Young Children Designing and Defining Space: Supporting the Co-Development of Mathematical Practices and Foundational Understandings of Space and Symmetry

By

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In honor of David W. Henderson,

who helped us experience what it means for mathematics to be alive.

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CHAPTER I

Introduction

Visualizing and reasoning about space are foundational to the epistemic enterprise of mathematics (Henderson & Taimina, 2005). However, current K-12 math curriculum largely neglects cultivating children's mathematical understandings of and visualizing of space. Given research suggesting that this neglect is determinantal to students' success in and to their understandings of the nature of mathematics (Cheng & Mix, 2012; Lehrer & Chazan, 1998; Wai, Lubinski, & Benbow, 2009), the National Research Council (2006) has called for the integration of spatial investigations into STEM classrooms. In response, researchers have increasingly explored how initial understandings and visualizations of space develop in early childhood, with an eye toward how these early experiences can inform the design of mathematics instruction (Clements & Sarama, 2007; Davis et al., 2015; Hawes, Moss, Caswell, Naqvi, & MacKinnon, 2017). These efforts have identified important resources young children bring to bear when they reason about space, including visually (de)composing and rotating structures and noticing and abstracting patterns. However, we still have little understanding of how these experiences and resources can be leveraged to co-develop children's mathematical practices and conceptions of space (Ma, 2016). This co-development is the central goal of recent instructional standards, such as the Common Core State Standards and the Next Generation Science Standards. Moreover, it is critical for students' coming to see and to use their experiences with space as resources for defining mathematical systems.

A central component of visualizing and conceiving of space is the development of a shared language and forms of representations that define and index salient features of shape and patterns (Goodwin, 1994; Senechal, 1990b). In mathematics, transformations are powerful analytical and theoretical tools used to describe and define invariant and variant patterns and properties of space (NCTM, 2014). Symmetry transformations have had a particularly generative role across areas of STEM. For example, developments in and applications of symmetries have resulted in new understandings of geometric systems, abstract algebra, crystal and molecular structures, and quantum theories (Kleiner, 1987; Senechal, 1990a; Stewart, 2013). Symmetries and their larger class of isometry transformations, though, have genetic roots in forms of cultural activities and play that engage with transformations as everyday physical motions (e.g., flips, turns, slides). For young children, these everyday motions permeate and are organizing actions across a broad set of activities from moving their bodies and objects in space to more representational activities, such as drawing, building, and navigating. Thus, children's early activities and senses of motion offer a potential starting point in early spatial math leaning that can afford new opportunities for expression and engagement with space (Lehrer & Chazan, 1998).

In line with renewed attention to the importance of and the role of transformations in supporting visualization in mathematics, the three papers in this dissertation address how learning ecologies can be designed to support young children in defining a mathematical system of space, beginning with core ideas of shape and isometric and symmetric transformation. Each paper addresses two central themes. First, they look at how young children's activities in space serve as epistemic resources in a key mathematical practice, definition, that is practical means for supporting the development of concepts of space (Kobiela & Lehrer, 2015). Second, they consider how introducing transformations into children's investigations and construction of space helps them to reorganize their visualization of structures as objects in the world into forms of mathematical patterns.

The first paper is a literature review that argues for the foundational role that symmetry can play in children's mathematical learning. I first draw on anthropological and historical accounts of cultural craft and mathematics to draw parallels between the epistemic role of symmetry across each context. This analysis reveals a long history of development and frequent interplay between mathematical inquiry and related activity in arts, sciences, and technologies. Attention to this historical scale illustrates how interchange among different cultural activities led to refinements and innovations in mathematical concepts of symmetry. This development across cultural activities situates mathematics as a ubiquitous human pursuit that children likely have access to through forms of cultural art and crafting activities. I then inquire about potential early roots of these forms of thinking in children's development and the activities that seem to foster conceptual and practical resources tied to symmetries. Drawing together these two scales of development, the paper concludes with a set of principles to guide the future design of learning ecologies that promise to extend the reach and grasp of children's understanding of the mathematical foundations of symmetry and space. These principles include providing children opportunities to understand relations between forms of isometry and symmetry (rather than treating them as isolated concepts), attending to the resources children have to reason about 3-D space and it's mathematical affordances, and situating instruction within coherent systems of locally meaningful activity that begin with children's existing spatial activities (i.e., construction and design). The design principals outlined in paper one informed two iterations of a design study reported in the next two papers.

The second paper presents the first iteration of a design study conducted in four 1st grade classrooms. Children participated in defining two forms of isometries and symmetries, rotations and reflections, in three dimensions and employed these forms of transformation as design tools for constructing and visualizing 3-D structures. The findings from this first iteration present a trajectory of how grade one students developed a mathematical system of 3D isometries and symmetries through the practice of defining. Central to this trajectory were teachers' uses of the variability in students' designs and descriptions of 3D structures to build a shared classroom history of definition. Children built on this shared history to assume increasingly active roles in authoring a local mathematical system of transformation. As children came to see isometry and symmetry transformations as tools to both define and visualize relations between 3-D structures, they also increasingly used these forms of transformations to guide their design of 3-D structures. Thus, children's defining of transformations served to extend their existing construction activities.

This first iteration highlighted important ways that children's construction activities served as grounds for defining 3-D structures and their transformations. In the second iteration, I built on this finding by placing construction at the center of instruction by using children's existing construction activities as a new starting point. Paper three reports on how this redesign resulted in children using informal motions and resources tied to construction to define classes of polyhedra. This new understanding of polyhedra and sense of motion as an epistemic resource supported the mathematization of these informal motions into conceptions of rotations as isometries and symmetries. Although this second iteration narrows in on rotations, the results of the study demonstrate how children's senses of motion and structures co-develop as they construct and define 3-D structures.

Together, these three papers contribute to understandings of how we can begin to broaden mathematics instruction in the early years to co-develop mathematical practices and conceptual systems that help extend children's spatial visualization. This broadening is important as it not only can support children in developing skills that are critical to success in mathematics, but it also may lead to more equitable STEM learning opportunities by opening new learning pathways (Henderson, 2006; Lehrer & Chazan, 1998). Exploring and visualizing space are central to young children's ways of knowing (Oudgenoeg-Paz, Boom, Volman, & Leseman, 2016; Piaget, 1970), yet, we neglect inviting children to use these experiences and intuitions as resources for engaging with powerful mathematical representations and concepts. Thus, understanding how children's everyday spatial experiences can be leveraged to support rich forms of mathematical engagement may afford children with new ways of entering into and relating to mathematics.

References

Cheng, Y. L., & Mix, K. S. (2014). Spatial training improves children's mathematics ability. Journal of Cognition and Development, 15(1), 2-11.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education*, 136-163.

Davis, B. (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. New York, NY: Routledge.

Goodwin, C. (1994). Professional vision. *American anthropologist*, *96*(3), 606-633.

Hawes, Z., Moss, J., Caswell, B., Naqvi, S., & MacKinnon, S. (2017). Enhancing Children's Spatial and Numerical Skills through a Dynamic Spatial Approach to Early Geometry Instruction: Effects of a 32-Week Intervention. *Cognition and Instruction*, 35(20), 1-29.

Henderson, D. W. (2006). Alive mathematical reasoning. In F. A. Rosamond & L. Copes (eds.), *Educational Transformations: Changing our lives through mathematics*. (pp 247-270). Bloomington, Indiana: Author House.

Henderson, D. W., & Taimina, D. (2005). *Experiencing geometry: Euclidean and non-Euclidean with history*. Prentice Hall.

Kleiner, I. (1986). The evolution of group theory: A brief survey. *Mathematics Magazine*, 195-215.

Kobiela, M., & Lehrer, R. (2015). The co-development of mathematical concepts and the practice of defining. Journal for Research in Mathematics Education, 46(4), 423-454.

Lehrer, R., & Chazan, D., Eds. (1998). *Designing learning environments for developing understanding of geometry and space*. Mahwah, NJ: Lawrence Erlbaum Associates.

National Research Council (2006). Learning to think spatially: GIS as a support system in the K 12 curriculum. Washington, DC: National Academic Press.

National Council of Teachers of Mathematics. (2014). Essential Understandings Series. Reston, VA: National Council of Teachers of Mathematics.

Ma, J. Y. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. Journal of the Learning Sciences, 25(3), 335-371.

Oudgenoeg-Paz, O., Leseman, P., & Volman, M. (2015). Exploration as a mediator of the relation between the attainment of motor milestones and the development of spatial cognition and spatial language. Developmental Psychology, 51(9), 1241.

Senechal, M. (1990a). Brief history of geometrical crystallography. *Historical atlas of crystallography*, 43-59.

Senechal, M. (1990b). Shape. In *On the shoulders of giants: New approaches to numeracy* (pp. 139-181). National Academy Press.

Piaget, J. (1970). Genetic epistemology. New York: W. W. Norton.

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, *101*(4), 817.

CHAPTER II

SYMMETRY AS A FOUNDATIONAL CONCEPT IN CHILDREN'S MATHEMATICAL LEARNING

Introduction: The Centrality of Symmetry and Isometry Across STEAM Communities

Children's mathematics education has typically focused on number and arithmetic operations, but there is increasing call to broaden children's experiences of mathematics to include investigation of geometry and space (NCTM, 2000). One source of this press is the articulation of standards for instruction in space and geometry beginning in early childhood (Common Core State Standards in Mathematics, 2010). Another source is the increasing recognition of the vital role of spatial visualization in STEM education, and the need to cultivate visualization practices systematically, rather than continuing to neglect them (NRC, 2006). However, despite renewed attention to the importance of spatial education, there is comparatively little consensus about core concepts and forms of mathematical experience about space that are both accessible to young children and consequential for learning to visualize and conceive of the nature of space.

Recently, the National Council of Teachers of Mathematics has identified symmetries and the larger system of isometric transformations as essential concepts that should be developed throughout K-12 mathematics education (NCTM, 2014). NCTM argues that symmetries and isometries serve as important anchoring concepts for continued conceptual mathematical learning. Additionally, prior research on children's geometry learning suggests that mathematical investigations of these concepts allow for more accessible entry points into mathematical visualization and epistemic practices, such as search for invariance, definition, conjecture, refutation, and proof (Lehrer & Chazan, 1998). Attention to the co-development or interplay of disciplinary, mathematical practices and concepts has gained greater attention in educational policy (see Common Core Mathematical Practice Standards). This attention aligns with views of disciplinary learning as an "epistemic endeavor," in which students gain access to disciplinary, collective ways of knowing, have agency in generating disciplinary knowledge, and are supported in developing productive mathematical dispositions (Lehrer, 2009). However, although standards documents outline the forms of practices that students should engage in, growth and change in these practices are not described, nor are their connection with content standards articulated. As a result, prospective pathways of development and productive means for supporting such development are at best only implicit.

In this paper, I aim to explore the potential suggested by the NCTM for isometry and symmetry as essential conceptual tools that could be profitably elaborated throughout the course of schooling. To situate the NCTM's call for change, I first present a historic case analysis of symmetry applications across a range of cultural activities. These cases provide illustrative examples of symmetry as a "concept-in-practice" (Hutchins, 2012), a term intended to suggest the mutual constitution of ways of knowing with knowledge. Additionally, I highlight intersections between these forms of cultural activity and development of the mathematics of symmetry. This broadened view of symmetry beyond traditionally recognized mathematics contexts fosters a greater understanding of mathematics as a form of cultural activity that emerges from everyday activity (Henderson & Taimina, 2005; Lave, Murtaugh, de la Rocha, 1984; Lehrer & Lesh, 2003). This analysis also suggests how educators might better leverage a wide set of cultural resources to introduce and cultivate children's spatial mathematics.

In the second part of the paper, I review studies of children's spatial reasoning to inform a sense of children's developing cognitive resources to engage in mathematical explorations of space and transformations. These forms of everyday visualization skills can serve as entry points into children's participation into mathematical forms of visualization practices. For example, children's initial noticing of the invariance within patterns and shapes can support the generation of conceptual and representational tools to explain this invariance. For example, Lehrer, Strom, and Confrey (2002) describe how children's everyday sense of similarity among classes of shapes served as an important resource to support new understandings of ratio and different representations of similarity as ratio across both Cartesian planes and algebraic expressions. The development and collective negotiation around the use and meaning of these tools are grounds for the development of visualization practices that influence or "discipline" (Stevens & Hall, 1998) students' perception—tuning how and what students notice about relationships of invariance among shapes. Thus, understanding how to leverage children's everyday resources to notice and reason about space is essential to inducting children into spatial forms of mathematical practice.

Finally, I review classroom studies of children's conceptions of isometry and symmetry to explore the extent to which these suggest that these forms of transformation provide mathematically productive means of considering space. This review also includes attention to the extent to which prior research has considered relations between mathematical practices and isometries and symmetries with the intent of examining how the designed activity structures support their co-development. Together these studies inform what forms of understandings of symmetries and isometries are feasible for young children. Additionally, they indicate how factors within the instructional design either afford or constrain students' mathematical engagement and conceptual understandings.

I conclude with a set of principles that can inform the future design of learning ecologies that capitalize on children's everyday resources and activities. I argue that these resources can serve as entry points into a child's geometry that positions transformations as core concepts that are co-constituted with mathematical practices. The outlined principles serve as a first step to developing a research base that can better inform how goals presented by NCTM and Common Core can be realized within classrooms.

Background

Symmetry is a unifying concept across and within disciplines and cultural activity, as it serves as a powerful tool that sustains important visualization practices, such as classification, spatial analysis, and representation (Senechal, 1990). These visually-based practices are often deeply intertwined with epistemic practices and knowledge systems, as they are shaped by and shape the forms of tools, concepts, and theories that dictate how one comes to see and represent the world (Gooding, 2006; Goodwin, 1994; Stevens & Hall, 1998) For example, in mathematics, the development of symmetry groups created coherence between different geometries, and symmetry has also played a vital role in conceptual and theory development in several domains of science, including crystallography and physics (Brading & Castellani, 2003; Senechal, 1989). The reach of symmetry extends beyond formal STEM disciplines into art and design. Many cultural communities have long used symmetry patterns in forms of artistic expression to communicate and represent cultural knowledge and values across generations (Washburn, 1999; Frame, 2004).

These brief allusions highlight the far-reaching range of symmetry, but as I will elaborate shortly, historical analyses reveal that the interconnections among these diverse forms of expression have played a central role in arriving at the current mathematical definition of symmetry. Thus, these examples illustrate how mathematical concepts get refined over time through the interplay of different cultural activities. Moreover, they position mathematics as an ongoing human pursuit—challenging views of mathematics as a discipline solely reserved for those culturally identified as mathematicians.

Finally, the interconnections across these examples also suggest that there may be a multitude of entry points and developmental pathways that leverage the unifying role of symmetry in children's everyday cultures of play and craft to support participation in mathematical activity (Lehrer & Chazan, 1998). Bishop (1988) argues that cultural mathematical systems arise from six universal activities—counting, locating, measuring, designing, playing, and explaining—that are sufficient to develop all systems of mathematics. He further proposes that if mathematics curricula are structured around commonplace activities, there is greater opportunity for equitable mathematics instruction in which individuals are positioned to "involve their own mathematical ideas whilst also permitting the 'international' [disciplinary] mathematical ideas" (p. 189). These mathematical ideas include not only those related to the technical elements of mathematics, but also consideration of different values and beliefs that are critical to negotiating a local epistemic culture within the classroom that sees the generation of mathematical knowledge as a collective endeavor (Knorr Cetina, 1999). Thus, identifying accessible entry points for students is facilitated by exploring a range of contexts that are productive for building conceptual understanding of symmetries and isometries. Taking this approach may also lead to more equitable mathematics instruction that privileges students' interests and cultural backgrounds as central to shaping a classroom's mathematical culture.

Defining Symmetry

The word symmetry derives from the ancient Greek word, summetria, meaning commensurable or of equal or proportional measure (Hon & Goldstein, 2008). However, wider use of summetria/symmetry in architecture and artistic expressions illustrate how the concept of symmetry is rooted in a universal affinity among humans to notice and represent harmony, balance, beauty, and unity (Brading & Castellani, 2013; Hargittai & Hargittai, 1994; Washburn & Crowe, 2004). This relation between symmetry and sentiments of unity and beauty continue to demonstrate our everyday sense of symmetry in relation to describing aesthetics within nature, art, dance, literature, and science. In many senses the aesthetic qualities of symmetry are also preserved in the technical definition of symmetry in formal sciences and mathematics. Brading and Castellani (2003) argue that the application of symmetry within scientific fields is almost always concerned with describing unity. Thus, this review uses the technical, mathematical definition of symmetry to provide a common language and analytical tool to examine symmetry use across cultural contexts.

Within the discipline of mathematics, symmetries are subsets of isometries. Isometries are transformations that preserve a planar figure's (or three-dimensional structure's) angles and lengths. Forms of isometries within a figure, include rotations and reflections, and for finite patterns, rotation and reflection are augmented by translation and glide. Isometries determine whether two figures are congruent. Figures are directly congruent if they can be rotated or translated so that the size, shape and orientation of both figures are identical. Reflections change a figure's orientation (i.e. handedness or chirality) but not affect the figure's size or shape; angles and distances are preserved. On the plane, the result of a reflection, the image, can be obtained by "flipping" the figure—a ½ turn rotation in 3-space. But in three dimensions, this is often not possible: A person's hands are mirror images, but one hand cannot be made directly congruent with the other by flipping it in 3-space. From a function-oriented perspective, symmetries are isometries that bring the points of a figure and associated portions of the plane onto other points on the figure and plane so that one cannot discern the difference between the figure and its transformed image (Henderson & Taimina, 2005). For instance, a square has reflection and rotation symmetries. One rotation symmetry of the square is a 90 degrees rotation about its midpoint because the image of the square is indistinguishable from the original square. In contrast, the image created by a rotation through 45 degrees about a square's midpoint is congruent with the figure but is not a symmetry. One can discern the difference between the figure and its 45-degree rotated image.

Symmetries are further classified by algebraic groups that describe the total set of symmetries of a particular object. A critical aspect of groups is that they are closed, so that every product or combination of symmetries within a group are also part of the group. For example, as

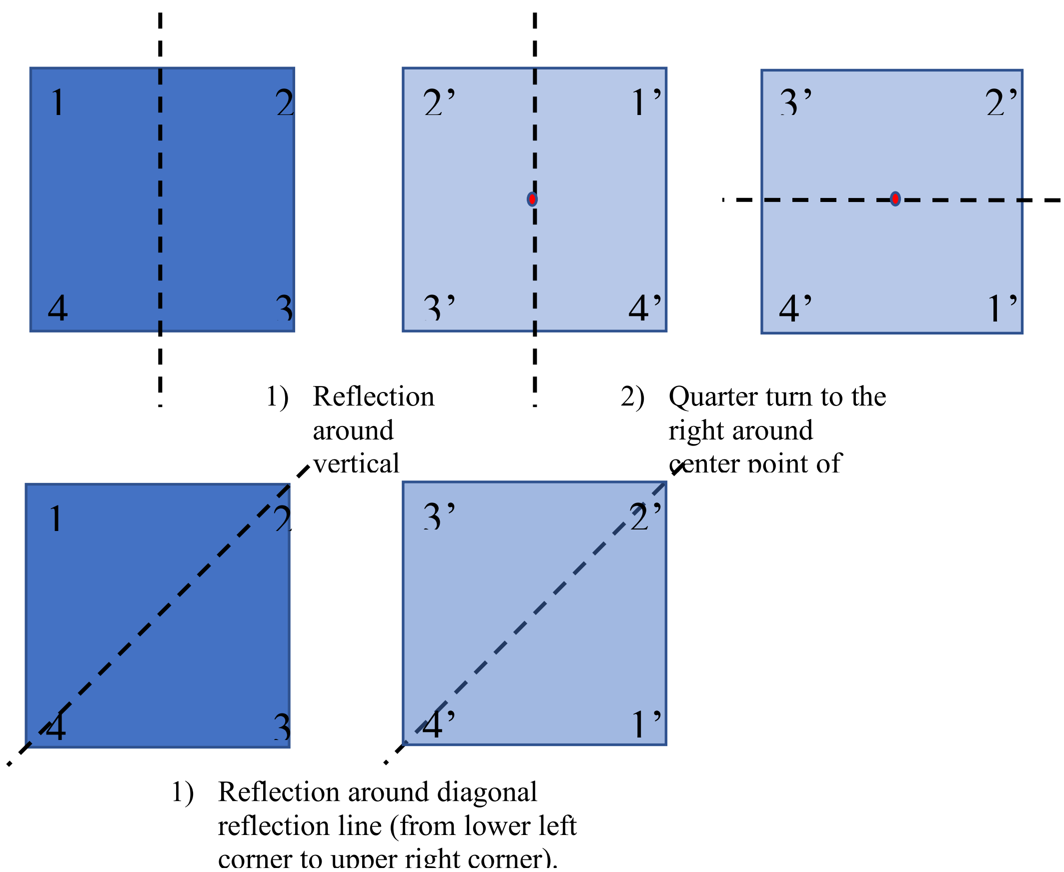


Figure 1: Equivalent symmetries of a square

illustrated in Figure 1, a reflection about a vertical reflection line through the midpoint of a square, followed by a quarter turn to the right about the center of a square is equivalent to a reflection about the diagonal (lower left corner to upper right corner) of the square. The symmetries through the center of a square make up a symmetry point group. Point groups do not change the location of the origin of an object (i.e., each isometry that passes through a line or point at the center of a square keeps to center point at a fixed location within the plane).

Objects such as finite and infinite patterns can have symmetry point groups and symmetry space groups. While symmetry point groups preserve the origin of a figure, space groups include translation and glide isometries—thus the origin of an object does not stay at the same location. Figure 2 shows the point group of a wallpaper pattern (only one central point of 1/2 turn symmetry and two perpendicular reflection lines through the center) and its space group (two sets of perpendicular reflection planes, two points of ½ turn symmetry, and glide symmetry

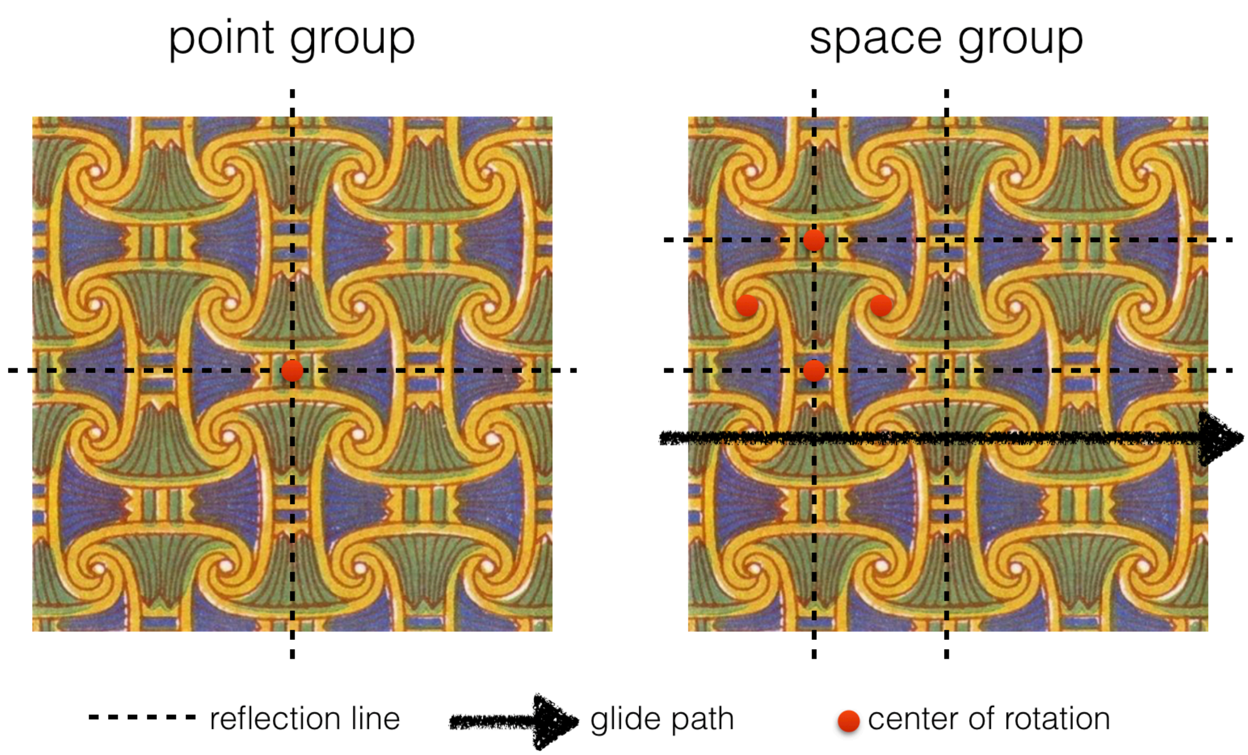


Figure 2: Example of symmetry point groups and space groups of a wallpaper pattern.

along a path through the points of rotation not at the intersection of perpendicular reflection planes). Symmetry group analysis aids in diagnosing the internal regularities within a pattern, and it also allows for comparison and classification across patterns that may initially look distinct but have similarities based on symmetries (Washburn & Crowe, 2004). Finally, the development and enumeration of symmetry groups have also further defined space and the tiling or packing of space by generating all of the possible unique regular configurations of space across 2D, 3D, and higher dimensional space.

The transformation-based definition of symmetry provides a productive framework to explore symmetry more broadly, however, not all instances of symmetry are generated with this lens of transformations. In fact, as described in the following section, the transformational group conception of symmetry did not develop until the mid to late 19th century. Despite the late development of transformational groups, in the remainder of this paper, I adopt this definition of symmetry because this definition of symmetry offers a language that has proven to identify connections and bridges together "seemingly disparate fields" (Hargittai and Hargittai, 1994).

Symmetry as a Concept-in-Practice Within and Across Cultures and Disciplines: A Historical Case Analysis

The development of mathematical concepts is shaped not only historically, within the professional community of mathematicians, but also every day, in the cultural and collective activity of everyday people (Henderson & Taimina, 2005). I adopt Hutchins's (2012) sense of *concepts-in-practice* to examine how symmetry derives meaning through its enactment within a variety of cultural practices and how its continuing conceptualization shapes practices. Hutchin's sense of *concepts-in-practice* stands in contrast to conceptions of concepts as within the mind of an individual. Instead Hutchins views concepts as "manifest in practice," referring to how concepts are distributed across individuals, guiding and arising from practices. For example, Hutchins (1995) studies how conceptual systems related to navigation are tied to the particular cultural practices of navigating ships and distributed across the actors involved in the navigation practices. Symmetry is a salient concept across multiple forms of both every day and more disciplinary activities. Tracing symmetry's entanglement in diverse forms of cultural practice provides a more detailed understanding of how symmetry has become refined as a concept over time. Of particular importance to the developmental trajectory of symmetry is its role in organizing overlapping activities that span multiple communities—giving rise to new forms of activities that allow for the collective engagement of individuals across these communities.

Within this section, I selected cases to illustrate the role of symmetry in shaping a community's visualization practices, especially classification, spatial analysis, and representation. Furthermore, the cases selected show how these symmetry-inspired visualization practices helped determine how practitioners across context come to see and understand space.

Case 1: The Development of Symmetry Groups within Mathematics and Crystallography

In mathematics, symmetry and isometry are used as tools for analyzing and classifying space and systems of geometry (Senechal, 1990; Henderson and Taimina, 2005). Developments of concepts of symmetries in the 19th century were an outcome of an explosion of different geometries (i.e. projective geometry, non-Euclidean geometries, differential geometry, algebraic geometry, and *n*-dimensional geometry) and the development of transformation groups within group theory. The extension of group theory to include transformation groups not only allowed for the unification of geometrical systems, but it also extended group theory to number theory, the theory of algebraic equations, the theory of differential equations, and function theory (Stewart, 2013; Kleiner, 1986). This development in mathematics, however, was bolstered by a contemporary development within the field of crystallography—the classification of crystals using symmetries to describe their internal structures. I briefly summarize the contemporary and intersecting developments within crystallography and the conceptualization of symmetry groups. In particular, I highlight how this development was supported by shifts in visualization practices of analysis, classification, and representation.

Due to the intricacies and patterns of crystalline forms, crystallography and mathematics have historically been closely intertwined. For example, Plato's fascination with crystals and the packing of solids led to his discovery of the five-regular polyhedra or platonic solids. However, prior to developments within crystallography throughout the 19th century, scientists disagreed about the importance of attending to the internal structure of crystals. This was due to the observed variability in their external forms found in nature and associated doubts that such irregularity would arise from internally consistent patterns. One proponent of studying the internal structure of crystals was Robert Hooke (Glazer, 2016). Figure 3a taken from Hooke's *Micrographia*, publish in 1665, shows Hooke's early drawings of crystals and the use of sphere packing to describe relations between the internal and external structure of crystal of urea and graphite. It took over another 100 years before a growing number of scientists developed theoretical and empirical evidence for the importance of the internal structures of crystals.

Nineteenth century developments within crystallography led to a consensus that crystals' internal structures should determine their classification. The first of these developments was the invention of the goniometer, a tool that could measure interfacial angles of three-dimensional crystals and polyhedra. This tool allowed for new practices of analysis that revealed a consistency in the structural elements composing crystals (Stewart, 2013 As a result, crystals increasingly were modeled as periodic packings of identical polyhedral elements defined by the measures of interfacial angles. As Figure 3b illustrates, in the early 19th century, Rene-Just Hauy generated models of crystals as regular packing of polyhedral elements.". Haüy's models helped persuade scientists that geometry was critical for the study of crystals. Additionally, they mark the first use of symmetry to describe the relation among the elements within crystals (Senechal, 1990). Haüy argued that all elements within a crystal were made up of the same polyhedral building blocks or "*molécules intégrantes*" arranged "symmetrically" around an axis.

> "[. . .] the manner in which Nature creates crystals is always obeying to the law of the greatest possible symmetry, in the sense that oppositely situated but corresponding parts are always equal in number, arrangement, and form of their faces." (Haüy as cited by Kubbinga, 2012).

This transcript articulates what Haüy called the symmetry law of crystals; however, contemporary mathematical theories of symmetry were still developing and based in matching faces and edges of polyhedra to determine the congruency of two polyhedra (Senechal, 1990).

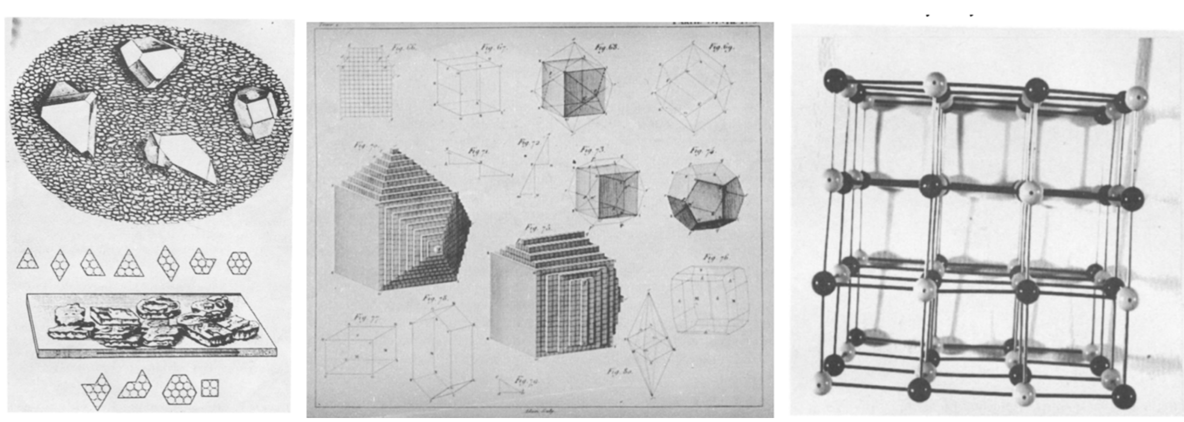


Figure 3a-c: (from left to right) a) Hooke's use of sphere packing to model the internal structure of crystals, b) Haüy's model of crystals as arrays of polyhedral elements, c) Bravais' model of crystals as point lattices (Senechal, 1989).

Haüy's student, Delafosse, continued to build upon his work, and further expanded studies of the internal structure of crystals. Hauy's symmetry axes were tied to the external symmetries of crystals, but Delafosse concluded instead that the building blocks of crystals were arranged in rows and planes. He stressed the importance of the symmetries of these internal planes and rows as critical to understanding variations in the external symmetries of crystals. He noted that within some crystal samples, the structures of these rows and planes resulted in a lower symmetry in the external form. As Figure 4 illustrates, the striations in a cubic crystal of boracite, reduces the number of symmetries found in a regular cube (Kubbinga, 2012).

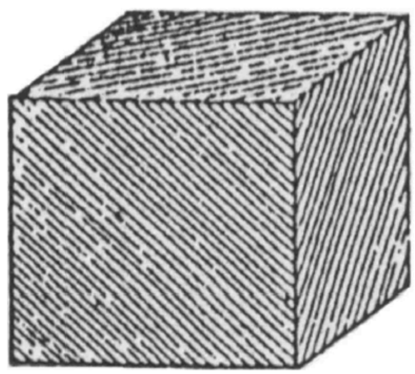


Figure 4: Cubic crystal of boracite with striations reducing symmetries of the cube.

Delafosse's work paved the way for Auguste Bravais, who recognized the importance of the rows and planes making up the internal structure of crystals. Bravais formalized Delafosse's ideas, first by shifting to modeling this internal structure as a system of regularly spaced points (figure 3c). Bravais' models were similar to Cauchy's use of points to model polyhedra (Senechal, 1990). This shift in representation allowed Bravais to determine all possible 14-point lattices within 3D space by allowing for easier application of rotations and reflections to points to find the unique symmetries for each lattice. Each point lattice or Bravais lattice is composed of a repeated system of points corresponding to the vertices and various bases (i.e. centers, faces) of a regular packing of parallelepipedon. From any point within a lattice, the lattice looks identical. Along with discovering these 14-point lattices, Bravais then was able to use these lattices to derive the 32-point groups of crystals that relate to crystals' external symmetry.

Bravais's lattices were noted by mathematicians, including Camille Jordan, who was concurrently exploring extensions of group theory to transformations (Senechal, 1990, Kleiner, 1986). Jordan extended Bravais's work by applying developments within group theory that included discrete and continuous (i.e. translations) groups of motions, outlining a subset of the space groups of crystals. While Jordan did not recognize the importance of these space groups to crystallography, in 1979 the crystallographer Leonhard Sohncke discovered Jordan's work and reinterpreted it using the existing crystallographic systems of points (Senechal, 1990). The space groups identified by Jordan and Sohncke did not account for all 32-point groups; thus, their lists were incomplete. However, by the early 1980's, Federov and Schoenflies independently expanded continuous motions to include improper motions (i.e., motions that require a combination of translations with either rotations or reflections). Discovery of these improper motions resulted in the final enumeration of all 230 3D space groups (Kubbinga, 2012).

This case suggests a rich interplay between investigations of mathematical and physical activity systems that give rise to conceptions of symmetries and their (group) structure. Changes in analytical tools and representations opened new ways to view and classify the structure of crystals and gave rise to new understandings of symmetries—as mathematical groups. Thus, the case illustrates how symmetry plays a critical role in visualization practices that have epistemic payoffs. The case that follows explores the role of symmetry in design practices.

