HAMILTONICITY AND STRUCTURE OF CLASSES OF MINOR-FREE GRAPHS

By

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Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

May, 2014

Nashville, Tennessee

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ACKNOWLEDGMENTS

First I would like to thank my adviser, Mark Ellingham, for all of his support and encouragement over the past four years. His time spent working with me and reading through my papers has truly made me a better mathematician. Next I would like to thank my fellow graduate students and friends who kept me sane and made my time in Nashville very enjoyable. Last I would like to thank my family for all of their love and support over the past five years and all the years before then.

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CHAPTER I

INTRODUCTION

1.1 Definitions

The main results of this dissertation are Hamiltonicity and structural results for graphs on surfaces and graphs with certain forbidden minors. We begin in the first chapter by providing relevant definitions and describing related known results. This provides context for the main results that follow. In Chapters II and IV, we prove results concerning Hamiltonicity of graphs on surfaces. In Chapter III, we outline notation and structural lemmas concerning $K_{2,t}$ minors. In Chapter V, we provide a complete characterization for a class of minor-free graphs. In Chapter VI, we discuss directions for future work.

Throughout this dissertation, let $G = (V(G), E(G))$ be a finite simple graph. A *path* in a graph G is a sequence of distinct vertices $v_1, v_2, ..., v_n$ such that $v_i v_{i+1} \in E(G)$ for $1 \leq i \leq n-1$. A path on three or more vertices together with the edge $v_n v_1$ is called a *cycle*. A *Hamilton cycle* or *path* is a cycle or path that includes every vertex of the graph. Not every graph contains a Hamilton cycle, and the next section as well as Chapters II and IV deal with results about restrictions that can be placed on graphs to ensure the existence of such a cycle, perhaps satisfying special conditions. One restriction involves the connectivity of a graph, which is defined as follows. A graph is *connected* if there is a path between any two vertices in the graph. If the removal of one vertex v and all of its incident edges from a connected graph results in a graph that is not connected, then v is a *cutvertex*. A set of vertices whose removal disconnects the graph is called a *cutset*. A graph is *k-connected* if the smallest cutset in a connected graph has size k or more, or if the graph is K_{k+1}. A graph is *bipartite* if its vertices can be divided into two disjoint parts, A and B, such that every edge in the graph is of the form ab where $a \in A$ and $b \in B$. A graph is *planar* if it can be drawn on the plane such that no edges cross. The equivalent definitions hold for *projective-planar* graphs, *toroidal* graphs, and graphs on the Klein bottle. A graph is *outerplanar* if it has a plane embedding in which all vertices are on the outer face. For a graph embedded on a surface with no crossing edges, the *face degree* of each face of the graph is the number of edges in the closed walks of the graph that make up the boundary of the face.

Chapters III through V deal with minors of graphs. A graph H is a *minor* of a graph G if H is isomorphic to a graph formed from G by contracting and deleting edges of G and deleting vertices of G. Another way to think of a k-vertex minor H of G is as a collection of pairwise disjoint subsets of the vertices of G, $(V_1, V_2, ..., V_k)$ where each V_i corresponds to a vertex $v_i \in V(H)$, $G[V_i]$ (the subgraph of G induced by the vertex set V_i) is connected for $1 \leq i \leq k$, and for each edge $v_iv_j \in E(H)$ there is an edge between a vertex of V_i and V_j in G. We will often identify minors in graphs by describing the sets $(V_1, V_2, ..., V_k)$. For each vertex $v \in V(H)$, the *branch set* of v is the set of vertices in G that contracts to v. A minor H of G is *rooted* at a vertex $x \in V(G)$ if x is in the branch set of a designated vertex of H. A graph is *H-minor-free* if it does not contain H as a minor.

1.2 Hamiltonicity Results for Graphs on Surfaces

We are now ready to look at results concerning Hamiltonicity of certain types of graphs. One of the earliest results of this kind is due to Whitney.

Theorem I.1 (Whitney, 1931 [31]). *Every 4-connected planar triangulation has a Hamilton cycle.*

Whitney's result was not strengthened for over twenty years. In 1956, Tutte finally removed the triangulation condition and in 1977, he published another paper that reorganized the proof.

Theorem I.2 (Tutte, 1956 [28]). *Every 4-connected planar graph has a Hamilton cycle.*

This result was later strengthened by Thomassen (with a minor correction by Chiba and Nishizeki [6]). A graph is *Hamilton-connected* if there is a Hamilton path between every two pair of vertices in the graph.

Theorem I.3 (Thomassen, 1983 [27]). *Every 4-connected planar graph is Hamilton-connected.*

Tutte's result in 1956 saw the introduction of what are now known as "Tutte cycles", structures within a graph that have many other useful applications. It is known that not all 3-connected planar graphs are Hamiltonian and even not all 3-connected triangulations of the plane are Hamiltonian [31] and these results will be discussed later. Hence Tutte's result cannot be strengthened by simply weakening the connectivity condition. Restricting the vertex degrees so that every vertex has degree three (cubic graphs) still does not guarantee Hamiltonicity for 3-connected planar graphs [29]. Barnette and Goodey are credited with conjectures that claim additional sufficient conditions for the Hamiltonicity of 3-connected, cubic, planar graphs.

Conjecture I.4 (Barnette, see [18], and Goodey [15]). *Every 3-connected, cubic, planar graph with all face degrees at most six is Hamiltonian.*

Conjecture I.5 (Barnette, see [16]). *Every 3-connected, cubic, bipartite, planar graph is Hamiltonian.*

In Chapter IV, we will see another condition on 3-connected planar graphs that guarantees Hamiltonicity. Now we consider graphs on other surfaces. Concerning graphs on the projective plane, Thomas and Yu have the following result:

Theorem I.6 (Thomas and Yu, 1994 [25]). *Every edge of a 4-connected projective-planar graph is contained in a Hamilton cycle.*

It is not true, however, that every 3-connected cubic projective-planar graph with face degree ≤ 6 is Hamiltonian as Barnette and Goodey conjectured for the plane. As a counterexample, take the Petersen graph embedded in the projective plane. Its face degrees are all five but the graph is not Hamiltonian. However, it has been shown by Ellingham and Zha that every 3-connected cubic projective-planar graph with face degrees three or six is in fact Hamiltonian (personal communication).

For graphs on the torus, Brunet and Richter proved the following:

Theorem I.7 (Brunet and Richter, 1995 [4]). *Every 5-connected toroidal triangulation is Hamiltonian.*

This result was later strengthened by Thomas and Yu.

Theorem I.8 (Thomas and Yu, 1997 [24]). *Every edge of a* 5*-connected toroidal graph is contained in a Hamilton cycle.*

For 4-connected toroidal graphs, Thomas, Yu, and Zang proved the existence of a Hamilton path [26]. The result for Hamilton cycles is still unknown and is a leading open conjecture due to both Grünbaum and Nash-Williams:

Conjecture I.9 (Grünbaum, 1970 [16], Nash-Williams, 1973 [21]). *Every* 4*-connected toroidal graph is Hamiltonian.*

If we place restrictions on the face and vertex degrees of toroidal graphs, then there are several Hamiltonicity results. A graph is *k-regular* if every vertex has degree k. The following three results are due to Altshuler.

Theorem I.10 (Altshuler, 1971 [1]). *Every* 6*-regular toroidal graph all of whose faces are triangles is Hamiltonian.*

Theorem I.11 (Altshuler, 1971 [1]). *Every* 4*-regular toroidal graph all of whose faces are quadrilaterals is Hamiltonian.*

Theorem I.12 (Altshuler, 1971 [1]). *Every cubic toroidal graph with an even number of faces all of which are hexagonal is Hamiltonian.*

Bouwer and Chernoff proved a related result:

Theorem I.13 (Bouwer and Chernoff, 1988 [2]). *Every* $\{6,3\}_{b,c}$ *toroidal graph is Hamiltonian.*

The $\{6,3\}_{b,c}$ toroidal graphs are a subclass of the cubic toroidal graphs with hexagonal faces. They satisfy certain symmetry conditions and include some graphs with an odd number of faces.

Another way to refer to a cubic toroidal graph all of whose faces are hexagons is as a "generalized honeycomb torus". Using this terminology, Yang et al. showed that Theorems I.12 and I.13 can be generalized.

Theorem I.14 (Yang et al., 2008 [32]). *Every cubic toroidal graph all of whose faces are hexagons is Hamiltonian.*

The next result was shown by Nakamoto, Ozeki, and Fujisawa, building on earlier work by Nakamoto and Ozeki [20].

Theorem I.15 (Nakamoto, Ozeki, Fujisawa [13]). *Every* 4*-connected graph on the torus with toughness exactly* 1 *is Hamiltonian.*

For graphs on the Klein bottle, Brunet, Nakamoto, and Negami proved the following result:

Theorem I.16 (Brunet, Nakamoto, Negami, 1999 [3]). *Every* 5*-connected triangulation on the Klein bottle is Hamiltonian.*

For more general surfaces, Duke proved that if graphs on a given surface are sufficiently highly connected, then they are Hamiltonian [12]. No fixed connectivity works for all surfaces, however. Duke's proof uses a well-known sufficient condition for the Hamiltonicity of any graph, due to Dirac, that each vertex of an n-vertex graph has degree $\geq n/2$ [11].

Instead of looking for Hamilton cycles in graphs, a related idea is to show that graphs have cycles of at least some minimum length. Chen and Yu proved that 3-connected planar, projective-planar, and toroidal graphs as well as graphs embeddable on the Klein bottle have cycles of length at least $cn^{\log_3 2}$ where n is the number of vertices [5].

In Chapter II, we discuss edge-Hamiltonicity for certain graphs on the torus, which is related to Conjecture I.9. The main result is Theorem II.1. Cases 2.2.4 and 2.2.5 are joint work with Mark Ellingham.

1.3 Structural Results for $K_{2,t}$-minor-free Graphs

The best-known result concerning minor-free graphs is Wagner's Theorem which was published in 1937.

Theorem I.17 (Wagner, 1937 [30])**.** *A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ as a minor.*

Another result of this type is Dirac's forbidden minor characterization of all K_4-minor-free graphs [11]. The forbidden minor characterization of outerplanar graphs is well-known and we provide a brief sketch of the proof here.

Theorem I.18. *A graph G is outerplanar if and only if G does not contain K_4 or $K_{2,3}$ as a minor.*

Proof. A graph is outerplanar if and only if each of its connected components is outerplanar so without loss of generality, assume G is connected. Furthermore, because K_4 and $K_{2,3}$ are 2-connected, if G contains either as a minor, then the minor would have to be in a block of G. Thus without loss of generality, assume G is 2-connected. Let G' be the graph formed from G by adding a vertex v adjacent to all vertices of G.

For the forward direction, assume G is outerplanar. Then G' is planar and thus by Wagner's Theorem contains no K_5 or $K_{3,3}$ minor. Therefore, it follows that G contains no K_4 or $K_{2,3}$ minor because v could be added to such a minor to give a K_5 or $K_{3,3}$ minor in G'.

For the reverse direction, assume G contains no K_4 or $K_{2,3}$ minor. If G' contains a K_5 or $K_{3,3}$ minor, then v must be in the minor. If we delete the branch set containing v, then the result is a K_4 or $K_{2,3}$ minor in G. Thus G' contains no K_5 or $K_{3,3}$ minor and therefore is planar. Now it follows that G is outerplanar. □

For graphs without rooted minors, Robertson and Seymour characterized all 3-connected 3-terminal rooted $K_{2,3}$-minor-free graphs [23] while Lino Demasi characterized all 3-connected 4-terminal planar rooted $K_{2,4}$-minor-free graphs [8]. For 3-connected graphs H with at most eleven edges, Ding and Liu describe the characterizations of all H-minor-free graphs [10].

In Chapter V, we will focus on $K_{2,4}$-minor-free graphs. There are several known structural results which apply specifically to these graphs. According to a result claimed by Dieng and Gavoille, every 2-connected $K_{2,4}$-minor-free graphs contains 2 vertices whose removal leaves the graph outerplanar [9]. Using this result, Streib and Young prove the following:

Theorem I.19 (Streib and Young, 2010 [22])**.** *Let G be a connected $K_{2,4}$-minor-free* *graph. Then the dimension of the minor poset of G is polynomial in $|E(G)|$.*

For more general $K_{2,t}$-minor-free graphs, Chudnovsky, Reed, and Seymour proved the following:

Theorem I.20 (Chudnovsky, Reed, and Seymour, 2011 [7]). *Let G be a $K_{2,t}$-minor-free graph with $|V(G)| = n$ and $t \geq 2$. Then $|E(G)| \leq (1/2)(t+1)(n-1)$.*

In Chapter III, we prove some general results about $K_{2,t}$-minor-free graphs. In Chapter IV, we prove that 3-connected, planar $K_{2,5}$-minor-free graphs are Hamiltonian. The main result is Theorem IV.3. In Chapter V, we provide a complete characterization of all $K_{2,4}$-minor-free graphs. The main results are Theorem V.7, Theorem V.11, and Theorem V.28. The work in these three chapters is joint with Mark Ellingham, Kenta Ozeki, and Shoichi Tsuchiya.

Chapter II

TOROIDAL RESULTS

We focus our attention now on Conjecture I.9, stating that all 4-connected toroidal graphs are Hamiltonian. As mentioned previously, Thomas and Yu showed that in the projective plane, there is a Hamilton cycle through any edge of a 4-connected graph in [25]. One might hope that this result could be extended to the torus, but in fact the statement is untrue for a general 4-connected toroidal graph. The Cartesian product of two even cycles embeds in the torus and gives a 4-connected quadrangulation. If you add an edge across one of the quadrangles, then you cannot find a Hamilton cycle through this edge. This example was observed by Thomassen in [27]. In fact, adding any number of edges to one side of the bipartition of this bipartite graph still yields no Hamilton cycle through these edges. This observation generalizes to any bipartite 4-regular quadrangulation of the torus. It is thus hard to extend the standard proof techniques, as used for example by Thomas and Yu [25], to show that 4-connected toroidal graphs are Hamiltonian.

One possible approach is to try to characterize situations where some edge is not on a Hamilton cycle. The following result is a step towards this. It shows that Thomassen's examples, and the generalizations using 4-regular bipartite quadrangulations, are critical in the sense that adding any edge on the other side of the bipartition restores the property that every edge is on a Hamilton cycle.

Theorem II.1. *Let G be a 4-connected, 4-regular, bipartite graph on the torus with*

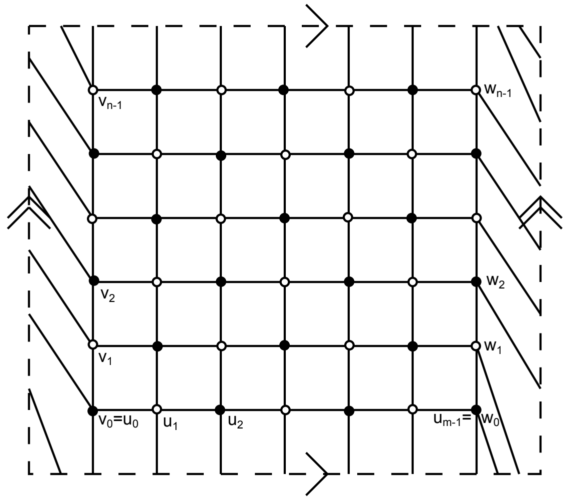


Figure 2.1

partition sets of white and black vertices. Suppose one edge, e_1, is added between two black vertices across a face. Then for any additional edge e_2 added between two white vertices and across a different face, there is a Hamilton cycle through e_1 and e_2. Thus the new graph has the property that for every edge e in the graph, there is a Hamilton cycle through e.

Proof. By Euler's formula, we know that all 4-regular, bipartite graphs on the torus are quadrangulations. These graphs are characterized by three parameters when drawn on the standard representation of the torus with vertical sides identified in parallel and horizontal sides identified in parallel: the height of the grid, the length of the grid, and the size of the shift. The graph is described as a grid because it is the Cartesian product of an even cycle and a path as shown in Figure 2.1.

The cycle is vertical and uses an edge that wraps around the horizontal sides of the torus representation. One copy of it is denoted by the vertices $v_0, v_1, ..., v_{n-1}$ in the figure. The length of this cycle gives the height of the grid. The path is horizontal and one copy of it is denoted by the vertices $u_0, u_1, ..., u_{m-1}$ in the figure. The length of this path gives the length of the grid. We denote by $w_0, w_1, ..., w_{n-1}$ the rightmost copy of the cycle in the grid. Then the amount of the shift is q where v_i is connected to w_{i+q} for indices taken mod n. This characterization is suggested by Altschuler in [1] and further elaborated on by Nakamoto and Negami in [19]. Without loss of generality, we can place e_1 in the bottom left corner of the grid because the grid can be shifted up or down or left or right without changing the parameterization. We next consider all possible placements of the edge e_2 between white vertices. When we refer to rows and columns throughout the proof, we mean rows of faces and columns of faces, not vertices. Let g_r and g_c denote the total number of rows and columns respectively in the grid excluding the row and column that wrap around the diagram. For the graph in Figure 2.1, $g_r = n - 1$ and $g_c = m - 1$. With this notation, we consider all cases $g_c \geq 0$, and $g_r \geq 3$ and odd. Note that g_r must be odd because our graphs are bipartite and there is no shift in connecting vertices vertically through the horizontal sides. Also g_r must be greater than one because if $g_r = 1$ then we have a multigraph. Number the rows from bottom to top starting at zero with the row that wraps around and the columns from left to right also starting at zero with the column that wraps around and includes a possible shift. Let e_r and e_c denote the row and column number respectively of the added edge e_2 between white vertices. Then $0 \leq e_r \leq g_r$, $0 \leq e_c \leq g_c$ and $(e_r, e_c) \neq (1, 1)$ since then e_2 would cross e_1. The proof is divided into two cases, each with two subcases. Case 1 restricts the placement of e_2 to rows and columns within the grid and Case 2 allows e_2 to be in the wraparound row or column. The subcases first consider larger grids and then cover the smaller cases. We outline the cases here:

Case 1. $e_r \geq 1$ *and* $e_c \geq 1$ *(therefore* $g_c \geq 1$*)*

Case 1.1. $g_r \geq 3$ *and* $g_c \geq 3$

Case 1.1.1. $e_r \geq 3$

Case 1.1.2. $e_r = 2$

Case 1.1.3. $e_r = 1$

Case 1.2. $g_r \geq 3$ *and* $g_c \leq 2$

Case 1.2.1. $g_c = 1$

Case 1.2.2. $g_c = 2$

Case 2. $e_r = 0$ *or* $e_c = 0$ *(or both)*

Case 2.1. $g_c \geq 1$

Case 2.1.1. $e_r = 0$, $g_r \geq 3$, *and* $e_c \geq 1$ *and even*

Case 2.1.2. $e_r = 0$, $g_r \geq 3$, *and* $e_c \geq 1$ *and odd*

Case 2.1.3. $e_c = 0$ *and* $g_c = 1$

Case 2.1.4. $e_c = 0$ *and* $g_c \geq 2$

Case 2.2. $g_c = 0$

Case 2.2 also has several subcases but they use different parameters that will be introduced later.

Case 1. $e_r \geq 1$ *and* $e_c \geq 1$

Note that necessarily we have $g_c \geq 1$.

Case 1.1. $g_r \geq 3$ *and* $g_c \geq 3$

Case 1.1.1. $e_r \geq 3$

This situation is shown in Figure 2.2. The picture on the left in Figure 2.2 is for $e_c > 1$ and odd while the picture on the right is for e_c even. The bold lines give the Hamilton cycle. Note that the scenario on the right still holds when $e_c = 2$. When $e_c = g_c$, we have similar pictures except that for e_c odd, we use a wraparound edge to connect the top right vertex with the bottom right vertex, and when e_c is even, we no longer need to use a wraparound edge. For any $e_r > 3$, the cycle follows the same up and down pattern through the columns starting at the left of the grid until it hits column e_c. It then follows the same back and forth pattern up the column for e_c odd and down the column for e_c even. Because g_r is always odd, the cycle will always reach every vertex in column e_c in this back and forth pattern. When $e_c = 1$, we have the Hamilton cycle as shown in Figure 2.3. Note that the parity of e_r is unimportant here and the same is true for later arguments.

Case 1.1.2. $e_r = 2$

If e_c is even, the picture on the right of Figure 2.2 also holds for $e_r = 2$. For e_c odd, we have the picture in Figure 2.4. When $e_c = 1$, we can extend the picture from Figure 2.3.

Case 1.1.3. $e_r = 1$

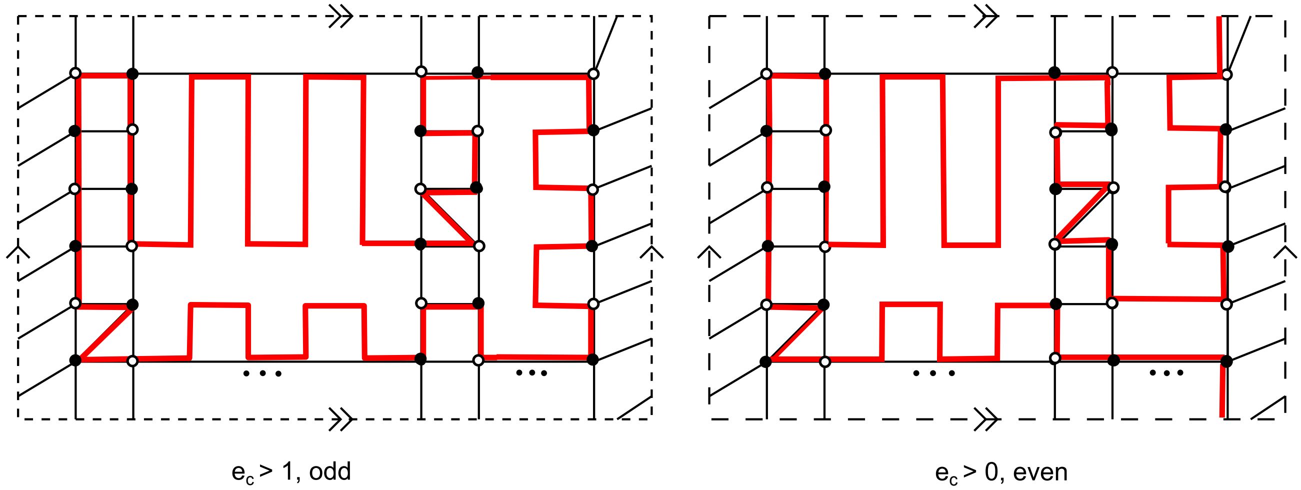


Figure 2.2 Case 1.1.1

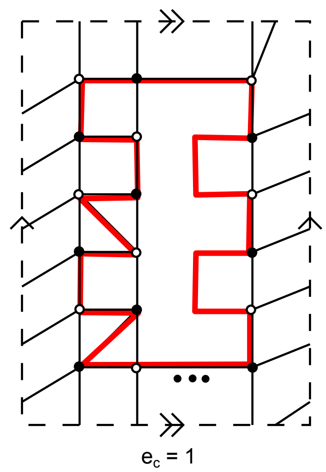
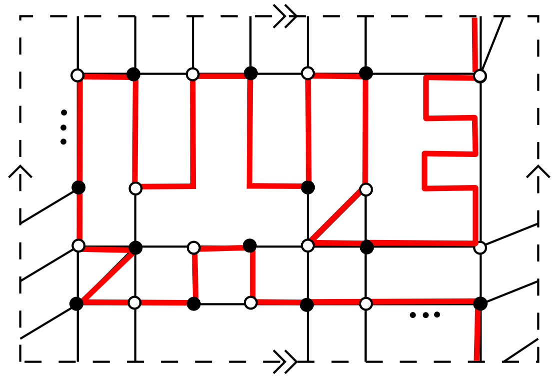
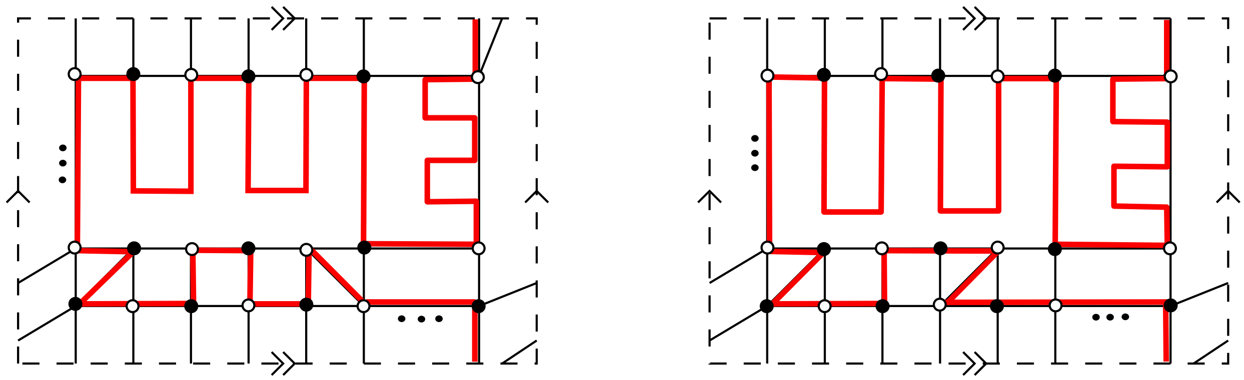


Figure 2.3 Case 1.1.1



$e_c > 3$, odd

Figure 2.4   Case 1.1.2



e_c odd e_c even

Figure 2.5   Case 1.1.3

This case is shown in Figure 2.5. On the left, e_c is odd and on the right $e_c > 2$ and even. Note that when $e_c = 2$, the picture on the right still holds. We cannot have $e_c = 1$ when $e_r = 1$ because then e_1 and e_2 would cross.

In the next subcase, we consider smaller grids.

Case 1.2. $g_c \leq 2$

Case 1.2.1. $g_c = 1$ *(and* $e_c = 1$*)*

This case is shown in Figure 2.6.

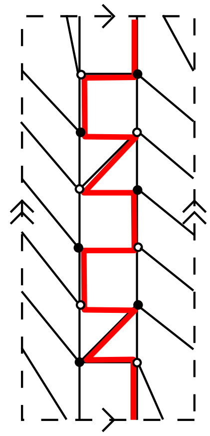


Figure 2.6 Case 1.2.1

Case 1.2.2. $g_c = 2$

This case is shown in Figure 2.7. On the left, $e_c = 1$, in the middle $e_c = 2$ and $e_r > 1$, and on the right, $e_c = 2$ and $e_r = 1$.

Case 2. *$e_r = 0$ or $e_c = 0$ (or both)*

We will consider two subcases of Case 2. The first covers grids with at least one column while the second considers grids with zero columns (of faces). Recall that our grid must have at least three rows.

Case 2.1. $g_c \geq 1$

Case 2.1.1. *$e_r = 0$, $g_r \geq 3$, and $e_c > 0$ and even*

In this case, we can shift the grid up one so that $e_r = 1$. Then we can shift to the right until $e_c = g_c$. Since $g_r > 1$, e_1 is one row above e_2 (not in the wraparound row or column) so reflecting this picture about a vertical line through the center of the grid and switching the colors of vertices reduces this situation to one already covered

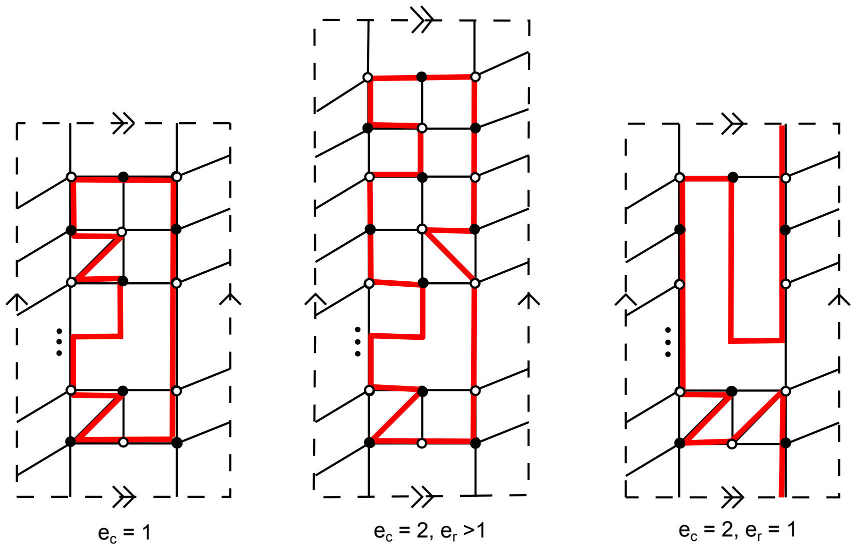


Figure 2.7 Case 1.2.2

previously with $e_r = 2$ and $e_c \geq 1$.

Case 2.1.2. $e_r = 0,\ g_r \geq 3,$ *and* $e_c > 0$ *and odd*

This case is shown in Figure 2.8. The picture only holds when $3 \leq e_c \leq g_c - 2$ or when $e_c = g_c$. In the latter case, the path just zigzags up the last column instead of going up and down and then back and forth through the columns after column e_c. Figure 2.9 shows the situation where $e_c = g_c - 1$. Note that these pictures hold even when $g_r = 3$. If $e_c = 1$, then we can shift the grid up and to the right until the edge between black vertices is in the upper right corner of the grid and we have a situation symmetric to $e_r = 2$ and $e_c = 1$ and considered previously in Case 1. Note that this translation and rotation works even for $g_c \leq 2$.

