, the equation $p_i(x) = 1$ is of the form $a_i x^{m_i} = 1$ for some $a_i \in A$ and $m_i \in \mathbb{Z}$.

Since $|\langle h\rangle \cap S| = \infty$, there exists $i \in \{1, \ldots, n\}$ and $k, l \in \mathbb{N}$ with $k \neq l$ such that $a_i(h^k)^{m_i} = 1$ and $a_i(h^l)^{m_i} = 1$. Then $h^{km_i} = h^{lm_i}$, and since h has infinite order, it must be that $m_i = 0$. But then $a_i = 1$ and the equation $p_i(x) = 1$ is trivially satisfied by all elements of A so that $S = A$ as desired. □

Corollary 2.3.2. *All proper algebraic subgroups of abelian groups are torsion.*

Proof. If $H \leq A$ is an algebraic subgroup, then for each $h \in H$, either h has finite order or we may apply Lemma 2.3.1 to h to conclude that $H = A$. □

Lemma 2.3.3. *Suppose that G is a group, A is an abelian hyperbolically embedded subgroup of G, S is an algebraic subset of G, H is a subgroup of G contained in S, and there exists $g \in G$ such that $H \leq g^{-1}Ag$. Then either H is torsion or $g^{-1}Ag \subseteq S$.*

Proof. Since the Zariski topology is invariant under conjugation by Remark 2.1.1 and since conjugation preserves the orders of group elements, we may assume that $g = 1$. As in the proof of Lemma 2.3.1, it suffices to prove the lemma in the case when S is the union of finitely many primitive solution sets.

Assume that H is not torsion. The majority of the proof below is similar to the proof of Theorem 2.2.11.

Let $p_1(x), \ldots, p_l(x) \in G * \langle x \rangle$ such that

$$H \subseteq S = \bigcup_{i=1}^{l} \{k \in G \,|\, p_i(k) = 1\}.$$

We may assume as in the the proof of Theorem 2.2.11 that $p_i(x)$ is of the form

$$p_i(x) = g_{1,i} x^{\alpha_{1,i}} g_{2,i} x^{\alpha_{2,i}} \ldots g_{m_i,i} x^{\alpha_{m_i,i}}$$

where $g_{1,i}, \ldots, g_{m_i,i} \in G$ and $\alpha_{1,i}, \ldots, \alpha_{m_i,i} \in \mathbb{Z}$, and if $m_i \geq 2$, then $g_{1,i}, \ldots, g_{m_i,i} \neq 1$ and $\alpha_{1,i}, \ldots, \alpha_{m_i,i} \neq 0$. For each $i \in \{1, \ldots, l\}$ and each $n \leq m_i$, define $Y_{i,n}$ as in the proof of Theorem 2.2.11. Then $A \hookrightarrow_h (G, X)$ for some $X \subseteq G$, and we may assume by Lemma 2.1.6 that X contains $\cup_{i=1}^{l} Y_{i,1}$

Let $\widehat{d_A}$ denote the relative metric in the Cayley graph $\Gamma(G, X \sqcup A)$ induced by the hyperbolic embedding $A \hookrightarrow_h (G, X)$. Since H contains some element h of infinite order and the metric space $(A, \widehat{d_A})$ is proper, there exists a strictly ascending sequence of natural numbers $\gamma_1, \gamma_2, \gamma_3, \ldots$ such that for any $\beta \in \mathbb{Z} \backslash \{0\}$ and any $i \in \mathbb{N}$,

$$\widehat{d_A}((h^{\gamma_i})^{\beta}, 1) > 2C \max\{m_1, \ldots, m_l\}$$

where C is the constant given by Lemma 2.1.8. Since for each $i \in \mathbb{N}$ we have that $h^{\gamma_i} \in H \subseteq S$, there exists $j \in \{1, \ldots, l\}$ and a subsequence of $\{\gamma_i\}_{i \in \mathbb{N}}$ whose corresponding h^{γ_i} satisfy the equation $p_j(x) = 1$. Without loss of generality, take $\{\gamma_i\}_{i \in \mathbb{N}}$ to be this subsequence.