Case 2: Symmetry in Cultural Artistic Expression and Anthropological Studies

Much of what is known about the role that symmetry plays in the artistic pattern design from different cultures come from anthropological studies. These studies apply Washburn and Crowe's (1988) adaption of group theory in mathematics to present an accessible theory of plane pattern analysis for the use of social scientists. Prior to the application of symmetry analysis, Washburn and Crowe (2004) note that anthropologists struggled to make sense of the geometrical patterns within nonrepresentational art. The then predominate analytical method utilizing features-and-themes often obstructed views of the regularity of pattern due to attention to decorative variations—i.e. change in use of decorative motifs or icons or color patterns inconsistent with the symmetries of the geometric patterns. For example, Washburn (2004) describes how previous features-and-theme analysis of the decoration on ceramics found in the Ica Valley of Peru highlight a change in motif in the ceramics of communities conquered by the Incans. However, later symmetry analysis of these motifs (figure 5) show a consistency in the symmetries used within these communities despite shifts in motif. Washburn suggests that this preservation of symmetry patterns serves as evidence that local communities preserve themselves in spite of external controls and influences. Thus, symmetry plane analysis has led to new visualization practices of anthropology, providing new conceptual tools (symmetry groups) and representational tools (e.g., a notational system as illustrated below) that have led to new ways of seeing or analyzing, classifying, and representing the patterns among these artistic expressions.

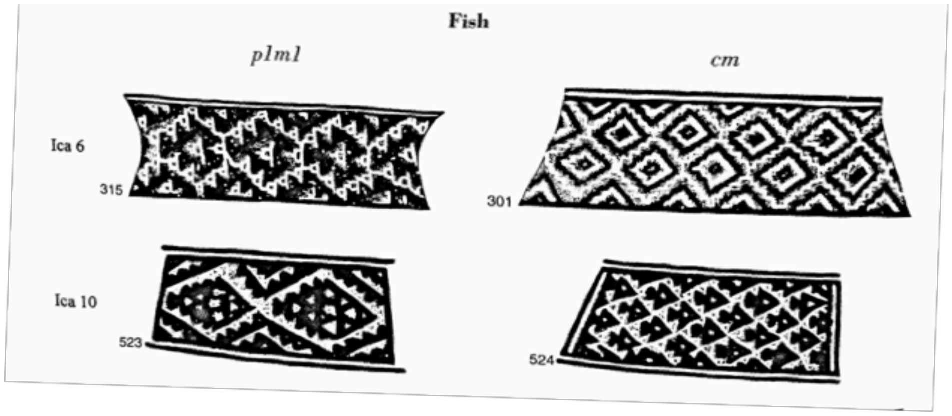


Figure 5: Consistency in Ica patterns based on symmetries over time despite change in form of motif. p1m1 (translation symmetry and reflection symmetry lines only in one direction), cm (two perpendicular reflection lines)

As a result of these changing visualization practices, anthropologists have made strides in characterizing design across cultural space and time. Additionally, they also have developed lenses that allow for more accurate study of design practices and shifts in cultural practices. For example, through interviews with individuals from the Hopi community about the decorative patterns used in their pottery (figure 6), Washburn (1999) learned that their use of an interlocking rotational symmetry motif represents communal beliefs about the relations between the cyclic nature of plant life and human life. Thus, within anthropological fields, symmetry has shaped classification practices, and it has become accepted as a tool to aid in generating explanations of cultural phenomena.

Figure 6: Puebloan pottery with rotational symmetry patterns (Washburn, 1999)

More prolonged studies of cultural crafting and art practices using symmetry plane analysis have also provided insight into the visualization practices that support the creation of designs and teaching of design patterns to younger generations within a community. In a study of Chinero textiles and Andean weaving practices, the Franquemonts discovered an intricate naming system for indexing different designs (2004). That naming system differentiates between basic design units. It also dictates how these units are put together to form patterns using isometric transformations. Figure 7 represents a basic design unit (chhili) composed by alternating between reflecting and rotating a triangular cell. This design unit is then combined with an identical unit through a "chongo" transformation or reflection to create a new pattern—chongo chhili. This naming and classification system has great consequence for weaving practices, as the size of a chunk of pattern that ever needs to be memorized is considerably less when the design is organized into small repeating chunks. Rather than remembering long rows of patterns, young Andean weavers first learn these basic units. After mastering these basic units, they then learn how to piece together units using different transformations. Thus, isometries and symmetries play a central role in shaping how this community of weavers classifies patterns (based on transformation), analyzes patterns (look for chunks and relations between chunks), and represents them (naming system that capture the relationship between parts and transformations).

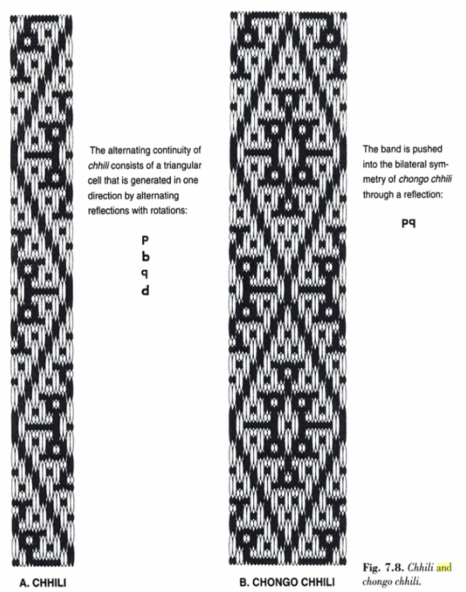


Figure 7: Chinero pattern and naming system representing how more complex designs are generated from units combined using a reflection.

The work of M.C. Escher is another rich example from art. Escher uses relations among symmetries to play with design variation. In fact, Schattschneider (2010), who has extensively studies the writings and work of Escher, states that most of Escher's inspiration came from the Islamic use of symmetry patterns he encountered on his multiple trips to the Alhambra in Granada, Spain. During the 1930's, Escher created sketches to recreate patterns he saw in the Alhambra. His brother, a geologist, quickly noted the relation between the periodic patterns of Escher's sketches and the periodic structure of crystal within crystallography. Subsequently, he sent Escher a paper written by Pólya, that outlined the four isometries of the plane (translations, reflections, rotations, and glides) and illustrations of the 17 plane symmetry groups that had previously been discovered by Federov. As previously mentioned, Federov was a mathematician and crystallographer who played a significant role in the development and application of symmetry group theory within crystallography (Senechal, 1990).

Figure 8: Example of Escher's use of color symmetry

Escher intensely studied Polya's illustrations and used them to guide his own design explorations and questions, which often mirrored those within mathematics. For examples, his sketchbooks show extensive classification systems for the tilings of quadrilateral and triangular systems. He developed his own notation system to identify how the edges of the tiles within his sketches of these systems were related as a way to eliminate repeated patterns. Additionally, Escher's explorations were among first systematic studies and applications of color symmetry. He experimented with how to color tilings in which symmetries bring like-color tiles to like-color tiles or bring all tiles to opposite colored tiles. An example of his use of color symmetry is shown in Figure 8. In this pattern, horizontal translation symmetries preserve color and vertical glide symmetries result in all yellow cats interchanging positions with black cats.

In spite of Escher's extensive engagement in mathematics, he never viewed himself as a mathematician and frequently claimed that he did not understand the very mathematics he was engaged in generating. As in the case of the Chinero weavers, Escher's classification of symmetry patterns and attention to the relations among symmetries were for the purpose of generating new designs. Today, mathematicians value Escher's artwork as highly accurate representations of abstract mathematical ideas.

Summary: Symmetry as a concept-in-practice across disciplinary and cultural activity

Looking across the range of cultural activities in which conceptions and applications of symmetry are generated and refined reveals that symmetry has historically played a consistent set of functions within visual practices. Although each activity uses different tools and representational systems to describe symmetry, those tools and systems are all used to describe and identify patterns, especially the variant and invariant aspects of those patterns. A second common feature among these activities is the tension between constraint and variation that either guides or motivates recurring activity. For example, within crystallography, this interplay between constraint and variation led to the exhaustive search of the number of 3D space groups possible. Escher created a personal system to generate novel patterns that relied on the regularities of quadrilateral tilings. Contemporary Chinero weavers explore new symmetry patterns that can be generated by combining established patterns given the constraints of the tools they use. It is these common features that result in a restructuring of activity into more mathematical activity. In each case, this process demonstrates how everyday activity leads to the generation of mathematical concepts and practices.

Identification of these common features has important implications for designing for young children's engagement in mathematical explorations of symmetries and isometries. In particular, they suggest features of activity systems that may be important if the desire is to build students' everyday activity into increasingly mathematical forms of activity. In mathematics, symmetry is most commonly used as a tool for classifying and analyzing space, whereas in other cultural practices, symmetry also serves as an important representation and design tool. There may be overlap between design activities and children's everyday activities (i.e., block play, drawing) that can be drawn upon to investigate symmetry and related visualization practices of classifying and analyzing. As in the cases just presented, this extension of symmetry for classifying and analyzing seems to be supported when there is sufficient constraint on the design possibilities. This balance between variety and constraint may encourage exploration of the possible variations within these constraints (i.e., how many unique tilings of the plane are there?). Thus, if design is used as an entry into mathematics, it is important to pay careful attention to how the activity balances constraint and openness sufficient to encourage exploration of possible variations. The cases also suggest that design may be a productive context for mathematical engagement beyond initial explorations of symmetry. There are many advanced questions that students can pursue, including questions about color symmetries and groups within other geometries. For example, in his later work, Escher also explored how to tile the hyperbolic plane (Schattschneider, 2010). Cases of craft, art, and crystallography suggest points of entrée to concepts and practices of symmetry and also connect to developmental pathways to more complex ideas. In the following section, I turn to developmental psychology studies that inform us about resources young children may have to begin engaging in visualization practices.

Children's Everyday Resources for Exploring Symmetries and Isometries

Because there is growing evidence that spatial reasoning predicts students' entry into and success in STEM fields, researchers are increasingly attending to relations between spatial reasoning and mathematics and science learning (Uttal & Cohen, 2013; Wai, Lubinski, & Benbow, 2009). However, little is known about how presumed general measures of spatial reasoning ability relate to the disciplinary visualization practices of professional scientists and mathematician, or to their epistemically based practices (Goodwin, 1994; Gooding, 2006; Stevens & Hall, 1998). Studies of expertise often find that the correlation between spatial reasoning measures and success within fields diminishes as participants attain greater expertise (Uttal & Cohen, 2012). These findings suggest that general visualization skills may be an initial resource for entrance into visually rich fields. However, general skills likely must become more "disciplined" (Stevens & Hall, 1998) as one participates more fully within a field and gains skill with discipline-specific tools and conceptual systems that mediate perception. Thus, although visualization may be central to mathematics and science fields, spatial reasoning measures are likely weak proxies for growth in mathematical visualization practices. In the following subsections, I explain that many of the measures of mental transformations that are featured in studies of spatial skills (i.e. mental rotation and reflections) fail to include important mathematical distinctions that are critical to differentiating among isometries. Nonetheless, in spite of these limitations, spatial reasoning measures may still identify some of the resources young children bring to bear to support the development of mathematical visualization practices.

There is some contest about the cognitive skills that constitute spatial reasoning (Davis et al., 2015). In this review, I focus on those skills that appear best aligned with supporting understandings of symmetries and isometries. These skills include 2D and 3D mental rotation, 2D bilateral reflection symmetry detection, and the discrimination of 2D mirror reversals. I conclude this review by coordinating these bodies of research to outline cognitive resources that young children are apparently developing that can be leveraged in mathematical explorations of symmetries and isometries. In addition, I highlight instructional scaffolds explored within these studies that may further encourage children to utilize and build on these resources.

Developing Perception of Reflection Symmetry and Identification of Mirror Images

Because symmetry detection is critical in many aspects of biological and social life, visual and cognitive scientists have devoted extensive research to determine how symmetry detection develops, how it is related to the physical structure of many animal visual systems, and how to best model symmetry detection. However, research findings repeatedly find that there are biological advantages to recognizing reflection symmetry, so most research has focused exclusively on perceiving and detecting reflection symmetry (Wagemans, 1995). In fact, in most studies, symmetry refers only to bilateral reflection symmetry. Thus, although we have some knowledge about the development of the cognitive underpinnings for reflection, we know much less about the development of thinking relevant to rotation and translation symmetry.

The studies that do include rotation and translation symmetry provide little insight into how perceiving these forms of symmetry develops. As a consequence, I do not extensively review these studies. However, Wagemans (1995) briefly reviews the handful of studies that compare the accuracy and speed of detection of rotation and/or translation symmetry, compared to reflection symmetry. He reports that among these studies, regardless of the form of stimuli used, the findings consistently show that adults are significantly more accurate and faster at detecting reflection symmetry than at identifying rotation and translation symmetry. In addition, the studies find no difference between adults' perception of rotation and translation symmetry. These findings support the theory that there is a biological predisposition to recognize reflection symmetry.

Reflection symmetry detection is regarded as a component of perception that helps organize the way we perceive and remember images and objects. In fact, studies show that symmetry shapes visual processing to such an extent that human mental representations of objects are more symmetrical than the true form of physical objects (Bruce & Morgan, 1975; Treder, 2010). There are a number of physiological, biological, and cognitive theories to account for this predisposition to recognize and represent reflection symmetrical objects. However, it seems evident that this ability emerges in infancy, is widespread, and is common across a number of animal species (Wagemans, 1997) Although some aspects of the perception of reflection symmetry appear to be innate, mirror reversals or chirality between objects that are important for processing reflection isometries, especially in 3D, do not seem to be as universal (Dehaene, Izard, and Pica, 2006; Kolinsky, Verhaeghe, Fernandes, Mengarda, Grimm-Caral, & Morais, 2011). Consistent with my interest in drawing implications for the design of learning ecologies, I next address factors that might be related to better coordination of reflection symmetry perception and the discrimination of objects and their mirror images. The ability to coordinate these two ways of visualizing could be important for learning to see reflection symmetries as a subset of isometries. I begin with an overview of what is currently understood about the development of reflection symmetry perception, beginning in infancy.

Studies on the development of infants' reflection symmetry perception find that infants are able to recognize only vertical reflection symmetry (Bornstein, Ferdinandsen, and Gross, 1981). However, Fisher, Ferdinandsen, and Bornstein (1981) suggest that this vertical symmetry detection in infants operates as a form of aesthetic preference rather than as an aid to visual possessing and generating mental representations of visual stimuli. In a study they conducted, four-month-old infants showed looking preferences for vertically symmetrical figures, in contrast with horizontally symmetrical and asymmetric figures. When a horizontally symmetrical figure and an asymmetric figure were paired, the infants showed no looking preference. In contrast, when the same infants were habituated to a vertically symmetrical figure and then shown a pair of figures that included the familiar figure and a novel vertically symmetrical figure, they did not look longer at the familiar figure, suggesting that they did not recognize it. This biological preference for vertical reflection has also been found among other animals, including but not limited to various species of birds, bees, and macaques (Beck, Pinsk, Kastner, 2005). This predisposition to notice vertical reflection symmetry also appears to play a role in shaping the continued development of reflection symmetry in childhood.

Studies suggest that young children are increasingly able to use symmetry to organize perceptual processing and the formation of mental representations of objects. Children's initial preference for vertically symmetrical objects may support earlier abilities to remember and recreate objects with vertical symmetry in comparison with objects that exemplify different reflection symmetry orientations. Bornstein and Stiles-Davis (1984) designed a set of cross-sectional studies to further explore the development of symmetry perception among four- to six-year-old children. In the first study, they familiarized children to a set of polygons. They then showed children pairs of closed polygon figures that included an asymmetric polygon and a polygon with either vertical, horizontal, or diagonal reflection symmetry. Within each pair trial, children were asked to identify which of the two was the familiar polygon. Four-year-olds were able to distinguish only between vertically symmetrical and asymmetrical figures. The five-year-olds could distinguish between vertically symmetrical and asymmetrical figures and horizontally symmetrical and asymmetrical figures. The six-year-olds could discriminate among all three lines of reflection symmetry. These findings are aligned with other studies that suggest a consistent developmental order to children's symmetry perception in which the distinction and mental representation of symmetrical stimuli begins first only with vertically symmetrical figures, followed by figures with horizontal symmetry, and finally by recognizing and remembering diagonally symmetrical figures (Treder, 2010).

This development seems to happen gradually and, based on findings from Bornstein and Stiles-Davis's second study (1984), it tied to an aesthetic preference that supports the ease of processing of a gestalt sense of wholeness. Bornstein and Stiles-Davis replicated their first design but replaced the closed polygon figures with dot drawings. In this case, only the oldest group participated. With this change to dot figures, six-year olds showed only a consistent ability to discriminate between vertically symmetric and asymmetric figures. The researchers concluded that children's ability to recognize reflection symmetry at different orientations seems initially limited to simple, closed figures.

These first two studies provided no insight into how the orientation of the reflection line influences perception. Thus, in a final study Bornstein and Stiles-Davis (1984) examined the extent to which the reflection symmetry line is used as a tool to organizational perception. They assessed four, five, six-year-olds. and adults' recreations of symmetrical dot drawings both for accuracy and process. All age groups were able to generate accurate recreations of simple dot drawings with vertical reflection symmetry. Moreover, all age groups used the vertical line of reflection to organize their drawing process. For example, participants typically mapped points across the axis of reflection by drawing a point to the left and then its corresponding point to right. Ability to use horizontal and diagonal reflection lines to organize drawing was related to both drawing accuracy and age. Accordingly, the five- and six-year-olds were more accurate than the four years olds in recreating dot figures that had horizontal reflection symmetry, and they also more consistently organized their drawing around the reflection line. The six-year-olds were better than both the five- and four-year-olds at using the diagonal reflection line to recreate drawings with diagonal reflection symmetry. These results suggest that the reflection line plays an increasingly central role in children's visual processing of reflection symmetrical figures. The mapping of opposite points that children displayed in their drawing processes also mirrors the mathematical definition of reflection symmetry—namely, a figure is symmetrical if points can be mapped directly onto the opposite points across the line of reflection. It may be that children's drawing can serve as a context that leverages everyday activity to generate mathematical understanding.

Bornstein and Stiles-Davis's (1984) findings depict a transition in children's early preference for symmetrical figures toward a more intuitive conceptual understanding of the invariant properties of figures that have reflection symmetry. Yet, it is unclear from this set of studies how children develop the ability to use reflection lines beyond those that are vertical. Some studies suggest that the superior performance on remembering and representing figures with vertical reflection symmetry may be caused by a priming effect. That is, because objects with vertical reflection symmetry are more frequent in our environment, greater attention to this form of symmetry is promoted. For example, Wenderoth (1994) found that when adults were shown blocks of figures that have an axis of reflection symmetry between horizontal and 45 degrees, they were less successful at discriminating between asymmetrical and vertically symmetrical figures. However, when figures were not blocked and there was an equal proportion of horizontally, vertically, and diagonally symmetric figures, participants were faster at recognizing the vertically symmetric figures. Thus, it could be that we simply receive less everyday exposure to figures that have other axes of reflection. As a result, our attention is primed to recognize vertical reflection symmetry more quickly.

It may be, therefore, that helping children improve at recognizing a range of symmetrical figures may be supported by providing time to explore figures and objects that vary in the orientation of their symmetrical lines of symmetry. However, further studies need to done to determine the effects of repeated exposure to figures with horizontal and diagonal reflection symmetry. A plausible alternative is that young children may not yet have developed sufficient conceptions of horizontal or diagonal to gain relevant perceptual information from repeated exposures to these figures. This may particularly be the case for diagonal symmetry. Olson (1970) found Across a number of studies of young children's (years 3-7) ability to construct the diagonal of a square using a checkerboard and checkers, Olson found that children's success was based on their concepts of relevant spatial attributes (1970). In particular, younger children (3-4-year-olds) often failed to construct the diagonal because they over-relied on topological cues of proximity, edges, and reference axes to guide their perception of the pattern (i.e., using the sides of the board to guide the construction of a straight line down the middle of the board). These topological features are irrelevant for constructing a diagonal, and often prevented the young children from recognizing the diagonal as a structured pattern. In contrast, the six-year-old children were almost always successful at constructing the diagonal; however, they often attended to more geometric attributes such as corners and straight. Olson further found that cueing children to examine how the diagonal started and ended at opposite corners served as a successful instructional scaffold only for those children at a transition stage (marked by correctly placing at least 2 consecutive checkers in the diagonal). The youngest children would often hear this cue and, in response, place two checkers in opposite corners--but complete the rest of the diagonal incorrectly. Although constructing the diagonal of a square and identifying a diagonal reflection plane are distinct tasks, Olson's (1970) findings suggest that the development of diagonals may not merely be a matter of increased exposure, but may, rather, depend on explicitly helping children attend to attributes of space that support an understanding of orientations other than vertical and horizontal.

All of the studies on children's perception of reflection symmetry use two-dimensional stimuli. Thus, little is known about how children perceive symmetry in three-space, in spite of the fact that the vast majority of symmetrical figures in their lives are three-dimensional. From a mathematics learning perspective, this gap is significant, especially because understandings of reflection symmetry and isometries in two-space do not necessarily generalize to these concepts in three-space. For example, children in these studies are often encouraged to think of visual folding—a form of rotation—to support their representation of reflection symmetrical figures. This strategy does not work reliably in three-space because reflections cannot always be modeled as rotations. Instead, symmetries and isometries in three-space require matching of parts of a structure that share the shape and size and that differ only in being oriented oppositely based on their positioning around a shared plane of reflection.

Studies on the recognition of mirror images as distinct from their non-transformed image provide some insight; however, these studies also use only 2D stimuli., They help illustrate challenges children encounter when using a sense of opposite orientation or mirror reversal as a distinguishing characteristic of figures only when these figures can be rotated in three-space into direct congruence. In particular, these studies highlight that using opposite orientation as a property to make form distinctions often arises only when this attention to orientation is consequential to the activity, as in reading and writing. For example, Casey (1984) found that preschool-aged children's performance at distinguishing between line drawings of 2D figures and their mirror image increased as they became more proficient with writing their names using correct letter form and orientation. Young children make little distinction between mirror images until they are at preschool age. This is likely because this is when they receive instruction that explicitly focuses their attention on the importance of orientation as a relevant property to help discriminate between shapes and forms.

Kolinsky and colleagues further support the conjecture that development is tied to exposure to activity in which orientation is meaningful, rather than reliant solely on maturation (2011). Kolinsky et al. assessed adults with varying literacy skills on their ability to use mirror image distinctions to guide the classification of figures. Participants in the study fell into three categories of literacy status—literate, ex-illiterate (that is, they received literacy classes as adults), and illiterate. The illiterate adults were worse than the other groups at using orientation as a categorization system to sort figures that were mirror images. There was no difference between the ex-illiterate and literate adults, providing further evidence that literacy instruction seems to promote attending to orientation.

Findings from these two studies suggest that writing helps to influence the perceptual attributes one learns to attend to and notice as important in representing distinction within that form of media. However, Olson (1970) points out that the use of these perceptual attributes may not automatically be generalized across different cultural forms of media. For example, he found that preschool children who could draw the diagonal of a square were often unsuccessful at constructing the diagonal using. Thus, Future studies need to explore scaffolds that support children in attending to orientation within and beyond literacy instruction, because scaffolds across contexts (i.e., writing, art, and mathematics) may differ in their effects, or, alternatively, may interact in beneficial ways to support learning. In the following subsection, I shift the focus to a different form of spatial reasoning, mental rotation.

The Development of Mental Rotation (Rotation Transformations)

Shepard and Metzler's (1971) study of adult mental rotation is considered a hallmark study of mental transformation. Their findings provided initial empirical evidence for parallels between mental transformation processes and manipulations in the physical world. They presented participants with pairs of 2D line drawings of asymmetric cube structures (Figure 9).

In each pair, one structure was the target figure and the other was either directly congruent through a rotation of various degrees or congruent only through a reflection and rotation. Participants were asked to determine whether the transformed structure was directly congruent to the target or a different 3D structure (mirror image). The pairs varied the angle of rotation of the transformed structure. Both response time and accuracy rate were measured and recorded.

Adults in the study were highly accurate at mentally rotating objects to make distinctions of direct congruency. In addition, there was a consistent positive linear relationship between response time and angle of rotation. That is, as the difference in angle of rotation between the target and transformed structures increased, adults took longer to make congruency decisions. Shepard and Metzler interpreted this finding as indicating that mental rotation closely mimics the process of physically rotating the object. Today, psychologists continue to provide additional evidence supporting these findings and interpretations, and they rely on the initial methodological design to guide the design of variants of the original task.

In the past two decades, developmental psychologists have pursued further study of children's development of mental rotation. The current body of studies suggests that this development begins in infancy and becomes increasingly flexible and accurate throughout childhood (Möhring and Frick, 2013; Frick, Daum, Walser, & Mast, 2009). However, methodological variations across the studies highlight potential experiences and scaffolds that play a critical role in children's development of mental transformation performance. Thus, the next section analyzes methodological variations among studies of infants and children's mental rotation abilities. and the ultimate aim is to draw implications that these variations might have for the activities and interactions we make accessible to children.

Studies on mental rotation in infants are relatively recent. The earliest studies appeared in

2008, and the collective findings are highly variable. Some have found an infant-male advantage on mental rotation skills (Moore & Johnson, 2011), but others find no evidence of gender difference among infants (Schwarzer, Freitag, & Buckel, 2010; Möhring & Frick, 2013). One proposed explanation is the extent to which studies attend to infants' developing motor abilities. In general, studies that demonstrate this sex difference appear to give little attention to infants' motor experience and skill. For example, Quinn and Liben (2008) report that among three-to-four-month-old infants, only male infants showed an ability to discriminate between 2D images that were congruent to an image through only rotation and those congruent through a reflection and rotation. Male infants gazed longer at the novel, mirror image figure and females showed no preference. In this study, infants first were visually habituated to a 2D target stimuli shown under various degrees of rotation. Next, the experimenter showed the infants a pair of figures that included one that was congruent through rotation and another congruent through a reflection and rotation (Figure 9). Each figure in a pair showed the same degree of rotation, and these degrees of rotation were different from those the infants were exposed to during habituation. This is a significant variation from Shepard and Meltzer's original task. Furthermore, the 2D stimuli raise some critique of mathematical validity. As mentioned in the mathematical introduction, in 2-space, reflections are rotations in 3-space. Thus, both figures in the pairs that Liben and Quinn (2008) presented to infants are mathematically congruent through rotation. It is difficult to know, based on their design, whether some infants showed no gaze preference because they correctly perceived the mirror image as congruent to the habituated figure. This possibility seems consistent with Shepard and Metzler's (1971) finding that the angle of the axis of rotation failed to affect adult mental rotation performance— response-times and accuracy rates of mental rotation were the same when the axis rotation was within the 2D plane or 3-space.

Moore and Johnson (2008 & 2011) provide some evidence in support of Quinn and

Figure 9: Stimuli used by Liben and Quinn (2008). Each pair (horizontal rows) are mathematically congruent.

Liben's (2008) finding of a male-infant advantage on mental rotation tasks. To avoid the validity issues entailed with 2D stimuli, Moore and Johnson (2008 & 2011) used videos of three-dimensional, asymmetric figures rotating around a fixed axis through a consistent turn angle. In their study, five-month-old and three-month-old infants were first habituated to a 3D structure by watching video of a structure rotating through the first 240 degrees of a whole turn rotation. Experimenters then measured infants' looking time at videos of either the habituated structure rotated through a portion of the final 120 degrees of a whole turn rotation or the mirror image of the structure rotated through the same portion of 120 degrees. Male infants at both three and five months could discriminate between the novel and familiar stimuli, whereas the female infants at the same ages showed no looking preference. Based on this set of studies using only mental rotation tasks with visual stimuli, the authors concluded that male infants show a consistent ability to discriminate between a rotated structure that is directly congruent to a habituated target structure and the rotated, mirror image of the same target structure. The researchers cited this finding as evidence that male infants are born with better mental rotation skills.

In a pair of studies, Schwarzer, Freitag, and Buckel (2010) and Möhring and Frick's

(2013) provide findings that are inconsistent with those of Quinn and Liben (2008) and Moore and Johnson (2008, 2011). In both studies, Schwarzer, et al. found that differences in motor experiences affect infants' mental rotation abilities; they also found no effect of gender. Schwarzer et al. (2010) used the same methods as those used by Quinn and Liben (2008) with nine-month-old infants; however, the images were of three-dimensional figures to compensate for the validity issues previously mentioned. They found that mental rotation was positively correlated with nine-month-olds' crawling ability. Both female and male infants who were proficient at crawling for more than two weeks prior to testing performed better than nine-month-olds who did not have crawling experience. In addition, Möhring and Frick (2013) studied how experience at manually exploring a three-dimensional object affected six-month-old infants' ability to mentally rotate the object. Infants were allowed to either manually explore or visually observe someone else mimic manual exploration of a novel 3D object. Then the infants were visually habituated to the object through a series of videos that showed the object rotated at predetermined angles. In the test phase, both groups of infants were shown the same object or its mirror image at a variety of novel degrees of rotation. Male and female infants in the manual exploration condition looked longer and at equal rates at the rotated mirror image. Neither male nor female infants in the looking condition showed any looking preference.

The findings from Schwarzer et al. (2010) and Möhring and Frick (2013) are consistent with a rich body of literature showing that manual exploration and motor experience enhance children's spatial structuring and representations. For example, to determine the relation between motor development and infants' 3D object exploration, Soska, Adolph, and Johnson (2010) used infants' ability to sit independently as a proxy for further developed motor control. After a group of self- and a group of non-self-sitting infants interacted with a 3D object, only the self-sitting infants reliably discriminated between 2D images that either accurately represented the target object as 3D or incorrectly represented the structure as flat. Further comparison of the infants' interactions revealed that the self-sitting infants were better able to explore the object by coordinating physical manipulation with the direction of their gaze—i.e., they were able to see themselves physically manipulate the object. Infants who could not sit independently did not reliably coordinate looking at the object with their physical manipulation of the object. These results, along with those of Schwarzer et al. (2010) and Möhring and Frick (2013), suggest that initial infant development of forms of mental representations and transformations, especially mental rotation, is coupled with the development of motor skill and coordination with visual input. These findings suggest two questions that deserve further research: a) to what extent does delayed motor skill development impact the development of forms of mental transformations? and b) how do material supports that aid infant physical object exploration influence the development of mental transformation skills?

The tie between motor experience and mental rotation continues to play a role in studies of mental rotation with young children and adults. In a cross-sectional study involving five-, eight-, eleven-year-olds, and adults, Frick, Daum, Walser, and Mast (2009) found that five- and eight-year-olds' mental rotation performances were slower when they were performing an interfering motor activity. In contrast, this interference had no impact on the performance of the eleven-year-olds and adults. It appears that as children age, their mental rotation skills may become less reliant on a coordinated physical manipulation. However, this study does not determine whether improvement in performance is due to improved mental rotation skill or to improvement at controlling the effects of the interfering stimuli. Moreover, even if improvement is tied to an increase in mental rotation skill, little is known about how and why these skills have improved.

A growing body of research looks more closely at factors that may provide insight into some of these questions, especially around the issue of what supports the development of greater mental rotation skills and other spatial reasoning skills. In particular, researchers are analyzing adult and child interactions during activities that require physical and mental manipulation of objects (e.g., in block and puzzle building and traditional mental rotation tasks). These studies consistently find that children who receive more exposure to and use language and gestures that highlight salient spatial information during these different transformation activities perform better on a measure of mental rotation (Ehrlich, Levine, and Goldin-Meadow, 2006; Pruden, Levine, and Huttenlocher, 2011). Thus, one factor that appears to improve children's mental rotation skills is the development and use of communication resources that help them to organize and highlight processes of spatial transformation.

Some researchers have tried to explore the utility of increasing children's exposure to and use of spatial language and gestures during spatial play as a way to improve children's performance on measures of spatial reasoning, including mental rotation. Unfortunately, many of these studies rely on very brief intervention periods, and none shows an influence on spatial skills other than on those directly related to block building and puzzle assembly. For example, Casey and colleagues (2008) designed a block-building intervention that was tied to storytelling for kindergarten students. The intervention was spaced over seven days, during which students worked in small groups to build structures to solve problems faced by the characters in the story (e.g., building a bridge to cross a river). At the end of the intervention period, students in the intervention condition performed significantly better on a block building assessment designed to measure the complexity of students' constructions. However, there was no improvement in their performance on a mental rotation task.

Similarly, Young, Cartmill, Levine, and Goldin-Meadow (2014) designed a two-day puzzle building intervention. Children were assigned to conditions that varied in adult use of spatial language and gesture, and the participants completed follow-up tasks in puzzle assembly and mental rotation. Compared to those in the other groups, those children who were provided with spatial language and gestures improved in their puzzle assembly accuracy and speed. Yet, as in Casey and colleague's study (2008), there was no improvement on children's mental rotation skills. In general, children provided with language and practice improve on those visual skills directly taught in the intervention but show no improvement in more generalizable spatial reasoning tasks. It has not been established whether this failure is because of the relatively short duration of the interventions or because the designs are failing to provide activity structures that support development. Previous studies have established that simple training that involves practicing mental rotation tasks improves on the performance of both children and adults on related assessments. Thus, perhaps interventions that encourage children to practice these skills during everyday activities may be effective. (Uttal & Cohen, 2012). One way to further explore this relationship might be to look beyond explicit training in gesture and language to examine how parents and children naturally engage in spatial play activities. Naturally occurring variations in activity structures may be found to correlate with children's mental rotation and related spatial reasoning skills.

Summary

Literature on the development of children's mental rotation skills and recognition of bilateral symmetry and mirror images has identified some of the resources that might equip young children to engage in mathematical explorations of symmetry and isometry as they enter their school years. However, the development of these resources is probably not due to maturation alone; it seems to be tied to experiential and interactional supports. In particular, these resources seem to develop when children are given frequent opportunities to engage with objects by manipulating and constructing them. Moreover, the benefit of these experiences may be contingent upon interactions with adults who provide children with language to describe the spatial relationships and properties of the objects being manipulated. More research is needed to determine how variation in spatial play activities affords or constrains this development. it is also likely that there is a wide range of cultural activities that young children engage in that can support spatial reasoning development, although so far, studies have emphasized drawing, block building, and puzzle building. By expanding the research to additional cultural activities, we may find more opportunities to study the contextualized development of children's spatial reasoning. Identifying complex, everyday learning contexts may also provide a greater understanding of the relationships between spatial language, gesture, physical manipulation experience, and spatial reasoning.

An important limitation of the literature is its failure to attend to distinctions among isometries that are critical for mathematicians and related cultural practitioners. In particular, no attention is given to differences between 3D and 2D rotation and reflection symmetries and isometries in three-space or to the differences between direct congruence and congruence in three-space. For example, mental rotation studies regard mirror images as "different," raising questions about the validity of the findings in relation to other studies that find that young children have a difficult time discriminating between mirror images. After all, if young children can recognize that the rotated mirror image of an object is distinct from the object's rotated direct copy, then they should have no trouble discriminating between mirror images and direct copies of objects when they are not rotated. More studies need to explore explanations for the difference in the findings between mental rotation studies and mirror image discrimination studies.

Another mathematical concept of symmetry unexplored in the developmental literature is children's understandings of figures with rotation symmetry. Current research often makes no distinction between different forms of symmetry; most studies assume that symmetry means bilateral reflection symmetry. We do not yet know if children are also able to use mental rotation of figures to identify figures with rotation symmetry, and whether children treat rotation symmetry as a salient attribute of shape.

