Characterizations of Graphs Without Certain Small Minors

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Chapter 1

Introduction

All graphs in this paper are finite and simple. Given a graph G, the vertex set of G is denoted $V(G)$ and the edge set is denoted $E(G)$. We write $v \sim w$ if there is an edge between v and w in G. For $v \in V(G)$, the *open neighborhood* of v in G is defined as $N_G(v) = \{w \in V(G) \mid v \sim w\}$. The *closed neighborhood* of v is defined as $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, we similarly define $N_G[S] = \bigcup_{v \in S} N_G[v]$ and $N_G(S) = N[S] \setminus S$. In each case, will leave off the subscript if the graph in question is clear. A *model* of H in G is a collection of disjoint subsets $\{B_v \subseteq V(G) \mid v \in V(H)\}$ where each B_v induces a connected subgraph of G and there exists an edge between vertices of B_v and B_w whenever $v \sim w$ in H. We refer to B_v as the *branch set* of v in this model. Then H is a *minor* of G, denoted $H \preccurlyeq G$, if there exists a model of H in G. Otherwise, we say that G is *H-minor-free*. A set S of graphs is called *minor-closed* if it has the property that if $G \in S$ and $H \preccurlyeq G$, then $H \in S$.

Given a graph G and an edge v_1v_2 in G, we *contract* v_1v_2 to form a graph G' by deleting v_1 and v_2 and replacing them with a new vertex v such that $N_G(\{v_1, v_2\}) = N_{G'}(v)$. An equivalent definition of graph minors is that H is a minor of G if a graph isomorphic to H can be obtained from G by a sequence of vertex deletions, edges deletions, and edge contractions.

Given a vertex v in G, we *split* v to form a graph G' by deleting v and replacing it with two new, adjacent vertices v_1 and v_2 such that $N_G(v) = N_{G'}(\{v_1, v_2\})$. This is not, in general, uniquely defined, as G' will depend on the choice of $N(v_1)$ and $N(v_2)$. Edge contraction and vertex splitting are inverse operations in the sense that, given two graphs G and H, H can be formed by contracting an edge v_1v_2 in G into the vertex v if and only if G can be formed from H by splitting v into v_1 and v_2.

The following lemma will be useful in later chapters, when we will use it to generate families of k-connected graphs.

Lemma 1.1. *Let G, G' be graphs with G' formed by splitting a vertex v of G into v_1 and v_2, where v_1 and v_2 each have degree at least k. If G is k-connected, then G' is also k-connected.*

Proof. Assume, to the contrary, that there exists a cut-set $S \subseteq V(G')$, $|S| < k$. We consider three cases. In each case, we see that if G' has connectivity less than k, then so must G. If neither v_1 nor v_2 are in S, then we can contract v_1v_2 to see that S is also a cut-set of G. If $\{v_1, v_2\} \subseteq S$, then $S \setminus \{v_1, v_2\} \cup \{v\}$ is a cut-set of G of size $|S| - 1$. Otherwise, without loss of generality, $v_1 \in S$ and $v_2 \notin S$. Because v_2 has at least k neighbors, there is at least one other vertex in its component after deleting S. Thus, $S \cup \{v_2\}$ is also a cut-set of G', so $S \setminus \{v_1\} \cup \{v\}$ is a cut-set of G of size $|S|$. □

Definition 1.2. Given two disjoint graphs G_1 and G_2, integers k and $0 \leq j \leq k$, and cliques $\{x_1, \ldots, x_j\} \subseteq V(G_1)$ and $\{y_1, \ldots, y_j\} \subseteq V(G_2)$, we form a *$k$-clique-sum* (or just *$k$-sum*) of G_1 and G_2 by identifying pairs of vertices $\{x_i, y_i\}$, then optionally deleting any number of edges from the resulting j-clique. When $j = 0$, this is equivalent to taking the disjoint union of G_1 and G_2.

Note that the resulting graph depends on the choice of clique in each graph, as well as the ordering of vertices in these cliques, so the k-sum of two graphs is not, in general, uniquely defined.

The theory of graph minors has seen deep and beautiful connections to topological graph theory (through, for example, obstructions to finding embeddings on surfaces), graph colorings (most notably through the Four Color Theorem and its generalization to Hadwiger's Conjecture), and theoretical computer science (providing insights into a multitude of interesting minor-closed families of graphs). In Chapter 2, we will explore a few significant results about graph minors, including Robertson and Seymour's graph structure theorem and their resolution of Wagner's conjecture as well as some structural results regarding $K_{2,t}$-minor-free graphs. Next, Chapter 3 describes a few algorithmic results in the field, including a bound on the complexity of determining if a graph contains some fixed minor, also due to Robertson and Seymour. This section also introduces a program we have written to find graph minors. It runs in exponential time asymptotically, but is fast enough on small graphs to exhaustively generate some minor-restricted families of graphs on up to tens of vertices. Chapters 4 and 5 contain characterizations of 4-connected $K_{2,5}$-minor-free graphs and planar 4-connected DW_6-minor-free graphs respectively. The former is joint work with Mark Ellingham, and the latter is joint work with John Maharry, Emily Marshall, and Liana Yepremyan. Finally, Chapter 6 describes a few different directions in which one might continue this research.

Chapter 2

Previous Work

2.1 Planar Graphs

Definition 2.1. A graph is called *planar* if it can be embedded in the plane without edges crossing.

If you wanted to convince someone that a graph G was planar, you could simply show them a crossing-free drawing of that graph, but it is less obvious how to demonstrate that G is non-planar. If G were planar, then any minor of G must also be planar (one can imagine performing vertex and edge deletions and edge contractions on an embedding of G to yield an embedding of any minor of G), so one way to demonstrate non-planarity is to find a minor of G which is known to be non-planar. One might therefore ask which non-planar graphs would this not work for, i.e. what graphs are non-planar, but have only planar minors?

In general, for any minor-closed set of graphs S, the *obstruction set* of S is the set of all minor-minimal graphs not in S. Hence, S is exactly the set of graphs containing no members of its obstruction set as minors.

One of the first major results in the field of graph minors was Wagner's Theorem, below, which relates the topological property of planarity to graph minors by finding the obstruction set for planar graphs.

Theorem 2.2 (Wagner)**.** *A graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.*

We therefore have a straightforward way of certifying that a graph is non-planar: finding one of these two minors.

Wagner also proved the following characterization of K_5-minor-free graphs.

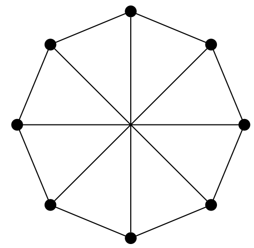


Figure 2.1: The 8-vertex Möbius ladder, used in Theorem 2.3.

Theorem 2.3 (Wagner, [28]). *A graph is K_5-minor-free if and only if it can be formed from 3-clique-sums of any number of graphs, each of which is either planar or the 8-vertex Möbius ladder V_8 (known as the Wagner graph, see Figure 2.1).*

Note that if a graph G is a 3-clique-sum of two graphs G_1 and G_2, then the at most three vertices at which G_1 and G_2 are glued together will form a cut-set, so G is at most 3-connected. Combined with the observation that V_8 is itself 3-connected, we have the following corollary.

Corollary 2.4. *A 4-connected graph is non-planar if and only if it has a K_5-minor.*

This says, loosely, that for sufficiently well connected planar graphs, we no longer need $K_{3,3}$ in the excluded minors characterization of planar graphs.

Similarly, we could look at the family of $K_{3,3}$-minor-free graphs. The following theorem appears in [27].

Theorem 2.5. *A graph is $K_{3,3}$-minor-free if and only if it can be formed from 2-clique-sums of any number of graphs, each of which is either planar or K_5.*

Corollary 2.6. *A 3-connected graph is non-planar if and only if it is K_5 or has a $K_{3,3}$-minor.*

Wagner's characterization of planar graphs raises an interesting question: does a similar characterization (by finding a finite obstruction set) exist for other minor-closed families of graphs? Wagner has stated that he did not conjecture a universal answer to this question, but the following still commonly bears his name.

Conjecture 2.7 (Wagner's Conjecture)**.** *Every minor-closed family of graphs has a finite obstruction set.*

This conjecture was eventually proven to be true by Neil Robertson and Paul Seymour. The next section will present two important theorems resulting from their work.

2.2 Robertson and Seymour's Graph Minor Project

2.2.1 Well-Quasi-Orderings

A *quasi-ordering* of a set X is a binary relation $\leq$ that is reflexive and transitive. If it is also antisymmetric, the relation forms a partial ordering of X. The graph minor relation ($\preccurlyeq$) is a quasi-ordering of the set of all graphs, which becomes a partial order if we consider isomorphism classes of graphs. If $\leq$ is a quasi-order on X and $x, y \in X$, then we say that $x < y$ if $x \leq y$ but $y \not\leq x$.

Definition 2.8. Let $\leq$ be a quasi-ordering of a set X. A sequence $x_1, x_2, \ldots$ of elements of X is called *good* if there exist $i < j$ such that $x_i \leq x_j$, and otherwise it is called *bad*. If every sequence is good, then $\leq$ is called a *well-quasi-ordering* of G.

Given a sequence $x_1, x_2, \ldots$ of elements of X and a quasi-ordering $\leq$, we can color each pair (x_i, x_j) with $i < j$ one of three colors depending on whether $x_i \leq x_j$, $x_i > x_j$, or x_i and x_j are incomparable. Then applying Ramsey's Theorem, we see that every sequence must contain either an infinite antichain (set of incomparable elements), a strictly decreasing subsequence, or a non-strictly increasing subsequence. Thus:

Proposition 2.9 ([7], Proposition 12.1.1)**.** *A quasi-ordering $\leq$ on X is a well-quasi-ordering if and only if X contains neither an infinite antichain nor an infinite strictly decreasing sequence $x_0 > x_1 > \cdots$.*

We are now able to state Robertson and Seymour's graph minor theorem:

Theorem 2.10 (Robertson and Seymour, [25])**.** *The graph minor relation is a well-quasi-ordering.*

In light of Proposition 2.9, this is equivalent to the statement that there are no infinite antichains under the graph minor relation, because there is clearly no infinite sequence of graphs with each a proper minor of the previous. For each minor-closed set S, the obstruction set of S is an antichain, so must be finite, so this theorem truly is a resolution of Wagner's Conjecture (Conjecture 2.7).

We finish off this subsection with an important first step towards the proof of Theorem 2.10, which we state but do not prove.

Theorem 2.11 (Kruskal [15])**.** *The set of finite trees is well-quasi-ordered by the topological-minor relation (and hence by the graph-minor relation as well).*

2.2.2 Tree Decomposition and Treewidth

It is possible to extend the proof of Theorem 2.11 to graphs that are sufficiently tree-like. To this end, Robertson and Seymour make extensive use of tree decompositions and the associated treewidth of graphs to, among other things, measure roughly how tree-like the structure of a given graph is. These concepts were introduced independently by Bertelè and Brioschi in [2] and Halin in [13].

Definition 2.12 (As seen in [19])**.** We inductively define the set of *k-trees* as follows. K_k is a k-tree, and a k-tree on n vertices is any graph that can be built from a k-tree on $n-1$ vertices by adding a new vertex that is adjacent to exactly k vertices, with those k vertices pairwise adjacent.

Note that by this definition, K_k is both a k-tree and a $(k-1)$-tree.

Each vertex that we add in this way extends a k-clique in the k-tree to a $(k+1)$-clique, and it is not possible to create a $(k+2)$-clique. We can thus view a k-tree $G \ncong K_k$ as the set of $(k+1)$-cliques in G glued together along k-cliques in a tree-like

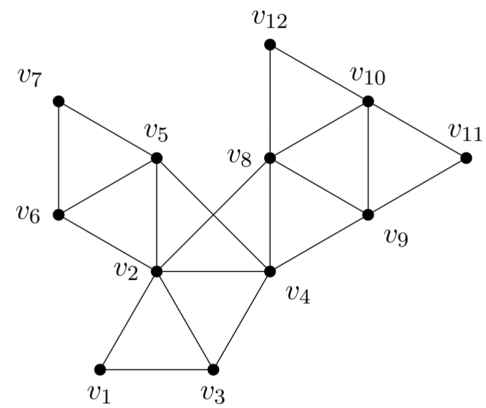


Figure 2.2: An example of a 2-tree, with the vertices labeled by the order in which they could be added as part of the construction in Definition 2.12.

structure. Specifically, start with K_{k+1} and a tree whose only vertex is associated to this $(k+1)$-clique. For each additional vertex v added to the k-tree, add to the tree a vertex t associated to the resulting $(k+1)$-clique and make t adjacent to one other vertex associated to a clique that contains the k neighbors of v.

Loosely, we can view a k-tree as a tree that has been "thickened" to allow k internally disjoint paths to pass along any path in the original tree. With this in mind, the following definition of the treewidth of a graph conveys how tree-like a graph is by measuring how much you would need to "thicken" a tree before it could contain the graph.

Definition 2.13. A graph G has *treewidth* at most k if it is a subgraph of some k-tree. The treewidth of G is denoted $\mathrm{tw}(G)$.

Equivalently, a graph has treewidth at most k if and only if it can be built from graphs $G_1, \ldots, G_n$ on at most $k+1$ vertices by clique-sums.

Lemma 2.14. *If $H \preccurlyeq G$, then* $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.

Graphs of treewidth at most k are sufficiently similar to trees that the proof of Theorem 2.11 can be extended to graphs of bounded treewidth.

Theorem 2.15. *For all integers k, the set of graphs of treewidth at most k is well-quasi-ordered by the graph-minor relation.*

2.2.3 Grids and Other Graphs with Large Treewidth

With Theorem 2.15 in hand, we know that an infinite sequence of graphs $G_1, G_2, \dots$ will have a good pair $G_i \preccurlyeq G_j$ with $i < j$ if there is any k such that $\mathrm{tw}(G_i) \leq k$ for all i, so if any bad sequence of graphs exists, then it must have unbounded treewidth. We therefore take this section to better understand graphs of large treewidth, and to discuss an important obstruction to having small treewidth.

The $r \times r$ *grid* is a graph with vertex set $\{(i, j) \mid i, j \in \{1, \dots, r\}\}$, with $(i_1, j_1) \sim (i_2, j_2)$ whenever $|i_1 - i_2| + |j_1 - j_2| = 1$ (so the $r \times r$ grid is the cartesian product of two paths on r vertices).

The $r \times r$ grid has treewidth exactly r, so by Lemma 2.14, any graph with a large grid as a minor must have large treewidth. Amazingly, though, having (sufficiently) large treewidth also forces a graph to have a large grid as a minor.

Theorem 2.16. *For every integer r there is an integer k such that every graph of treewidth at least k has an $r \times r$ grid minor.*

Corollary 2.17. *A minor-closed family of graphs has bounded treewidth if and only if it excludes at least one planar graph.*

Proof. If a minor-closed family $\mathcal{F}$ contains all planar graphs, then in particular it will contain grids of all sizes, so will have unbounded treewidth. Conversely, let $\mathcal{F}$ have unbounded treewidth and consider any planar H. Then H will be a minor of the $r \times r$ grid for sufficiently large r, and $\mathcal{F}$ will contain a graph with treewidth large enough to force an $r \times r$ grid-minor. Because $\mathcal{F}$ is closed under taking minors, $H \in \mathcal{F}$. $\square$

This takes us one step closer to a proof of Theorem 2.10: an infinite sequence of graphs $G_1, G_2, \dots$ will be good if it contains some graph (without loss of generality

G_1) that is planar. This is because either $G_1 \preccurlyeq G_i$ for some $i > 1$, or all of $G_2, G_3, \ldots$ are G_1-minor-free, and hence of bounded treewidth. Thus, by Theorem 2.15, it will contain a good pair.

2.2.4 The Structure Theorem and Graph Minor Theorem

In the previous subsection, we described a structure common to all graphs which exclude a fixed planar graph H as a minor, namely that they have bounded treewidth (so can be built from clique-sums of graphs of bounded size). We then showed how this structure is used to show that an infinite sequence of graphs containing a planar graph will be good.

In this subsection, we describe Robertson and Seymour's Graph Structure Theorem (Theorem 2.19 below), which provies a structure common to the set of H-minor-free graphs for *any* fixed graph H. This will allow us to finish off our discussion of the proof of Theorem 2.10 by appealing to an argument similar to one we have seen previously in this section: given any infinite sequence of graphs $G_1, \ldots$, either there is a good pair including G_1 (in which case we are done) or all other graphs are G_1-minor free. We will then use the Structure Theorem to describe the remaining graphs in this sequence. The proof of Theorem 2.10 is then completed by showing that graphs with this structure are themselves well-quasi-ordered.

We will restrict our attention to the structure of K_n-minor-free graphs, because if H is a graph on n vertices, then the set of H-minor-free graphs is, in particular, K_n-minor-free. Therefore, any structural description of the latter will still describe, if a bit more coarsely, the former.

We have already seen, in Theorem 2.3, one example of a structure resulting from excluding the non-planar K_5 as a minor. Planar graphs were an important component of that characterization, because planar graphs must be K_5-minor-free. Analogously, one structural reason that a graph G would not have a K_n-minor is if G can be

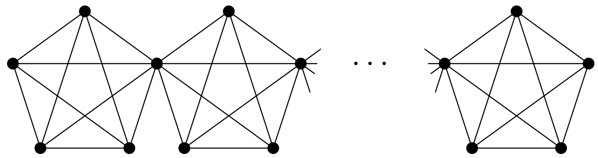


Figure 2.3: A graph of this form can have arbitrarily large genus, but will not contain K_6 as a minor.

embedded on a surface that K_n cannot be. This is to say, if G has too low a genus, where the genus of a graph is the smallest genus surface on which the graph can be embedded without edge crossings.

Unfortunately, while having a small genus is enough to ensure that a graph does not have a large complete minor, the reverse is not true. A graph can have arbitrarily large genus without introducing large complete minors. Figure 2.3 shows a family of graphs which can have arbitrarily large genus, even though no member contains even a K_6-minor. However, the graphs in this family *are* formed by gluing together graphs of low genus, so just as in Theorem 2.3, we might want our structure theorem to allow graphs that are built up by clique-summing together graphs of bounded genus, to capture K_n-minor-free graphs that may have arbitrarily large genus.

This is not yet sufficient to form all K_n-minor-free graphs, because there are other ways to greatly increase the genus of a graph without necessarily introducing a large complete minor. An *apex graph* is any graph formed from a planar graph by adding a single vertex, adjacent to any number of the other vertices. Because planar graphs are K_5-minor-free (by Theorem 2.2), all apex graphs are K_6-minor-free, yet they can have arbitrarily large genus. We might therefore need our structure theorem for K_n-minor-free graphs to allow a certain number of these apex vertices to be added.

It is also possible to add a certain amount of "fringe" around a face of an embedded graph in such a way that can arbitrarily increase the genus of a graph without necessarily introducing a large K_n-minor. Graphs constructed from a planar graph

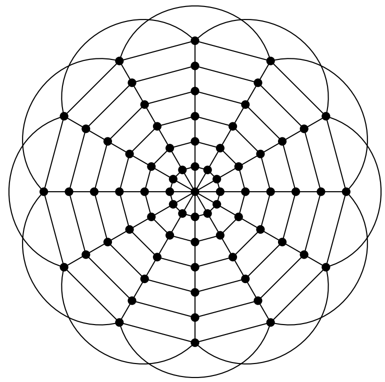


Figure 2.4: Adding a vortex to a planar graph to increase the genus without introducing a much larger complete-graph minor.

by adding edges to the outside face as in Figure 2.4, can have a K_7-minor, but will be K_8-minor-free, see [24]. Such graphs can have arbitrarily large genus, though.

Let G be a graph embedded on a surface Σ, and fix a face with boundary cycle $v_1, \ldots, v_m$. A circular interval of this cycle is any set $\{v_i, v_{i+1}, \ldots, v_{i+l}\}$ for any i and $l < m$, where subscripts are taken modulo m. We add a *vortex* to G as follows. Fix any set Λ of circular intervals from this cycle. For each interval $I \in \Lambda$, add a vertex adjacent to any subset of I. You may then add edges between any pairs of vertices corresponding to intervals from Λ with nonempty intersection.

The *depth* of the vortex is the maximum number of intervals of Λ that any vertex v_i belongs to.