Let Q be the subset of all $q(x) \in G * \langle x \rangle$ of the form

$$q(x) = h_1 x^{\beta_1} h_2 x^{\beta_2} \dots h_N x^{\beta_N}$$

for some $N \leq m_j$ and $\beta_1, \dots, \beta_N \in \mathbb{Z}$ such that

(a) $h_1, h_2, \dots, h_N \in Y_{j,N}$.

(b) $q(x) = 1$ is A-equivalent to $p_j(x) = 1$ (see Definition 2.2.10).

Certainly Q is non-empty since $p_j(x)$ satisfies conditions (a) and (b). Let $q(x)$ be an element of Q whose corresponding value of N is minimal.

Since

$$\widehat{d_A}((h^{\gamma_1})^{\beta}, 1) > 2C \max\{m_1, \dots, m_l\},$$

we may argue as in the proof of Theorem 2.2.11 that if $N \neq 1$ for $q(x)$, then there exists $r \in \{1, \dots, N\}$ for which h_r is in A. Without loss of generality, $r = 2$. Letting

$$q'(x) = (h_1 h_2) x^{\beta_1 + \beta_2} \dots h_N x^{\beta_N},$$

we see that since A is abelian, $q'(x)$ is in Q, which contradicts the minimality of N. Thus it must be that $N = 1$ for $q(x)$.

Since $N = 1$, we have $h_1 (h^{\gamma_1})^{\beta_1} = 1$ so that $h_1 \in A$. Then $q(x) = 1$ is an equation with coefficients in A which is A-equivalent to $p_j(x) = 1$, so $\{k \in G \mid p_j(k) = 1\} \cap A$ is an algebraic subset of A. Since $h \in A$ has infinite order and $h^{\gamma_i} \in \{k \in G \mid p_j(k) = 1\} \cap A$ for each $i \in \mathbb{N}$, we have that

$$|\langle h \rangle \cap \{k \in G \mid p_j(k) = 1\} \cap A| = \infty,$$

so $\{k \in G \mid p_j(k) = 1\} \cap A = A$ by Lemma 2.3.1. Thus $A \subseteq S$ as desired. $\square$

We can now prove Corollary 1.2.10:

Proof of Corollary 1.2.10. Let $\mathscr{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda$. By [27, Thm. 5.2], for any finite relative generating set X of G, the action of G on $\Gamma(G, X \sqcup \mathscr{H})$ is acylindrical. If H is non-elementary, then G admits a non-elementary acylindrical action on a hyperbolic space and is thus acylindrically hyperbolic, so $H = G$ by Theorem 1.2.8. Otherwise, the action of H on $\Gamma(G, X \sqcup \mathscr{H})$ is elementary. If this is the case, then by Theorem 2.1.13, H is either elliptic or virtually cyclic (and hence cyclic).

So now let H be elliptic. We want to show that $H \leq g^{-1} H_\lambda g$ for some $g \in G, \lambda \in \Lambda$. Since G is torsion-free, we have by [29, Thm. 1.14] that every element of G which is not conjugate to an element of H_λ for some $\lambda \in \Lambda$ acts on $\Gamma(G, X \sqcup \mathscr{H})$ with unbounded orbits. Hence there exists $a \in (H \cap H_\lambda) \backslash \{1\}$ for some $\lambda \in \Lambda$ (unless $H = \{1\}$, in which case we are done). It suffices to show that every $b \in H$ belongs to the same H_λ.

If there exists $b \in H \backslash H_\lambda$, then by Lemma 2.1.6 we may assume that $b \in X$ so that $d_{X \sqcup \mathscr{H}}(1, b) = 1$. By [**?** , Lem. 4.4], there exists a finite subset $F_\lambda \subseteq H_\lambda$ such that if $h \in H_\lambda \backslash F_\lambda$, $b \in G \backslash H_\lambda$, and $d_{X \sqcup \mathscr{H}}(1, b) = 1$, then bh acts on $\Gamma(G, X \sqcup \mathscr{H})$ with unbounded orbits. Since a is non-trivial and thus of infinite order, we have that for large enough n, $a^n \notin F_\lambda$ so that ba^n acts on $\Gamma(G, X \sqcup \mathscr{H})$ with unbounded orbits. This contradicts the assumption that H is elliptic. This completes the proof of the first statement of Corollary 1.2.10.