Visualization practices play a pivotal role in mathematics and the generation of new mathematical knowledge. Spatial reasoning may help support the development of mathematically specific forms of visualization. Yet, measures of spatial reasoning are limited as assessments of mathematical concepts; their exclusive use constrains our ability to produce insights into the development of children's mathematical understanding and visualization practices. Although researchers have reported correlations between measures of spatial reasoning and mathematical performance, spatial reasoning tasks should not be positioned as mathematical activities. Further research needs to focus both on how spatial reasoning skills are related to everyday forms of visualization practices and how they may be related to more disciplinary forms of these practices.

Instructional Designs that Support Children's Co-Development of Mathematical Practices and Concepts of Symmetries and Isometries

NCTM and Common Core's emphasis on the co-constitution of mathematical practices and concepts marks an important shift towards sociocultural views of learning. Within this sociocultural framing, learning is seen as embedded in collective forms of practice that are context dependent, informed and shaped by the material and representational tools available, and constrained by individual and shared histories of members within the collective (Lave and Wenger, 1991). Mathematical learning, then, is tied to mathematical practices and concepts. In order for these practices and concepts to develop, students need to be engaged in recurring activity systems that encourage the negotiation and refinement of local mathematical practices that are the primary route by which knowledge and concepts are generated and shared (Lehrer, 2009).

With this sociocultural shift, there has also been a move towards organizing instruction around "big ideas," or properties used for identifying core kinds of mathematical objects, rather than individual concepts. Big ideas can serve as generative and shared resources that allow students to participate in creating and expanding their local mathematical system (Schifter, Russell, and Bastable, 1999). Transformations, including isometries and symmetries, are an example of a big idea within mathematics, as they can help define and classify mathematical objects through the analysis of variant and invariant properties. Thus, a critical aspect of transformations is their importance as tools within epistemic mathematical practices, such as defining, that extend definition and classification to the visual realm.

Transformations also require and support students' engagement in visualization practices. Proponents of integrating and centering spatial mathematics within K-12 mathematics curriculum have long advocated for the central role that visualization plays in aiding in developing new mathematical insights, representational forms, practices, and concepts (Eisenberg & Dreyfus, 1991; Lehrer & Chazan, 1998). This argument has recently received renewed attention, as a growing body of research continues to report predictive relationships between students' spatial reasoning abilities (i.e. mental rotation, mental spatial structuring) and later access and success in STEM disciplines (Wai, Lubinski, & Benbow, 2008; Uttal & Cohen, 2012). Yet, despite these findings, less is known about how these every day, informal visualization skills develop into more disciplinary forms of visualization practices and become integrated with and support epistemic practices. Given the goals outlined by NCTM and Common Core to support students in co-developing mathematical concepts and practices, future instructional designs need to focus more closely on the development of more mathematical visualization practices and their coordination with epistemic practices.

Within this section, I review the limited number of research programs that engage young children (PreK-3) in mathematics instruction that rely on symmetries and isometries to serve resources to inform the design of future learning ecologies. In particular, I look at the range of isometries and symmetries students are exposed to and the forms of activities designed to engage students in developing conceptual understandings of these transformations. Finally, I examine how these activities afford the co-development of conceptual understanding and mathematical practices—including both visualization practices and epistemic practices. To organize this review, I have identified three approaches to integrating isometries and symmetries into early elementary curriculum based on each approach's systematic treatment of isometries and transformations as a class of transformations tied to mathematical practices.

Approach 1: Informal Use of Isometries and Reflection Symmetry as Analytic Tools to Solve Geometric Puzzles

The National Research Council (2006) identified a need for greater integration of spatial thinking into the curriculum; however, there is little consensus about how to best accomplish this objective. The approach adopted by two research teams, Moss, Hawes, Naqvi, and Caswell (2015) and Sarama and Clements (2004), is to infuse spatial reasoning into the mathematics curriculum by designing activities that encourage students to practice mentally representing and transforming geometric shapes.

Within these programs, isometries are introduced as everyday motions (i.e., flips, turns, and slides) and symmetries are limited to bilateral horizontal and vertical reflection symmetry. The range of activities includes composing and decomposing 2D shapes, completing bilateral reflection symmetric patterns with pre-specified reflection lines, and identifying congruent 2D shapes and 3D objects. Both of these programs found that this range of activities supports improvement on a number of measures closely related to those featured in the instruction. For example, Hawes and colleagues (2017) report that after a 32-week-long intervention, students in K-2, showed gains on items including the naming of 2D shapes, the composition and decomposition of 2D shapes, paper-folding, and 2D mental rotation. Similarly, Clements and colleagues (2011) report that Pre-K students improved on their abilities to decompose and compose shapes, identify shapes, and compare shapes based on properties (i.e. sides and angles).

Although these improvements are promising, they do not convey much about growth in students' mathematical understandings of isometries and reflection symmetry as transformations in which properties of size and shape are invariant. Moreover, in these interventions students do not explore how transformations can be further specified by amount of turn angle and distance between an image and pre-image, number of and orientation of reflection lines. Finally, students never explore the relationships among these transformations—i.e., how is a "flip" or reflection isometry related to reflection symmetry or what compositions of flips are equivalent to turns or slides? Thus, connections between activities from a conceptual standpoint are left unarticulated, and the sequencing of activities is based on the complexity of the shapes and objects students are asked to transform. A strength of this approach is that isometries and reflection symmetry are developed as tools to aid in solving geometric puzzles, but the importance of relations among isometries and symmetries is not systematically supported. Nor does there appear to be any systematic attention to engaging children in mathematical practices other than puzzle solution.

Approach 2: Isometries and Symmetries as Tools to Analyze Invariance

The second approach is motivated by articulations of mathematics as the science of patterns and structure, in which the goals of mathematics are to explain the variant and invariant nature of patterns and structures (Devlin, 2012). In alignment with this view, Mulligan and Mitchellmore (2016) developed an early elementary mathematics program to support students in visualizing, modeling, representing, and making generalizations about mathematical structures and patterns.

Unlike the previously reviewed approach, the activities within this program are designed to follow a developmental sequence within domain areas of K-2 mathematics. Within the symmetry and transformation "pathway," students first are introduced to 2D reflection symmetry in the context of folding familiar and geometric symmetrical shapes. Then, students explore translations; however, doing so occurs only after students receive exposure to repeated patterns. The activity begins with students giving directions for walking in a straight line and then shifts to generalizing this experience to describe strip patterns with translation symmetry. Next, students move to rotation isometries—making distinctions between quarter, half, and whole turns—before they apply these turns to think about rotation symmetry. Within the articulation of these activities, there is a clear sense of how different forms of symmetries and isometries should be sequenced within the program. Yet, it is unclear how the sequencing of these activities is consequential for students' understanding, as they do not support students in coming to see these forms of isometries and symmetries as related beyond the connection between rotation isometries and symmetries. This isolation of concepts marked by brief activities that do not seem to coherently build on one another is unlikely to support students in engaging in mathematical practices in which generating knowledge and connections within the developing mathematical system is taken as a shared goal. Thus, the development of visualization practices within these activities leads to conceptual understandings of isometries and symmetries as isolated forms of transformation. The structure and pattern emphasis does not appear to extend to conjectures about the relationships among transformations.

Approach 3: Isometries and Symmetries as a System of Transformations

The final approach, illustrated by Lehrer and colleagues (1998), contrasts to the previous two approaches in that it featured children's engagement with a local craft of quilting to investigate 2D isometries and symmetries rather than isolated and short design activities. This study was embedded in a larger study of children's geometry that built from children's embodied experiences of space; thus, the students and teacher had developed significant shared mathematical resources. There was also an established mathematical culture that privileged students' asking mathematical questions, posing conjectures, and empirically investigating these conjectures; thus, students engaged in epistemic practices—taking an active role in constructing mathematical knowledge.

Within the quilting and isometry and symmetry study, students engaged in a 5-week-long exploration of the relationships between multiple forms of 2D isometry and symmetry as they asked questions that originally served the purpose of expanding the variety of quilt designs produced in the classroom. After each student designed a paper quilt by manipulating an individual core square using isometric transformations (rotation, vertical reflection, horizontal reflection) to create a two-by-two square quilt, students shared their quilts so their peers could consider how they went about making their quilts. To facilitate the sharing of designs, students generated a notational system to describe the isometries used to create their quilts. As students explored the quilt designs of peers and the representation of the isometries used, they conjectured that their notational system included distinction between flips or reflections (i.e., right flips and left flips) that were inconsequential to the final design. To test this conjecture, they examined a number of student-generated quilts that used right flips and left flips, and determined that although the movements looked distinct, the results were equivalent. Through this empirical investigation, students revised their notation system to remove distinctions between these equivalent isometries. They also invented and defined a new form of reflection—a diagonal reflection—that was not part of their original set of isometries. Children's invention and focus on equivalence demonstrates a moment in which their shared visualization practices—analyzing patterns, representing patterns, and classifying patterns-- supported engagement in epistemic practices—definition of equivalence, conjecture and empirical demonstration. These discoveries then opened opportunities for new conceptual understanding about the relationships among isometries. Students' investigations of quilts eventually led them to discover relationships between symmetries (both turn and reflection) and isometries as a consequence of the different patterns in students' starting core squares. For example, students noticed that an even number of repetitions of a reflection resulted in the original state and that two reflections were equivalent to a rotation.

This study illustrates that providing children with conceptual tools, such as symmetry, congruency, and isometry, to apply in everyday activities, can support the co-development mathematical concepts and epistemic mathematical practices. Students actively engaged in a range of these practices, including design, analysis, representation, and classification. Not only is the structuring of activity important for the application of spatial tools, but the development of socio-mathematical norms is also critical. Socio-mathematical norms are tacit, mutually agreed classroom values and expectations that guide and inform the roles and structure of discourse that constitute productive participation in mathematical practices. These norms are critical for guiding and informing the roles and forms of discourse that constitute productive participation in mathematical practices (Yackel & Cobb, 1996). Because these norms do not simply appear and, indeed, are often unlike the norms usually in play in classrooms, attention to supporting and fostering disciplinary norms requires intensive work by teachers. Jacobson and Lehrer conducted a study that compared teachers with and without extensive professional development about children's geometry thinking and understandings of space. They found that the teachers' knowledge and practices in relation to the content and epistemological goals played a significant role in shaping their classrooms' mathematical culture and students' conceptual understandings of isometries and symmetries within the quilting context. Compared to other instructional programs, Lehrer and colleagues' design may not be easily implemented within a short period of time. Teachers need to undergo a significant shift, not only in their mathematical/conceptual understanding, but also in their values and beliefs about the epistemic nature of mathematics and teaching practices.

Summary: Instructional Designs

In this review, I began by articulating a view of isometries and symmetries as a concept-in-practice. Adopting this view has consequences for the forms of learning environments that one would design to support the systematic development of transformations in early elementary grades. Across forms of activity, isometries and symmetries are powerful tools that support the visualization practices of classification, spatial analysis, and representation. The coordination of these visualization practices leads to new forms of activity and they generation of new concepts and knowledge as they become integrated with larger systems of knowing or epistemic practices.

NCTM has named isometries and symmetries as big ideas across K-12 mathematics. Moreover, developmental studies have suggested that young children have robust everyday visualization practices, that employ in formal isometries to make sense of the structure and patterns in their everyday world. And yet, there is a limited body of research on how to support students' systematic development of mathematical understandings of isometries and symmetries that articulates the relationships between these transformations. The existing studies do provide some insights into the accessibility of these concepts for young students and some initial visions of how these ideas might co-develop with mathematical visualization and epistemic practices. These studies consistently show that young children can easily grasp initial understandings of 2D isometries (reflections, translations, and rotations) when they are represented as every day motions. In addition, consistent with the studies of children's perception, children consistently are able to analyze and design patterns that have reflection symmetry. Although Mulligan and Mitchellmore (2016) and Lehrer and colleagues' (1998) instructional programs also include rotation and translation symmetry, research on the development of these ideas is relatively sparse., Given students grasp of translation and turn isometries, more studies are needed that extend symmetries beyond reflection symmetry. Other than those of Lehrer and colleagues (1998), none of the studies I reviewed ask students to investigate the relationships between different isometries and symmetries. Instead, each type of transformation is presented as an isolated concept rather than as a component of an overarching system of isometries and symmetries.

These studies also reveal the role that mathematical practices play in supporting students to participate in generating shared understandings of transformations. Specifically, Lehrer and colleagues (1998) show how visualization practices co-develop alongside epistemic practices, when children's activity supports empirical explorations and generalizations of pattern design that relies upon multiple isometries and symmetries. Although the other studies encourage the development of visualization practices, especially spatial analysis, the brief duration of activities and the treatment of isometries and symmetries as isolated transformations limits the role of and emergence of related forms of mathematical practice, such as classification, representation, conjecture, and empirical proof. If we aim to develop classrooms in which mathematical conceptual understanding and mathematical practices develop together and support each other, it will be necessary to design classrooms where investigating the relationships between concepts is valued and encouraged through the establishment of an epistemic culture.

Finally, a limitation of each of the studies reviewed is that they privilege 2D isometries and symmetry and neglect 3D counterparts. Many of students' everyday activities engage them with transformations in 3D, and distinctions and relationships among the transformations are also more prevalent in 3D. For example, rotation and reflection isometries are critical for establishing distinctions between congruence and direct congruence of asymmetric 3D structures, but this distinction is easily ignored in 2D. Thus, future studies also should investigate the accessibility of 3D isometries and symmetries to young children.

Discussion

NCTM and the Common Core's emphasis on including spatial mathematics as a more central component of K-12 mathematics education is occurring at the same time as a renewed attention to the importance of cultivating students' spatial visualization skills to prepare them for participation in STEM fields. Studies have found students' visualization skills to be predictors of their success within STEM fields (Wai, Lubinski, & Benbow, 2009); moreover, studies of professional practice reveal that disciplinary visual practices are deeply tied to epistemic practices (Gooding, 2008). Thus, as educational policies increasingly adopt practice views of disciplinary learning, there is a growing need to articulate how to foster disciplinary ways of knowing and seeing within mathematics. NCTM and the Common Core State Standards provide an initial motivation but leave much to be articulated about what constitutes and how to support the development of a child-appropriate geometry that privileges the co-development of concepts and visual and epistemic mathematical practices. In this paper, I explored the potential of positioning symmetries and isometries as foundational concepts in a child's geometry that may begin to address the goals set by NCTM and the Common Core.

Symmetries and isometries are generative concepts across domains of mathematics and science. Additionally, they are widely used across a range of cultural forms of activity, especially design. Through the presentations of historical cases that highlight symmetry as a *concept-in-practice*, I exemplified the relationship between the development of symmetry and visualization practices across varied cultural activities. Although the majority of educational and psychological studies frame isometries and symmetries as tools primarily for the visual practice of spatial analysis, the development of symmetry within science, mathematics, and design is intricately tied to an interplay among not only the practice of analysis, but in addition, classification and representation. For example, as new representations developed to model crystals, they opened new ways to visualize and analyze crystals—expanding the classification system of crystals and the field's understandings of relations among different symmetries (Senechal, 1990). Within design, it is often the search for a way to expand the variations of design within a constrained system, such as tiling a plane, that motivate the development of naming or classifying systems and ways of analyzing patterns that help guide the creation of new designs (Schattschneider, 2010). In this sense, forms of visualization practices are varied and derive their form and function from their relation with other forms of practice (Rouse, 2007). Visualization practices are often deployed in service of epistemic practices—in which the goal of expanding existing systems of knowledge requires refining or developing new ways of seeing and conceptualizing space. Thus, the development of symmetry is deeply tied to engaging in mathematical epistemic practices.

The view of symmetry as a concept embedded in a system of visualization and epistemic practices has several implications for the design of future learning environments for supporting young students' development of a mathematics of space and mathematical practices. First, symmetries and isometries deserve a role more central than as mere tools for spatial analysis. Moving beyond analysis creates greater opportunities to explore relations among different isometries and symmetries, as the development of classification representational systems open considerations of the variations within the constraints of a designed or physical space. The instructional programs reviewed in this paper show that it is feasible to engage young children in explorations of isometries and symmetries as isolated forms of transformation via activities of analysis. However, Lehrer and colleagues (1998) provide evidence that young children can participate in generating a mathematical system of symmetries and isometries when the learning ecology supports engagement within a wider range of epistemic and visualization practices that are rooted in everyday forms of activity. These researchers illustrate by recounting how one instructional context, 2D paper quilting, supported the co-development of student's mathematical practices and concepts of isometries and symmetries. Design activities seem particularly well suited for this goal, because design is a representational practice that affords a playful search for variance within a constrained system that can create an authentic need for a shared system of classifying and analyzing designs. Examples from M.C. Escher and cultural crafting practices lend further support to this conjecture. Future instructional studies should explore the range of design activities that can support the co-development of mathematical practices and concepts of isometries and symmetries. It would be particularly helpful to begin to identify the common features across design contexts that are most supportive of this co-development.

As we attend to widening the range of productive design activities that highlight isometries and symmetries, there is a second implication for the design of instructional activities. Important developments in the conceptualization of symmetry from a historical perspective have taken place in the context of 3D-space. As I have recounted, crystalline structures are 3D, and so are most crafting activities, such as basket making. This is of consequence because certain distinctions between forms of isometries and symmetries are more prevalent within 3-space, especially those that relate to difference forms of congruence and rotation and reflections. Existing instructional design have not yet explored children's understandings of 3D isometries and symmetries. This is ironic because studies of children's developing spatial reasoning skills suggest that they have a number of resources to support understanding of 3D transformations; moreover, children live and act within a three-dimensional world. Productive design contexts that may introducing young children to 3D isometries and symmetries include the design of 3D crystalline structures and other crafting activities, including quilting with fabric that requires conceiving of the transformations within quilting patterns as 3D.

Symmetries and isometries are a form of transformation within mathematics that has affinity with everyday forms of cultural activity. This affinity can provide an entry point into mathematics, as the development of mathematical ways of knowing are rooted in and grow out of children's everyday ways of knowing (Lehrer & Lesh, 2003). Symmetries and isometries are also highly generative concepts, in that they open pathways to explore further forms of mathematical transformation. Thus, further studies that examine children's development of isometries and symmetries are needed to continue expanding the possibilities of supporting a mathematics of space that privileges the range of mathematical transformations and related concepts.

References

Beck, D. M., Pinsk, M. A., & Kastner, S. (2005). Symmetry perception in humans and macaques. *Trends in cognitive sciences*, *9*(9), 405-406.

Bishop, A. J. (1988). Mathematics education in its cultural context. *Mathematics education and culture*, 19(2), 179-191.

Bornstein, M. H., & Stiles-Davis, J. (1984). Discrimination and memory for symmetry in young children. *Developmental Psychology*, *20*(4), 637.

Bornstein, M. H., Ferdinandsen, K., & Gross, C. G. (1981). Perception of symmetry in infancy. *Developmental Psychology*, *17*(1), 82.

Brading, K., & Castellani, E. (Eds.). (2003). Symmetries in physics: philosophical reflections. Cambridge, U.K.: Cambridge University Press.

Bruce, V. G., & Morgan, M. J. (1975). Violations of symmetry and repetition in visual patterns. *Perception*, *4*(3), 239-249.

Bryant, P. E. (1973). Discrimination of mirror images by young children. *Journal of Comparative and Physiological Psychology*, *82*(3), 415.

Casey, M. B. (1984). Individual differences in use of left–right visual cues: A reexamination of mirror-image confusions in preschoolers. *Developmental psychology*, *20*(4), 551.

Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction*, *26*(3), 269-309.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education*, 136-163.

Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: the case of geometry. *Journal of mathematics teacher education*, *14*(2), 133-148.

Cuoco, A., Goldenberg, E.P., & Mark, J. (1996). Habits of mind: An organizing principle for a mathematics curriculum. *Journal of Mathematical Behavior, 15*(4), 375-402.

Dehaene, S., Izard, V., Pica, P., & Spelke, E. (2006). Core knowledge of geometry in an Amazonian indigene group. *Science*, *311*(5759), 381-384.

Dehaene, S., Nakamura, K., Jobert, A., Kuroki, C., Ogawa, S., & Cohen, L. (2010). Why do children make mirror errors in reading? Neural correlates of mirror invariance in the visual word form area. *Neuroimage*, *49*(2), 1837-1848.

Davis, B. (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. New York, NY: Routledge.

Devlin, K. (2012). *Introduction to mathematical thinking.* Palo Alto, CA: Keith Devlin.

Ehrlich, S. B., Levine, S. C., & Goldin-Meadow, S. (2006). The importance of gesture in children's spatial reasoning. *Developmental psychology*, *42*(6), 1259.

Eisenberg, T., & Dreyfus, T. (1991, February). On the reluctance to visualize in mathematics. In Zimmermann, W. & Cunningham, S. (Eds.), *Visualization in teaching and learning mathematics* (pp. 25-37). Mathematical Association of America.

Ferrara, K., Hirsh-Pasek, K., Newcombe, N. S., Golinkoff, R. M., & Lam, W. S. (2011). Block talk: Spatial language during block play. *Mind, Brain, and Education*, *5*(3), 143-151.

Fisher, C. B., Ferdinandsen, K., & Bornstein, M. H. (1981). The role of symmetry in infant form discrimination. *Child development*, 457-462.

Frame, M. (2004). Motion pictures: symmetry as animator, classifier, and syntax in the Nasca embroideries of Peru. In Washburn, D. & Crowe, D. (Eds.), *Symmetry comes of age: the role of pattern in culture* (pp. 133-176). Seattle, WA: University of Washington Press, Seattle.

Frame, M. (1986). The visual images of fabric structures in ancient Peruvian art. In *The Junius B. Bird Conference on Andean Textiles* (pp. 47-80).

Franquemont, E.M. & Franquemont, C.R. (2004). Tanka, chongo, kutij: Structure of the world through cloth. In Washburn, D. K., & Crowe, D. W. (Eds.), *Symmetry comes of age: the role of pattern in culture* (pp. 177-214). Seattle, WA: University of Washington Press.

Frick, A., Daum, M. M., Walser, S., & Mast, F. W. (2009). Motor processes in children's mental rotation. *Journal of Cognition and Development*, *10*(1-2), 18-40.

Ginsburg, H. P., & Seo, K. H. (1999). Mathematics in children's thinking. *Mathematical thinking and learning*, *1*(2), 113-129.

Glazer, A. M. (2016). *Crystallography: A Very Short Introduction*. Oxford: Oxford University Press.

Goldenberg, E. P., Cuoco, A. A., & Mark, J. (1998). A role for geometry in general education. *Designing learning environments for developing understanding of geometry and space*, *1*, 3-44.

Gooding, D. C. (2006). From phenomenology to field theory: Faraday's visual reasoning. *Perspectives on Science*, 14 (1), 40-85.

Goodwin, C. (1994). Professional vision. *American anthropologist*, *96*(3), 606-633.

Hargittai, I., & Hargittai, M. (1994). *Symmetry: A unifying concept*. Bolinas, CA: Shelter Publications, Inc.

Hawes, Z., Moss, J., Caswell, B., Naqvi, S., & MacKinnon, S. (2017). Enhancing Children's Spatial and Numerical Skills through a Dynamic Spatial Approach to Early Geometry Instruction: Effects of a 32-Week Intervention. *Cognition and Instruction*, 35(20), 1-29.

Henderson, D. W., & Taimina, D. (2005). *Experiencing geometry: Euclidean and non-Euclidean with history*. Prentice Hall.

Hon, G., & Goldstein, B. R. (2008). From summetria to symmetry: The making of a revolutionary scientific concept. Chicago, IL: Springer Science & Business Media.

Hutchins, E. (2012). Concepts in practice as sources of order. Mind, Culture, and Activity, 19(3), 314–323.

Jacobson, C., & Lehrer, R. (2000). Teacher appropriation and student learning of geometry through design. *Journal for Research in Mathematics Education*, 71-88.

Kleiner, I. (1986). The evolution of group theory: A brief survey. *Mathematics Magazine*, 195-215.

Knorr Cetina, K. D. (1999). Epistemic cultures: How the sciences make knowledge. Cambridge: Harvard University Press.

Kolinsky, R., Verhaeghe, A., Fernandes, T., Mengarda, E. J., Grimm-Cabral, L., & Morais, J. (2011). Enantiomorphy through the looking glass: literacy effects on mirror-image discrimination. *Journal of Experimental Psychology: General*, 140(2), 210.

Kubbinga, H. (2012). Crystallography from Haüy to Laue: controversies on the molecular and atomistic nature of solids. *Acta crystallographica section A: Foundations of crystallography*, *68*(1), 3-29.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

Lave, J., Murtaugh, M., & de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B. Rogoff & J. Lave (Eds.), Everyday cognition: Its development in social context (pp. 67-94). Cambridge, MA: Harvard University Press.

Lehrer, R. (2009). Designing to develop disciplinary knowledge: Modeling natural systems. *American Psychologist*, *64(8),* 759-771.

Lehrer, R., & Chazan, D., Eds. (1998). *Designing learning environments for developing understanding of geometry and space*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lehrer, R., Jacobson, C., Thoyre, G., Kemeny, V., Strom, D., Horvath, J., ... & Koehler, M. (1998). Developing understanding of geometry and space in the primary grades. *Designing learning environments for developing understanding of geometry and space*, 169-200.

Lehrer, R., & Lesh, R. (2003). Mathematical learning. In W. Reynolds & G. Miller (Eds.), Comprehensive handbook of psychology, Volume 7 (p. 357-391). New York: John Wiley.

Lehrer, R., Strom, D., & Confrey, J. (2002). Grounding metaphors and inscriptional resonance: Children's emerging understanding of mathematical similarity. Cognition and Instruction, 20(3), 359-398.

Levine, S. C., Jordan, N. C., & Huttenlocher, J. (1992). Development of calculation abilities in young children. *Journal of Experimental Child Psychology*, *53*(1), 72-103.

Möhring, W., & Frick, A. (2013). Touching Up Mental Rotation: Effects of Manual Experience on 6‐Month‐Old Infants' Mental Object Rotation. *Child development*, *84*(5), 1554-1565.

Moore, D. S., & Johnson, S. P. (2008). Mental rotation in human infants a sex difference. *Psychological Science*, *19*(11), 1063-1066.

Moore, D. S., & Johnson, S. P. (2011). Mental Rotation of Dynamic, Three‐Dimensional Stimuli by 3‐Month‐Old Infants. *Infancy*, *16*(4), 435-445.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: a case study. *ZDM*, *47*(3), 377-390.

Mulligan, J. & Mitchelmore, M. (2016). *Pattern and structure mathematics awareness program.* Australia: Australian Council of Education Research.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2014). *Essential Understandings Series*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices & Council of Chief State School

Officers. (2010). Common core state standards for mathematics. Washington, DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

National Research Council (2006). Learning to think spatially: GIS as a support system in the K-12 curriculum. Washington, DC: National Academic Press.

Olson, D. R. (2013). *Cognitive development: The child's acquisition of diagonality*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Pruden, S. M., Levine, S. C., & Huttenlocher, J. (2011). Children's spatial thinking: does talk about the spatial world matter? *Developmental Science*,*14*(6), 1417-1430.

Quinn, P. C., & Liben, L. S. (2008). A sex difference in mental rotation in young infants. *Psychological Science*, *19*(11), 1067-1070.

Rouse, J. (2007). Practice theory. In Turner, S. & Risjord, M(Eds.), *Handbook of the philosophy, history, and social studies of science 15: Anthropology and sociology* (pp. 639-862). Dordrecht, Netherlands: Elsevier.

Sarama, J., & Clements, D. H. (2004). Building blocks for early childhood mathematics. *Early Childhood Research Quarterly*, *19*(1), 181-189.

Schattschneider, D. (2010). The mathematical side of MC Escher. *Notices of AMS*, *57*, 706-718.

Schifter, D., Russell, S. J., & Bastable, V. (1999). Teaching to the big ideas. In Solomon, M.Z. (Ed.), *The diagnostic teacher: Constructing new approaches to professional development* (pp. 22-47). New York, NY: Teachers College Press.

Schwartz, D. L., & Holton, D. L. (2000). Tool use and the effect of action on the imagination. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *26*(6), 1655.

Schwarzer, G., Freitag, C., Buckel, R., & Lofruthe, A. (2013). Crawling is Associated with Mental Rotation Ability by 9-Month-Old Infants. *Infancy*, *18*(3), 432-441.

Senechal, M. (1989). Symmetry revisited. *Computers & Mathematics with Applications*, *17*(1), 1-12.

Senechal, M. (1990). Brief history of geometrical crystallography. *Historical atlas of crystallography*, 43-59.

Shepard, R. N., & Metzler, J. (1971). Mental Rotation of Three-Dimensional Objects. *Science*, *171*, 701-703.

Soska, K. C., Adolph, K. E., & Johnson, S. P. (2010). Systems in development: motor skill acquisition facilitates three-dimensional object completion. *Developmental psychology*, *46*(1), 129.

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. In Lampert, L. & Blunk M.L. (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107-149). Cambridge, UK: Cambridge University Press.

Stewart, I. (2013). *Symmetry: A Very Short Introduction*. Oxford: Oxford University Press.

Treder, M. S. (2010). Behind the looking-glass: A review on human symmetry perception. *Symmetry*, *2*(3), 1510-1543.

Uttal, D. H., & Cohen, C. A. (2012). Spatial thinking and STEM education: When, why, and how? In Ross, B.H. (Ed.), *Psychology of learning and motivation: Advances in research and theory* (*57*, pp, 147-182). Oxford: Elsevier, Inc.

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., Newcombe, N. S., Filipowicz, A. T., & Chang, A. (2014). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematical skills. *Child development*,*85*(3), 1062-1076.

Wagemans, J. (1997). Characteristics and models of human symmetry detection. *Trends in cognitive sciences*, *1*(9), 346-352.

Wagemans, J. (1995). Detection of visual symmetries. *Spatial vision*, *9*(1), 9-32.

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, *101*(4), 817.

Washburn, D. (1999). Perceptual anthropology: The cultural salience of symmetry. *American anthropologist*, *101*(3), 547-562.

Washburn, D. (2004). Symmetry analysis of Ica Valley ceramics: Insights into Ica-Inca interactions. In Washburn, D. K., & Crowe, D. W. (Eds.), *Symmetry comes of age: the role of pattern in culture* (pp. 215-231). Seattle, WA: University of Washington Press.

Washburn, D. K., & Crowe, D. W. (Eds.). (2004). *Symmetry comes of age: the role of pattern in culture*. Seattle, WA: University of Washington Press.

Washburn, D. K., & Crowe, D. W. (1988). *Symmetries of culture: Theory and practice of plane pattern analysis*. Seattle, WA: University of Washington Press.

Wenderoth, P. (1994). The salience of vertical symmetry. *Perception*, *23*(2), 221-236.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in mathematics education*, 458-477.

Young, C. J., Cartmill, E. A., Levine, S. C., & Goldin-Meadow, S. Gesture and Speech Input are Interlocking Pieces: The Development of Children's Jigsaw Puzzle Assembly Ability

Chapter III

FOSTERING YOUNG CHILDREN'S VISUALIZATION OF 3-D STRUCTURES THROUGH THE CO-DEVELOPMENT OF MATHEMATICAL DEFINING PRACTICES AND CONCEPTUAL SYSTEMS OF ISOMETRIES AND SYMMETRIES

Introduction

Visualizing and conceiving space is consequential for participation in fields ranging from art and design to STEM professions (Henderson & Taimina, 2005; Wai, Lubinski, & Benbow, 2008). It is also central to children's ways of developing and coming to know about the social and physical world around them (Oudgenoeg-Paz, Leseman, & Volman, 2015; Piaget, 1970). Yet, much of schooling neglects cultivating students' understanding of space (NRC, 2006), a neglect that is explicitly addressed by contemporary calls to expand the spectrum of education to include the mathematics of space and geometry beginning in early childhood and elementary school (Clements, Sarama, & DiBiase, 2004; NGA & CCSSO, 2010). Although there are potentially many productive ways to cultivate children's understanding of space and geometry (Lehrer & Chazan, 1998), recent research signifies the importance of transformations as conceptual tools for children to analyze and understand the visual structure of patterns and shapes (Clements & Sarama, 2007; Hawes, Moss, Caswell, Naqvi, & MacKinnon, 2017; Lehrer & Slovin, 2016; Mulligan & Mitchelmore, 2016; Ng & Sinclair, 2015). This emphasis on transformation is aligned with its historic development as a means for affording new ways of visualizing. For example, describing the internal structure of crystalline structures with isometries (congruence preserving transformations) and symmetries (a subclass of isometries that preserve self-congruence) led to new representations of crystals and new theories of atomic interactions. Further, investigations of crystalline structures prompted the invention of new forms of periodic symmetry (e.g., glides, screws), so that both the science of crystals and the mathematics of isometry and symmetry were refashioned (Senechal, 1990).

Of course, children do not participate in these historic developments, but isometries' affinities with everyday forms of motion (flipping, turning, and sliding) that are often manifested in children's playful construction (e.g., block building, puzzle assembly) suggest potential genetic pathways for mathematizing these forms of motion (Casey et al., 2008; Kamii, Miyakawa, & Kato, 2004; Lehrer et al,. 1998; Verdine et al., 2014). Children's playful construction and exploration of motion also promote their propensities to develop language and gestures to communicate about increasingly complex spatial relations (Ehrlich, Levine, & Goldin-Meadow, 2006; Ferrera, Hirsch-Pasek, Newcombe, Golinkoff, & Sallcross Lam 2011; Young, Cartmill, Levine, & Goldin-Meadow, 2014). Integrating construction activities into early childhood classrooms can help develop these potential seeds of transformation (Casey et al., 2004; Clements & Sarama, 2007; Hawes et al., 2017). For example, Hawes and colleagues (2017) worked with K-2 teachers to develop a year-long dynamic geometry program that included a variety of transformation based 2D and 3D geometric puzzles that teachers then integrated into their mathematics instruction. Children that participated in the program showed greater growth in their spatial language and abilities to mentally rotate and compose 2D figures compared to children in control classrooms.

In this study, we aimed to build on this research base by investigating ways to expand children's grasp of isometry and symmetry to include three-dimensional structures, a topic that is comparatively neglected in the research literature (Hawes, LeFevre, Xu, & Bruce, 2015). Considering transformations of three-dimensional structures offers prospective continuities between children's movements and spatial language in the world and the mathematics of rotation and reflection. Yet, there are potential challenges as well. In particular, reflections result in a change in orientation that cannot be readily enacted through a physical motion, such as flipping. For example, the reflection of one's right hand in a mirror results in an opposite orientation that cannot be reproduced by flipping one's hand. Moreover, the chiral aspect of some 3D objects makes anticipating the outcome of a rotation of 3D objects harder for young children compared to similar tasks with 2D objects (Hawes et al., 2015; Jansen et al., 2013). Later, we outline the forms of material and instructional supports we designed to help overcome these challenges related to orientation and chirality.

Reorganizing Children's Construction Activities to Support Mathematical Development

In designing an instructional ecology to support children's development of ideas and procedures of transformation. we were informed by a practice view of knowledge development (Lampert, 1990). In this view, disciplinary concepts, such as isometry and symmetry, and disciplinarily endorsed forms of activity, such as searching for invariants in light of variation, co-originate (Rouse, 2015). As new concepts emerge from activity, goals and ways of participating in activities are renegotiated. Unfortunately, to our knowledge, few studies of children's introduction to transformation provide analyses of classroom interaction that can inform our understandings of how the dual relation between concept and practice develops. Nor do they suggest aspects of the learning ecology, such as tasks and tools, instructional supports, and the emergent, local mathematical community that interact to support this development (Yackle & Cobb, 1996). These forms of insights are increasingly necessary since current standards represented by the *Common Core State Standards in Mathematics* and *Next Generation Science Standards* emphasize that students should have opportunities to understand how their participation in mathematical practices provides the means for the production and revision of mathematical ideas (NGA & CCSSO, 2010).