Definition 2.18. A graph G is said to be *k-nearly embeddable* on a surface Σ if it can be formed by adding at most k vortices of depth at most k to a graph embeddable on Σ, then adding at most k apex vertices.

The following theorem then states, roughly, that vortices and apex vertices and bounded-size clique-sums are the only ways in which a graph can have arbitrarily high genus without necessarily introducing a large complete minor.

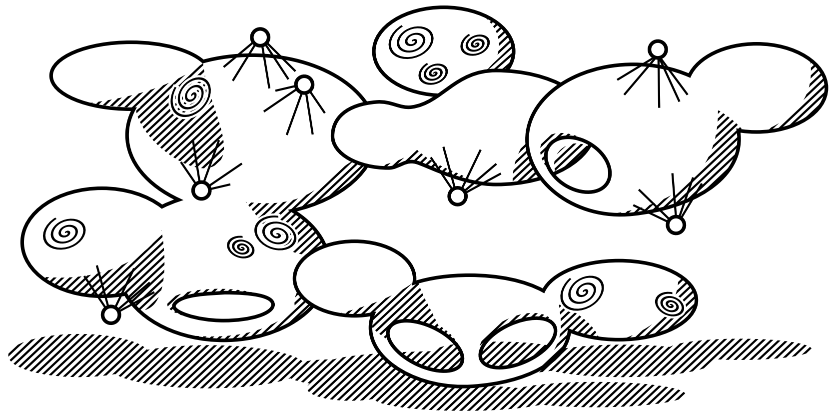


Figure 2.5: An illustration of the structure promised by Theorem 2.19, drawn by Felix Reidl [21].

Theorem 2.19 (Robertson and Seymour, [24]). *For every n there exists k such that every K_n-minor-free graph G can be built by k-clique-summing graphs that are k-nearly embeddable on a surface on which K_n does not embed.*

This theorem only describes a common structure; it is not a complete characterization of K_n-minor-free graphs. There will be many graphs with this structure which *do* contain the minor in question. This result is, however, "best possible" in the sense that the structure is necessary for being H-minor-free and sufficient for being H'-minor-free for some larger H', see [24].

To finish off the proof of Theorem 2.10, Robertson and Seymour needed to show that graphs that were k-nearly embeddable in any fixed surface were well-quasi-ordered, and that this could be lifted to graphs formed from these by taking clique-sums. Then given any infinite sequence of graphs $G_1, \ldots$, either $G_1 \preccurlyeq G_i$ for some $i > 1$ or the graphs $G_2, \ldots$ are G_1-minor-free, and hence $K_{|V(G_1)|}$-minor-free. Thus, they are formed from clique-sums of k-nearly embeddable graphs. In either case, this will be a good sequence, so the set of finite graphs is well-quasi-ordered by the graph minor relation.

2.3 Graphs Without $K_{2,t}$ as a Minor

Apart from the very rough structure promised by Theorem 2.19, the specific restrictions that result from excluding a given graph H as a minor are not well understood, and an exact characterization of the family of H-minor-free graphs is only known for a few graphs H.

This section focuses on excluded minor theorems for $K_{2,t}$-minor-free graphs for different t, setting the stage for the characterization of 4-connected $K_{2,5}$-minor-free graphs, presented in Chapter 4.

2.3.1 Outerplanar and $K_{2,3}$-Minor-Free Graphs

Definition 2.20. A graph is called *outerplanar* if it has a planar embedding in which all of its vertices are on a single face (usually taken to be the outside face).

Theorem 2.21. *A graph G is outerplanar if and only if it is K_4-minor-free and $K_{2,3}$-minor-free.*

Proof. Let G be an outerplanar graph and let G' be the graph formed by adding a vertex adjacent to all other vertices of G. If we imagine placing this new vertex in the outside face of an outerplanar embedding of G, we see that G' is still planar. However, if G had either $K_{2,3}$ or K_4 as a minor, then G' would have $K_{3,3}$ or K_5 as a minor, respectively, which it cannot.

Similarly, if G is any graph without either $K_{2,3}$ or K_4 as a minor, then if we add a new vertex adjacent to every other vertex in G, the resulting graph G' cannot have either a $K_{3,3}$- or K_5-minor, so will be planar. Deleting this vertex from a planar embedding of G' gives an outerplanar embedding of G. □

Corollary 2.22 (to Corollary 2.6)**.** *If G is 2-connected and $K_{2,3}$-minor-free, then G is either outerplanar or K_4.*

Recently, Ellingham, Marshall, Ozeki, and Tsuchiya gave a complete characterization of $K_{2,4}$-minor-free graphs [9].

2.3.2 Edge-Density for $K_{2,t}$-Minor-Free Graphs

Chudnovsky, Reed, and Seymour in [5] proved the following bound on the number of edges in an n-vertex graph with no $K_{2,t}$-minor.

Theorem 2.23. *For any $t \geq 2$ and any graph G on $n > 0$ vertices with no $K_{2,t}$-minor,*

$$|E(G)| \leq \frac{1}{2}(t+1)(n-1).$$

This extends a result of Myers in [17], who had shown that Theorem 2.23 holds for $t \geq 10^{29}$.

The authors also give the following construction to show that this result is best possible for n, t such that $t \mid n-1$: form a $K_{2,t}$-minor-free graph by taking the disjoint union of $\frac{n-1}{t}$ copies of K_t and adding a single vertex adjacent to all other vertices. Such a graph will have exactly $\frac{1}{2}(t+1)(n-1)$ edges.

The analogous construction for general $K_{s,t}$ is also optimal for $s = 3$ and sufficiently large t, is not known to be optimal for $s = 4, 5$, and is known not to be optimal for $s \geq 6$.

Their bound can be improved somewhat if one restricts to more highly connected graphs.

Theorem 2.24. *For every $t \geq 0$, there exists $c(t) \geq 0$ such that every 3-connected n-vertex graph with no $K_{2,t}$-minor has at most $\frac{5n}{2} + c(t)$ edges.*

They also describe a family of 4-connected $K_{2,5}$-minor-free graphs with $\frac{5n}{2}$ edges (namely $C_{\frac{n}{2}}[K_2]$, the notation for which we will define in Chapter 4, where we also show that these are, in a sense, the prototypical members of this family). They use this to demonstrate that the coefficient of the linear term in this bound cannot be improved for 4-connected $K_{2,t}$-minor-free graphs.

2.3.3 On the Structure of $K_{2,t}$-Minor-Free Graphs

Guoli Ding, in the unpublished article [8], gives a structural description of $K_{2,t}$-minor-free graphs. As with the structure theorem of Robertson and Seymour, his description is not a complete characterization of this family; while all graphs without $K_{2,t}$ as a minor (for any given t) will have the promised structure, there will also be many graphs with this structure that do have $K_{2,t}$-minors. The results described in Chapter 4 can be seen as a refinement to this structural description, giving a complete characterization in the particular case of 4-connected $K_{2,5}$-minor-free graphs.

Roughly speaking, Theorem 2.29 below states that for any t, all 2-connected $K_{2,t}$-minor-free graphs can be built by gluing together graphs of bounded size (depending on t) which can be augmented by adding "strips" and "fans" of arbitrary size. This is analogous to the following result about $K_{1,t}$-minor-free graphs.

Theorem 2.25 (Robertson and Seymour, [22]). *There exists a function $f(t)$ such that every component of a $K_{1,t}$-minor-free graph is a subdivision of a graph on at most $f(t)$ vertices.*

Before we can formally state Theorem 2.29, we will need a few definitions.

First, consider a graph G with a Hamiltonian cycle C, which Ding calls the *reference cycle*. Any edge outside of C is called a *chord*, and two chords ab and cd with distinct endpoints are said to *cross* if their endpoints appear in the order a, c, b, d around C. We call G a *type-I* graph if every chord crosses at most one other chord and if, for each pair of chords ab and cd that do cross, either ac and bd are both edges of C or both ad and bc are. Any 2-connected outerplanar graph is type-I, with the reference cycle given by the walk around the outer face. Indeed, type-I graphs can be thought of as a generalization of 2-connected outerplanar graphs in which chords are allowed to cross, but only in very restricted ways. Guoli Ding refers to the set of all type-I graphs as $\mathcal{P}$ (not to be confused with the family of graphs that will be defined in Chapter 4).

Definition 2.26. Let H be a type-I graph with reference cycle C such that there exist distinct edges $ab, cd \in E(C)$ so that all chords connect the two paths of $C \setminus \{ab, cd\}$.

If ab and cd do not share an endpoint, then $H \setminus \{ab, cd\}$ is referred to as a *strip* with corners a, b, c, and d.

If ab and cd do share an endpoint, say $b = c$, then $H \setminus \{ab, cd\}$ is referred to as a *fan* with corners a, b, and d.

The number of chords across C is called the *length* of the strip or fan.

Definition 2.27. We can *add* a strip or fan to a graph G by identifying the corners of the strip or fan with distinct vertices of G. An *augmentation* of G is any graph obtained by adding strips and fans to disjoint sets of vertices of G.

Define $\mathcal{B}_m$ to be the set of all graphs on at most m vertices, $\mathcal{A}_m$ to be the set of all augmentations of graphs in $\mathcal{B}_m$, and $\mathcal{A}'_m$ to be the set of all graphs obtained by augmenting a graph in $\mathcal{B}_m$ with strips (but not fans).

Definition 2.28. Consider any two graphs G_1 and G_2 and vertices $z_1 \in V(G_1)$ and $z_2 \in V(G_2)$ of degree exactly two. For $i \in \{1, 2\}$, let x_i and y_i be the two vertices adjacent to z_i. Define the *modified-2-sum* of G_1 and G_2 over z_1 and z_2 to be the simple graph formed by deleting z_1 and z_2 and identifying x_1 with x_2 and y_1 with y_2.

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are families of graphs, then the *modified-2-sum* of $\mathcal{G}_1$ with $\mathcal{G}_2$ is the set of graphs obtainable by taking a modified-two-sum of a graph in $\mathcal{G}_1$ with any number of graphs in $\mathcal{G}_2$. Define $\mathcal{G}^1$ to be G, and $\mathcal{G}^k$ to be the modified-2-sum of $\mathcal{G}^{k-1}$ with $\mathcal{G}$ for $k \geq 2$.

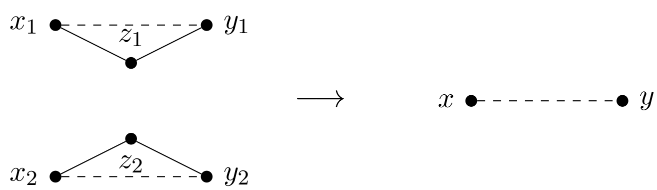


Figure 2.6: A modified-2-sum.

Guoli Ding refers to this operation simply as *2-summing*, but because it differs from the usual definition of a clique sum (Definition 1.2), we will always use the descriptor *modified* when referring to this operation.

Ding defines $\mathcal{K}_r := \{K_{r,s} \mid s \geq r\}$, and says that a class of k-connected graphs is *minor-closed* if every k-connected minor of a member is also a member. We can now state the main theorem of his paper.

Theorem 2.29. *Let $\mathcal{G}$ be a minor-closed class of 2-connected graphs. Then $\mathcal{K}_2 \not\subseteq \mathcal{G}$ if and only if $\mathcal{G} \subseteq (\mathcal{P} \cup \mathcal{A}_m)^m$ for some m.*

Note that $\mathcal{K}_2 \not\subseteq \mathcal{G}$ if and only if there exists some t such that every graph in $\mathcal{G}$ is $K_{2,t}$-minor-free. In particular, for each t this describes a structure common to the set of $K_{2,t}$-minor-free graphs.

This theorem can also be used to describe all $K_{2,t}$-minor-free graphs: a graph is $K_{2,t}$-minor free only if all blocks (maximal 2-connected subgraphs) satisfy the above theorem, because $K_{2,t}$ is 2-connected, so the branch sets for the minor can be assumed to all live inside a single block.

Corollary 2.30. *Let $\mathcal{G}$ be a minor-closed class of k-connected graphs.*

1. *If $k = 3$, then $\mathcal{K}_2 \not\subseteq \mathcal{G}$ if and only if $\mathcal{G} \subseteq \mathcal{A}_m$ for some m.*

2. *If $k = 4$, then $\mathcal{K}_2 \not\subseteq \mathcal{G}$ if and only if $\mathcal{G} \subseteq \mathcal{A}'_m$ for some m.*

3. *If $k = 5$, then $\mathcal{K}_2 \not\subseteq \mathcal{G}$ if and only if $\mathcal{G} \subseteq \mathcal{B}_m$ for some m.*

The $k = 3$ case follows from Theorem 2.29 because the modified-2-sum of two graphs on more than 3 vertices will be at most 2-connected (the two joined vertices form a 2-cut), and sufficiently large graphs in $\mathcal{P}$ will have connectivity exactly 2. The $k = 4$ case follows because the corners of an added fan will form a 3-cut if the fan introduces any new vertices. The $k = 5$ case follows because the corners of an added strip will form a 4-cut if the strip introduces any new vertices.

This corollary itself has some important consequences.

Corollary 2.31. *Fix any $t > 0$. Then there exists $m = m(t)$ such that every graph G satisfying at least one of*

- *G is 3-connected, $|V(G)| \geq m$ and $\delta(G) \geq 6$*
- *G is 4-connected and $\delta(G) \geq m$*
- *G is 5-connected and $|V(G)| \geq m$*

will have a $K_{2,t}$-minor.

In particular, this states that there are only finitely many 5-connected $K_{2,t}$-minor-free graphs for each t.

Chapter 3

Algorithmic Aspects of Graph Minor Theory

3.1 Theoretical Results

We now turn our attention to the problem of determining, for any given graphs G and H, whether $H \preccurlyeq G$. If both graphs are given as input to this algorithm, then the problem is NP-complete (such an algorithm can be used to solve the Hamilton Cycle problem on a graph G on n vertices by determining if $C_n \preccurlyeq G$). It remains NP-complete to determine whether a graph has a K_n-minor, when n is part of the input, see [10]. We might hope that minor-containment could be solved more efficiently for particular graphs H, and indeed this is the case.

A simple graph is K_3-minor-free if and only if it is a forest, which can be identified very quickly. Bruce Reed and Zhentao Li present in [20] a linear time algorithm to find K_5-minors, which returns a model of the minor if it exists or a decomposition into the structure described in Wagner's characterization of K_5-minor-free graphs, Theorem 2.3. It is also possible to determine planarity in linear time, producing either an embedding or a $K_{3,3}$- or K_5-minor as appropriate. See, for instance, [3]. Indeed, Robertson and Seymour were able to show (in [23]) that for *any* fixed graph H, it is possible to determine whether or not H is a minor of a graph G in $O(|V(G)|^3)$ time. This hides a highly super-exponential dependence on $|V(H)|$, however, making their algorithm very inefficient in practice.

Their result is still very important, especially to theoretical computer science. Combined with their Theorem 2.10, we see that inclusion in any minor-closed family can be tested in $O(n^3)$ time, as it can be solved by testing for a finite number of excluded minors, each of which can be done in cubic time. Without first knowing the obstruction set for that family, though, these theorems would only prove the existence of a cubic algorithm to test inclusion; they would not allow you to actually construct one. This is of particular relevance, because the obstruction set for even familiar minor-closed families can be quite large. There are 35 forbidden minors for the set of graphs embeddable on the projective plane [1], and 16,629 known forbidden minors for toroidal graphs [12].

Just the existence of such an algorithm can be useful, however. The problem of determining if a graph is knotlessly embeddable (i.e. if it can be embedded in $\mathbb{R}^3$ such that no cycle induces a non-trivial knot) was previously not known to be decidable, but because this property is closed under taking minors [18], it is not only decidable but can be solved in cubic time.

3.2 Practical Graph Minor Containment

In our research on 4-connected $K_{2,5}$-minor-free graphs (Chapter 4) and planar 4-connected DW_6-minor-free graphs (Chapter 5), it has been very helpful to explicitly construct the family in question up to some number of vertices. To do so in the former case, we made use of a program written by Mark Ellingham to test for $K_{2,t}$-minors, which takes advantage of the fact that the branch sets for the vertices in the second partite set can be assumed to be singletons. The program checks each t-tuple of vertices S in a graph G and determines if there exist two connected subgraphs of $G \setminus S$ which are both adjacent to each of vertex of S. When it was no longer feasible to construct all graphs on n vertices and filter down by connectivity and $K_{2,5}$-minor-freeness, we wrote a program which would take in every 4-connected $K_{2,5}$-minor-free graph on $n-1$ vertices and then would, for each graph, split every vertex in every way that preserves 4-connectedness (Lemma 1.1), add to this list the contraction-minimal 4-connected graphs described in Theorem 4.9, and test each resulting graph for $K_{2,5}$-minors. While this process could certainly be made more efficient, it was sufficient for our purposes.

To experiment with families of graphs excluding other minors, notably DW_6, we needed a program capable of determining whether $H \preccurlyeq G$ for arbitrary graphs H and G. While the algorithm described by Robertson and Seymour is quite impractical, it would not be hard to create an exponential time program which would be effective at least for reasonably small graphs. As far as we are aware, the only publicly available program with this capability is the `minor` function of the `Graph` class in SageMath [26], an open source mathematics library. Its implementation converts minor containment into an integer linear program, then uses existing linear programming solvers to determine feasibility. This program was too slow, however, to be used to test a large number of even fairly small graphs. There are also some libraries for working with matroid minors, for example Macek by Petr Hliněný [14].

Efficiently finding graph minors is of interest even outside of mathematics. To encode a problem into D-Wave's quantum computers, it is necessary to find a minor in a (potentially very large) graph of qubit interactions. This motivated the heuristic algorithm presented in [4], which is effective on graph with hundreds of vertices, but is not guaranteed to find a minor if it exists.

We therefore designed and implemented a new program, called canary, to find graph minors. The source code and documentation for the program are available at `https://github.com/JZacharyG/canary`. Canary was used to confirm the previously generated family of 4-connected $K_{2,5}$-minor-free graphs on up to 17 vertices, as well as to generate all planar 4-connected DW_6-minor-free graphs on up to 20 vertices. The latter family was later confirmed by hand.

Canary works, roughly, as follows. Fix simple graphs G and H on n and m vertices respectively. Fix an ordering of the vertices of each graph $g_1, \ldots, g_n$ and $h_1, \ldots, h_m$. We build up a model for H as a minor of G through a depth first search; at every point in the search tree we maintain a model for a subgraph H' of H with $V(H') = \{h_1, \ldots, h_k\}$ for some k, along with all edges h_ih_j with $i, j < k$ and some subset of edges h_ih_k with $i < k$. Then at each level in the search tree, we attempt to extend our model by either incorporating another edge from h_k to a previous vertex, or (if there are no more such edges to add) beginning a branch set for h_{k+1}.

In the latter case, we select a vertex of G not currently part of the model and assign it to be part of the branch set for h_{k+1} (we will say that it is the *root* of this branch set). This node in the search tree, then, will have a child for each possible root. Upon fixing a root for this branch set, we will find a way of extending the current model to a model for all of H, if any extension placing this root vertex in the branch set for h_{k+1} exists. If we fail to find such an extension and have to move on to another possible root, then we can safely assume that any previously checked roots are not part of this branch set.

When extending the model to include an edge h_ih_k, we must find a path connecting the branch sets of h_i and h_k. Given such a path, it does not matter exactly which vertices along it are assigned to h_i and which to h_k, so we instead note that these vertices will belong to one of the two branch sets (we say that these vertices are *semiassigned* to h_i and to h_k). If we are later able to determine to which branch set a vertex in a path belongs, then that will force everything before/after it as well. In effect, we have a path and two cutoffs, dividing the path into pieces assigned to h_i, either, and h_k.

There are a few ways in which we are able to prune the search tree, which mostly have to do with either ruling out potential roots or requiring that paths be chosen minimally.

For more information, the full source code is available at `https://github.com/JZacharyG/canary`.