To prove the second statement of the corollary, assume that H is conjugate to a subgroup of some abelian H_λ. Then $H_\lambda \hookrightarrow_h G$, and since H is torsion-free, we have by Lemma 2.3.3 that $H = \{1\}$ or H is conjugate to H_λ. □

2.3.3 Ascending chains of algebraic subgroups

In general, ascending chains of algebraic subgroups need not stabilize, even in acylindrically hyperbolic groups. For example, consider the following construction:

Example 2.3.4. Consider a group with presentation $\langle X | R \rangle$ that contains an infinite ascending chain of subgroups

$$K_1 \leq K_2 \leq \dots$$

where for each $i \in \mathbb{N}$, $K_i \neq K_{i+1}$. Let

$$H = \langle X, a, t_1, t_2, \dots \,|\, R, [X,a] = 1, [K_1, t_1] = 1, [K_2, t_2] = 1, \dots \rangle.$$

Then $G = H * \mathbb{Z}$ is acylindrically hyperbolic with non-elementary acylindrical action on the Bass-Serre tree corresponding to the graph of groups with vertex groups H and $\mathbb{Z}$ and trivial edge group. Furthermore, for each $i \in \mathbb{N}$,

$$K_i = \{x \in H * \mathbb{Z} \,|\, [x, t_i] = 1\} \cap \{x \in H * \mathbb{Z} \,|\, [x, a] = 1\}$$

so that $K_1 \leq K_2 \leq \dots$ is an ascending chain of algebraic subgroups of G that does not stabilize.

However, ascending chains of algebraic subgroups in acylindrically hyperbolic groups are subject to the property given by Corollary 1.2.11, whose proof is given below:

Proof of Corollary 1.2.11. Assume that condition (a) fails. Then there exists a hyperbolic space S such that G acts on S acylindrically and, by Theorem 2.1.13, $\cup_{i \in \mathbb{N}} H_i$ contains an element h which is loxodromic with respect to this action. Hence there exists some $I \in \mathbb{N}$ such that $h \in H_I$. Then exactly one of two things is true: either $\langle h \rangle \leq H_i \leq E_G(h)$ for all $i \geq I$, where $\langle h \rangle$ is a finite index subgroup of $E_G(h)$ (in which case the chain clearly stabilizes), or there exists $I' \geq I$ such that H_i acts non-elementarily on S for all $i \geq I'$.

In the latter case, we have by Theorem 1.2.8 that for each $i \geq I'$ there exists a finite subgroup $K_i \leq G$ such that

$$C_G(K_i) \leq H_i \leq N_G(K_i),$$

and in particular, we saw in the proof of Theorem 1.2.8 that $K_i = K_G(H_i)$. So in fact we have that

$$C_G(K_G(H_i)) \leq H_i \leq N_G(K_G(H_i)),$$

where $C_G(K_G(H_i))$ is a finite index subgroup of $N_G(K_G(H_i))$ since $K_G(H_i)$ is finite. Moreover, by definition, $K_G(H_{i+1}) \leq K_G(H_i)$ for all $i \geq I'$, and since $K_G(H_{I'})$ is finite, the sequence $\{K_G(H_i)\}_{i \geq I'}$ stabilizes. Hence there exists $I'' \geq I'$ such that

$$C_G(K_G(H_{I''})) \leq H_i \leq N_G(K_G(H_{I''}))$$

for all $i \geq I''$, where $C_G(K_G(H_{I''}))$ is a finite index subgroup of $N_G(K_G(H_{I''}))$, so the chain $H_1 \leq H_2 \leq H_3 \leq \dots$ must stabilize. □

It should be noted that the group G in Example 2.3.4 has an infinite ascending chain of algebraic subgroups because its free factor H also has such a chain. Is it the case then, that for relatively hyperbolic groups, all examples of infinite ascending chains of algebraic subgroups arise in a similar manner? In other words,

Question 2.3.5. Let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a collection of groups with the property that for each $\lambda \in \Lambda$, every ascending chain of algebraic subgroups in H_λ stabilizes, and let G be a relatively hyperbolic group with peripheral subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$. Is it true that every chain of algebraic subgroups of G stabilizes?