Despite a paucity of such analyses, Lehrer and colleagues (1998) indicate that engaging children in design and construction can support the co-development of mathematical practices and concepts of isometry and symmetry in two-dimensions. Drawing on relations between craft and the mathematics of transformation (e.g., Beyer, 1999; Washburn & Crowe, 1988), in the Lehrer et al. (1998) study, third grade children employed reflections (flips) and rotations (turns) as construction tools to design and produce quilts. As they considered variations in the symmetries of their quilt designs, children explored relations between properties (color, presence or absence of symmetries) of the core generating cell used to design the quilt and the resulting number and type of symmetries in the quilt. For instance, students found that the number of design possibilities was greatest for core cells that were asymmetric. Children's exploration of these symmetry relations was supported by their concurrent engagement in practices of defining, representing, conjecturing, and generalizing. Additionally, children's sense of what, why, and how they were designing evolved in relation to new roles and goals introduced by these mathematical practices. Thus, children's design activities were reorganized as they developed concepts of symmetry and isometry, and their design activities were coordinated with new forms and ways of participating in mathematical practice.

To design for supporting the co-development of concepts of isometry and symmetry in three-space and participation in mathematical practice, we elected to initially focus on the practice of defining. Defining includes generating and negotiating the grounds of entities that ground the development of mathematical systems in both professional practice (Lakatos, 1976) and in classrooms (Kobiela & Lehrer, 2015). Our guiding conjecture was that engaging young children in the practice of defining would position them in new roles to negotiate a shared way of communicating about and visualizing 3D isometries and symmetries of objects during construction. In turn, as they articulated meaningful definitions of isometry and symmetry, variation in their articulations would likely lead to more refined and elaborated collective conceptions of isometry and symmetry. Despite the relative absence of opportunities for children throughout schooling to engage in the practice of defining, we anticipated that engagement with this form of mathematical practice would be firmly within their grasp. Young children have extensive experience with negotiating the meaning, purpose, and roles in creating a wide variety of representational forms during play and schooling. For example, Rowe's (2008) analysis of two-year-olds' early writing experiences with adults demonstrates that young children are forceful and active participants in shaping and communicating the intentionality of their early written forms to adults and to peers. However, while potentially accessible to children, engagement in the practice of defining also rests on how a teacher supports children to participate in aspects of defining. As indicated by Kobiela & Lehrer (2015), teachers must be aware of, communicate, and model ways of engaging in defining, and they must be strategic in selecting examples and supporting students to test whether or not a particular example is consistent with verbal or written articulations of the defined object. Hence, we anticipated that variations in teacher supports would influence the extent to which children were able to leverage this form of mathematical practice to develop understandings of isometry and symmetry.

In sum, by investigating the co-development of young children's participation in defining and their emerging concepts of symmetry and isometry, we aimed to characterize how students came to visualize 3-space through a lens of mathematical transformation and how they made sense of and participated in the mathematical practice of defining. Our analysis and findings that follow were guided by the following questions:

1) How does the practice of defining influence young children's conceptions of isometry and symmetry?
2) How does variability in instructional support affect the learning ecology and young children's co-development of concepts and practice?
3) How do young children's construction and visualization of 3D structures become reorganized to include concepts of isometries and symmetries of objects?

Methods

Study Context

This study was part of a larger ongoing research project aimed at designing a K-5 spatial mathematics sequence based on core ideas of measure and transformation. The research team worked in collaboration with a group of voluntary teachers at a public elementary school located in a rural Southeastern town of the U.S. The school serves students from primarily low income families with diverse cultural and linguistic backgrounds including mostly recent Pacific Islander and Latino immigrant families and rural White families.

At the time of this study, teachers within the project had participated in classroom studies and professional development for four years. At the outset of the study, we notified the four participating 1st grade teachers in the larger project of this study's goals, asking for voluntary participation. All four teachers agreed to have their classrooms participate. Each classroom had 22-24 students between the ages of six and seven. Given the voluntary nature of the sampling procedure, these teachers are not meant to be representative of all teachers, but rather, they serve as a subset of teachers interested in integrating geometry into their existing mathematics curriculum. Additionally these teachers' prior practices demonstrated a commitment to eliciting and responding to student thinking. Nevertheless, these teachers are representative of teachers who are novices at teaching geometric transformations, particularly 3D isometries and symmetries, and supporting children's practice of defining.

Phases of Inquiry

Because this study utilized design research (Cobb et al., 2003), the phases of inquiry were broken into iterative chunks of instructional design, design implementation, and retrospective analysis. This current report focuses on the first design cycle. In the discussion, we explain how the analyses and findings of the focal retrospective analysis provide implications for a future instructional redesign and implementation.

Starting points of the instructional design. The initial design of conjectures and instructional design began with a review of the research about young students' understandings of isometries and symmetries. Additionally, we examined how isometric transformation is historically aligned with cultural forms of activity, such as craft and design (that use transformations in 3-space) as well as what came to be called abstract algebra (Wongkamalasai, 2018). We were also informed by mathematical analysis of the distinction between direct (e.g. via rotations or translations only) and indirect (needing reflections) congruence (Henderson & Taimina, 2005). This led to a central design conjecture about the potential fruitfulness of situating activities within 3D-space and design. 3D space has historically helped to make important conceptual distinctions between rotations and reflections (Kubbinga, 2012; Senechal, 1990), and, as previously discussed, design contexts open possibilities of leveraging students' existing everyday activity into more mathematical forms of activity.

Defining isometries and symmetries. Conceptions of symmetry are rooted in a human affinity to notice and represent senses of balance and unity. This is true even of the concept of symmetries as transformations used in mathematics and physics (Brading & Castellani, 2003).

However, although everyday notions of symmetry as balance can be captured in everyday language and gesture, symmetry transformations are tied to specialized forms of mathematical language, representations and tools (Ng & Sinclair, 2015; Senechal, 1990). Thus, our design was guided by the mathematical definition of symmetries as a subset of isometric transformations, albeit here we only refer to finite symmetries.

Isometries include transformations that preserve an object's angles and lengths and consequently, the object's overall size and shape. Thus, isometries are transformation that can be used to determine whether two objects are congruent. Forms of isometries include rotations, reflections, translations, and combinations of these isometries (i.e., glides and screws). We decided to focus only on rotation and reflection isometries during this instructional design because they are tied to the finite rotation and reflection symmetries most prevalent in children's 3D constructions.

Rotation isometries are rotations of a particular magnitude or angle around an axis of rotation. All points of an object are rotated the same degree and about the same rotation axis. Thus, rotations not only preserve an object's size and shape, but also its orientation and direct congruence. In contrast, 3D reflection isometries are a mapping of the points comprising an object across a plane of reflection so that the reflected points are the same magnitude of distance from the reflection plane, but in the opposite direction. Thus, reflections only preserve congruence, meaning an object's size and shape remain invariant but its orientation is reversed. Mirrors are an everyday instantiation of reflection planes and a way to experience reflections' change in orientation that we hoped to explore with children. Many everyday objects also demonstrate this property of mirror reflection, including pairs of hands and feet.

The rotation symmetries and reflection symmetries of an object are isometries that, when enacted, preserve an object's self-congruence. From an everyday sense, this means that one cannot discern a difference in the position, size, and shape between the object and it's transformed image without a way to notate where each point on the object started. Often when children are introduced to symmetries they are disconnected from isometries; however, we conjectured that by supporting students to develop a mathematical system of isometries and symmetries, they would develop a richer mathematical language to reason about and define symmetries of objects using properties of invariance that go beyond just everyday senses of balance and "same".

Activity structure and instructional sequencing. Because our goal was to support students to develop a mathematical system of transformation, meaning that they could articulate both relations and differences between transformations, we decided to first help children develop a mathematical language to describe properties that are most relevant to isometries – size, shape, and orientation—and an understanding of how these properties can be used to describe transformations. In prior work with children, we noted that most students did not have a language to describe relations of congruence that they discovered after rotating or reflecting structures into accordance beyond stating that two structures "looked the same." Thus, we decided to start the instructional unit with problematizing this issue of communication in the context of a familiar activity—building an exact copy or directly congruent structure of a target structure. We then introduced rotation and reflection isometries as a way to challenge possible static conceptions of these properties; thus, isometries were initially treated as tools to test relations of direct congruence and eventually (mirror) congruence. Once we established concepts of rotation and reflection isometry and congruence, we introduced reflection and rotation symmetries of objects. We were intentional about foregrounding children's developing conceptions of isometries and congruence as resources for sense-making in initial symmetry activities. Figure 1 shows the ordering of construction tasks and associated concepts throughout the unit within the larger activity structure that recurred throughout the lesson. We did not just want students to construct examples of mathematical concepts, but rather we wanted their constructions to be positioned as mathematical objects for collective analysis and defining. Thus, we conjectured that this aspect of the activity structure would be critical to introducing new goals and roles associated with defining and would help students coordinate their existing construction activities with the class's mathematical activity (Lehrer, Jacobson, Kemeny, & Strom, 1999; Saxe, 2002).

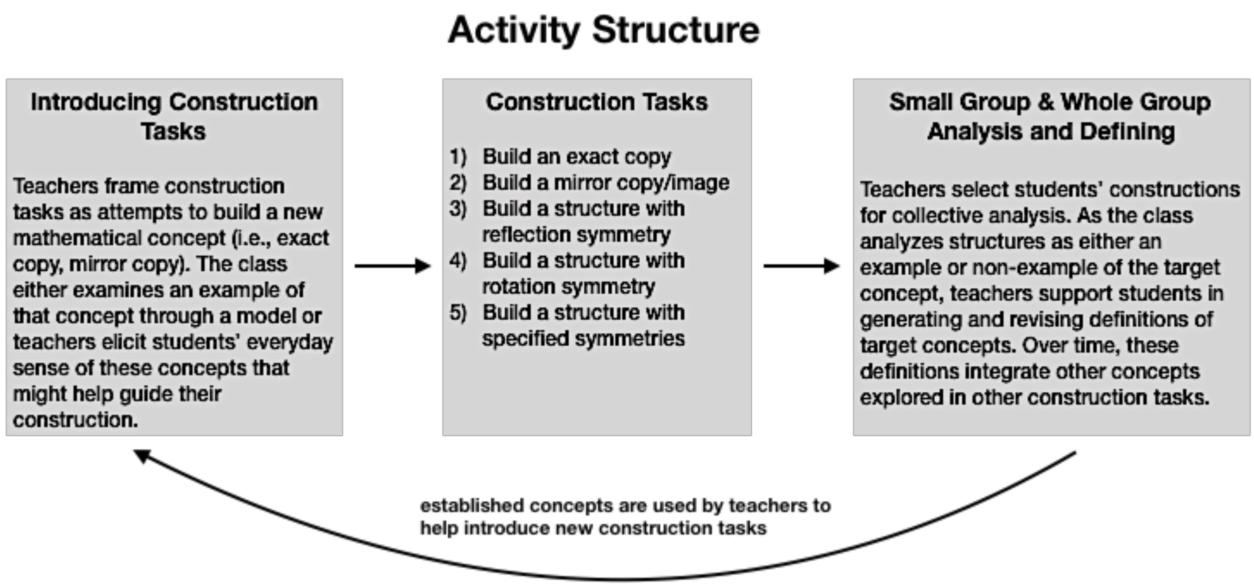


Figure 5: Design of mathematical activity, including the sequence of construction tasks.

Implementation of Instructional Design. The implementation of the initial design allowed us to test conjectures around the affordances of 3D isometries and symmetries and the activity structures supported by design tasks. This phase of inquiry started with five consecutive instructional days in January of the school year, followed by interviews with ten students. Towards the end of the school year in May, we extended instruction for three more days in one teacher's (Mrs. P) classroom (End of year testing precluded inclusion of all classrooms). We used these three days to both consolidate and extend concepts established in January and to introduce symmetry constraints for design challenges involving 3D structures (e.g., a structure with ½ turn rotation symmetry but without reflection symmetry) as a means for promoting relational thinking about isometries and symmetries of objects. Following completion of this second phase of instruction, we concluded with a final round of interviews with 15 of Mrs. P's students.

Returning now to the initial phase of design, Figure 6 depicts the instructional sequence for the first five days of instruction as it was enacted within the classroom with notes about major conceptual developments we intended to support. Because of the flexibility of teachers' schedules, we were able to observe and to provide instructional supports for each classroom back to back. This allowed for small refinements to the design across the classrooms. Analysis was ongoing throughout this phase, and it informed changes to the instructional design and highlighted key aspects of instruction that appeared consequential for future analyses.

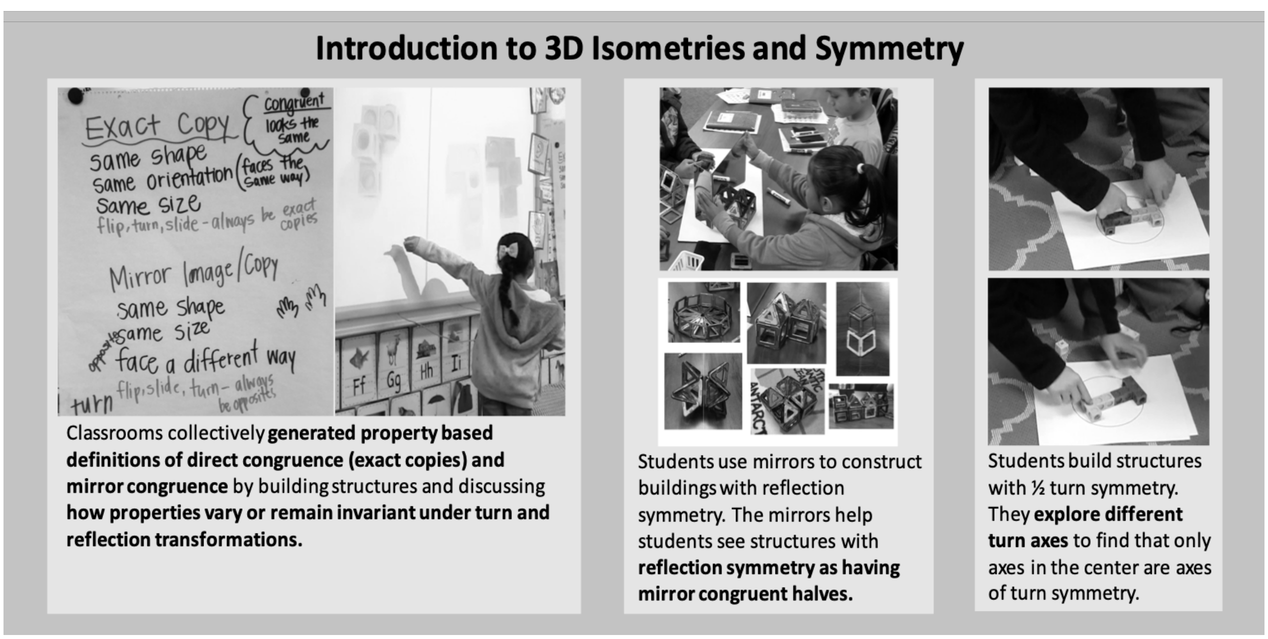


Figure 6: Instructional activities for days 1 through 5 of instruction.

In May (the second phase), students again investigated reflection and rotation symmetries but we incorporated two significant changes to tasks based on our initial analysis of the video and interview data from January. The first change was the introduction of a 3D CAD system, Tinkercad (Autodesk®), to aid in the analysis of the symmetries of objects. We opted to

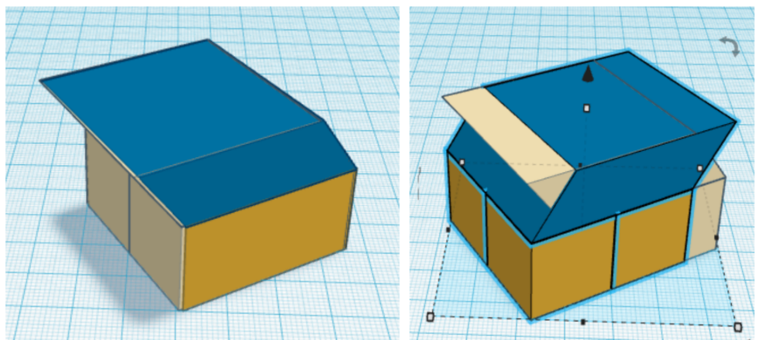


Figure 7: Example of a structure in Tinkercad with no half turn symmetry. The image and pre-image of the structure do not overlap after the half turn. Autodesk screen shots reprinted courtesy of Autodesk, Inc.

introduce this program because unlike in physical space, this digital space allows 3D structures to be superimposed. Thus, the user can leave a tracing of the pre-image of a structure as she rotates or reflects a structure. We conjectured that Tinkercad would allow for an easier visual testing of symmetry because the pre-image and transformed image exactly align when the transformation is a symmetry. Conversely, as in Figure 7, when the transformation is not a symmetry of the object, the pre-image and transformed image do not align. Hence, Tinkercad supports visualization of symmetries as direct mappings. We hoped that this additional form of visualization would help students further distinguish between isometries and symmetries. For example, as we later describe more completely, we noted that some students during the initial part of instruction struggled with differentiating between the enactment of a half turn and a half turn symmetry of an object.

The second change introduced during the final phase of instruction entailed a shift from testing for an object's symmetries post-hoc to challenging students to incorporate symmetries as constraints to guide and revise their 3D constructions (e.g., designing a structure with *only* half turn symmetry). We conjectured that such design under constraint would prompt students to visualize and anticipate reflection planes and rotation axes, and to anticipate relations between symmetries (e.g., would multiple planes of reflection also imply a rotation?). This conjecture was further explored in the final set of student interviews.

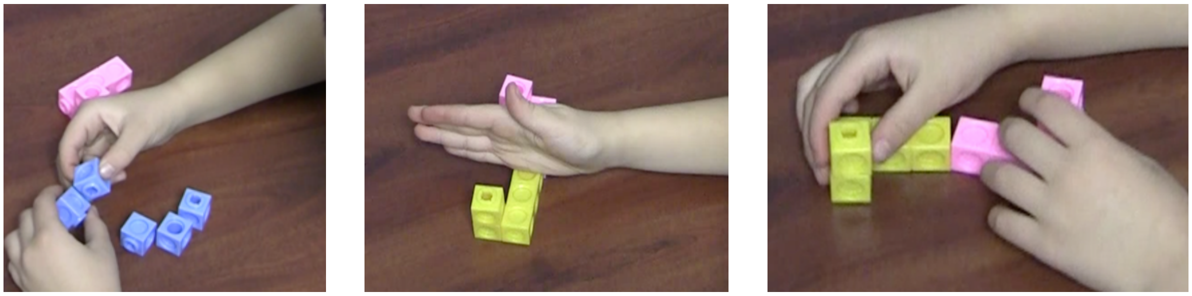
Classroom data collection. Throughout the implementation phase in the field, we acted as participant-observers in each of the classrooms. Teachers and students perceived our roles as teacher assistants and instructional coaches. Teachers often paused instruction to ask questions about what to do next or requested that one of us jump in when they were unsure of how to make sense of student thinking. Children were accustomed to researchers observing and participating in their classrooms from previous work in the research program. They were comfortable asking us for assistance, but they treated their teachers as the lead instructor.

Due to our active roles within each of the classrooms, our ability to record thick descriptions of classroom interactions was limited; however, we kept brief field notes—documenting aspects of the design that seemed to work well, fruitful teacher and student innovations, and aspects of the design that were changed on the spot. After each instructional period, we debriefed with teachers and shared our field notes in order to get their perspective on aspects of instruction we noticed and aspects that they felt were salient that we left out. These field notes served as a document to determine how instruction should proceed or change the following day. At the end of each instructional day, M.W. met with all four teachers to discuss collectively what teachers noticed across the classrooms, to address questions that they had, and to plan what changes to the design needed to take place the following day.

In addition to field notes, we also collected video data. We were interested in capturing the development of conceptual understandings and mathematical practices within classroom interaction; thus, video recording of whole class discussions and instruction were captured in each of the classrooms. We were intentional in these videos to capture a wide lens of both teacher and student interaction, rather than focusing solely on the moves of the teacher. In addition to whole class video, in each of the classrooms we selected a small group of students (3 to 4 students) that we followed consistently throughout the study. These small groups were selected based on the teacher's judgement of students' likelihood of being comfortable talking with the researchers about their thinking. All classroom videos were transcribed for both talk, notable changes in prosody, and gesture. Gesture was included based on initial noticing about the importance of gesture in supporting whole class discussions and the uptake of gestures across students. The inclusion of gesture is consistent with studies of the roles that gesture plays in helping create and negotiate both individual and joint understandings in settings that span everyday conversation, classrooms, and professional work (Goodwin; 2012; Streeck; 2009) These videos and transcripts served as the primary source of data for the analyses presented within this report.

Clinical interviews. As mentioned previously, at the end of each phase of instruction (January, May) we conducted and videotaped interviews with students to gain insight into the variability among students' conceptual understandings—offering a more individual view of development compared to that captured from video of the whole class. We conducted the first set of interviews with 10 students (2-3 per class) who we selected from the focal small groups. The goal of these interviews was to get a sense of students' understandings of individual forms of isometries and symmetries—including both procedural development (i.e. could they identify and construct structures using isometries and symmetries) and conceptual development (i.e. how did they explain concepts). In May, we interviewed all of Mrs. P's students that participated in both phases of instruction. In total, we interviewed 15 out of a possible 20 students (three students did not have parental consent and two students were absent). The final set of interviews was aimed at looking at how students' concepts of rotation and reflection symmetries supported their abilities to visualize and anticipate reflection planes and rotation axes, and at students' abilities to relate and coordinate between symmetries.

The January student interview included four construction tasks that were similar to tasks students experienced in the classroom. In the first two tasks, we asked students to construct a directly congruent structure and a mirror image of a novel asymmetric structure (Figure 8a). During the last two tasks, we presented students with a five cube asymmetric structure and five loose cubes. We next asked them to use the five blocks to complete the building so that it first had reflection symmetry (Figure 8b) and then rotation symmetry (Figure 8c). After each task, the interviewer (M.W.) asked students to explain how they knew their structure was an example of the corresponding mathematical concept. Students were encouraged to revise their construction if during their explanation they realized they made a mistake. For example, when trying to demonstrate that they made a structure with half turn symmetry, students often noted they made a mistake after rotating their structures and corrected their structures.



A. B. C.

Figure 8: Post interview tasks. A) Students first used loose cubes to be an directly congruent and mirror congruent structure to a given asymmetry structure. B) Next, students used the yellow cubes to complete a structure so the resulting structure had mirror symmetry. C) Finally, students used the yellow cubes to complete a structure to the resulting structure had ½ turn symmetry.

The May interview required students to complete four iterations of a construction using two congruent, asymmetric 3D structures made using Magformers. We asked students to arrange the two preconstructed, congruent parts into a structure with 1) only half turn symmetry, 2) only reflection symmetry, 3) both half turn symmetry and reflection symmetry, and 4) no symmetry. After completing each construction task, the interview (M.W.) asked students to describe the symmetries of their structure. There was a mirror available for students to demonstrate where the symmetrical reflection planes in their structure were located. Again, if students noted a mistake in their structure during their explanation, M.W. invited students to correct their structure. After students completed and explained each structure, M.W. helped them photograph the structure using a photo collage app (Pic Collage) that students were familiar with from other classroom activities (Figure 9). If students were unable to complete the tasks, the corresponding cell in the collage was left blank. As students proceeded through the tasks, they also were able to use the collage as a record of what structures they had already constructed. We report the analysis of both sets of clinical interviews in the following section.

Figure 9: Two example of students' construction from May clinical interviews captured using the Pic Collage app.

Retrospective analysis of instructional design. The third phase of inquiry that we describe in detail was the retrospective analysis of this first implementation of the instructional design. This phase of analysis allowed for discerning how elements of the design shaped children's investigations and the interactions that took place in the classroom, and how concepts and participation in practices of defining were co-constituted.

Analysis of clinical interviews. For the retrospective analysis of interview data, we first coded each response to a task across both sets of interviews as correct or incorrect. For correct responses, we also made the following distinctions based on number of attempts and supports: correct on first attempt, correct on second attempt with self-correction, and correct after three or more attempts with some interviewer prompting (e.g., Can you turn your structure to help? What do you think we would see in the mirror?). Responses were coded as incorrect when students made several attempts to complete a task but could not find a satisfactory solution (e.g., I can't figure out how to make one with no symmetry) or when a student presented an incorrect structure and attempted to describe how it met all the constraints of the task. Because we were also interested in students' explanations of concepts of congruence, isometry, and symmetry, we documented students' explanations and looked for common forms of explanations and difficulties. This latter analysis, particularly of the January interviews, helped inform instructional changes made in May. We report the results of the clinical interviews in the findings section.

Analysis of classroom video data. For the retrospective analysis of video data, we followed Lincoln and Guba's (1985) constant comparative method. Because we are most interested in how defining practices and concepts develop in interaction, we defined the unit of analysis as episodes of defining. Each episode was bounded by the introduction or problematizing of a concept and ended with agreement of a new definition or refinement of a definition.

After identifying and creating multi-modal transcripts for defining episodes across the classrooms, our next step of analysis was to characterize each defining episode. We used Kobiela and Lehrer's (2015) framework of aspects of defining to characterize the defining activities and roles enacted by teachers and students. The aspects of defining most prominent in our coding were a) proposing definitions, b) describing properties, c) evaluating examples and non-examples of concepts, d) constructing definitional explanations, f) revising definitions, and g) establishing and reasoning about systematic relations. In addition to coding for these aspects of defining, we also coded for who was positioned in defining roles (teacher vs. student); what concepts were being established, refined, or related; and the classroom collective sense of concepts at different time points.

Next, we used open-coding to analyze how aspects of the learning ecology supported classroom defining practices and conceptual development. We paid particular attention to verbal and non-verbal (i.e. gestures and use of tools) discourse moves used by the teacher and students in subsequent turns of talk, the ways in which physical and conceptual tools were used, how this tool use became shared, and how tasks helped coordinate physical tools and concepts. By comparing similar defining episodes across the classrooms, we refined our codes to those aspects of the learning ecology that appeared most consequential for students' co-development of concepts and practices.

Finally, with this refined coding system, we re-analyzed video from two of the classrooms, Mrs. P's and Mrs. H's. We chose these two classrooms because they were the last two classrooms we observed each day; thus, the implementation of the instructional design was typically the most refined as we were able to make small revisions based on the first two classrooms. Consequently, these two classrooms are also the source of our illustrative episodes in the findings section. We next discuss how this final coding scheme helped us understand how the instructional design supported the co-development of conceptual understandings of 3D isometries and symmetries of objects and mathematical defining, and how this co-development yielded new ways of visualizing and constructing 3D structures.

Findings

In this section, we describe how aspects of the learning ecology, including the tasks, tools, teachers, and the classroom's emerging mathematical defining practices, shaped students' conceptual understandings of 3D isometries and symmetries. We also explain how students' ways of visualizing and constructing 3D structures using symmetries and isometries developed through these interactions and supported the co-development of definitions and concepts. We include results from the clinical interviews to triangulate between video analysis of collective activity and individual understandings.

Our first finding relates to how the practice of defining influenced children's concepts of isometry and symmetry and the ensuing trajectory of this conceptual development. As teachers helped establish and define a mathematical system that included relations and distinctions between forms of transformations and congruence, students had opportunities to refine and negotiate shared conceptions of isometry and symmetry. As we later illustrate, these refinements occurred during definitional episodes in which students established important distinctions between rotation and reflection isometries, related symmetries to isometries as particular cases of isometries, and related rotation and reflection symmetries. In each of these definitional episodes, students had greater opportunities to coordinate multiple aspects of defining that went beyond simply describing properties of examples and non-examples to include relations among transformations. The shift from analyzing examples as isolated cases to reflecting on relations among them signaled a transition toward attention to a mathematical system. Children's individual responses to the clinical interviews appear to corroborate the conceptual transitions evident in the analysis of collective interaction and provide further evidence of the feasibility of engaging young children in 3-D transformations.

The second finding describes aspects of teachers' practices that orchestrated tasks, conceptual and physical tools, and the roles and goals of defining. In particular, we identified two key aspects of teachers' practices that were consequential for children's co-development of concepts and practices and the reorganization of their construction activities. The first aspect consisted of teachers' tracing and recording the classroom's history of definition throughout the unit. One way that teachers accomplished this was by maintaining and anchoring discussions to charts that acted as a written history of definitions and the classroom's mathematical system.

However, beyond just referencing these charts as a way to maintain this historical thread of defining, teachers also strategically used forms of discourse moves to tie students' ideas to and to position them as authors of this shared history. Two common discourse moves used by teachers were revoicing, the restatement and positioning of students' ideas (O'Connor & Michaels, 1993) and prolepsis, the posing of questions or statements that both presume and shape shared understandings and goals (Stone, 1993; Stone & Wertsch, 1984). Although prior work has demonstrated the importance of these forms of discourse moves in helping students build mathematical meaning, the cases we later describe demonstrate how teachers tailored these moves to support a particular practice—defining—within a new (to them) domain of mathematics. A second key aspect of teachers' practices was their leveraging of students' multiple ways of using physical tools (i.e., mirrors, construction materials, and turn axes) into opportunities to negotiate collective ways of seeing or visualizing. During these negotiations, teachers again pulled in the class's history of defining to encourage students to coordinate concepts of and a language of transformation with their ways of seeing. They also often amplified and highlighted aspects of the visual field through gesturing and modeling tool use. During these interactions, teachers and students collectively developed a localized and socially organized way of seeing that aligned with their emerging mathematical goals (Goodwin, 1994).

The third finding relates to how aspects of construction tasks during both phases of instruction (January, May) helped reorganize students' construction and visualization of 3D structures to include considerations of isometries and symmetries. This finding provides insights into the affordances of construction activities to support young children's mathematical development. The first of these aspects was the large variability among structures analyzed and constructed by students. This variability enriched moments of group analysis and evaluation of structures as containing or representing physical instantiations of concepts. These moments of the group's analysis of a range of structures also offered opportunities to revise or extend concepts and ways of using visualization tools. For example, students' construction of structures with multiple symmetries encouraged students to consider varying the location of and the coordination of reflection planes and turn axes. A second aspect of the construction tasks critical to reorganizing students' goals of constructing was the gradual introduction of symmetries as a design constraint. During the first part of instruction, students mainly analyzed their structures for symmetries after construction; however in the final part of instruction in May, students were encouraged to consider the symmetries of their structure during design. By challenging students to design structures with particular types and numbers of symmetries, some students developed greater understanding of relations between reflection and rotation symmetries. These students displayed new forms of agency by using symmetries to articulate design preferences and to invent ways to represent the symmetries in their designs.

To help illustrate these specific findings and to demonstrate how students' concepts and practices of defining co-developed in relation to an evolving mathematical system and aspects of the learning ecology, we turn now to situating these findings within episodes drawn from the three major parts of the instructional unit— comparing rotation and reflection isometries, relating symmetries and isometries, and designing 3D structures using symmetries as constraints. For each of these episodes we describe analysis of events that shaped students' co-development of concepts and defining practice, emphasizing the aspects of teachers' practices and the construction tasks previously outlined.

Developing Two Senses of Congruence By Analyzing Isometries Using Invariant and Variant Properties

The first part of instruction was rooted in what we anticipated would be a familiar 3D construction activity for students—constructing copies or congruent structures of a target structure. By electing to focus only on target structures that were 3D asymmetric structures, we hoped that the variability in students' attention to the property of orientation would help problematize what counts as an "exact copy". Our goal was to introduce rotation and reflection isometries as tools to help establish two different forms of copy or congruence—direct congruence and mirror congruence.

Establishing an initial definition for exact copies. During the first two days of instruction, students considered what makes structures exact copies or directly congruent. Teachers first showed students an asymmetric five-cube structure (Figure 10) and asked students to build an exact copy of the structure. When asked why their structure was an exact copy, students initially stated that it was because their structure looked the same as the original structure. Teachers brought in a number of examples and non-examples (e.g., structures with the same shape but different size, structures with a different shape) to help move students from the general sense of "looking the same" to describe what properties were the same and to point out how these properties can help negotiate shared understandings. For example, in the following dialogue, Mrs. H helped students consider the property of size in relation to exact copies. It became apparent that not all students interpreted exact copies as being the same size. For these students, they may have been thinking of same in relation to similarity rather than direct congruence. The ambiguity around “same” played an important role in initial moments of defining.

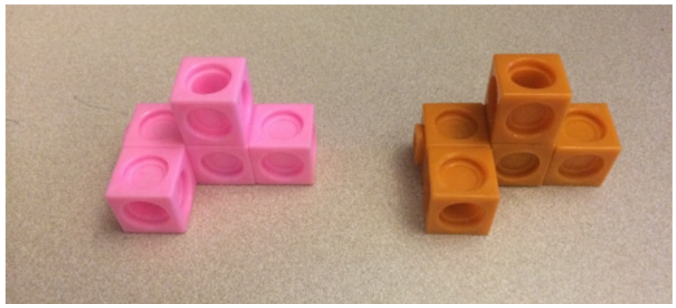


Figure 10: Asymmetric cube structure (left) shown to students and an exact copy constructed by a student (right).

1. Mrs. H: Matt said something else, and I think I heard some other people say it too, he said something about the height, and then you said something else, you called it something else, well it's the same…
2. Matt: It's the same height, well this makes it the same height and this makes it the same (points out corresponding pieces on each structure).
3. Mrs. H: What else could I say instead of height? They're the same number or is there another way to talk about the way it looks? Ben?
4. Ben: Um
5. Mrs. H: Is one bigger than the other (looking at structures in Figure 6)?
6. Ben: Yes?
7. Mrs. H: There is? So if it's a copy, does that mean it can be bigger than the original?
8. Class: No, yes

    […discussion among students, Mrs. H brings in a new structure that is larger than the structures in Figure 6]

9. Angel: Because even though its bigger than the other one, it will still look the same

10. Mrs. H: So it’s bigger than the other one but it still looks the same, okay, well, is it still *exactly* the same or is it just the same shape?

11. Matt: It's the same shape but it’s not the exact same thing because it's bigger.

12. Mrs. H: So it wouldn't be the same size but it would be the same shape. So let's talk about what we want to call a copy, what do we want our class definition […lists students names], we're going to come up with a definition of copy and we have to decide okay, what is that really cause we had a difference of opinion, Angel was saying it could be bigger as long as it’s the same shape and some of you are saying, nope, it has to be the same size and the same shape.