Chapter 4

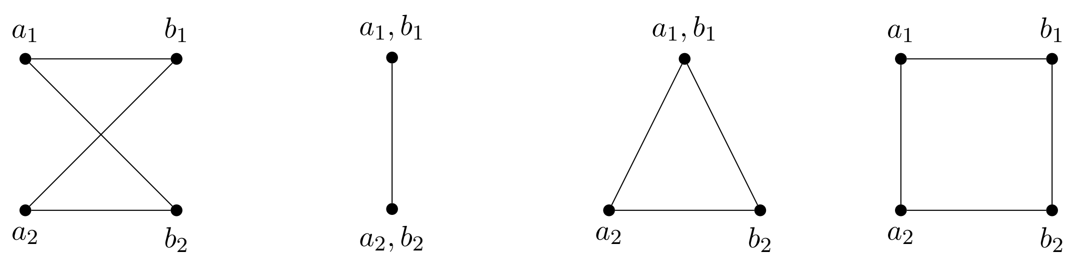
Characterization and Enumeration of 4-Connected $K_{2,5}$-Minor-Free Graphs

This chapter provides a complete structural characterization of 4-connected $K_{2,5}$-minor-free graphs. The following is joint work with Mark Ellingham.

If $K_{2,t} \preccurlyeq G$, then there is a model of $K_{2,t}$ in G in which the branch set of every vertex in the partite set of size t is a singleton. It will often be convenient to work with such a model, or to assume that a model is of this form, so we say a $K_{2,t}$-minor in a graph G is *given by* $(R_1, R_2; S)$ if R_1, R_2, and S are disjoint subsets of $V(G)$ such that R_1 and R_2 both induce connected subgraphs of G, $S \subseteq N(R_1) \cap N(R_2)$, and $|S| = t$.

4.1 Preliminary Definitions

Define a X-, I-, Δ-, or Q-type piece to be a copy of one of the graphs shown in Figure 4.1. The following definitions and lemmas describe how these pieces can be glued together, through $\mathcal{Q}$-seqeuences, to form a family of graphs. We will see in Theorem 4.12 that, on at least 9 vertices, these are exactly the 4-connected $K_{2,5}$-minor-free graphs.



(a) An X-type piece. (b) An I-type piece. (c) A Δ-type piece. (d) A Q-type piece.

Figure 4.1: The four building blocks referred to in Part 2 of Theorem 4.12.

Definition 4.1. Consider a cyclic sequence $P_1, \ldots, P_m$, $m \geq 1$, of X-, I-, Δ-, and Q-type pieces. We shall refer to vertex a_1 in piece P_j as a_1^j, and similarly for the vertices a_2, b_1, and b_2 in each piece. The index of a piece in such a sequence should always be taken only up to congruence modulo m. We can construct one or more simple graphs from this sequence as follows. For each j, with $1 \leq j \leq m$, glue P_j to P_{j+1} by identifying either $b_1^j = a_1^{j+1}$ and $b_2^j = a_2^{j+1}$, or $b_1^j = a_2^{j+1}$ and $b_2^j = a_1^{j+1}$. Let $\mathcal{P}$ denote the set of all graphs that can be built in this way, and let $\mathcal{Q} := \{G \in \mathcal{P} \mid \delta(G) \geq 4\}$.

We may assume that the sequence of pieces from which we build a graph in $\mathcal{P}$ has all I-type pieces appear immediately between two X-type pieces (unless the graph is K_2), because in any other context the edge added by the I would have already been present, so removing it from the sequence and gluing everything else as before would yield an isomorphic graph. We call a cyclic sequence of pieces with this restriction, along with a choice of how to glue each consecutive pair of pieces, a $\mathcal{P}$-sequence. Given a $\mathcal{P}$-sequence corresponding to a graph G and any vertex $v \in V(G)$, $P(v)$ denotes the circular interval of pieces in this sequence that contain v.

We now define a restriction on $\mathcal{P}$-sequences which is equivalent to the corresponding graph being in $\mathcal{Q}$ (see Lemma 4.5).

Definition 4.2. A *$\mathcal{Q}$-sequence* is a $\mathcal{P}$-sequence $P_1, \ldots, P_m$ in which each Q-type piece and each I-type piece appears immediately between X-type pieces, and in which consecutive Δ-type pieces (say P_j and P_{j+1}) are glued with opposing orientations, i.e. with $b_1^j = a_2^{j+1}$ and $b_2^j = a_1^{j+1}$.

For an example of a $\mathcal{Q}$-sequence and its corresponding graph, see Figure 4.2.

Observation 4.3. *Consider any $\mathcal{Q}$-sequence $P_1, \ldots, P_m$ corresponding to a graph G and any vertex v of G. If G has at least six vertices, and hence at least 3 pieces, then $P(v)$ must be one of the following or a reflection thereof: XX, XIX, QX, ΔX, $\Delta^2 X$,*

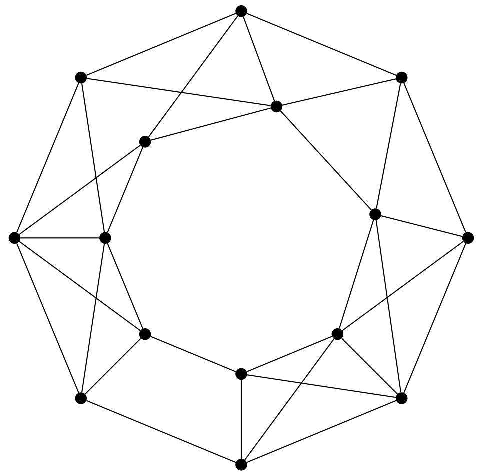


Figure 4.2: The graph corresponding to the $\mathcal{Q}$-sequence $\Delta\Delta\Delta XIXQXIXX$.

Δ^3, or $X\Delta X$; see Figure 4.5. As such, $2 \leq |P(v)| \leq 3$ and $5 \leq |V(P(v))| \leq 7$. Note also that if $u \sim v$ then the edge uv is contained in some piece, so $P(u) \cap P(v) \neq \emptyset$.

Lemma 4.4. *Let G be a graph corresponding to a $\mathcal{Q}$-sequence $P_1, \ldots, P_m$, and fix any vertex v of G. Let $P(v) = P_i, \ldots, P_j$. If G has at least 7 vertices, then $\{a_1^i, a_2^i, b_1^j, b_2^j\}$ are all distinct vertices and hence all vertices of $P(v)$ (as they appear in Figure 4.5) are distinct.*

Proof. If, to the contrary, there exists some v' with $v' \in \{a_1^i, a_2^i\}$ and $v' \in \{b_1^j, b_2^j\}$, then $P(v')$ contains the interval $P_j, \ldots, P_i$. Because each vertex is in at most three pieces, either $j = i$ or $j + 1 \equiv i \mod m$ and there are no pieces that do not contain v, or $j + 2 \equiv i \mod m$ and P_{j+1} is either I- or Δ-type. In any of these cases, all vertices of G are contained in some piece containing v. In general, $|V(P(v))| \leq 7$, but this counts v' twice, so $|V(G)| \leq 6$. □

Lemma 4.5. *Suppose $G \in \mathcal{P}$, with $|V(G)| \geq 7$. Then the following are equivalent:*

1. *$G \in \mathcal{Q}$.*

2. *G can be formed by gluing together a $\mathcal{Q}$-sequence as described in Definition 4.2.*

Proof. For the equivalence of these two conditions, we need only look at each possible way of gluing two pieces together, and examine the degree of the two vertices common to those pieces. It is clear that either way of gluing a Q-type piece to either a Δ- or Q-type piece would result in a vertex of degree 3, and if two Δ-type pieces are glued together with $b_2^j = a_2^{j+1}$, then this vertex will have degree 3. Thus, any graph in $\mathcal{Q}$ must be formed from a $\mathcal{Q}$-sequence.

Conversely, take any $\mathcal{Q}$-sequence, $P_1, \ldots, P_m$ corresponding to a graph G on at least seven vertices, and let v be any vertex of G. By Lemma 4.4, all vertices of $P(v)$ are distinct, so any of possibilities for $P(v)$ will give v degree at least 4, as can be seen in Figure 4.5. □

Two $\mathcal{Q}$-sequences on the same cyclic sequence of pieces will give isomorphic graphs, so we will, for the remainder of the paper, not distinguish between these superficially different gluings when talking about a $\mathcal{Q}$-sequence. To see this, note that consecutive pieces are either both Δ-type pieces, and their gluing is forced, or at least one is an X-type piece, and so the two ways of gluing them would yield isomorphic graphs.

Furthermore, the decomposition of a graph in $\mathcal{Q}$ (with at least 13 vertices) into a $\mathcal{Q}$-sequence is unique; see Lemma 4.14 below.

Definition 4.6. The lexicographic product of two graphs G and H, denoted $G[H]$ is a graph with vertex set $V(G[H]) = V(G) \times V(H)$ and an edge between (g_1, h_1) and (g_2, h_2) whenever $g_1 \sim g_2$, or $g_1 = g_2$ and $h_1 \sim h_2$. We will only use this notation to refer to $C_n[K_2]$. In this case, where C_n is the cycle $v_1, \ldots, v_n$, we refer to the pair of vertices associated with v_i as $T_i = \{v_{i,1}, v_{i,2}\}$. See Figure 4.3.

Lemma 4.7. *Each graph $G \in \mathcal{P}$ is a minor of $C_n[K_2]$ for some n.*

Proof. Let $G \in \mathcal{P}$ be formed by gluing together the $\mathcal{P}$-sequence $P_1, \ldots, P_m$, as described in Definition 4.1. The result is clearly true if $m = 1$. Otherwise, we will

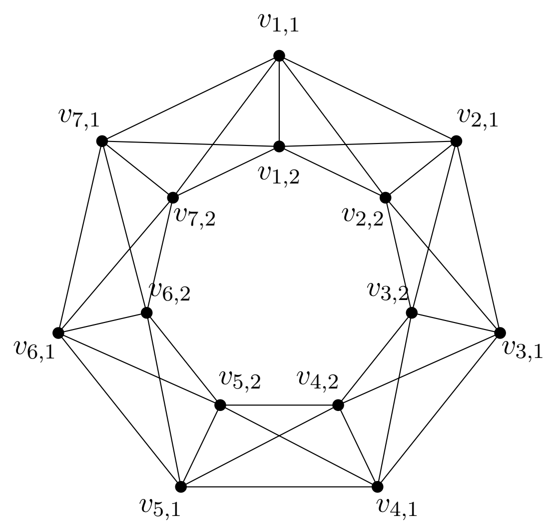


Figure 4.3: A labeled drawing of $C_7[K_2]$.

realize G as a minor of $C_{2m}[K_2]$ by performing the following two operations for each i, $1 \le i \le m$. First, if P_i is...

- X-type, then delete the edges $v_{2i-1,1}v_{2i-1,2}$ and $v_{2i,1}v_{2i,2}$.

- I-type, then contract the edges $v_{2i-1,1}v_{2i,1}$ and $v_{2i-1,2}v_{2i,2}$.

- Δ-type, then contract the edge $v_{2i-1,1}v_{2i,1}$.

- Q-type, then delete the edges $v_{2i-1,1}v_{2i,2}$ and $v_{2i-1,2}v_{2i,1}$.

Next, if P_j is glued to P_{j+1} by identifying...

- $b_1^j = a_1^{j+1}$ and $b_2^j = a_2^{j+1}$, then delete the edges $v_{2i,1}v_{2i+1,2}$ and $v_{2i,2}v_{2i+1,1}$ and contract $v_{2i,1}v_{2i+1,1}$ and $v_{2i,2}v_{2i+1,2}$.

- $b_1^j = a_2^{j+1}$ and $b_2^j = a_1^{j+1}$, then delete the edges $v_{2i,1}v_{2i+1,1}$ and $v_{2i,2}v_{2i+1,2}$ and contract $v_{2i,1}v_{2i+1,2}$ and $v_{2i,2}v_{2i+1,1}$.

Thus, G is a minor of $C_n[K_2]$ for some n. $\square$

4.2 Characterization

The following definition, theorem, and lemma are necessary for the inductive argument in the first part of the proof of Theorem 4.12.

Definition 4.8. The *square* of a graph G, denoted G^2, is a graph on the same vertex set as G, with $v_1 \sim v_2$ in G^2 if v_1 and v_2 are at distance at most 2 in G.

Theorem 4.9 (Fontet [11] and Martinov [16])**.** *A 4-connected simple graph G has no edge e such that G/e is 4-connected if and only if G is the square of a cycle of length at least 5, or the line graph of a cyclically-4-edge-connected cubic graph.*

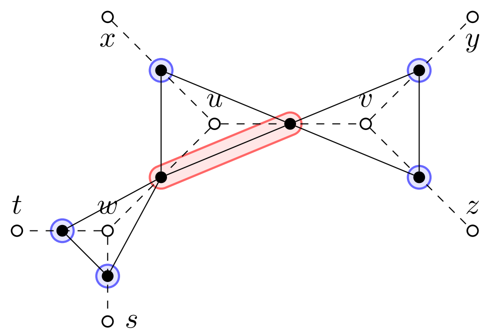


Figure 4.4: A neighborhood around uv in H (shown with dashed edges), along with the corresponding portion of $\mathrm{L}(H)$ (shown with solid edges), highlighting the $K_{2,5}$ minor, with the branch set of the remaining vertex given by the entire rest of the line graph.

Lemma 4.10. *For all 3-connected cubic graphs $H \ncong K_4$, $\mathrm{L}(H)$ has a $K_{2,5}$-minor.*

Proof. Let $H \ncong K_4$ be a 3-connected cubic graph. Every such graph must have some edge, uv, not in a triangle. Let w, x and y, z be the (necessarily distinct) neighbors of u and v, respectively. Although the two neighbors of w other than u may not be distinct from x, y, and z, call them s and t, as in Figure 4.4.

Because H is 3-connected, $H \setminus \{v, w\}$ is connected. Note that u cannot be a cut vertex of $H \setminus \{u, w\}$, because it has degree 1 in this graph, so $H \setminus \{u, v, w\}$ is connected. Because $x \neq y$, $H \setminus \{u, v, w\}$ must contain an edge, so its edges will induce a nonempty connected subgraph of $L(H)$ which avoids the vertices corresponding to edges uv, ux, uw, vy, vz, ws, and wt. Then $L(H)$ has a $K_{2,5}$ minor given by $(\{uv, uw\}, \mathrm{L}(H) \setminus N_{\mathrm{L}(H)}[uv, uw]; N_{\mathrm{L}(H)}(uv, uw))$. See Figure 4.4. □

Every cyclically-4-edge-connected graph is, in particular, 3-connected, and $\mathrm{L}(K_4) \cong C_6^2$. This, combined with Lemma 4.10, gives the following corollary to Theorem 4.9.

Corollary 4.11. *Every 4-connected $K_{2,5}$-minor-free graph on n vertices is either C_n^2 or is obtained from a 4-connected $K_{2,5}$-minor-free graph on $n-1$ vertices by splitting a vertex.*

This corollary allows us to generate the n-vertex graphs in this family from those on $n-1$ vertices by splitting every vertex of every graph in each way that preserves 4-connectivity (Lemma 1.1), testing the resulting graphs for $K_{2,5}$-minors, then adding in C_n^2.

Theorem 4.12. *For a simple graph G with $|V(G)| \geq 9$, the following are equivalent:*

1. *G is 4-connected and $K_{2,5}$-minor-free.*

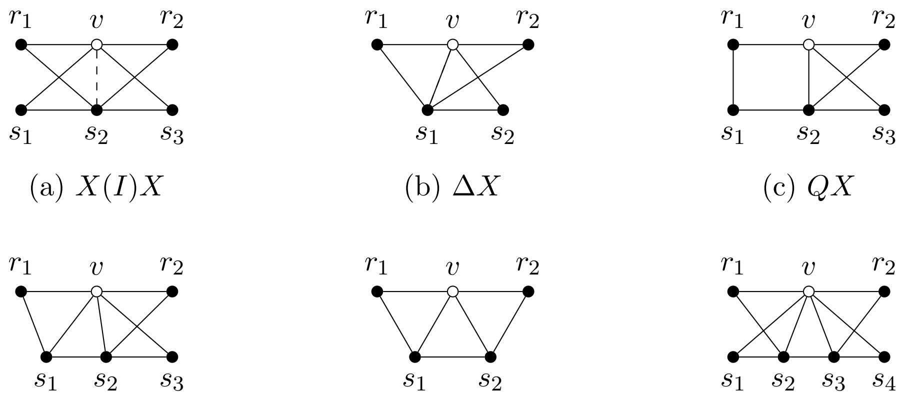
2. *$G \in \mathcal{Q}$*

3. *G is a 4-connected minor of $C_n[K_2]$ for some n.*

Proof ($1 \implies 2$). Through an exhaustive computer search to generate this family on up to 12 vertices using Corollary 4.11, it has been confirmed that all 4-connected $K_{2,5}$-minor-free graphs on between 9 and 12 vertices are in $\mathcal{Q}$. The rest of the proof proceeds by induction on the number of vertices.

Let G be an arbitrary 4-connected $K_{2,5}$-minor-free graph with $|V(G)| = n \geq 13$. If $G \cong C_n^2$, then we can obtain G by gluing together the $\mathcal{Q}$-sequence $\Delta^n = \Delta, \Delta, \ldots, \Delta$.

Otherwise, by Corollary 4.11, G must contain an edge v_1v_2 whose contraction yields a 4-connected graph G'. Let $v \in V(G')$ be the vertex formed by contracting



(a) $X(I)X$ (b) ΔX (c) QX

(d) $\Delta\Delta X$ (e) $\Delta\Delta\Delta$ (f) $X\Delta X$

Figure 4.5: Given a $\mathcal{Q}$-sequence and a vertex v of the corresponding graph, these are all possible intervals $P(v)$ of pieces containing v, up to reflections. Vertices are labeled as in $P_j, \ldots, P_m$ in G', as described in the first part of the proof of Theorem 4.12.

v_1v_2. Note that G' is still $K_{2,5}$-minor-free and $|V(G')| = n - 1$, so by the induction hypothesis, G' is built from a $\mathcal{Q}$-sequence $P_1, \ldots, P_m$. Label the vertices of P_j with $a_1^j, a_2^j, b_1^j, b_2^j$ as in Definition 4.1.

By cyclically shifting the sequence of pieces, we can assume that $P_j, \ldots, P_m$ are the only pieces containing v in this sequence. Because each vertex of G' is contained in either two or three pieces, $m - j \in \{1, 2\}$. This immediately implies that neither P_j nor P_m is I-type, and that if either is Δ-type, v must not be $a_1 = b_1$ in that piece.

Then v is either b_1^j or b_2^j; define r_1 to be a_1^j or a_2^j respectively. Similarly, v is either a_1^m or a_2^m; define r_2 to be b_1^m or b_2^m respectively. In particular, $v \neq r_1, r_2$. The vertices $(V(P_j) \cup \cdots \cup V(P_m)) \setminus \{v, r_1, r_2\}$ induce a path $s_1, \ldots, s_k$, labeled so that $s_1 \in V(P_j)$ and $s_k \in V(P_m)$ with $2 \leq k \leq 4$ (see Figure 4.5). With these definitions, $N(v_1), N(v_2) \subseteq \{v_1, v_2, r_1, r_2, s_1, \ldots, s_k\}$, $\{r_1, s_1\} = \{a_1^j, a_2^j\}$, and $\{r_2, s_k\} = \{b_1^m, b_2^m\}$.

Our choices of r_1 and r_2 are not uniquely defined. In particular, reversing the sequence of pieces $P_1, \ldots, P_m$ would yield a graph isomorphic to G', but with the role of P_j and P_m, and hence r_1 and r_2, swapped. This would also reverse the labeling of the path $s_1, \ldots, s_k$. Because our labeling of v_1 and v_2 was arbitrary, we can assume

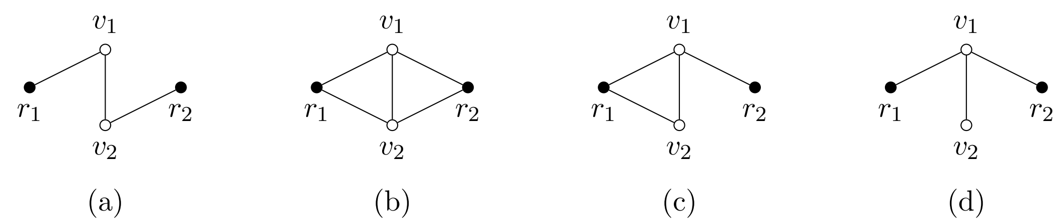


Figure 4.6: The 4 possible ways in which v_1, v_2, r_1, and r_2 can be connected in G.

that v_1, v_2, r_1, and r_2 are chosen so that they are connected in one of the four ways shown in Figure 4.6.