Case 2.1.3. $e_c = 0,\ g_c = 1$

This case is shown in Figure 2.10. There are two different pictures depending on

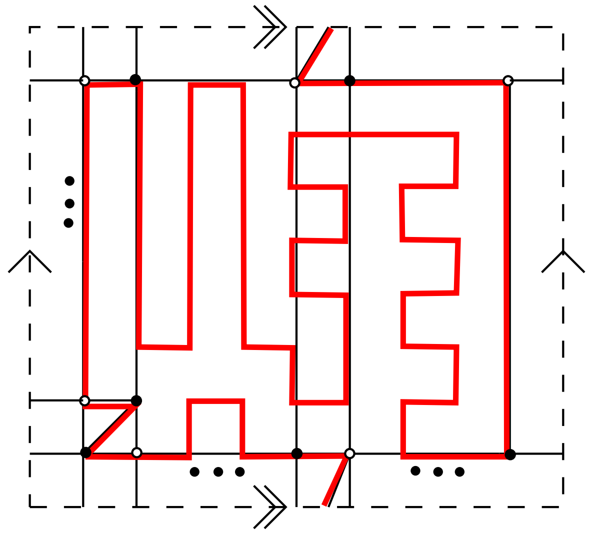


Figure 2.8 Case 2.1.2

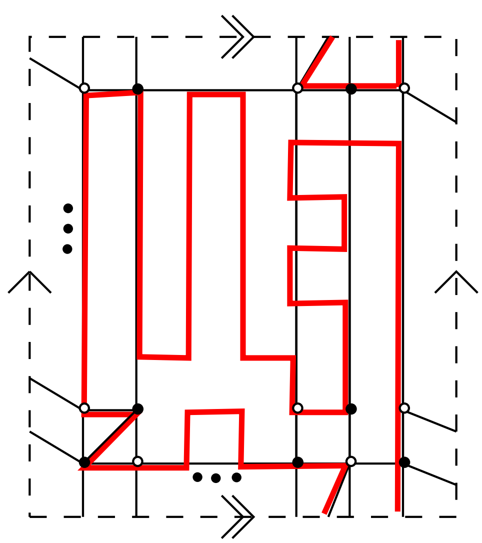


Figure 2.9 Case 2.1.2

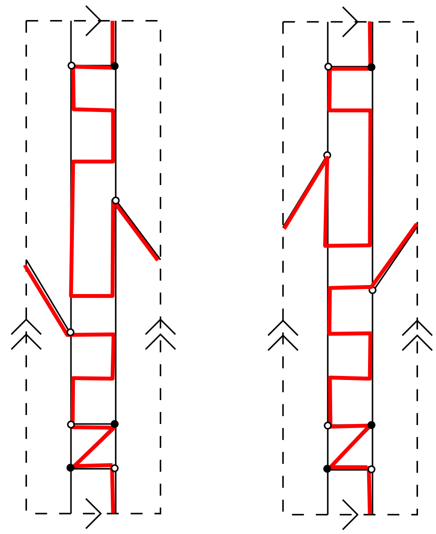


Figure 2.10 Case 2.1.3

the position of e_2. If $e_2 = (v_i, w_j)$, then the picture on the left is for $i < j$ and the picture on the right is for $i > j > 0$; note the picture on the left also works for $j = 0$: treat this as $j = n$.

Case 2.1.4. $e_c = 0$ *and* $g_c \geq 2$

When $g_c \geq 2$, the situation can be reduced to one already covered above. Shift the grid to the right until the edge between black vertices is in the last column before the wrap around column. Then shift up until the edge between black vertices is in the top row before the wrap around row. Now $e_c = g_c - 1 \neq 0$ and rotating this picture 180° results in a situation considered previously in which $e_c = 2$. If e_r is now zero, then this case is covered in Case 2 otherwise it is covered in Case 1.

Case 2.2. $g_c = 0$

For this case it is difficult to use the standard representation. Figure 2.11 shows an example of a cycle in this representation. With only one column of vertices, the cycle must use many of the shifted edges which makes it difficult to follow. Thus for this case we will use two different representations of our graph. An example of the first is shown in Figure 2.12. Instead of one column of vertices with the shifted edges connected around the horizontal sides of the standard torus representation, we draw two columns of vertices so now the shifted edges are the horizontal edges connecting the two columns. Note that now some vertices appear more than once in our drawing and not all edges are shown. Added edges e_1 and e_2 in the graph connect vertices at distance $q-1$ or $q+1 \pmod{n}$ where q is the shift. Note that q must be odd and $n = g_r + 1$ must be even since the graph is bipartite, and also $q \geq 3$ to avoid multiple edges. We will refer to an edge that connects vertices at a distance of $q+1$ or $q-1$ as a $q+1$ edge or a $q-1$ edge respectively. If we have at least one $q-1$ edge, without loss of generality we will label the vertices so that $e_1 = (1, q)$ where vertices 1 and q are both black. Here and for the rest of this Chapter, we will use the notation (m, n) for the edge between vertices labeled m and n. If both edges are $q+1$ edges, then without loss of generality, we will draw e_1 as connecting vertex 1 and vertex $q+2$ where again both are black.

In Figure 2.12 we have 16 vertices and $q = 7$ so the vertex labeled 0 is connected to the vertex labeled 7. Observe that with these drawings, a grid with n vertices and a shift of q has the same underlying graph as a grid with n vertices and a shift of $n-q$. Thus without loss of generality, we will choose a shift of q where $q \leq n/2 - 1$. Note that we cannot have $q = n/2$ because then we would have multiple edges between the vertices labeled i and $i + n/2$. In many situations, we can depict e_1 and e_2 in

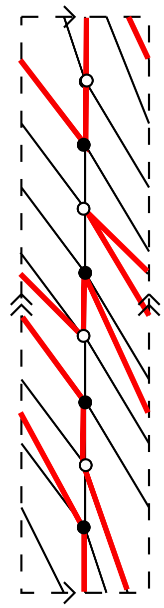


Figure 2.11

the diagram without needing to repeat every vertex twice, so the top vertex in the column on the left is the vertex labeled $n-1$ for a graph with n vertices. Ending at this vertex ensures each vertex is shown at least once since $q \leq n/2-1$. An example of this situation is shown on the left of Figure 2.12. If e_2 is not in this drawing, then we can draw more vertices above the edge $(n-1, n-q-1)$ until each vertex appears exactly twice and then every possible added edge will be in this column of faces, unless $e_2 = (0, q-1)$ in which case we shift the picture as described later. For this case, we draw the section of the graph that extends above the edge $(n-1, n-q-1)$ next to and on the right of the column of faces already present to make the drawings more compact. This situation is shown on the right of Figure 2.12. Note that by moving the top section of the column down and creating two columns of faces, we no longer have each vertex appearing twice. In particular, the vertices that are adjacent to faces from both columns only appear once. Now we consider the location of e_2. Suppose the edge connects vertices v_1 and v_2. Let $a = \min\{v_1, v_2\}$.

Case 2.2.1. $0 < a < q$ *and* e_2 *connects* a *with the vertex labeled* $a - (q \pm 1)$ *(mod* n*)*

This is the situation where we need to draw each vertex twice to fit both e_1 and e_2 in the diagram. We draw two columns of faces as described previously. Figure 2.13 illustrates the situation with a $q-1$ edge e_1 on the left and a $q+1$ edge in the middle. Note that the picture on the left requires $n-q-1 \geq q+1$ which is always true because n is even so $q \leq n/2 - 1$. The picture in the middle requires $n - q - 1 \geq q + 2$. This inequality is true within our given constraints on n and q except when $n = 2q + 2$, so the picture on the right shows this case. The pictures show e_2 as a $q + 1$ edge. If e_2 is a $q - 1$ edge, then the cycle follows a very similar pattern with the only difference being that the edge is angled the other way. Note that the picture holds even when $a = q - 1$ or $a = 2$. The situation where $0 < a = q - 1$ and a is the endpoint of a $q - 1$ edge does not happen since if $(0, q - 1)$ is an edge, then we would take $a = 0$.

Case 2.2.2. $0 < a < q$ *and* e_2 *connects* a *with the vertex labeled* $a + (q \pm 1)$

This case is shown in Figure 2.14 (i) and (ii). Part (i) is when the e_1 is $q - 1$ edge and part (ii) is for a $q + 1$ edge. We show e_2 as a $q + 1$ edge but a very similar cycle works when e_2 is a $q - 1$ edge. Notice that all pictures in this figure require $a + q \leq n - 1$ but this is always true by the choice of a.

Case 2.2.3. $q < a < 2q$

This case is shown in Figure 2.14(iii) and (iv) for e_2 as a $q - 1$ edge again with

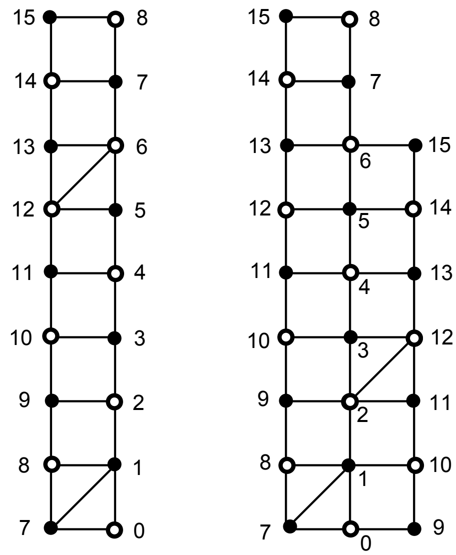


Figure 2.12

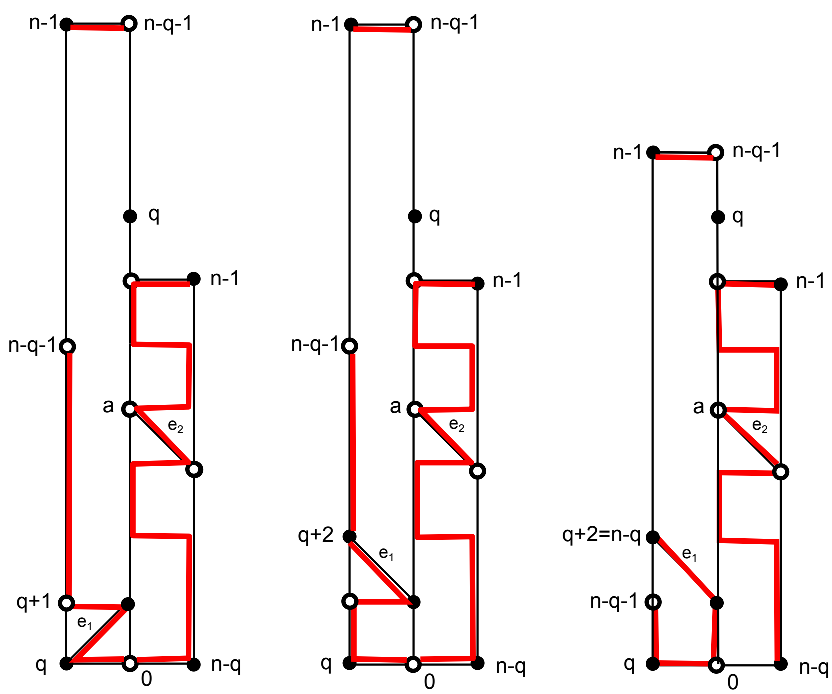


Figure 2.13

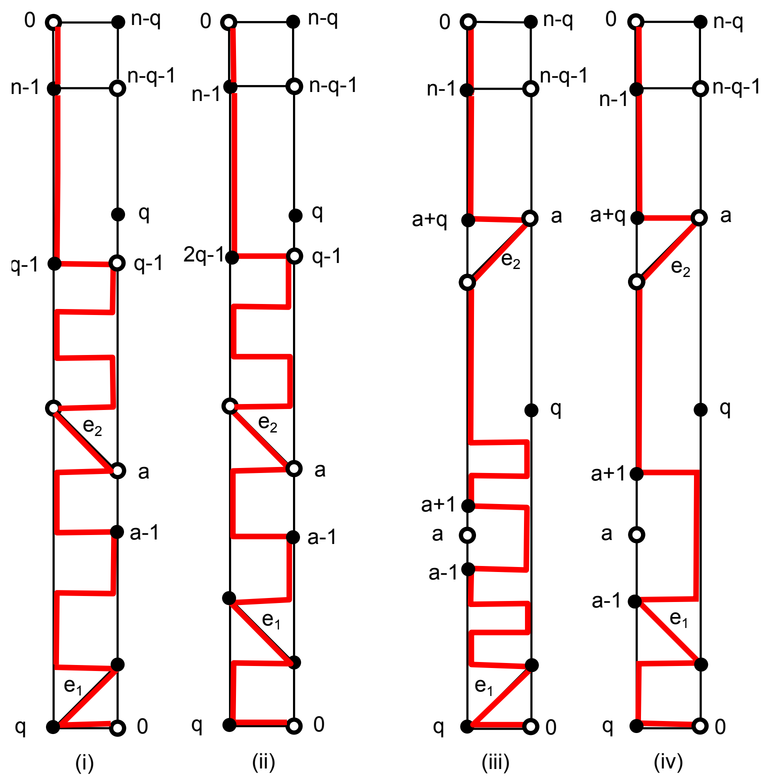

0
n-q
n-1
n-q-1
q
q-1
q-1
e2
a
a-1
e1
q
0
(i)
0
n-q
n-1
n-q-1
q
2q-1
q-1
e2
a
a-1
e1
q
0
(ii)
0
n-q
n-1
n-q-1
a+q
a
e2
q
a+1
a
a-1
e1
q
0
(iii)
0
n-q
n-1
n-q-1
a+q
a
e2
q
a+1
a
a-1
e1
q
0
(iv)

Figure 2.14

different cases depending on the type of e_1. When e_2 is a $q+1$ edge, only minor changes are required.

For the two remaining cases we will use a new representation of the graph. We will draw the n vertices as points along a circle so the shifted edges are now chords of the circle. Every vertex is shown exactly once and only the chords used in the Hamilton cycle are drawn. The general pattern is to divide the vertices up into cycles by taking $q+1$ consecutive vertices along the circle and closing up the cycle with a chord. The added edges e_1 and e_2 give cycles of length $q+2$ or q for a $q+1$ edge or a $q-1$ edge respectively. Next these cycles are connected by choosing two consecutive vertices in one cycle, removing the edge between them from the cycle, and instead adding in the chords that connect these vertices to vertices in the next cycle. Figure 2.15 illustrates this process for a graph with 16 vertices, a shift of 5, and two added $q-1$ edges.

We have three cycles here, say A, B, and C where A includes e_2, B includes e_1, and C is the third cycle. To describe this process of connecting cycles in detail, we begin with some notation. Let the edges (v_{2i+1}, v_{2i+2}) in the graph be *odd edges* and the edges (v_{2i}, v_{2i+1}) be *even edges* for $0 \leq i \leq (n-1)/2$. Thus each edge is labeled by the parity of its lower numbered index. Suppose we start with an odd edge in A, say (v_i, v_{i+1}). Then if we take chords from each of these vertices, we can reach the even edge (v_{i+q}, v_{i+q+1}). If this even edge is in B, then we say the odd edge (v_i, v_{i+1}) is a *forwards-linking edge* because we can use it to link cycle A forwards to cycle B by replacing the edges (v_i, v_{i+1}) and (v_{i+q}, v_{i+q+1}) with the appropriate chords. Similarly, we say the even edge (v_{i+q}, v_{i+q+1}) in B is a *backwards-linking edge.*

Consider a general situation where we have a number of cycles formed by chords of length $q-1, q$, or $q+1$, with no intervening vertices. Observe that in general the first two edges of every cycle R are always backwards-linking edges and the last two edges of R are always forwards-linking edges (these four edges are not necessarily distinct). To see this, consider a cycle $R = v_i, v_{i+1}, ..., v_j$. Then the vertex v_{i-q} must be in the cycle immediately prior to R because this cycle has length at least q so it at least contains vertices $v_{i-1}, v_{i-2}, ..., v_{i-q}$. It follows that (v_i, v_{i+1}) and also (v_{i+1}, v_{i+2}) are both backwards-linking edges. Similarly, the vertex v_{j+q} must be in the cycle immediately following R because this cycle has length at least q and thus contains at least the vertices $v_{j+1}, v_{j+2}, ..., v_{j+q}$. It follows that (v_{j-1}, v_j) and also (v_{j-2}, v_{j-1}) are both forwards-linking edges. These observations now ensure that every cycle has a backwards-linking edge and a forwards-linking edge of each parity.

In the example in Figure 2.15, the number of vertices is such that every vertex fits in a cycle with none left over. This will not always be the case, however, and we will have to connect vertices into the cycles in much the same way as we connect cycles to each other.

Case 2.2.4. $a = 0$

We consider two additional subcases.

Case 2.2.4.1. $a = 0$ *is connected to the vertex labeled* $-(q \pm 1)$

We have the situation shown in Figure 2.16 on the left. The example shown has 16 vertices, a shift of 5, and one $q+1$ edge and one $q-1$ edge. Also there are only two cycles as the number of vertices between vertex $q+2$ and vertex $-(q-1)$ is less than $q+1$. Let S be this set of vertices outside of both cycles. Note that the number of

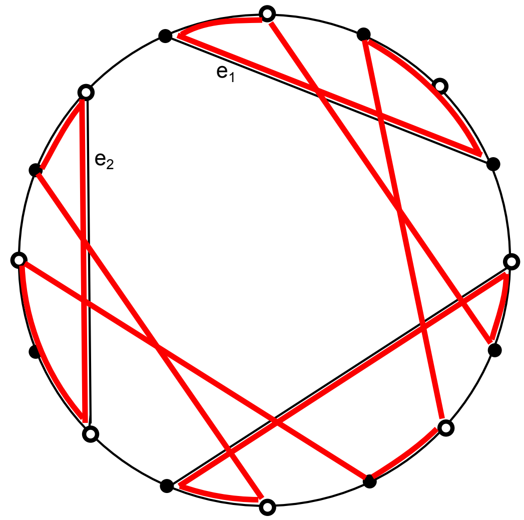


Figure 2.15

vertices in S must always be even since it is a path of vertices between a black vertex and a white vertex. If there are not enough vertices in S to create another cycle, then there are at most $q-1$ vertices, and this is the situation shown in the example. We have two cycles to start: C_1 which connects vertices $0, -1, -2, ..., -(q-1)$ and C_2 which connects vertices $1, 2, ..., q+2$. To create a Hamilton cycle, we must connect C_1 and C_2 and also add in the vertices from S.

By the observation made earlier, we know that C_1 has a forwards-linking edge and a backwards-linking edge of each parity. Let (v_i, v_{i+1}) be an odd forwards-linking edge of C_1. Then (v_{i+q}, v_{i+q+1}) is an even backwards-linking edge of C_2. We connect C_1 and C_2 by replacing the odd forwards-linking edge (v_i, v_{i+1}) in C_1 and the even backwards-linking edge (v_{i+q}, v_{i+q+1}) in C_2 with the chords used to join these edges. The picture on the left of Figure 2.16 illustrates this process. The odd edge and even edge we removed to connect the cycles are labeled as odd edge 1 and even edge 1 respectively. Now it remains to join the vertices of S into the cycle. The vertex v_l such that v_l is in S and v_{l-1} is in C_2 has even index since e_1 joins vertex 1 to either vertex q or $q+2$. Thus (v_l, v_{l+1}) is an even edge and if we follow chords from its two endpoints back into C_2, we reach the endpoints of an odd edge. In particular we reach vertices 1 and 2 if e_1 is a $q-1$ edge and vertices 3 and 4 if e_1 is a $q+1$ edge. In either case, it is clear that we reach vertices in C_2 and we can join v_l and v_{l+1} into C_2 by removing the odd edge in C_2 and replacing it with the appropriate chords and the even edge (v_l, v_{l+1}). In Figure 2.16 on the left, the odd edge and even edge involved in this process are labeled as odd edge 2 and even edge 2 respectively. The remaining vertices of S are joined into the cycle pairwise in the same way. We take an even edge between two adjacent vertices in S, follow the chords back to an odd edge in C_2, and replace the odd edge with the two chords and the even edge. Because $|S| \leq q-1$, the vertex of highest index in S is always joined by a chord to a vertex in C_2 so we can join in all vertices of S by this process. Note also that when picking up vertices of S, we only remove odd edges from C_2 and when we joined C_1 and C_2, we removed an even edge from C_2 so these processes will not interfere with each other.

If $|S| \geq q+1$, then we have more than two cycles. We take vertex v_l in S such that v_{l-1} is in C_2 and form the cycle $v_l, v_{l+1}, ..., v_{l+q}, v_l$. We continue forming cycles in this way with no intervening vertices until no more can be formed without intersecting C_1. Denote these new cycles in order $D_1, D_2, ..., D_k$. Now if cycles C_1, C_2, and D_1 through D_k cover all vertices of the graph, then we can just join adjacent cycles by the process described previously and we are done. We use odd forwards-linking edges and even backwards-linking edges so that the process of linking a cycle with its forward neighbor will not interfere with linking a cycle with its backwards neighbor. Now suppose the cycles C_1, C_2, and D_1 through D_k do not cover all the vertices. Denote by S' the set of all vertices not in C_1, C_2, or any of the D_i. To form a Hamilton cycle, we must join all of the cycles $C_1, C_2, D_1, D_2, ..., D_k$ and also join in the vertices of S'. We join the cycles exactly as before. We use odd forwards-linking edges and even backwards-linking edges. Then we use the cycle D_k to pick up the vertices of S'. Let v_r be such that $v_r \in S'$ but $v_{r-1} \in D_k$. Observe that r must be even. Then (v_r, v_{r+1}) is an even edge and we can follow chords backwards to D_k to get endpoints of an odd edge. We remove this odd edge and replace it with the chords and the edge (v_r, v_{r+1}). The rest of the vertices of S' are joined in pairwise in the exact same way. Each pair results in the removal of an odd edge from D_k and we know that D_k was joined to D_{k-1} by the removal of an even edge so these processes will not interfere. An example is shown on the right of Figure 2.16. Here, $k = 2$ and $|S'| = 4$. Two odd edges of D_k are removed to join in the vertices of S' and one even edge is removed to join D_k with D_{k-1}.

Case 2.2.4.2. *$a = 0$ is connected to the vertex labeled $q \pm 1$*

This case can be reduced to one covered previously. If 0 is connected to $q+1$, then we must have $e_1 = (1, q+2)$ so that e_1 and e_2 do not cross. Now if we renumber the vertices by adding one to every vertex, we now have $e_1 = (1, q+2)$ and $e_2 = (2, q+3)$ and switching the vertex colors gives a situation covered in Figure 2.14 (ii). If 0 is connected to $q-1$ then either $e_1 = (1, q)$ or $e_1 = (1, q+2)$. Again we can renumber the vertices by adding one so that now $e_2 = (1, q)$ and $e_1 = (2, q+1)$ or $(2, q+3)$ and

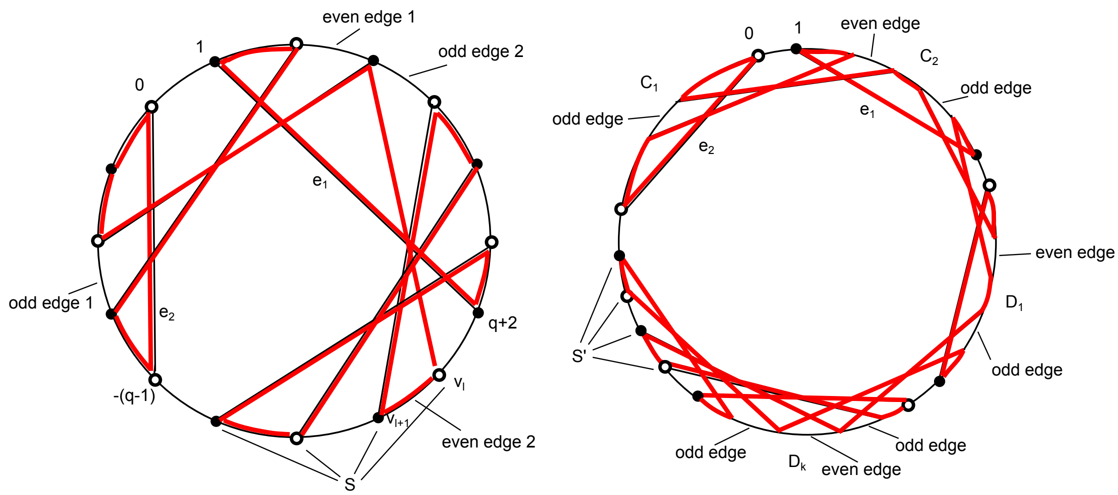


Figure 2.16

switching the vertex colors gives a situation covered in Figure 2.14 (i).

Case 2.2.5. $2q \leq a$

The arguments used in this case apply for $a \geq q+3$ and since $q \geq 3$, this covers $a \geq 2q$ for all possible values of q. We list the case as $2q \leq a$ instead of $q+3 \leq a$ because we covered $q < a < 2q$ in a previous case. The situation in this case is very similar to Case 2.2.4 except now when we form C_1 using the chord e_2 and C_2 using the chord e_1, C_1 and C_2 are no longer adjacent. We may have vertices in a set $S = \{q+1, q+2, ..., a\}$ (or $\{q+3, q+4, ..., a\}$ if e_1 is a $q+1$ edge) and also in another set, say T where $T = \{a+q+2, a+q+3, ..., n-1, 0\}$ if e_2 is a $q+1$ edge and $\{a+q, a+q+1, ..., n-1, 0\}$ if e_2 is a $q-1$ edge. But now as before, we can form cycles from the vertices in T if $|T| \geq q+1$. Say we form cycles $E_1, E_2, ..., E_p$ starting immediately after C_1. Denote by T' all remaining vertices of T that are not formed into cycles. Next we form as many cycles as we can with the vertices in S

without intersecting C_1, say $D_1, D_2, ..., D_k$ starting immediately after C_2. Denote by S' all remaining vertices of S that are not formed into cycles.

Suppose first that $S' \neq \emptyset$. To form a Hamilton cycle H, we must join the cycles $C_1, C_2, D_1, ..., D_k, E_1, ...E_p$ and also the vertices from T' and S'. We begin forming H by first joining cycles C_2, D_1, D_2... D_2, using odd forwards-linking edges and even backwards-linking edges. Now we add the vertices of S' into H by joining them pairwise into the cycle D_k. Even edges are added between the vertices of S' and odd edges of D_k are removed. The next step is to join C_1 to H. We do this by taking chords from the vertices $a-1$ and $a-2$, which belong to S', into C_1. Note that $(a-2, a-1)$ is an even edge so chords from these vertices will join endpoints of an odd edge in C_1. The edge $(a-2, a-1)$ was added to our cycle when we joined vertices $a-1$ and $a-2$ into D_k so now we can remove this edge and also the odd edge $(a-2+q, a-1+q)$ in C_1 and replace them by the appropriate chords to join C_1 into H. Observe now that we have removed an odd edge of C_1 to connect it backwards into H. Previously, odd edges were linked with chords going forwards. We want to maintain this reversal through the joining of cycles $C_1, E_1, ..., E_p$ because now the set T' will begin with an odd vertex whereas S' began with an even vertex. This means that we will need to remove even edges of E_p to join in vertices of T' so we want to connect E_p to E_{p-1} with an odd backwards-linking edge and an even forwards-linking edge so the processes do not interfere. The vertices of T' are joined pairwise into E_p in the same way S' is joined into D_k except now we remove even edges of E_p and add in odd edges between consecutive vertices of T'.

Now suppose S' is empty. We first join the vertices of T' into E_p as just described.

Then we join E_p to E_{p-1}, E_{p-1} to E_{p-2}, etc. by using odd backwards-linking edges and even forwards-linking edges. The difference is that we now continue this process all the way back through C_2 instead of reversing the roles of odd and even edges. $\square$

Chapter III

GENERAL RESULTS FOR $K_{2,t}$ MINORS

In this chapter, we discuss the idea of a standard $K_{2,t}$ minor that will be used in the next two chapters. We also look at several results concerning rooted $K_{2,2}$-minor-free graphs and the interactions between $K_{2,t}$ minors and separations in graphs. To discuss these ideas, we also provide additional definitions.

Recall that one way to think of a k-vertex minor H of G is as a collection of pairwise disjoint subsets of the vertices of G, $(V_1, V_2, ..., V_k)$ where each V_i corresponds to a vertex $v_i \in V(H)$, $G[V_i]$ (the subgraph of G induced by the vertex set V_i) is connected for $1 \leq i \leq k$, and for each edge $v_iv_j \in E(H)$ there is an edge between a vertex of V_i and V_j in G. We call this an *edge-based model* of H in G. More generally, we may allow there to be a path P_{ij} rather than an edge between V_i and V_j in G if $v_iv_j \in E(H)$. We then require that each path P_{ij} be internally disjoint from all other such paths and from $V_1, ...V_k$. We call this a *path-based model* of H in G.

Let $\{a_1, a_2, b_1, b_2, ..., b_t\}$ be the vertex set of $K_{2,t}$ with $\deg(a_1) = \deg(a_2) = t$, a_1 not adjacent to a_2, and $\deg(b_i) = 2$ for $1 \leq i \leq t$. In a graph G with $K_{2,t}$ as a minor, let R_1 and R_2 be the branch sets of a_1 and a_2 in an edge-based model of $K_{2,t}$. Suppose B is the branch set of b_i for some i. Then there is a path $v_1v_2...v_k$, $k \geq 3$, with $v_1 \in R_1$, $v_k \in R_2$, and $v_i \in B$ for $2 \leq i \leq k-1$. Let $B' = \{v_2\}$ and let $R_2' = R_2 \cup \{v_3, ..., v_{k-1}\}$. We can replace B with B' and R_2 with R_2' and still have an edge-based model of $K_{2,t}$. Note that if B includes vertices not in the path $v_2v_3...v_{k-1}$, replacing B with B' still results in an edge-based model of $K_{2,t}$. Hence without loss of generality we may assume that every branch set of a vertex b_i for $1 \leq i \leq t$ is a single vertex. Let $S = \{s_1, s_2, ..., s_t\}$ be the set of such vertices in G. We say $(R_1, R_2; S)$ represents a *standard (edge-based) $K_{2,t}$ minor*. Observe that G contains a $K_{2,t}$ minor if and only if G contains a standard $K_{2,t}$ minor. Note that the standard model also applies for $K_{2,t}$ minors rooted at two vertices in the branch sets of the vertices in the part of size two.

A *k-separation* in a graph G is a pair (H, K) of edge-disjoint subgraphs of G with $G = H \cup K$, $|V(H) \cap V(K)| = k$, $V(H) - V(K) \neq \emptyset$, and $V(K) - V(H) \neq \emptyset$.