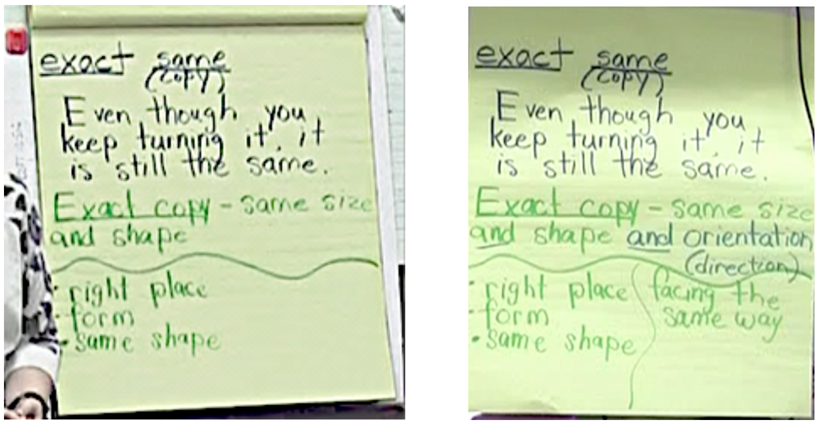
In this episode Mrs. H used familiar instructional discourse strategies, revoicing and prolepsis. The interplay between these discourse strategies helped animate students in new defining roles and communicate the goal of defining to the class. The episodes began with Mrs. H revoicing Matt’s attention to the property of size from a previous conversation. By revisiting this conversation, Mrs. H signaled that Matt’s attention to size is an important property to consider when evaluating exact copies, but rather than directly stating this, she posed a proleptic question to the class in turn 7. We coded this as an instance of prolepsis because Mrs. H acted as if there was already a shared understanding that properties, such as size, can be used as defining criteria of exact copies. Angel’s response in line 9 demonstrated an attempt to infer the underlying meaning of Mrs. H’s question, but he ultimately does not share Mrs. H’s sense of same size as a defining property of exact copies. This breakdown highlighted a lack of shared understanding and mismatch in goals. Mrs. H then qualified her question in turn 10 by asking whether the two structures of different sizes are still *exactly* the same or just the same shape. By restating her question, Mrs. H communicated that the previous response was not what she was looking for, and offered greater context to help shape a different response. Stone (1996) argues that prolepsis can help gradually build common understanding or ground between speakers so they come to a shared understanding of the goal of the interaction. Thus, while Mrs. H's original question presumes that students already know the goal of defining, the extended interaction helps the classroom jointly make sense of this goal.

Mrs. H ended the exchange by revoicing Matt and Angel's ideas, positioning them in opposition. By doing so, Mrs. H highlighted this disagreement and took the opportunity to reframe the episode from a lens of defining, She made explicit that their goal was to come to a definition that they could all agree on. Through prolepsis and revoicing, Mrs. H was able to involve more students in conversation and as participants in defining. Because defining is based on negotiation, this multitude of voices and ideas is critical; thus, she was then able to highlight this negotiation as important to their epistemic goal (Kobiela & Lehrer, 2015), rather than as a simple disagreement to be repaired in the conduct of everyday conversation (Schegloff et al., 1977).

At the conclusion of this discussion, Mrs. H revisited the properties that students' decided were necessary for structures to share in order to be exact copies: shape (form) and size. She first wrote *exact copy* and asked, "exact copy, alright, so we said that it's the same what? As students listed each property, Mrs. H not only included this in their definition of exact copy, but she also made explicit that these properties are ones that they had jointly agreed on, making statements like, "and then we said that…" and "and what did shape mean?" Mrs. H's inscriptions on the chart and appeal to the classroom's shared history were important for communicating the social negotiation of definitions and students' roles in shaping those definitions. The next section further demonstrates how Mrs. H's use of these inscriptions of definition represented and anchored the classroom's continuing mathematical activity in the history of definitions across days of instructions.

Comparing rotations and reflections to reason about orientation. Once classrooms established an initial definition for exact copies that included properties of size and shape, we next wanted students to attend to how rotation and reflection isometries effect relations of direct congruence and consequently, to what properties change and stay the same. Early on in the lesson, students suggested that even if you turn a structure, it is still an exact copy because you can always turn the structure back to its original position. However, some students were not convinced of this based on their reliance on static visual comparison. For example, one student suggested that turning does change an object because when you turn a square it becomes a diamond. Other students argued that turning a structure makes it "look different." In the following case, we demonstrate how students refined their senses of orientation through construction activities that highlighted invariant and variant properties of rotation and reflection isometries.

In Mrs. H's classroom, she started the second day of instruction by showing students two structures from the previous day that students had agreed were exact copies based on the properties of size and shape. Mrs. H started by recounting how the class came to agree that both structures were exact copies. During this episode Mrs. H reminded students of their discovery that no matter how Mrs. H turned her structure, the student holding the second structure (an exact copy) could turn his structure to "match" Mrs. H's structure. She concluded this recounting by providing students with a new word to describe this relation—orientation. She then modeled revising their definition of exact copies by adding [same] orientation to their chart (Figure 11). Next, Mrs. H asked students what they thought orientation meant. One student suggested a definition of "facing the same way" that seemed to resonate with the class. However, while students agreed about this sense of orientation, it became clear in the next interaction that students had different interpretations of "facing the same way."

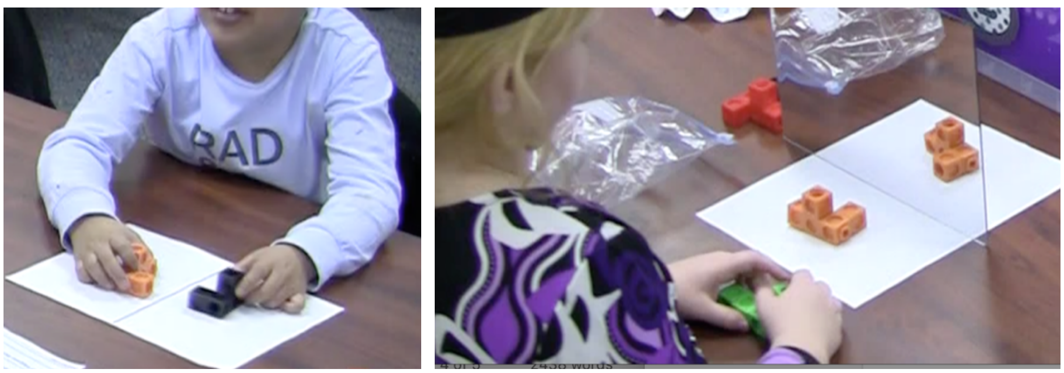


A. B.

Figure 11: Definition charts for exact copies. A) Chart before discussion of orientation B) Chart revised to include conceptions about orientation

Mrs. H proceeded by rebuilding one of two directly congruent structures into a mirror copy (physically changing the orientation of the structure). She then asked the class whether the two structures had the same orientation, or "were they facing the same way". Students responded with a mix of yes and no. Some students seemed to think that "facing the same way" meant that the fronts of the structures pointed the same direction. In contrast, other students looked at the correspondence of individual pieces, stating that the mirror image has a piece that was "on the wrong side." We next show how introducing reflections helped clarify this sense of orientation and helped distinguish exact copies from mirror copies or direct congruence from mirror congruence.

In a second construction task, we first asked students to build what they thought the original five cube structure would look like in a mirror. To help students make sense of this task, we provided students with a sheet of paper with a line drawn across the middle to mark where the mirror should go. Students placed the original structure on one side of the line and built the mirror image on the opposite side of the line or plane of reflection (Figure 8a). After students built their prediction of the mirror image, we passed out portrait mirrors to check their prediction. Students placed the mirrors on the drawn line and compared what they built to what they saw in the mirror (Figure 12b).



A. B.

Figure 12: Building mirror images. A) Students first built a prediction of what the orange building would look like in a mirror. B) Students used the mirror to check whether their prediction of the mirror image were correct.

Despite the designed supports in this task, students' sense of mirrors and their interpretations of the task varied. We had anticipated that students would readily see mirrors as tools to produce reflections, but it became evident that some students had a hard time comparing what they saw in the mirror to structures outside the mirror. For example, when students compared an exact copy of the original structure to the reflection image of the original structure, students often did not see a difference, or they would ignore what they saw in the mirror and compare their prediction to the original structure. However, other students seemed to better grasp this task. As these students checked their predictions, we noticed that they often tried to rotate their constructed prediction to match the image in the mirror. They also tended to spend time playing with the mirrors by pointing at the mirror. For students' struggling to harness the mirror as a conceptual tool, teachers and researchers highlighted the mirror's effect on orientation by engaging students in noticing how their fingers pointed back at them." We found that these forms of student interactions and instructional supports served as important resources for clarifying the task and use of mirrors as reflection tools in the ensuing whole group discussion.

Mrs. H started the conversation by making ties to their previous exploration and definition of exact copies. She brought back the class's chart that had the definition of exact copies, signaling to students that this prior history may be a resource for their current conversation. Although not apparent to students yet, Mrs. H was trying to outline their existing mathematical system, and she later used this chart to help expand the system to include a new form of congruence, mirror congruence, and a new form of transformation, reflection. After highlighting students' work with exact copies recorded on the chart, she asked students to share what they had discovered about mirror images or mirror copies. She first asked students how they built their predictions of the mirror image with the intent of making use of students' sense of what happens when they look in a mirror. She elicited a number of students' thoughts. For example, one student, Cassie started by sharing that she built her prediction of the mirror image to be an exact copy of the original structure. Mrs. H then brought in another student, Ben, who accurately built a mirror image. When she asked Ben how he built his prediction of the mirror image, he explained that, "mirrors, when you look to something, um they're different, they look different." Mrs. H posed this sense of mirror images looking "different" to the class.

1. Mrs. H: Who else thinks that? When you look in a mirror, when you put something in a mirror it looks different?
2. Class: (a few hands raised)
3. Mrs. H: How do they look different to you Lila?
4. Lila: Because when there's two blocks it only looks like one block
5. Mrs. H: You think that’s what it looks like. Anybody else? What do you think, Parker?
6. Parker: The mirror is looking at me and if you point in the mirror and it’s back to you (gestures pointing into a mirror).
7. Mrs. H: So Parker says the mirror is looking at me and he says sometimes when you point at the mirror, it points back at you. So he said the mirror (picks up a mirror, Figure 13) is looking at me and he said when you point at the mirror (points into the mirror).
8. Parker: It points back to you.
9. Mrs. H: It points back to him, to us. Do you see that?

Figure 13: Mrs. H demonstrating how the mirror image of a pointing finger changes orientation or "points back to you."

Mrs. H attempted to make the use of the mirror as a tool to produce reflections explicit to the class. In turn 7 she accomplished this through physically reenacting and narrating students’ use of the mirror. By having students explain what actually happened in the mirror (i.e., it points back to you), Mrs. H highlighted a new way of visualizing by using mirrors. Mrs. H was the last classroom we worked in each day, thus while we found that this highlighting to be consequential, it was not consistently implemented across the other classrooms. As a result, some students continued to ignore the image in the mirror and used the mirrors as a partition rather than as a reflection tool.

Once the class established the function of the mirrors in supporting their visualization of mirror images, Mrs. H turned back to students' constructions. She showed one pair of students' accurate prediction of the mirror image of the original structure. But, before using the mirror to verify whether this pair of students' prediction was correct, Mrs. H first asked students to evaluate the prediction by deciding whether the prediction was a direct copy of the original structure or something different.

1. Mrs. H: What can I do to decide if the prediction is an exact copy, how can I tell? How can I find out? Serena?
2. Serena: You can look at them and see if they're the same
3. Mrs. H: How can I look at them to see if they're the same? Can you show me? […]
4. Serena: You can look at them and if they're kind of wrong you can make them.
5. Mrs. H: Well how can I tell if they're wrong? What can I do to this to find out?
6. Serena: You can turn it
7. Mrs. H: How? Can you show me?
8. Serena: If it's facing this way you can turn it this way (twists just a portion of the building to change the orientation)
9. Mrs. H: Oops, I know they can turn that way but we have to pretend that they're stuck like that okay (meaning that although the cube construction could be disassembled in

parts, the class would pretend otherwise). So you can't be turning like this because that's not the purpose of this. We're going to pretend like it's stuck. Alright? So you could do that, that's very true but we're going to pretend like it's stuck. Since I can't do it that way, what can I do? And if you're on the carpet right now I want you looking, I want you looking to see what you think. How can I find out, what do I need to do? To see if this is the same to see if it's an exact copy or not.

10. Rain: You can take them like this (turns whole building, trying to align the buildings into direct congruence) and see
11. Mrs. H: Oh, see what she did? She said well you can turn them both like this and see if that's an exact copy or not. So if I do that ... so look here, is that an exact copy?

We found it notable that Mrs. H started this exchange in turn 1 by explicitly asking for a procedure or way to analyze the structures rather than just asking whether the structure were exact copies. Her attention to a procedure for visually analyzing relations of direct congruence helped refine the class's sense of rotations, but it also helped position rotations as a mathematical transformation or tool for spatial analysis. In particular, rotations were clarified as turns of an entire structure rather than individual components. Thus, this is also another moment demonstrating the importance of Mrs. H's attempts to make ways of visualizing explicit to the class. By acknowledging a sense of rotation as rotating individual components of the structure, Mrs. H highlighted this as one possible interpretation of rotation. However, she juxtaposed this with a rotation of the entire structure, pretending as if the pieces were stuck together. When Rain demonstrated rotating the entire structure, Mrs. H highlighted this movement in turn 11, stating, "oh, see what she did." She then provided further narration of the movement. This episode concluded with Mrs. H asking students again whether they thought the predicted mirror image was a direct copy of the original structure. Students all agreed that it was not, and Mrs. H supported them in justifying this by going through their defining properties of exact copies. In sum, Mrs. H helped establish rotations as a form of mathematical transformation by emphasizing rotations as a tool to analyze relations of direct congruence and coordinating this with the class's definition of exact copies.

In the final exchange of this episode, Mrs. H asked students, "if [the predicted structure] is not an exact copy, is it a mirror image and what does that mean, what does that look like?" Students collectively evaluated the status of the predicted structure using the physical mirror to verify that it matched the image of the original structure in the mirror. Mrs. H then labelled the structure as a mirror image and had students compare the original structure to the constructed mirror image. Students immediately pointed out that the structures had different orientations because "the pointy piece is on the wrong side… it's on the right instead of the left." Mrs. H then asked students what properties were still the same in order to draw a relation between direct copies and mirror images or mirror copies. Students agreed that the mirror image was the same size and the same shape as the original structure. Mrs. H concluded by taking the opportunity to guide students in forming a classroom definition for mirror copies.

1. Mrs. H: Okay, so I'm going to write that part down. We agree, a mirror image is still the same what?
2. Class: Size
3. Mrs. H: And same?
4. Class: Shape

5. Mrs. H: So let's look at what exact copy was (pointing to definition), we have same size, same shape. Mirror image, same size, same shape but is it the same direction? The same orientation?
6. Class: No!

Although the initial start of the definition for mirror images followed a similar sequence to that establishing the classroom definition for exact copies, Mrs. H was intentional about weaving between the developing definition and the established definition for exact copies in turn 5. This discourse move helped establish relations between direct copies and mirror copies and, as such, expanded the mathematical system. Also, the gradual development of orientation as a property to distinguish mirror images and exact copies helped clarify initial confusions around the meaning of orientation (Note: We intentionally did not bring in examples of structures where the mirror image is directly congruent, i.e. any structure with reflection symmetry). Finally, we also observed a shift in students' ways of evaluating examples and non-examples, an important development in their defining practice. Where they initially used words like "same" and "different," they started to more consistently use properties and definitions to evaluate examples and non-examples of direct and mirror congruence.

Summary: The co-development of isometry concepts, defining practice, and visualization. At the onset of the instructional sequence, students' sense of congruence was tacit, using non-specific judgement of same or different based on static one-to-one comparisons. As teachers introduced examples and non-examples of congruent structures to help highlight salient properties, they also structured discussions in ways that helped animate students within the practice of defining. Teachers ways of acting as if students were engaged in defining in conjunction with moments of explicitly verbalizing the goals of defining or maintaining a written record of definitions helped gradually build joint understandings of the goals and means of defining. We also noted that as we introduced rotations, reflections, and tools to support the visualization of these transformation, teachers highlighted different ways of visualizing these and tied these visualizations to definitions of exact copies and mirror copies. In this way, ways of visualizing new concepts of reflections and rotations were negotiated by referring to the existing mathematical system. At the conclusion of this first part of instruction, students began to increasingly use the classroom's shared definitions of exact copies and mirror copies as tools to help support the evaluation of examples and non-examples. They also used new concepts of reflections and rotations as tools to evaluate these two forms of copies.

Introducing Reflection and Rotations Symmetries as Isometries that Preserve Self-Congruence

The second part of instruction in January (phase 1) built on the classroom's definitions of exact copies and mirror copies, and their developing senses of rotations and reflection isometries. Here our goal was to have students develop conceptual understandings of symmetries as special forms or a subset of rotations and reflections that preserve self-congruence—a structure looks exactly the same before and after the transformation.

Establishing reflection symmetries as a subset of reflection isometries: "The mirror completes the other half!". Up until this point in instruction, students had considered reflections where the reflection plane, represented by the mirrors, were located outside of the structures. Because we decided to look only at finite symmetries of objects or symmetries with a fixed point in this unit, our first goal in supporting students' understandings of reflection symmetries was to have students explore reflections where the plane of reflection was located inside the structure. We started by showing students two structures— one with no symmetry and one with vertical bilateral reflection symmetry (Figure 14a & b). Despite previous findings that suggest that children have a natural preference for reflection symmetry (Bornstein & Stiles-Davis, 1984), children showed no preference for either structure. Instead, they described the structures by comparing the sizes of the structures and the number and shape of individual parts making up each building. Because we wanted to elicit students' informal sense of symmetry, we then asked students if they noticed if either of the buildings was made by putting together copies of parts of the building. Students misinterpreted this question and thought that we were asking whether the two structures were copies based on the structure of the activities the previous day. Thus, even though we observed children employ reflection symmetry in their own designs, symmetry, even in more informal senses, was not yet a conceptual tool that students used to analyze and compare structures.

A. B. C.

Figure 14: A) Structure without reflection symmetry. B) Structure with bilateral reflection symmetry. C) Structure split in half showing the two have opposite orientations.

We next showed students the two buildings split into two parts along similar planes and asked which building was made up of mirror copies. Comparing the parts of both buildings, students identified that the parts belonging to the building with reflection symmetry were the same. The teachers pulled back in their definitions for mirror copies and asked students to describe what was the same about them. Students then stated that the parts were the same size, the same shape, but the orientations were opposite because hollow spaces or openings were pointing opposite directions (Figure 10c). Teachers brought in the mirrors as a tool to test whether the parts were truly mirror images—simultaneously modeling placing the reflection plane or mirror inside the structure (Figure 15). The following exchange demonstrates students' reaction to Mrs. H's demonstration of using the mirror to test for mirror or reflection symmetry and the introduction of the concept of symmetry.

Figure 15: Using the mirror to test for reflection symmetry within a 3D structure.

1. Mrs. H: So let's find out for sure.... so looking at it like this, here are the two halves. I'm going to put the mirror just like dividing right down the middle, so if I put it like this (see Figure 11).
2. Parker: And it's already done!
3. Matt: It's just the same thing
4. Mrs. H: It's the same as what?
5. Matt: As... this one (points to the original, intact building)
6. Mrs. H: So it's the same as that, the same as, he said it was the same as this (holds up whole building)

7. Mrs. H: Alright, so that's called […checks with M.W. if it's okay to introduce the term symmetry]. It's called mirror symmetry because…when I put the half in the mirror and the mirror divides the two in half, it uses it right here, what you see is the same on both sides here. Kind of like the finger thing (points in the mirror), do you see the other side in the mirror coming out at you?

Students' initial noticing of reflection symmetries were full of excitement and surprise; however, their conceptions were disconnected from previously established concepts of congruence. To help clarify for students what constitutes a finite reflection symmetry and make connections to concepts of congruence, we next had students analyze a collection of preselected structures. This set of structures included structures with no symmetry and structures with reflection symmetry where the orientation of the symmetrical reflection plane varied. By including structures with no reflection symmetry or non-examples, we also hoped that this activity would help students differentiate between planes of reflection that can be placed anywhere inside or outside a structure from planes of reflection symmetry. For this activity, we opted to use Polydrons as construction materials because their rigid connections allow for stable structures when split into two halves. The structures were placed at students' tables in pairs—one structure was whole or intact and the other was a copy of this structure split in half to indicate a plane of reflection.

During the whole group conversation that followed the analysis activity, students shared their observations when using the mirrors to represent reflection planes through the center of the structures. This conversation was critical in building on the initial observation that the mirror could be used to "complete the other half" of a building to describing when this was and was not the case. In Mrs. H's classroom, this clarification was established by asking students whether the mirror “completed” the other half for *all* the structures or for just some of the structures. Students expressed that it happened only for some structures. Mrs. H then supported students in describing why this occurred by comparing the parts that made up each structure. One student, Parker explained that on one of the asymmetric structures the sides separated by the mirror were different sizes, so when you put the mirror in between them, the mirror did not show the other half. To help demonstrate Parker’s observation, Mrs. H brought in an example of a structure with reflection symmetry, and she asked whether this structure would have reflection symmetry by evaluating the two parts.

1. Mrs. H: Okay and when it does have reflection symmetry, what is that... so when it does have reflection symmetry, the halves what?
2. Class: Are together
3. Mrs. H: It’s the same what? It has the same
4. Class: Size
5. Mrs. H: So it has the same size, "the halves have the same size"
6. Jenna: Shape *and size*
7. Mrs. H: Size and shape
8. Author 3: What's different about the halves? Can we hold up one that does have reflection symmetry?
9. Mrs. H: Uh.... so we know that the halves are the same size and the halves are the same shape when we see reflection symmetry, right? Do you see that? We know that that the halves have the same size and that the halves have the same shape, but what is different about this (holds out)
10. Jenna: When you open them, they are opposites.

11. Mrs. H: When you open they're what?
12. Jenna: Opposites
13. Class: Opposites
14. Mrs. H: They're opposites, absolutely. So they're opposite, we want to use the word orientation, right? Okay so reflection symmetry, the halves have the same size and the same shape but they are what?
15. Class: Opposite
16. Mrs. H: Orientation

Here the use of examples and non-examples helped move students beyond just saying the halves on the sides of the mirror are “the same” to connecting to their sense of congruence to describe what properties were the same. It also helped clarify that not all reflection planes placed inside a structure represent a reflection symmetry. This clarification positioned students to establish reflection symmetry as a particular case of reflection isometries as “when the mirror completes the other half of a structure.”

However, despite our intention of providing students examples of reflection symmetry with varied planes of reflection, students did not remark on this as notable. They also did not consider that the mirrors could be placed at positions other than where we provided a split between halves of each structure. This is likely because for each structure made of Polydrons, there was only one possible way for the reflection plane to physically fit between parts of the structures. Thus, to support students in attending to the position of the reflection planes and the number of symmetrical reflection planes, in the next activity, where we asked students to construct their own structures with reflection symmetry, we switched to Magformers. In contrast to the rigid connections of Polydrons, Magformers are connected by magnets. The magnet connections allowed students to experiment with placing the mirror at various locations in their structures without having to break the structures apart. We highlight a case from a focal small group of students analyzing the different structures at their tables to demonstrate how the variability in their own constructions helped further support students' understanding of reflection symmetry and planes of reflection.

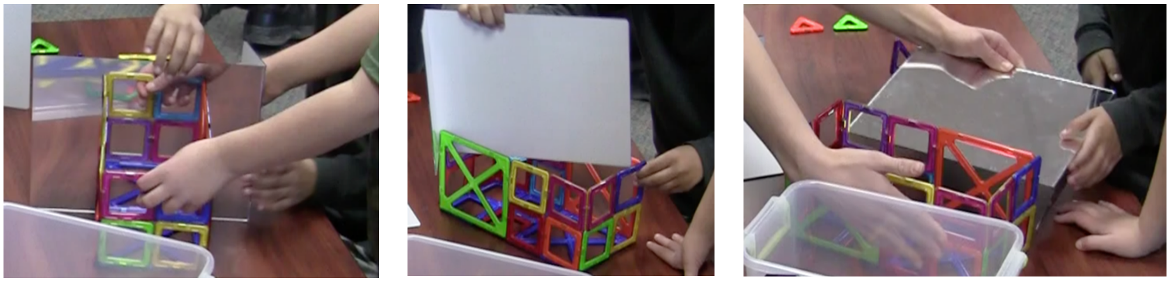
A. B. C.

Figure 16: Students' structures with reflection symmetry. A) Student indicates a position where he could place a mirror to test for reflection symmetry. B) Students' first attempt at counting the number of planes of reflection symmetry in their structure. C) Using the mirror to help coordinate counting with the analysis of reflection symmetries.

After students had time to construct their own structures with reflection symmetry in their small groups, they used mirrors to analyze different planes of reflection within the structure. During one small group interaction facilitated by M.W., she placed the structure shown in Figure 16 in the center of the table for the group's collective attention. She then asked students where the mirror could go to show a reflection symmetry. Students quickly exclaimed that the mirror can "go anywhere" accompanied with a gesture to opposite edges, marking where the mirror would intersect the structure (Figure 16a). M.W. asked for further clarification posing the question, "well, *anywhere?"* She then modeled placing the mirror between non-opposite edges (an asymmetrical reflection plane) and asked if placing the mirror at this location would count as a symmetry. The students objected by comparing the image in the mirror to the original structure. They then corrected the placement of the mirror to a position between opposite edges.

Next, M.W. asked the students to count how many symmetries the structure has. Despite students' gesture of the mirror as intersecting two edges, students proceeded by having one student count the number of edges as a second student used his finger to mark the beginning of the count (Figure 16b). M.W. supported this new form of counting by asking them to use the mirror to represent the reflection plane (Figure 16c). Students then recounted using the mirror to support their counting, helping them coordinate an existing procedure of counting with their emerging understanding of reflection planes. As a result, a new form of analysis procedure developed of not only identifying if a structure has a plane of reflection symmetry, but identifying *how many* planes of reflection planes a structure has.

Immediately following this interaction, this same group of students independently decided to apply this new counting procedure to a second structure, a rectangular prism (Figure 13). By experimenting with the total number of planes of reflection symmetry this new structure had, students further clarified the difference between planes of reflection versus symmetrical planes of reflection. For example, the students first thought that a reflection plane through a diagonal of the structure was a symmetry because the mirror partitioned the structure into two equal halves (Figure 13b); however, upon closer inspection of the image in the mirror, students noted that this image did not match the original structure. Finally, counting the planes of reflection also helped expand students' sense of the possible locations of reflection planes. Figure13c shows students considering a new way of holding the mirror, parallel to the table, and this helped them find an additional plane of reflection symmetry at a less obvious angle.

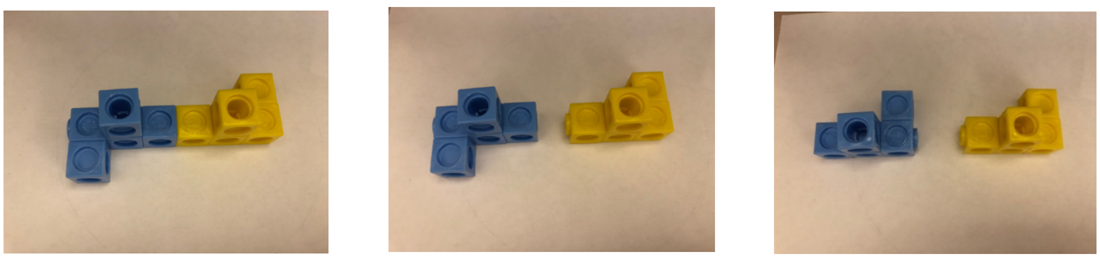


A. B. C.

Figure 17: Students using a new procedure of counting the total number of reflection planes in a rectangular prism. A) Students first show a plane of reflection symmetry through the center of a face, perpendicular to the table. B) Students check for reflection symmetry through diagonally opposite edges. C) Students consider changing the orientation of the mirror in relation to the table and test for a reflection symmetry through the center of faces that are parallel to the table.

The variability among structures built by students was consequential for supporting greater relations and distinctions between reflections and reflection symmetries as some structures challenged students to re-evaluate what it meant for the parts of a structure on either side of the reflection plane to be the "same." In particular, some structures, like the rectangular prism, helped clarify what it meant for parts of a structure on either side of a reflection plane to be the "same" versus to be mirror congruent. Other structures invited students to "see" many planes of reflection symmetry and supported a new procedure of counting reflection planes. This counting of reflection planes helped students understand that structures can have multiple reflection symmetries. After establishing this new counting procedure, students were more likely to test new positions for reflection planes that perhaps are not as immediately visible. Next, we describe students' exploration of rotation symmetry, and highlight how students' defining work around reflection symmetry helped students anticipate relations between concepts.

Rotation symmetries as a subset of rotation isometries: "You can't even tell that you turned it!". The final activity with turn symmetries followed a similar sequence to the reflection symmetry activities; however, we first had to introduce students to degrees or magnitudes of rotation. We accomplished this by connecting rotations to physical turns of the body. For example, we first asked students what a whole turn looks like. Students turned their bodies and teachers helped students connect a full turn to completing a circle. Students then turned their bodies in a half circle, and teachers labeled this as a half turn or one half of a whole turn. In the unit, we limited students' analysis to half turns and half turn symmetry, as students had not yet done much work with angles and rotations.



A. B. C.

Figure 18: Initial structure shown to students as an example of ½ turn symmetry. A. Teacher showing students the outcome of a half turn. B. Teacher asking students to evaluate the congruent parts making up the structure. C. Students noting the two parts of the structure can be rotated into direct congruence.

Once students could identify full turns and half turns, the teacher showed students a structure with half turn rotation symmetry constructed out of multi-link cubes (Figure 18a *structure was a single color). Students closed their eyes as the teacher turned the structure in a half turn around an axis of symmetry; however, we did not at this point introduce the turn axis. As students opened their eyes, the teacher asked them to guess how they moved the structure. Students exclaimed that the teacher had not moved the structure at all. Students then closed their eyes again as the teacher repeated the turn. Students now had mixed responses and guessed that she either did not move the structure or that she did a full turn. Teachers finally asked students to watch how she moved the structure, modeling rotating the structure in a half turn. Students were surprised and noted that the structure was turned a half turn. When asked what students noticed about the half turn, students noted that the structure looked exactly the same or like the structure wasn’t moved at all. One student, Mario explained, “It looks like you didn't move it but you did, you turned it half around so it looks like it’s exactly the same but actually the one [part] that’s on right is actually on the left, its actually on the left now.” Teachers took up students’ notion of turning the structure a half turn without anyone being able to notice as the classroom definition for rotation symmetry. In Mrs. P’s classroom, she next changed the color of one half of the structure, modeled the half turn symmetry, and asked students what about the structure allowed for the structure to have turn symmetry (Figure 14).

1. Mrs. P: Watch again, I’m going to turn it half of a circle again, okay Kirsten? I’m just going to turn it, so watch the way it looks. Put that in your brain just like the sight words we do okay and now keep your brain on that. Ready? I’m turning it half of a turn.
2. Class: Same!
3. Mrs. P: It looks the same! Did you guys see me move it?
4. Class: Yeah
5. Mrs. P: I moved the building, so I wonder how it looks the same every time I turn it half of a turn, how does it look the same? Okay, Kelly, what are thinking? I’m not sure if that was a rhetorical question or not but what are you thinking?
6. Kelly: Um, I’m thinking that this one [part] is upside down and that one [the other part] is not upside down, and when you turn it that one goes upside down. But if that one isn’t upside down, then it won’t be the same.

Kelly’s explanation is similar to the sense of thinking of turn symmetry as a direct mapping, where all points of the structure map directly onto a second point of points. She appealed to this by explaining that the parts of the structure switched places or parts that were right side up turned upside down and vice versa. Mrs. P struggled to follow this reason and suggested in the next turn of talk to take the parts of the structure apart.

7. Mrs. P: Okay, so you're thinking if I took it apart maybe? Okay (see Figure 18b)
8. Kelly: Yeah, um if you took it apart and you um, if you put um…
9. Mrs. P: What do you notice about these two buildings that I put together? Katie?
10. Katie: They're the same
11. Mrs. P: What do you mean, what kind of same? They're congruent, right? They look the same, but they're the same how, like what kind of copy?
12. Katie: Um the same size and the same shape
13. Mrs. P: Okay, are they facing the same way?
14. Katie: No
15. Mrs. P: Is there a way that I can move one of them to make them
16. Class: Yeah
17. Mrs. P: Mario, how can I move it?
18. Mario: You can do a half circle
19. Mrs. P: A half turn, a half circle and make them look the same (turns one structure in half a circle, Figure 14c). So, these, what kind of copies are these?
20. Class: Same, Exact [copies]

Similar to Mrs. H's move when introducing reflection symmetry, Mrs. P was trying to help students make a systematic relation between congruence and symmetry. However, it's notable that students appeared to need less support when comparing structures based on congruent parts. We attribute this moment of students' establishing mathematical relations to their previous developments in the practice of defining from which new understandings of congruence and isometries emerged. This moment also highlights how the practice of defining and related goals of establishing mathematical relations helped position concepts as tools to support shared ways of seeing or visualizing. Finally we take this moment as evidence that the mathematical system itself was becoming more stable for students as they now positioned concepts as a resource for further mathematical reasoning.

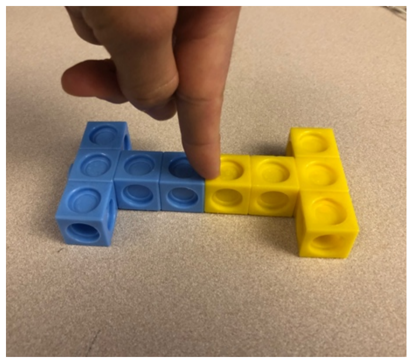
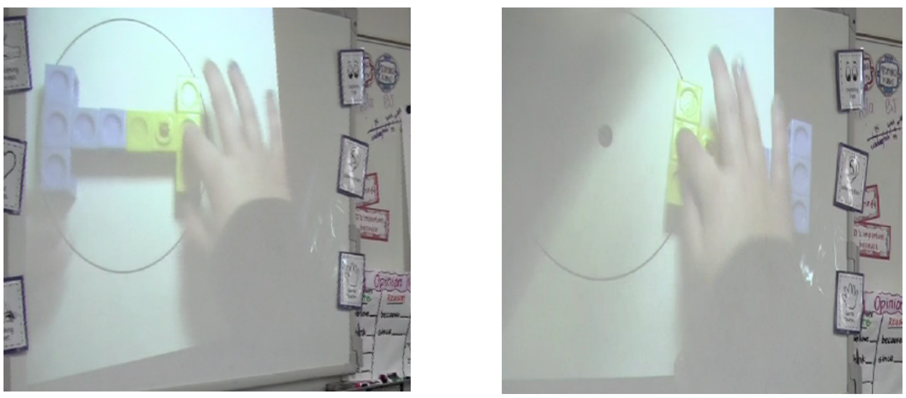


Figure 19: Student's structure with ½ turn symmetry about a turn axis indicated by pointer finger.

After this initial sense of turn symmetry was established in the classroom, students designed their own structures with half turn symmetry. To help scaffold this construction activity for students, we asked students to build two directly congruent five-cube structures and to attach these structures to have half turn symmetry. Students then analyzed their structures by rotating their structures a half of a turn. Once students had constructed their structures and tested them for half turn symmetry, teachers introduced the turn axis as a tool to communicate how students rotated their structure. In both Mrs. P and Mrs. H's classroom the need for a way to communicate how to turn a structure to see turn symmetry arose from the variability among students' structures—some structures only had one axis of turn symmetry and other structures had multiple axes of turn symmetry. For example, in Mrs. P's classroom, she first had one student, Katie, come demonstrate how she placed her finger in the center of the structure (acting as a turn axis, Figure 19) and rotated the structure around her finger. Mrs. P tagged this as a turn axis. To help demonstrate how the placement of the turn axis affects how a structure will rotate, Mrs. P also introduced a new tool- a circle outline printed on a sheet of white paper. Placing Katie's structure on top of the paper, Mrs. P asked students to notice where the structure moved in relation to the circle as Katie turned the structure one half of a turn around a turn axis placed at the center. Students noted that the structure stayed in the same location, but that the sides of the structure switched places.