We need to show that the subgraph of G corresponding to $P_j, \ldots, P_m$ can also be decomposed into X-, I-, Δ-, and Q-type pieces, implying that $G \in \mathcal{Q}$.

To this end, we first show that v_1, v_2, r_1, and r_2 must be connected as in Figure 4.6a, i.e. that v_1 (and hence v_2) cannot be adjacent to both r_1 and r_2. Assume to the contrary that $v_1 \sim r_1$ and $v_1 \sim r_2$. We will find a $K_{2,5}$-minor given by $(R_1, R_2; S)$ with R_1 and R_2 adjacent to v_1, v_2, and s_{i^*} (for some i^* to be determined), as well as two other vertices elsewhere in the graph.

If v_2 is adjacent to r_1 (i.e. v_1, v_2, r_1, and r_2 are connected as is shown in Figures 4.6b or 4.6c), then choose i^* to be minimum so that s_{i^*} is adjacent to v_2. Because v_2 is adjacent to at least 3 vertices in $N_{G'}(v)$, we must have either $v_2 \sim r_2$, $v_2 \sim s_i$ for some $i > i^*$, or both.

Otherwise v_1, v_2, r_1, and r_2 are connected as shown in Figure 4.6d, in which case v_2 has at least three neighbors in $s_1, \ldots, s_k$; let s_{i^*} be the second such neighbor of v_2.

We can now finish describing the $K_{2,5}$-minor by defining the branch sets R_1 and R_2 and the remaining two vertices in S.

Define j^- to be maximum such that $|V(P_{j^-}) \cup \cdots \cup V(P_{j-1})| \geq 5$. Because each piece contributes either zero, one, or two new vertices, we in fact have $5 \leq |V(P_{j^-}) \cup \cdots \cup V(P_{j-1})| \leq 6$. Such a j^- will exist, because G' has at least 12 vertices and at most seven will be contained in $P(v)$, with two of these vertices shared with P_{j-1}. All possible sequences $P_{j^-}, \ldots, P_{j-1}$ are shown in Figure 4.7.

With this, define

$$S := \{v_1, v_2, s_{i^*}, a_1^{j^-}, a_2^{j^-}\}$$
$$R_1 := \{s_i \mid i < i^*\} \cup V(P_{j^-}) \cup \cdots \cup V(P_{j-1}) \setminus S$$
$$R_2 := \{s_i \mid i > i^*\} \cup V(P_1) \cup \cdots \cup V(P_{j^- -1}) \setminus S.$$

Thus, for each sequence of pieces $P_{j^-}, \ldots, P_{j-1}$ in Figure 4.7, R_1 consists of the set surrounded by a thicker line plus possibly a path consisting of $\{s_i \mid i < i^*\}$, and S consists of $\{v_1, v_2, s_{i^*}\}$ plus the two vertices circled with a thin line. From this, we see that R_1 will induce a connected subgraph (even if $i^* = 1$ so that $s_1 \notin R_1$) and will be adjacent to $a_1^{j^-}$ and $a_2^{j^-}$. Because it contains r_1, some vertex adjacent to s_1, and all s_i for $i < i^*$, R_1 will also be adjacent to v_1, v_2, and s_{i^*}.

Next, note that G' has at least 12 vertices, $|V(P_{j^-}) \cup \cdots \cup V(P_m)| \leq 11$ (see Figures 4.5, and 4.7), and this set of pieces has exactly 4 vertices in common with $V(P_1) \cup \cdots \cup V(P_{j^- -1})$. Thus, $|V(P_1) \cup \cdots \cup V(P_{j^- -1})| \geq 5$, so by similar reasoning to the above, R_2 will induce a connected subgraph (even if $i^* = k$ so that $s_k \notin R_2$), and it will be adjacent to $a_1^{j^-}$ and $a_2^{j^-}$. Because it contains r_2 and all s_i for $i > i^*$, R_2 will also be adjacent to v_1, v_2, and s_{i^*}.

Thus, if v_1, v_2, r_1, and r_2 were connected as in Figures 4.6b, 4.6c, or 4.6d, then G would have a $K_{2,5}$-minor given by $(R_1, R_2; S)$, a contradiction, so we see that neither v_1 nor v_2 can be adjacent to both r_1 and r_2 in G. These vertices must therefore be connected as is shown in Figure 4.6a.

Observation 4.13. *We will now assume, without loss of generality, that $r_1 \sim v_1$ and $r_2 \sim v_2$ in G. If P_j is X-type, let $r_1' = s_1$. Then we may apply the same reasoning to*

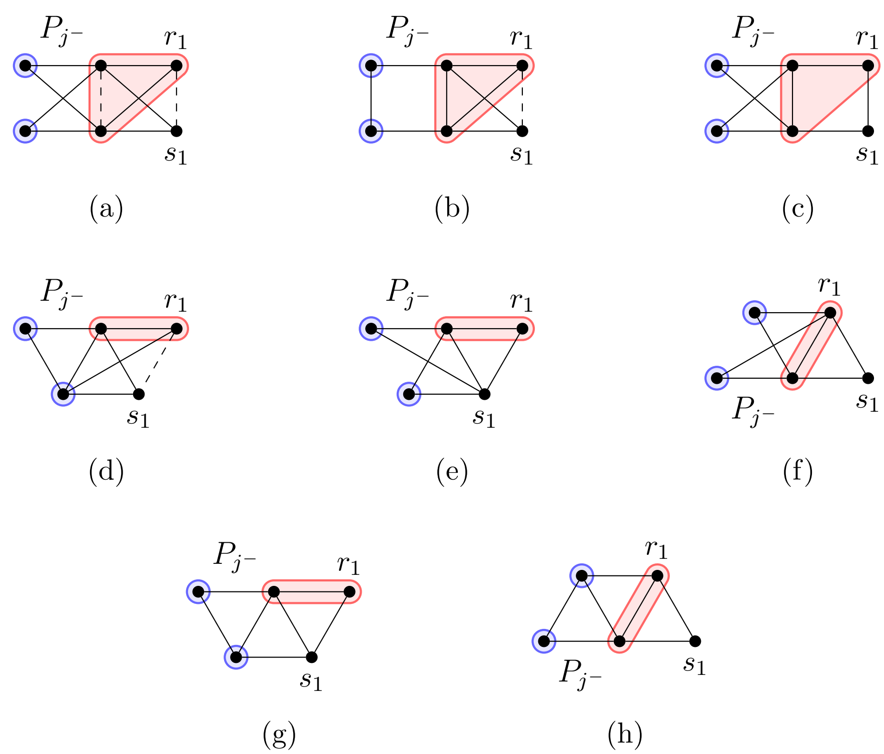


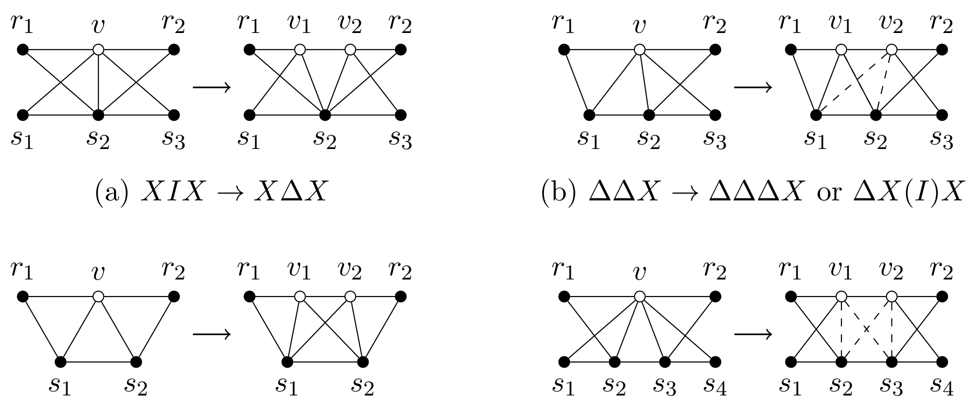
Figure 4.7: All possible sequences of pieces $P_{j^-}, \ldots, P_{j-1}$.

r_1', r_2 as to r_1, r_2, so we may assume that r_1' is adjacent to v_1 but not v_2. Similarly, if P_m is X-type we may assume that $r_2' = s_k$ is adjacent to v_2 but not v_1.

We now show that for every possible sequence $P(v) = P_j, \ldots, P_m$ of pieces in G', the corresponding subgraph in G must also have been decomposable into a sequence of pieces. See Figure 4.8. The arguments made are symmetric, so we need only check, for example, one of QX and XQ.

(XX, ΔX, QX) If the sequence $P_j, \ldots, P_m$ is of type XX, ΔX, or QX, then v has degree 4, so by Observation 4.13 v_1 can have degree at most 3, which contradicts the 4-connectedness of G. There is no way to split v in any of these sequences to yield a 4-connected $K_{2,5}$-minor-free graph.

(XIX) By Observation 4.13 and 4-connectivity, we must have $v_1, v_2 \sim s_2$, giving an $X\Delta X$ in G.



(a) $XIX \to X\Delta X$

(b) $\Delta\Delta X \to \Delta\Delta\Delta X$ or $\Delta X(I)X$

(c) $\Delta\Delta\Delta \to \Delta X\Delta$

(d) $X\Delta X \to X\Delta\Delta X, X(I)X(I)X, XQX$

Figure 4.8: The left-hand-side of each subfigure shows a possible sequence of pieces $P_j, \ldots, P_m$ in G' containing v. The right hand side shows the corresponding portion of G, with forced edges solid and optional edges dashed.

($\Delta^2 X$) Because $v_1 \not\sim s_3$, we must have $v_1 \sim s_1, s_2$ (so that v_1 has sufficient degree). Then v_2 must be adjacent to one or both of s_1 and s_2, yielding either a $\Delta\Delta\Delta X$ or a $\Delta X(I)X$.

(Δ^3) Note that v_1 and v_2 each need to have degree at least 4, so must be adjacent to both s_1 and s_2 (because they are not adjacent to r_2 or r_1 respectively). This sequence in G' can only result from a $\Delta X\Delta$ in G.

($X\Delta X$) By Observation 4.13 and degree constraints, we see that v_1 and v_2 are each adjacent to one or both of s_2 and s_3 and symmetrically each of s_2, s_3 is adjacent to one or both of v_1, v_2. Thus, this could only have resulted from an $X\Delta\Delta X$, an $X(I)X(I)X$, or an XQX in G.

In each case, we see that for G to have yielded a $G' \in \mathcal{Q}$, it too must be in $\mathcal{Q}$. $\square$

For the remaining two parts of the main proof, we will refer to the vertex labeling of $C_n[K_2]$ described in Definition 4.6. See also Figure 4.3.

Proof (2 $\implies$ 3). Consider an arbitrary graph $G \in \mathcal{Q}$ on at least 9 vertices, defined by the $\mathcal{Q}$-sequence $P_1, \ldots, P_m$. We have already shown in Lemma 4.7 that each graph in $\mathcal{P}$ is a minor of $C_n[K_2]$ for sufficiently large n, so all that remains to be shown is that that G is 4-connected.

If every piece is Δ-type, then $G \cong C_m^2$, which is 4-connected. Otherwise, this sequence will contain at least one X-type piece, so we may assume that P_m is X-type. Consider the union of $\{a_1^i b_1^i \mid 1 \leq i < m, a_1^i \neq b_1^i\} \cup \{a_2^i b_2^i \mid 1 \leq i < m, a_2^i \neq b_2^i\}$ with either $\{a_1^m b_1^m, a_2^m b_2^m\}$ or $\{a_1^m b_2^m, a_2^m b_1^m\}$ so that the edges form two disjoint cycles C_1 and C_2 instead of one cycle. Each vertex of G will be adjacent to at least two vertices on the other cycle. Now consider any $S \subseteq V(G)$ with $|S \cap V(C_1)| \leq 1$. Then S cannot be a cut-set of G, because C_1 is still connected, and every remaining vertex of C_2 is still adjacent to some vertex of C_1. Thus, any cut-set must contain at least two vertices from C_1, and similarly must contain at least 2 vertices from C_2. Thus, G is 4-connected. $\square$

Proof (3 $\implies$ 1). It suffices to show that $C_n[K_2]$ is $K_{2,5}$-minor-free for all $n \geq 3$, so assume to the contrary that this is not the case.

Consider the smallest n such that $C_n[K_2]$ contains a $K_{2,5}$-minor, and let that minor be given by $(R_1, R_2; S)$ so as to minimize $|R_1| + |R_2|$. Because $|V(K_{2,5})| = 7$, $n \geq 4$. Recall that $T_i = \{v_{i,1}, v_{i,2}\}$.

Claim 4.13.1. $|R_1|, |R_2| > 1$.

If, without loss of generality, $R_1 = \{v_{1,1}\}$, then we must have $S = N(v_{1,1})$. But then $N(v_{1,2}) \subseteq R_1 \cup S$, so R_2 cannot be adjacent to $v_{1,2} \in S$, a contradiction.

Claim 4.13.2. *For all i, $T_i \not\subseteq R_1$ and $T_i \not\subseteq R_2$.*

If, without loss of generality, $T_1 \subseteq R_1$, then $(R_1 \setminus \{v_{1,2}\}, R_2; S)$ must also give a $K_{2,5}$-minor, because $N(T_1) \subseteq N(v_{1,1})$. This contradicts the minimality of $|R_1| + |R_2|$.

Claim 4.13.3. *For all i, $T_i \cap S \neq \emptyset$.*

Proof. If, without loss of generality, the pair T_1 contains no vertex of S, then we will find a pair of disjoint edges, $v_{1,1}w_1$ and $v_{1,2}w_2$ for some w_1 and w_2, which we can contract to give a $K_{2,5}$-minor in $C_{n-1}[K_2]$.

If $T_1 \cap (R_1 \cup R_2) = \emptyset$, then take $w_1 = v_{2,1}$ and $w_2 = v_{2,2}$. Otherwise, without loss of generality, assume $v_{1,1} \in R_1$. Then by Claim 4.13.2, $v_{1,2} \notin R_1$, and so by Claim 4.13.1, we can choose $w_1 \in \{v_{2,1}, v_{2,2}, v_{n,1}, v_{n,2}\}$ such that $w_1 \in R_1$. If $v_{1,2}$ is part of any branch set, we must have $v_{1,2} \in R_2$. In this case, we can similarly find $w_2 \in \{v_{2,1}, v_{2,2}, v_{n,1}, v_{n,2}\}$ such that $w_2 \in R_2$ (and so $w_2 \neq w_1$). Otherwise, let w_2 be any vertex in $\{v_{2,1}, v_{2,2}, v_{n,1}, v_{n,2}\} \setminus \{w_1\}$.

For each $j \in \{1, 2\}$, $v_{1,j}$ is either in the same branch set as w_j or no branch set at all. We can therefore contract the edge $v_{1,j}w_j$, assigning the resulting vertex to the branch set containing w_j if such a branch set exists, and the resulting graph will still have a $K_{2,5}$-minor.

If $\{w_1, w_2\} = T_2$ (similarly T_n), then the graph formed by contracting $v_{1,1}w_1$ and $v_{1,2}w_2$ is exactly $C_{n-1}[K_2]$, and otherwise it is a subgraph of $C_{n-1}[K_2]$. In either case, this contradicts the minimality of n. □

Thus, $4 \leq n \leq 5$. Note, however, that $n \neq 4$, because $C_4[K_2]$ has only eight vertices, so by Claim 4.13.1 it is $K_{2,5}$-minor-free.

Finally, $n \neq 5$. With at least one vertex of each T_i in S, we see that $|R_1|+|R_2| \leq 5$. Combining this with Claim 4.13.1, we can take $|R_1| = 2$ without loss of generality. If, without loss of generality, $R_1 \subseteq T_1 \cup T_2$ then R_1 is not adjacent to the vertex of $T_4 \cap S$, a contradiction. Thus, $C_n[K_2]$ is $K_{2,5}$-minor-free for all n. □

4.3 Enumeration

This section provides a generating function for the number of unlabeled 4-connected $K_{2,5}$-minor-free graphs, weighted by the number of vertices, as well as an asymptotic estimate of this number, summarized in Theorem 4.15. To do so, we first show that, for any $n \geq 13$, there is a bijection between 4-connected $K_{2,5}$-minor-free graphs (i.e. graphs in $\mathcal{Q}$, by Theorem 4.12) on n vertices and $\mathcal{Q}$-sequences which correspond to a graph on n vertices, up to rotation and reflection of the sequence. As we have already shown in Lemma 4.5 that every graph in $\mathcal{Q}$ on at least 6 vertices can be built from some $\mathcal{Q}$-sequence and that each $\mathcal{Q}$-sequence defines a single isomorphism class in $\mathcal{Q}$, it only remains to show that each sufficiently large graph in $\mathcal{Q}$ can only be built from a single $\mathcal{Q}$-sequence.

Lemma 4.14. *Each graph $G \in \mathcal{Q}$ with $|V(G)| \geq 13$ is built from a unique $\mathcal{Q}$-sequence, up to reflections and cyclic shifts.*

Proof. Fix an arbitrary graph $G \in \mathcal{Q}$ on at least 13 vertices. Given a $\mathcal{Q}$-sequence corresponding to G, we can assign each edge of G to one of the following classes, based on the role that it plays in this $\mathcal{Q}$-sequence: I (the edge of an I), Δ-rail (a edge of a Δ contained in only this piece), Q-rail (an edge of a Q whose endpoints are not contained in the same X), Δ^2 (an edge common to two Δ's), ΔX (an edge connecting a Δ to an X), QX (an edge connecting a Q to an X), XXX (an edge of an X appearing between two other X's), YXX (an edge of an X appearing between an X and a piece that is not an X), or YXY (an edge of an X appearing between two non-X pieces). We now endeavor to show, for every edge e, that this class can be determined by looking at the local structure of G, and hence that e must be assigned to the same class in every $\mathcal{Q}$-sequence corresponding to G.

Claim 4.14.1. *Fix any $\mathcal{Q}$-sequence for G. Let v_1, v_2, and v_3 be vertices of G such that each pair is contained in some piece not containing the third. Then G has at most 12 vertices.*

Proof. In this case, the three circular intervals $P(v_1)$, $P(v_2)$, and $P(v_3)$ together cover every piece in the $\mathcal{Q}$-sequence. Because each vertex is in at most 3 pieces, and each pair of these vertices are contained in some common piece, the $\mathcal{Q}$-sequence contains at most 6 pieces. Each piece contributes at most two vertices to the graph, so G has at most 12 vertices. $\square$

Thus, because G has at least 13 vertices, any triple of vertices which has each pair contained in some common piece (if they form a triangle in G, for instance) will in fact have all three contained in some piece.

Given an edge vw contained in the triangle uvw, this implies that u is in a piece containing both v and w. If v and w are contained in only one common piece, then u will also be contained in this piece; if it is an X-type piece, then the type (either X or something else) of the neighboring pieces will determine if the edges a_1a_2 and b_1b_2 are present, but if it is any other type of piece, then the edges between vertices in this piece are unaffected by the rest of the $\mathcal{Q}$-sequence. If v and w are contained in more than one piece then one of them, say P_i, also contains u. The edges uv and uw must each be of the form $a^i_j b^i_k$ for some j and k, and the presence of these edges is unaffected by pieces other than P_i. Thus, the number of triangles containing an edge e, denoted $\tau(e)$, is a well-defined function of its class.

Claim 4.14.2. *Fix any $\mathcal{Q}$-sequence for G. If $|V(G)| \geq 9$, then each Q-rail-edge is in a unique induced 4-cycle that contains another Q-rail-edge.*

Proof. Let v_1v_2 be a Q-rail-edge. Because $|V(G)| \geq 4$ this $\mathcal{Q}$-sequence has more than 2 pieces, so the X-type pieces on either side of the Q-type piece containing v_1v_2 are not the same piece. Thus, this Q-type piece is itself such an induced 4-cycle.

Now let v_1, v_2, v_3, v_4 be any induced 4-cycle in G in which v_1v_2 and v_3v_4 are both Q-rail-edges. Based on the way we assemble $\mathcal{Q}$-sequences, Q-rail-edges can only be incident to QX-edges or edges contained in an X (YXY-edges or YXX-edges). If v_2v_3 (or analogously v_1v_4) is contained in an X-type piece, then v_1v_2 and v_3v_4 are incident with different sides of this X, and so these edges belong to distinct Q-type pieces. In this case, v_1v_4 must also be contained in an X-type piece, so G must be the graph corresponding to $XQXQ$, which has eight vertices.