Lemma III.1. *Suppose (H, K) is a 2-separation in a graph G with $V(H) \cap V(K) = \{x, y\}$. If G contains a $K_{2,t}$ minor $(R_1, R_2; S)$ with $t \geq 3$, then one of the following holds:*

(i) there exists a $K_{2,t}$ minor in $H + xy$

(ii) there exists a $K_{2,t}$ minor in $K + xy$

(iii) $x \in R_1$ and $y \in R_2$ (or vice versa)

Proof. Let $H' = H - \{x, y\}$ and $K' = K - \{x, y\}$. We consider the location of a minor with respect to x and y in G and assume we are not in the situation in (iii). We have some subset of $\{x, y\}$ either in R_1 or in R_2; the situations are symmetric so we consider the former. Observe that because the subgraph induced by R_i is connected for $i = 1, 2$, if $x, y \notin R_i$, then either $R_i \subseteq V(H')$ or $R_i \subseteq V(K')$.

First suppose $x, y \in R_1$. Without loss of generality, assume $R_2 \subseteq V(H')$; it follows that $S \subseteq V(H')$. Let $R_1' = R_1 - V(K')$; then $(R_1', R_2; S)$ is a $K_{2,t}$ minor in $H + xy$

and we have (i).

Next assume $x \in R_1$ and $y \notin R_1 \cup R_2$. Without loss of generality, assume $R_2 \subseteq V(H')$; it follows that $S \subseteq V(H') \cup \{y\}$. Let $R_1' = R_1 \cap V(H)$. Then $(R_1', R_2; S)$ is a $K_{2,t}$ minor in $H + xy$ and we have (i).

Finally assume $x, y \notin R_1 \cup R_2$. Without loss of generality, assume $R_1 \subseteq V(H')$; it follows that $S \subseteq V(H)$ and thus $R_2 \subseteq V(H')$. Now $(R_1, R_2; S)$ is a $K_{2,t}$ minor in $H + xy$ and we have (i). □

By a $K_{2,t}$ minor $(R_1, R_2; S)$ rooted at x and y, we mean $x \in R_1$ and $y \in R_2$. If part (iii) of Lemma III.1 holds, then the $K_{2,t}$ minor splits into two minors, K_{2,t_1} and K_{2,t_2}, both rooted at x and y where $t_1 + t_2 = t$. In particular for $K_{2,4}$ and $K_{2,5}$ minors, we will be concerned with rooted $K_{2,2}$ minors; we will describe the structure of graphs without rooted $K_{2,2}$ minors. Lino Demasi provides a description of $K_{2,2}$-minor-free graphs with all four vertices rooted in terms of disjoint paths in Lemma 2.2.2 of his thesis [8].

An *xy-outerplane embedding* is an embedding of a connected graph G in a closed disk D such that a Hamilton xy-path is contained in the boundary of D for $x, y \in V(G)$. The Hamilton xy-path is called the *outer path*. A graph is *xy-outerplanar*, or *path-outerplanar*, if it has an xy-outerplane embedding. Observe:

(1) Suppose G is xy-outerplanar, H is yz-outerplanar, and $V(G) \cap V(H) = \{y\}$. Then $G \cup H$ has an xz-outerplane embedding.

(2) If G has an xy-outerplane embedding, then $G + xy$ also has an xy-outerplane embedding.

A *block* of a graph G is a maximal connected subgraph of G without a cutvertex. Blocks are either 2-connected, K_2, or an isolated vertex. The *block-cutvertex tree* of G is a tree whose vertices are the blocks of G; two vertices are adjacent if the corresponding blocks in G intersect. We have the following lemmas:

Lemma III.2. *Suppose $G' = G + xy$ is a block. Then G has no $K_{2,2}$ minor rooted at x and y if and only if G is xy-outerplanar.*

Proof. ($\Leftarrow$): Assume an xy-outerplane embedding of G. Add a vertex z to G' in the outer face and adjacent to x and y; the resulting graph G'' is still outerplanar. If G' has a $K_{2,2}$ minor rooted at x and y, then G'' has a $K_{2,3}$ minor which is a contradiction since outerplanar graphs are $K_{2,3}$-minor-free.
($\Rightarrow$): Proceed by induction on $|E(G)|$. The base case for G is K_2 which has no $K_{2,2}$ minor rooted at x and y and is clearly xy-outerplanar. Now assume the claim holds for all graphs on $m \geq 1$ edges and suppose $|E(G)| = m + 1$ (and hence G' is 2-connected).

First assume there is a cutvertex v in G. Then G must consist of more than one block and since G' is 2-connected, x and y must be in separate blocks. Furthermore, since G' is 2-connected, the block-cutvertex tree of G must be a path $B_1v_1B_2v_2...v_{k-1}B_k$ where each B_i is a block of G, each v_i is a cutvertex in G, and $x \in B_1$, $y \in B_k$. Define $v_0 = x$ and $v_k = y$. Because G has no $K_{2,2}$ minor rooted at x and y, the block B_i of G has no $K_{2,2}$ minor rooted at v_{i-1} and v_i for $1 \leq i \leq k$. We thus can apply the inductive hypothesis to each block; each block B_i is $v_{i-1}v_i$-outerplanar. By Observation (1), the outerplane embeddings of the blocks can then be combined

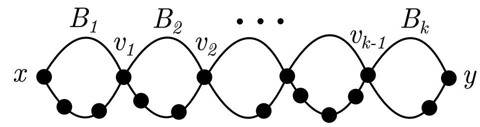


Figure 3.1

in such a way as to create an xy-outerplane embedding of G (see Figure 3.1).

Now assume there is no cutvertex in G (G is 2-connected). We delete an edge $e = u_1u_2$ in G, $e \neq xy$. Without loss of generality, suppose $x \neq u_1$. The graph $G - e$ clearly still has no $K_{2,2}$ minor rooted at x and y. We consider three cases:

Case 1. *The graph $G - e$ is 2-connected.*

By induction, $G - e$ has an xy-outerplane embedding. If we cannot add e to an xy-outerplane embedding of $G - e$ and create an xy-outerplane embedding of G, then there must be some edge w_1w_2 such that u_1, w_1, u_2, w_2 occur in that order along the outer path of G' (see Figure 3.2). This situation, however, yields the $K_{2,2}$ minor rooted at x and y and shown in Figure 3.2. Thus we can conclude G is xy-outerplanar.

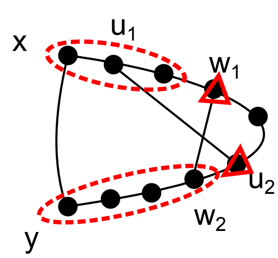


Figure 3.2

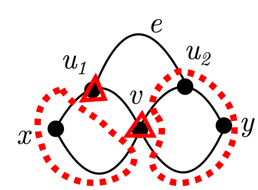


Figure 3.3

Case 2. *There is a cutvertex v in $G - e$ that separates x and y.*

The cutvertex v must also separate u_1 and u_2 because otherwise it would also be a cutvertex in G which is a contradiction since G is 2-connected. Assume that in $G - e$,

x and u_1 are in the same component and y and u_2 are in the same component. In this situation we can find a rooted $K_{2,2}$ minor: we take $S = \{v, u_1\}$ (see Figure 3.3). Then $x \in R_1$ and $y \in R_2$. We know there is a path from x to v which does not include u_1 in $G - e$ because otherwise u_1 would be a cutvertex of G. This path is included in R_1.

Case 3. *There is a cutvertex v in $G - e$ that does not separate x and y.*

Because there is no cutvertex in G, the block-cutvertex tree of $G-e$ must again be a path $B_1v_1B_2v_2...v_{k-1}B_k$ where each B_i is a block of $G-e$ and each v_i is a cutvertex in $G-e$. Note $k \geq 2$. Suppose $u_1 \in B_1$, $u_2 \in B_k$, and $x, y \in B_a$ for some $a \in \{2, ..., k\}$ (the situation $a = 1$ is symmetric to $a = k$ so without loss of generality, we exclude it). Note $u_1, u_2 \neq v_i$ for any i since then v_i would be a cutvertex in G. The block B_a is 2-connected with no $K_{2,2}$ minor rooted at x and y so B_a is xy-outerplanar; without loss of generality, suppose x, v_{a-1}, v_a, y occur in that order along the outer path of B_a. Then v_{a-1} and v_a are consecutive on the outer path because otherwise G has a $K_{2,2}$ minor rooted at x and y as shown in Figure 3.4. We have $x, v_{a-1} \in R_1$ and $y, v_a \in R_2$. The minor still exists even when $v_a = u_2$ (or symmetrically $v_{a-1} = u_1$) or $x = v_{a-1}$ (or symmetrically $y = v_a$). If B_i has a $K_{2,2}$ minor rooted at v_{i-1} and v_i for $i \in \{1, 2, ..., k\} - \{a\}$ (where $v_0 = u_1$ and $v_k = u_2$), then G has a $K_{2,2}$ minor rooted at x and y. Figure 3.5 illustrates the case with $i < a$, and $i > a$ is symmetric. Note the minor exists even when $a = k$ with u_2 playing the role of v_a. We now can apply the inductive hypothesis to all of the blocks; each block B_i has an $v_{i-1}v_i$-outerplane embedding. Then we can position the blocks in such a way as to create an xy-outerplane embedding of G as shown in Figure 3.6. Observation (1) allows us to arrange the blocks $B_{a-1}B_{a-2}, ..., B_1B_kB_{k-1}...B_{a+1}$ appropriately.

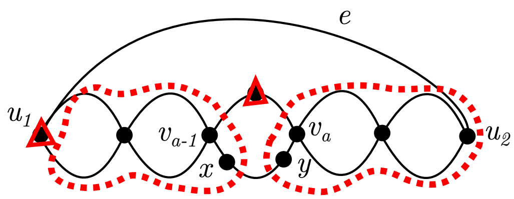


Figure 3.4

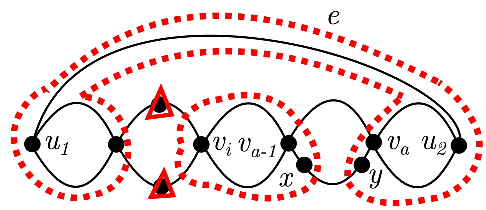


Figure 3.5

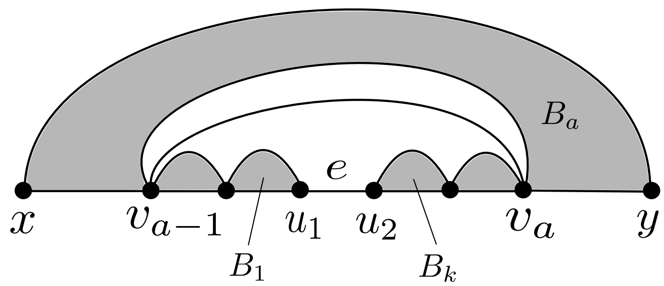


Figure 3.6

$\square$

At the end of Chapter IV, we use another result concerning $K_{2,t}$ minors and separations stated here:

Lemma III.3. *Suppose (H, K) is a 3-separation in a graph G with $V(H) \cap V(K) = \{x_1, x_2, x_3\}$. If G contains a standard $K_{2,t}$ minor (R_1, R_2, S) with $t \geq 4$, then one of the following holds:*

(i) There exists a $K_{2,t}$ minor in $H + x_1x_2 + x_2x_3 + x_1x_3$

(ii) There exists a $K_{2,t}$ minor in $K + x_1x_2 + x_2x_3 + x_1x_3$

(iii) $x_i \in R_1$ and $x_j \in R_2$ for some $i \neq j$

Proof. Let $H' = H - \{x_1, x_2, x_3\}$ and $K' = K - \{x_1, x_2, x_3\}$. We consider the location of a minor with respect to x_1, x_2 and x_3 in G and assume we do not have the situation in (iii). Then one of $R_1 \cap \{x_1, x_2, x_3\}$ or $R_2 \cap \{x_1, x_2, x_3\}$ is empty. Without loss of generality, assume $R_2 \cap \{x_1, x_2, x_3\} = \emptyset$. Then either $R_2 \subseteq V(H')$ or $R_2 \subseteq V(K')$ so without loss of generality, assume $R_2 \subseteq V(H')$. It follows that $S \subseteq V(H)$. Now if $R_1 \cap \{x_1, x_2, x_3\} \neq \emptyset$, then $((R_1 \cap V(H)) \cup (\{x_1, x_2, x_3\} - S), R_2; S)$ is a $K_{2,t}$ minor in $H + x_1x_2 + x_2x_3 + x_1x_3$ and we have (i). If $R_1 \cap \{x_1, x_2, x_3\} = \emptyset$, then the original minor is a minor in H and we have (i) again. $\square$

In Chapter IV, the following results concerning Hamilton paths in outerplanar and path-outerplanar graphs will be helpful.

Lemma III.4. *Let G be a 2-connected outerplanar graph. Let $x \in V(G)$ and let xy be an edge on the outer cycle Z of G. Then for some vertex t with $deg_G(t) = 2$, there exists a Hamilton path $xy...t$ in G.*

Proof. Fix a forward direction on Z and denote by v_1Zv_2 the forward path from v_1 to v_2 on Z. Proceed by induction on $|V(G)|$. In the base case, $G = K_3$ and the result is clear. Now assume the lemma holds for all graphs with at most $n-1$ vertices and assume $|V(G)| = n \geq 4$. Assume y follows x on Z and let $w \neq y$ be the other neighbor of x on Z. If $\deg_G(w) = 2$, then we take $t = w$ and xZw is a desired Hamilton path in G. Otherwise let v be a neighbor of w such that $vw \notin E(Z)$ (possibly $v = y$). Let G' be the subgraph of G induced by vZw; G' is a 2-connected vw-outerplanar graph with $|V(G')| \leq n-1$. By the inductive hypothesis, there exists a Hamilton path $Q = vw...t$ in G' where $\deg_{G'}(t) = \deg_G(t) = 2$. Now $xZv \cup Q$ is the desired path in G. $\square$

Corollary III.5. *Let G be an xy-outerplanar graph with outer path $x = v_1 v_2 \ldots v_n = y$, $n \geq 2$. Let k be such that $1 \leq k \leq n-1$. Then there exist two vertex disjoint paths $P = x \ldots v_s$ and $Q = y \ldots v_t$ such that $V(G) = V(P \cup Q)$ and the following two symmetric conditions hold:*

(i) $s \leq k$ and either $v_s = x$ or $deg_G(v_s) = 2$,

(ii) $t \geq k+1$ and either $v_t = y$ or $deg_G(v_t) = 2$.

Proof. Let ℓ be the largest integer such that $\ell \leq k$ and $\deg_G(v_\ell) = 2$; if no such ℓ exists choose $\ell = 1$. Let m be the smallest integer such that $m \geq k+1$ and $\deg_G(v_m) = 2$; if no such m exists, choose $m = n$. Suppose $m = \ell + 1$; then $P = v_1 v_2 \ldots v_\ell$ and $Q = v_m v_{m+1} \ldots v_n$ are desired paths. Now suppose $m \neq \ell + 1$ and let $I = \{\ell+1, \ell+2, \ldots, m-1\}$. Note that $\deg_G(v_i) \geq 3$ for all $i \in I$. Suppose there is a vertex v_i with $i \in I$ such that v_i is adjacent to v_j with $j \in I$ and $j \neq i \pm 1$. Without loss of generality, assume $j > i$. Then the subgraph of G induced by $v_i v_{i+1} \ldots v_j$ is 2-connected and outerplanar with $v_i v_j$ on the outer face so by Lemma III.4, there is a degree two vertex v_r with $\ell + 1 \leq r \leq m$. Because G is planar, v_r must also be degree two in G. This contradicts the choice of v_ℓ and v_m however, so no such v_i and v_j exist. Hence for all $i \in I$, v_i has a neighbor v_j with either $j \geq m+1$ or $j \leq \ell - 1$.

Suppose all vertices v_i with $i \in I$ have a neighbor v_j with $j \geq m+1$. Then let r be such that v_r is a neighbor of $v_{\ell+1}$ and $r \geq m+1$. By Lemma III.4 there is a path $R = v_r v_{\ell+1} \ldots v_t$ where v_t is a degree two vertex with $t \geq m \geq k+1$ and R includes all vertices v_i with $\ell + 1 \leq i \leq r$. Then $P = v_1 v_2 \ldots v_\ell$ and $Q = v_n v_{n-1} \ldots v_r \cup R$ are the desired paths. Symmetrically if all vertices v_i with $i \in I$ have a neighbor v_j with

$j \leq \ell - 1$, then we can find two desired paths.

Otherwise there is an integer $r \leq m-2$ such that $r = \max\{i \in I : v_i$ is adjacent to v_j with $j \leq \ell - 1\}$. Now since G is planar, for all $i \in I$ with $i \geq r+1$, v_i is adjacent to a vertex v_j where $j \geq m+1$. Let p be such that v_p is adjacent to v_r and $p \leq \ell - 1$. By Lemma III.4, there is a path $R = v_p v_r ... v_s$ where v_s is a degree two vertex with $s \leq k$ and R includes all vertices v_i with $p \leq i \leq r$. Now $v_1 v_2 ... v_{p-1} R$ is a desired path P. Symmetrically, take a neighbor v_q of v_{r+1} with $q \geq m+1$ and apply Lemma III.4 again to find a desired path Q. □

Corollary III.6. *Let G be an xy-outerplanar graph with $|V(G)| \geq 3$. Then for some vertex t with $deg_G(t) = 2$, there exists a Hamilton path $x...t$ in $G - y$.*

Proof. Let $k = n - 1$ in Corollary III.5 and the result follows. □

Chapter IV

HAMILTONICITY OF 3-CONNECTED, PLANAR, $K_{2,5}$-MINOR-FREE GRAPHS

In proving the main result of this chapter, the following results concerning 3-connectivity will be helpful. The first is a consequence of Theorem 7.2 in the paper by Halin.

Theorem IV.1 (Halin [17])**.** *Let G be a 3-connected graph with $|V(G)| \geq 5$. Then for every $v \in V(G)$ with deg$(v) = 3$, there is an edge e incident with v such that G/e is 3-connected.*

Lemma IV.2. *Let G be a 3-connected graph and suppose (H, K) is a 3-separation in G with $V(H) \cap V(G) = \{x, y, z\}$. Suppose $K' = K - V(H)$ is nonempty and connected, each of x, y, and z is adjacent to a vertex of K', and H is 2-connected. Let G' be the graph formed from G by contracting K' to a single vertex. Then G' is 3-connected. Furthermore, for every cycle Z' in G' there is a cycle Z in G with $|V(Z)| \geq |V(Z')|$.*

Proof. Let v be the vertex in G' formed from contracting K'. We claim that every pair of vertices in G' has three vertex-disjoint paths between them. By Menger's Theorem, it will follow that G' is 3-connected. We consider five different types of pairs of vertices.

First suppose $w_1, w_2 \in V(H) - \{x, y, z\}$; there are three internally disjoint paths from w_1 to w_2 in G: P_1, P_2, and P_3. If $V(P_i) \cap V(K') = \emptyset$ for $i = 1, 2, 3$, then

P_1, P_2, and P_3 are the desired paths in G'. If $V(P_i) \cap V(K') \neq \emptyset$ for some i, then $|V(P_i) \cap \{x, y, z\}| \geq 2$ since $\{x, y, z\}$ separates K' from H. Thus $V(P_i) \cap V(K') \neq \emptyset$ for at most one i. Suppose $V(P_1) \cap V(K') \neq \emptyset$. Then all vertices of $V(P_1) \cap V(K')$ are in a single subpath of P_1 which we replace by v to form a new path P_1'. Now P_1', P_2, and P_3 are the desired paths in G'.

Second consider $w_1 \in V(H) - \{x, y, z\}$ and $w_2 \in \{x, y, z\}$, say $w_2 = x$. If there are not three internally disjoint paths between w_1 and x in G', then there is a 2-cut $\{u_1, u_2\}$ that separates w_1 and x. Since there is no 2-cut in G, one of u_1 and u_2 must be v, say $u_2 = v$. Now, however, u_1 is a cutvertex in H which is a contradiction since H is 2-connected. Hence there are three internally disjoint paths between w_1 and x.

Third consider $w_1, w_2 \in \{x, y, z\}$, say $w_1 = x$ and $w_2 = y$. Because H is 2-connected, there are two internally disjoint paths P_1 and P_2 from x to y in H. Take $P_3 = xvy$. Then P_1, P_2, and P_3 are the desired paths in G'.

Fourth consider $w_1 \in V(H) - \{x, y, z\}$ and v. For any $w_2 \in V(K')$, there are three internally disjoint paths P_1, P_2, and P_3 from w_2 to w_1 in G. Without loss of generality, say $x \in V(P_1)$, $y \in V(P_2)$, and $z \in V(P_3)$. Form P_1' from P_1 by replacing the subpath $w_2...x$ with vx, form P_2' from P_2 by replacing the subpath $w_2...y$ with vy, and finally form P_3' from P_3 by replacing the subpath $w_2...z$ with vz. Now P_1', P_2', and P_3' are the desired paths in G'.

Finally consider $w_1 \in \{x, y, z\}$, say $w_1 = x$, and v. By a consequence of Menger's Theorem, there are internally disjoint paths from $\{y, z\}$ to x in H, say $P_1 = y...x$ and $P_2 = z...x$. Then $P_1' = vP_1$, $P_2' = vP_2$, and $P_3 = vx$ are the desired paths in G'.

Let Z' be a cycle in G'. If $vx, vy, vz \notin E(Z')$, then Z' is also a cycle in G. If

$\{vx, vy, vz\} \cap E(Z') \neq \emptyset$, then $|\{vx, vy, vz\} \cap E(Z')| = 2$ and without loss of generality, say $vx, vy \in E(Z')$. Then form Z from Z' by replacing the subpath xvy with a path from x to y via K'; such a path necessarily exists because K' is connected and each of x, y, and z is adjacent to a vertex of K'. Now $|V(Z)| \geq |V(Z')|$. $\square$

For a proper subgraph H of G, an *H-bridge* is a subgraph of G induced by one of the following: all edges of a component C of $G - V(H)$ together with all edges connecting C to H, or an edge xy with $x, y \in V(H)$.

We now state and prove the main theorem:

Theorem IV.3. *Let G be a 3-connected, planar, $K_{2,5}$-minor-free graph. Then G is Hamiltonian.*

Theorem IV.3 is proved by assuming G is not Hamiltonian, taking a longest cycle C in G and finding a contradiction with either a longer cycle or a $K_{2,5}$ minor.

Proof. Assume that G is not Hamiltonian and assume G is represented as a plane graph. Let C be a longest non-Hamilton cycle in G. Fix a forward direction on C that will be shown as clockwise in the figures. Denote by x^+ the vertex directly after the vertex x on C and by x^- the vertex directly before x. Define $C[x, y]$ to be the forward subpath of C from x to y which includes x and y. If $x = y$ then $C[x, y] = \{x\}$. Define $C(x, y) = C[x, y] - \{x, y\}$, $C(x, y] = C[x, y] - x$, and $C[x, y) = C[x, y] - y$. Define $[x, y] = V(C[x, y])$ and define (x, y) etc. similarly. Define $G[x, y]$ to be the subgraph of G induced by $[x, y]$ and define $G(x, y)$ etc. similarly. We say a vertex z is *between* x and y if $z \in (x, y)$. For any subpath P of C, we have an *$x - v$ jump out of P* if x is an internal vertex of P, $v \in V(C) - V(P)$, and there is a path from x to

v in G which intersects C only at x and v. Because G is 3-connected, there must be at least one jump out of $C[x, y]$ whenever x and y are not consecutive on C. A jump out of $[x, y]$ is understood to mean a jump out of $C[x, y]$.

Let D be a component of $G - V(C)$ with the most neighbors on C. Let $U = \{u_0, u_1, ..., u_{k-1}\}$ be the neighbors of D along C in forward order. Because G is 3-connected, $k \geq 3$. Let $S_i = C[u_{i-1}, u_i]$ with subscripts taken modulo k. We call these special paths *sectors* and a jump out of S_i for any i is called a *sector jump*; note that sector jumps do not intersect D. Let $U_i = (u_i, u_{i+1})$. If $U_i = \emptyset$ for some i, then there is a cycle longer than C: replace S_i with a path from u_{i-1} to u_i through D. Thus $U_i \neq \emptyset$ for all i and there is a sector jump out of every sector.

For a vertex $x \in V(C)$, define $\sigma(x)$ as follows: $\sigma(x) = i$ when $x \in U_i$ and $\sigma(x) = i + \frac{1}{2}$ when $x = u_i$. Define the *length* of a sector jump $x - y$, as $\min\{|\sigma(x) - \sigma(y)|, k - |\sigma(x) - \sigma(y)|\}$.

Claim 1. *For every sector jump $x - y$ of length greater than 1, there is a sector jump $x_1 - y_1$ of length 1 with $x_1, y_1 \in [x, y]$ and another sector jump $x_2 - y_2$ of length 1 with $x_2, y_2 \in [y, x]$.*

For any sector jump $u - v$, define the *linear length* as $|\sigma(u) - \sigma(v)|$. We claim that for any jump $x' - y'$ of linear length $\ell' > 1$, there is a jump $x'' - y''$ with $x'', y'' \in [x', y']$ such that $x'' - y''$ has linear length $\ell'' < \ell'$. There is a sector U_j such that $U_j \subset (x', y')$. Let $x'' - y''$ be any jump out of U_j. If $x'' - y''$ does not intersect $x' - y'$, then necessarily by planarity, $x'' - y''$ has linear length $\ell'' < \ell'$. If $x'' - y''$ intersects $x' - y'$, then $x'' - y'$ is a jump with linear length strictly less than ℓ' and we take this one. We may repeat this process until we reach a jump $x^* - y^*$ with $x^*, y^* \in [x', y']$ of linear length 1, and hence also length 1.

If we relabel $\{u_0, u_1, ..., u_{k-1}\}$ keeping the same cyclic order so that $x \in U_0$ and repeatedly apply the previous paragraph beginning with the jump $x - y$, we obtain the required jump $x_1 - y_1$. Similarly, relabeling so that $y \in U_0$ yields the jump $x_2 - y_2$. This completes the proof of Claim 1.

Claim 2. $k \leq 3$.

Assume that $k \geq 4$. If there is a bridge of C with attachments in the interiors of four or more sectors, then there is a $K_{2,5}$ minor similar to the one shown in Figure 4.1. When $k \geq 5$, there can be a bridge with attachments in the interior of three sectors that are not all in a row; then there is a $K_{2,5}$ minor similar to the one shown in Figure 4.2. If there are two bridges each with attachments in the interiors of three consecutive sectors, then there is a $K_{2,5}$ minor as shown in Figure 4.3.

Now suppose there is one bridge B with attachments in the interiors of three consecutive sectors, say $s_1 \in S_1$, $s_2 \in S_2$, and $s_3 \in S_3$. Then since $k \geq 4$, $s_1 - s_3$ is a jump of length greater than 1. Therefore by Claim 1, there is a sector jump $x - y$ of length 1 with endpoints between s_3 and s_1 along C, and with an endpoint in at most one of S_1 or S_3. Now there is a $K_{2,5}$ minor as shown in Figure 4.4. Now all bridges have attachments in the interiors of at most two sectors and therefore jumps of length 1 cannot intersect jumps of length at least 2, or any other jump of length 1 whose ends are not in the same two sectors.

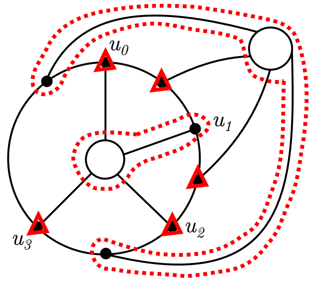


Figure 4.1

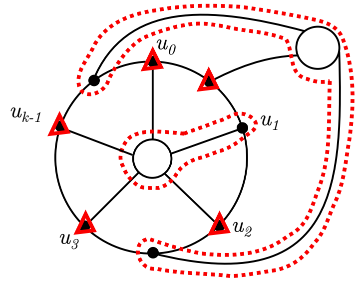


Figure 4.2

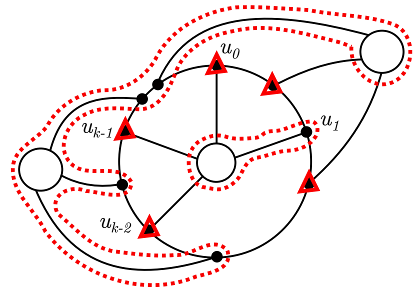


Figure 4.3

Case 2.1. *There is a jump of length at least* 2.

Let $x - y$ be a jump of length ≥ 2. Then by Claim 1, there is a sector jump of length 1, $x_1 - y_1$, between x and y and a sector jump of length 1, $x_2 - y_2$, between y and x. Let x, x_1, y_1, y, y_2, x_2 appear around C in that forward order. If x, x_1, and x_2 are all in distinct sectors, then there is a $K_{2,5}$ minor as shown in Figure 4.5. If x, x_1, and x_2 are all in the same sector, then there is a minor symmetric to the one shown in Figure 4.5; x, x_1, and x_2 switch roles with y, y_1, and y_2. If exactly two of x, x_1, and x_2 are in the same sector, say x and x_1, then we have the situation shown in Figure 4.6.

When $k \geq 5$, at least one of the vertices labeled u_i, u_j, and u_m must be present. With u_i, there is a $K_{2,5}$ minor as shown in Figure 4.7. With u_m, there is a $K_{2,5}$ minor symmetric to the one in Figure 4.7; $x_1 - y_1$ and $x_2 - y_2$ switch roles and u_0, u_m play the roles of u_i, u_2. With u_j, there is a $K_{2,5}$ minor as shown in Figure 4.8.