One possible route towards a positive answer is to try to prove that

(a) If H is a subgroup of G acting eliptically on a relative Cayley graph corresponding to the collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$, then either A is finite or A is conjugate into H_λ for some $\lambda \in \Lambda$, and

(b) if A is an algebraic subgroup of G such that $A \leq H_\lambda$ for some $\lambda \in \Lambda$, then A is algebraic in H_λ.

These items, together with Corollary 1.2.11, would show that every chain stabilizes.

Chapter 3

A MIXED IDENTITY-FREE ELEMENTARY AMENABLE GROUP

3.1 Project 2 Preliminaries

In this section, we provide the remaining preliminaries necessary to construct the group stipulated in Theorem 1.3.5.

3.1.1 Mixed identities

Given a group G, the following remark allows us to simplify the problem of showing that G is MIF by considering only mixed identities arising from elements of $G * \langle x \rangle$.

Remark 3.1.1. Observe that for any nontrivial element $g \in G$, the extension of the identity map on G to a map

$$\iota : G * F(x_1, x_2, \dots) \to G * \langle x \rangle$$

given by sending $x_i \mapsto x^i g x^i$ is an embedding. As a result, G satisfies a nontrivial mixed identity $u = 1$ for some $u \in G * F(x_1, \dots, x_n)$ if and only if G satisfies a nontrivial (single-variable) mixed identity $v = 1$ for some $v \in G * \langle x \rangle$.

3.1.2 An outline of the Ol'shanskii-Osin-Sapir construction

In [26], Ol'shanskii, Osin, and Sapir provide a construction of elementary amenable groups satisfying the direct limit characterization of lacunary hyperbolicity in Definition 1.3.4. As the main result of Project 2 arises as a modification of this construction, we present a brief overview the construction below, highlighting the key ingredients necessary to make this modification.

Remark 3.1.2. It should be noted that while the goal of the construction in [26] is a direct limit satisfying the conditions of Definition 1.3.4, the guarantee of these conditions is an optional element of the direct limit construction that we present in this chapter. (See Remark 3.2.6 for a modification of our construction which ensures that these conditions are met so that the resulting group is lacunary hyperbolic.)

Given a prime number p and a non-decreasing sequence $\mathbf{c}$ of natural numbers $c_1 \leq c_2 \leq \ldots$, define the following. Let $A_0 = A_0(p)$ be the group

$$A_0 = \langle a_i,\, i \in \mathbb{Z} \,|\, a_i^p = 1,\, i \in \mathbb{Z} \rangle$$

and let R_n denote the collection of relations of the form

$$[\ldots[a_{i_0}, a_{i_1}], \ldots, a_{i_{c_n}}] = 1$$

for all commutators with $\max\limits_{j,k} |i_j - i_k| \leq n$. For $n \in \mathbb{N}$, define $A_n = A_n(p, c_1, \ldots, c_n)$ by

$$A_n = \langle A_0 \,|\, \cup_{i=1}^{n} R_i \rangle$$

or equivalently,

$$A_n = \langle A_{n-1} \,|\, R_n \rangle.$$

Let $A = A(p, \mathbf{c})$ be the group

$$A = \langle A_0 \,|\, \cup_{i=1}^{\infty} R_i \rangle.$$

Notice firstly that the group A is a locally nilpotent group generated by elements of order p and is thus a locally finite p-group. Indeed, every finitely generated subgroup of A is contained in a subgroup $B = \langle a_{-N}, \ldots, a_N \rangle$ for some $N \in \mathbb{N}$. Since B is a nilpotent group generated by finitely many elements of finite order, it is a finite group, and moreover, since all of its (nontrivial) generators are of order p, B is its own unique Sylow p-subgroup and is thus a p group.

Secondly, observe that A_n and A admit automorphisms φ_n and φ respectively, both given by extending the map on generators $a_i \to a_{i+1}$, $i \in \mathbb{Z}$. Define the group $G = G(p, \mathbf{c})$

to be the extension of A by φ, i.e.

$$G = \langle A, t \,|\, t a_i t^{-1} = \varphi(a_i), i \in \mathbb{Z} \rangle.$$

Then G is clearly 2-generated ($G = \langle a_0, t \rangle$), and since G is (locally finite p-group)-by-(infinite cyclic), it is elementary amenable.