Otherwise, v_2v_3 and v_1v_4 are QX-edges, in which case v_1, v_2, v_3, and v_4 must be the vertices of a single Q-type piece (because if there was any other piece involved, some edge in this cycle would be contained in an X-type piece).

Thus, if G has at least nine vertices, each Q-rail-edge is contained in exactly one induced 4-cycle with another Q-rail-edge, namely the Q-type piece to which it belongs. $\square$

For each edge, we now use these two claims to determine, based on the local structure in G, the single class to which this edge must be assigned in every $\mathcal{Q}$-sequence for G. Note that we can first identify Δ- and Q-rail-edges unambiguously, so it makes sense to refer to these when classifying other edges, despite not having a particular $\mathcal{Q}$-sequence in mind.

XXX: $\tau(e) = 0$, all adjacent edges have $\tau \leq 1$.

Q-rail: $\tau(e) = 0$, some adjacent edge has $\tau \geq 2$.

Δ-rail: $\tau(e) = 1$, the other edges of this triangle each have $\tau \geq 2$.

YXX: $\tau(e) = 1$, some other edge of this triangle has $\tau = 1$.

Δ^2: $\tau(e) = 2$, in a triangle with a Δ-rail-edge.

QX: $\tau(e) = 2$, in an induced 4-cycle with two Q-rail-edges, as in Claim 4.14.2.

YXY: $\tau(e) = 2$, not in an induced 4-cycle with two Q-rail-edges, nor a triangle with a Δ-rail-edge.

ΔX: $\tau(e) = 3$.

I: $\tau(e) = 4$.

Claim 4.14.3. *If G has at least 11 vertices and $u, v \in V(G)$ are not contained in a common piece, then $|V(P(u)) \cap V(P(v))| \leq 3$, so in particular u and v have at most three common neighbors.*

Proof. Consider vertices u and v not in a common piece. Any vertex in $V(P(u)) \cap V(P(v))$ must be either a_1 or a_2 from the first piece in $P(u)$ or b_1 or b_2 from the last piece of $P(u)$. Thus $|V(P(u)) \cap V(P(v))| \leq 4$. If $|V(P(u)) \cap V(P(v))| = 4$, then all vertices of G are contained in $P(u) \cup P(v)$, so G has at most $7 + 7 - 4 = 10$ vertices. Otherwise, this set contains at most 3 vertices. □

Claim 4.14.4. *If G has at least 13 vertices and $C = v_1v_2v_3v_4$ is a 4-cycle whose edges each belong to an X-type piece, and in which each pair of non-consecutive vertices of C is either adjacent or has four distinct common neighbors in G, then the edges of C form an X-type piece in every $\mathcal{Q}$-sequence for G.*

Proof. Consider any two consecutive edges in such a cycle, without loss of generality v_1v_2 and v_2v_3. Being adjacent, v_1 and v_2 are contained in a common piece, as are v_2 and v_3. Either because they are adjacent or by Claim 4.14.3, v_1 and v_3 must also be contained in some common piece, so by Claim 4.14.1 there is some piece containing all three of these vertices. By Lemma 4.4, for each edge contained in an X-type-piece there is exactly one piece containing both of its endpoints, so the edges v_1v_2 and v_2v_3 must belong to the same X-type piece. Applying the same reasoning to the pairs v_2v_3 and v_3v_4, and then to v_3v_4 and v_4v_1, we see that the edges in this cycle are all contained in a single X-type piece, as desired. □

Considering any particular $\mathcal{Q}$-sequence for G, and any XXX-, YXX-, or YXY-edge, the edges of the X-type-piece containing this edge will form such a 4-cycle.

We can now use only these classes and the structure of G to find the vertex set and type of each piece in a $\mathcal{Q}$-sequence for G, showing that this does not depend on our choice of $\mathcal{Q}$-sequence. Each I-edge belongs to an I-type piece on the edge's endpoints. Each Δ-rail-edge is in exactly one triangle, the vertices of which are exactly the vertices of associated Δ-type piece. Each Q-rail-edge is contained in the Q-type piece defined by the unique induced 4-cycle described in Claim 4.14.2. Each XXX-, YXX-, and YXY-edge will be contained in the X-type-piece given by a 4-cycle described in Claim 4.14.4.

Now consider any two $\mathcal{Q}$-sequences corresponding to G. We already know that the every piece in one sequence can be associated to a piece in the other which is of the same type and corresponds to the same subgraph of G. To see that these pieces must also be arranged in the same order, temporarily delete the I-type pieces from each $\mathcal{Q}$-sequence. Then, in either sequence, two pieces appear consecutively if and only if they share exactly two vertices of G, so these I-deleted sequences are identical. We can now add back all I-type pieces, which in each $\mathcal{Q}$-sequence must be located between the two X-type pieces containing both vertices of the I. This is unambiguous as long as there are more than two non-I-type pieces, which there must be because $|V(G)| \geq 13$. Thus, given any graph in Q on at least 13 vertices, there is a unique $\mathcal{Q}$-sequence corresponding to that graph. □

Theorem 4.15. *The number g_n of n-vertex 4-connected $K_{2,5}$-minor-free graphs, up to isomorphism, is equal to the coefficient of x^n in the generating function*

$$g(x) = -1 - x - 3x^2 - 2x^3 - 6x^4 - 3x^5 - 8x^6 + 5x^8 + \frac{1}{1-x} + \frac{2f(x) + f(x^2) + f(x)^2}{4 - 4f(x^2)} + \sum_{k=1}^{\infty} \frac{\varphi(k)}{2k} \log\left(\frac{1}{1 - f(x^k)}\right)$$

where $f(x) = x^2(1 + x^2 + \frac{1}{1-x})$. The asymptotic growth of these coefficients is

$$g_n \sim \frac{\alpha^n}{2n} \quad \text{as } n \to \infty,$$

where $\alpha \approx 1.85855898$ is the largest root of $1 - x + x^2 - 2x^3 - x^4 + x^5$.

Proof. We can now create a generating function for $\mathcal{Q}$-sequences. There is exactly one $\mathcal{Q}$-sequence on m pieces with no X-type piece, namely a sequence of m Δ-type pieces, corresponding to the graph C_m^2. For all other $\mathcal{Q}$-sequences, we can split the sequence into chunks by cutting the sequence after each X-type piece. This gives the following possible chunks: $\tilde{I} = IX$, $\tilde{Q} = QX$, and $\tilde{\Delta}^n = \Delta^n X$, for any $n \geq 0$. (So consecutive X-type pieces would be broken into $\tilde{\Delta}^0$ chunks.) These chunks contribute 2, 4, and $2+n$ vertices to the corresponding graph, respectively, as the last two vertices of each chunk are counted as part of the next chunk. Thus, $f(x) = x^2(1 + x^2 + \frac{1}{1-x})$ is a generating function for the number of distinct chunks, weighted by the number of vertices that they contribute to the corresponding graph.

We can describe any $\mathcal{Q}$-sequence with at least one X-type piece, and hence any isomorphism class of graphs in $\mathcal{Q}$ on at least 9 vertices except for C_n^2, by a sequence of chunks, and two sequences of chunks will yield the same graph exactly when the sequences are equivalent up to reflections and cyclic shifts.

These symmetries of a cyclic sequence of length k correspond to the dihedral group of order $2k$ if $k \geq 3$ and to the unique groups of order 1 and 2 for $k = 1$ and $k = 2$ respectively. The *cycle index* of a permutation group which acts on a set of size n is a polynomial in $a_1, \ldots, a_n$ which captures the cycle structure of each permutation in the group, with the variable a_i corresponding to cycles of length i. For every $k \geq 1$, the cycle index of the corresponding group is given by

$$\left(\frac{1}{2k}\sum_{i|k}\varphi(i)a_i^{\frac{k}{i}}\right) + \begin{cases} \frac{1}{2}a_1a_2^{\frac{k-1}{2}} & k \text{ is odd} \\ \frac{1}{4}a_1^2a_2^{\frac{k-2}{2}} + \frac{1}{4}a_2^{\frac{k}{2}} & k \text{ is even} \end{cases},$$

where φ is Euler's totient function, with $\varphi(i)$ giving the number of positive integers $m \leq i$ such that $\gcd(m, i) = 1$.

The first term comes from rotations which can be decomposed into $\frac{k}{i}$ disjoint cycles, each of length i. To see that there are exactly $\varphi(i)$ such rotations, consider any fixed i (where $i \mid k$) and let $g = \frac{k}{i}$. A rotation by $j \leq k$ positions will decompose into $\gcd(j,k)$ cycles of length $\frac{k}{\gcd(j,k)}$. This rotation will thus have cycles of length i exactly when $\gcd(j,k) = g$. If we let $j = gm$ for some integer $m \leq i$ and substitute, we see that this happens exactly when $\gcd(gm, gi) = g \iff \gcd(m,i) = 1$.

The second term comes from reflections. For odd k each reflection fixes one element, while for even k half of the reflections fix two elements and the other half fix none. In each case, the remaining elements appear in cycles of length two.

By Pólya's Theorem (in the form stated in [6], Theorem 5.1), we obtain a generating function for the number of orbits of cyclic sequences of chunks, weighted by the number of vertices in the corresponding graph, by substituting $f(x^i)$ for each a_i and then summing over all k. We can turn this into a generating function for $\mathcal{Q}$-sequences by adding $\frac{1}{1-x}$, adding one to the coefficient of each x^n to account for Δ^n.

By Theorem 4.12 and Lemma 4.14, this will also be a generating function for isomorphism classes of 4-connected $K_{2,5}$-minor-free graphs, weighted by the number of vertices, valid for graphs on at least 13 vertices. To give the correct count for smaller graphs, and to independently verify a few larger coefficients, all 4-connected $K_{2,5}$-minor-free graphs on up to 17 vertices were generated. The coefficients on between 9 and 12 vertices were already correct, because the conclusion of Lemma 4.14 is in fact true for all graphs in $\mathcal{Q}$ on at least 9 vertices. The coefficients for smaller graphs are corrected by adding a polynomial of degree eight. This gives

$$g(x) = \underbrace{-1 - x - 3x^2 - 2x^3 - 6x^4 - 3x^5 - 8x^6 + 5x^8 + \frac{1}{1-x}}_{\text{A}} +$$
$$\underbrace{\sum_{k=1}^{\infty} \frac{1}{2k} \sum_{i|k} \varphi(i) f(x^i)^{\frac{k}{i}}}_{\text{B}} + \underbrace{\sum_{k=1}^{\infty} \left(\frac{f(x^2)^{k-1} f(x)}{2} + \frac{f(x^2)^k}{4} + \frac{f(x^2)^{k-1} f(x)^2}{4} \right)}_{\text{C}}.$$

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18
count	1	3	9	27	21	47	53	113	142	283	404	770	1179	2196

n	19	20	21	22	23	24	25	26	27
count	3558	6510	10950	19839	34209	61714	108100	194630	344623

n	28	29	30	31	32	33	34
count	620320	1106135	1992406	3570396	6438014	11578137	20903848

n	35	36	37	38	39	40
count	37694548	68145909	123142351	222909873	403503068	731310276

Table 4.1: The exact number of isomorphism classes of 4-connected $K_{2,5}$-minor-free graphs on n vertices for $5 \leq n \leq 40$.

Table 4.1 gives the number of isomorphism classes of 4-connected $K_{2,5}$-minor-free graphs on up to 40 vertices, found by computing the power series expansion of g, with, again, the number on up to 17 vertices independently verified through an exhaustive generation of members of this family.

We can simplify B as follows:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2k} \sum_{i|k} \varphi(i) f(x^i)^{\frac{k}{i}} &= \sum_{i=1}^{\infty} \sum_{k\geq 1, i|k} \frac{\varphi(i)}{2k} f(x^i)^{\frac{k}{i}} \\
&= \sum_{i=1}^{\infty} \sum_{j\geq 1} \frac{\varphi(i)}{2ij} f(x^i)^j \\
&= \sum_{i=1}^{\infty} \frac{\varphi(i)}{2i} \sum_{j\geq 1} \frac{f(x^i)^j}{j} \\
&= \sum_{i=1}^{\infty} \frac{\varphi(i)}{2i} \log\left(\frac{1}{1-f(x^i)}\right).
\end{aligned}$$

Furthermore, note that C is simply a geometric series. Thus, we have

$$\begin{aligned}
g(x) = {} & \underbrace{-1 - x - 3x^2 - 2x^3 - 6x^4 - 3x^5 - 8x^6 + 5x^8 + \frac{1}{1-x}}_{\text{A}} + \\
& \underbrace{\sum_{k=1}^{\infty} \frac{\varphi(k)}{2k} \log\left(\frac{1}{1-f(x^k)}\right)}_{\text{B}} + \underbrace{\frac{2f(x) + f(x^2) + f(x)^2}{4 - 4f(x^2)}}_{\text{C}}.
\end{aligned}$$

To complete the proof of Theorem 4.15, we now examine the asymptotic growth of the coefficients of this generating function. Before we begin, consider the following expression.

$$\frac{1}{1-f(x)} = \frac{1-x}{1-x-2x^2+x^3-x^4+x^5}$$

To help determine the form of a partial fraction decomposition later in this analysis, we point out that the denominator of $(1-f(x))^{-1}$, $b(x) = 1-x-2x^2+x^3-x^4+x^5$, has five distinct roots. We will mostly refer to the reciprocals of these roots, (equivalently, the roots of $x^5 b(x^{-1}) = 1-x+x^2-2x^3-x^4+x^5$, the polynomial formed by reversing the sequence of coefficients in b), which are approximately

$$\alpha \approx 1.85855898 \qquad \beta \approx -1.43965 \qquad \gamma \approx 0.711655$$
$$\delta \approx -0.065282 + 0.721738i \qquad \overline{\delta} \approx -0.065282 - 0.721738i\ .$$

Note that A adds one to all sufficiently large coefficients, so does not have an effect on their asymptotic growth.

We now examine the $k = 1$ term of B, which will turn out to dominate the asymptotic growth of this family. For all $n \geq 1$ we have

$$\begin{aligned}
&n[x^n]\frac{\varphi(1)}{2}\log\frac{1}{1-f(x)} \\
&= \frac{1}{2}[x^{n-1}]\frac{d}{dx}\log\frac{1}{1-f(x)} \\
&= \frac{1}{2}[x^{n-1}]\frac{-f'(x)}{1-f(x)} \\
&= \frac{1}{2}[x^{n-1}]\frac{x(4-5x+6x^2-8x^3+4x^4)}{(1-x)(1-x-2x^2+x^3-x^4+x^5)}
\end{aligned}$$

The numerator and denominator of this rational function have no common factors, and the denominator has no repeated roots, and so yields the following partial fraction decomposition for some non-zero constants $b_1, \ldots, b_6$.

$$\begin{aligned}
&= \frac{1}{2}[x^{n-1}]\left(\frac{b_1}{1-x} + \frac{b_2}{1-\alpha x} + \frac{b_3}{1-\beta x} + \frac{b_4}{1-\gamma x} + \frac{b_5}{1-\delta x} + \frac{b_6}{1-\overline{\delta} x}\right) \\
&= \frac{b_1}{2} + \frac{b_2\alpha^{n-1}}{2} + \frac{b_3\beta^{n-1}}{2} + \frac{b_4\gamma^{n-1}}{2} + \frac{b_5\delta^{n-1}}{2} + \frac{b_6\overline{\delta}^{n-1}}{2}
\end{aligned}$$

Because $|\alpha| > 1, |\beta|, |\gamma|, |\delta|, |\overline{\delta}|$, this expression will be dominated by the second term, provided $b_2 \neq 0$. We will therefore explicitly compute b_2 by taking the above partial fraction decomposition, multiplying through by $q(x) = (1-x)(1-x-2x^2+x^3-x^4+x^5)$, and taking the limit as $x \to \alpha^{-1}$.

$$\begin{aligned}
\frac{x(4-5x+6x^2-8x^3+4x^4)}{(1-x)(1-x-2x^2+x^3-x^4+x^5)} &= \frac{x(4-5x+6x^2-8x^3+4x^4)}{q(x)} \\
&= \left(\frac{b_1}{1-x} + \frac{b_2}{1-\alpha x} + \frac{b_3}{1-\beta x} + \frac{b_4}{1-\gamma x} + \frac{b_5}{1-\delta x} + \frac{b_6}{1-\overline{\delta} x}\right)
\end{aligned}$$

$$\begin{aligned}
&\alpha^{-1}(4\alpha^{-4} - 8\alpha^{-3} + 6\alpha^{-2} - 5\alpha^{-1} + 4) \\
&\quad = \lim_{x\to\alpha^{-1}}\left(\frac{b_1 q(x)}{1-x} + \frac{b_2 q(x)}{1-\alpha x} + \frac{b_3 q(x)}{1-\beta x} + \frac{b_4 q(x)}{1-\gamma x} + \frac{b_5 q(x)}{1-\delta x} + \frac{b_6 q(x)}{1-\overline{\delta} x}\right) \\
&\quad = \lim_{x\to\alpha^{-1}} \frac{b_2 q(x)}{1-\alpha x} = \lim_{x\to\alpha^{-1}} \frac{b_2(q(x)-q(\alpha^{-1}))}{-\alpha(x-\alpha^{-1})} = \frac{b_2 q'(\alpha^{-1})}{-\alpha}
\end{aligned}$$

We now show that this equation is satisfied by $b_2 = \alpha$. This is the only solution because α^{-1} is not a double root of q, and hence $q'(\alpha^{-1}) \neq 0$.

$$
\begin{aligned}
&(4\alpha^{-4} - 8\alpha^{-3} + 6\alpha^{-2} - 5\alpha^{-1} + 4) + b_2 q'(\alpha^{-1}) \\
&= (4\alpha^{-4} - 8\alpha^{-3} + 6\alpha^{-2} - 5\alpha^{-1} + 4) \\
&\qquad + \alpha(-6\alpha^{-5} + 10\alpha^{-4} - 8\alpha^{-3} + 9\alpha^{-2} - 2\alpha^{-1} - 2) \\
&= \alpha^{-4}((4 - 8\alpha + 6\alpha^2 - 5\alpha^3 + 4\alpha^4) + (-6 + 10\alpha - 8\alpha^2 + 9\alpha^3 - 2\alpha^4 - 2\alpha^5)) \\
&= -2\alpha^{-4}(1 - \alpha + \alpha^2 - 2\alpha^3 - \alpha^4 + \alpha^5) \\
&= -2\alpha^{-4}(0) = 0
\end{aligned}
$$

Thus,

$$[x^n]\frac{\varphi(1)}{2}\log\frac{1}{1-f(x)} \sim \frac{b_2\alpha^{n-1}}{2n} = \frac{\alpha^n}{2n} \text{ as } n \to \infty. \tag{4.1}$$

We now bound the remaining terms of B. Note that f has nonnegative coefficients, and hence so does $\log(1/(1-f(x^k)))$.

$$
\begin{aligned}
&[x^n]\sum_{k=2}^{\infty}\frac{\varphi(k)}{2k}\log\left(\frac{1}{1-f(x^k)}\right) \\
&\leq [x^n]\sum_{k=2}^{\infty}\log\left(\frac{1}{1-f(x^k)}\right) = \sum_{k=2}^{\infty}[x^n]\log\left(\frac{1}{1-f(x^k)}\right) \\
&= \sum_{\substack{k\geq 2\\ k|n}}[x^n]\log\left(\frac{1}{1-f(x^k)}\right) \quad (\text{because}[x^n]\log\left(\frac{1}{1-f(x^k)}\right) = 0 \text{ if } k \nmid n) \\
&= \sum_{\substack{k\geq 2\\ k|n}}[x^{n/k}]\log\left(\frac{1}{1-f(x)}\right) = \sum_{\substack{1\leq t\leq n/2\\ t|n}}[x^t]\log\left(\frac{1}{1-f(x)}\right) \\
&\leq \sum_{t=1}^{\lfloor n/2\rfloor}[x^t]\log\left(\frac{1}{1-f(x)}\right) \leq \sum_{t=1}^{\lfloor n/2\rfloor} b^*\frac{\alpha^t}{t} \quad (\text{by Equation 4.1, for some } b^*) \\
&\leq \sum_{t=1}^{\lfloor n/2\rfloor} b^*\alpha^t \leq b^*\frac{\alpha^{n/2+1}-1}{\alpha-1} = O(\alpha^{n/2})
\end{aligned}
$$