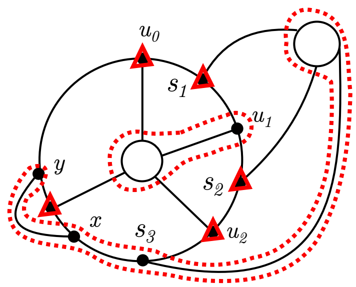


Figure 4.4

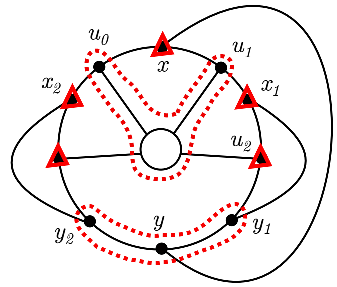


Figure 4.5

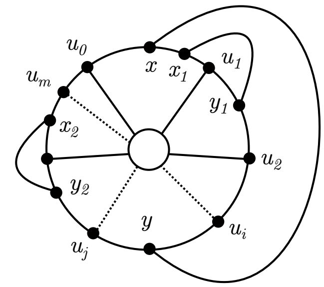


Figure 4.6

Now $k = 4$, so $x - y$ is a sector jump of length two with $x \in U_0$ and $y \in U_2$. Then by Claim 1 there are sector jumps of length 1 out of both U_1 and U_3, which cannot intersect the jump $x - y$. If these jumps both jump into U_0 or symmetrically U_2, then there is a $K_{2,5}$ minor as shown in Figure 4.9. Hence without loss of generality, there must be a $U_1 - U_0$ jump and a $U_3 - U_2$ jump. Let $x_1 - x_0$ be the $U_1 - U_0$ jump with x_0 closest to x (and possibly equal to x) and x_1 closest to u_1. Let $x_3 - x_2$ be the $U_3 - U_2$ with x_2 closest to y (and possibly equal to y) and x_3 closest to u_3. If $(x_0, u_0) = \emptyset$ and $(x_1, u_1) = \emptyset$, then there is a longer cycle as in Figure 4.10. If $(x_0, u_0) \neq \emptyset$, then there is a $K_{2,5}$ minor as in Figure 4.11; hence $(x_1, u_1) \neq \emptyset$. Let $r - r'$ be a jump out of $C[x_1, u_1]$. Because of the choice of $x_1 - x_0$, $r' \notin [x, x_0]$ and $r - r'$ does not intersect $x_1 - x_0$. Hence there are two options for r': $r' \in U_2$ or $r' \in [u_0, x_1)$. If $r' \in U_2$, then there are three sector jumps of length 1 and a $K_{2,5}$ minor similar to the one shown in Figure 4.12 exists. Thus $r' \in [u_0, x_1)$. By symmetric arguments, there is a jump $s - s'$ with $s \in (x_3, u_3)$ and $s' \in [u_2, x_3)$. Now there is a $K_{2,5}$ minor as shown in Figure 4.13; in the figure, $r' \neq u_0$ and $s' \neq u_2$ but a similar minor exists if $r' = u_0$ or $s' = u_2$.

Case 2.2. *All jumps have length at most* $1\frac{1}{2}$.

Assume first that $k \geq 5$. Since all sector jumps have length at most $1\frac{1}{2}$, by Claim 1 we can conclude there is a sector jump of length 1 out of every sector. Hence there are at least three sector jumps of length 1 no two of which have endpoints in the same two sectors and there is a $K_{2,5}$ minor similar to the one shown in Figure 4.12.

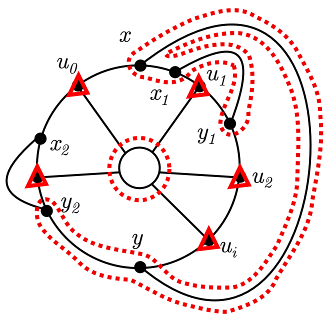


Figure 4.7

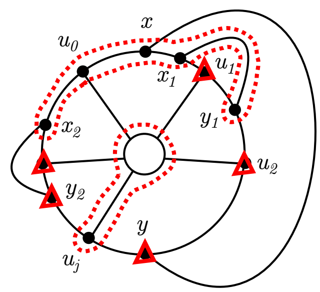


Figure 4.8

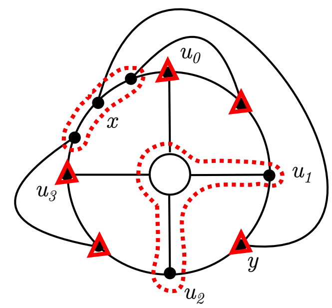


Figure 4.9

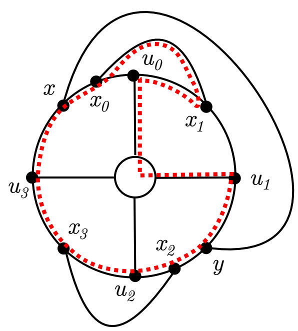


Figure 4.10

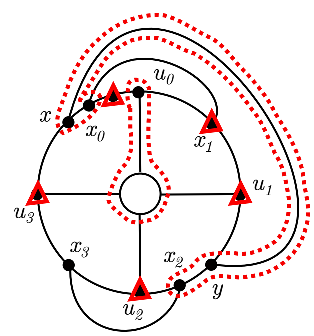


Figure 4.11

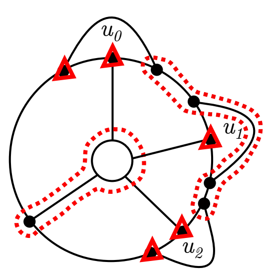


Figure 4.12

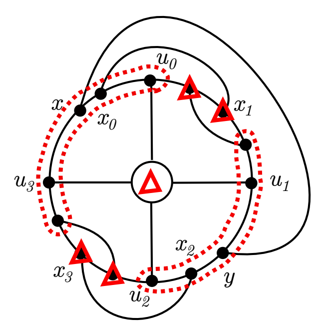


Figure 4.13

Now $k = 4$. Without loss of generality, assume there are sector jumps $S_0 - S_1$ and $S_2 - S_3$. If there is a sector jump $S_1 - S_2$ or $S_3 - S_0$, then there is a $K_{2,5}$ minor similar to the one shown in Figure 4.12, so there are no such jumps. Let $x_0 - x_1$ be the sector jump $S_0 - S_1$ such that x_0 is closest to u_3 and x_1 is closest to u_1. Similarly, let $x_2 - x_3$ be the sector jump $S_2 - S_3$ such that x_2 is closest to u_1 and x_3 is closest to u_3. If $(x_0, u_0) = \emptyset$ and $(x_1, u_1) = \emptyset$, then there is a longer cycle similar to the one shown in Figure 4.10. There are three additional possible longer cycles symmetric to the one shown and hence we consider four pairs of sets of vertices along C: $A_0 = (x_0, u_0)$ and $B_1 = (x_1, u_1)$, $B_0 = (u_3, x_0)$ and $A_1 = (u_0, x_1)$, $B_2 = (u_1, x_2)$ and $A_3 = (u_2, x_3)$, and $A_2 = (x_2, u_2)$ and $B_3 = (x_3, u_3)$. At least one set from each pair must be nonempty to avoid a longer cycle. If $A_i \neq \emptyset$ for at least three i, then there is a $K_{2,5}$ minor as shown in Figure 4.14 or symmetric to this one. Thus $A_i \neq \emptyset$ for at most two i and therefore $B_j \neq \emptyset$ for at least two j. Note that because of the choice of $x_0 - x_1$, no jump out of B_0 or B_1 can intersect $x_0 - x_1$ and similarly because of the choice of $x_2 - x_3$, no jump out of B_2 or B_3 can intersect $x_2 - x_3$.

If there is a $K_{2,2}$ minor in $G[u_3, u_1]$ rooted at u_3 and u_1 and another $K_{2,2}$ minor in $G[u_1, u_3]$ rooted at u_1 and u_3, then there is a standard $K_{2,5}$ minor $(R_1, R_2; S)$ in G: S consists of a vertex of D, two vertices reachable by both u_3 and u_1 in $G[u_3, u_1]$ and two vertices reachable by both u_3 and u_1 in $G[u_1, u_3]$. Any jump leaving one of the A_i or B_j that is inside C creates such a rooted $K_{2,2}$ minor. Hence if there is any such inside jump in either S_0 or S_1, then there cannot be such an inside jump in S_2 or S_3 and vice versa. Let a *bad pair* of jumps be two inside jumps that create two rooted $K_{2,2}$ minors and hence a $K_{2,5}$ minor as just described.

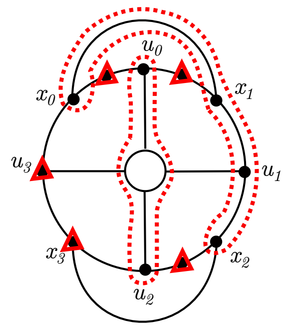


Figure 4.14

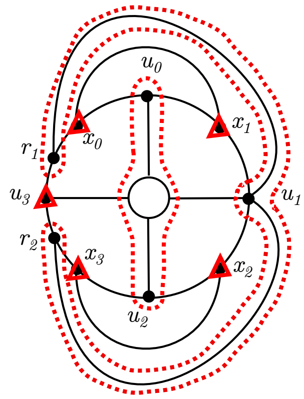


Figure 4.15

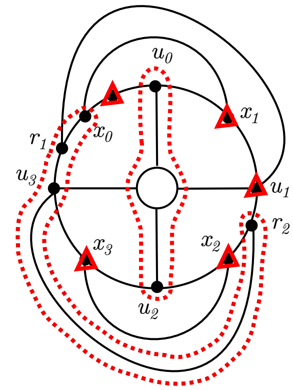


Figure 4.16

Suppose there are sector jumps of length $1\frac{1}{2}$ out of two nonempty B_j. Then by planarity of G, up to symmetry there are two options for the two jumps: $r_1 - u_1$ and $r_2 - u_1$ with $r_1 \in B_0$ and $r_2 \in B_3$ or $r_1 - u_1$ and $r_2 - u_3$ with $r_1 \in B_0$ and $r_2 \in B_2$. First suppose the former. Then there is a $K_{2,5}$ minor as shown in Figure 4.15. Next suppose the latter. Then if both B_1 and B_3 are nonempty, by planarity and the choice of $x_0 - x_1$ and $x_2 - x_3$, each must have an inside jump out of them; then there is a bad pair of jumps. Hence at most one is nonempty and therefore one of A_0 and A_2 is nonempty. Without loss of generality, suppose $A_0 \neq \emptyset$; then there is a $K_{2,5}$ minor as shown in Figure 4.16. Now at most one B_j has jumps of length $1\frac{1}{2}$ out of it.

Since $B_j \neq \emptyset$ for at least two j and at most one B_j has jumps of length $1\frac{1}{2}$ out of it, some nonempty B_j, say B_0, has no jump of length $1\frac{1}{2}$, and hence has an inside jump. Observe now that if $G[u_3, x_0]$ contains a $K_{2,2}$ minor rooted at u_3 and x_0, then there is a $K_{2,5}$ minor as shown in Figure 4.17.

Consider the structure of the A_i. If A_0 is empty, then all jumps from B_0 must to go u_0. If A_1 is also empty, then apply Corollary III.6 to $G[u_3, x_0]$ to get a path $P = x_0 ... t$ such that $V(P) = (u_3, x_0]$ and t is a degree two vertex in $G[u_3, x_0]$. Because all jumps from B_0 go to u_0, t must be adjacent to u_0. Now using P, there is a longer cycle shown in Figure 4.18. The thick shaded line between u_3 and x_0 in the figure represents the path P. Hence either A_0 or A_1 is nonempty.

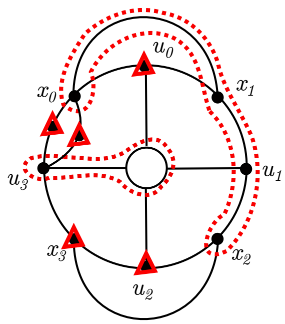


Figure 4.17

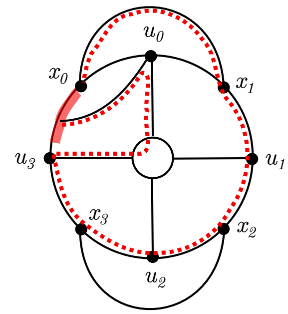


Figure 4.18

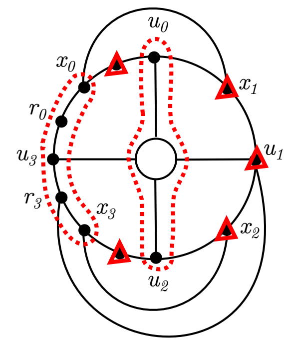


Figure 4.19

Because at most two A_i are nonempty and at least one of A_0 and A_1 is nonempty, at most one of A_2 and A_3 is nonempty. Therefore at least one of B_2 and B_3 is nonempty. Because there is an inside jump in S_0, there cannot be an inside jump in S_2 or S_3 else there would be a bad pair of jumps. Hence there must be a sector jump of length $1\frac{1}{2}$ out of either B_2 or B_3.

Suppose first there is a sector jump $r_3 - u_1$ with $r_3 \in B_3$. There cannot be an inside jump out of B_2 and there cannot be another jump of length $1\frac{1}{2}$ out of B_2 hence $B_2 = \emptyset$ and therefore $A_3 \neq \emptyset$. If $A_0 \neq \emptyset$, then there is a $K_{2,5}$ minor as shown in Figure 4.19. If $A_1 \neq \emptyset$, then there is a $K_{2,5}$ minor as shown in Figure 4.20.

Next suppose there is a sector jump $r_2 - u_3$ with $r_2 \in B_2$. Then since the minors in Figures 4.19 and 4.20 do not use the edge r_0u_0, symmetric minors exist in this situation when either A_0 or A_1 is nonempty. This ends the proof of Claim 2.

Henceforth we assume $k = 3$.

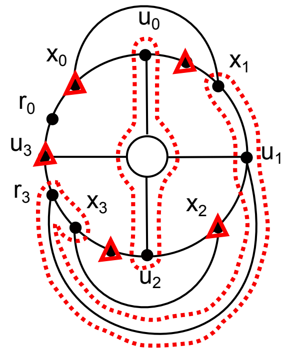


Figure 4.20

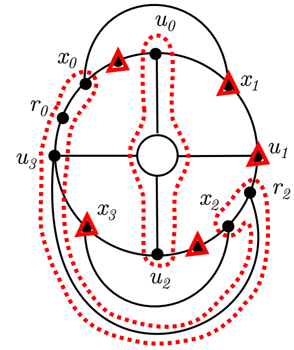


Figure 4.21

Claim 3. *Without loss of generality, we may assume all bridges of C other than the bridge containing D are single edges, and D is a single degree three vertex.*

Let $D = D_0, D_1, D_2, ..., D_\alpha$ be the components of $G - V(C)$ and denote by D_i^+ the bridge of C corresponding to D_i for each i, $0 \le i \le \alpha$. Because G is 3-connected and D is a component of $G - V(C)$ with the most neighbors along C, $|V(D_i^+) \cap V(C)| = 3$ for all i. Let $G = G_0$ and let $H_0 = G_0 - V(D_0)$. If H_0 is not 2-connected, then there is a cutvertex u. Now $u \notin V(C)$ and $V(C)$ must be entirely in one component of $H_0 - u$. Since the attachment vertices of D_0^+ are all on C, vertices of D_0^+ are only adjacent to vertices on one side of the cut. Hence u is also a cutvertex in $G_0 = G$, which is a contradiction. Thus H_0 is 2-connected. Now let $G_1 = G_0/E(D_0)$. Then by Lemma IV.2, G_1 is 3-connected.

Repeat this process for $i = 0, 1, ..., \alpha$. For each i, form G_{i+1} from G_i by contracting D_i^+ to a single vertex d_i. Let $G' = G_{\alpha+1}$ which is 3-connected. At each step, apply the second part of Lemma IV.2 to conclude that any cycle Z' in G' corresponds to a cycle Z in G with $|V(Z)| \ge |V(Z')|$.

By Theorem IV.1, there is an edge e_1 incident with d_1 such that the graph formed from G' by contracting e_1 is 3-connected. Let G'_1 be this graph. Any cycle Z'' in G'_1 corresponds to a cycle Z' in G' with $|V(Z')| \geq |V(Z'')|$. To see this, let x, y, and z be the neighbors of d_1 in G' and suppose $e_1 = d_1 z$. Call the vertex that results from the contraction z. If $xz, yz \notin E(Z'')$, then take $Z' = Z''$. If $|\{xz, yz\} \cap E(Z'')| = 1$, say $xz \in E(Z'')$, form Z' from Z'' by replacing xz with the path xd_1z. If $xz, yz \in E(Z'')$, form Z' from Z'' by replacing the subpath xzy with xd_1y. Note in all cases, $|V(Z')| \geq |V(Z'')|$. Apply Theorem IV.1 repeatedly to find contractible edges e_{i+1} incident with d_{i+1}, and continue this process of forming new graphs by contracting these edges. For $1 \leq i \leq \alpha - 1$, let G'_{i+1} be the graph formed from G'_i by contracting the edge e_{i+1} incident with d_{i+1}. Now any cycle W in G'_α corresponds to a cycle Z' in G' and hence a cycle Z in G with $|V(Z)| \geq |V(Z')| \geq |V(W)|$. Furthermore, since G'_α is a minor of G, any $K_{2,5}$ minor in G'_α corresponds to a $K_{2,5}$ minor in G. Thus without loss of generality, take $G = G'_\alpha$; this proves Claim 3.

We are now in the general situation an example of which is shown in Figure 4.22. There are three sectors labeled S_0, S_1, and S_2. Let $t_0 - t_1$ be the $S_0 - S_1$ sector jump (if any $S_0 - S_1$ sector jump exists) with t_0 closest to u_2 and t_1 closest to u_1. Similarly let $t_2 = t_3$ be the $S_1 - S_2$ sector jump (if any $S_1 - S_2$ sector jump exists) with t_2 closest to u_0 and t_3 closest to u_2. Use $t_4 - t_5$ for $S_2 - S_0$ sector jumps. Because every sector must have a jump out of it, there are at least two sector jumps; without loss of generality, assume there are sector jumps $t_0 - t_1$ and $t_2 - t_3$. Define $X_0 = (t_0, u_0)$, $X_1 = (u_0, t_1)$, $X_2 = (t_2, u_1)$, $X_3 = (u_1, t_3)$, $X_4 = (t_4, u_2)$, and $X_5 = (u_2, t_5)$. Note these sets are not defined when the necessary t_i vertices do not exist.

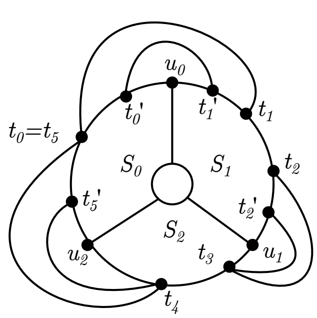


Figure 4.22

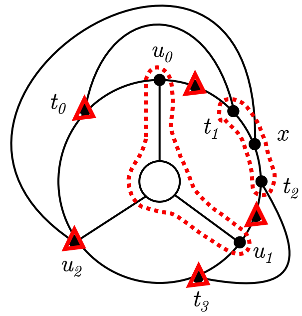


Figure 4.23

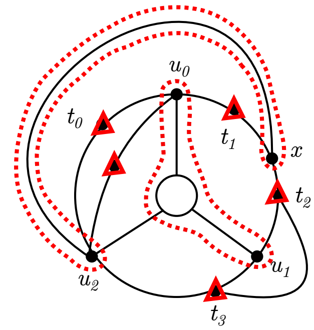


Figure 4.24

Claim 4. *There are no sector jumps $u_2 - x$ where $x \in (t_1, t_2)$.*

Let $u_2 - x$ be a sector jump with $x \in (t_1, t_2)$. If $(u_0, t_1) \neq \emptyset$ and $(t_2, u_1) \neq \emptyset$, then there is a $K_{2,5}$ minor as shown in Figure 4.23. Now at least one of (u_0, t_1) and (t_2, u_1) is empty and without loss of generality, assume $(u_0, t_1) = \emptyset$. If there is a $K_{2,2}$ minor in $G[u_2, u_0]$ rooted at u_2 and u_0, then there is a $K_{2,5}$ minor as shown in Figure 4.24. Now all jumps out of (u_2, u_0) must go to t_1. Apply Corollary III.6 to $G[u_2, u_0]$ to find a path $P = u_0 ... t$ such that $V(P) = (u_2, u_0]$ and t is a degree two vertex in $G[u_2, u_0]$ and hence must be adjacent to t_1. Now using P, there is a longer cycle as shown in Figure 4.25. This completes the proof of Claim 4.

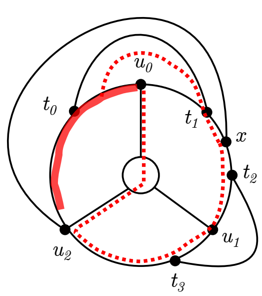


Figure 4.25

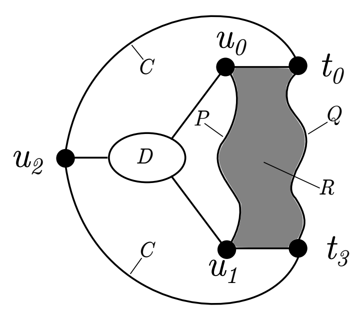


Figure 4.26

Claim 5. *Either $t_0 \neq u_0^-$ or $t_3 \neq u_1^+$ (X_0 and X_3 cannot both be empty).*

Assume that $t_0 = u_0^-$ and $t_3 = u_1^+$. Consider the representation of the graph shown in Figure 4.26 and focus on the portion of the graph in the shaded region $R = G[t_0, t_3]$. If either u_0u_1 or t_0t_3 exist, remove them. The graph is still 3-connected because u_0 and u_1 and t_0 and t_3 each have three vertex disjoint paths between them without these edges as can be seen in the figure and hence we have not created a 2-cut. Let P be the path from u_0 to u_1 along the outer face of R and Q be the path from t_0 to t_3 along the outer face of R; note all vertices of R are enclosed by these paths together with u_0t_0 and u_1t_3. P and Q are both paths, without repeated vertices, because any repeated vertex would be a cutvertex in G but G is 3-connected. Additionally, $|V(P)| \geq 3$ because we have removed the edge u_0u_1 if it existed and $|V(Q)| \geq 3$ because $t_1, t_2 \in V(Q)$ (and possibly $t_1 = t_2$).

The paths P and Q may intersect but only in limited ways. If P and Q intersect at two non-consecutive vertices on C, then using Claim 4 these two vertices would form a 2-cut in G. Hence there are three possibilities for P and Q: $V(P) \cap V(Q) = \{x, x^+\}$, $V(P) \cap V(Q) = \{x\}$, or $V(P) \cap V(Q) = \emptyset$.

First assume $V(P) \cap V(Q) = \{x, x^+\}$. We will show that there is a longer cycle as shown in Figure 4.27. Let $R_1 = G[t_0, x]$ and $R_2 = G[x^+, t_3]$. Note that since $t_0t_1, t_2t_3 \in E(G)$, $t_1 \in V(R_1)$ and $t_2 \in V(R_2)$. We will construct two paths P_1 and Q_1 such that $P_1 = u_0...x$ and $Q_1 = t_0...x$, $V(P_1 \cup Q_1) = V(R_1)$, and $V(P_1) \cap V(Q_1) = x$. Represent the portion of P in R_1 as $(u_0 = p_0)p_1...p_{r-1}(p_r = x)$ and similarly represent the part of Q in R_1 as $(t_0 = q_0)q_1...q_{s-1}(q_s = x)$. If $s = 1$ (i.e. $t_0x \in E(G)$), then let $P_1 = t_0x$ and $Q_1 = C[u_0, x]$. Hence assume $s \geq 2$.

Now let $P_1 = u_0...x$ and $Q_1 = t_0...x$ be paths in R_1 disjoint except at x, such that $|V(Q_1)| \geq 3$ and $t_1 \in V(Q_1)$. Such paths necessarily exist since we can take $P_1 = p_0...p_r$ and $Q_1 = q_0...q_s$. Additionally assume $|V(P_1) \cup V(Q_1)|$ is maximum. Suppose $V(P_1 \cup Q_1) \neq V(R_1)$ and let K be a component of $R_1 - V(P_1 \cup Q_1)$. Because G is 3-connected, K must have three neighbors. By planarity of G, K must have three neighbors in $V(P_1 \cup Q_1) \cup \{u_2\}$. By Claim 4, there is no sector jump from u_2 to any vertex of R except possibly t_1 or t_2 and $t_1, t_2 \notin V(K)$ so K has three neighbors in $V(P_1) \cup V(Q_1)$. If either t_1 or t_2 is in K, while they may be adjacent to u_2, they each have three additional neighbors in R, hence K must have three neighbors in $V(P_1 \cup Q_1)$. Furthermore, K must have two neighbors in one of P_1 or Q_1. Suppose K is adjacent to w_1 and w_2 on either P_1 or Q_1. If w_1 and w_2 are consecutive on either P_1 or Q_1, then there is a longer path P_1 or Q_1: replace the edge w_1w_2 with a path from w_1 to w_2 through K. Hence w_1 and w_2 are not consecutive.

Suppose $w_1, w_2 \in V(Q_1)$ and let w_3 be a vertex between them along Q_1. The vertex w_3 together with a vertex from K form a $K_{2,2}$ minor in R_1 rooted at t_0 and x and hence there is a $K_{2,5}$ minor in G as shown in Figure 4.28. Now suppose $w_1, w_2 \in V(P_1)$ and let w_3 again be a vertex between them. The vertex w_3 together with a vertex from K form a $K_{2,2}$ minor in R_1 rooted at u_0 and x. Now take an interior vertex of Q_1, which exists because $|V(Q_1)| \geq 3$, to form a $K_{2,3}$ minor in R_1 rooted at x and t_0 and hence a $K_{2,5}$ minor in G similar to the one shown in Figure 4.28. Thus no such component K exists, $V(P_1 \cup Q_1) = V(R_1)$, and P_1 and Q_1 are desired paths in R_1.

By symmetric arguments, there is a path $P_2 = u_1...x^+$ in R_2 and a path $Q_2 = t_3...x^+$ in R_2 such that $V(P_2 \cup Q_2) = V(R_2)$ and $V(P_2) \cap V(Q_2) = x^+$ and hence there is a longer cycle as shown in Figure 4.27.

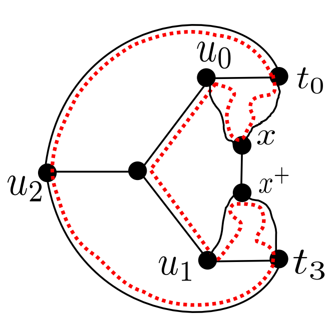


Figure 4.27

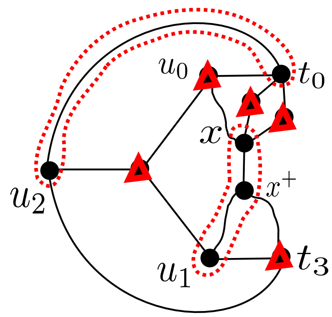


Figure 4.28

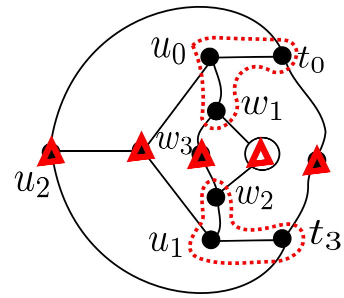


Figure 4.29

Now assume $V(P) \cap V(Q) = \{x\}$. We will show that this case can be reduced to the previous one in which $V(P) \cap V(Q) = \{x, x^+\}$. First assume $x^+ = u_1$. Then there is a longer cycle similar to the one shown in Figure 4.27: the subpath from u_1 to t_3 through x^+ becomes the edge u_1t_3 since $u_1 = x^+$. Now let $R_1 = G[t_0, x]$ and $R_2 = G[x^+, t_3]$. Then $t_1 \in V(R_1)$ and $t_2 \in V(R_2) \cup \{x\}$. We claim there are internally vertex-disjoint paths $x^+ ... u_1$ and $x^+ ... t_3$ in R_2, namely segments of the outer face of R_2. Suppose not and assume v is the first intersection vertex along C from x^+ of the two paths along the outer face of R_2. Since t_2 is adjacent to t_3, $t_2 \notin \{x, v\}$; also $t_2 \notin V(R_2)$ so by Claim 4, $\{v, x\}$ is a 2-cut in G separating x^+ and $C(v, x)$. Delete all edges xz where $z \in V(R_2) - \{x^+\}$, and now we can apply the arguments of the previous case with $V(P) \cap V(Q) = \{x, x^+\}$. (If $t_2 = x$, we drop the condition $t_2 \in V(Q_2)$ which just ensures that any component K of $R_2 - V(P_2 \cup Q_2)$ does not have a jump from u_2.)

Finally suppose $V(P) \cap V(Q) = \emptyset$. Let P' and Q' be disjoint paths in R from u_0 to u_1 and from t_0 to t_3 respectively such that $|V(P')|, |V(Q')| \geq 3$. Such paths necessarily exist because we can take $P' = P$ and $Q' = Q$. Assume additionally that $|V(P') \cup V(Q')|$ is maximum. Suppose $V(P') \cup V(Q') \neq V(R)$ and let K be a component of $R - V(P' \cup Q')$. Because G is 3-connected, K has three neighbors. Furthermore, because G is planar, K must have three neighbors in $V(P' \cup Q') \cup \{u_2\}$. By Claim 4, there are no jumps from u_2 to any vertex of R except possibly t_1 and t_2, and $t_1, t_2 \notin K$. Hence K must have three neighbors in $V(P' \cup Q')$. Without loss of generality, suppose K is adjacent to w_1 and w_2 where $w_1, w_2 \in V(P')$. If w_1 and w_2 are consecutive along P', then there is a path longer than P' from u_0 to u_1: replace w_1w_2 on P' with a path from w_1 to w_2 through K. Hence w_1 and w_2 are not consecutive and there is a vertex w_3 between them on P'. Now there is a $K_{2,5}$ minor in G as shown in Figure 4.29; the vertex on Q' is necessarily there because $|V(Q')| \geq 3$. Hence $V(P') \cup V(Q') = V(R)$.

Now we construct a longer cycle using P' and Q'. If $(u_2, t_0) = \emptyset$ or symmetrically $(t_3, u_2) = \emptyset$, then there is a longer cycle as shown in Figure 4.30; hence $(u_2, t_0) \neq \emptyset$ and $(t_3, u_2) \neq \emptyset$. If there is a $K_{2,2}$ minor in $G[u_2, t_0]$ rooted at u_2 and t_0, then there is a $K_{2,5}$ minor as shown in Figure 4.31. Suppose that all jumps out of $[u_2, t_0]$ go to u_0. Then apply Corollary III.6 to $G[u_2, t_0]$ to find a path $P = t_0 ... t$ such that $V(P) = (u_2, t_0]$ and t is a degree two vertex in $G(u_2, t_0)$ and hence must be adjacent to u_0 by our assumption. Now using P there is a longer cycle similar to the one shown in Figure 4.30: replace t_0u_0 by $P \cup tu_0$. Now all jumps out of $[u_2, t_0]$ do not all go to u_0 and there is a jump $x_1 - x_2$ with $x_2 \in [t_3, u_2)$; without loss of generality, let $x_1 - x_2$ be the $[u_2, t_0] - [t_3, u_2)$ jump with x_1 closest to but not equal to t_0 and x_2 closest to (and possibly equal to) t_3.