Similarly, for $n \in \mathbb{N} \cup \{0\}$, define the group $G_n = G_n(p, c_1, \ldots, c_n)$ to be

$$G_n = \langle A_n, t \,|\, t a_i t^{-1} = \varphi_n(a_i), i \in \mathbb{Z} \rangle \tag{3.1}$$

and observe that for each n, the natural quotient map $A_n \twoheadrightarrow A_{n+1}$ extends via the identity on t to a map $\varepsilon_n : G_n \twoheadrightarrow G_{n+1}$ so that the group G is the direct limit of the sequence

$$G_0 \overset{\varepsilon_0}{\twoheadrightarrow} G_1 \overset{\varepsilon_1}{\twoheadrightarrow} G_2 \overset{\varepsilon_2}{\twoheadrightarrow} \ldots \tag{3.2}$$

For each $n \in \mathbb{N}$, define S_n to be the generating set $\{a_0, t\}$ of G_n. In this framework, the key result in the construction from [26] is the following lemma:

Lemma 3.1.3. *([26, Lem. 3.24]) The groups G_n are hyperbolic, and, provided the sequence $\boldsymbol{c}$ grows fast enough, the sequence (3.2) (with generating sets $S_n = \{a_0, t\}$ for each G_n) satisfies all conditions of Definition 1.3.4 so that the direct limit of this sequence, $G(p, \boldsymbol{c})$, is lacunary hyperbolic.*

Remark 3.1.4. In particular, the proof of this lemma shows that given a finite subset $\mathscr{F}_n$ of G_n, any sufficiently large choice of c_{n+1} guarantees that the map $\varepsilon_n : G_n \twoheadrightarrow G_{n+1}$ is injective on $\mathscr{F}_n$.

3.2 Proof of the main result

Since for any prime number p and any non-decreasing sequence $\mathbf{c}$ of natural numbers the group $G(p, \mathbf{c})$ is 2-generated and elementary amenable, the proof of Theorem 1.3.5 is reduced to the following proposition.

Proposition 3.2.1. *Given a prime number p, there exists a non-decreasing sequence $\boldsymbol{c}$ of natural numbers $c_1 \leq c_2 \leq \dots$ such that the group $G(p, \boldsymbol{c})$ defined in Subsection 3.1.2 is MIF.*

To prove this proposition, we will need two lemmas and the following definition.

Definition 3.2.2. Let G be a non-elementary (i.e. not virtually cyclic) hyperbolic group. Then G contains a unique, maximal finite normal subgroup called the *finite radical* of G. (Existence of the finite radical follows from [25, Prop. 1].)

The following lemma is a simplification of [17, Cor. 1.7].

Lemma 3.2.3. *Let G be a non-elementary hyperbolic group with trivial finite radical. Then G is MIF.*

Remark 3.2.4. It should be noted that trivial finite radical is a necessary condition for a non-elementary hyperbolic group to be MIF. Indeed, for any group G with nontrivial finite normal subgroup N of size n and any $g \in N \setminus \{1\}$, G satisfies the nontrivial mixed identity $[x^{n!}, g] = 1$.

Lemma 3.2.5. *For each $n \in \mathbb{N}$, the groups G_n defined by (3.1) are non-elementary hyperbolic with trivial finite radical.*

Proof. For each $n \in \mathbb{N}$, the group G_n admits an epimorphism onto the wreath product $(\mathbb{Z}/p\mathbb{Z}) \operatorname{wr} \mathbb{Z}$ (given by adding the relations $[a_i, a_j]$ for all pairs $i, j \in \mathbb{Z}$) and so G_n is not virtually cyclic. The group G_n is hyperbolic by Lemma 3.1.3.

To see that G_n has no nontrivial finite normal subgroups, first observe that any element g of G_n may be written as

$$g = g_1 t^{\alpha_1} g_2 t^{\alpha_2} \dots g_k t^{\alpha_k},$$

where $g_1, g_2, \ldots, g_k \in A_n$ and $\alpha_1, \alpha_2 \ldots, \alpha_k \in \mathbb{Z}$. Rewriting the element as

$$g = g_1 \left(t^{\beta_1} g_2 t^{-\beta_1}\right) \left((t^{\beta_2} g_3 t^{-(\beta_2)}\right) \ldots \left(t^{\beta_{k-1}} g_k t^{-\beta_{k-1}}\right) t^{\beta_k},$$

where $\beta_j = \sum_{i=1}^{j} \alpha_i$, we observe that each of the elements $g_1, \left(t^{\beta_1} g_2 t^{-\beta_1}\right), \ldots, \left(t^{\beta_{k-1}} g_k t^{-\beta_{k-1}}\right)$ is an element of A_n so that $g = at^{\beta_k}$ for some element $a \in A_n$.