To simplify the analysis of C, we first point out that

$$
\begin{aligned}
0 &\leq [x^n]f(x) = [x^n](2x^2 + x^3 + 2x^4 + x^5 + x^6 + \ldots) \\
&\leq [x^n]\frac{2}{1-x} = [x^n](2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + \ldots)
\end{aligned}
$$

for all n. We will also use the following properties of generating functions with non-negative coefficients: if $0 \leq [x^n]g_1(x) \leq [x^n]g_2(x)$ and $0 \leq [x^n]h(x)$ for all n then $0 \leq [x^n]g_1(x^k) \leq [x^n]g_2(x^k)$, $0 \leq [x^n]g_1(x)^k \leq [x^n]g_2(x)^k$, and $0 \leq [x^n]g_1(x)h(x) \leq [x^n]g_2(x)h(x)$ for all n. Thus, we can bound the contribution from C as follows:

$$
\begin{aligned}
&[x^n]\frac{2f(x) + f(x^2) + f(x)^2}{4 - 4f(x^2)} \\
&\leq [x^n]\frac{1}{1 - f(x^2)}\left(\frac{1}{1-x} + \frac{1}{2(1-x^2)} + \frac{1}{(1-x)^2}\right) \\
&= [x^n]\frac{1}{1 - f(x^2)}\left(\frac{5 + x - 2x^2}{2(1-x)^2(1+x)}\right) \\
&= [x^n]\left(\frac{c_1}{1-x} + \frac{c_2}{(1-x)^2} + \frac{c_3}{1+x} + \frac{c_4}{1-\sqrt{\alpha}x} + \frac{c_5}{1+\sqrt{\alpha}x}\right. \\
&\qquad + \frac{c_6}{1-\sqrt{\beta}x} + \frac{c_7}{1+\sqrt{\beta}x} + \frac{c_8}{1-\sqrt{\gamma}x} + \frac{c_9}{1+\sqrt{\gamma}x} \\
&\qquad \left. + \frac{c_{10}}{1-\sqrt{\delta}x} + \frac{c_{11}}{1+\sqrt{\delta}x} + \frac{c_{12}}{1-\sqrt{\bar{\delta}}x} + \frac{c_{13}}{1+\sqrt{\bar{\delta}}x}\right) \\
&= O(\alpha^{n/2})
\end{aligned}
$$

Thus, $[x^n]g(x) \sim \frac{\alpha^n}{2n}$, completing the proof of Theorem 4.15. $\square$

Chapter 5

Characterization of Planar 4-Connected DW_6-minor-free Graphs

Define the *double wheel* on $n+2$ vertices to be the join of C_n with $\overline{K}_2$ (see Figure 5.1). We refer to the vertices of C_n as the *rim* of the double wheel and to the vertices of $\overline{K}_2$ as the *hub*. This section presents a complete characterization of planar 4-connected DW_6-minor-free graphs. Because $K_{2,6}$ is a subgraph of DW_6, it is possible to deduce from this result a characterization of planar 4-connected $K_{2,6}$-minor-free graphs. The work is joint with John Maharry, Emily Marshall, and Liana Yepremyan.

We begin by defining the six structural families of graphs which are referred to in Theorem 5.1. An example of a graph in each of these families is shown in Figure 5.2. Recall from Definition 4.8 that P_m^2 denotes the square of the path on $v_1, \ldots, v_m$. We say that a graph G has a *strict-P_m^2-subgraph* if it contains P_m^2 in such a way that $N_G(v_j) = \{v_{j-2}, v_{j-1}, v_{j+1}, v_{j+2}\}$, i.e., no vertex outside of this subgraph is incident with v_j, for all j with $3 \leq j \leq m-2$. It will be necessary for the proof of Theorem 5.2 to locate strict-P_m^2-subgraphs in these families, so we point those out as well.

- $\mathcal{C} = \{C_{2n}^2 \mid n \geq 3\}$ where C_{2n}^2 denotes the square of the cycle $(w_1, v_1, w_2, v_2, \ldots, w_n, v_n)$ on $2n$ vertices.

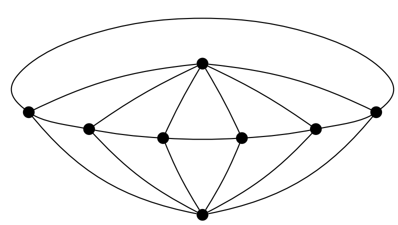


Figure 5.1: The double wheel graph DW_6.

- $\mathcal{D}^0 = \{D^0_{2n,i} \mid n \geq 4, 3 \leq i \leq n-1\}$ where $D^0_{2n,i}$ denotes the graph formed by squaring the cycle $(w_1, v_1, w_2, v_2, \ldots, w_n, v_n)$ on $2n$ vertices and adding the edge $v_1 v_i$. Note that $D^0_{2n,i}$ is isomorphic to $D^0_{2n,n-i+2}$.

  $D^0_{2n,i}$ contains a strict-P_m-subgraph on $w_1, v_1, w_2, \ldots, v_i, w_{i+1}$ with $m = 2i+1$ and on $w_i, v_i, w_{i+1}, \ldots, v_1, w_2$ with $m = 2(n-i+2)+1$.

- $\mathcal{D}^1 = \{D^1_{2n+1,i} \mid n \geq 3, 2 \leq i \leq n-1\}$ where $D^1_{2n+1,i}$ denotes the graph formed by squaring the cycle $(w_1, v_1, w_2, v_2, \ldots, w_n, v_n)$ on $2n$ vertices, subdividing $v_n v_1$ with the vertex t_1, and adding the edges $t_1 v_i$ and $t_1 w_1$. Note $D^1_{2n+1,i} \cong D^1_{2n+1,n-i+1}$.

  $D^1_{2n+1,i}$ contains a strict-P_m-subgraph on $t_1, w_1, v_1, \ldots, v_i, w_{i+1}$ with $m = 2i+2$ and on $w_i, v_i, w_{i+1}, \ldots, v_n, w_1, t_1$ with $m = 2(n-i+1)+2$.

- $\mathcal{D}^2 = \{D^0_{2n+2,i} \mid n \geq 3, 2 \leq i \leq n\}$ where $D^2_{2n+2,i}$ denotes the graph formed by squaring the cycle $(w_1, v_1, w_2, v_2, \ldots, w_n, v_n)$ on $2n$ vertices, subdividing $v_n v_1$ with the vertex t_1 and $v_{i-1} v_i$ with t_i, and adding the edges $t_1 t_i$, $t_1 w_1$, and $t_i w_i$. Note $D^2_{2n+2,i} \cong D^2_{2n+2,n-i+2}$.

  $D^2_{2n+2,i}$ contains a strict-P_m-subgraph on $t_1, w_1, v_1, \ldots, v_{i-1}, w_i, t_i$ with $m = 2i+1$ and on $t_i, w_i, v_i, \ldots, v_n, w_1, t_1$ with $m = 2(n-i+2)+1$.

- $\mathcal{X}_e = \{X^+_{2n}, X^-_{2n} \mid n \geq 4\}$ where X^+_{2n} denotes the graph formed by squaring the cycle $(w_2, v_1, w_3, v_n, w_4, v_{n-1}, \ldots, w_n, v_3, w_1, v_2)$ on $2n$ vertices and adding the edges $v_2 v_n$ and $w_2 w_n$, and X^-_{2n} is formed from X^+_{2n} by deleting the edge $v_2 w_2$.

  $X^{\pm}_{2n}$ contains a strict-P_m-subgraph on $v_3, w_n, v_4, \ldots, v_n, w_3$ with $m = 2n-4$.

- $\mathcal{X}_o = \{X^+_{2n+1}, X^-_{2n+1} | n \geq 4\}$ where X^-_{2n+1} denotes the graph formed by squaring the cycle $(w_1, v_1, w_2, v_2, \ldots, w_n, v_n)$ on $2n$ vertices, subdividing the edge $v_1 v_n$ with vertex t, and adding the edges $t v_{n-1}$ and $t v_2$, and X^+_{2n+1} is formed from X^-_{2n+1} by adding the edge $t w_1$.

$X^{\pm}_{2n+1}$ contains a strict-P_m-subgraph on $w_2, v_2, w_3, \ldots, v_{n-1}, w_n$ with $m = 2n-3$.

Let $\mathcal{F} = \mathcal{C} \cup \mathcal{D}^0 \cup \mathcal{D}^1 \cup \mathcal{D}^2 \cup \mathcal{X}_e \cup \mathcal{X}_o \cup \{Y, Z\}$, where Y and Z are the two graphs on nine vertices shown in Figure 5.3. The main theorem of this chapter is the following.

Theorem 5.1. *The set of planar 4-connected DW_6-minor-free graphs is exactly $\mathcal{F}$.*

We prove this theorem in two steps. First, in Theorem 5.2, we show that all graphs in $\mathcal{F}$ are DW_6-minor-free. Next, we show that this set suffices by proving that it contains the even squared cycles and is closed under an operation that generates all other planar 4-connected DW_6-minor-free graphs.

Theorem 5.2. *Every graph in $\mathcal{F}$ is DW_6-minor-free.*

Before proving Theorem 5.2, we give the following lemma which bounds the size of strict-P_m^2-subgraphs in a minimum counterexample to this theorem.

Lemma 5.3. *Assume that some graph in $\mathcal{C} \cup \mathcal{D}^0 \cup \mathcal{D}^1 \cup \mathcal{D}^2 \cup \mathcal{X}_e \cup \mathcal{X}_o$ has a DW_6-minor. Let G be such a graph on the minimum number of vertices and let P_m be the m-vertex path graph on $v_1, \ldots, v_m$, labeled in order along the path (see Figure 5.4). If G contains a strict-P_m^2-subgraph then $m \leq 8$.*

Proof. Assume, to the contrary, that a minimum counterexample G contains a strict-P_m^2-subgraph for some $m \geq 9$. Fix a model of DW_6 in this graph. Because G is connected, we may assume that every vertex in G is assigned to some branch set in this model. For any vertex v in this graph, let $\mathrm{bs}(v)$ denote the set of vertices in the same branch set as v.

Claim 5.3.1. *For $3 \leq j \leq m-2$, the vertices v_{j-1} and v_{j+1} are not in the same branch set.*

Assume v_{j-1} and v_{j+1} are in the same branch set. If $v_j \in \mathrm{bs}(v_{j-1}) = \mathrm{bs}(v_{j+1})$, then we can contract $v_{j-1}v_{j+1}$ (assigning the resulting vertex to $\mathrm{bs}(v_{j+1})$) and v_jv_{j+2}

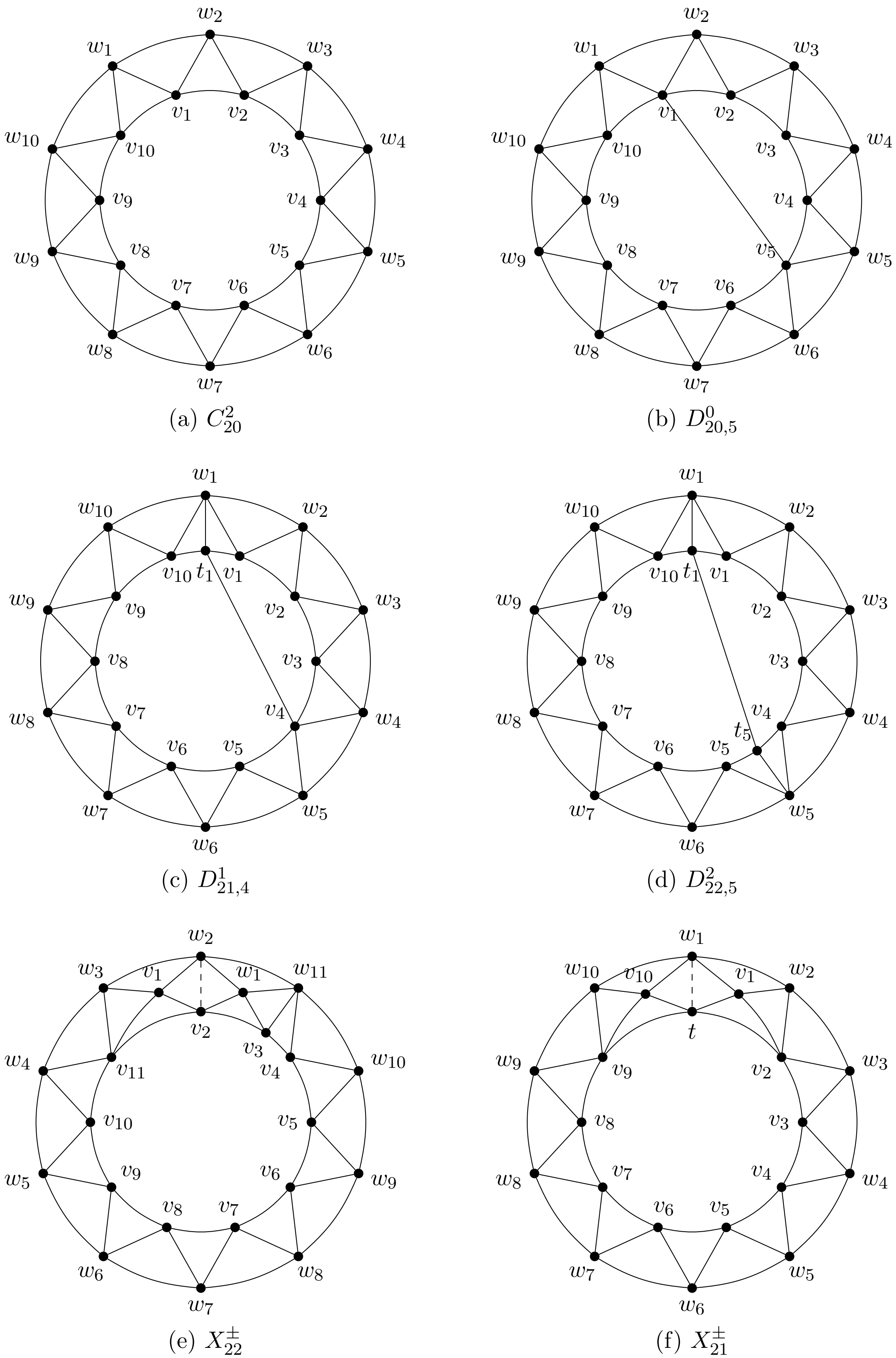


Figure 5.2: A representative from each of the six families described at the beginning of this chapter.

(a) Y (b) Z

Figure 5.3: The two exceptional graphs mentioned in Theorem 5.1.

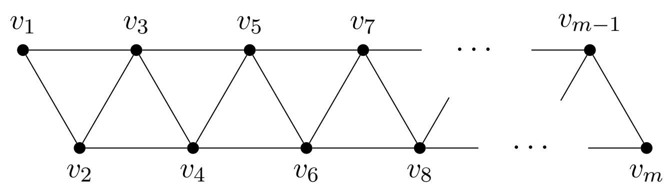


Figure 5.4: P_m^2, the square of the path $v_1, \ldots, v_m$.

(assigning the resulting vertex to $\mathrm{bs}(v_{j+2})$). The resulting graph will still have a DW_6-minor (because $N[\{v_{j-1}, v_j, v_{j+1}\}] = N[\{v_{j-1}, v_{j+1}\}]$ and $N[\{v_{j+2}\}] \subseteq N[\{v_j, v_{j+2}\}]$). However, contracting these two edges will give a smaller graph within the same structural family, contradicting the minimality of our counterexample. Similarly, if $v_j \in \mathrm{bs}(v_{j+2})$, then we could contract the edges $v_{j-1}v_{j+1}$ and v_jv_{j+2} to find a smaller counterexample, and symmetrically $v_j \notin \mathrm{bs}(v_{j-2})$. Thus, $\mathrm{bs}(v_j) = \{v_j\}$, but then this branch set is adjacent to at most 3 other branch sets, a contradiction (DW_6 has minimum degree 4).

Claim 5.3.2. *The vertices v_j and v_{j+1} are not in the same branch set (for $4 \leq j \leq m-4$).*

Let $4 \leq j < m-4$ and assume $\mathrm{bs}(v_j) = \mathrm{bs}(v_{j+1})$ (the case when $j = m-4$ follows from that of $j = 4$ by symmetry). By Claim 5.3.1, $v_{j-2}, v_{j-1}, v_{j+2}, v_{j+3} \notin \mathrm{bs}(v_j) = \mathrm{bs}(v_{j+1})$, and $v_{j+4} \notin \mathrm{bs}(v_{j+2})$. Because $\mathrm{bs}(v_j)$ must touch at least four other branch

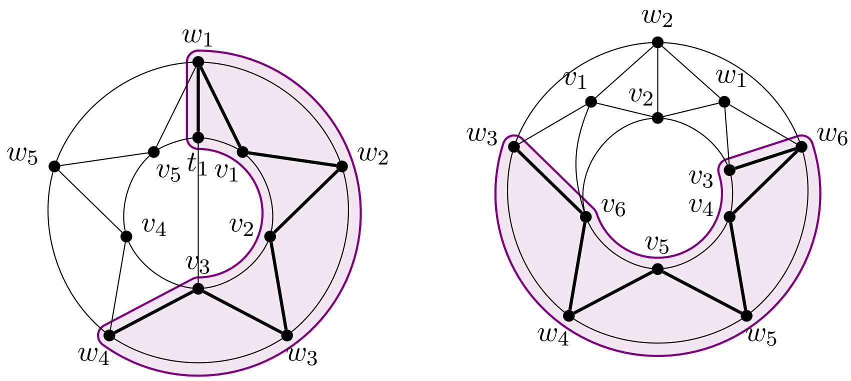


Figure 5.5: Strict-P_8^2-subgraphs in $D^1_{2\cdot 8-5,3}$ and X^+_{8+4}, with the corresponding P_8 emphasized.

sets, v_{j+2} and v_{j+3} belong to distinct branch sets. But then $\mathrm{bs}(v_{j+2})$ can touch at most three other branch sets, a contradiction because DW_6 has minimum degree 4.

By Claim 5.3.1 for $j = 4, 6$ and Claim 5.3.2 for $j = 4, 5$, $\mathrm{bs}(v_5) = \{v_5\}$. Applying Claim 5.3.1 for $j = 3, 7$, we also see that both v_4 and v_6 must be the only vertex in their respective branch sets. Because $\mathrm{bs}(v_4)$, $\mathrm{bs}(v_5)$, and $\mathrm{bs}(v_6)$ each touch at most four other branch sets, they must correspond to vertices of degree 4. Hence, the 4 branch sets adjacent to each must correspond to edges in DW_6. These branch sets are pairwise adjacent, however, and DW_6 does not contain a triangle of degree 4 vertices, a contradiction. Thus, we have $m \leq 8$. $\square$

Proof (Theorem 5.2). By Lemma 5.3 and the discussion at the beginning of the chapter about lengths of strict-P_m^2-subgraphs in members of these families, a minimum counterexample to this theorem would be a minor of either $D^1_{2\cdot 8-5,3}$ or X^+_{8+4}, and one can readily confirm that these two graphs do not have a DW_6-minor. See Figure 5.5 for an illustration of P_8^2 in each of these candidate graphs.

Because DW_6 has two vertices of degree 6 and $\Delta(Y) = \Delta(Z) = 5$, we must contract at least two edges of Y (similarly Z) in any model of DW_6, were such a minor to exist. However, because Y and Z each have only nine vertices, any way of

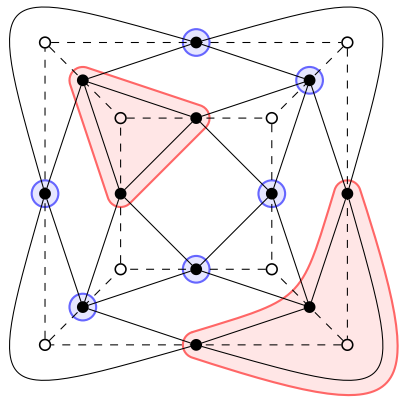


Figure 5.6: The line graph of the cube contains DW_6 as a minor.

performing these contractions would result in a graph too small to contain a DW_6-minor. Thus, every graph in $\mathcal{F}$ is DW_6-minor-free. $\square$

Recall from Theorem 4.9 that if a 4-connected graph cannot be formed by splitting a vertex of a 4-connected graph with one fewer vertex, then it is either a squared cycle or the line graph of cubic cyclically-4-edge-connected graph. Given a cubic cyclically-4-edge-connected graph G, and two vertex disjoint edges e and f, the operation of subdividing e and f and adding an edge connecting the new vertices is called *adding a handle* to G.