If there is a jump out of $[u_2, x_1]$ to $(x_1, u_0]$, then there is a $K_{2,5}$ minor as shown in Figure 4.32. A symmetric minor exists if there is a jump out of $[x_2, u_2]$ to $[u_1, x_2)$. Hence the only vertex that is the endpoint of a jump out of $C[x_2, x_1]$ is u_2. If there is a $K_{2,2}$ minor in $G[x_2, x_1]$ rooted at x_2 and x_1, then there is a $K_{2,5}$ minor as shown in Figure 4.31. Furthermore, $G[x_2, x_1]$ is 2-connected because it is a circuit graph in a 3-connected graph. Now apply Lemma III.4 to $G[x_2, x_1]$ to find a path $P_1 = x_2x_1...t$ where $V(P_1) = [x_2, x_1]$ and t is a degree two vertex in $G(x_2, x_1)$ and hence must be u_2. If there is a $K_{2,2}$ minor in $G[x_1, t_0]$ rooted at x_1 and t_0, then there is a $K_{2,5}$ minor similar to the one shown in Figure 4.31. Now apply Corollary III.6 to $G[x_1, t_0]$ to find a path $P_2 = t_0...s$ where $V(P_2) = (x_1, t_0]$ and s is a degree two vertex in $G[x_1, t_0]$ and hence must be adjacent to u_0. Using P_1 and P_2, there is a longer cycle as shown in Figure 4.33.

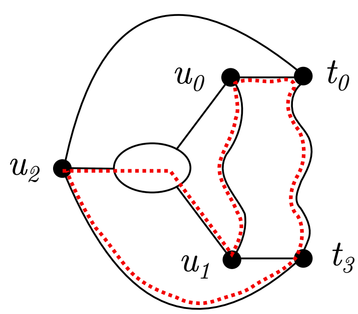


Figure 4.30

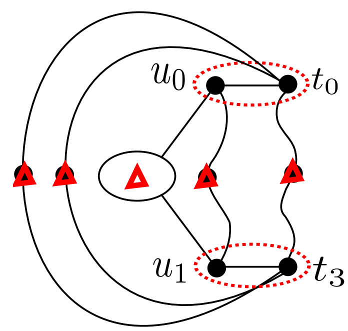


Figure 4.31

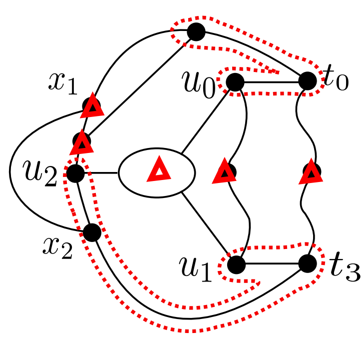


Figure 4.32

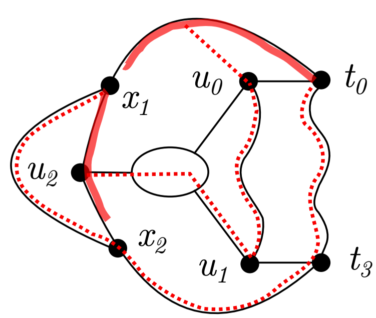


Figure 4.33

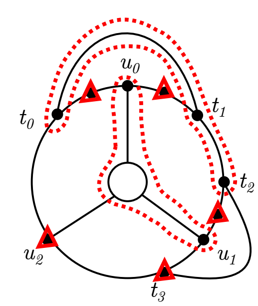


Figure 4.34

This completes the proof of Claim 5.

Claim 6. *Either $t_1 = u_0^+$ or $t_2 = u_1^-$ (at least one of X_1 and X_2 is empty).*

Assume that $t_1 \neq u_0^+$ and $t_2 \neq u_1^-$. By Claim 5, either $t_0 \neq u_0^-$ or $t_3 \neq u_1^+$. Without loss of generality, suppose $t_0 \neq u_0^-$. Then there is a $K_{2,5}$ minor shown in Figure 4.34.

Claim 7. *At most two pairs of sectors have jumps between them.*

Assume that there are three sector jumps $t_0 - t_1$, $t_2 - t_3$, and $t_4 - t_5$ where possibly $t_0 = t_5$, $t_1 = t_2$, or $t_3 = t_4$. By Claim 5, X_0 and X_3 cannot both be empty and symmetrically, X_1 and X_4 cannot both be empty and X_2 and X_5 cannot both be empty. Hence $X_i \neq \emptyset$ for at least three i. By Claim 6, at least one of X_1 and X_2 is empty and symmetrically, at least one of X_3 and X_4 is empty and at least one of X_5 and X_0 is empty. Hence $X_i \neq \emptyset$ for exactly three i. Furthermore, the nonempty X_i must be rotationally symmetric about C. Without loss of generality, suppose X_0, X_2, and X_4 are nonempty and X_1, X_3, and X_5 are empty.

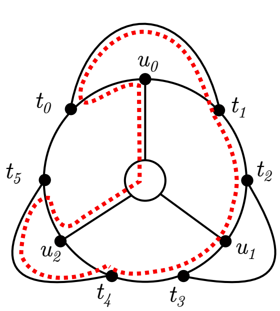


Figure 4.35

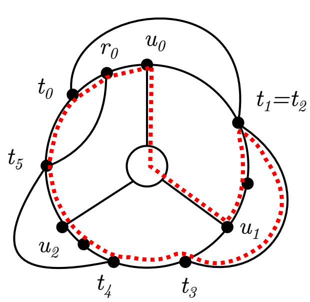


Figure 4.36

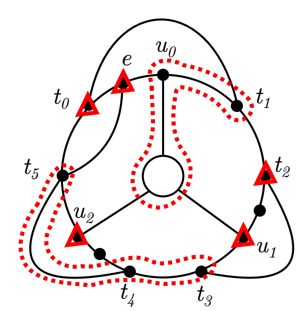


Figure 4.37

If $t_1 = t_2$, then there is a longer cycle as shown in Figure 4.36. A symmetric longer cycle exists if $t_3 = t_4$ or if $t_5 = t_0$ hence these vertices must be distinct. Now consider a jump $r_0 - r_0'$ out of X_0. There are three options for r_0': $r_0' \in [t_5, t_0)$, $r_0' = t_1$, or $r_0' = u_2$. Suppose first that $r_0' \in [t_5, t_0)$. Since $t_1 \neq t_2$, there is a $K_{2,5}$ minor as shown in Figure 4.37.

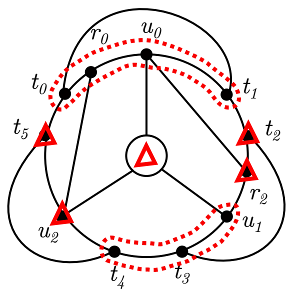


Figure 4.38

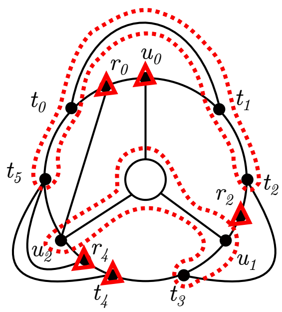


Figure 4.39

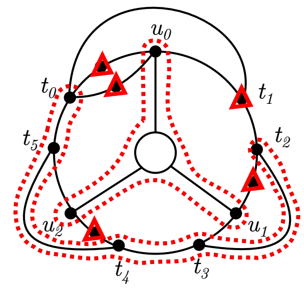


Figure 4.40

Now r_0' is either t_1 or u_2 and symmetrically for a jump $r_2 - r_2'$ out of X_2, r_2' is either u_0 or t_3 and for a jump $r_4 - r_4'$ out of X_4, r_4' is either u_1 or t_5. If at least two of r_0', r_2' and r_4' are u_i, then there is a $K_{2,5}$ minor similar to the one shown in Figure 4.38. If only one of r_0', r_2', and r_4' is a u_i, then there is a $K_{2,5}$ minor shown in Figure 4.39. Hence all three jumps must be outside C: $r_0' = t_1$, $r_2' = t_3$, and $r_4' = t_5$. If there is a $K_{2,2}$ minor in $G[t_0, u_0]$ rooted at t_0 and u_0, then there is a $K_{2,5}$ minor as shown in Figure 4.40. Hence there is no such rooted $K_{2,2}$ minor and symmetrically, there are no rooted $K_{2,2}$ minors in $G[t_2, u_1]$ or $G[t_4, u_2]$. Because all jumps from X_4 go to t_5, we can apply Corollary III.6 to $G[t_4, u_2]$ and find a path $P = t_4 ... t$ where t is adjacent to t_5 and $[t_4, u_2) = V(P)$. Now if $(t_5, t_0) = \emptyset$, then there is a longer cycle similar to the one shown in Figure 4.35: replace the edge t_5t_4 by $P \cup tt_5$. Hence $(t_5, t_0) \neq \emptyset$ and symmetrically $(t_1, t_2) \neq \emptyset$ and $(t_3, t_4) \neq \emptyset$.

Let $y - y'$ be a jump out of $[t_5, t_0]$. There are three possibilities for y': $y' \in (t_0, u_0)$,

$y' = u_0$, or $y' = u_2$. If $y' \in (t_0, u_0)$, then there is a $K_{2,5}$ minor similar to the one shown in Figure 4.37. If $y' = u_0$, there is a $K_{2,5}$ minor shown in Figure 4.41; a similar minor exists when $y' = u_2$ (replacing edge yu_0 by yu_2). This completes the proof of Claim 7.

Henceforth we assume there are jumps $t_0 - t_1$ and $t_2 - t_3$, but not $t_4 - t_5$.

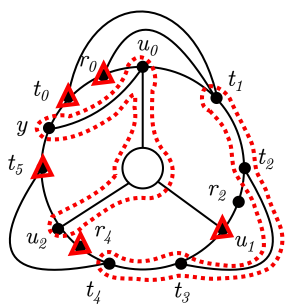


Figure 4.41

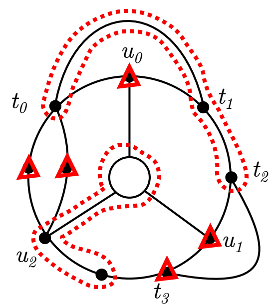


Figure 4.42

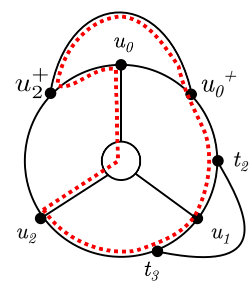


Figure 4.43

The graph in Figure 4.43 shows a longer cycle that exists if $t_0 = u_2^+$ and t_0 is adjacent to u_0^+. There is a symmetric longer cycle if $t_3 = u_2^-$, and t_3 is adjacent to u_1^-.

Claim 8. *Either $t_1 \neq u_0^+$ or $t_2 \neq u_1^-$ (at most one of X_1 and X_2 is empty).*

Assume that $t_1 = u_0^+$ and $t_2 = u_1^-$. If $t_0 = u_2^+$, then there is a longer cycle as shown in Figure 4.43. Symmetrically, if $t_3 = u_2^-$, there is a longer cycle. Hence $t_0 \neq u_2^+$ and $t_3 \neq u_2^-$. If there is a $K_{2,2}$ minor in $G[u_2, t_0]$ rooted at u_2 and t_0, then there is a $K_{2,5}$ minor shown in Figure 4.42; hence $G[u_2, t_0]$ is u_2t_0-outerplanar. If there is a $K_{2,2}$ minor in $G[t_0, u_0]$ rooted at t_0 and u_0, then there is a $K_{2,5}$ minor as shown in Figure 4.44; hence $G[t_0, u_0]$ is t_0u_0-outerplanar. All jumps out of $C[u_2, t_0]$ must go to either $C(t_0, u_0)$ or to u_0.

Suppose first that all jumps go to u_0. Then all jumps out of $C[t_0, u_0]$ must go to t_1 since jumps to u_2 are blocked by planarity. By Corollary III.6 applied to $G[u_2, t_0]$, there is a path $P_1 = t_0 ... t$ such that $V(P_1) = (u_2, t_0]$ and t is a degree two vertex in $G[u_2, t_0]$ and therefore is adjacent to u_0. Similarly if $(t_0, u_0) \neq \emptyset$ by Corollary III.6, there is a path $P_2 = t_0 ... s$ such that $V(P_2) = [t_0, u_0)$ and s is a degree two vertex in $G[t_0, u_0]$ or $s = t_0$ and therefore is adjacent to t_1. Using P_1 and P_2, there is a longer cycle as shown in Figure 4.45.

Now there is some jump out of $C[u_2, t_0]$ that goes to $C(t_0, u_0)$. All jumps out of $C[t_3, u_2]$ must go to $C[u_1, t_3)$ and there is a $K_{2,5}$ minor similar to the one shown in Figure 4.46. This concludes the proof of Claim 8.

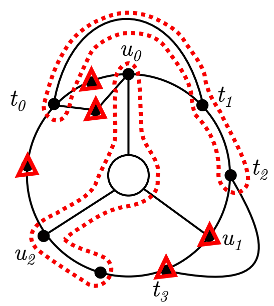


Figure 4.44

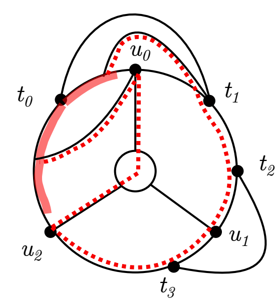


Figure 4.45

Now by Claims 6 and 8, exactly one of X_1 and X_2 is empty. Without loss of generality, assume $X_1 = \emptyset$ and $X_2 \neq \emptyset$. Hence $t_1 = u_0^+$ and $t_2 \neq u_1^-$. If $t_0 = u_2^+$, then there is a longer cycle as in Figure 4.43; hence $t_0 \neq u_2^+$. As in the proof of Claim 8, we can show that there are no rooted $K_{2,2}$ minors along $C[u_2, t_0]$ or $C[t_0, u_0]$ using Figures 4.42 and 4.44. Hence if all jumps out of $C[u_2, t_0]$ go to u_0, then we again get a longer cycle as in Figure 4.45.

Thus there is a jump $r - r'$ with $r \in (u_2, t_0)$ and $r' \in X_0$. We now focus on the structure of the other two sectors. If there is a $K_{2,2}$ minor in $G[t_1, u_2]$ rooted at t_1 and u_2, then there is a $K_{2,5}$ minor as shown in Figure 4.46. It follows that if $[a, b] \subseteq [t_1, u_2]$, then $G[a, b]$ has no $K_{2,2}$ minor rooted at a and b, so is ab-outerplanar. Suppose $(t_3, u_2) \neq \emptyset$ and let $s - s'$ be a jump out of $C[t_3, u_2]$. By Claim 7, $s' \notin (u_2, t_0]$ hence $s \in [u_1, t_3)$. Now there is a $K_{2,5}$ minor similar to the one shown in Figure 4.46. Thus $(t_3, u_2) = \emptyset$ and $t_3 = u_2^-$.

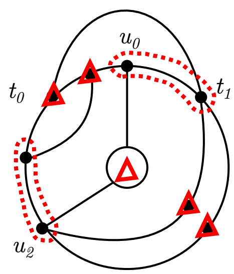


Figure 4.46

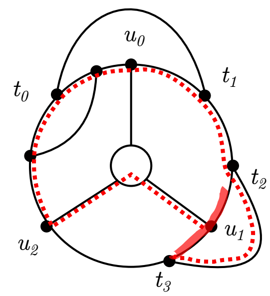


Figure 4.47

If there is a jump from $C(u_1, t_3)$ to u_2, then the minor similar to the one in Figure 4.46 still exists so all jumps out of $C[u_1, t_3]$ must go to $C[t_2, u_1)$. We now focus on jumps out of X_2. Let Y be the set of vertices in X_2 that jump to $X_3 \cup \{t_3\}$ and Z be the set of vertices in X_2 that jump to $[u_0, t_2)$. We consider three cases: $Z = \emptyset$, $Y \subseteq Z$, and $Z \neq \emptyset$ and $Y - Z \neq \emptyset$.

First suppose $Z = \emptyset$ so all jumps out of X_2 go to $X_3 \cup \{t_3\}$. $G[t_2, t_3]$ is t_2t_3-outerplanar. Furthermore $G[t_2, t_3]$ is 2-connected because it is the graph inside a cycle in a 3-connected planar graph; see Lemma 2 of [14]. Hence apply Lemma III.4 to $G[t_2, t_3]$ to find a path $P = t_2t_3...t$ such that $V(P) = [t_2, t_3]$ and t is degree two in $G[t_2, t_3]$; $t = u_1$ because u_1 is the only degree two vertex in $G[t_2, t_3]$ besides possible t_2 and t_3. Now using P, there is a longer cycle as shown in Figure 4.47.

Second suppose $Y \subseteq Z$ so every vertex with a jump out of X_2 jumps to $[u_0, t_2)$. If $t_1 \neq t_2$, then there is a $K_{2,5}$ minor shown in Figure 4.48; hence $t_1 = t_2$ and therefore all jumps from X_2 to $[u_0, t_2)$ go to u_0. Now by Corollary III.6 applied to $G[t_2, u_1]$, there is a path $P = t_2 \ldots t$ such that $V(P) = [t_2, u_1)$ and t is degree two in $G[t_2, u_1]$, therefore has a jump out of X_2, and therefore is adjacent to u_0. Now using P, there is a longer cycle shown in Figure 4.49.

Finally suppose $Z \neq \emptyset$ and $Y - Z \neq \emptyset$. Note the minor in Figure 4.48 exists here as well if $t_1 \neq t_2$ so $t_1 = t_2$ and there are jumps from Z to u_0. Let y be the first vertex of $Y - Z$. If there is a vertex $z \in (y, u_1) \cap Z$, then there is a $K_{2,5}$ minor shown in Figure 4.50. Otherwise there is $z \in (t_2, y) \cap Z$. If y jumps to $(u_1^+, t_3]$, then there is a $K_{2,5}$ minor shown in Figure 4.51; hence y jumps to u_1^+. Now apply Corollary III.6 to $G[t_2, y]$ to find a path $P = t_2 \ldots t$ such that $V(P) = [t_2, y)$ and t is degree two in $G[t_2, y]$ or $t = t_2$ and hence is adjacent to u_0. Now using P, there is a longer cycle shown in Figure 4.52.

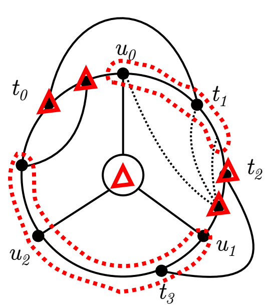


Figure 4.48

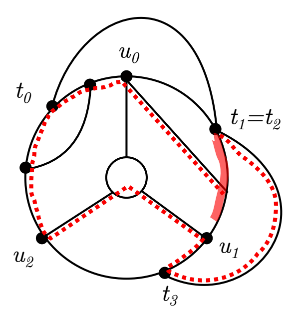


Figure 4.49

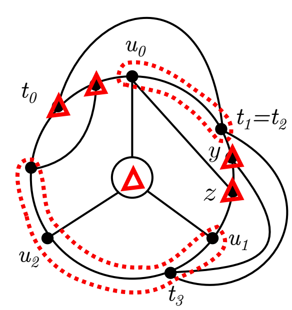


Figure 4.50

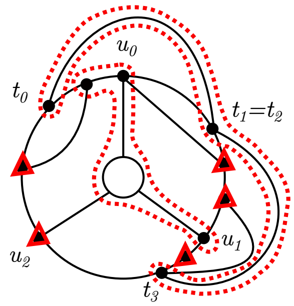


Figure 4.51

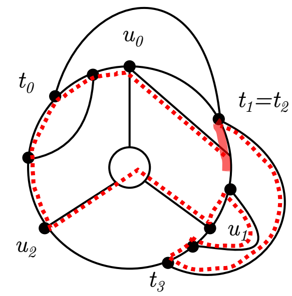


Figure 4.52

There are no remaining possibilities for X_1 and X_2 and hence the proof is complete.

□

A natural next step is to consider the same result for $K_{2,6}$-minor-free graphs. It is not true, however, that all 3-connected planar $K_{2,6}$-minor-free graphs are Hamiltonian. In fact, we can construct an infinite family of 3-connected planar $K_{2,6}$-minor-free graphs.

Lemma IV.4. *The graph shown in Figure 4.53 is a 3-connected, planar, non-Hamilton, $K_{2,6}$-minor-free graph for all values of $k \geq 1$.*

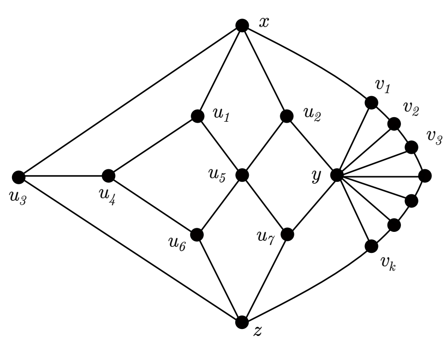


Figure 4.53

Proof. Let G be the graph in Figure 4.53. Then the graph formed from G by contracting all of the vertices labeled v_i to a single vertex v is known as the Herschel graph, the smallest 3-connected, planar, non-Hamiltonian graph. Suppose G has a Hamilton cycle C. If $v_1, v_2, ..., v_k$ appear consecutively along C, then we can form a Hamilton cycle in the Herschel graph by contracting this portion of C to a single vertex v. If $v_1, v_2, ..., v_k$ do not appear consecutively, then since y is the only neighbor of $v_2, ..., v_{k-1}$ outside of the other vertices v_i, there must be some j, $1 \leq j \leq k-1$, such that $x, v_1, ..., v_j, y, v_{j+1}, ..., v_k, z$ appear in that consecutive order in C. We consider the location of u_2 along C. Since $\deg(u_2) = 3$ and the edge u_2y is not in C, we must have u_5, u_2, x appearing in that consecutive order along C. Similarly for the vertex u_7, we can conclude that u_5, u_7, z must appear in that consecutive order along C. But now $C = zu_7u_5u_2xv_1...v_jyv_{j+1}...v_kz$ and C is not Hamiltonian. Thus G has no Hamilton cycle C.

To see that G is $K_{2,6}$-minor-free, we observe that (H, K) is a 3-separation in G where H is the graph on the left of Figure 4.54 and K is the graph on the right.

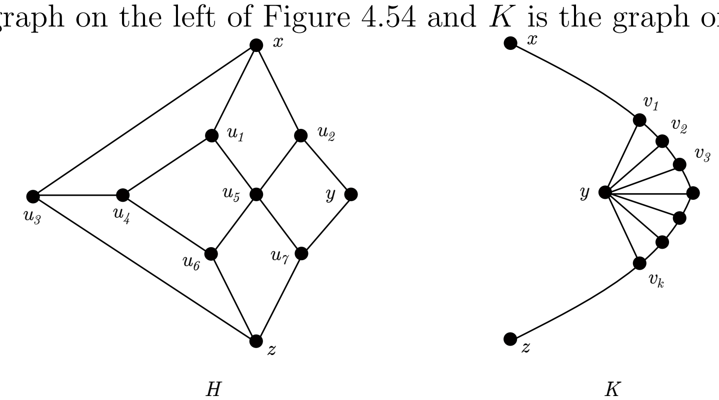


Figure 4.54 H and K

We prove several claims.

Claim 1. *The graph $\bar{H} = H + xy + yz + xz$ has no $K_{2,6}$ minor.*

Suppose (R_1, R_2, S) is a $K_{2,6}$ minor in $\bar{H}$. Then since $|V(\bar{H})| = 10$ and no vertex has degree six, we must have $|R_1| = |R_2| = 2$. Then R_1 and R_2 must consist of either two adjacent vertices of degree at least four or contain a vertex of degree five. Suppose to start that $x \in R_1$; then the other vertex of R_1 must be a neighbor of x. If $R_1 = \{x, u_3\}$, $R_1 = \{x, u_2\}$, or $R_1 = \{x, y\}$, then R_1 does not have six distinct neighbors and hence we cannot form R_1 with $|R_1| = 2$. If $R_1 = \{x, u_1\}$, then $S = \{u_4, u_5, u_3, u_2, y, z\}$ and $R_2 = \{u_6, u_7\}$ but neither u_6 nor u_7 has degree at least four. If $R_1 = \{x, z\}$, then $S = \{u_3, u_1, u_2, u_6, u_7, y\}$ and $R_2 = \{u_4, u_5\}$ but now u_4 and u_5 are not adjacent and hence R_2 is not connected. Thus $x \notin R_1$ and hence symmetrically $x \notin R_2$. By symmetric arguments, $z \notin R_i$ for $i = 1, 2$. The vertices x and z are the only degree five vertices so now R_1 and R_2 must both consist of adjacent degree four vertices. There are only two degree four vertices, however, (y and u_5) so we cannot form R_1 and R_2. Thus $\bar{H}$ is $K_{2,6}$-minor-free.

Claim 2. *The graph $\bar{K} = K + xy + yz + xz$ has no $K_{2,6}$ minor.*

The graph $\bar{K} - y$ is outerplanar and thus contains no $K_{2,3}$ minor (and hence no $K_{2,6}$) minor. Therefore if $\bar{K}$ contains a $K_{2,6}$ minor, then y must be in the minor. If $y \in S$, then $\bar{K} - y$ must contain a $K_{2,5}$ minor but again $\bar{K} - y$ is outerplanar and contains no $K_{2,3}$ and hence no $K_{2,5}$ minor. If $y \in R_i$ for $i = 1$ or 2, then $\bar{K} - y$ must contain a $K_{1,6}$ minor but $\bar{K} - y$ is a cycle and hence contains no $K_{1,6}$ minor. Thus $\bar{K}$ is $K_{2,6}$-minor-free.

Now by Lemma III.3, Claim 1, and Claim 2, if G contains a $K_{2,6}$ minor $(R_1, R_2; S)$, then we must have one of x, y, and z in R_1 and one of x, y, and z in R_2. Without loss of generality, suppose $x \in R_1$.

First assume $y \in R_2$. Since $\bar{H}$ is $K_{2,6}$-minor-free, $v_i \in S$ for at least one i. Suppose to start $v_i \in S$ for only one i. Then there must be a $K_{2,5}$ minor $(R_1', R_2'; S')$ in $H+yz$ rooted at x and y. Since $|V(H)| = 10$ and no vertex of $H + yz$ has degree five or more, both $|R_1'| \geq 2$ and $|R_2'| \geq 2$ and thus either $|R_1'| = 2$ or $|R_2'| = 2$. Without loss of generality, suppose $|R_1'| = 2$. In order to have at least five neighbors, R_1' must contain a degree four vertex and since $(R_1', R_2'; S')$ is rooted at x and y, either $x \in R_1'$ or $y \in R_1'$, say $y \in R_1'$. The degree four vertices are u_5 and z and since u_5 is not adjacent to either y or x, we must have $R_1' = \{z, y\}$. Now, however, R_1' does not have five distinct neighbors in $H + yz$ and there is no $K_{2,5}$ minor $(R_1', R_2'; S')$.

Now $v_i, v_j \in S$ for $i \neq j$. In order to have R_1 adjacent to v_i and v_j, $z \in R_1$. Now there must be a $K_{2,4}$ minor $(R_1', R_2'; S')$ in $H + xz$ with $x, z \in R_1'$ and $y \in R_2'$. Because $v_i, v_j \in S$ and $y \in R_2$, x and z are not connected in R_1 using a path in K, thus $(R_1', R_2'; S')$ must exist in H alone. Since $\deg_H(y) = 2$ and y is not adjacent to a degree four vertex, $|R_2'| \geq 3$ in order to ensure R_2' has at least four neighbors. Since $x, z \in R_1'$ and x and z are not adjacent, $|R_1'| \geq 3$. Since $|V(H)| = 10$ and $|S'| = 4$, $|R_1'| = |R_2'| = 3$. Thus $R_1' = \{x, z, u_3\}$ and therefore S contains four of u_1, u_2, u_4, u_6, and u_7 and $u_5 \notin S$ so $u_5 \in R_2'$. S cannot contain both u_2 and u_7 since one of these vertices must be in R_2' thus $\{u_1, u_4, u_6\} \subset S$. But now R_2' does not reach u_4. Thus there is no $K_{2,4}$ minor $(R_1', R_2'; S')$. We cannot have $v_i \in S$ for three distinct i since then one v_i would be reachable from y but not from x or z.

Now we must have $z \in R_2$ (and $y \notin R_2$ and symmetrically, $y \notin R_1$). Again since $\bar{H}$ is $K_{2,6}$-minor-free, $v_i \in S$ for at least one i. At most one v_i is reachable by both x and z without using y, however, so there is exactly one v_i in S. Hence there must be a $K_{2,5}$ minor $(R_1', R_2'; S')$ in $H + xy + yz$ rooted at x and z with $y \notin R_1' \cup R_2'$. Since $|V(H)| = 10$ and no vertex of H has degree five or more, either $|R_1'| = 2$ or $|R_2'| = 2$. Without loss of generality, suppose $|R_1'| = 2$ and further suppose $x \in R_1'$. Then the other vertex in R_1' must be a neighbor of x and the two vertices together must have five distinct neighbors in $H + xy + yz$ other than z. None of the pairs x and u_3, x and u_2, and x and y together have five such neighbors. If $R_1' = \{x, u_1\}$, then $S' = \{u_3, u_4, u_5, u_2, y\}$ and $R_2' \subseteq \{z, u_6, u_7\}$ but then R_2' is not adjacent to u_2. Thus we cannot form R_1' and there is no $K_{2,5}$ minor $(R_1', R_2'; S')$. $\square$

Chapter V

A CHARACTERIZATION OF $K_{2,4}$-MINOR-FREE GRAPHS

In this chapter, we provide a complete characterization of all $K_{2,4}$-minor-free graphs. We start by defining a class of graphs and describing several small examples which together make up all 3-connected $K_{2,4}$-minor-free graphs. We begin with 3-connected graphs because all 4-connected graphs on at least six vertices have a $K_{2,4}$-minor. This is obvious for complete graphs. Otherwise, a pair of nonadjacent vertices and the four internally disjoint paths between them guaranteed by Menger's Theorem yield a $K_{2,4}$ minor. In Section 2 we extend the characterization to 2-connected graphs. The generalization to all graphs follows because a graph that is not 2-connected is $K_{2,4}$-minor-free if and only if each of its blocks is $K_{2,4}$-minor-free.