Now observe that if $\beta_k = 0$, then $g \in A_n$, and if $\beta_k \neq 0$, then the image of g in G_n mod the normal closure of A_n is of infinite order, and thus g is of infinite order. Thus all finite-order elements of G_n are contained in A_n. So if H is a finite subgroup of G_n, then $H \leq \langle a_{-N}, \ldots, a_N \rangle$ for some $N \in \mathbb{N}$. But such a subgroup cannot be both normal in G_n and nontrivial, since $H \cap t^{-(2N+1)} H t^{2N+1} = \{1\}$. □

We can now prove Proposition 3.2.1.

Proof of Proposition 3.2.1. Set $c_1 = 1$ and define the set $\mathscr{F}_1 = \{1_{G_1}\}$. Fix an enumeration $\{w_i(x)\}_{i \in \mathbb{N}}$ of the elements of $G_0 * \langle x \rangle$.

Now given c_k, the resulting group G_k, and a finite subset $\mathscr{F}_k \subseteq G_k$, choose c_{k+1} and $\mathscr{F}_{k+1}$ as follows. Let

$$\pi_k : G_0 * \langle x \rangle \to G_k * \langle x \rangle$$

be the extension of the homomorphism $\varepsilon_{k-1} \circ \ldots \circ \varepsilon_0 : G_0 \to G_k$ given by sending $x \mapsto x$. Let $w_{k,G_k}(x)$ denote the image of $w_k(x)$ under π_k. If $w_{k,G_k}(x) \neq 1$, first observe that by by Lemma 3.2.5, G_k is non-elementary hyperbolic with trivial finite radical. By Lemma 3.2.3, G_k is MIF, so there exists $g_k \in G_k$ such that $w_{k,G_k}(g_k) \neq 1$. In this case, add $w_{k,G_k}(g_k)$ to $\mathscr{F}_k$.

Now choose c_{k+1} large enough so that the resulting ε_k is injective on $\mathscr{F}_k$. (This is possible by Remark 3.1.4.) Define

$$\mathscr{F}_{k+1} = \varepsilon_k(\mathscr{F}_k).$$

After choosing a sequence $\mathbf{c}$ in the above manner, It remains to show that the group $G = G(p,\mathbf{c})$ is MIF.

For each $k \in \mathbb{N}$ define the homomorphism

$$\sigma_k : G_k * \langle x \rangle \to G * \langle x \rangle$$

to be the extension of the natural quotient map $G_k \to G$ given by sending $x \mapsto x$. To see that G is MIF, first observe that if $w(x) \in (G * \langle x \rangle)\setminus\{1\}$, then there exists some $i \in \mathbb{N}$ such that $\sigma_0(w_i(x)) = w(x)$, and furthermore, since $w(x)$ is nontrivial, $w_{i,G_i}(x) \neq 1$. By construction, $(\varepsilon_k \circ \ldots \circ \varepsilon_i)(w_{i,G_i}(g_i))$ is in $\mathscr{F}_{k+1}\setminus\{1\}$ for every $k \geq i$, so in particular, $\sigma_i(w_{i,G_i}(g_i)) \neq 1$. Since $\sigma_i(w_{i,G_i}(x)) = w(x)$ and σ_i is a homomorphism, we have that $\sigma_i(w_{i,G_i}(g_i)) = w(\sigma_i(g_i))$. Thus $w(\sigma_i(g_i)) \neq 1$, and so $w(x) = 1$ is not a mixed identity on G. Since $w(x)$ was arbitrary, this shows that G is MIF. □