Lemma 5.4 ([29]). *All cubic cyclically-4-edge-connected graphs other than K_4 can be built from either $K_{3,3}$ or the cube by adding handles.*

Observation 5.5. *If G is obtained from H by repeatedly adding handles, then $\mathrm{L}(H) \preccurlyeq \mathrm{L}(G)$.*

Corollary 5.6. *Consider any cyclically-4-edge-connected graph $G \ncong K_4$. If G is formed by adding handles to $K_{3,3}$, then $\mathrm{L}(G)$ is nonplanar, because $\mathrm{L}(K_{3,3})$ is nonplanar. Otherwise, G is formed by adding handles to the cube, so $DW_6 \preccurlyeq \mathrm{L}(cube) \preccurlyeq \mathrm{L}(G)$ (see Figure 5.6).*

Additionally, $\mathrm{L}(K_4) \cong C_6^2$*, and* C_{2n+1}^2 *is nonplanar for each* $n \geq 2$*. Thus, every planar 4-connected* DW_6*-minor-free graph can be built by performing a sequence of vertex splits starting with the square of an even cycle.*

Given a vertex v with degree 4 or 5 in a planar 4-connected graph G, we will now define notation to distinguish between the possible splits of v that will preserve planarity and 4-connectedness. As shown in Lemma 1.1, each split of v into v' and v'' that makes each adjacent to at least three neighbors of v will result in a 4-connected graph. For the graph to remain planar, the vertices in $N(v') \setminus \{v''\}$ must appear consecutively around v in the (unique) planar embedding of G, and similarly for v''. If the degree of v is 4, then these conditions can only be met by making two non-consecutive neighbors of v adjacent to both v' and v'', with the others adjacent to one. If the degree of v is 5, then we can have either one neighbor or two non-consecutive neighbors of v adjacent to both, and the rest adjacent to exactly one of the two in the way that preserves planarity. We therefore write v-w or v-uw to refer to the splits in which v' and v'' are both adjacent to only w, or to only u and w, respectively. This is well defined, up to swapping the labels of v' and v''. If v has degree 5, then v-w is a subgraph of v-uw, so in particular, if the former has a DW_6-minor, the latter will as well.

Proof (Theorem 5.1). We have already proven in Theorem 5.2 that every graph in $\mathcal{F}$ is DW_6-minor-free. One can readily check that C_{2n}^2, Y, Z and 4-connected. Each graph $D_{2n,i}^0$ contains C_{2n}^2 as an induced subgraph, so is 4-connected. The graph $D_{2n+1,i}^1$ can be formed from a split of $D_{2n,i}^0$ $(i \neq 2)$ or C_{2n}^2 $(i = 2)$, $D_{2n+2,i}^2$ from a split of $D_{2n+1,i}^1$ or $D_{2n+1,i-1}^1$, $X_{2n}^{\pm}$ from a split of $D_{2n-1,2}^1$, and $X_{2n+1}^{\pm}$ from a split of $D_{2n,3}^0$, where each split preserves 4-connectivity by Lemma 1.1. See the case analysis below for details. Thus, all graphs in $\mathcal{F}$ are 4-connected.

From Corollary 5.6, we know that all planar 4-connected DW_6-minor-free graphs are either even squared cycles or can be built by splitting a vertex of a planar 4-

connected DW_6-minor-free graph on one fewer vertex. Thus, to show that these are the only 4-connected planar graphs without a DW_6-minor, it suffices to show that $\mathcal{F}$, which contains the even squared cycles, is closed under those vertex splits which preserve planarity and 4-connectivity and do not introduce a DW_6-minor.

Splits of Small Graphs

All relevant splits of the following graphs have been checked by hand and confirmed by computer, with the following results.

$D^1_{7,2}$: Splits will give $D^0_{8,3}$, X^+_8, or will introduce a DW_6-minor.

$D^0_{8,3}$: Splits will give $D^1_{9,2}$, $X^{\pm}_9$, or will introduce a DW_6-minor.

X^-_8: This graph is isomorphic to $D^0_{8,3}$.

X^+_8: Splits will give X^+_9, Y, Z, or will introduce a DW_6-minor.

X^-_9: Splits will give X^-_{10} or will introduce a DW_6-minor.

X^+_9: Splits will give X^+_{10} or will introduce a DW_6-minor.

Y: All splits give DW_6-minors.

Z: All splits give DW_6-minors.

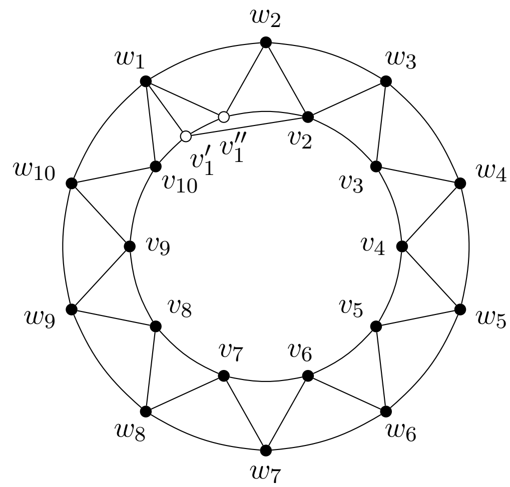
Splits of C^2_{2n}

Graphs in $\mathcal{C}$ are vertex-transitive, so it suffices to look only at splits of v_1.

Splitting v_1 (similarly any other vertex) of C^2_{2n}:

Case 1: Split v_1-v_2w_1 gives $D^1_{2n+1,2}$ (see Figure 5.7a).

Case 2: Split v_1-v_nw_2 is equivalent to split v_1-v_2w_1.



(a) v_1-v_2w_1 gives $D^1_{2n+1,2}$

Figure 5.7: Splitting v_1 (similarly any other vertex) of C^2_{2n}

Splits of $D^0_{2n,i}$

We will break into separate cases for splits of the vertices v_1 (symmetric to v_i), v_j (for $j \notin \{1, i\}$), and w_j. We will assume that $n \geq 5$; for $n = 4$, see the section at the beginning of the proof.

Splitting v_1 (equivalently v_i) of $D^0_{2n,i}$:

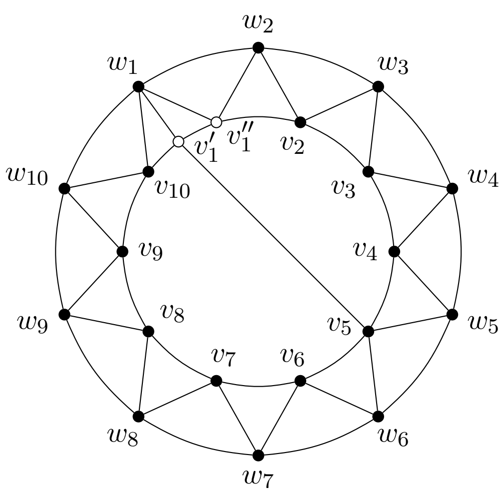
Case 1: Split v_1-w_1 gives $D^1_{2n+1,i}$ (see Figure 5.8a). Adding a second edge to v_2 is covered in Case 3, and instead adding a second edge to v_i gives a DW_6-minor by Case 5.

Case 2: Split v_1-w_2 is equivalent to the split v_1-w_1 in $D^0_{2n,n+2-i}$.

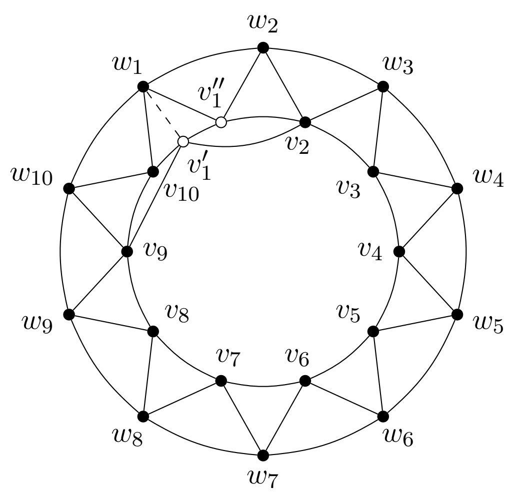
Case 3: Split v_1-v_2 gives a DW_6-minor if $i \neq n-1$ (see Figure 5.8c). If $i = n-1$ then this split gives X^-_{2n+1}, and adding a second edge to w_1 gives X^+_{2n+1} (see Figure 5.8b). Instead adding a second edge to v_n gives a DW_6-minor, as it contains the split v_1-v_n which is isomorphic to the split v_1-v_2 in $D^0_{2n,3}$ by Case 4 (note that $3 \neq n-1$ as $n \geq 5$).

Case 4: Split v_1-v_n is equivalent to the split v_1-v_2 in $D^0_{2n,n+2-i}$.

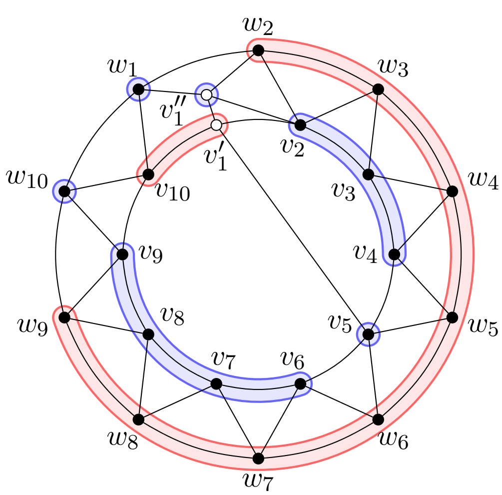
Case 5: Split v_1-v_i gives a DW_6-minor (see Figure 5.8d).



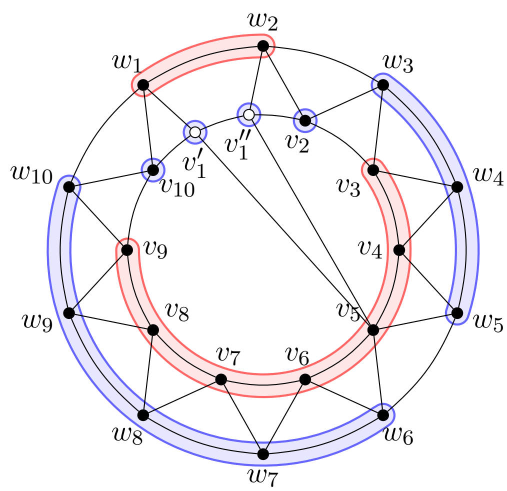
(a) Split v_1-w_1 gives $D^1_{2n+1,i}$



(b) Split v_1-$v_2(w_1)$, $i = n-1$, gives X^-_{2n+1} and X^+_{2n+1}



(c) Split v_1-v_2, $i \neq n-1$ has a DW_6-minor. Hub: $\{v_1', v_n\}$ and $\{w_2, \ldots, w_{n-1}\}$. Rim: $\{w_n\}$, $\{w_1\}$, $\{v_1''\}$, $\{v_2, \ldots, v_{i-1}\}$, $\{v_i\}$, and $\{v_{i+1}, \ldots, v_{n-1}\}$.



(d) Split v_1-v_i has a DW_6-minor. Hub: $\{v_3, \ldots, v_{n-1}\}$ and $\{w_1, w_2\}$. Rim: $\{v_1'\}$, $\{v_1''\}$, $\{v_2\}$, $\{w_3, \ldots, w_i\}$, $\{w_{i+1}, \ldots, w_n\}$, and $\{v_n\}$.

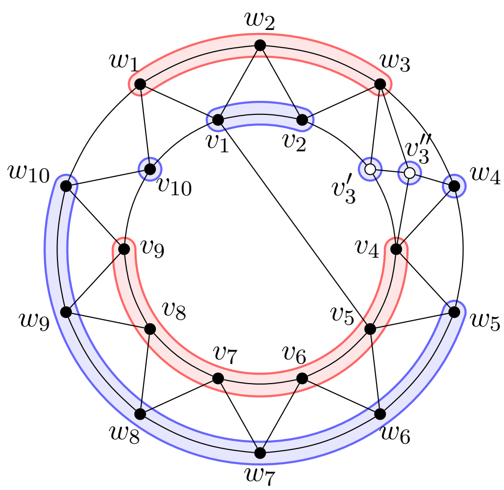
Figure 5.8: Splitting v_1 (equivalently v_i) of $D^0_{2n,i}$

Splitting v_j of $D^0_{2n,i}$ for $j \notin \{1, i\}$:

By symmetry, it suffices to assume that $j < i$.

Case 1: Split v_j-$v_{j+1}w_j$ gives a DW_6-minor (see Figure 5.9a).

Case 2: Split v_j-$v_{j-1}w_{j+1}$ gives a DW_6-minor. Considering the automorphism swapping v_1 and v_i, this case is equivalent to the split $v_{j'}$-$v_{j'+1}w_{j'}$ where $j' = i + 1 - j$, covered in Case 1.



(a) Split v_j-$v_{j+1}w_j$ has a DW_6-minor. Hub: $\{w_1, \ldots, w_j\}$ and $\{v_{j+1}, \ldots, v_{n-1}\}$. Rim: $\{v_1, \ldots, v_{j-1}\}$, $\{v_j'\}$, $\{v_j''\}$, $\{w_{j+1}\}$, $\{w_{j+2}, \ldots, w_n\}$, and $\{v_n\}$.

Figure 5.9: Splitting v_j of $D^0_{2n,i}$ for $j \notin \{1, i\}$

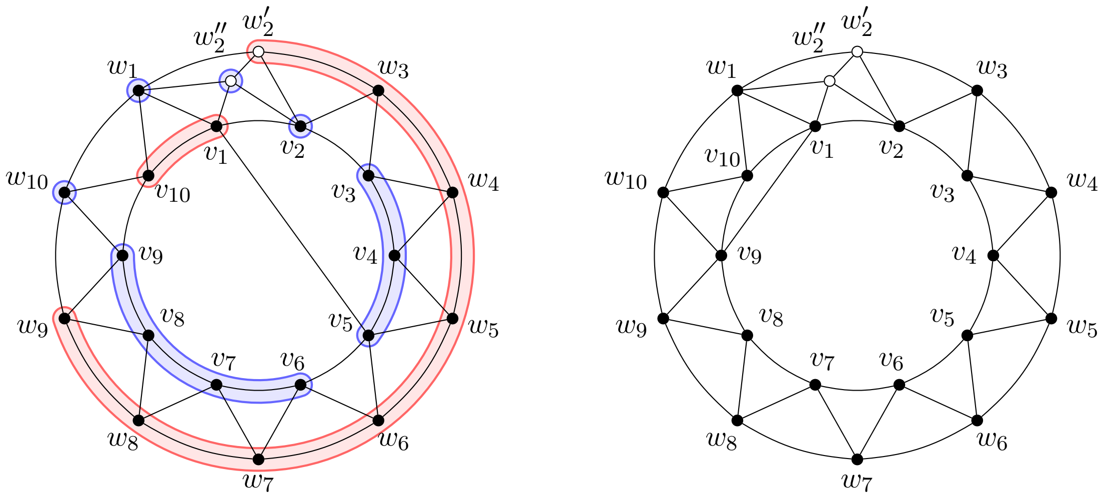
Splitting w_j of $D^0_{2n,i}$:

By symmetry, it suffices to assume that $2 \leq j \leq i$.

Case 1: Split w_2-v_2w_1 has a DW_6-minor if $i \neq n - 1$ (see Figure 5.10a) and gives X^+_{2n+1} if $i = n - 1$ (see Figure 5.10b).

Case 2: Split w_j-v_jw_{j-1} for $j \neq 2$ is isomorphic the split $v_{j'}$-$v_{j'-1}w_{j'+1}$ where $j' = j-1$, covered in Case 2 of the splits of v_j in $D^0_{2n,i}$, and thus gives a DW_6-minor.

Case 3: Split w_j-$v_{j-1}w_{j+1}$ is equivalent to the split $w_{j'}$-$v_{j'}w_{j'-1}$ where $j' = i+2-j$, covered in Cases 1 and 2, from the automorphism swapping v_1 and v_i.



(a) Split w_2-v_2w_1, $i \neq n-1$ has a DW_6-minor.
Hub: $\{v_1, v_n\}$ and $\{w_2', w_3, \ldots, w_{n-1}\}$.
Rim: $\{w_n\}$, $\{w_1\}$, $\{w_2''\}$, $\{v_2\}$, $\{v_3, \ldots, v_i\}$, and $\{v_{i+1}, \ldots, v_{n-1}\}$.

(b) Split w_2-v_2w_1, $i = n-1$, gives X^+_{2n+1}.

Figure 5.10: Splitting w_j of $D^0_{2n,i}$

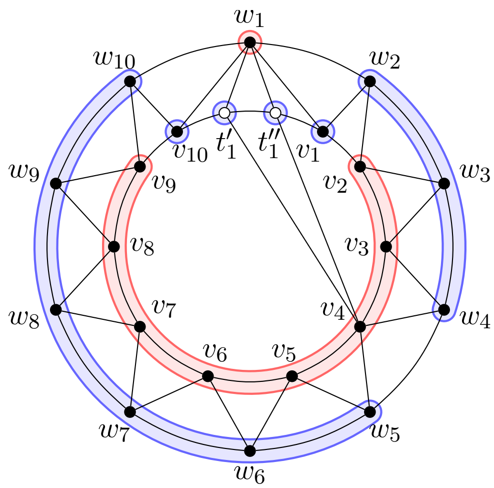
Splits of $D^1_{2n+1,i}$

We will break into separate cases for splits of the vertices t_1, v_i, w_1, v_j (for $j \neq i$), and w_j (for $j \neq 1$). We will assume that $n \geq 4$; for $n = 3$, see the section at the beginning of the proof.

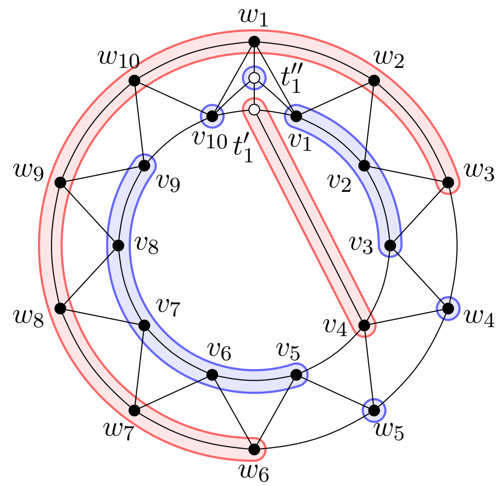
Splitting t_1 of $D^1_{2n+1,i}$:

Case 1: Split t_1-v_iw_1 gives a DW_6-minor (see Figure 5.11a).

Case 2: Split t_1-v_1v_n gives a DW_6-minor (see Figure 5.11b, in which we assume, by symmetry, that $i \neq n-1$).



(a) Split t_1-v_iw_1 has a DW_6-minor.
Hub: $\{w_1\}$ and $\{v_2, \ldots, v_{n-1}\}$.
Rim: $\{v_n\}$, $\{t_1'\}$, $\{t_1''\}$, $\{v_1\}$, $\{w_2, \ldots, w_i\}$, and $\{w_{i+1}, \ldots, w_n\}$.



(b) Split t_1-v_1v_n has a DW_6-minor.
Hub: $\{t_1', v_i\}$ and $\{w_1, \ldots, w_{i-1}, w_{i+2}, \ldots, w_n\}$.
Rim: $\{v_n\}$, $\{t_1''\}$, $\{v_1, \ldots, v_{i-1}\}$, $\{w_i\}$, $\{w_{i+1}\}$, and $\{v_{i+1}, \ldots, v_{n-1}\}$.