5.1 The 3-connected Case

All graphs G with $|V(G)| < 6$ are trivially $K_{2,4}$-minor-free; the 3-connected ones are K_5, $K_5 - e$, $K_5 - 2K_2$, and K_4. To describe the graphs with $|V(G)| \geq 6$, first we define a class of graphs and identify those that are 3-connected and $K_{2,4}$-minor-free. We then look at some small graphs that do not fit into this class. Finally, we show that every 3-connected $K_{2,4}$-minor-free graph is one of these we have described.

5.1.1 A Class of Graphs $G^{(+)}_{n,r,s}$

For $n \geq 3$, $0 \leq r, s \leq n-3$, let $G_{n,r,s}$ consist of a spanning path $v_1v_2...v_n$ which we call the *spine* and edges v_1v_{n-i} for $1 \leq i \leq r$ and v_nv_{1+j} for $1 \leq j \leq s$. The graph $G^{+}_{n,r,s}$ is $G_{n,r,s} + v_1v_n$; we call v_1v_n the *plus edge*. Examples are shown in Figures 5.1 and 5.2. Note $G^{(+)}_{n,r,s} \cong G^{(+)}_{n,s,r}$. Hence, for simplicity we assume $r \leq s$ throughout unless otherwise stated.

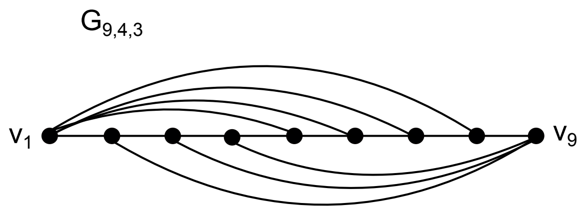


Figure 5.1

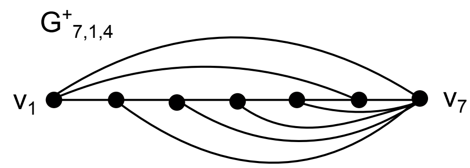


Figure 5.2

Consider $G_{n,r,s}$ with $n \geq 4$. Observe that $G_{n,1,s}$ and symmetrically $G_{n,r,1}$ are not 3-connected. We claim the following:

Lemma V.1. *For $n \geq 4$, $G^{(+)}_{n,r,s}$ is 3-connected if and only if (i) $r = 1$, $s \geq n-3$, and the plus edge is present (or symmetrically $s = 1$, $r \geq n-3$, and the plus edge is present) or (ii) $r, s \geq 2$ and $r + s \geq n-2$.*

Proof. To prove the forward direction, assume G is 3-connected and first suppose $r = 1$. Then if the plus edge is not present, then v_1 has degree two and $\{v_2, v_{n-1}\}$ is a 2-cut. Similarly if $s \leq n-4$, then v_{n-2} has degree two and $\{v_{n-3}, v_{n-1}\}$ is a 2-cut. Next suppose $r, s \geq 2$. If $r + s \leq n-3$, then there is necessarily a degree two vertex v_i with $4 \leq i \leq n-3$ and hence a 2-cut in G.

To prove the reverse direction, assume G is not 3-connected and consider a possible 2-cut. The vertices v_1 and v_2 do not form a 2-cut because $G - \{v_1v_2\}$ is a path. Similarly, v_{n-1} and v_n and v_1 and v_n do not form 2-cuts. The vertices v_1 and v_i with $2 < i < n$ do not form a 2-cut because $G - \{v_1, v_i\}$ contains a path $v_{i+1}v_{i+2}...v_nv_2v_3...v_{i-1}$. Similarly, v_i and v_n do not form a 2-cut for $1 < i < n-1$.

Finally consider two vertices v_i, v_j, with $1 < i < j < n$. First assume $j = i + 1$. If $n = 4$, then $i = 2$, $j = 3$ and $r = s = 1$ and therefore the plus edge v_1v_4 is present; hence $G - \{v_2, v_3\}$ is connected and $\{v_2, v_3\}$ is not a 2-cut. Now $n \geq 5$ so either $j \neq n-1$ or $i \neq 2$. Without loss of generality, say $j \neq n-1$; then $v_1v_2...v_{i-1}$ and $v_{j+1}v_{j+2}...v_n$ are connected because v_1 is adjacent to v_{n-1}. Next assume $j \neq i+1$. Then there is a vertex between v_i and v_j and since $r + s \geq n-2$, all vertices between v_i and v_j must be adjacent to v_1 or v_n. In particular, $v_{i+1} \neq v_j$ must be adjacent to either v_1 or v_n. The two situations are similar, so without loss of generality, assume v_{i+1} is adjacent to v_1. When $i \neq 2$, $v_1v_2...v_{i-1}$, $v_{i+1}v_{i+2}...v_{j-1}$, and $v_{j+1}v_{j+2}...v_n$ are all connected because v_n is adjacent to v_2. When $i = 2$, then $v_1v_2...v_{i-1}$, $v_{i+1}v_{i+2}...v_{j-1}$, and $v_{j+1}v_{j+2}...v_n$ are all connected because either v_n is adjacent to v_1 or v_n is adjacent to v_{i+1} (since either $s = 1$ forcing the plus edge or $s \geq 2$). □

Lemma V.2. *Let $G = G^{(+)}_{n,r,s}$ with $n \geq 6$ and $r + s \leq n-1$. If G has a standard $K_{2,4}$ minor $(R_1, R_2; S)$, then $v_1 \in R_1$ and $v_n \in R_2$ (or vice versa).*

Proof. The graph $G - v_1$ is outerplanar and thus has no $K_{2,3}$ minor. Therefore, if G has a $K_{2,4}$ minor, then v_1 must be included in the minor. By symmetry, v_n must also be included in the minor. We cannot have $v_1 \in S$ because then the outerplanar graph $G - v_1$ would have a $K_{2,3}$ minor. Symmetrically, $v_n \notin S$. If $v_1, v_n \in R_i$, then $G - \{v_1, v_n\}$ must have a $K_{1,4}$ minor, but $G - \{v_1, v_n\}$ is a path and there is no $K_{1,4}$

minor in a path. The only remaining possibility is $v_1 \in R_1$ and $v_n \in R_2$ (or vice versa). $\square$

Lemma V.3. *For $n \geq 6$, $G^{(+)}_{n,r,s}$ is $K_{2,4}$-minor-free if and only if $r+s \leq n-1$.*

Proof. To prove the forward direction, suppose $r+s \geq n$. Then there are vertices v_i and v_{i+1} such that both v_1 and v_n are adjacent to both v_i and v_{i+1} and $3 \leq i \leq v_{n-3}$. Then there is a $K_{2,4}$ minor $(R_1, R_2; S)$ in G: let $S = \{v_2, v_i, v_{i+1}, v_{n-1}\}$, $R_1 = v_1$, and $R_2 = v_n$.

Now suppose that $r+s \leq n-1$. Let $A = \{v_{n-r}, v_{n-r+1}, ..., v_{n-1}\} = N(v_1) - v_2$ and $B = \{v_2, v_3, ..., v_{s+1}\} = N(v_n) - v_{n-1}$ (which intersect only if $v_{n-r} = v_{s+1}$). Suppose G has a $K_{2,4}$ minor $(R_1, R_2; S)$. Then by Lemma V.2, $v_1 \in R_1$ and $v_n \in R_2$. We consider the makeup of S. Suppose $\{s_1, s_2, s_3\} \subseteq S \cap A$, in that order along the spine. Now since $\{v_1, s_1, s_3\} \subseteq R_1 \cup \{s_1, s_3\}$ separates s_2 and v_n, and $v_n \in R_2$, we cannot have R_2 adjacent to s_2, which is a contradiction. Thus $|S \cap A| \leq 3$. Symmetrically, $|S \cap B| \leq 3$. We must have $s_1, s_2 \in S \cap A$ and $s_3, s_4 \in S \cap B$ in the order s_4, s_3, s_2, s_1 along the spine. Since $v_n \in R_2$, there must be a $v_n - s_2$ path in $G - \{v_1, s_1, s_3, s_4\}$. Then since v_{s+1} is a cutvertex separating v_n and s_2 in $G - \{v_1, s_1, s_3, s_4\}$, we have $v_{s+1} \in R_2$. Now there must also be an $v_1 - s_3$ path in $G - \{v_n, v_{s+1}, s_4\}$ but no such path exists. Thus there is no $K_{2,4}$ minor. $\square$

All graphs on fewer than six vertices are necessarily $K_{2,4}$-minor-free. The 3-connected ones are K_5, K_5-e, K_5-2K_2, and K_4. Three of these graphs are isomorphic to graphs in the family $G^{(+)}_{n,r,s}$: $K_5 - e \cong G^+_{5,2,2}$, $K_5 - 2K_2 \cong G^+_{5,1,2}$, and $K_4 \cong G^+_{4,1,1}$.

Denote by $\mathcal{G}$ the set of all graphs $G_{n,r,s}$ or $G^+_{n,r,s}$ that are 3-connected and $K_{2,4}$-

minor-free. Then by Lemmas V.1 and V.3 and the discussion of 3-connected $K_{2,4}$ -minor-free graphs on fewer than six vertices:

$$\mathcal{G} = \{G^{+}_{n,1,n-3} : n \geq 4\} \cup \{G^{(+)}_{n,r,s} : n \geq 5,\ 2 \leq r \leq s \leq n-3,\ r+s = n-1 \text{ or } n-2\}$$

There are some isomorphisms between graphs in $\mathcal{G}$ and also symmetries within certain graphs of the class. The graph $G^{+}_{n,1,n-3} = K_1 + C_{n-1}$ is a wheel with hub v_n and has the obvious symmetries. In $G_{n,2,n-4}$, there is an automorphism that swaps v_1 and v_{n-2} and fixes v_n. To see this, consider Figure 5.3. The graph in the figure without the dotted edges e_1 and e_2 is $G_{9,2,5}$. Define a mapping σ (corresponding to reflection about a vertical axis in the figure) such that σ fixes v_{n-1} and v_n and $\sigma(v_i) = v_{n-1-i}$ for $1 \leq i \leq n-2$. In general, the map σ is an involution and an automorphism of $G_{n,2,n-4}$.

With the edge e_1, the graph is $G^{+}_{9,2,5}$ and with e_2, the graph is $G_{9,2,6}$. Thus in general, σ is an isomorphism from $G^{+}_{n,2,n-4}$ to $G_{n,2,n-3}$ which maps e_1 to e_2. With both edges e_1 and e_2, the graph is $G^{+}_{9,2,6}$ and hence in general σ is an automorphism of $G^{+}_{n,2,n-3}$. Now in general σ maps the spine $P = v_1v_2...v_n$ to the path $\sigma(P) = v_{n-2}v_{n-3}...v_2v_1v_{n-1}v_n$. When $r = 2$, we call this the *second spine*. In $G_{n,2,n-4}$ and $G^{+}_{n,2,n-3}$, the second spine is the image of the spine under an automorphism, and in one of $G^{+}_{n,2,n-4}$ and $G_{n,2,n-3}$, it is the image under an isomorphism of the spine in the other graph.

Finally, $G_{6,2,2}$ is vertex-transitive and is isomorphic to the triangular prism. These symmetries and isomorphisms will be important later on, particularly in Section 5.2

when we discuss which edges of $G \in \mathcal{G}$ can be subdivided without creating a $K_{2,4}$ minor.

Up to isomorphism, $\mathcal{G}$ contains one graph with $n = 4$ and $2n - 8$ graphs for each $n \geq 5$.

5.1.2 Small Cases

There are nine examples of small graphs G, $G \notin \mathcal{G}$, that are 3-connected and $K_{2,4}$-minor-free. They are shown in Figure 5.6. The first is K_5 which is the final 3-connected graph that has less than six vertices and so is necessarily $K_{2,4}$-minor-free.

Lemma V.4. *The graph C^+ is $K_{2,4}$-minor-free.*

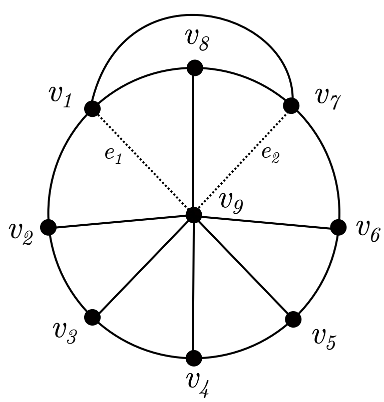


Figure 5.3

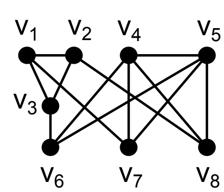


Figure 5.4 C^+

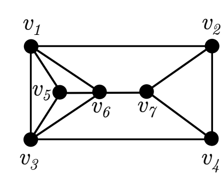


Figure 5.5 D

Proof. Consider C^+ with vertices labeled as in Figure 5.4. $|V(C^+)| = 8$. Suppose there is a $K_{2,4}$ minor $(R_1, R_2; S)$ in C^+ and suppose $|R_1| = 1$. Then R_1 must be either v_4 or v_5 since these are the only vertices of degree four. Say, without loss of generality, $R_1 = \{v_4\}$. Then $S = \{v_5, v_6, v_7, v_8\}$, and R_2 must be a subset of $\{v_1, v_2, v_3\}$. None of these three vertices are adjacent to v_5, however, so we cannot have R_2 adjacent to

v_5 and thus we cannot have $|R_1| = 1$, or symmetrically, $|R_2| = 1$. Thus $|R_1| \geq 2$ and $|R_2| \geq 2$ and since $|C| = 8$, $|R_1| = |R_2| = 2$.

Let T be a triangle with a set N of neighbors with $|N| = 3$. Suppose $R_1 \subseteq V(T)$. Then we would have $N \subseteq S$ along with the third vertex t of T, but N separates t from the rest of the graph so R_2 cannot be adjacent to t. Thus R_1 (or symmetrically R_2) cannot consist of two vertices in a triangle with only three neighbors. In C^+, we have the following triples of vertices which form such triangles: $\{v_1, v_2, v_3\}$, $\{v_4, v_5, v_6\}$, $\{v_4, v_5, v_7\}$, and $\{v_4, v_5, v_8\}$. The only remaining pairs of adjacent vertices that could make up R_1 or R_2 are $\{v_3, v_6\}$, $\{v_2, v_8\}$, and $\{v_1, v_7\}$ where all three cases are symmetric. If $R_1 = \{v_3, v_6\}$, then R_2 must be $\{v_7, v_8\}$ but this set is not an option for R_2. □

Corollary V.5. *All minors of C^+ are $K_{2,4}$-minor-free.*

The graphs C, B^+, B, A^+ (contract v_1v_7 and v_2v_8), A, and $K_{3,3}$ in Figure 5.6 are minors of C^+ and hence are all $K_{2,4}$-minor-free.

Consider D with vertices labeled as shown in Figure 5.5.

Lemma V.6. *The graph D is $K_{2,4}$-minor-free.*

Proof. Suppose there is a $K_{2,4}$ minor $(R_1, R_2; S)$ in D. Since $|V(D)| = 7$, at least one of R_1 or R_2 must consist of a single vertex of degree four. There are three degree four vertices: v_1, v_3, and v_6. Suppose $R_1 = \{v_6\}$; then $S = \{v_1, v_3, v_5, v_7\}$ and $R_2 = \{v_2, v_4\}$. But neither v_2 nor v_4 is adjacent to v_5. By symmetric arguments, $R_1 \neq \{v_3\}$ and $R_1 \neq \{v_1\}$ and thus there is no $K_{2,4}$ minor. □

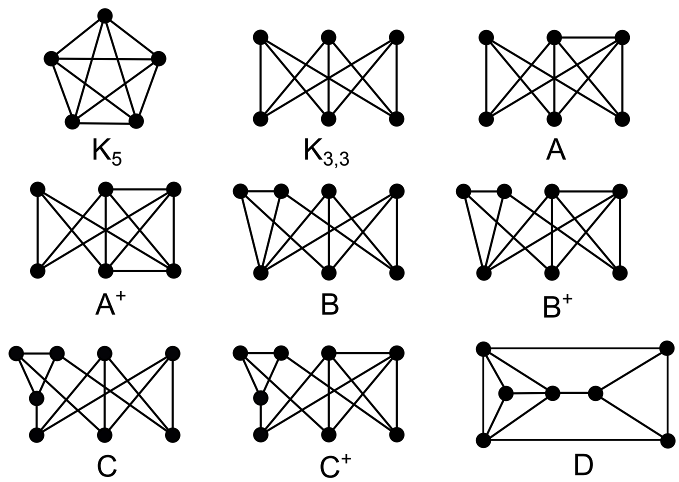


Figure 5.6

5.1.3 Main Theorem

Theorem V.7. *Let G be a 3-connected graph. Then G is $K_{2,4}$-minor-free if and only if $G \in \mathcal{G}$ or G is one of the nine small exceptions shown in Figure 5.6.*

Our original proof of this theorem examined the structure of 3-connected $K_{2,4}$-minor-free graphs relative to a longest non-Hamilton cycle in the graph. We constructed a case analysis based on possible structures and either derived a contradiction with a longer non-Hamilton cycle or a $K_{2,4}$ minor, or found a desired graph. Recent results of Ding and Liu [10] shorten our proof, so we present the new shorter version here. First we explain their notation. Denote by Oct$\backslash e$ the graph obtained from the octahedron by removing one edge, and denote the cube by Q_3, shown in Figure 5.8. Denote by V_8 the graph shown in Figure 5.7. A *3-sum* of two 3-connected graphs G_1 and G_2 is a graph G obtained by identifying a triangle of G_1 with a triangle of G_2 and possibly deleting some of the edges of the common triangle as long as no degree two vertices are created. Any 2-cut in G would lead to a 2-cut in either G_1 or G_2 so

G is 3-connected. An example is the graph denoted K_5^{Δ} shown in Figure 5.9 which is a 3-sum of K_5 and a triangular prism. A *common 3-sum* of three or more graphs is formed by specifying one triangle in each graph and identifying all as a single triangle called the *common triangle*; again edges of the common triangle may be deleted as long as no degree two vertices are created. Let $\mathcal{S}$ be the set of all graphs formed by taking common 3-sums of wheels and triangular prisms. Note that all graphs in $\mathcal{S}$ are 3-connected. We have the following result due to Ding and Liu.

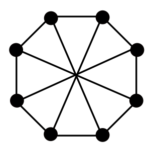


Figure 5.7

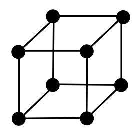


Figure 5.8

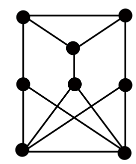


Figure 5.9 K_5^{Δ}

Theorem V.8 (Ding, Liu [10]). *The family of 3-connected Oct$\backslash e$-minor-free graphs consists of graphs in $\mathcal{S}$ and 3-connected minors of V_8, Q_3, and K_5^{Δ}.*

Oct$\backslash e$ contains $K_{2,4}$ as a minor so all 3-connected $K_{2,4}$-minor-free graphs must lie inside the family described in Theorem V.8. In particular, we must consider 3-connected minors of V_8, Q_3, and K_5^{Δ} and also members of $\mathcal{S}$.

Proof of Theorem IV.3. Lemmas V.3, V.4, V.5, and V.6 give the reverse direction of the proof so it remains to show the forward direction. In particular, we show that the graphs listed are the only $K_{2,4}$-minor-free graphs that are 3-connected minors of V_8, Q_3, and K_5^{Δ} or are members of $\mathcal{S}$.

We begin by determining which members of $\mathcal{S}$ are $K_{2,4}$-minor-free. Any common 3-sum of four wheels contains a $K_{3,4}$ minor (the three vertices of the common triangle form the part of size three). In fact any common 3-sum of any four or more graphs contains a $K_{3,4}$ minor. Thus we consider common 3-sums of at most three graphs. We begin by looking at how many wheels can be in the common 3-sum. Denote by W_n the wheel on n vertices.

First consider a common 3-sum of three wheels, W_k, W_ℓ, and W_m. For $k = \ell = 5$ and $m = 4$, since all vertices of $W_4 = K_4$ are equivalent, there are two ways up to symmetry to form a common 3-sum (disregarding the possible existence of the edges of the common triangle): the centers of the two wheels are either identified or not identified. Both ways result in a $K_{2,4}$ minor; Figure 5.10 shows the minor for each case. The dotted edges are the edges of the common triangle which may or may not be present in the common 3-sum. Since graphs with $k \geq 5$, $\ell \geq 5$, and $m \geq 4$ all have one of these two graphs as a minor, these graphs also have $K_{2,4}$ minors and hence only one of k, ℓ, m can be greater than 4. When $k = 6$, $\ell = m = 4$, there is again a $K_{2,4}$ minor shown in Figure 5.11. All graphs with $k > 6$ and $\ell = m = 4$ have this graph as a minor and hence also have a $K_{2,4}$ minor. For $k = 5$, $\ell = m = 4$, we have the graphs shown in Figure 5.12. With none of the dotted edges of the common triangle, this graph is $K_{2,4}$-minor-free and is isomorphic to the graph B. With e_1 (or symmetrically e_2), the graph has the $K_{2,4}$ minor shown on the left of the figure. With e_3, the graph has the $K_{2,4}$ minor shown on the right of the figure. Hence $k, \ell, m \leq 4$. For $k = \ell = m = 4$, we have the graph shown in Figure 5.13. With any two of the dotted edges, the graph has the $K_{2,4}$ minor shown in the figure for e_1 and e_2. With none of the edges, the graph is isomorphic to $K_{3,3}$. With any one dotted edge, the graph is isomorphic to A. Henceforth we can consider common 3-sums with at most two wheels.

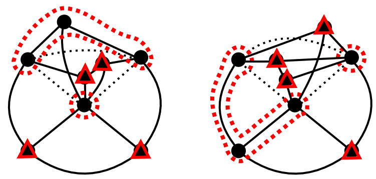


Figure 5.10

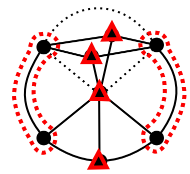


Figure 5.11

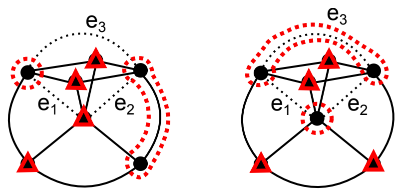


Figure 5.12

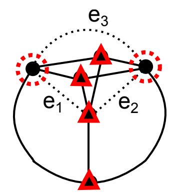


Figure 5.13

Next consider common 3-sums with two wheels and begin with a common 3-sum of two wheels and a prism. If the wheels are W_k and W_4 with $k \geq 5$, then all common 3-sums have the $K_{2,4}$ minor shown in Figure 5.14. If both wheels are W_4, then we have the graph shown in Figure 5.15. With the edge labeled e_1 (or symmetrically e_2 or e_3), we have the $K_{2,4}$ minor shown in the figure. With none of the dotted edges, the graph is isomorphic to C.

Now consider a common 3-sum of two wheels W_k and W_ℓ in which the centers of the wheels are not identified (and $k, \ell \geq 5$). The case in which either k or ℓ is 4 is also covered here because in W_4, any vertex can be considered as a center or non-center as appropriate. We have the graph shown in Figure 5.16. At least one of the edges labeled e_1 and e_2 must be present in the common 3-sum to ensure there are no degree two vertices. Let $n = k + \ell - 3$. With e_1 and e_2, the graph is isomorphic to $G_{n,k-2,\ell-2}$. With e_1 (or symmetrically e_2), the graph is isomorphic to either $G_{n,k-3,\ell-2}$ or $G_{n,k-2,\ell-3}$. In all cases e_3 is the optional plus edge. The spine is shown in the figure as the thick, highlighted path. Hence we have all graphs in $\mathcal{G}$ with at least five vertices.

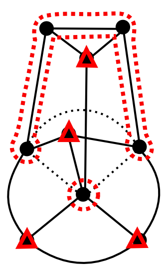


Figure 5.14

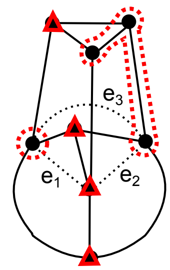


Figure 5.15

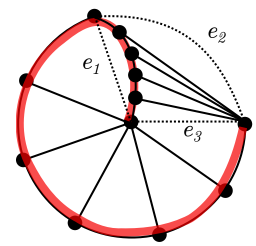


Figure 5.16

Now suppose the centers of W_k and W_ℓ are identified in the common 3-sum. For $k, \ell = 5$, we have the graph shown in Figure 5.17; a common 3-sum of any two wheels with $k, \ell \geq 5$ have this graph as a minor. With the edge labeled e_1, the graph has the $K_{2,4}$ minor shown. Without e_1, both e_2 and e_3 must be present to ensure there are no vertices of degree two. Then the graph is a wheel. In W_4, all vertices are symmetric so a common 3-sum of W_k and W_4 for any $k \geq 4$ was considered in the previous case in which the centers of the wheels were not identified. Henceforth we consider common 3-sums with at most one wheel.

Now consider common 3-sums that include two prisms and begin with a common 3-sum of two prisms and one wheel. We have the graph in Figure 5.18 with the $K_{2,4}$ minor shown; the figure shows the minor for W_4, and a common 3-sum of two prisms and any larger wheel has this graph as a minor. Now consider a common 3-sum of two prisms. We have the graph in Figure 5.19. At least two of the three dotted edges are needed to ensure there are no degree two vertices and so we have the $K_{2,4}$ minor shown. In a common 3-sum of three prisms, the dotted edges do not need to be present to ensure 3-connectivity. However, instead of using one of the dotted edges in the $K_{2,4}$ minor as in Figure 5.19, we can use a path between these two vertices through the third prism. Hence a $K_{2,4}$ minor similar to the one shown in Figure 5.19 exists in a common 3-sum of three prisms. Henceforth we consider common 3-sums with at most one prism.

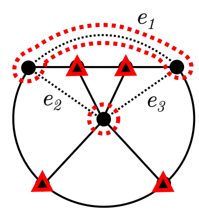


Figure 5.17

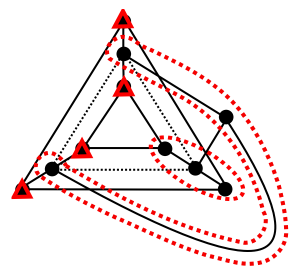


Figure 5.18

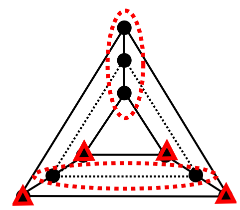


Figure 5.19

Consider a common 3-sum of a wheel W_k and a prism. Up to symmetry, there is one common 3-sum for $k \geq 5$, shown in Figure 5.20 for $k = 5$; any common 3-sum of W_k and a prism with $k \geq 6$ has this graph as a minor. At least one of the edges e_1 and e_2 must be present to ensure there are no vertices of degree two so we have the $K_{2,4}$ minor shown. When $k = 4$, we have the graph shown in Figure 5.21. Two of the three dotted edges must be present to ensure there are no degree two vertices. With all three edges, the graph is isomorphic to D. With any two of the three, the graph is isomorphic to $G_{7,3,2}$.

Finally, consider common 3-sums of a single graph. The wheel W_k ($k \geq 4$) is isomorphic to the graph $G^+_{k,1,k-3}$ and the triangular prism is isomorphic to the graph $G_{6,2,2}$.

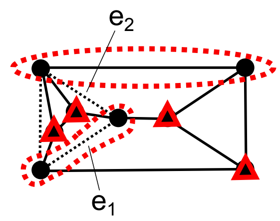


Figure 5.20

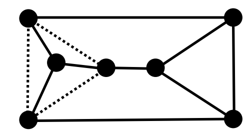


Figure 5.21

Next we look at 3-connected minors of V_8. Once we obtain a minor that has six vertices, we do not need to consider further minors formed by edge contraction because all graphs on fewer than six vertices are trivially $K_{2,4}$-minor-free. Furthermore, if no set of edges and vertices can be deleted to form a 3-connected $K_{2,4}$-minor-free graph, then any minor of interest involves at least one edge contraction so without loss of generality, we will first consider edge contractions followed by either additional edge contractions or deletions.

V_8 itself has a $K_{2,4}$ minor $(R_1, R_2; S)$: take two adjacent vertices that are not consecutive on the outer cycle in Figure 5.7 as R_1 and their four neighbors as S. The graph is 3-regular so the deletion of any set of edges or vertices results in a graph that is not 3-connected. Thus any 3-connected, $K_{2,4}$-minor-free minor of V_8 must result from at least one edge contraction so first we consider edge contractions. Up to symmetry, there are two contractions to consider. The first is shown in Figure 5.22 and is isomorphic to the graph B. We further consider minors of this graph. The deletion of any edge results in a graph that is not 3-connected since all edges are incident with a degree three vertex. Up to symmetry, there are six edge contractions to consider: v_1v_5, v_1v_7, v_6v_7, v_5v_6, v_3v_7, and v_4v_5. Contracting v_1v_5 or v_4v_5 both result in a vertex of degree two so the graph is not 3-connected; the graphs also have six vertices so we do not consider further minors. Contracting v_1v_7 results in a graph isomorphic to $W_6 = G^+_{6,1,3}$, and deleting any edge of $G^+_{6,1,3}$ results in a graph that is not 3-connected. Contracting v_6v_7 or v_3v_7 both result in graphs isomorphic to $G^+_{6,2,2}$. The edge corresponding to the plus can be deleted to give $G_{6,2,2}$. Finally contracting v_5v_6 results in a graph isomorphic to A. Only one edge can be deleted from A and the result is $K_{3,3}$.

The second edge contraction up to symmetry in V_8 results in the graph shown in Figure 5.23 and contains a $K_{2,4}$ minor so we must further consider minors of this graph. Every edge is adjacent to a degree three vertex so the deletion of any edge results in a 2-connected graph. Hence we first consider contracting edges. Up to symmetry, there are four edge contractions to consider: v_1v_2, v_3v_4, v_2v_6, and v_3v_7. Contracting v_3v_4 results in a graph that are not 3-connected; the graphs also have six vertices so we do not need to consider further minors.