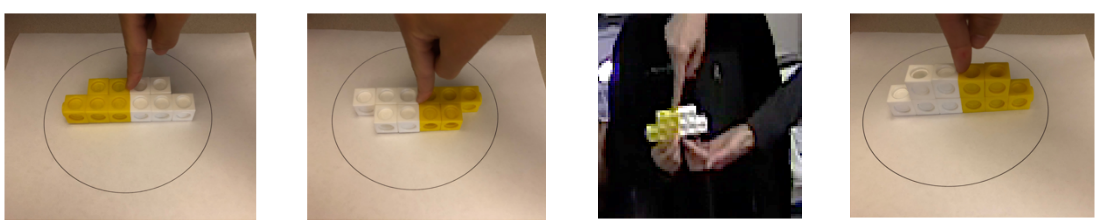


A . B.

Figure 20: Example of a non-symmetrical turn axis. A. Student first indicated a new turn axis with her pointer finger at the edge of the structure. B. The result of a half turn around the indicated turn axis.

To help further demonstrate the utility of specifying the location of the turn axis, M.W. then asked what would happen if Katie moved her finger to the edge of the structure (a non-example of an axis of turn symmetry, Figure 20a). Students predicted that the structure would still end up in the exact same location. After Katie moved the structure around this new turn axis, students were surprised that now the structure was outside the circle (Figure 20b). Mrs. P then asked students if this would be a place where the structure has turn symmetry based on their previous sense of turn symmetry as not being able to tell that anyone had turned the structure. Most students said no; however, some students mentioned that you could just do a full turn. Rather than taking this up, Mrs. P decided to constrain turn symmetry to half turn symmetry, clarifying that she was only talking about the half turn. She then took this opportunity to revise the class definition of turn symmetry, summarizing this episode by stating "so that's a special thing about turn symmetry, it has to land on itself."

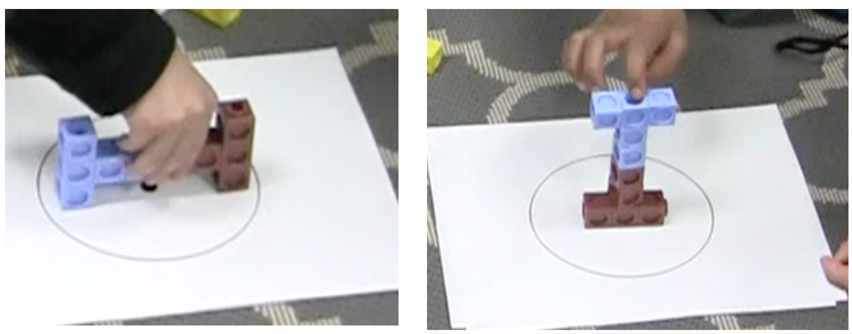
The class then looked at one more structure constructed by a student, Mario. Mario came to the front of the classroom and placed his structure in the center of the circle outline; however, he laid the structure so that axis perpendicular to the paper was not an axis of turn symmetry (Figure 21a). Mario rotated his structure, placing his finger in the center of the structure as the turn axis, but hesitated because his structure no longer appeared to have half turn symmetry (Figure 21b). Mrs. P noted this and marked it as tricky. She then invited him to show the class how he rotated the structure without the circle outline and turn axis. Mario turned the structure using two hands and demonstrated that his structure did have half turn symmetry as Mrs. P helped indicate the turn axis using her fingers (Figure 21c). Next, Mario placed the structure back in the center of the circle so that the axis of turn symmetry was now perpendicular to the circle. This was an important moment, as it highlighted for students that turn axes can go through any side of the structure, and this can change how the structure rotates.



A. B. C. D.

Figure 21: Student searching for symmetrical turn axis. A) Students' structure before turning. B) Students' structure after turning around non-symmetrical turn axis indicated by pointer finger. C) Teacher helping students identify the turn axis of symmetry. D) Student modeling the turn axis of symmetry.

After demonstrating how to use the turn axis to analyze their structures, students went back and experimented using turn axes on their own structures. Students noted that by varying the turn axis based on the ways they placed the structure on the circle outline, they were able to find some structures with more than one axis of turn symmetry (Figure 22). Thus, similar to students' analysis of their structure with reflection symmetry, students also noted that counting the number of symmetrical turn axes helped them compare and describe the variability across the different structures constructed by students.



A. B.

Figure 22: Student searching for and counting multiple turn axes of symmetry. The turn axes are indicated by his pointer finger and the black center dot on the paper.

Summary: The co-development of isometry and symmetry concepts, defining practice, and visualization. Although students' designs prior to instruction often incorporated symmetries, it was evident that students did not yet have a shared concept and language to identify symmetries as salient aspects of their own and others' designs. However, by introducing new tools—reflection planes and turn axes—to support a new way to analyze 3D structures, teachers helped students use their developing mathematical system of congruence and isometries to define the rotation and reflection symmetries of objects. This coordination between developing conceptual tools and physical tools opened new practices of visual analysis.

Additionally, the variability among students' own structures that served as examples and non-examples of developing concepts of symmetry were critical to supporting this coordination and refinements of students' understanding. By the end of these five days of instruction, students not only started to use their classroom definitions to evaluate examples of rotation and reflection symmetries, but they also started to use their system of definitions to begin establishing relations between isometries and symmetries. In this way, students not only were engaging in defining for the purpose of shared communication, but they also were beginning to understand the epistemic affordance of defining—to expand their local mathematical system.

January cognitive interviews. These conclusions about the collective level of conceptual change were buttressed by children's (n =10) responses to the first clinical interview. All students built an exact copy or directly congruent (100% 1st attempt) and mirror copy (80% 1st attempt, 20% 2nd attempt) of a target asymmetric structure. 70% of students successfully constructed a structure with rotation symmetry (40% 1st attempt; 30% 2nd attempt) and 80% of students successfully constructed structures with reflection symmetry (40% 1st attempt, 40% 2nd attempt). The three students not successful at building a structure with rotation symmetry interpreted all half turns as a half turn symmetry. Students not successful constructing a structure with reflection symmetry confused the reflection plane as a tool to divide a structure into two parts rather than two mirror congruent parts. The results of these post interviews helped inform the design of the final three days of instruction. In particular, we decided we needed to provide students with additional visual aids to help them to better relate isometries and symmetries and to design structures with symmetries as constraints on design.

Constructing 3D Structures Using Reflection and Rotation Symmetries as Design Constraints

The final part of instruction occurred approximately three and half months after the conclusion of the previous five days of instruction. Due to end of the year testing, we were only able to continue instruction in Mrs. P's classroom. Our goal for these final three days of instruction was to support further growth in students' understandings of relations among symmetries. The initial day and a half served as a review of reflection and rotation symmetries and as an introduction to coordinating these forms of symmetries during analysis. During this time, students analyzed a set of pre-constructed 3D structures for both reflection and rotation symmetry. The set of structures included structures with both symmetries, only one type of symmetry, and no symmetry. This was a novel task, as in the previous instructional period, students only analyzed structures for one type of symmetry. As previously mentioned, after the first five days of instruction some students took half turns to be synonymous with turn symmetry. We attributed this confusion partly to students' difficulty of coordinating tracking of the magnitude of turns (a new concept for students) and maintaining an image of what the structure looked like before turning. Similarly, some students understood planes of reflection symmetry to be places within the structure that they could insert a mirror to break the structure into two parts. These students did not coordinate the image in the mirror with the structure as a whole.

To help students coordinate pre-images and transformed images of structures with reflection planes and half turns, we introduced Tinkercad as a visual tool. We used Tinkercad to show students a computer rendering of the physical structures they had analyzed for different symmetries. However, because students senses of half turns were based on physical turns of their body, we also tried to better leverage this resource by positioning a Legoman on top of the 3D structures. Thus, when the structure was rotated a half turn in Tinkercad, the Legoman also turned half of a turn (Figure 23). This visual support seemed crucial for some students, as it was easier for them to determine when the Legoman did a half turn versus identifying an exact half turn of the 3D structures. This instructional support is made more evident in the following exchange where students are analyzing the structure in Figure 23 for half turn symmetry.

1. Mrs. P: Let's look at the rotation with Ms. Megan.
2. M.W.: Before we turn it, what do you think, do you think it will have half turn symmetry?
3. Mrs. P: Don't just yell, do you think it's [the building] going to land back on itself? Look at where Mr. Lego man is facing and if he turns and faces the back is it going to look the same?
4. Class: No, yes
5. Mrs. P: Ryan's saying he thinks so.
6. Student: I think so too
7. Mrs. P: Okay, let's see. Turn and now he's facing the back. Did it land back on itself?
8. Class: Yes
9. Mrs. P: Look, Alex, I'll turn it. Look at how it looks, the two roofs are right here and if I turn it a half turn where are the two roofs?
10. Student: Right here again
11. Student: The same place
12. Daniel: It turned and landed right back where it was!

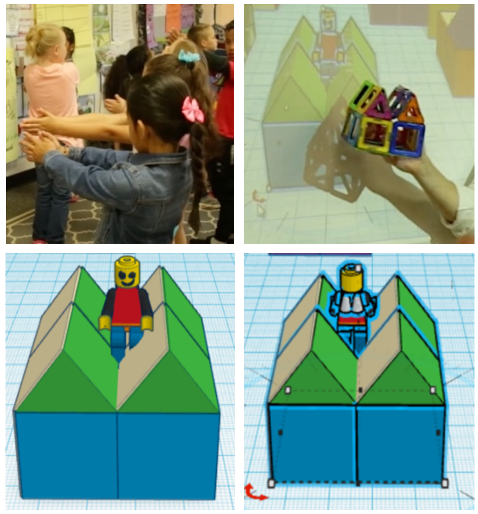


Figure 23: Examples of symmetry analysis supported by Tinkercad. A Legoman was placed on top of each structure to help students connect their body sense of half turns. Autodesk screen shots

While teachers introduced the idea of a structure "landing back on itself" after a turn as evidence of half turn symmetry in January, this phase now was connected to a more literal and visual instantiation. Previously students had to compare the location and shape of the pre-image of the structure to its transformed image, and they had to simultaneously track a half turn. We found that the visualization provided by Tinkercad helped both offload and break down some of these cognitive tasks. In particular, by focusing students' attention to what direction Mr. Lego's faced (the front or the back) as evidence of a half turn, they now were able to focus more on the relation of the parts between the pre-image and transformed image of the structure. For example, the discussion in line 9 of where the "two roofs" started and landed later became a resource when they went back to analyzing their structures without the aid of Tinkercad. Students talked much more about where individual parts started and ended after a half turn which seemed to be more helpful in organizing their analysis than judging whether the whole structure looked "the same."

At the conclusion of analyzing structures for reflection symmetries and rotation symmetries, Mrs. P highlighted the structures that had both forms of symmetry. She then asked students what they noticed about this subset of structures, encouraging students to pose possible relations between reflection symmetries and rotation symmetries.

1. Mrs. P: Okay, what do you notice about the buildings that have both symmetries? Because buildings, we just learned, hands down, we learned buildings can have one symmetry, they can have a reflection symmetry or maybe they don't have any symmetry at all. But what do you guys notice about the ones that have both? First off, which ones do have both? Stella, which one? Which one has both? Number
2. Student: 1
3. Stella: Number 5
4. Mrs. P: Okay so number 1 and number 5 are this one and this one (see Figure 24). What do you notice about the ones that have both symmetries? What do you notice Daniel?
5. Daniel: That this one [on the left] almost looks like that one [on the right] but the [left] one has squares and [the right]one has triangles [on the tops].
6. Mrs. P: Okay, he noticed that both look the same. Thanks Daniel...
7. Student: He said one has triangles and one has squares
8. Mrs. P: Okay they almost look the same. You mean Daniel that they just look symmetrical? They look congruent?
9. Sophia: They're actually alike because if you put the mirror here and here, here and here (gesturing to planes of reflection symmetry, Figure 24).
10. Student: You can make a plus sign (cross arms to make two perpendicular reflection planes)

11. Mrs. P: They make a plus sign, so that was two.

Figure 24: Student uses her hands to gesture to the perpendicular planes of reflection symmetry in the structures the teacher is holding.

While some students, as represented by Daniel, still wanted to compare individual parts of the structures, other students, such as Sophia, used the classroom's shared attention and understanding of reflection symmetry and planes of reflection to explain the similarity between the structures. As students became more attuned to the usefulness of making definitional explanations, they also opened up opportunities to establish further systematic relations between concepts. In this case, Sophia pointed out that the structures they found had both reflection and rotation symmetries had two planes of reflection symmetry that were perpendicular to each other. Sophia's explanation was then taken up by a shared gesture where students used their arms to represent a "plus sign" or two perpendicular planes. This conversation then concluded with the class proposing a conjecture that in order for structures to have both rotation and reflection symmetry, the structure needed to have at least two planes of (perpendicular) reflection symmetry.

After supporting students in analyzing 3D structure for both reflection and rotation symmetry, we posed design challenges for students that positioned symmetries as design constraints, again with the intention of supporting student consideration of relations among symmetries. Specifically we asked students to construct structures with only rotation symmetry, only reflection symmetry, and both forms of symmetry. This was a challenging task for some students, as they were still learning to coordinate transformations with the pre-image and transformed image. These students tended to approach these construction tasks by building and analyzing for symmetries post-hoc. Other students who seemed to have a better command of this, more readily used symmetries as a design tool. For example, Figure 25a shows one student's construction of a structure with only half turn symmetry. When M.W. probed about her structure, she explained how she decided to place triangles in diagonally opposite corners to make it easier for others to see her structure had half turn symmetry. M.W. then asked how changing the placement of the triangles could change the symmetry of the structure, and the student demonstrated that placing the triangles directly opposite each other resulted in reflection symmetry and no turn symmetry (Figure 25b). Finally, when M.W. asked what she would need to do to make the structure have both turn symmetry and reflection symmetry, she took both triangles off the structure and demonstrated both symmetries by rotating the structure and gesturing to the planes of reflection symmetry (Figure 25c). Other students using symmetries as design tools similarly explained how during construction, they imagined building around reflection planes and turn axes. This was in contrast to students' previous construction activities where they only considered the symmetry of their structures after completion.

Although the group of students using symmetries as a design tool did not represent a majority of the class, we saw this development as a promising outcome of this first iteration that refinements in the design might make more robust. In particular, we found that the students that were more disposed to compare structures using symmetries and that used symmetries as design tools were better able to compare and maintain a mental image of a structure's pre-image and image after transformation. Children's articulation of these images tended to incorporate visual tools (reflection planes and rotation axes) and key elements on the structure (like the triangles in Figure 21) that they attended to before, during, and after transformations. This ways of notating or highlighting salient parts of structures organized around a center plane or axis seemed to help students reduce the complexity of the image they had to remember. In future iterations, we hope to explore how to make this way of tagging points of the structure more widely available to students.

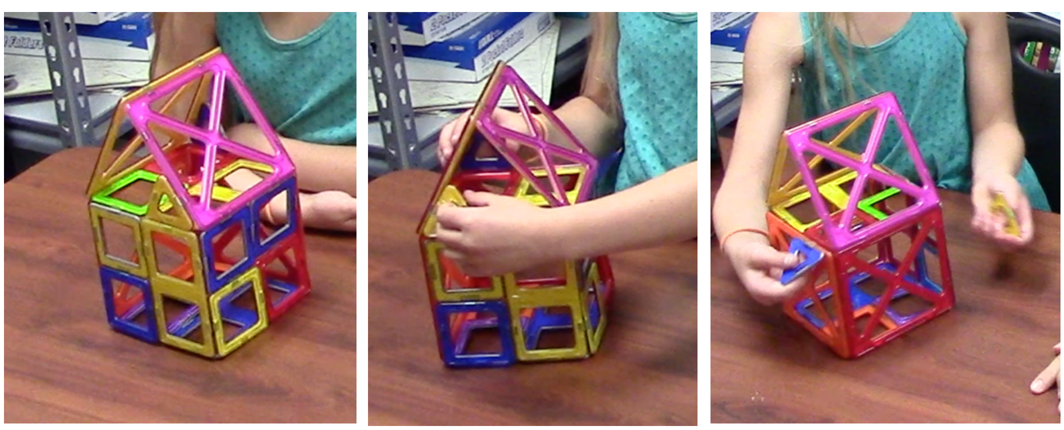


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Figure 25: Students' use of symmetries as a design constraint. A) Student place triangles in diagonally opposite corners to notate turn symmetry. B) Student moved triangles to directly opposite corners to change the structure to only have reflection symmetry. C) Student removed the triangle pieces to demonstrate how the structure could have turn symmetry and reflection symmetry.

Summary: The co-development of isometry concepts, defining practice, and visualization. Asking students to consider both reflection and rotation symmetries as they analyzed and constructed 3D structures typically generated many opportunities for students' to refine their understanding of individual symmetries. For some students, these activities also allowed them to consider relations between rotation and reflection symmetries and to coordinate their visualization of turn axes and reflection planes. Finally, as students began to see the utility of the mathematical system of isometries and symmetries, they also became more disposed to use symmetries to guide their own designs and hold each other accountable to using their shared system to evaluate and compare 3D structures.

May clinical interviews. The refinements in children's conceptions about rotation symmetries and reflection symmetries observed during the second phase of instruction were also evident in the final set of clinical interview (n=15). All students successfully constructed and justified structures with *only* reflection symmetry (86.7% 1st attempt; 13.3% 2nd attempt) and with reflection symmetry and half turn symmetry (100% 1st attempt). 86.7% of students accurately built an asymmetric structure on their first (66.7%) or second attempt (20%). Students had a more difficult time putting together the two congruent parts to make structures with *only* half turn symmetry. 46.7% of students were successful building a structure with only rotation symmetry on their first (33%) or second attempt (13.3%); however, after providing students with a prompt of asking students to hold one part constant and then attaching the second part, an additional 40% of students solved this problem after three or more attempts.

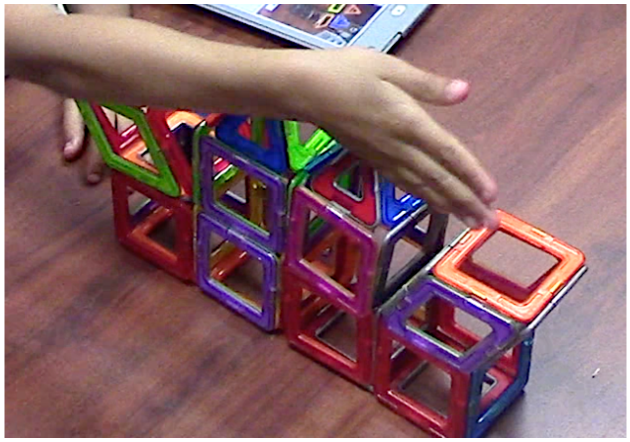

Figure 26: Student modeling difficult to see plane of reflection symmetry using his hand. Many students struggled to see this plane of reflection because it was not a place they could insert a physical mirror.

In contrast to the January interviews, all students were able to accurately model and explain half turn symmetry. Where some students struggled to distinguish between a half turn and a half turn symmetry in January, during the May interview students now demonstrated turning each structure a half turn and highlighted how individual components of the structure appeared to stay in the same position or how congruent parts "switched places." This provides further evidence that the introduction of the Tinkercad visualization may have helped students to develop strategies to coordinate a half turn and the visual analysis of turn symmetry. Finally, students also improved in their abilities to anticipate the placement of symmetrical reflection planes; however, 40% of students struggled to find a second plane of reflection symmetry in their structures with both reflection and rotation symmetry. These students in particular struggled to find the plane of reflection that could not be physically modeled with the mirror (You would have to saw through the Magformer pieces, see Figure 26). Thus, this is a limitation of using Magformers that will need to be addressed in future iterations, perhaps with a similar 3D CAD visual aid used for rotations.

Discussion

As we consider ways to "spatialize" curriculum so that students develop the visualization skills necessary to participate in professional STEM practices, generating a more robust geometry curriculum in the early grades is a necessary first step. Our results suggest that investigating 3D isometries and symmetries might be a fruitful starting place not only for students' conceptual development, but also as an accessible entry point for the foundational mathematical practice of defining. Furthermore, we found that as students' increasingly participated in generating an emergent mathematical system of transformation, they also learned to coordinate concepts of transformation with construction activities and tools that encouraged new forms of visual analysis and definitions of 3D structures. This coordination of concepts, mathematical practice, and forms of visual analysis supported students to develop shared and disciplinary ways of visualizing space that may be generative for future mathematics learning.

Developing Defining Practices and Concepts of Isometry and Symmetry

In this first iteration of the design we were interested in understanding the co-emergence of participation in the practice of defining and the development of a conceptual system for isometry and symmetry. Note that this conceptual system included ideas anchored to particular systems of inscription and material means (i.e., charts of definitions, construction materials). We found that students developed understandings of isometry and symmetry as they participated in defining these mathematical entities. As they gained greater understanding of how to participate in defining, students' conceptual development came to encompass relations between isometries and symmetries, and for some, relations among symmetries. This has important implication for continuing efforts to articulate new pathways of mathematics learning oriented towards space and geometry and the forms of early geometry experiences we design for young children.

Beyond just designing individual tasks and activities that expose students to different transformations, we need to consider how to help students build a more coherent set of relations among transformations—a mathematical system. Furthermore, our study suggests that engaging students in defining transformations and their relations as a tool to describe patterns and change and invariance may support students to become more disposed to view mathematical objects through a dynamic lens of transformation.

In our analysis of students' developing conceptions of congruence and isometries, we offer one potential starting point for young children's development of transformation that begins with 3D isometries and symmetries. In particular, we traced how students' tacit judgements of same/different relations became more property based through the classroom's joint endeavor in defining direct and mirror congruence using rotation and reflection isometries. One of the critical aspects of the development of concepts of congruence and isometries was the defining of orientation by comparing the results of rotations and reflections, and these differences were more salient with the aid of mirrors and chiral 3D structures. Students' sense of relations between congruence and isometries were later instrumental to establishing definitions of finite 3D rotation and reflection symmetries. In particular, students' were able to use their definitions of direct congruence and mirror congruence to support their analysis of structure with symmetries. As a result of students' developing practices of and dispositions towards defining, students' refined their sense of symmetries as "looking the same" to include notions of symmetries as a subset of isometries that result in self-congruent images. Students understandings of symmetries' relation to isometries and congruence were further supported in the final sequence of activities that encouraged them to coordinate rotation symmetries and reflection symmetries.

We see this description of students' development of a system of transformation as an initial step in generating a trajectory related to this development, but also a more robust understandings of what might constitute a child's geometry. As with any description of development, we acknowledge that students' development was contingent upon the designed set of activities, tools, instructional supports, and prior history of teachers and students. Additionally, we limited instruction to only 3D isometries and symmetries; thus, there are still questions about how students' understandings of these concepts relate to 2-space, and how these understandings might serve as a foundation for the learning of a broader set of transformations and their larger implications for students' conceptual understandings of space.

Supporting an Emergent Classroom Practice of Defining and Conceptual System of Transformation

Analysis of the function and forms of instructional supports were central to our understanding of how students came to co-develop a practice of defining and concepts of 3D isometry and symmetry. In particular, teachers' practices were instrumental to supporting a shared history and purpose of defining across days of instruction that were critical to students' participation in this foundational form of mathematical practice and to students' emerging understanding of relations among isometries. Prominent and recurrent teaching practices that helped establish this shared history included orchestrating and inscribing moments of definition invention and revision through strategic forms of discourse (i.e., prolepsis and revoicing) and establishing the collective visualization of different isometries and symmetries by coordinating a careful selection of examples and non-examples with physical tools of analysis. These moments of defining were further amplified by the provision of tools and tasks that supported students to construct a large variety of structures that offered opportunities for elaboration and revision of what might be invariant. Although other studies have pointed to the importance of teachers' practices, especially their discourse moves, in supporting classroom discussions and helping build students' mathematical sense making (Cazden, 2001; O'Connor & Michaels, 1993; Stein & Smith, 2011), we provided a more local account of how particular constituents of these practices interact over the course of an instructional unit to establish a local mathematical system and collective practice of defining. For example, teachers' selection of subsets of students' structures as examples and non-examples of developing concepts surfaced different ideas from students that teachers then strategically put in relation each other through revoicing, gesturing, and modeling use of tools and ways of visualizing. Ng and Sinclair (2015) study of grade 1 students' learning of 2-D reflection symmetry similarly noted the importance of teacher's gesture and press for spatial language to explain patterns of invariance. Thus, as we work with teachers to adopt more robust geometry curriculum, we need to attend to how their existing practices need to be tuned to children's thinking around geometry and spatial concepts that are often novel to both teachers and students.

In the current study, much of this tuning of teachers' practices occurred with the explicit support of researchers in the classroom. For example, researchers suggested which student constructions to present during whole class discussions and what types of revisions to definitions to press for and highlight. This is a necessary part of design research; however, as we come to better understand the implications for these types decisions in the classroom, we hope to integrate this decision making knowledge into teachers' professional development and instructional guides.

Reorganizing 3D construction activities by positioning symmetries as design constraints

During our last three days of instruction we began to explore the affordances of using 3D structure design as a context to support the coordination of rotation symmetries and reflection symmetries. As students considered ways to incorporate symmetries as an aspect of their design, this opened up new forms of mathematical activity. Thus, this study provides further evidence that contexts of design offer accessible entrance into mathematical practices. In particular, we found students inventing ways to notate or represent symmetries by adding symmetrical elements into their design. Students also seemed to articulate symmetries as part of their design aesthetic by choosing to alter their structures based on preferences for particular types of symmetry. In future iterations of this study, we hope to explore further how to position symmetries as tools for design, and how this might provide students with ways to experience mathematics as a forum for personal agency and aesthetic.

Finally, while we found that there was great utility in using Magformers and multilink cubes as forms of construction materials, future iterations might also explore the affordances of other materials. The Magformers in particular allowed students to construct a wide variety of structures, to easily experiment with making small revisions to their structures, and to test for some planes of reflection. However, they also may have constrained students' senses of where one could place a reflection plane within a structure. For example, in the May clinical interview, it became clear that some students did not consider placing reflection planes in positions that could not be physically tested with the mirror. One form of design tool that we believe might have great utility in supporting students' understandings of transformation are 3D CAD programs. We briefly experimented with using Tinkercad to help students visualize rotation symmetries, and while this seemed to support students in coordinating rotations and analysis of rotation symmetries, students did not get to experience Tinkercad as a design tool or a visual aid for reflection symmetry. Thus, our current insights into using 3D CAD programs are limited but hold promise as a future research opportunity.

References

Beyer, J. (1999). *Designing tesselations: The secrets of interlocking patterns*. McGraw-Hill.

Bornstein, M. H., & Stiles-Davis, J. (1984). Discrimination and memory for symmetry in young children. Developmental Psychology, 20(4), 637.

Brading, K., & Castellani, E. (Eds.). (2003). Symmetries in physics: philosophical reflections. Cambridge, U.K.: Cambridge University Press.

Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction*, *26*(3), 269-309.

Cazden, C. (2001). *Classroom discourse: The language of teaching and learning* (2nd edition). Portsmouth, NH: Heinemann.

Clements, D. H., Sarama, J., & DiBiase, A. M. (2003). *Engaging young children in mathematics: Standards for early childhood mathematics education*. Routledge.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education*, 136-163.

Cobb, P., Confrey, J., diSessa, A. A., Lehrer, R., & Schauble, L. (2003) Design experiments in education research. Educational Researcher, 32(1), 9.

Ehrlich, S. B., Levine, S. C., & Goldin-Meadow, S. (2006). The importance of gesture in children's spatial reasoning. *Developmental psychology*, *42*(6), 1259.

Ferrara, K., Hirsh-Pasek, K., Newcombe, N. S., Golinkoff, R. M., & Lam, W. S. (2011). Block talk: Spatial language during block play. *Mind, Brain, and Education*, *5*(3), 143-151.

Goodwin, C. (1994). Professional vision. *American anthropologist*, *96*(3), 606-633.

Goodwin, C. (2013). The co-operative, transformative organization of human action and knowledge. Journal of pragmatics, 46(1), 8-23.

Hawes, Z., LeFevre, J. A., Xu, C., & Bruce, C. D. (2015). Mental rotation with tangible three-dimensional objects: A new measure sensitive to developmental differences in 4-to 8-year-old children. Mind, Brain, and Education, 9(1), 10-18.

Hawes, Z., Moss, J., Caswell, B., Naqvi, S., & MacKinnon, S. (2017). Enhancing Children's Spatial and Numerical Skills through a Dynamic Spatial Approach to Early Geometry Instruction: Effects of a 32-Week Intervention. *Cognition and Instruction*, 35(20), 1-29.

Henderson, D. W., & Taimina, D. (2005). *Experiencing geometry: Euclidean and non-Euclidean with history*. Prentice Hall.

Jansen, P., Schmelter, A., Quaiser-Pohl, C., Neuburger, S., & Heil, M. (2013). Mental rotation performance in primary school age children: Are there gender differences in chronometric tests? Cognitive Development, 28, 51–62

Kamii, C., Miyakawa, Y., & Kato, Y. (2004). The development of logico-mathematical knowledge in a block-building activity at ages 1–4. Journal of Research in Childhood Education, 19(1), 44-57.

Kobiela, M., & Lehrer, R. (2015). The codevelopment of mathematical concepts and the practice of defining. Journal for Research in Mathematics Education, 46(4), 423-454.

Kubbinga, H. (2012). Crystallography from Haüy to Laue: controversies on the molecular and atomistic nature of solids. *Acta crystallographica section A: Foundations of crystallography*, *68*(1), 3-29.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, United Kingdom: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29-63.

Lehrer, R., & Chazan, D., Eds. (1998). *Designing learning environments for developing understanding of geometry and space*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lehrer, R., Jacobson, C., Kemeny, V., & Strom, D. (1999). Building on children's intuitions to develop mathematical understanding of space. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 63–87). Mahwah, NJ: Lawrence Erlbaum Associates.

Lehrer, R., Jacobson, C., Thoyre, G., Kemeny, V., Strom, D., Horvath, J., & Koehler, M. (1998). Developing understanding of geometry and space in the primary grades. *Designing learning environments for developing understanding of geometry and space*, 169-200.

Lehrer, R., Slovin, H., Dougherty, B. J., & Zbiek, R. M. (2014). Developing Essential Understanding of Geometry and Measurement for Teaching Mathematics in Grades 3-5. Reston: National Council of Teachers of Mathematics.

Levine, S. C., Jordan, N. C., & Huttenlocher, J. (1992). Development of calculation abilities in young children. *Journal of Experimental Child Psychology*, *53*(1), 72-103.

Lincoln, Y. S., & Guba, E. G. (1985). Naturalistic inquiry. Beverly Hills, CA: Sage. [pp. 36-45; 298-331]

Mulligan, J. & Mitchelmore, M. (2016). *Pattern and structure mathematics awareness program*. Australia: Australian Council of Education Research.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

National Research Council (2006). Learning to think spatially: GIS as a support system in the K-12 curriculum. Washington, DC: National Academic Press.

Ng, O. L., & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. ZDM, 47(3), 421-434.

O'Connor & Michaels (1993). Aligning academic task and participation status through revoicing: Analysis of a classroom discourse strategy. Anthropology and Education Quarterly, 24(4), 318-335.

Rowe, D. W. (2008). The social construction of intentionality: Two-year-olds' and adults' participation at a preschool writing center. Research in the Teaching of English, 387-434.

Saxe, G. B. (2002). Children's developing mathematics in collective practices: A framework for analysis. Journal of the Learning Sciences, 11(2-3), 275-300.

Senechal, M. (1990). Brief history of geometrical crystallography. *Historical atlas of crystallography*, 43-59.

Schegloff, E. A., Jefferson, G., & Sacks, H. (1977). The preference for self-correction in the organization of repair in conversation. Language, 53(2), 361-382.

Stein, M. K., & Smith, M. (2011). 5 Practices for Orchestrating Productive Mathematics Discussions. National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA.

Stone, C. A. (1993). What is missing in the metaphor of scaffolding. Contexts for learning: Sociocultural dynamics in children's development, 169-183.

Stone, C. A., & Wertsch, J. V. (1984). A social interactional analysis of learning disabilities remediation. Journal of Learning Disabilities, 17(4), 194-199.

Streeck, J. (2009). Gesturecraft: The manu-facture of meaning. Amsterdam, Netherlands: John Benjamins.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). The child's conception of geometry. New York: Harper and Row.

Piaget, J. (1970). Genetic epistemology. New York: W. W. Norton.

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., Newcombe, N. S., Filipowicz, A. T., & Chang, A. (2014). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematical skills. *Child development*,*85*(3), 1062-1076.

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, *101*(4), 817.

Washburn, D. K., & Crowe, D. W. (1988). *Symmetries of culture: Theory and practice of plane pattern analysis*. Seattle, WA: University of Washington Press.

Wongkamalasai, M. (2018). *Young children designing and defining space: Supporting the co-development of mathematical practices and foundational understandings of space and symmetry* (Unpublish doctoral dissertation). Vanderbilt University, Nashville, TN.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in mathematics education*, 458-477.

Young, C. J., Cartmill, E. A., Levine, S. C., & Goldin-Meadow, S. (2014). Gesture and Speech Input are Interlocking Pieces: The Development of Children's Jigsaw Puzzle Assembly Ability. In Proceedings of the Annual Meeting of the Cognitive Science Society (Vol. 36, No. 36).

Chapter IV

POSTIONING MOTIONS AND TRANSFORMATIONS AS EPISTEMIC RESOURCES IN YOUNG CHILDREN'S DEFINING OF POLYHEDRA

Introduction

Investigations of space using motions have historically served as the grounds for generating and expanding mathematical systems and understandings (Henderson & Taimina, 2005; Senechal, 1990b). Current school mathematics, however, largely neglects engaging children in spatial investigations where their informal senses of motion can support new understandings about space and structures. This turn away from more spatial approaches to mathematics has potentially severe consequences for children's mathematical learning. First, it implicitly communicates to children that their experiences moving in and interacting with space are not valued forms of mathematical resources (Civil, 2002; Foucault, 1972). Second, it presents children with an impoverished view of the nature of mathematics and influences how they come to identify with mathematics (Henderson, 2006; Gutierrez, 2018). Third, it neglects opportunities to cultivate children's abilities to visualize space dynamically, not as a mere collection of static properties (Lehrer, Jenkins, & Osana, 1998). Spatial visualization not only supports mathematical engagement but also impacts children's immediate and long-term achievement across domains of mathematics (Cheng & Mix, 2012; Wai, Lubinski, & Benbow, 2009). In response to these consequences, we argue that spatial approaches towards mathematics that leverage and develop children's intuitions of and visualization of motion in space should be a more central component of K-12 mathematics.

This paper presents a design study from a first-grade classroom where we aimed to support children in using their informal senses of motions and conceptions of space to define 3-D structures. In the second half of instruction, we looked to build on children's developing practice of defining and their emerging conceptions of 3-D structures by mathematizing one form of motion, turns, into mathematical conceptions of rotation isometries and symmetries. Although children use motions to investigate and conceptualize aspects of their everyday environments (Oudgenoeg-Paz, Boom, Volman, & Leseman, 2016; Piaget, 1960 & 1970), we still have little sense of how to design learning ecologies that leverage these resources to support children in engaging in mathematical practices (Ma, 2016). Thus, this study contributes to our knowledge of how to position investigations of space as grounds for helping children experience the co-origination of mathematical practice and conceptual development, beginning in early childhood. In the current iteration, we opted to begin with the practice of defining because this practice serves as the grounds by which mathematical concepts get negotiated and systematically related, and it is foundational to other forms of mathematical practice (i.e. conjecture, representation, refutation, and proof) (Lakatos, 1976). In the presentation of our results, we address two research questions:

1. How do young children's informal senses of motion act as epistemic tools in the defining of 3-D structures?
2. How do children's mathematical senses of motion and structure co-develop?