Remark 3.2.6. The above construction may be modified so that the resulting group $G(p,\mathbf{c})$ is lacunary hyperbolic. To do so, begin by fixing a function $f : \mathbb{N} \to \mathbb{N}$ so that $n = o(f(n))$. Then, during the step at which c_{k+1} is to be chosen, make the following alteration. Let δ_k denote the hyperbolicity constant of G_k with respect to the generating set $S_k = \{a_0, t\}$, and let $r(\varepsilon_k)$ denote the injectivity radius of ε_k with respect to the generating set S_k. Now choose c_{k+1} large enough so that in addition to being injective on $\mathscr{F}_k$, the resulting ε_k satisfies $r(\varepsilon_k) \geq f(\delta_k)$. Define

$$\mathscr{F}_{k+1} = \varepsilon_k(\mathscr{F}_k)$$

as before, and proceed in the same way. Then, after choosing the sequence $\mathbf{c}$, observe that the resulting group $G(p,\mathbf{c})$ is lacunary hyperbolic by Lemma 3.1.3.

To prove Theorem 1.3.6, we need the following result.

Lemma 3.2.7. *([17, Prop. 5.4(c)]) Every nontrivial subnormal subgroup of a MIF group G is also MIF.*

Remark 3.2.8. In particular, the proof of [17, Prop. 5.4(c)] notes that if N is a normal subgroup of G satisfying the nontrivial mixed identity $w(x) = 1$ for some $w(x) \in N * \langle x \rangle$, then for any $n \in N \backslash \{1\}$, G satisfies the nontrivial mixed identity $w([x,n]) = 1$.

Proof of Theorem 1.3.6. Given a prime number p, Proposition 3.2.1 yields a sequence $\mathbf{c}$ of natural numbers such that the group $G(p,\mathbf{c})$ is MIF. Now observe that the (nontrivial) locally finite p-group $A(p,\mathbf{c})$ is normal in $G(p,\mathbf{c})$, so by Lemma 3.2.7, $A(p,\mathbf{c})$ is MIF. $\square$

3.3 Notable examples of finitely generated amenable groups which are not MIF

In this section, we examine two other reasonable candidates for examples of finitely generated MIF amenable groups and explain why they fail to be MIF.

3.3.1 The Grigorchuk group

Given a binary rooted tree T_2, we may think of the nodes of T_2 as finite binary strings (where the root is represented by the empty string). The *Grigorchuk group G* is defined to be the subgroup of $Aut(T_2)$ generated by elements a, b, c, and d whose actions on binary strings $w \in \{0,1\}^*$ are as follows.

$$\begin{array}{ll} a(0w) = 1w & a(1w) = 0w \\ b(0w) = 0a(w) & b(1w) = 1c(w) \\ c(0w) = 0a(w) & c(1w) = 1d(w) \\ d(0w) = 0w & d(1w) = 1b(w) \end{array}$$

The Grigorchuk group was initially constructed in [11] and was shown in [12] to be the first known example of a finitely generated group with intermediate growth. Notably, it is an example of a group which is amenable but not elementary amenable. (Amenability follows from subexponential growth, while Chou shows in [6] that elementary amenable groups have either polynomial or exponential growth.)

Proposition 3.3.1. *The Grigorchuck group satisfies the nontrivial mixed identity*

$$[[[[x,b],d],d],ada] = 1.$$

Proof. To see that the Grigorchuk group satisfies a nontrivial mixed identity, first consider the subgroup $H = \langle b,c,d,aba,aca,ada\rangle$ which is the normal subgroup of index 2 stabilizing the first level of T_2. The subgroup H admits a monomorphism $\varphi : H \hookrightarrow G\times G$ given by sending

$$\begin{aligned} \varphi(b) &= (a,c) & \qquad \varphi(aba) &= (c,a) \\ \varphi(c) &= (a,d) & \qquad \varphi(aca) &= (d,a) \\ \varphi(d) &= (1,b) & \qquad \varphi(ada) &= (b,1) \end{aligned}$$

(For references, see [9, Ch. 8, #13-14].) Observe that φ maps the H-conjugates of ada into the first copy of G and the H-conjugates of d into the second copy of G. Hence, if K is the normal closure in H of the subgroup $\langle d,ada\rangle$, then $\varphi(K)$ is a nontrivial direct product, and since φ is injective, K itself decomposes as a nontrivial direct product where the two direct factors are the H-conjugates of d and ada respectively. Hence (as in Example 1.3.1) K satisfies the nontrivial mixed identity $[[x,d],ada] = 1$. Since K is subnormal in G, we may apply Remark 3.2.8 twice to obtain that G satisfies the nontrivial mixed identity $[[[[x,b],d],d],ada] = 1$. $\square$