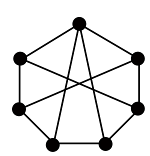


Figure 5.22

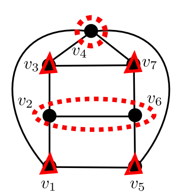


Figure 5.23

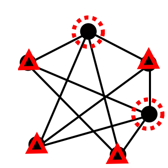


Figure 5.24

Contracting the edge v_1v_2 results in a graph isomorphic to $G^+_{6,2,2}$. The edge corresponding to the plus can be deleted to result in $G_{6,2,2}$. Contracting v_3v_7 results in a graph isomorphic to $G_{6,2,2}$; deleting any edge of this graph results in a graph that is not 3-connected. Contracting v_2v_6 yields a graph with a $K_{2,4}$ minor as shown in Figure 5.24. Deleting any edge from this graph results in a graph that is not

3-connected.

Now consider 3-connected minors of Q_3. Q_3 itself has a $K_{2,4}$ minor $(R_1, R_2; S)$: take any two adjacent vertices as R_1 and their four neighbors as S. The graph is 3-regular so the deletion of any edge results in a graph that is not 3-connected. Hence any 3-connected $K_{2,4}$-minor-free minor of Q_3 must come from at least one edge contraction so we first consider edge contractions. However, all edges are symmetric and the contraction of any edge results in a graph isomorphic to the one shown in Figure 5.23. This graph was already fully analyzed so we are done with Q_3.

Finally we consider K_5^{Δ} as in Figure 5.9. This graph is isomorphic to C^+ so it is $K_{2,4}$-minor-free. Deleting the edge v_7v_8 results in the graph C. Up to symmetry, there are four edge contractions of K_5^{Δ} to consider: v_1v_2, v_3v_5, v_4v_8, and v_7v_8. Contracting v_1v_2 results in a degree two vertex; contracting an edge incident with this vertex results in a graph isomorphic to A. Only one edge can be deleted from A without creating a 2-connected graph and the result is $K_{3,3}$. Contracting v_7v_8 results in a graph with three degree two vertices; at least three edge contractions are needed to yield a 3-connected graph but then the graph will have fewer than six vertices. Contracting the edge v_3v_5 results in the graph B^+. Contracting the edge v_4v_8 results in the graph $G_{7,2,3}$. Hence we consider further minors of these three graphs: C, B^+, and $G_{7,2,3}$.

First consider $G_{7,2,3}$ with spine $v_1v_2v_3v_4v_5v_6v_7$. Deletion of any edge results in a graph that is not 3-connected. Contraction of any edge not on the spine or the edge v_5v_6 results in a degree two vertex and hence a graph that is not 3-connected; these graphs have six vertices so we do not consider further minors. Graphs in $\mathcal{G}$

are closed under contracting spine edges. We look at the graphs resulting from these contractions. Contracting v_6v_7 results in a graph isomorphic to $G^+_{6,1,3}$; deleting any edge of this graph results in a graph that is not 3-connected. Contracting v_4v_5 results in a graph isomorphic to $G_{6,3,2}$; deleting any edge of this graph results in a graph that is not 3-connected. Contracting v_3v_4 or v_2v_3 result in graphs isomorphic to $G_{6,2,2}$; deleting any edge of this graph results in a graph that is not 3-connected. Finally, contracting v_1v_2 results in a graph isomorphic to $G^+_{6,2,2}$; the edge corresponding to the plus can be deleting resulting in $G_{6,2,2}$.

Now consider B^+ with vertices labeled as in Figure 5.25. Deleting the edge v_6v_7 results in a graph isomorphic to B. We already considered minors of B when it occurred as a minor of V_8. Deleting the edge v_3v_6 (or symmetrically v_3v_7) results in a graph isomorphic to $G_{7,3,2}$ and we have already considered further minors of this graph. Up to symmetry, there are six edge contractions to consider: v_1v_2, v_1v_3, v_1v_4, v_3v_6, v_4v_6, and v_6v_7. Contracting v_1v_3 or v_6v_7 all result in graphs that are not 3-connected. Contracting v_1v_2 results in a graph isomorphic to A and contracting v_1v_4 results in a graph isomorphic to the graph A^+. One edge can be deleted from A^+ to result in A and one edge can be deleted from A to result in $K_{3,3}$. Contracting v_3v_6 results in a graph isomorphic to $W_6 = G^+_{6,1,3}$; deleting any edge of this graph results in a graph that is not 3-connected. Finally, contracting v_4v_6 results in a graph isomorphic to $G^+_{6,2,2}$. The edge corresponding to the plus can be deleted resulting in $G_{6,2,2}$.

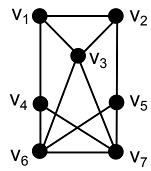


Figure 5.25

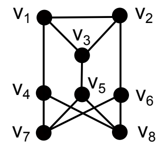


Figure 5.26

Finally, consider C with vertices labeled as in Figure 5.26. Deleting any edge of C results in a graph that is not 3-connected. Up to symmetry, there are three edge contractions to consider: v_1v_2, v_1v_4, and v_4v_7. Contracting v_1v_2 results in a graph with a degree two vertex; contracting an edge incident with this vertex results in a graph isomorphic to $K_{3,3}$. Contracting v_1v_4 results in a graph isomorphic to B; further minors of B have already been considered. Contracting v_4v_7 results in a graph isomorphic to $G_{7,3,2}$; further minors of $G_{7,3,2}$ have already been considered.

We have now shown that all 3-connected $K_{2,4}$-minor-free graphs that are in $\mathcal{S}$ or are minors of V_8, Q_3, or K_5^δ are all members of $\mathcal{G}$ or are the small cases in Figure 5.6. Thus the proof is complete. $\square$

In the same paper, Ding and Liu prove the following result where $K_{3,3}^\ddagger$ is the graph $K_{3,3}$ with two additional edges added on the same side of the bipartition:

Theorem V.9 (Ding and Liu [10]). *The family of all 3-connected $K_{3,3}^\ddagger$-minor-free graphs consists of 3-connected planar graphs and 3-connected minors of the three graphs shown in Figure 5.27.*

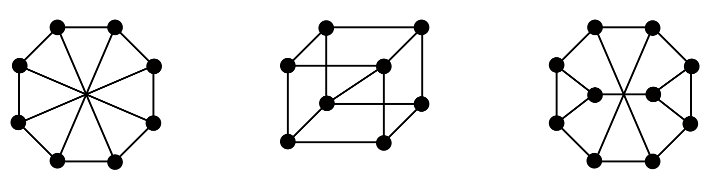


Figure 5.27

It is worthwhile to observe that because $K_{2,4}$ is a minor of $K_{3,3^{\ddagger}}$, $K_{2,4}$-minor-free graphs must be a subset of the graphs described in Theorem V.9. This theorem can be combined with Theorem V.8 to conclude that for large enough graphs, all $K_{2,4}$-minor-free graphs must be planar and members of $\mathcal{S}$ and hence only common 3-sums of two wheels or just one wheel or just one prism are possible. The analysis required for the small cases is not simplified by using this theorem, however, so we provide the full analysis using only Theorem V.8.

5.2 The 2-connected Case

In order to describe the structure of 2-connected $K_{2,4}$-minor-free graphs, we need the following lemma:

Lemma V.10. *Let z be a degree two vertex in a graph G with neighbors x and y. Let G' be the graph formed from G by replacing the path xzy with an xy-outerplanar graph on at least 3 vertices. Then G is $K_{2,t}$-minor-free if and only if G' is $K_{2,t}$-minor-free, for $t \geq 3$.*

Proof. ($\Leftarrow$): G is a minor of G' so if G' is $K_{2,t}$-minor-free then so is G.

($\Rightarrow$): Let $H = G - z$. Let K be the xy-outerplanar graph in G'. Then (H, K) is a 2-separation in G' with $V(H) \cap V(K) = \{x, y\}$. Because G is $K_{2,t}$-minor-free, we know that $H + xy$ is $K_{2,t}$-minor-free and also there is no $K_{2,3}$ minor in H rooted at x and y. Because $K + xy$ is outerplanar, $K + xy$ is $K_{2,t}$-minor-free. Thus by Lemma III.1, if G' has a $K_{2,t}$ minor, then $x \in R_1$ and $y \in R_2$. If $|S \cap V(K)| \geq 2$, then K has a $K_{2,2}$ minor rooted at x and y which contradicts Lemma III.2. Thus $|S \cap V(H)| \geq 3$ but now we have a $K_{2,3}$ minor rooted at x and y in H which is a contradiction. Hence G' is $K_{2,t}$-minor-free. □

We can now describe the structure of 2-connected $K_{2,4}$-minor-free graphs. Let G be a 2-connected graph with a 2-cut $\{x, y\}$. If $G - \{x, y\}$ has four or more components, then G has a $K_{2,4}$ minor: let $x \in R_1$, $y \in R_2$, and let S consist of one vertex from each of the four components. Thus we assume $G - \{x, y\}$ has at most three components. A set of edges F in a $K_{2,4}$-minor-free graph G is *subdividable* if the graph formed from G by subdividing all edges of F is $K_{2,4}$-minor-free.

Theorem V.11. *Let G be a 2-connected graph. Then G is $K_{2,4}$-minor-free if and only if G is one of the following:*

(i) an outerplanar graph,

(ii) three nontrivial xy-outerplanar graphs joined together at the vertices x and y, with or without the edge xy,

(iii) a 3-connected $K_{2,4}$-minor-free graph G' with each edge x_iy_i in a subdividable set of edges $\{x_1y_1, x_2y_2, \ldots, x_ky_k\}$ replaced by an x_iy_i-outerplanar graph.

Proof. ($\Leftarrow$): All outerplanar graphs are $K_{2,4}$-minor-free since they are $K_{2,3}$-minor-free. To show that a graph G in (ii) is $K_{2,4}$-minor-free, we use Lemma V.10. G is $K_{2,4}$-minor-free if the graph formed from G by replacing each of the three outerplanar pieces with a single vertex is $K_{2,4}$-minor-free. This graph is either $K_{2,3}$ or $K_{1,1,3}$ and is thus $K_{2,4}$-minor-free. We use Lemma V.10 again to show that graphs in (iii) are $K_{2,4}$-minor-free. Let G' be a graph formed from a 3-connected $K_{2,4}$-minor-free graph by subdividing some set of subdividable edges. G' is still $K_{2,4}$-minor-free by the definition of subdividable edges. Now replace the degree two vertex in each subdivided edge xy with an xy-outerplanar graph and by Lemma V.10, the resulting graph is still $K_{2,4}$-minor-free.

($\Rightarrow$): We proceed by induction on $|V(G)|$. If G is 3-connected, then (iii) holds so suppose G is K_3 or has a 2-cut $\{x, y\}$. For the base case, take $n = 3$ or 4; a connectivity 2 $K_{2,4}$-minor-free graph on three or four vertices is K_3, $K_{1,1,2}$ or C_4 all of which are outerplanar and thus are in (i). As discussed above, we know that $G - \{x, y\}$ has at most three components. Suppose there are exactly three components. If some $\{x, y\}$-bridge is not xy-outerplanar, then we have a $K_{2,4}$ minor: by Lemma III.2, there is a $K_{2,2}$ minor rooted at x and y in this bridge to which we may add one vertex from each of the two remaining components of $G - \{x, y\}$. Thus all three $\{x, y\}$-bridges must be xy-outerplanar and we have (ii).

Now assume $G - \{x, y\}$ consists of two components. If neither $\{x, y\}$-bridge is xy-outerplanar, then G contains a $K_{2,4}$ minor. If both $\{x, y\}$-bridges are xy-outerplanar then the whole graph is outerplanar and we have (i). Hence one bridge, H, is not xy-outerplanar and one bridge, K, is xy-outerplanar. Now form a graph G' from G by replacing K with a single edge xy. $|V(G')| < |V(G)|$ and thus by induction, G' is either in (i), (ii), or (iii). Because $G' = H + xy$ and H is not xy-outerplanar, G' is not outerplanar and hence not in (i). If G' is in (ii), then there is a 2-cut $\{u, v\}$ in G such that $G - \{u, v\}$ consists of three components and thus G is also in (ii) by the argument above.

Now assume G' is in (iii); G' is a 3-connected $K_{2,4}$-minor-free graph with subdividable edges replaced by path-outerplanar pieces. Suppose first that xy is not part of one of the path-outerplanar pieces. Because G is $K_{2,4}$-minor-free, the graph formed by contracting $K - \{x, y\}$ to a single vertex adjacent to both x and y is also $K_{2,4}$-minor-free. This graph is isomorphic to the one formed from G' by subdividing xy. Hence xy is a subdividable edge in G'. Now by Lemma V.10, replacing this subdivided edge with an xy-outerplanar graph results in a graph that is still $K_{2,4}$-minor-free. Hence G is in (iii).

Next suppose xy is part of one of the outerplanar pieces, say a uv-outerplanar graph F. Then we need to look at where the edge xy lies in F. If xy is on the outer path of F, then replacing xy with an xy-outerplanar graph results in a new graph that is still uv-outerplanar. Thus G is again in (iii). If xy is not on the outer path of F, then we can show $G - \{x, y\}$ consists of three components which will be a contradiction since we are assuming $G - \{x, y\}$ consists of two components. Since F is uv-outerplanar, all of its vertices are on the outer path. Order the path from u to v so that u, x, y, v appear in that forward order. Then since xy is not on the outer path, there must be a vertex w between x and y along the path. Because $xy \in E(F)$, w cannot be adjacent to any vertex before x along the path including u or any vertex after y along the path including v. Thus w is in a separate component from u and v in $G' - \{x, y\}$. These two components are both distinct from $K - \{x, y\}$ in G so $G - \{x, y\}$ has three components which is a contradiction. $\square$

To complete the 2-connected case, it remains to find all sets of subdividable edges F in part (iii) of Theorem V.11 for each 3-connected $K_{2,4}$-minor-free graph. Note that if a set of edges is subdividable, then all subsets of that set are also subdividable. In proving that sets are subdividable, the following lemma will be helpful.

Lemma V.12. *Let G be a $K_{2,t}$-minor-free graph with $t \geq 3$. Let G' be the graph formed from G by subdividing an edge with a vertex x. Then if G' has a $K_{2,t}$ minor, $x \in S$.*

Proof. Because G is $K_{2,t}$-minor-free, if G' has a $K_{2,t}$ minor, then x must be in the minor. Suppose $(R_1, R_2; S)$ is a $K_{2,t}$ minor in G' and suppose $x \in R_1$. Then since $\deg(x) = 2$ and $t \geq 3$, we cannot have $R_1 = \{x\}$. One or both of the neighbors of x must also be in R_1. Then $(R_1 - \{x\}, R_2; S)$ is a $K_{2,t}$ minor in G which is a contradiction. Symmetrically $x \notin R_2$ and thus $x \in S$. □

We will state the maximal subdividable sets of edges in each graph. We start with graphs in $\mathcal{G}$ with $n \geq 6$. Recall the spine is the path $v_1 v_2 ... v_n$ and when $r = 2$, the second spine is $v_{n-2} v_{n-3} ... v_1 v_{n-1} v_n$.

Theorem V.13. *Consider $G^{(+)}_{n,r,s}$ with $r \leq s$ and $n \geq 6$.*

(i) $r = 1$: The wheel $G^{+}_{n,1,n-3}$ has $n-1$ maximal subdividable sets of edges. Each one includes all edges of the rim as well as one of the spokes.

(ii) $r = 2$: For $G_{6,2,2}$, there are six maximal subdividable sets of edges. All are symmetric and correspond to the edge set of the spine under an automorphism of $G_{6,2,2}$, the triangular prism.

For $G_{7,2,3}$, there are three maximal subdividable sets of edges: the set $\{v_1v_2, v_4v_5, v_6v_7, v_3v_7\}$, the edge set of the spine, and the edge set of the second spine.

For $G_{6,2,3}$, there are three maximal subdividable sets of edges: the edge set of the path $v_4v_3v_6v_5v_1v_2$, the edge set of the spine, and the edge set of the second spine. Under *the isomorphism σ, these correspond to three maximal subdividable sets of edges in $G^+_{6,2,2}$, namely the edge set of the path $v_1v_2v_6v_5v_4v_3$, the edge set of the second spine, and the edge set of the spine, respectively.*

For $G^{(+)}_{n,2,s}$ with $n \geq 8$ and $G^+_{7,2,3}$, there are two maximal subdividable sets of edges: the edge set of the spine and the edge set of the second spine.

(iii) $r \geq 3$: The edge set of the spine $v_1v_2...v_n$ is the only maximal subdividable set of edges.

Symmetric sets of edges in $G^{(+)}_{n,r,s}$ with $r > s$ are also maximal subdividable sets.

Proof. We first show that the graph formed by subdividing all of the edges in each claimed subdividable set of edges is $K_{2,4}$-minor-free. For the wheel $G^+_{n,1,n-3}$, subdividing all edges of the rim and one spoke results in a graph which is a subgraph of $G_{2n,2,2n-4}$ and hence $K_{2,4}$-minor-free. Note that the graph formed by subdividing all edges of the spine in $G^{(+)}_{n,r,s}$ is a subgraph of another graph in $\mathcal{G}$ with $2n-1$ vertices and thus is $K_{2,4}$-minor-free. This observation holds even when $r = 2$. Recall for $G_{n,2,n-4}$ and $G^+_{n,2,n-3}$, there is an automorphism σ that reverses the path $v_1v_2...v_{n-2}$ and fixes v_{n-1} and v_n. The spine maps to the second spine $v_nv_{n-1}v_1v_2...v_{n-2}$. Hence for these graphs, the edge set of the second spine is a subdividable set of edges. Recall also that $G^+_{n,2,n-4}$ is isomorphic to $G_{n,2,n-3}$; the spine in $G^+_{n,2,n-4}$ maps to the second spine in $G_{n,2,n-3}$ and vice versa. Hence for graphs of the form $G^+_{n,2,n-4}$ or $G_{n,2,n-3}$, the edge set of the second spine is subdividable. Note that because of the symmetry of $G^+_{6,2,2}$, there are two isomorphisms from it to $G_{6,2,3}$: one maps the spine to the second spine in $G_{6,2,3}$ and the other maps the spine to $v_4v_3v_6v_5v_1v_2$. Hence $G_{6,2,3}$ has two sets of subdividable edges in addition to the spine.

All sets listed in the statement of the theorem have now been covered except for the set $\{v_1v_2, v_4v_5, v_6v_7, v_3v_7\}$ in $G_{7,2,3}$. Let G' be the graph formed from $G_{7,2,3}$ by subdividing all edges of this set. Denote by x_{ij} the vertex subdividing the edge v_iv_j. Then $V(G') = V(G) \cup \{x_{12}, x_{45}, x_{67}, x_{37}\}$. Note that $G' - x_{37}$ is a subgraph of a graph in $\mathcal{G}$ and thus is $K_{2,4}$-minor-free. Therefore, by Lemma V.14, if G' has a $K_{2,4}$ minor $(R_1, R_2; S)$, then $x_{37} \in S$. Without loss of generality, assume $v_3 \in R_1$ and $v_7 \in R_2$. We consider the makeup of R_1. Suppose first that $v_4 \in R_1$ and $v_2 \notin R_1$. Because the vertices of R_1 must have at least four distinct neighbors, we must also have $x_{45}, v_5 \in R_1$. Now $G' - \{v_3, v_4, x_{45}, v_5, x_{37}\}$ is a cycle and thus R_2 can have at most two additional neighbors which implies $|S| \leq 3$, a contradiction. Next suppose $v_2 \in R_1$ and $v_4 \notin R_1$. Again because the vertices of R_1 must have at least four distinct neighbors, $x_{12}, v_1 \in R_1$. Now $G' - \{v_3, v_2, x_{12}, v_1, x_{37}\}$ is again a cycle and thus R_2 can have at most two additional neighbors which implies $|S| \leq 3$, a contradiction. Finally suppose $v_2, v_4 \in R_1$. Then we must have either $v_2, x_{12} \in R_1$ or $x_{45}, v_5 \in R_1$. In either case the graph without $\{v_2, v_3, v_4, x_{37}\}$ and these vertices is a path and again we cannot form S of size four. Hence there is no $K_{2,4}$ minor.

Now we show that the sets of edges listed are maximal and are the only subdividable sets. Begin with the wheel $G^+_{n,1,n-3}$. All edges of the rim are in each set so we consider the spokes. If we subdivide two adjacent spokes, we have the $K_{2,4}$ minor shown in Figure 5.28. A similar minor exists if we subdivide nonadjacent spokes as long as $n \geq 6$. Hence we cannot divide two spokes and the sets listed are maximal and are the only subdividable sets of edges.

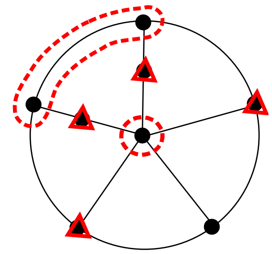


Figure 5.28

Now assume $r, s \geq 2$. For this portion of the proof, we remove the assumption that $r \leq s$. We then consider the subdivision of edges of the form $v_1 v_{n-i}$ for $0 \leq i \leq r$, and edges $v_n v_{1+j}$ for $0 \leq j \leq s$ are handled similarly. Denote by $G^{(+)}_{n,r,s} \circ v_i v_j$ the graph formed from $G^{(+)}_{n,r,s}$ by subdividing the edge $v_i v_j$ with the vertex x_{ij}. We consider two cases. The first, Case A, is shown in Figure 5.29. The graph is $G^{+}_{5,2,2} \circ v_1 v_5$ and has the $K_{2,4}$ minor shown. Because all graphs $G^{(+)}_{n,r,s}$ with $n \geq 6$ and $r, s \geq 2$ except for $G_{6,2,2}$ have $G^{+}_{5,2,2}$ as a minor, this $K_{2,4}$ minor exists in general for other members of $\mathcal{G}$ with subdivided edges. In particular, the minor exists in $G^{(+)}_{n,r,s} \circ v_1 v_{n-i}$ for $0 \leq i \leq r-2$, provided $s \geq 2$. We form $G^{+}_{5,2,2} \circ v_1 v_5$ as a minor from $G^{(+)}_{n,r,s} \circ v_1 v_{n-i}$ so that the $K_{2,4}$ minor shown exists by contracting all edges of the paths $v_3 v_4 ... v_{n-i-3}$ and $v_{n-i} v_{n-i+1} ... v_n$ and deleting multiple edges.

The second case, Case B, is shown in Figure 5.30. The graph is $G^{+}_{5,2,2} \circ v_1 v_3$ and has the $K_{2,4}$ shown. Note that the minor does not use the edge $v_2 v_5$. As with Case A, this minor exists in many larger graphs that have $G^{+}_{5,2,2}$ as a minor. We list them here:

(B1) in $G^{+}_{n,r,s} \circ v_1 v_{n-i}$ with $s \geq 2$ and $2 \leq i \leq r$

(B2) in $G_{n,r,s} \circ v_1 v_{n-i}$ with $s \geq 2$ and $3 \leq i \leq r$

(B3) in $G_{n,r,s} \circ v_1 v_{n-2}$ with $s \geq 3$

For graphs in (B1), form $G^+_{5,2,2} \circ v_1 v_3$ as a minor from $G^+_{n,r,s} \circ v_1 v_{n-i}$ so that the $K_{2,4}$ minor shown still exists by contracting all edges of the paths $v_3 v_4 ... v_{n-i}$ and $v_{n-i+1} v_{n-i+2} ... v_{n-1}$ and deleting multiple edges as well as the edge $v_1 v_3$ if it is present after contraction. Similarly for graphs in (B2), contract all edges of the paths $v_3 v_4 ... v_{n-i}$ and $v_{n-i+2} v_{n-i+3} ... v_n$ and delete multiple edges and $v_1 v_3$. For graphs in (B3), contract $v_1 v_2$ and all edges of the path $v_4 v_5 ... v_{n-2}$ and delete multiple edges and $v_1 v_3$. By symmetry, Case B covers the subdivision of $v_n v_{1+i}$ in situations symmetric to those described.

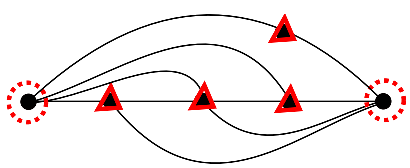


Figure 5.29 Case A

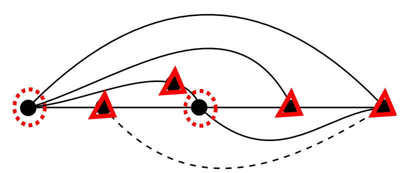


Figure 5.30 Case B

For $G^{(+)}_{n,r,s}$ with $r, s \geq 3$, Case A covers the subdivision of the edges $v_1 v_n, v_1 v_{n-1}, ...,$ $v_1 v_{n-r+2}$ and Case B covers subdivision of the edges $v_1 v_{n-r}, v_1 v_{n-r+1}, ..., v_1 v_{n-2}$. Symmetrically all edges adjacent to v_n that are outside the spine are covered and hence the spine is the only maximal subdividable set of edges.

Now either $r = 2$ or $s = 2$. Suppose to start that $r \geq 3$ and $s = 2$ and consider $G^{(+)}_{n,r,2}$. Then Case A covers subdivision of the edges $v_1 v_n, v_1 v_{n-1}, ..., v_1 v_{n-r+2}$ and Case B covers $v_1 v_{n-r}, v_1 v_{n-r+1}, ..., v_1 v_{n-2}$ in $G^+_{n,r,2}$, and $v_1 v_{n-r}, v_1 v_{n-r+1}, ..., v_1 v_{n-3}$ (and also $v_1 v_{n-2}$ using B3 if $s \geq 3$) in $G_{n,r,2}$. All nonspine edges $v_1 v_j$ are covered except for $v_1 v_{n-2}$ when $r = 3$ and $s = 2$ and there is no plus edge. We consider these edges separately. By symmetry, in $G^{(+)}_{n,2,s}$ with $r = 2$ and $s \geq 3$, this argument covers all nonspine edges v_nv_j except for v_nv_3 when $r = 2$ and $s = 3$ (so $n = 6$ or 7) and there is no plus edge.

Now suppose $r = 2$ and $s \geq 3$ and consider $G^{(+)}_{n,2,s}$. Case A covers subdivision of v_1v_n and Case B covers $v_1v_{n-r}, ..., v_1v_{n-2}$ (use B3 if there is no plus edge). The edge v_1v_{n-1} is subdividable in $G^{(+)}_{n,2,s}$. It can be subdivided along with all edges of the spine except for $v_{n-2}v_{n-1}$. Subdividing v_1v_{n-1} and $v_{n-2}v_{n-1}$ in $G^{(+)}_{n,2,s}$ with $r, s \geq 2$ and $n \geq 6$ results in the $K_{2,4}$ minor shown in Figure 5.31 (contract v_nv_{n-1}, and all edges of $v_3v_4...v_{n-3}$ when $n \geq 7$).

At this point we reinstate the assumption that $r \leq s$. If we do not have $(r, s) = (2, 2)$ or $(2, 3)$, the previous arguments show that the only subdividable edge not on the spine is v_1v_{n-1}, and this edge cannot be subdivided along with $v_{n-2}v_{n-1}$. Hence the edge set of the spine and the edge set of the second spine are maximal subdividable sets of edges in $G^{(+)}_{n,2,s}$ and are the only ones.

Next we examine the small cases. We only need to look at graphs with $r = 2$ and $s = 3$ and no plus edge, and with $r = s = 2$.

Start with $G_{7,2,3}$. From above, the nonspine edges that are subdividable are v_3v_7 and v_1v_6. We know v_3v_7 is subdividable along with v_6v_7, v_4v_5, and v_1v_2. We claim that v_3v_7 is not subdividable together with any of the other spine edges, v_5v_6, v_3v_4, and v_2v_3. Subdividing v_5v_6 and v_3v_7 creates a $K_{2,4}$ minor as shown in Case B. Case B shows the minor in $G_{7,3,2}$ when v_2v_3 and v_1v_5 are subdivided: contract v_1v_2 and both edges of the path $v_3v_4v_5$ and delete multiple edges. A symmetric minor exists in $G_{7,2,3}$ when v_5v_6 and v_3v_7 are subdivided. Subdividing v_3v_4 and v_3v_7 creates a $K_{2,4}$ minor as shown in Figure 5.31. The figure shows a minor in $G_{7,3,2}$ when v_4v_5 and v_1v_5 are subdivided: contract both edges of $v_5v_6v_7$ and delete multiple edges. A symmetric minor exists in $G_{7,2,3}$ when v_3v_4 and v_3v_7 are subdivided. Finally subdividing v_2v_3 and v_3v_7 creates a $K_{2,4}$ minor as shown in Figure 5.32. The figure shows a minor in $G_{7,3,2}$ when v_1v_5 and v_5v_6 are subdivided: contract v_3v_4 and delete the edge v_1v_3 after contraction. A symmetric minor exists in $G_{7,2,3}$ when v_2v_3 and v_3v_7 are subdivided. We have shown which edges of the spine v_1v_6 is subdividable with, namely the edges of the second spine. Hence it remains to show that we cannot subdivide v_3v_7 and v_1v_6 at the same time. Doing so results in the $K_{2,4}$ minor shown in Figure 5.33. Hence $\{v_1v_2, v_3v_4, v_6v_7, v_1v_5\}$ and the edge set of the spine and second spine are maximal subdividable set of edges and are the only ones in $G_{7,2,3}$.