Design Conjectures

Although naming and labeling 3-D structures is a common activity in early math instruction, these activities are often constrained to a narrow set of structures that are presented as static objects with specified orientations and properties. As a result, children develop overly restrictive and tacit conceptions of 3-D structures and their properties (Resnick, Verdine, Golinkoff, & Hirsch-Pasek, 2016). In this section, we introduce considerations that guided our redesign of children's typical mathematical activities involving 3-D structures to include considerations of motion and definition.

Our instructional design is grounded in sociohistorical and interactional theories of learning (Cole & Engström, 1993; Erikson, 2004). These perspectives emphasize that learning and development occur over cumulative, microgenetic moments where individuals with shared and unique histories are engaged in social practices. These moments of interaction are shaped by established and shifting frames of participation that influence how individuals make use of and recognize semiotic resources in their environment to help them work towards shared goals or ends (Goodwin, 2017). Thus, what gets taken as a resource and what function these resources serve are defined by individuals' framing of activity and forms of shared practices. Given our aims of positioning children's experiences with space and motion as resources to define 3-D structures, our considerations focused on how to establish this new practice of defining and how to get children to see physical and embodied senses of motions as semiotic resources within this practice. To this end, we drew upon three principles to design instruction, as follows.

Building from children's existing practices and resources. Children routinely engage in investigating and conceptualizing aspects of space and structures across a range of activities and context. From interactions with adults and peers around these spatial activities, children begin to develop spatial language that helps to index objects and their features (i.e. size, topology, and part) and the positions in space (Ferrara, Hirsh-Pasek, Newcombe, Golinkoff, & Lam, 2011; Pruden, Levine, & Huttenlocher, 2011; Young, Cartmill, Levine, & Goldin-Meadow, 2014). These interactions also support children in using motions and construction to compose new structures and to visualize how objects move in space (Casey et al., 2008; Clements & Sarama; 2007; Hawes, Moss, Caswell, Naqvi, & MacKinnon, 2017). We thus began our instructional unit with having children construct and describe 3-D structures. Although this activity of describing and constructing 3-D structures is distinct from mathematical defining, our goal was to first elicit a shared language to begin talking about 3-D structures that could be repositioned and extended as semiotic resources in subsequent defining activities. We conjectured that this shared language would help children index and communicate about properties of structure across a range of examples.

Defining to transform 3-D structures into mathematical patterns. The second consideration that guided our design was disciplining children's propensities to recognize and extend patterns. In much of schooling, patterns are treated as if they are self-evident and based solely in perceptual experience. However, studies of professional practice illustrate how visualizing patterns and structures is a complex social enterprise requiring established networks of tools, concepts, and disciplined actors (Goodwin, 1994; Latour, 1999; Stevens & Hall, 1998). Geometric objects are also spatial patterns that are defined by networks of invariant properties (Senechal, 1990b). Thus, students must develop disciplined ways of seeing 3-D structures using mathematical concepts and tools in order to conceive of and visualize these structures as geometric patterns rather than as objects in the world.

In mathematics, one way this disciplining of the perception of structures and patterns occurs is through engagement in the practice of defining (Kobiela & Lehrer, 2015; Lehrer, Jacobson, Kemeny, and Strom, 1999). Defining geometric objects serves to distinguish invariant from variant properties, articulates the meaning of these properties, and establishes systematic relations among properties. However, developing children's and even adults' practices of defining present instructional challenges and can be difficult to establish in classrooms. For example, it is often difficult for students to overcome use of prototype reasoning and they also tend to overgeneralize properties of select examples as defining properties (i.e. all prisms must have square faces) (Ambrose, 2009; Lo & Cox, 2018). Thus, our design was guided by instructional activities and supports that have proved to be consequential to developing children's defining practice. These include engaging children in classifying and constructing examples and non-examples of mathematical objects that are carefully selected to highlight invariant patterns; maintaining a written record of defining work; promoting a technical and shared language; modeling and animating children in aspects of defining; articulating the goals of defining; and negotiating shared ways of seeing (Kobiela & Lehrer, 2015; Lehrer et al., 1999; Wongkamalasai, Henderson, and Lehrer, 2018). Because many of these are specific to the implementation of instruction and are responsive to children mathematical activity in the moment, we highlight how these instructional supports were enacted by teachers in classroom episodes in the results section.

Using and Developing Senses of Motions to Support Defining. Finally, we considered the growing consensus around the foundational role that enacted and embodied motions play in mathematical thinking and generation of new conceptual understandings (Henderson & Taimina, 2005; Lakoff & Nunez, 2001; Alibali & Nathan; 2011). Although research highlights many ways embodied senses of motion are enacted in mathematical activities, we focus on the role that enacted and gestured motions can play as analytical tools (Goodwin, 2017; Streeck, 2009). Animating objects during analysis can highlight new properties by changing the perspective of and salient properties of objects. Furthermore, by conceptualizing motions as forms of mathematical transformations, motions help to define invariant properties across objects and patterns (NCTM, 2015).

During instruction, we initially encouraged children to physically and visually imagine moving structures in informal ways to support their analysis and defining of 3-D structures. However, in the second phase we aimed to mathematize one sense of motion, turning, into mathematical conceptions of rotation isometries and symmetries. We conjectured that these new conceptions of rotations would extend children's analysis of 3-D structures and support children in defining new properties and relations. Although these concepts are not typically part of early elementary math curriculum, children readily experience rotations as everyday motions, and we have found in a previous study that they are largely accessible to children (Wongkamalasai, et al., 2018). Additionally, symmetries are a powerful and generative concept, and thus, developing these ideas early may support mathematics learning more broadly.

Methods

Context of Study

This study was conducted in collaboration with Mrs. B, a first-grade teacher at a rural public school serving an economically, culturally, and linguistically diverse community. Mrs. B's classroom had 21 students, 48% of students were classified as ELL. At the time of the study, Mrs. B was in her tenth-year teaching and was recommended as a collaborator by the school's math coach based on her flexibility and openness to trying new things in her classroom. Like most early elementary teachers, Mrs. B's math instruction was heavily focused on number, and she was less familiar with teaching geometry beyond traditional activities of naming and labeling types and properties of shapes. Throughout the study, Mrs. B and M.W. acted as co-instructors. Mrs. B took on the role of the lead teacher across the majority of lessons, and M.W. provided assistance with students, routinely helped guide classroom discussion and questioning, and helped Mrs. B make instructional decisions about fruitful next steps.

Instructional Design

Instruction consisted of two phases spread out over 13 days of 75-90 minute sessions. In phase one, children constructed 3-D structures and engaged in analyzing these to develop definitions for two classes of polyhedra, pyramids and prisms. (Note: we constrained prisms to only right prisms). Our focus on pyramids and prisms emerged from their recurrence in children's own constructions. Additionally, they were valued forms of geometric structures in the school's math curriculum; thus, Mrs. B emphasized that this focus was aligned with her existing instructional goals. During phase two of instruction, children extended their conceptions of pyramids and prisms by considering how structures were related based on properties of direct congruence and half turn symmetries. Congruence and symmetries opened up considerations of turns as rotation isometries that preserve lengths and angles and that have centers and magnitudes. Figure 1 presents a summary of the sequencing of mathematical concepts and activities across the instructional unit.

Throughout the unit, lessons followed a similar activity structure—students first described and constructed 3D structures. Teachers then used the variability inherent in the class's structures to create an example set for students to collectively analyze for similarities and differences and to classify and sort into groups that were similar in some way. From this sorting and comparison, the class made generalizations about the rules (i.e. properties) guiding their sorting. These rules served as the class's definitions of polyhedra and mathematical relationships. More specific description of how this activity structure organized each lesson are presented along with the result. In the following sections, we outline three principals that were central to guiding instructional decisions and activities in the moment.

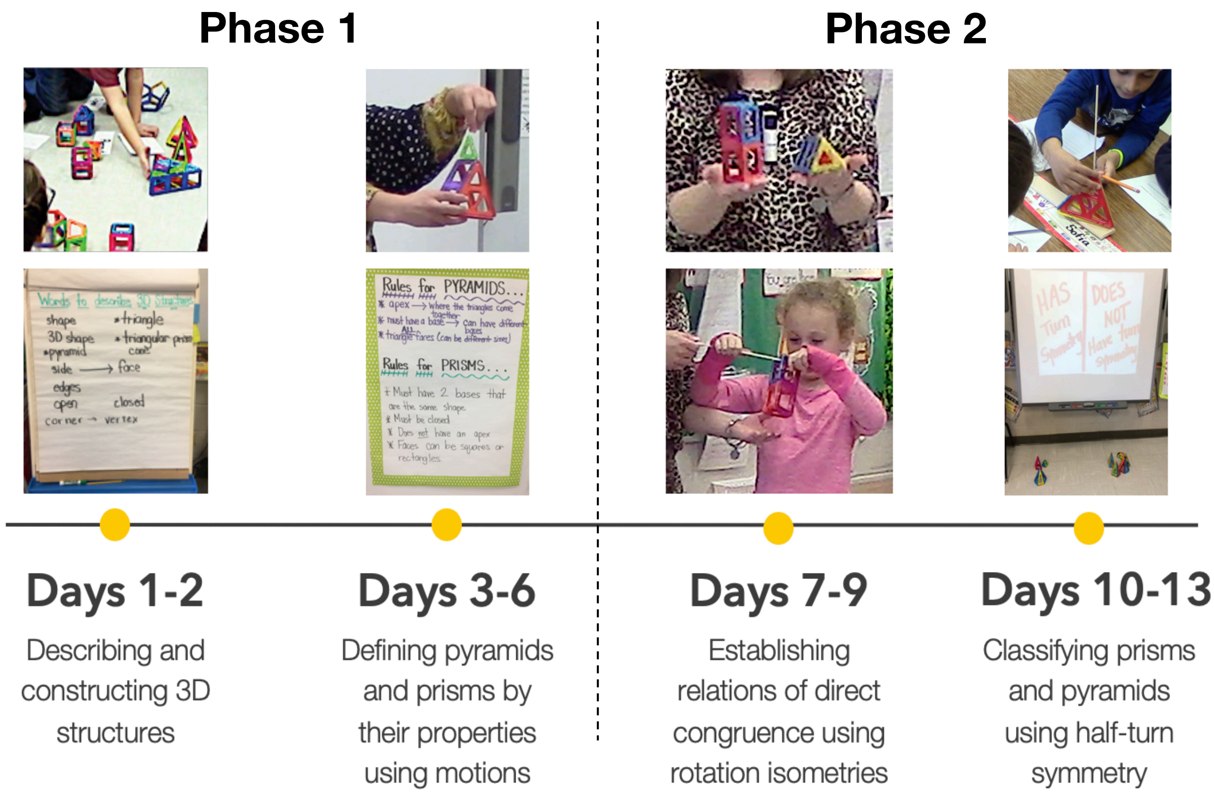

Figure 27: Instructional sequence of mathematical concepts and activities

Inducting students into the practice of defining. Teachers orchestrated whole class discussions in ways that animated students as participants in and that held them accountable to the goals of mathematical defining. Thus, although discussions often started by students describing properties of 3-D structures, teachers made public records of these descriptions and emphasized that the class needed to both agree on the meaning of each element of the record and whether all elements belonged in the record. These records also represented the shared technical language developed by the class. Teachers routinely oriented children to this technical language and asked children to restate ideas that included more informal language using the class's technical language. Holding children accountable to this technical language was important to establishing relations across activities and to building systematic relations between concepts.

Because discussions revolved around sets of structures, teachers' selections of which examples and non-examples to include were critical to what mathematical properties and concepts children discussed. Teachers intentionally selected structures that acted to make particular properties more salient or to problematize more tacit conceptions.

3-D structures as constructed and mobile. Because we wanted children to use motions as a tool to support their analysis and defining of 3-D structures, children used Magformers (magnetized polygons) to construct polyhedra and other structures. This allowed teachers to encourage children to attend to how they went about making their structures during whole class discussions, and to see structures as objects that can be picked up, moved, decomposed, and combined. We also avoided presenting objects as having a specified orientation; thus, rather than setting objects on tables, teachers and students routinely held structures in their hands and changed how the objects were oriented in their hands.

Expanding definitional imagination through constructive play. As children became more familiar with definitions of pyramids and prisms, we positioned these definitions as constraints to guide innovative constructions. We asked students to construct novel members of each class of polyhedra. Thus, construction was reorganized as a mathematical practice that served to support children's use of imagination in mathematics and to generate wider example sets that were consequential to extending children's defining of these forms of polyhedra.

Data Collection & Analysis

Classroom activity. We recorded all instructional sessions—capturing a wide angle of whole group discussions and the interactions of a consistent group of children during small group activities. In addition, after each day of instruction and after debriefing and planning with Mrs. B, M.W. generated field notes outlining what had happened in the classroom, major changes made to instruction, insights on productive instructional supports, and Mrs. B's general sense of what was happening.

Retrospective analysis of classroom activity. We started our analysis of the classroom video by first creating a content log of the activities, concepts, major developments, and notable interactions for the 13 days of instruction. This content log was based on rough transcriptions made during an initial pass through the video and field notes. We used the content log to select episodes that became the focus of analysis. Because we were interested in how motions supported defining 3-D structures, we selected episodes that spanned the instructional unit where a student's or teacher's appeal to motion contributed to, clarified or expanded a definition. We considered an appeal to motion as instances where teachers or students explicitly talked about moving a structure. This talk was often accompanied by an enactment of the motion, either through physical action or gesture. After this initial selection of episodes, we then conducted two phases of analysis that were guided by our research questions: 1) How do young children's informal senses of motion act as epistemic tools in the defining of 3-D structures? 2) How do children's mathematical senses of motion and structure co-develop?

The first phase of coding focused on characterizing the different functions children's senses of motions played in supporting the defining of 3-D structures. Using the constant comparative method (Glaser & Strauss, 1967) we first engaged in open and focused coding to characterize children and teachers' appeals to motion, attending to what motions were discussed or enacted. From this analysis, three categories of motions and actions emerged—rotations, assembly, and collapsing. We then applied Kobiela and Lehrer's (2015) framework of aspects of defining to determine what function the different motions played in defining.

During the second phase of analysis, we selected five episodes analyzed in phase one for more fine-grained analysis to understand how children's senses of motion and 3-D structures co-developed as they engaged in defining. This analysis followed methods of interaction analysis and was informed by Goodwin's (2018, 2013) theory of co-operative action and Streeck's (2009) ecologies of gestures (e.g. ceiving and haptic gestures). Thus, we looked at how teachers and students' mathematical meanings were co-constructed through the coordination, reuse, and layering of language, gesture, actions, and tools across turns of talk. We paid particular attention to how different actions on objects got taken up by students and teachers as resources and how they introduced new resources into interactions as the class's practice of defining became more established.

Individual assessment. We conducted two sets of clinical interviews—the first set after day nine (n=19) and the second set at the end of instruction (n=14). We interviewed all children who had parental consent; five students left the school after the first round of interviews or were absent.

The first interview consisted of six items in which we presented children with a 3-D structure that they labeled as a pyramid, prism, or neither. The structures included two prisms, two pyramids, and two structures that were neither. Students also had to explain why they assigned a structure to particular group. Because we were interested in how students' applied definitions to support their reasoning, rather than their ability to recall definitions, students had a copy of the rules for prisms and pyramids that the class had generated. We invited students to use this list as a tool to support their reasoning, asking students whether the rules could help them if they did not assume to do so themselves. We scored each item as correct or incorrect and then characterized students' reasoning as either using definitions as a tool for explanation or as based on informal judgement, such as appeal to prototypes (e.g., "it doesn't look like a prism").

The end of unit interview consisted of seven items. For the first three items, students looked at a before and after picture depicting how a square pyramid was rotated. The pyramid was rotated either a half turn, quarter turn, or quarter turn symmetry (Figure 2). Students had to identify on a physical model the turn axis was and to provide the magnitude of rotation. The final four items looked at student's ability to identify axes of half turn symmetry. We showed students four structures (prism, pyramid, octahedron, and 3D "Z") and asked students to identify all the symmetrical turn axes for each structure. We scored the first three items based on the accuracy of students' identification of the turn axis and the magnitude of rotation. On the last four items, we counted how many axes of half turn symmetry the student found that were unique (i.e. if a structure has a symmetrical turn axis through each pair of opposite vertices, we only counted one of those axes as unique).

Figure 28: Isometry items from end of unit interview, students identified in the before image where the turn axis was placed in the structure and how much of a turn was performed to achieve the after image.

Results

This section presents the results of the retrospective analysis in three sections. The first section describes how children and teachers' appeals to informal senses of motions functioned as epistemic tools to extend children's initial practice of naming and labeling shapes to a mathematical practice of defining classes of polyhedra. The second section looks at how children's locally defined system of polyhedra supported them in developing senses of turning motions into conceptions of rotation isometries and symmetries. These developed senses of rotations then acted as new epistemic tools to further analyze and define polyhedra; thus together, these first two sections illustrate the co-development of children's senses of structure and motion. The third section presents the results of the cognitive interviews to provide a more individual look at children's developed senses of polyhedra and rotations.

The Epistemic Function of Children's Informal Senses of Motion in Defining Polyhedra

During the first phase of instruction, introducing motions into children's existing practice of naming and labeling shapes opened up opportunities for children to see new relations and distinctions between 3-D structures and their properties. These opportunities afforded new forms of participation, promoted and sustained by teachers, that aligned with aspects of mathematical defining (Kobiela & Lehrer, 2015). Using select classroom episodes, I demonstrate three ways that appeals to motion helped establish this new epistemic practice and new conceptions of polyhedra. Specifically, appeals to motion helped 1) elicit attention to new properties of structures, 2) negotiate meanings or shared definitions of properties, and c) establish systematic relationships between properties that then served as the basis for the classroom's definitions of prisms and pyramids.

Eliciting attention to new properties of 3-D structures. The start of this instructional unit was not the first encounter children had with the names and properties of geometric structures. We sought to leverage children's prior experiences naming shapes as the grounds for the collective development of an initial shared language of description. To begin eliciting this descriptive language, we first showed students two nets, one of a tetrahedron (regular triangular pyramid) and another of a tetrahedron with a missing face; we then folded the nets into their 3-D configurations. As children described the similarities and differences between these two structures, we found that they had a fairly robust spatial language to describe 3-D structures, including both names of familiar 3-D and 2-D shapes (triangle, cone, pyramid, triangular prism) and properties (sides, corners). To preserve these descriptions as shared linguistic resources that could be later reflected on, we generated a visible record that was continually added onto throughout the unit.

This initial set of words, however, operated as a disconnected set of labels for objects and parts of objects in the world rather than as systematically related concepts. For example, children routinely described how the two structures resembled objects in their world, stating, "they look like ice cream cones" or "they look like pyramids." They also would interchange these labels unproblematically and listed properties of the structures without considering relations between them. At the end of day one, we invited children to write descriptions for 3-D structures of their own making using the words generated by the class. Some students wrote down as many words that applied to their structure, not attending to redundancy or conflicting descriptions, and others only wrote that their structure was a 3-D shape.

To help children begin to see how their language could operate as an important resource to classify, define, and make systematic relations and distinctions between 3-D objects, we began to problematize children's interchanging use of names for classes of structures. The first motion episode occurred on the first day of instruction and was initiated by M.W. After generating a list of words to describe the two tetrahedrons, M.W., referred to the class's record of descriptions and highlighted that the structures have been given three different names.[1]

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[1] Transcripts presented throughout the paper follow Jeffersonian conventions, and they are numbered based on turns of talk. To conserve space, some turns of talk are omitted, but the numbering is each turn is preserved.

1.	M.W.:	So I heard that it looks like a pyramid, that it's a pyramid, a triangular prism, and maybe a cone. And I wonder, when we build if we are going to be able to decide more, I tried to ask someone what is a triangular prism and someone told me that it's made up of triangles. And then someone told me it was a pyramid and they also told me that that was made out of triangles. And then we had someone say that it was a cone, and then I took a piece of paper and I said okay, here's my piece of paper and I'm going to fold this into *((rolls paper into a cone))*, what is this going to make?
2.	Ben:	A cone
3.	Timothy:	It looks like an ice cream cone
4.	M.W.:	A cone. So we have three maybe different names that maybe those are called, but we'll have to decide what we want to call them […]

M.W.'s act of constructing a cone from a single sheet of paper introduced a counter-example to a pyramid that children readily accepted as a cone. Introducing this counter-example helped tangibly contextualize her proleptic[2] statement in turn 4, "so we have three different names that maybe those [pyramids] are called, but we'll have to decide what we want to call them." By making this statement, M.W. indicated that mathematical objects have specific names, and coming to agreement on those criteria for those names in one goal of the practice of defining. Thus, although children were not yet engaged in the practice of defining, M.W. began to assert a new frame of participation for children's mathematical activity. Additionally, by constructing the cone, M.W. also modeled for students that, beyond naming, construction is a valued form of mathematical activity within this new practice of defining. Construction introduced a new and expanded example set that could challenge and invite further conceptualizations.

However, rather than treating these constructed objects as static entities for visual

[2] According to Stone (1996), prolepsis is a mechanism used in scaffolding that gradually builds common ground or situational understandings. This building of common ground occurs because of the speaker presuming an aspect of the context that the listener then has to work to understand. In a previous study (Wongkamalasai et al., 2018), we argue that prolepsis is an important instructional support for inducting children into the practice of defining.

comparison, M.W. continued the interaction by inviting children to imagine moving the paper cone and the pyramids by rolling them in their hands.

4.	M.W.	[…] But if we go back to thinking about this cone what do you notice, when I hold this, what does it feel like when I wrap my hand around this *((rolls paper cone around cupped left hand))*?
5.	Student:	[Flat]
6.	M.W.:	That if I took my hand around this *((rolls pyramid around cupped left hand))*
7.	Student:	Smooth
8.	M.W.:	Smoo::th. If I do that same thing on here, does that feel the same?
9.	Class:	No!
10.	M.W.:	Why not?
11.	Rafat:	That has holes!
12.	M.W.:	Well it has holes
13.	Molly:	There are edges, and that doesn't have any edges
14.	Rafat:	That has edges!
15.	Ben:	That has one edge!
16.	M.W.	What do you mean by an edge? […] I don't know what you mean

The physical and simulated action of rolling the cone and pyramids in the palm of one's hand was positioned as a new semiotic resource to aid in describing structures and extending children's descriptions to more topological considerations (e.g. smooth). However, rather than accepting the new description of smoothness as an important property in itself, in turn 10 M.W. pressed on students to explain what about the pyramids makes them not smooth. In this way, she positioned motion as a tool that not only elicits new properties, but also as a tool to consider structural relations between perceptual features and action (i.e. what about a structure allows an action to occur?). By focusing students' attention on relations between a structure and its motions, children attended to a new defining property of 3-D structures, edges (turns 13-15). M.W. then took another opportunity to induct children into the practice of defining by explicitly asking for a definition for edge (turn 16). The episode concluded with Molly proposing a definition of edges that is specific to the Magformers ("where the magnets touch"), which Mrs. B restated in more general terms in turn 21 as "where the sides meet."

17.	Mrs. B:	Ms. Molly, that was a great word that she used, what are you talking about edges, what do you mean when you say edges
18.	Molly:	It's where the um
19.	Mrs. B:	Can you come up here and point. We're just going to have Ms. Molly come up here and point to what she's talking about when she's talking about an edge.
20.	Molly:	The long parts where the magnets touch, meet um it's the edge and um that's the part that makes it not so smooth and that *((pointing to cone))* doesn't have any edges
21.	Mrs. B:	Oh, so Molly was saying that these parts where the sides meet, that those are the edges. […] Wonderful and edges is a term that mathematicians often use when they are describing shapes or when they are describing the structures that they create as well.

Although motions might be a resource in children's everyday sensemaking about space, these same resources may not yet be recognized in children's' existing frames of what is relevant to the doing of mathematics. Thus, the teacher, M.W., in this interaction played a significant role in helping create a new participation framework of defining where both construction and appeals to motions, like rotations, are recognized as epistemic tools. In this case, construction helped expand example sets that lead to new comparisons, and motions elicited attention to new properties of structures (edges). In addition to reframing the roll of motion and construction in support mathematical activity, M.W. also used prolepsis to prime students' attention to a repurposing of their existing spatial language. At this point in instruction, this repurposing of language is not yet established; however, in the remaining episodes we illustrate how children's spatial language gets reorganized into a mathematical system to describe and conceptualize aspects of 3-D structures and space.

Negotiating shared definitions for properties of 3-D structures. Often when children are introduced to properties of geometric structures, the definitions of these properties are treated as tacit and perceptually self-evident. For example, although many children know that a triangle is a shape with three sides and three corners or vertices, we rarely engage children in explicitly defining what a side is. Lehrer and colleagues (1998) have demonstrated that these tacit definitions of properties often lead children to develop overly narrow images of geometric structures. Thus, rather than just focusing on helping children define particular classes of polyhedra, we also supported children in defining the properties that would eventually expand their imaginations about potential members of two classes of polyhedra—pyramids and prisms.

In the previous episode, we demonstrated how motions elicited new topological descriptions and properties of structures. In this second episode, motions helped children consider 3-D structures, in this case two examples of pyramids, from an unfamiliar perspective that opened a space for children to consider and reconcile two competing definitions for apex. This episode comes from the third day of instruction. We showed students a square pyramid, selected because it conformed to students' images of pyramids, and a hexagonal pyramid, selected because most students likely would not regard it as a pyramid. We asked students to come up with a list of similarities and differences of these two structures as the first step towards generating a definition for pyramids. After students generated this list, M.W. asked whether everyone agreed with all the statements on the list, and understood the meaning of each statement. We start this exchange with M.W. asking students whether all students agreed that both structures were "pointy" and what the class meant by pointy.

3	M.W.:	=Do we all agree about that they're both pointy? (3.0) And what do we mean by pointy? What are we looking at when we say it's pointy?
4	Molly:	The tips on the top *((points to pyramids))*
5	Ben:	[THE TIPS]
6	Class:	[THE TIPS] [THE TOPS]
7	Mrs. B:	=Was there a word that we used ? (.) maybe to talk about-

8	Ben:	CORNER (.) Vertex I mean
10	Mrs. B:	Vertex? (sec) Okay so similarities (.) they both have a =how many vertex (.) or vertice::s do they ha::ve?
11	Molly:	That one *((hexagonal pyramid))* only has one
12	Kailyn:	[I KNOW (.) I KNOW]
13	Mrs. B:	Molly says one
14	Ben:	That one *((pointing toward pyramids))* one has five right there (.) and that one has one, (.) and that one has one
15	Mrs. B:	Why would this one only have one *((holding out hexagonal pyramid))* ? and why would this one *((holding out square pyramid))* have five?
16	Kailyn	BECAUSE IT'S DIFFERENT
17	Mrs. B:	Molly point to where you're talking about (.) the vertex *((holding out both pyramids to Molly))*
18	Molly:	It only has one on the top *((using index gesture to point to apex of hexagonal pyramid))*
19	Mrs. B:	What about this structure? *((holds up and shakes square pyramid))*
20	Molly:	Um it has ones on the bottom *((uses index finger to outline a square corresponding with the vertices of the square base of pyramid)))* and the top *((points to apex))*
21	Ben:	That one has five *((pointing to square pyramid))*.
22	Rafat:	AND THAT ONE ALSO HAS SOME *((pointing to hexagonal pyramid))*

Perhaps unsurprisingly, students first equated the description of pointy with the "tips" or apex of the pyramids; however, because we had previously inscribed students' spatial language, Mrs. B was able to support children in reusing this language as a semiotic resource in the present. This appeal to previously established "mathematical" words resulted in children labeling tips as corners or vertices, which had more expansive meanings than "tips". Thus, in turn 10, Mrs. B was able to ask children how many vertices each pyramid had. Although we can only conjecture about why, Molly proclaimed that the hexagonal pyramid only had one vertex. (We conjecture that this is because of the contrast between the angle of the apex versus all the other vertices on the pyramids). While the class ended up correcting Molly's statement by counting out the number of vertices on each pyramid, M.W. continued the interaction by revisiting Molly's claim to bring students' attention back to the apex of each pyramid.

60	M.W.:	So Molly said (.) she said we::ll that one has one (1.5) Point to the vertex that you were talking about Molly
61	Molly:	*((points to apex of hexagonal pyramid))*
62	M.W.:	What do you notice? (.) is different about that vertex (.) compared to a::ll the other vertices on that structure?
63	Molly:	It's kind of flat *((holding out RH with fingers together and extended))*
64	M.W.:	What do you mean it's kind of flat?
65	Molly:	Like um (2.0) this part *((pointing to apex of hexagonal pyramid))* um *((take pyramid into her hands from Mrs. B))* these are pointy *((indicating vertices on base by cupping hand around base with LF))* and this one *((moves both hands up towards apex and grasps apex with all fingers like a pinching gesture of RH))* is sort of flat because the tops of the triangles all meet together there.
95	M.W.:	So that pointy part is where all the triangular faces do what?
96	Student:	Meet together
97	M.W.:	Meet together (.) So mathematicians (.) you know what? They call that pointy vertex of a pyramid like you guys have been saying that it kind of looks different than the other vertices doesn't it?
98	Ben:	Yeah
99	M.W.:	When you look at it it kind of looks different, you guys are describing it as pointy? Mathematicians also thinks that that vertex looks different and they call that vertex (.) they have a special name for it (.) they call it an (.) apex
100	Mrs. B:	O::h I'm going to put (.) I'm going to put that up here on our chart because Ms. Megan just gave us a ver::y <interesting> and helpful word (.) She said apex (.) Can anyone tell me again what Ms. Megan said about what that apex was? (2.0) Mathematicians call it an apex because it's where what happens? What happens Mr. Antonio?
101	Antonio:	The top

In turn 65, as Molly tried to explain why the apex of the hexagonal pyramid looked different, she begins by taking the pyramid into her hand and cups the base of the pyramid to indicate the position of the six vertices that are not the apex. This initial handling of the pyramid helped to mark the apex as a distinct type of vertex. She then moved her hand to the top of the pyramid where she grabbed and pinched the apex. This action seemed to cue Molly's attention to the structural composition of the apex and supported her conceptualizing it as where "the tops of the triangles all meet together there"[3]. M.W. helped to further distinguish this top vertex as different by introducing a new term, apex. Furthermore, in line with the class's emerging defining practice, this new term also created a need for an agreed upon definition. In turn 100, Mrs. B opened up this defining to students. Antonio's proposal that the apex is the "top" of the pyramids made it apparent that students needed further support in reconciling Molly's description of the apex with their previous identification of the top or tip of the pyramids. To help children come to a shared definition for apex, M.W. and Mrs. B turned the square pyramid so that it rested on a triangle face instead of the base.

107	Mrs. B:	What if I sat it like this *((turns square pyramid so that it is resting on a triangle face))*
108	M.W:	Where's the apex, did the apex change?
109	Rafat:	The apex is right here *((points to apex))* because *((flips pyramid so apex is facing to ceiling))* whenever you tilt it the apex keeps on changing *((moves pyramid around to it rests on different parts))* so if you change it the apex will move with it (.) it will change the spot where it will be
110-126		[students trying to interpret and revoice Rafat's statement]
127	Ben:	Wait, it's actually up here (*(points to two vertices on top with L and R index fingers))*
128	Rafat:	Yeah it's actually right here
129	M.W.:	O::h okay Rafat . Why did you just change your mind ? Or if someone disagrees . Okay Brinley go up and point to the structure .
132	Mrs. B:	Where do you think it is Ms. Brinley?
133	Brinley:	*((points to apex))*
134	Mrs. B:	O::H so Brinley went back to where Molly said (.) Brinley why do you think it's right here?
135	Brinley:	Because if you turn it around *((flips structure so that it's back to apex on top))* this is where they all meet *((gestures grabbing and dragging triangle sides together at the apex))*
136	M.W.:	That's where what all meet ?
137	Brinley:	The triangles (.) they all meet *((turns pyramid back to resting on triangle*

[3] Streeck (2009) and Goldin-Meadow & Beilock (2010) have previously demonstrated how gestures support ceiving and the gathering of new information through touch that then gets taken up in speech.

		face)) and if you turn it around *((rotates pyramid back to apex on top with RH hand while using left hand to point to and track the apex during rotation))* it's still going to be there *((rotates pyramid back onto face while still using L index finger to trace the position of the apex))*
138	M.W.:	Okay (.) if where Rafat and Ben pointed to why don't you think that is the apex?
139	Brinley:	Because not all of these triangles *((uses LH to cup around all the tops of triangle near apex))* don't meet there

By turning the pyramid so that the apex was no longer positioned at the top, the teachers used motion to provide a new perspective or vantage point of the pyramid. This helped problematize students' competing senses of apex as "the top" and as the vertex "where all triangle faces meet together". Additionally, by demonstrating that the structures are movable rather than static, students began to see structures with less defined orientations (i.e. a structure can be a pyramid even if it's not sitting on its base). This more dynamic view of structures afforded Rafat to use motion to state an initial claim that the apex does not change because it moves with the structure (turn 109). However this appeal to motion was not yet coordinated with a definition for apex, as he quickly abandons his idea in turn 128 after Ben claims that the apex is now the two vertices closest to the ceiling. These two competing claims created contest among students, which Brinley helped to resolve by further developing Rafat's original claim.

In turn 135, Brinley modified Rafat's explanation and animation of the apex moving with the pyramid by layering it with Molly's definition of apex as where all the triangles meet together both verbally and gesturally. Brinley's lamination or coordination of different semiotic resources (Molly's definition, Rafat's rotation, and the top of the pyramid) created an argument for the replacement of students' sense of apex as "the top" with Madison's appeal to "where all the triangles meet together." Thus, motion played a critical role in the classroom's definition of a new property, apex. It provided a new tool for analysis, and because it was situated within the classroom's developing frame of defining, it also helped students craft definitional explanations. In sum, this episode illustrates how a property can be defined in a way that preserves its character despite changes in orientation resulting from its motion.

Establishing systematics relations between properties of 3-D structures. Geometric definitions are based on a network of properties. This network view of properties allows for the systematic relating of different forms of geometric objects (e.g. squares as special cases of rectangles) and for distinguishing necessary and sufficient sets of properties from other sets of properties. In this third episode, we explain how children's senses of rolling supported them in establishing relations between defining properties of prisms, a significant shift from their original written description of structures as disconnected lists of properties. These relations became central to the class's eventual definition of prisms.

This third episode occurred on the fifth day of instruction during which children were comparing prisms. During the previous day of instruction, one student noted that a hexagonal prism could roll more easily than a rectangular prism, and we decided to try to mine the potential of this insight for expanding children's conceptions of prisms. We asked children to build their own hexagonal prisms, but because some children ran out of hexagons, they improvised by constructed pentagonal and octagonal prisms, stating that they looked similar to the hexagonal prism. This moment of improvisation revealed that children seemed to have some tacit, procedural notions about prisms, thus, we hoped to leverage and build upon these initial understanding by developing a language to define prisms.

After students built either a hexagonal, pentagonal, or octagonal prism, M.W. asked what new similarities and differences the class noticed. Students responded by restating that the structures can roll. To help students start attending to properties of structures that allow for this rolling, M.W. set up a contrast by modeling two different ways of rolling the prisms—around the square faces and base-over-base. She then asked students what makes the structures more "rollable" the first way (around the faces) versus the second way (base-over-base).