3.3.2 Akhmedov's construction of amenable groups with infinite girth

Given a finitely generated group G, the *girth* of G is defined to be the infimum of all $n\in\mathbb{N}$ such that for every finite generating set S of G, the Cayley graph $\Gamma(G,S)$ contains a cycle of length at most n without self-intersections (see [39]). In [1], Akhmedov details the construction of a finitely generated amenable group of infinite girth which does not satisfy any nontrivial identity. Such a group is a promising candidate for a finitely generated MIF amenable group not only because it is already identity-free, but also because by [17, Prop. 5.4(d)], infinite girth is a requisite property for finitely generated MIF groups. However, the group constructed in [1] is a nontrivial (restricted) wreath product, and all such groups satisfy a nontrivial mixed identity. Indeed, for any wreath product (restricted or unrestricted), we have the following.

Proposition 3.3.2. *Let G be a wreath product of two nontrivial groups, and let $A \times B$ be any decomposition of the base of the wreath product into a direct product of nontrivial groups A and B. Then for any $a \in A \backslash \{1\}$ and any $b \in B \backslash \{1\}$, G satisfies the nontrivial mixed identity $[[[x,a],a],b] = 1$.*

Proof. As in Example 1.3.1, the base $A \times B$ of the wreath product G satisfies the nontrivial mixed identity $[[x,a],b] = 1$. Since the base is normal in G, we can apply Remark 3.2.8 to obtain that G satisfies the nontrivial mixed identity $[[[x,a],a],b] = 1$. □

3.4 Results in the context of Hull-Osin paper

In [17], Hull and Osin ask the following question:

Question 3.4.1. ([17, Q. 6.5]) Does there exist a countable amenable highly transitive subgroup $A \leq Sym(\mathbb{N})$ such that $A \cap Alt(\mathbb{N}) = \{1\}$?

It should be noted that here, ”highly transitive” refers specifically to the natural action of $Sym(\mathbb{N})$ on $\mathbb{N}$ being highly transitive. Recall the Wielandt (or Jordan-Wielandt) Theorem [15, Thm. 5.2], which states that every infinite primitive subgroup of $Sym(\mathbb{N})$ containing a non-trivial permutation with finite support contains $Alt(\mathbb{N})$. In the above question, A is a countable highly transitive subgroup of $Sym(\mathbb{N})$ and thus an infinite primitive subgroup. Therefore, the Wielandt Theorem yields an alternative; either A contains $Alt(\mathbb{N})$ or $A \cap Alt(\mathbb{N}) = \{1\}$.

Additionally, for a subgroup of $A \leq Sym(\mathbb{N})$, the containment of $Alt(\mathbb{N})$ in A is itself a sufficient condition to guarantee that that action of A on $\mathbb{N}$ is highly transitive. (Indeed, for each $k \in \mathbb{N}$ and any two k-tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ of distinct elements of $\mathbb{N}$, the element $(a_1\ b_1)\ldots(a_k\ b_k) \in Alt(\mathbb{N})$ sends a_i to b_i for $i = 1, \ldots, k$.) As a result, Question 3.4.1 essentially asks whether all countable amenable highly transitive subgroups of $Sym(\mathbb{N})$ contain $Alt(\mathbb{N})$ (and are thus highly transitive in a somewhat trivial way) or if there exist other examples which are (necessarily) mutually exclusive with $Alt(\mathbb{N})$.

The construction of countable MIF amenable groups in Theorem 1.3.5 serves as a first step towards answering the above question. Indeed, together with the Wielandt Theorem, Theorem 1.3.2 implies that any example group providing an affirmative answer to Question 3.4.1 must necessarily be MIF like the groups we constructed. If one is to attempt to show that some of these groups are highly transitive, however, then some additional machinery may be necessary. For example, one may have to prove some sort of highly transitive action analogue to [17, Thm. 3.6] compatible with the direct limit construction so that the resulting group is given a highly transitive action (though success in such a case would bypass the need to explicitly guarantee the MIF property as in the Theorem 1.3.5 construction).

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