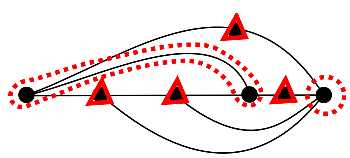


Figure 5.31

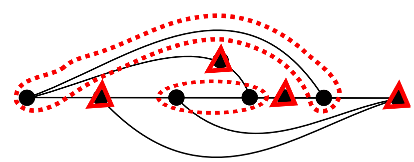


Figure 5.32

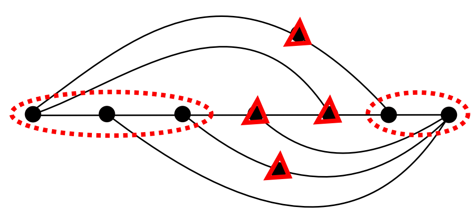


Figure 5.33

Next consider $G_{6,2,3}$. The nonspine edges that are subdividable are v_1v_5 and v_3v_6. We know v_3v_6 is subdividable along with v_3v_4, v_5v_6, v_1v_5, and v_1v_2. We claim that v_3v_6 is not subdividable together with the other spine edges, v_2v_3 and v_4v_5. Subdividing v_4v_5 and v_3v_6 creates a $K_{2,4}$ minor as in Case B. Case B shows a minor in $G_{6,3,2}$ when v_2v_3 and v_1v_4 are subdivided: contract v_1v_2 and v_3v_4. A symmetric minor exists

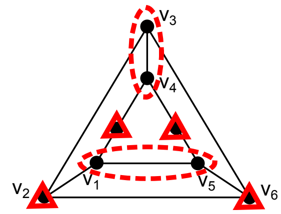


Figure 5.34

in $G_{6,2,3}$ when v_4v_5 and v_3v_6 are subdivided. Subdividing v_2v_3 and v_3v_6 creates a $K_{2,4}$ minor as shown in Figure 5.32. The figure shows a minor when v_1v_4 and v_4v_5 are subdivided in $G_{6,3,2}$ and a symmetric minor exists in $G_{6,2,3}$ when v_2v_3 and v_3v_6 are subdivided. We have already shown which spine edges v_1v_5 can be subdivided with, namely the edges of the second spine. We may subdivide both v_1v_5 and v_3v_6; then we get the maximal set already analyzed above. Hence the edge set of the path $v_4v_3v_6v_5v_1v_2$ and the edge set of the spine and second spine are the maximal subdividable sets of edges in $G_{6,3,2}$.

Because $G^+_{6,2,2}$ is isomorphic to $G_{6,2,3}$, we do not need to analyze this graph separately. The result follows from the analysis just completed for $G_{6,2,3}$.

Finally consider $G_{6,2,2}$; Cases A and B do not cover any edges as every edge is subdividable in this graph. From Figure 5.34, we can see there are two sets of similar edges: $E_1 = \{v_2v_3, v_3v_6, v_2v_6, v_1v_4, v_4v_5, v_1v_5\}$ and $E_2 = \{v_1v_2, v_3v_4, v_5v_6\}$. Each of the six symmetric subdividable sets of edges contains all edges of E_2 and two non-adjacent edges of E_1. If three edges of E_1 are subdivided, then necessarily two are adjacent and we have a $K_{2,4}$ minor symmetric to the one shown in Figure 5.34. Hence the subdividable sets given are maximal and are the only subdividable sets for $G_{6,2,2}$.

□

It remains to describe the maximal subdividable sets of edges in the small cases. The following lemma and corollary will be helpful.

Lemma V.14. *Let G be a graph and let G' be the graph formed from G by subdividing an edge uv with a vertex x. If G' has a standard $K_{2,t}$ minor $(R_1, R_2; S)$ with $x \notin S$ where $t \geq 3$, then G also has a $K_{2,t}$ minor.*

Proof. If $x \notin R_i$ for $i = 1, 2$, then $(R_1, R_2; S)$ is a $K_{2,t}$ minor in G. If $x \in R_i$ for some i, then without loss of generality, say $x \in R_1$. Let $R_1' = R_1 - \{x\}$. Now $(R_1', R_2; S)$ is a $K_{2,t}$ minor in G. □

Corollary V.15. *Let G be a $K_{2,t}$-minor free graph for $t \geq 3$. Let G' be the graph formed from G by subdividing an edge with a vertex x. If G' has a standard $K_{2,t}$ minor $(R_1, R_2; S)$, then $x \in S$.*

Throughout the proofs we will frequently use the fact that in a standard $K_{2,4}$ minor, R_i must contain either a vertex of degree at least four, or two vertices of degree three. We will consider graphs formed from 3-connected graphs by subdividing edges. Call the vertices of the 3-connected graph *original* vertices and the vertices of the subdivided edges *new* vertices.

Lemma V.16. *The set $\{b_1c_1, b_2c_2, b_3c_3\}$ is a subdividable set of edges in C^+ and C where vertices are labeled as in Figure 5.35.*

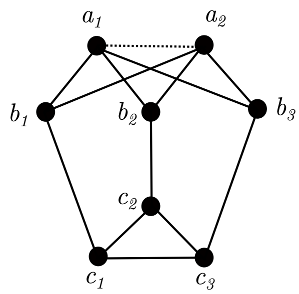


Figure 5.35

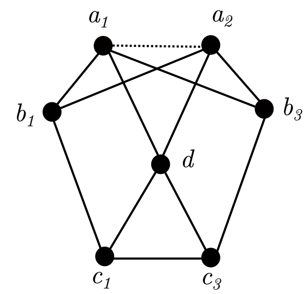


Figure 5.36

Proof. Consider the embedding of C^+ shown in Figure 5.35 with the edges of $\{b_1c_1, b_2c_2, b_3c_3\}$ subdivided. Let x_i be the vertex subdividing the edge b_ic_i for $i = 1, 2, 3$. Let G' be the graph with these three edges subdivided. Then because C^+ is $K_{2,4}$-minor free and by repeated applications of Corollary V.15, if G' has a $K_{2,4}$ minor and thus a standard $K_{2,4}$ minor, then one of x_1, x_2, or x_3 must be in S. Suppose $x_1 \in S$ and without loss of generality, suppose $b_1 \in R_1$ and $c_1 \in R_2$. Then since R_2 must contain at least two vertices of degree three, one of c_2 and c_3 must be in R_2. The vertices c_1, c_2, and c_3 form a triangle, however, so in order to form an R_2 with four distinct neighbors, we must additionally have at least one of b_2 or b_3 in R_2. Without loss of generality, assume $b_2 \in R_2$. Then without loss of generality, either $c_3, x_3, b_3, a_2 \in R_2$ or $c_2, x_2 \in R_2$. If the former holds, then $|R_2| \geq 6$ and since $|R_1| \geq 2$, we cannot form S of size four. Hence $c_2, x_2 \in R_2$. Now we must have at least two vertices in R_1 as well; one of a_1 and a_2 must be in R_1. Hence, $\{c_1, c_2, x_2, b_2\} \subseteq R_2$ and if $\{c_1, c_2, x_2, b_2\} = R_2$, then R_2 no longer has four distinct neighbors in $G' - (R_1 \cup R_2)$ so we must have $|R_2| \geq 5$. Now with $|R_1| \geq 2$, we have $|V(G') - (R_1 \cup R_2)| = 4$ and hence $b_3, x_3 \in S$. But since $\deg(x_3) = 2$ and one of its neighbors is in S, then we cannot form R_1 and R_2 both adjacent to $x_3 \in S$. Thus we have no $K_{2,4}$ minor with

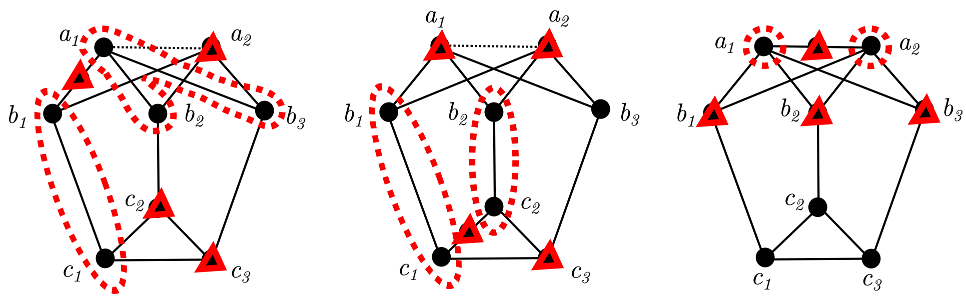


Figure 5.37

$x_1 \in S$. From the graph in Figure 5.35, we can see that the situations with $x_2 \in S$ and $x_3 \in S$ are symmetric and thus G' is $K_{2,4}$-minor free. $\square$

Lemma V.17. *The set $\{b_1c_1, b_2c_2, b_3c_3\}$ is a maximal set of subdividable edges in C^+ and C and is the only one.*

Proof. By Lemma V.16, the set is subdividable so it remains to show it is maximal and the only one. If we subdivide a_1b_1, then there is a $K_{2,4}$ minor as shown in the left of Figure 5.37. The edges a_ib_j are symmetric for $i = 1, 2$ and $j = 1, 2, 3$ so subdividing any one results in a minor symmetric to the one shown. If we subdivide the edge c_1c_2, then there is a $K_{2,4}$ minor as shown in the middle of Figure 5.37. The edges c_ic_j are all symmetric for $i \neq j$ so subdividing any one results in a minor symmetric to the one shown. Finally subdividing the edge a_1a_2 in C^+ results in the $K_{2,4}$ minor shown on the right in Figure 5.37. Thus no set containing a_1a_2, c_ic_j for $i \neq j$, or a_ib_j for $i = 1, 2$, $j = 1, 2, 3$ is subdividable. The only set excluding all of these edges is $\{b_1c_1, b_2c_2, b_3c_3\}$ and hence it is maximal and is the only subdividable set.

$\square$

Lemma V.18. *The set $\{b_1c_1, b_3c_3\}$ is the only maximal set of subdividable edges in*

B^+ and B where vertices are labeled as in Figure 5.36.

Proof. Contracting the edge b_2c_2 in C^+ or C results in the graph B^+ or B, respectively. Hence since b_1c_1 and b_3c_3 are subdividable in C^+, they are also subdividable in B^+ and B. To show that the set is maximal, consider subdivision of other edges. Here we can extend the minors found in Figure 5.37 to find a standard $K_{2,4}$ minor in B^+ or B. In B^+ and B, the vertices b_2 and c_2 are replaced by a single vertex d. Thus for the minor on the left of Figure 5.37, d replaces c_2 in S and for the minor on the right, d replaces b_2 in S. For the minor in the middle, $R_i = \{b_2, c_2\}$ is replaced by $V(R_i') = \{d\}$ which covers subdividing c_1d or c_3d; for subdividing c_1c_3 swap the roles of b_2, c_2 with b_3, c_3 before contracting b_2c_2. Thus no edges other than b_1c_1 and b_3c_3 are subdividable in B^+ and B and $\{b_1c_1, b_3c_3\}$ is a maximal set of subdividable edges and is the only one. $\square$

Lemma V.19. *The three symmetric sets $\{v_1v_2, v_3v_4, v_6v_7, v_5v_6\}$, $\{v_1v_2, v_3v_4, v_6v_7, v_3v_5\}$, and $\{v_1v_2, v_3v_4, v_6v_7, v_1v_5\}$ are subdividable sets of edges in D.*

Proof. Let D' be the graph formed from D by subdividing each edge of the set $\{v_1v_2, v_3v_4, v_6v_7, v_5v_6\}$ with the vertices x_1, x_2, x_3, x_4, respectively. Suppose D' has a $K_{2,4}$ minor $(R_1, R_2; S)$. Since D is $K_{2,4}$-minor free, at least one of the vertices x_i must be in the minor. If none of the x_i are in S, then by repeated applications of Lemma V.14, there is a $K_{2,4}$ minor in D which is a contradiction. Thus at least one x_i must be in S.

Suppose first that $x_1 \in S$; without loss of generality, $v_1 \in R_1$ and $v_2 \in R_2$. Then $D - x_1$ must contain a standard $K_{2,3}$ minor $(R_1, R_2; S - x_1)$ rooted at v_1 and v_2. We consider the location of the three vertices of $S' = S - x_1$. Let $P_1 = v_5x_4v_6x_3v_7$ and $P_2 = v_3x_2v_4$. If $|S' \cap V(P_1)| \geq 2$, then since $v_2 \in R_2$ and v_2 is only adjacent to v_7 in P_1, in order to reach all vertices of S', R_2 must contain all of P_2. Hence $S' \cap V(P_2) = \emptyset$ and therefore $|S' \cap V(P_1)| = 3$. Now in order for R_1 to reach all vertices of S', R_1 must contain at least one vertex of P_1; in particular, for R_1 to have three distinct neighbors on P_1, $v_6 \in R_1$. Now, however, R_2 can no longer have three neighbors on P_1 and hence cannot reach all of S'. Thus $|S' \cap V(P_1)| \leq 1$ and therefore $|S' \cap V(P_2)| \geq 2$. In order for R_1 to reach two vertices of P_2, R_1 must contain $\{v_6, x_3, v_7\}$. Now, however, R_2 cannot reach two vertices of P_2. Thus we cannot have $|S' \cap V(P_2)| \geq 2$ and there is no $K_{2,3}$ minor in $D - x_1$ rooted at v_1 and v_2. Hence we cannot form a $K_{2,4}$ minor with $x_1 \in S$ or symmetrically $x_2 \in S$.

Now suppose $x_3 \in S$; without loss of generality, $v_7 \in R_1$ and $v_6 \in R_2$. R_1 must contain at least two original vertices and since v_2, v_7, and v_4 are all of degree three and form a triangle, it must contain one of v_1 or v_3. Assume without loss of generality $v_3 \in R_1$. Then R_2 must contain another original vertex since v_6 no longer has four neighbors outside of R_1. With $v_3 \in R_1$, there must be a $v_7 ... v_3$ path in R_1 and hence either $v_2, v_1 \in R_1$ or $v_4 \in R_1$. If $v_2, v_1 \in R_1$, then $v_5 \in R_2$ but now R_2 does not have four neighbors in $V(D') - (R_1 \cup R_2)$. Thus we must have $v_4 \in R_1$ (and hence $x_2 \in R_1$). If $v_2 \in R_2$, then v_1 and x_1 must also be in R_2 and now $|V(D') - (R_1 \cup R_2)| = 3$ so we cannot form S. If $v_1 \in R_2$ then necessarily $S = \{x_4, v_5, x_3, v_2\}$ since we know $x_1 \notin S$, but now R_1 cannot be adjacent to x_4 since its two neighbors are in R_2 and S. Hence we cannot form a $K_{2,4}$ minor with $x_3 \in S$.

Finally suppose $x_4 \in S$. Then $D' - x_4$ must contain a $K_{2,3}$ minor $(R_1, R_2; S - x_4)$

rooted at v_5 and v_6. Without loss of generality, suppose $v_5 \in R_1$ and $v_6 \in R_2$. We know that D' does not contain a $K_{2,4}$ minor with x_1, x_2, or x_3 in S and thus $D' - x_4$ does not contain a $K_{2,3}$ minor rooted at v_5 and v_6 with x_1, x_2, or x_3 in $S' = S - x_4$. We consider the location of S'. If $v_1, v_3 \in S'$, then since $v_5 \in R_1$, R_1 cannot be connected and adjacent to the third vertex of S'. Thus at most one of v_1 and v_3 can be in S' and therefore at least two of v_2, v_4, and v_7 are in S'.

If $v_2, v_4 \in S'$, then since $v_6 \in R_2$ and $v_5 \in R_1$, in order for R_1 to reach v_2 and v_4, we must have $v_1, v_3 \in R_1$. Then in order for R_2 to reach v_2 and v_4, we must have $x_3, v_7 \in R_2$. Now, however, we cannot form S' with three vertices. If $v_2, v_7 \in S'$, then in order for R_2 to reach v_2 and v_7, we must have either $v_1, x_1 \in R_2$ or $v_3, x_2, v_4 \in R_2$. The former forces $v_3, x_2, v_4 \in R_1$ but then we cannot form S' of size three. In the latter case, we cannot form a connected R_1 adjacent to $v_7 \in S'$. The situation with $v_4, v_7 \in S'$ is symmetric to the one just considered and hence there is no $K_{2,3}$ minor in $D' - x_4$ rooted at v_5 and v_6. Therefore there is no $K_{2,4}$ minor in D' with $x_4 \in S$. Thus D' is $K_{2,4}$-minor free.

The sets $\{v_1v_2, v_3v_4, v_6v_7, v_3v_5\}$ and $\{v_1v_2, v_3v_4, v_6v_7, v_1v_5\}$ are symmetric to $\{v_1v_2$ $, v_3v_4, v_6v_7, v_5v_6\}$ so they are also subdividable by symmetric arguments. □

Lemma V.20. *The three symmetric sets* $\{v_1v_2, v_3v_4, v_6v_7, v_5v_6\}$, $\{v_1v_2, v_3v_4, v_6v_7, v_3v_5\}$, *and* $\{v_1v_2, v_3v_4, v_6v_7, v_1v_5\}$ *are maximal sets of subdividable edges in* D *and are the only maximal subdividable sets.*

Proof. By Lemma V.19, the sets are subdividable so it remains to show they are maximal and are the only ones. If we subdivide v_1v_3, then there is a $K_{2,4}$ minor as

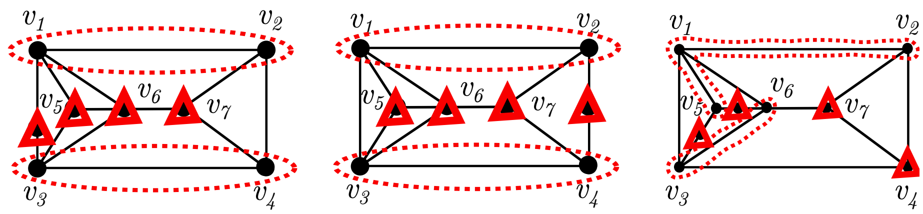


Figure 5.38

shown on the left in Figure 5.38. Symmetric minors exist if we subdivide v_1v_6 or v_3v_6 hence no subdividable set can include any of these edges. If we subdivide v_2v_4, then there is a $K_{2,4}$ minor as shown in the middle of Figure 5.38. Symmetric minors exist if we subdivide v_2v_7 or v_4v_7 hence no subdividable set can include any of these edges. Finally, if we subdivide two edges incident with v_5, then there is a $K_{2,4}$ minor as shown on the right of Figure 5.38. Therefore no subdividable set can include two of the edges v_1v_5, v_5v_6, and v_3v_5. There are three symmetric sets which include only one of these three edges and none of the edges $v_1v_3, v_1v_6, v_3v_6, v_2v_4, v_2v_7$, and v_4v_7 and they are precisely the sets listed in the statement of the lemma. Thus these sets are maximal and are the only such sets.

□

Lemma V.21. *The set $\{de, b_1e, b_3e\}$ is the only set of maximal subdividable edges in A where vertices are labeled as in Figure 5.39.*

Proof. First we show the set is subdividable. Let G' be the graph formed from A by subdividing de, b_1e, and b_3e. Then if G' has a standard $K_{2,4}$ minor, R_1 and R_2 must consist of a degree four vertex or two degree three vertices. Suppose $R_1 = \{a_1\}$. Then $S = \{b_1, d, a_2, b_3\}$ but now e is the only remaining original vertex for R_2 so we cannot have R_1 or R_2 consisting of a single degree four vertex (the situation with a_2 is symmetric to this one). Hence R_1 and R_2 must contain at least two original vertices. All three new vertices are adjacent to e so e must be in R_1 or R_2, say R_1, and then one of b_1, d, b_3 must also be in R_1. Thus one of the new vertices is also in R_1 so now we must have exactly two new vertices in S, two original vertices in S, and hence $|R_1| = |R_2| = 2$. The only pairs of adjacent original vertices that are adjacent to two new vertices, however, all include e so we cannot form two such pairs and hence we cannot form R_1 and R_2. Thus G' is $K_{2,4}$-minor-free.

If we subdivide a_1a_2, then there is a $K_{2,4}$ minor as shown on the left in Figure 5.40. If we subdivide a_1b_1, then there is a $K_{2,4}$ minor as shown on the right in Figure 5.40. Symmetric minors exist if we subdivide $a_1d, a_1b_3, a_2b_1, a_2d$, or a_2b_3. Thus no subdividable set of edges can contain any of these edges and the only set that excludes all of them is $\{de, b_1e, b_3e\}$. Hence this set is maximal and is the only one.

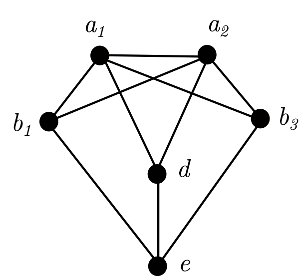


Figure 5.39

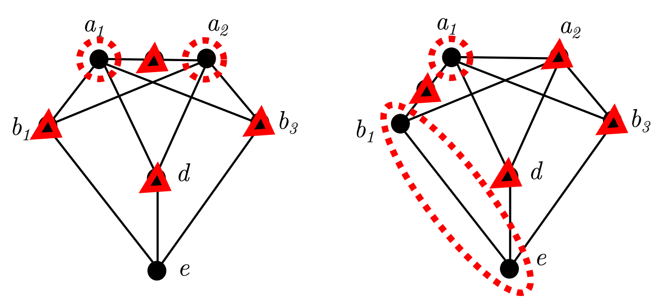


Figure 5.40

∎

Lemma V.22. *The edge de is the only subdividable edge in A^+ where edges are labeled as in Figure 5.41.*

Proof. Let G' be the graph formed from A^+ by subdividing the edge de with a vertex x. Then since $|V(G')| = 7$, if G' has a standard $K_{2,4}$ minor, either R_1 or R_2 must be a single vertex of degree four; without loss of generality let $R_1 = \{a_1\}$. Then $S = \{b_1, d, a_2, b_3\}$ and hence $R_2 = \{x, e\}$ but $(R_1, R_2; S)$ is not a $K_{2,4}$ minor. Hence R_1 and R_2 cannot consist of a single vertex so there is no $K_{2,4}$ minor in G'.

Because A is a minor of A^+, a spanning subgraph in fact, the $K_{2,4}$ minors shown in Figure 5.40 exist in A^+ as well. Thus $a_1a_2, a_1b_1, a_1d, a_1b_3, a_2b_1, a_2d$, and a_2b_3 cannot be subdivided in A^+. Edges b_1e and b_3e are similar to a_1d in A^+ so they also cannot be subdivided. Hence de is the only subdividable edge in A^+.

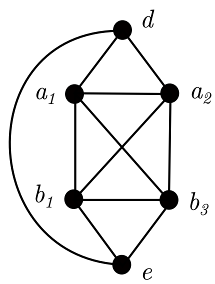


Figure 5.41

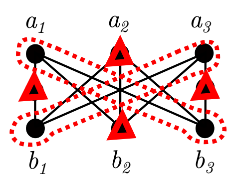


Figure 5.42

□

Lemma V.23. *Up to symmetry, the set $\{a_1b_1, a_1b_2, a_1b_3\}$ is the only maximal subdividable set in $K_{3,3}$.*

Proof. $K_{3,3}$ is a minor of the graph A; delete the edge a_1a_2 and relabel e as a_3 and d as b_2. By Lemma V.21, the edges a_1b_1, a_1d, and a_1b_3 are subdividable in A. These edges correspond to the ones labeled a_1b_1, a_1b_2, and a_1b_3 in $K_{3,3}$ in Figure 5.42 so they are also subdividable. If we subdivide two edges not incident with the same vertex, then there is a $K_{2,4}$ minor similar to the one shown in Figure 5.42. Hence no subdividable set can contain two edges incident with the same vertex so up to symmetric, $\{a_1b_1, a_1b_2, a_1b_3\}$ is maximal and is the only one. □

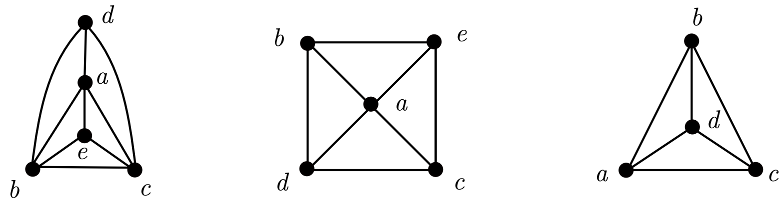


Figure 5.43

For the graph $K_5 - e$, we consider the picture on the left in Figure 5.43.

Lemma V.24. *The sets* $\{ad, ae, bd, cd\}$ *and* $\{ad, ae, bd, ce\}$ *are subdividable sets of edges in* $K_5 - e$*.*

Proof. Observe that $K_5 - e$ is isomorphic to $G^+_{5,2,2}$. We can label the vertices so that the spine is $bdaec$. Hence $\{ad, ae, bd, ce\}$ is a subdividable set of edges. $K_5 - e$ is also a minor of D: contract the triangle $(v_2v_4v_7)$ to the vertex d. Now the set $\{ad, ae, bd, cd\}$ corresponds to the set $\{v_6v_7, v_5v_6, v_1v_2, v_3v_4\}$ in D and thus is subdividable. □

Lemma V.25. *Up to symmetry, the sets* $\{ad, ae, bd, cd\}$ *and* $\{ad, ae, bd, ce\}$ *are maximal sets of subdividable edges in* $K_5 - e$ *and are the only ones.*

Proof. By Lemma V.24, the sets are subdividable so it remains to show they are maximal. If we subdivide the edge bc with a vertex x we have a $K_{2,4}$ minor with $R_1 = \{b\}$, $R_2 = \{c\}$ and $S = \{x, e, a, d\}$. Thus bc, ab, and symmetrically ac cannot be subdivided. Each of the six other edges is individually subdividable so we consider combinations of subdivisions. There are three symmetric 4-cycles that do not contain any of the edges ab, bc, and ac: $adbea$, $adcea$, and $bdceb$. If we subdivide all edges of a 4-cycle, then there is a $K_{2,4}$ minor as shown in Figure 5.44. The six individually subdividable edges form a $K_{2,3}$ and it can be seen that the maximal sets that do not

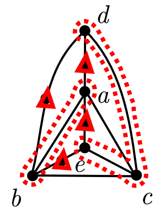


Figure 5.44

contain a 4-cycle are precisely the sets symmetric to those in Lemma V.24. Therefore these sets are exactly the maximal subdividable sets. □

Lemma V.26. *In $K_5 - 2K_2$, the sets $\{ad, ae, bd, cd\}$, $\{ad, ae, bd, ce\}$, and $\{ad, bd, cd, ce, be\}$ are the only maximal subdividable sets up to symmetry.*

Proof. By Lemma V.24, $\{ad, ae, bd, cd\}$ and $\{ad, ae, bd, ce\}$ are subdividable sets in $K_5 - e$ so because $K_5 - 2K_2$ is a minor of $K_5 - e$, the sets are also subdividable in $K_5 - 2K_2$. $K_5 - 2K_2$ is a wheel so by the same arguments as in Theorem V.13 (i), the set $\{ad, bd, cd, ce, be\}$ and three symmetric copies of this set are subdividable because they correspond to all edges of the rim and one spoke.

The minor shown in Figure 5.44 when all edges of the 4-cycle *adbea* are subdivided exists in $K_5 - 2K_2$ as well since it does not use the edge bc. Hence we cannot subdivide all edges of the 4-cycles *adbea*, *abeca*, *aecda*, and *acdba*. If we subdivide two consecutive edges incident with a, then there is a $K_{2,5}$ minor as shown in Figure 5.45. So we may use at most two edges incident with a. If we use only one edge incident with a then we may use all other edges as in the third set. If we use two edges incident with a, then they must be opposite, say ad and ae, and we can add any edges not incident with a as long as they do not complete a second path from d to e; this gives the first two sets. $\square$

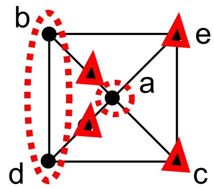


Figure 5.45

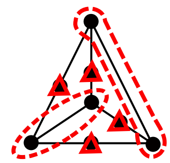


Figure 5.46

Lemma V.27. *In K_4, the set of four edges, three incident with a single vertex and one additional edge, is a maximal subdividable set. There are 12 such symmetric sets in K_4.*

Proof. Let G' be the graph formed from K_4 by subdividing all four edges in one of the symmetric sets listed in the lemma. If there is a standard $K_{2,4}$ minor in G', then R_1 and R_2 must each consist of two original vertices and S must consist of the four new vertices. One original vertex is only adjacent to new vertices, however, so this vertex cannot be paired with another original vertex to form an R_i without also including a new vertex. Hence there is no $K_{2,4}$ minor.

Figure 5.46 shows a $K_{2,4}$ minor when all edges of a 4-cycle are subdivided. Hence the only way to avoid taking four edges of a 4-cycle is to take three edges incident with a single vertex and any additional edge as stated in the lemma. Thus these sets are maximal and are the only ones. $\square$

As mentioned earlier, a graph G is $K_{2,4}$-minor-free if and only if each of its blocks is $K_{2,4}$-minor-free, so we can state our overall result as follows.

Theorem V.28. *A graph is $K_{2,4}$-minor-free if and only if each of its blocks is de-*

scribed by Theorem V.11, where for Theorem V.11 (iii), the 3*-connected graphs are given in Theorem V.7 and the subdividable sets are described in Theorem V.13 and Lemmas V.17, V.18, V.20, V.21, V.22, V.23, V.25, V.26, and V.27.*

Chapter VI

FUTURE WORK

One possible future direction is to extend the result of Theorem II.1 to graphs on the Klein bottle. There are classifications of 4-connected, 4-regular graphs on the Klein bottle and although they are more complicated than those of the torus, they might be useful for proving a similar result.

The results for minor-free graphs in Chapters IV and V lead to several future directions. One idea is to provide a complete characterization for $K_{2,5}$-minor-free graphs. If this proves too difficult, then characterizing planar $K_{2,5}$-minor-free graphs is perhaps more feasible. Presumably, this proof would be aided by a characterization of rooted 2-terminal $K_{2,3}$-minor-free graphs which is another result to consider.

Another idea is to characterize all H-minor-free graphs for other small graphs H of connectivity 2 in addition to $K_{2,4}$. The results of Ding and Liu in [10] characterize H-minor-free graphs for many small 3-connected graphs H so this idea is a natural next step.

In regards to Hamiltonicity, we may be able to show that 3-connected planar $K_{2,6}$-minor-free graphs are Hamiltonian except for a family of well-characterized exceptions. One family of exceptions is described in Lemma IV.4, and based on computer results by Gordon Royle, it appears that all other exceptions may be related to these (private communication).

One final problem concerning forbidden minors and suggested by David Wood involves a class of graphs called subhamiltonian planar graphs. These graphs are planar graphs in which every minor of the graph is a subgraph of a Hamiltonian planar graph. The class of graphs is minor-closed and thus by the Robertson and Seymour Graph Minor Theorem has a forbidden minor characterization. The question then is to determine the forbidden minor characterization.

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