23	M.W.:	If we're thinking about them being rollable When you say their rollable (.) what's making them <u>ro:lla:ble</u>? That I just demonstrated is <u>not</u> rollable the-
24	Ben:	[THE SQUARES]
25	M.W.:	What about the squares?
26	Ben:	The squares are flat↓ *((uses RH extended with fingers together to sweep across imaginary flat surface))* and when it's more flat (.) it makes it roll like a ball *((moving both hands as if tracing the continuous rolling of a ball))*
27	Rafat:	[YEAH and they need something <u>flat</u> to make it roll like a <u>ball</u> because a ball has something flat to make it roll]
28	M.W.:	A ball is <u>flat</u>?
29	Ben:	Yeah
30	Joey:	No::
31	M.W.:	It is?
32	Joey:	No it's not flat like this
33	Ben:	Yeah it's like smoo::th *((circles RH continuously))* all the way around
34	M.W.:	OH (.) It's <u>smoo::th *((mimes cupping hands around a ball))*</u>
35	Rafat:	Yeah and that's why is can roll really good
38	Molly:	The um (.) it's made out of a circle they just bent it to have straight edges
39	M:	(pause) It's made out of (.) What do you mean↑ it's ma::de out of a circle? *((uses both hands to make a circle))*
40	Molly:	Well um it was originally a circle it's just like someone bent it
41	M:	=What part is made like a circle? *((picks up hexagonal prism))* You mean like a::ll of this is made like a circle?
42	Timothy:	NO
43	Molly:	No like the
44	Timothy:	=It's actually like *((grabs prism from M.W.'s hands))*
45	Molly:	The orange part *((referring to hexagon base))* was made out of circle they just it's like somebody bent it

By introducing two different senses of rolling, students began to attend to individual components of the prisms, the square faces. As M.W. pressed them to explain what about the squares made the prisms "rollable," students compared the prisms to a familiar object that rolls, a ball. Although it's unclear why the students' appealed to the squares and balls being flat, they eventually arrived at "smooth" accompanied with a gesturing of circles. This appeal to smoothness and circles was then used by Molly in turn 38 to further distinguish the bases from the square faces. Here Molly used the gestural reference to circle to generate a description of the bases as circles someone "bent to have straight edges."

Next, M.W. asked students if there's a way to label the parts that are bent circles in order to generate a language to differentiate the bases from the faces of the prisms. Students proposed to call these parts end or side faces. In turn 57, M.W. took up students' proposal of end faces but proposes that the class reuse a word already part of their shared mathematical language, bases. Now that the bases were isolated components of the prisms, M.W. asked students to discuss what is the same about the bases across all the examples of prisms.

57	M.W.:	What if we call these (.) just like you guys called this a base what if we call these bases or ends . What if we also call these bases of this shape. So when you guys talk about these parts that are more circles (.) how bout we call these bases (.) so what can we say is the same about the bases of these structures?
58	Molly:	They look different?
59	Rafat:	They're all smooth
60	Molly:	They look different because they're different shapes for the um bases that um are holding it together
61	M.W.:	When you talk about the parts that hold it together, what are you talking about?
61-71		[Students discussing what parts the bases hold together]
72	Rafat:	That this one she's talking about that these are all connected together *((traces around the square faces positioned like a wheel))* and these two *((holds the bases as if mimicking how the bases are pressing against the squares))* are holding the squares
73	Ben:	Yeah so if you took those off *((gestures removing the bases as if they were then ends of a wheel rather than top and bottom))* they would just fall down flat *((makes a collapsing gesture with both hands))*.
74	M.W.:	O::h
81	M.W.:	Okay, so we're saying that these two bases on the end *((grabbing prism by bases like a wheel as if squeezing bases together to hold squares in place))*

		do something important, they kind of hold *((grasps prism tighter))* everything together.

In turn 60, Molly provided a structural function that was similar across all the bases—they hold the prisms together. Rafat helped extend Molly's notion by decomposing the prism into squares that are connected in a ring and two supporting bases. He accomplished this by reusing a previous gesture of circles to highlight the connected square faces and layering a new gesture of squeezing the bases together onto Molly's appeal to "holding together." In turn 73, Ben compliments Rafat and Molly's contributions by offering a visual scene of the prism collapsing after removing the bases to further emphasize their structural importance. Finally, to help articulate this complex collaborative meaning making for others, M.W. restated that the two bases serve an important function, they hold everything together. This episode concludes with students listing two new similarities between the set of prisms – they all have two matching bases and they all have square faces that these bases hold together.

In contrast to the previous two episodes, the appeal to rolling prisms was initiated by children rather than a teacher. We take this an indication that children were developing a greater sense of motion as an epistemic resource in their new defining practice. The act of rolling prisms and articulating what made this rolling possible led children to decompose prisms into two critical properties—bases and prisms—and to consider their structural relation. Later on this structural relation between the bases and faces of prisms supported children in expanding their class of prisms to also include those with rectangle faces.

Summary: Motions as Epistemic Resources in the Defining 3-D Structures. The three episodes during the first phase of the instruction illustrate how everyday senses of motion supported children in defining two classes of polyhedra (pyramids and prisms) and their related properties. Motions took on greater status as epistemic resources as children developed a new participation frame of defining. This new frame of mathematical activity gave motions distinct functions within defining; motions helped to elicit new properties, to negotiate shared definitions of properties, and to establish systematic relation between properties. As children became more familiar with the practice of defining and the supportive role of motions, they also became more inclined to initiate and use appeals of motion to support their mathematical sense making. These episodes also demonstrate the critical role that teachers played in establishing a new practice of defining and in modeling how both motion and construction are valued resources in this practice.

Co-Developing Children's Conceptions of Rotation Isometries and Symmetries and 3-D Structures

Informal motions greatly supported children's defining of pyramids and prisms; however, in the second phase of instruction, we looked to develop these ideas of motion into mathematical ideas of rotation isometries and symmetries. Isometries and symmetries have historically played a key role in the conceptualization of spatial structures (Senechal, 1990a; Washburn & Crowe, 1988); thus, we also hoped that by developing these conceptions of rotations, children would see new patterns and relations between pyramids and prisms. In this section we explain two key ways in which children's history of development of conceptions of pyramids and prisms helped spur mathematization of informal senses of motion, and how this mathematization afforded new considerations of 3-D structures.

Indexing and conceptualizing magnitudes and centers of rotation. Initial classroom considerations of rotations as isometries emerged as students justified why rotations could be used as a test to determine whether two polyhedra were directly congruent on day 7. Mrs. B hid a structure (a triangular prism) in a bag and children generated and selected three questions about properties of the structure to guide their construction of a prediction of Mrs. B's hidden structure. As children compared their predictions with Mrs. B's structure, they narrowed in on two structures that had the same bases and sides as Mrs. B's structure, but the two structures appeared to be different sizes because they were oriented in different ways. (One structure was standing on a triangle face, and the other was standing on a rectangular face). Children debated whether these two structures could both be direct copies of Mrs. B's based on their size; however, some students convinced others that their sizes were the same if you turned one of them.

This initial conceptualizing of rotations as a test for direct congruence supported children in identifying invariant properties of rotations—size and shape. However, at this point, children did not yet have a language to describe how one would have to rotate a structure to show direct congruence beyond physically enacting the turn. To fill this gap. we introduced rotation centers, or axes of rotation, and magnitudes of rotation (constrained to half turns and whole turns) as tools to support their talk about rotations. This reconceptualizing of rotations also opened opportunities for children to distinguish between any half turn and half turn symmetries. Given that these are not concepts often introduced to young children, we initially encountered a number of instructional design challenges in supporting children's understanding of axes and magnitudes of rotation. We briefly touch on these challenges next as the subsequent changes we made to the instructional design informed the analysis and our understandings of how children's notions of rotations developed. We then present a classroom episode to demonstrate how children's previous efforts to define polyhedra supported children in developing conceptions of and visualizing rotation symmetries.

A first challenge we encountered was related to establishing turn axes. Children's embodied senses of rotating objects impeded their ability to incorporate using wooden rods as axes of rotation as they experimented rotating various 3-D structures. Rather than treating the rods as static axes that structures rotated around, children tended to use the rods as levers. To help support children in adopting a new way of enacting turns with the rods as fixed axes, we made a number of design changes. First, we introduced a familiar image of rotations with a visible center, the turning of hands on a clock. Children readily connected the pin at the center of clock holding the hands as a turn axis, and they explained how the arms of a clock traveled around in a circle. We then had children embody the rotation of the clock using their bodies to enact the motion of the hands of the clock. Children's turning of their bodies helped them to gain a new perspective of turns as having a center (grounded in the center of their bodies) and a constantly changing heading (represented by the pointing of outstretched hands). This attention to a change of heading also opened up consideration of magnitudes of rotation. As children rotated their bodies, we asked them to distinguish between a half turn and a whole turn. Children had no difficulty enacting half turns and whole turns. When we pressed children to explain how they knew they made a half turn, they appealed to the numbers of the clock and described how when they turned their bodies it was like their hands went from pointing at the 12 to the 6. Thus, children laminated different semiotic resources made available by introducing rotations of a clock and their bodies to further conceptualize rotations as having a center and magnitude.

However, despite this development, when children moved back to rotating 3-D structures around the wooden rods, they still had a hard time coordinating this new sense of rotation with the enactment of a turn outside their own bodies. (They continued using the rods as levers). Additionally, they struggled tracking the changing heading of the structures, thus, they struggled distinguishing half turns and whole turns from all other magnitudes of rotation. This led us to make a second design change. Because children seemed to have a hard time keeping the rods in a static position, we provided children with a plank of wood with pegs holes (Figure 3) to both hold the rods and to make the center of rotation more visible. A second affordance of introducing this plank of wood was that it helped partition space in half. Children worked with a partner that sat directly opposite of them (Figure 3), and as children engaged with this new tool, they started adopting their bodies as landmarks similar to the 6 and 12 on the clock. For example, as children explained how they identified half turns with this new tool, they repeatedly referred to how the structure started facing them and then moved to face their partner. (Note: Because the Magformers had large gaps that made it difficult to keep the turn axes stable, we also taped over the center of the Magformers and created different holes that the turn axes could be placed in. We also made holes in the frames of the Magformers to allow children to test axes that were not physically possible.)

Figure 29: Redesign of tools to support children's conceptions of rotation axis.

After introducing this new tool, we introduced half turn symmetries by having children compare the outcome of half turns of structures with and without half turn symmetry. This comparison led to an initial conception of half turn symmetries as a half turn where "a structure still looks the same." To help further establish this sense of half turn symmetries, we asked students to try and build prisms and pyramids with and without half turn symmetry.

In the following classroom episode, we demonstrate how children's developed conceptions of and language to describe pyramids and prisms helped to index rotations and differentiate half turns from half turn symmetries. This episode occurred on the last day of instruction after students had constructed prisms and sorted them into two groups—prisms with and without half turn symmetry. At the start of the interaction, M.W. selected a triangular prism a student, Hope, constructed for whole group analysis and discussion. Hope initially identified this prism as having *no* half turn symmetry, and M.W. invited her to demonstrate how she analyzed the structure to reach her conclusion.

1	M.W.:	So what kind of prism do you have?
2	Hope:	I have um a
3	M.W.:	What are the bases?
4	Hope:	Triangles
5	M.W.:	Okay so now we're looking at a prism that has triangle bases (.) Okay so you're saying that this one did <u>not</u> have turn symmetry (.) Where did you put your turn axis to check? Where are you putting it?
6	Hope:	*((places turn axis in the middle of the triangle bases *see axis 1 in Figure 4*, the prism is under a document camera so students are seeing what Hope is doing on a projected screen))*
7	M.W.:	In the middle of what?
8	Hope:	Um the base
9	M.W.:	In the middle of the base okay so right now let's look (.) Let's make sure we can track where it is (.) So right now what's at the bottom of the screen?
10	Class:	AN EDGE
11	M.W.:	An edge okay so turn it in a half turn
12	Hope:	*((rotates structures in half turn))*
13	M.W.:	Okay and <u>now</u> what's at the bottom of the screen?
14	Class:	A CORNER
15	M.W.:	A corner or we call that a
16	Class:	VERTEX
17	M.W.:	A vertex (.) Okay so was that a half turn symmetry?
18	Class:	NO
19	M.W.:	No okay did you try and put your turn axis anywhere else?
20	Hope:	Um I tried it we tried to put it through here *((places axis through vertex and*

		*a triangle base *see axis 2 in Figure 4*))* but it didn't work
21	M.W.:	Okay so now you're putting it through a vertex and a base, okay
22	Hope:	And we turned it like this *((rotates triangular prism in a half circle))*
23	M.W.:	So let's look at this (.) so I want to see up here does anyone else see where this could have half turn symmetry?
24	Class:	YEAH
25	M.W.:	So Rafat do you think that you can find another place? Where do you think? So we're thinking that maybe this does have half turn symmetry but where would the axis go?
26	Rafat:	*((places axis through center of rectangular face and the opposite edge *see axis 3 in Figure 4*))*
27	M.W.:	Okay so now where does he have the turn axis? Where would you describe where the turn axis is now?
28	Jose:	The edge and the face
29	M.W.:	Okay so one of the edges and one of the faces (.) So now let's look (.) So here's a new way we're going to put it like this (.) okay and let's look what do we see at the bottom of the screen?
30	Class:	The edge and a triangle face
31	M.W.:	Okay so now let's turn it a half of a turn
32	Brinley:	*((rotates structure around a half turn symmetry))*
33	Class:	It has half turn symmetry!

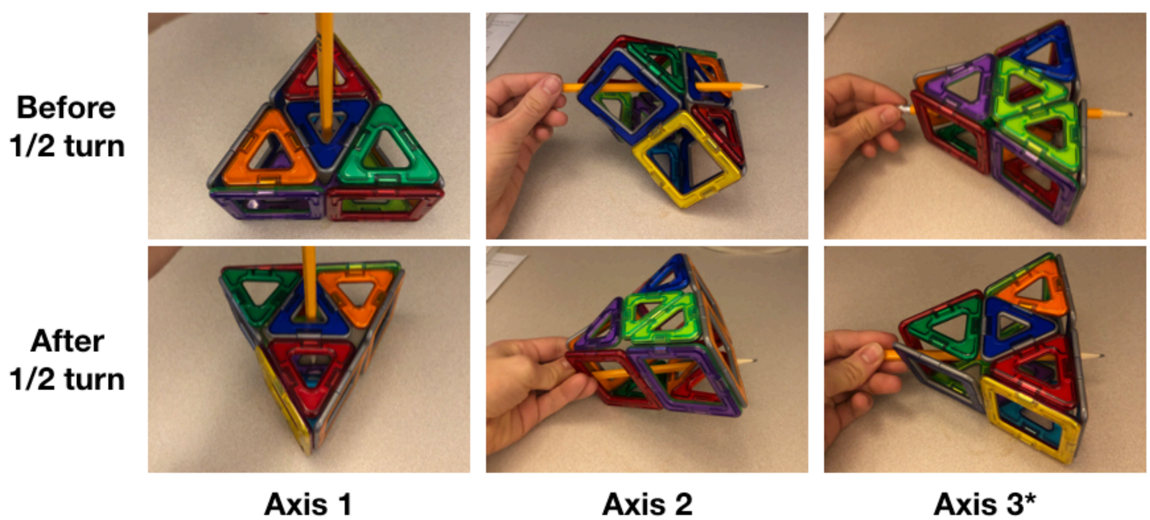

Figure 30: Different turn axes tested by children for 1/2 turn symmetry. Children used the properties of the prism to index where they placed the turn axis. Axis 3 shows students' eventual identification of an axis of rotation symmetry.

As Hope demonstrated how she analyzed her triangular prism for half turn symmetry, M.W. scaffolded the interaction by explicitly asking Hope and the class to use their knowledge and shared descriptive language of 3-D structures to articulate the placement of the turn axis and also to index parts of the structure before and after a half turn. This indexing of aspects of the structures in relation to the half turn helped establish a shared field of vision and clarify the meaning of “still looks the same” in the classroom’s initial definition of a symmetry (i.e., same means the same parts of a structure switch places).

Beyond these two functions though, this indexing practice also helped students make sense of M.W.’s question in turn 19, “did you try and put your turn axis anywhere else?” This question acted to extend children’s analysis of half turn symmetries to include the testing of multiple turn axes and a way to communicate what counts as a unique and untested axis. After Hope struggled to find an axis of turn symmetry, this episode concluded with M.W. opening this new activity to the whole class. This led to Rafat’s identification of a new symmetrical axis of rotation, indexed by Jose, and a reclassification of the triangular prism as a structure *with* half turn symmetry. In sum, indexing the parts of structures and the placement of the turn axis supported children in further understanding half turn symmetries by opening up a new activity of analyzing multiple turn axes and enriching children’s notion of “same”.

Enhancing the saliency of structural patterns and relations. Children became quite adept at analyzing structures for half turn symmetry and searching for multiple axes of symmetry. This was in part because of the indexing practice just described, but this symmetry analysis also became refined in relation to the variability among structures students constructed and analyzed. During the final three days of instruction, we intentionally constrained children’s constructions to different pyramids and prisms, so that children could start to consider relations between these two classes of polyhedra and their rotation symmetries. In these activities, children's expanded notions of pyramids and pyramids developed during the first phase of instruction, were critical to increasing the variability of polyhedra for the group's consideration. Figure 5 shows the various pyramids and prisms constructed and analyzed by students for half turn symmetries; students constructions included polyhedra with various bases and with variable locations and number of symmetrical turn axes. Thus, as demonstrated in the previous episode, this variability afforded children to extend their symmetry analysis to include considering the location and count of symmetrical ½ turn axes rather than just determining whether a structure had at least one axis of half turn symmetry.

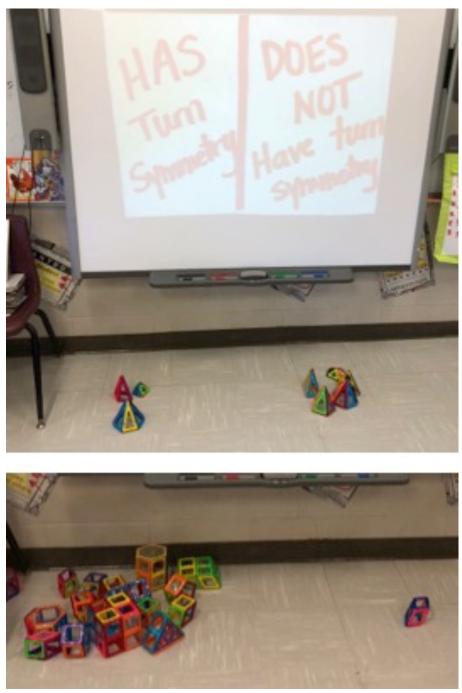

Figure 31: Children's pyramids and prisms sorted using ½ turn symmetries. *Note: Children misclassified the oblique rhombic prism as a structure with no ½ turn symmetry.

A second consequence of the wide example set generated by students was the emergence of a salient pattern in children's grouping of pyramids and prisms. Children constructed far more prisms than pyramids with half turn symmetry. This pattern opened up a new mathematical question about the relation between polyhedra and half-turn symmetries—why are there more prisms with half turn symmetry than pyramids? In the concluding episode of the instructional unit, teachers posed this question to students to provide students with an opportunity to make conjectures that would help establish systematic relations between their new conceptual understandings of polyhedra and rotations. In the following interaction, Mrs. B just pulled the class back together after students spent time discussing the question in small groups.

1	Mrs. B:	Antonio and Hope had a really good discussion about why they thought there were more prisms with half turn symmetry than pyramid (.) Explain to us (.) What were you guys saying?
2	Hope:	Because the turn axis could only go through here *((places turn axis through apex of hexagonal prism))* on the pyramid (.) the only place you could but the axis in the pyramids is through the apex.
3	Mrs. B:	So you're saying with the pyramid (.) in order for us to test whether or not it had turn symmetry where did we have to put that turn axis?
4	Hope:	Right here *((presents pyramid with axis through apex))*
5	Mrs. B:	What about a prism? Antonio what were you saying about a prism? *((picks up a hexagonal prism))*
6	Antonio:	You could put it through the bases or the faces
7	Mrs. B:	Okay so look table three (.) Antonio said that you could put it through the bases (.) so I could put it right through here *((places axis through center of hexagon bases))* through the center of my bases to show turn symmetry? He said that I could put it through the faces *((places axis through opposite square faces))* (.) And where else could I put it?
8	Antonio:	The edges
9	Mrs. B:	He said through the edges, Allie do you agree that if I put it through the edges that that will show half turn symmetry? So they said that the reason we had more prisms was because there were more places to put the turn axis (.) On those lovely pyramids it seemed like the only place where we could position the turn axis so it would have half turn symmetry was through the what?
10	Class:	THE APEX

In this exchange, Hope and Antonio coordinate and assemble their developed indexing practice and conceptions of the defining properties of both classes of polyhedra and rotations to compose generalizations about pyramids and prisms and their possible rotation symmetries. Thus, the coordination of children's spatial analysis and concepts of 3-D structures and rotations helped to further position motions, in this case rotation isometries and symmetries, as an epistemic resource. Now children's developed sense of a mathematical motion helped to make systematic relations between pyramids and prisms. This systematic relation was further articulated by Mrs. B in turn 9. Her revoicing of Hope and Angel's generalizations positioned them as jointly making a conjecture to explain why the pattern of there being more prisms with half turn symmetry emerged.

This episode also demonstrates the affordance of co-developing children's senses of motion and structure. Hope and Antonio's generalization were possible because of the classroom's prior history of defining these classes of polyhedra. In particular, this established mathematical system of polyhedra helped to focus students' attention on invariant properties across the variety of structures composing their example sets. For example, children likely would not been able to abstractly conceive of the set of prisms as all having two bases. Thus, children's co-developing notions of 3-D structures and motions both buttressed their understandings of each of these mathematical concepts, but it also opened new mathematical relations that would not be possible without the class's history of mathematical work across the 13 days of instruction.

Summary: The Co-Development of Conceptions of Rotations and 3-D Structures. Although young children are rarely introduced to rotation isometries and symmetries beyond their approximation instantiated in everyday senses of turning, the second phase of instruction demonstrated that these concepts are firmly within the grasp of young children. However, the challenges we encountered during instruction revealed important insights into the form of instructional and material supported necessary for children's learning. In particular, although children routinely experience turning objects and turning their bodies, these experiences have different conceptual affordances. We had to redesign materials to leverage these different affordances of children's embodied experiences with these two kinds of turn to first establish rotation isometries as having a center and magnitude.

Children's initial sense of center and magnitudes of rotation were further developed and extended to investigate rotation symmetries by appeals to their conceptions of pyramids and prisms developed during the first phase of instruction. First, children's use of properties of pyramids and prisms served as a way to index a) the change of heading or magnitudes of turns, b) where parts of structures started and ended after a rotation, and c) the location of the axis of rotation. This indexing enriched children's use of rotations as a tool for spatial analysis, and it also helped children understand that structures can have multiple axes of rotation symmetry. Second, children's expanded notions of pyramid and prism provided an important source of variability into children's analysis of rotation symmetries. This variability helped to make patterns between pyramids, prisms, and rotation symmetries more salient. Children's conceptions of these structures, however, also helped children attend to the invariance of this pattern and to establish new systematic relations between pyramids and prisms. In this way, while informal motions initially helped define classes and properties of 3-D structures, the formalizing of these motions by appealing to this conceptual system of isometries and symmetries opened opportunities to expand children's understandings of 3-D structures.

An Individual Look at Children's Conceptions of 3-D Structures and Rotations

In this section, we give a brief summary of the results of the one-on-one cognitive interviews to provide a more individual view of children's developed conceptions of pyramids, prisms, and rotation isometries and symmetries. We acknowledge that many of the episodes presented in the previous two sections are often dominated by a handful of students. Although this is a limitation of the current study, we hope to use insights gained from the classroom analysis to broaden participation in future iterations of the design study. However, we also note that introducing a new participation framework into mathematics classrooms takes time to develop. So although participation was narrow in the first phase of instruction, by the end of the unit, we saw a significant increase in participation by more students in whole group discussions. Still, we had concerns about what individual children made out of the defining activities across the instruction unit, thus the cognitive interviews helped monitor learning at an individual level.

Children's Conceptions of Pyramids and Prisms. Results from the first cognitive interviews focused on the classification of polyhedra as a pyramid, prism, or neither are reported in Table 2. All students correctly classified the regular polyhedra (i.e. triangular pyramid, cube, and octagonal pyramid). The majority of students also justified their classifications by appealing to their shared definitions, explaining how a structure did or did not satisfy all the rules for a particular class. Thus, although students made some classification errors on the more ambiguous cases, they still carefully weighed the different defining properties for each case and showed hesitancy in their classification rather than looking for the most obvious feature as grounds for classification.

Table 1: Results from 1st Clinical Interview- Classifying Polyhedra (N =19)

3D Structure	Correct	Use of Definition Explanation
Triangular pyramid	100%	100% (systematic)
Rhombic prism	62%	100% (systematic)
Hexagonal anti-prism (neither)	86%	95% (systematic) 5% (informal)
Cube (prism)	100%	100% (systematic)
Octagonal pyramid	100%	100% (systematic)
Rhombus base w/ triangle faces (neither)	86%	71% (systematic)

Children's Conceptions and Visualization of Rotation Isometries and Symmetries. Results from the second cognitive interview focused on rotation isometries and symmetries. They suggest that students developed new understandings and ways of visualizing turns as rotation isometries and symmetries (see Tables 2 and 3 for summary statistics). Students were quite adept at identifying both the center of rotations and their magnitudes. Additionally, on the symmetry items, students also displayed understandings about distinctions between half turns and half turn symmetries. Students were not as successful findings the symmetrical turn axis on the 3-D "Z" shape compared to the canonical polyhedra. We attribute this to the complexity of the shape, but we also noted in interviews that children had a harder time indexing parts of the 3-D "Z." We found during classroom instruction the indexing of parts of structures was an important part of children's symmetry analysis; thus, children might have struggled applying their language generated to index polyhedra to this new form.

Table 2: Results from 2nd Clinical Interview: Describing Isometries of Square Pyramid (N=14)

Type of turn	Identification of turn axis	Identification of turn magnitude
½ turn	100%	100%
¼ turn	85.7%	78.6%
¼ turn symmetry	100%	7.1% (Whole turn)
		64.2% (WT + ½ turn)
		28.6% (WT+ ½ turn + ¼ turn)

Table 3: Results from 2nd Clinical Interview: Analysis of ½ Turn Symmetries (N=19)

3-D Structure	Correct Identification of Axes of Turn Symmetry
Octagonal Pyramid	100%
Octagonal Prism	100% (75.4% only 2 out of 3 unique axes)
Octahedron	92.9% (64.3% only 2 out of 3 unique axes)
3-D "Z"	64.3%

Summary: Individual's understandings of 3-D structures and rotations. The results of the cognitive interview suggest that individual children developed greater understandings of 3-D structures and rotations as isometries and symmetries from the collective defining and analysis activities in the classroom. This was true even for students who did not often visibly participate in the interactions highlighted in our analysis of episodes. Thus, although more research is needed to determine how to expand children's participation in the practice of defining, being an active observer of these interactions may also contribute to children's mathematical learning.

Discussion

Children's informal and developing senses of motion served as resources for their participation in the practice of defining which became a tool for the exercise of mathematical imagination. As children analyzed structures using informal motions and explaining what about structures allowed them to move (e.g., to roll), teachers and students attended to new properties, negotiated ways of communicating about these properties, and learned about relations among these new and previously considered properties. Children's definitions and conceptions of polyhedra then served as important resources in mathematizing everyday experiences of turns as rotation isometries and symmetries. Properties of prisms and pyramids helped children index centers and magnitudes of rotation, and these aspects of rotations in turn created new opportunities for considering relations between properties of 3-D structures and their symmetries.

In conclusion, this study demonstrates three benefits of incorporating and developing children's senses of motion into mathematical investigations of space. First, motions can support young children in developing a mathematical practice of defining. Second, when motions are articulated as isometries and symmetries, they further extend children's understandings of mathematical structures and their properties. Third, incorporating informal and mathematical forms of motions into children's analysis of 3-D structures supported mathematical visualization. Results from the classroom data and cognitive interview demonstrate that children were adept at predicting outcomes of rotations and at discriminating which rotations, based on their centers and magnitude, would result in a symmetry. Given the growing attention to cultivating children's visualization skills, this study provides an image of how these skills can be cultivated in ways that are disciplinary specific and mathematically productive.

Although children come to mathematics classroom with many experiences of motion, we found that children do not readily position these experiences as resources on their own. Teachers had to help position motions as a resource for defining by first initiating appeals to motion during moments of defining and using motions to pose definitional questions (i.e. what makes structures roll, why do some structure feel smooth when you roll them in your hand). In this iteration, much of the work was done by a researcher rather than the classroom teacher; thus, implementing these types of supports likely requires teachers to develop greater understandings of the role that motions play in mathematical thinking and a deeper understanding of geometric structures. During planning sessions, Mrs. B often voiced that she was unsure of what types of questions to ask children to support their defining, and she often reflected on her own gaps of knowledge. We propose as a next step further analysis of the instructional supports implemented during instruction that supported children's use of motions to define 3-D structures. Greater understanding of these supports will help inform what forms of mathematical content knowledge are needed to teach early geometry and the design of future professional development.

Throughout the instructional unit, children were also engaged in designing and constructing 3-D structures using Magformers. To get a sense of children's views of why we might be constructing in math class, we asked them at the start of instruction why building might be a good way to learn math. Students' responses ranged from being uncertain to making appeals to number. For example, one student said that the Magformers would make good counters to model addition and subtraction problems. We took this as evidence that, despite being six- and seven-years-old, these children already had limited views of what constitutes mathematics. As children developed greater understandings of pyramids and prisms, children's construction activities became reorganized into playful explorations of variability and constraints. This served to widen the example sets of structures available for the class's collective analysis, and at the conclusion of instruction, this served to make salient patterns between structures and their symmetries. Thus, construction was also an important resource in children's defining practice. During the end of unit interviews, we asked some students to revisit the question of why constructing is a good way to learn math. This time, a student told us that, "building helps you think of structures that you didn't even know existed." Although we cannot make claims that this is how all students viewed construction by the end of the unit, we present this student's quote to end with an argument for the importance of cultivating children's sense of the role that imagination plays in mathematics.

References

Alibali, Martha W., and Mitchell J. Nathan. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures." Journal of the learning sciences 21.2 (2012): 247-286.

Ambrose, R., & Kenehan, G. (2009). Children's evolving understanding of polyhedra in the classroom. *Mathematical Thinking and Learning*, *11*(3), 158-176.

Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction*, *26*(3), 269-309.

Civil, M. (2002). Everyday mathematics, mathematicians' mathematics, and school mathematics: Can we bring them together? Journal for Research in Mathematics Education, 11, 40–62.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *Journal for Research in Mathematics Education*, 136-163.

Cole, M. & Engström, Y. (1993). A cultural-historical approach to distributed cognition. *Distributed Cognitions:* In Salomon, G (Ed.). *Psychological and Educational Considerations.* Cambridge University Press.

Goldin-Meadow, S., & Beilock, S. L. (2010). Action's influence on thought: The case of gesture. Perspectives on psychological science, 5(6), 664-674.

Goodwin, C. (2017). *Co-Operative Action.* Cambridge University Press.

Goodwin, C. (1994). Professional vision. *American anthropologist*, *96*(3), 606-633.

Goodwin, C. (2013). The co-operative, transformative organization of human action and knowledge. Journal of pragmatics, 46(1), 8-23.

Gutierrez, R. (2018). Why we need to rehumanize mathematics. *Annual Perspectives in Mathematics Education: Rehumanizing Mathematics for Students Who Are Black, Latinx, and Indigenous.* NCTM.

Ferrara, K., Hirsh-Pasek, K., Newcombe, N. S., Golinkoff, R. M., & Lam, W. S. (2011). Block talk: Spatial language during block play. *Mind, Brain, and Education*, *5*(3), 143-151.

Foucault, M. 1972. The Archeology of Knowledge. New York: Pantheon Books.

Hawes, Z., Moss, J., Caswell, B., Naqvi, S., & MacKinnon, S. (2017). Enhancing Children's Spatial and Numerical Skills through a Dynamic Spatial Approach to Early Geometry

Instruction: Effects of a 32-Week Intervention. *Cognition and Instruction*, 35(20), 1-29.

Henderson, D. W. (2006). Alive mathematical reasoning. In F. A. Rosamond & L. Copes (eds.), Educational Transformations: Changing our lives through mathematics. (pp 247-270). Bloomington, Indiana: Author House.

Henderson, D. W., & Taimina, D. (2005). *Experiencing geometry: Euclidean and non-Euclidean with history*. Prentice Hall.

Kobiela, M., & Lehrer, R. (2015). The co-development of mathematical concepts and the practice of defining. Journal for Research in Mathematics Education, 46(4), 423-454.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer& D. Chazan Eds), Designing learning environments for developing understanding of geometry and space (pp. 137-167). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, United Kingdom: Cambridge University Press.

Latour, B. (1999). Pandora's hope. Essays on the reality of science studies. Cambridge, MA: Harvard University Press.

Lehrer, R., Jacobson, C., Kemeny, V., & Strom, D. (1999). Building on children's intuitions to develop mathematical understanding of space. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 63–87). Mahwah, NJ: Lawrence Erlbaum Associates.

Ma, J. Y. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. Journal of the Learning Sciences, 25(3), 335-371.

Miller, H. E., Vlach, H. A., & Simmering, V. R. (2017). Producing spatial words is not enough: Understanding the relation between language and spatial cognition. *Child development*, *88*(6), 1966-1982.

National Council of Teachers of Mathematics. (2014). Essential Understandings Series. Reston, VA: National Council of Teachers of Mathematics.

Oudgenoeg-Paz, O., Leseman, P., & Volman, M. (2015). Exploration as a mediator of the relation between the attainment of motor milestones and the development of spatial cognition and spatial language. Developmental Psychology, 51(9), 1241.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). The child's conception of geometry. New York: Harper and Row.

Piaget, J. (1970). Genetic epistemology. New York: W. W. Norton.

Resnick, I., Verdine, B. N., Golinkoff, R., & Hirsh-Pasek, K. (2016). Geometric toys in the attic? A corpus analysis of early exposure to geometric shapes. *Early Childhood Research Quarterly*, *36*, 358-365.

Senechal, M. (1990a). Brief history of geometrical crystallography. Historical atlas of crystallography, 43-59.

Senechal, M. (1990b). Shape. In *On the shoulders of giants: New approaches to numeracy* (pp. 139-181). National Academy Press.

Stone, C. A. (1993). What is missing in the metaphor of scaffolding. Contexts for learning: Sociocultural dynamics in children's development, 169-183.

Streeck, J. (2009). Gesturecraft: The manu-facture of meaning. Amsterdam, Netherlands: John Benjamins.

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., Newcombe, N. S., Filipowicz, A. T., & Chang, A. (2014). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematical skills. *Child development*,*85*(3), 1062-1076.

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, *101*(4), 817.

Washburn, D. K., & Crowe, D. W. (1988). Symmetries of culture: Theory and practice of plane pattern analysis. Seattle, WA: University of Washington Press.

Young, C. J., Cartmill, E. A., Levine, S. C., & Goldin-Meadow, S. (2014). Gesture and Speech Input are Interlocking Pieces: The Development of Children's Jigsaw Puzzle Assembly Ability. In Proceedings of the Annual Meeting of the Cognitive Science Society (Vol. 36, No. 36).