Figure 5.11: Splitting t_1 of $D^1_{2n+1,i}$

Splitting v_i of $D^1_{2n+1,i}$:

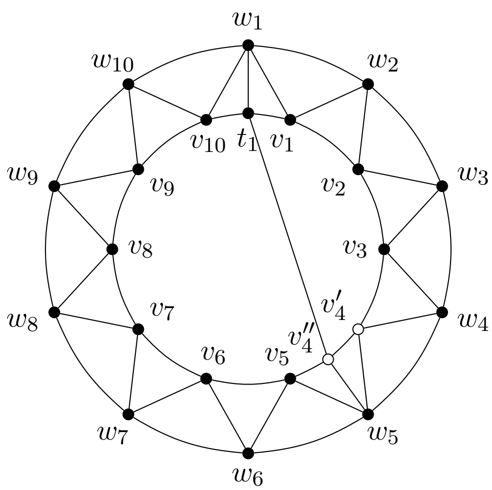
Case 1: Split v_i-w_{i+1} gives $D^2_{2n+2,i+1}$ (see Figure 5.12a). Adding a second edge to v_{i-1} is covered in Case 3, and instead adding a second edge to t_1 is covered in Case 5.

Case 2: Split v_i-w_i is equivalent to the split $v_{i'}$-$w_{i'+1}$ in $D^1_{2n+1,i'}$, where $i' = n-i+1$.

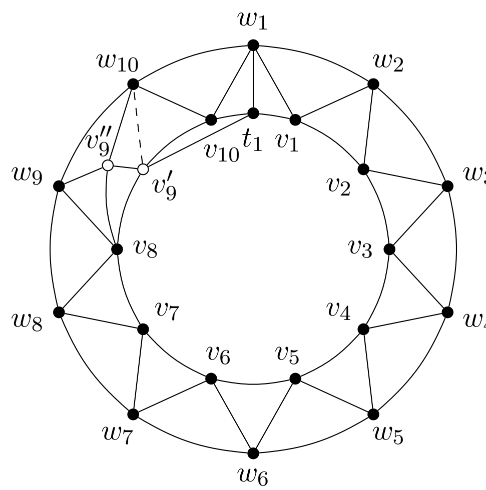
Case 3: Split v_i-v_{i-1} gives a DW_6-minor if $i \neq n-1$ (see Figure 5.12c). If $i = n-1$, then this split gives $X^-_{2(n+1)}$, and adding the edge $v_i'w_{i+1}$ gives $X^+_{2(n+1)}$ (see Figure 5.12b). Instead adding the edge $v_i''v_n$ gives a DW_6-minor, as it contains the split v_i-v_n which is isomorphic to the split v_2-v_1 in $D^0_{2n+1,2}$ by Case 4 (note that $2 \neq n-1$ as $n \geq 4$).

Case 4: Split v_i-v_{i+1} is equivalent to the split $v_{i'}$-$v_{i'-1}$ in $D^1_{2n+1,i'}$, where $i' = n-i+1$.

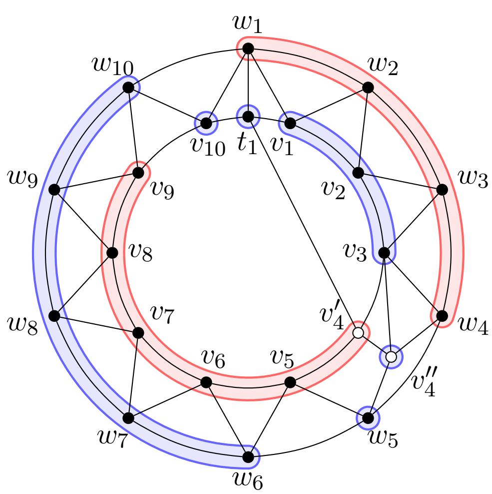
Case 5: Split v_i-t_1 gives a DW_6-minor (see Figure 5.12d in which we assume, by symmetry, that $i \neq n-1$).



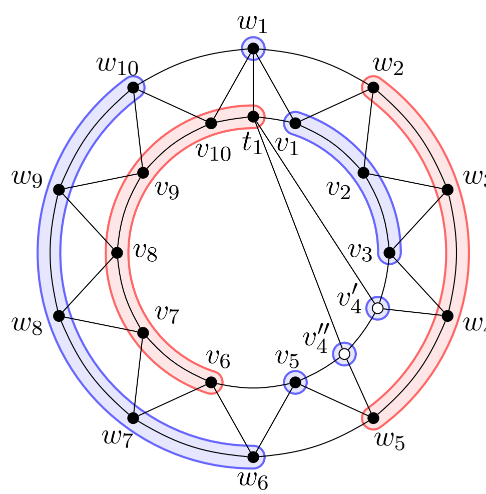
(a) Split v_i-w_{i+1} gives $D^2_{2n+2,i+1}$.



(b) Split v_i-$v_{i-1}(w_{i+1})$, $i = n-1$, gives $X^{\pm}_{2(n+1)}$.



(c) Split v_i-v_{i-1}, $i \neq n-1$, has a DW_6-minor.
Hub: $\{w_1, \ldots, w_i\}$ and $\{v_i', v_{i+1}, \ldots, v_{n-1}\}$.
Rim: $\{t_1\}$, $\{v_1, \ldots, v_{i-1}\}$, $\{v_i''\}$, $\{w_{i+1}\}$, $\{w_{i+2}, \ldots, w_n\}$, and $\{v_n\}$.



(d) Split v_i-t_1 has a DW_6-minor.
Hub: $\{w_2, \ldots, w_i, w_{i+1}\}$ and $\{v_{i+2}, \ldots, v_n, t_1\}$.
Rim: $\{v_1, \ldots, v_{i-1}\}$, $\{v_i'\}$, $\{v_i''\}$, $\{v_{i+1}\}$, $\{w_{i+2}, \ldots, w_n\}$, and $\{w_1\}$.

Figure 5.12: Splitting v_i of $D^1_{2n+1,i}$

Splitting w_1 of $D^1_{2n+1,i}$:

Case 1: Split w_1-v_n gives a DW_6-minor if $i \neq 2$ (see Figure 5.13a). If $i = 2$ then there is an automorphism swapping w_1 and v_i, which makes this split equivalent to the split v_2-w_3, covered in Case 1 of the splittings of v_i in $D^2_{2n+1,2}$. In particular, this split will give $D^2_{2(n+1),3}$, and both refinements of this split give a DW_6-minor.

Case 2: Split w_1-v_1 is equivalent to the split w_1-v_n in $D^1_{2n+1,n-i+1}$.

Case 3: Split w_1-w_2 gives a DW_6-minor if $i \neq n-1$ (see Figure 5.13b). If $i = n-1$, then this split gives $X^-_{2(n+1)}$, and adding a second edge to t_1 gives $X^+_{2(n+1)}$ (see Figure 5.13c). Adding instead a second edge to v_n gives a DW_6-minor, as it contains the split w_1-v_n (note that $i = n-1 \neq 2$ because $n \geq 4$).

Case 4: Split w_1-w_n is equivalent to the split w_1-w_2 in $D^1_{2n+1,i'}$, where $i' = n-i+1$.

Case 5: Split w_1-t_1 gives $D^0_{2(n+1),i+1}$ (see Figure 5.13d). Adding a second edge to either w_2 or w_n is covered in Cases 3 and 4 respectively.

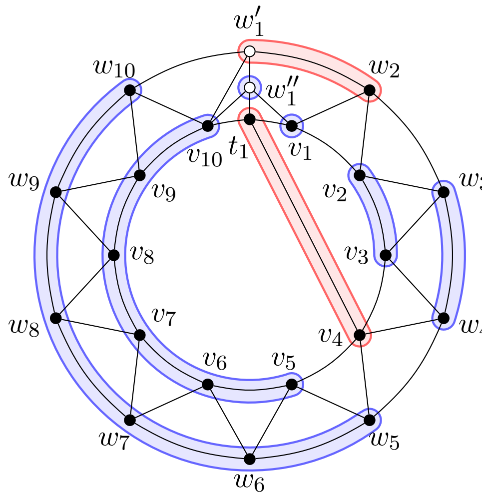
Splitting v_j of $D^1_{2n+1,i}$, for $j \neq i$:

By symmetry, it suffices to assume that $j < i$.

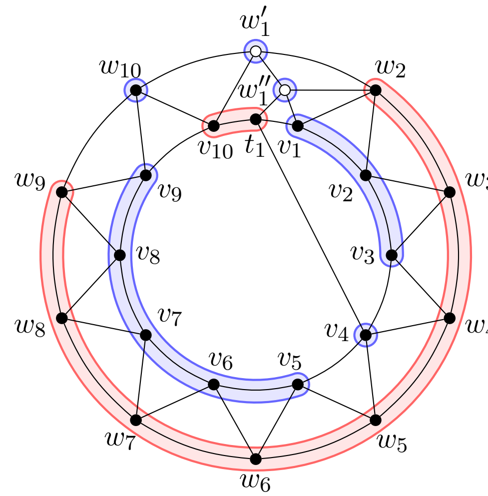
Case 1: Split v_1-t_1w_2 gives a DW_6-minor if $i \neq n-1$ (see Figure 5.14a) and $X^+_{2(n+1)}$ if $i = n-1$ (see Figure 5.14b).

Case 2: Split v_j-$v_{j-1}w_{j+1}$, for $j \neq 1$, gives a DW_6-minor (see Figure 5.14c).

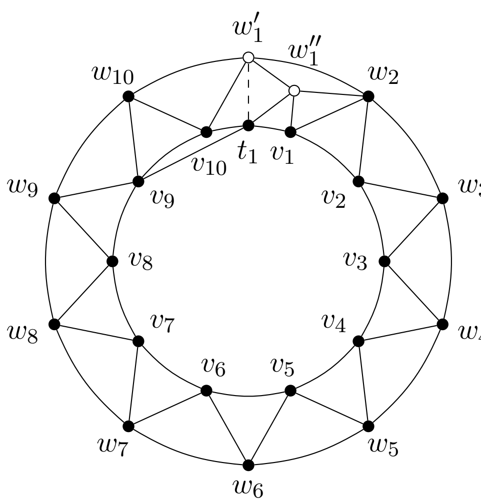
Case 3: Split v_j-$v_{j+1}w_j$ gives a DW_6-minor (see Figure 5.14d).



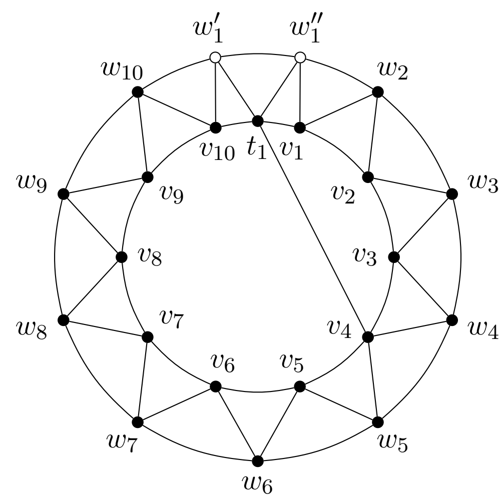
(a) Split w_1-v_n, $i \neq 2$ has a DW_6-minor. Hub: $\{w_1', w_2\}$ and $\{t_1, v_i\}$. Rim: $\{v_1\}$, $\{v_2, \ldots, v_{i-1}\}$, $\{w_3, \ldots, w_i\}$, $\{w_{i+1}, \ldots, w_n\}$, $\{v_{i+1}, \ldots, v_n\}$, and $\{w_1''\}$.



(b) Split w_1-w_2, $i \neq n-1$ has a DW_6-minor. Hub: $\{v_n, t_1\}$ and $\{w_2, \ldots, w_{n-1}\}$. Rim: $\{w_1'\}$, $\{w_1''\}$, $\{v_1, \ldots, v_{i-1}\}$, $\{v_i\}$, $\{v_{i+1}, \ldots, v_{n-1}\}$, and $\{w_n\}$.

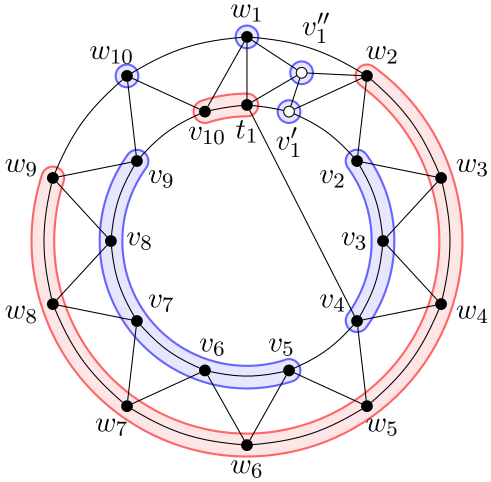


(c) Split w_1-$w_2(t_1)$, $i = n-1$, gives $X^{\pm}_{2(n+1)}$.

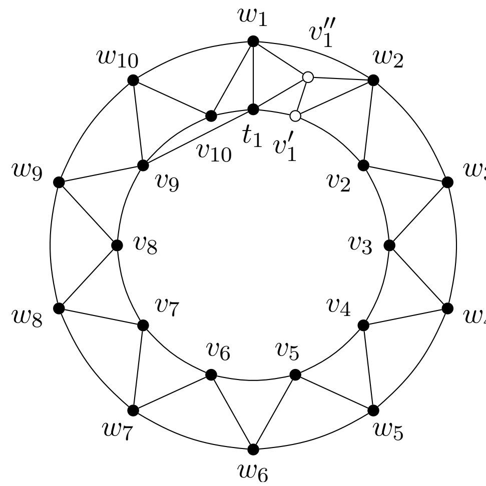


(d) Split w_1-t_1 gives $D^0_{2(n+1),i+1}$.

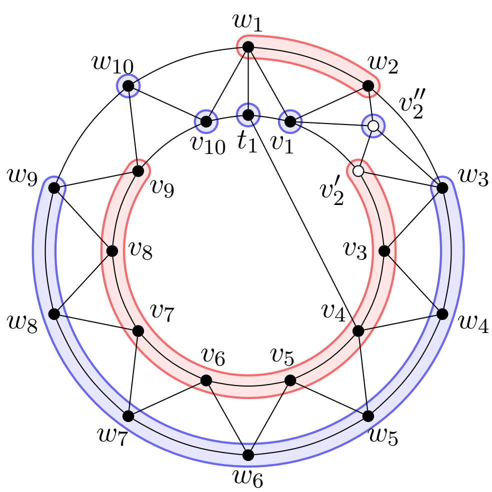
Figure 5.13: Splitting w_1 of $D^1_{2n+1,i}$



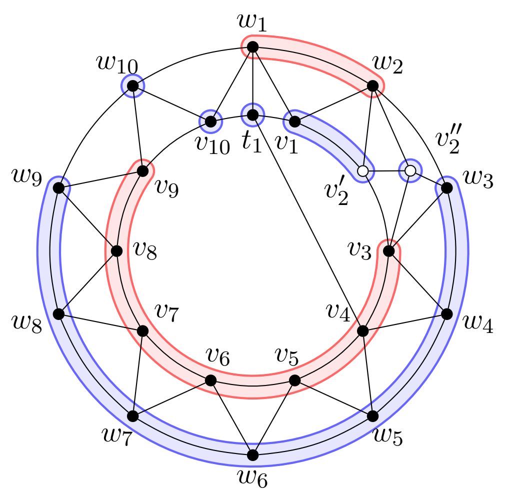
(a) Split v_1-t_1w_2, $i \neq n-1$, has a DW_6-minor.
Hub: $\{v_n, t_1\}$ and $\{w_2, \ldots, w_{n-1}\}$.
Rim: $\{w_n\}$, $\{w_1\}$, $\{v_1''\}$, $\{v_1'\}$, $\{v_2, \ldots, v_i\}$, and $\{v_{i+1}, \ldots, v_{n-1}\}$.



(b) Split v_1-t_1w_2, $i = n-1$, gives $X^+_{2(n+1)}$.



(c) Split v_j-$v_{j-1}w_{j+1}$, $j \neq 1$, has a DW_6-minor.
Hub: $\{w_1, \ldots, w_j\}$ and $\{v_j', \ldots, v_{n-1}\}$.
Rim: $\{w_n\}$, $\{v_n\}$, $\{t_1\}$, $\{v_1, \ldots, v_{j-1}\}$, $\{v_j''\}$, and $\{w_{j+1}, \ldots, w_{n-1}\}$.



(d) Split v_j-$v_{j+1}w_j$ has a DW_6-minor.
Hub: $\{w_1, \ldots, w_j\}$ and $\{v_{j+1}, \ldots, v_{n-1}\}$.
Rim: $\{w_n\}$, $\{v_n\}$, $\{t_1\}$, $\{v_1, \ldots, v_j'\}$, $\{v_j''\}$, and $\{w_{j+1}, \ldots, w_{n-1}\}$.

Figure 5.14: Splitting v_j of $D^1_{2n+1,i}$, for $j \neq i$

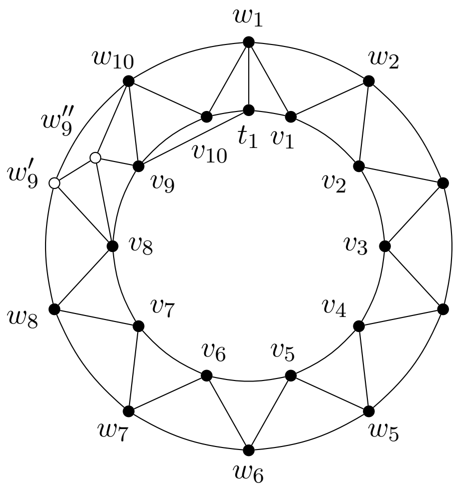
Splitting w_j of $D^1_{2n+1,i}$, $j \neq 1$:

By symmetry, it suffices to assume that $2 \leq j \leq i$.

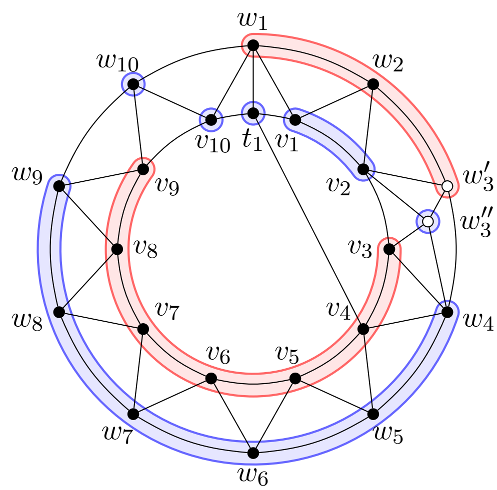
Case 1: Split w_{n-1}-$v_{n-2}w_n$ gives X^+_{2n+2} (see Figure 5.15a). Note that $j = n-1$ implies $i = n - 1$.

Case 2: Split w_j-$v_{j-1}w_{j+1}$, for $j \neq n - 1$, gives a DW_6-minor (see Figure 5.15b).

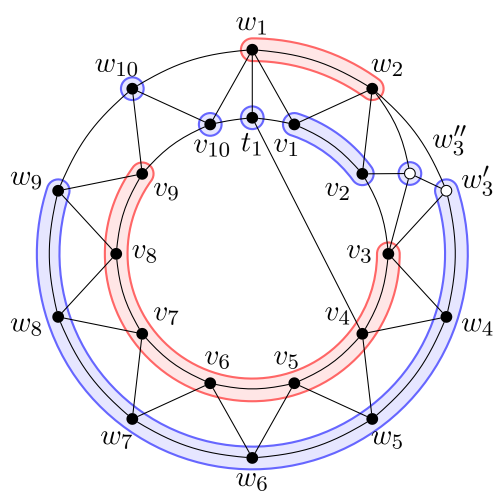
Case 3: Split w_j-v_jw_{j-1} gives a DW_6-minor (see Figure 5.15c).



(a) Split w_{n-1}-$v_{n-2}w_n$ (hence $i = n-1$) gives X^+_{2n+2}.



(b) Split w_j-$v_{j-1}w_{j+1}$, $j \neq n-1$ has a DW_6-minor.
Hub: $\{w_1, \ldots, w_j'\}$ and $\{v_j, \ldots, v_{n-1}\}$.
Rim: $\{w_n\}$, $\{v_n\}$, $\{t_1\}$, $\{v_1, \ldots, v_{j-1}\}$, $\{w_j''\}$, and $\{w_{j+1}, \ldots, w_{n-1}\}$.



(c) Split w_j-v_jw_{j-1} has a DW_6-minor.
Hub: $\{w_1, \ldots, w_{j-1}\}$ and $\{v_j, \ldots, v_{n-1}\}$.
Rim: $\{w_n\}$, $\{v_n\}$, $\{t_1\}$, $\{v_1, \ldots, v_{j-1}\}$, $\{w_j''\}$, and $\{w_j', \ldots, w_{n-1}\}$.

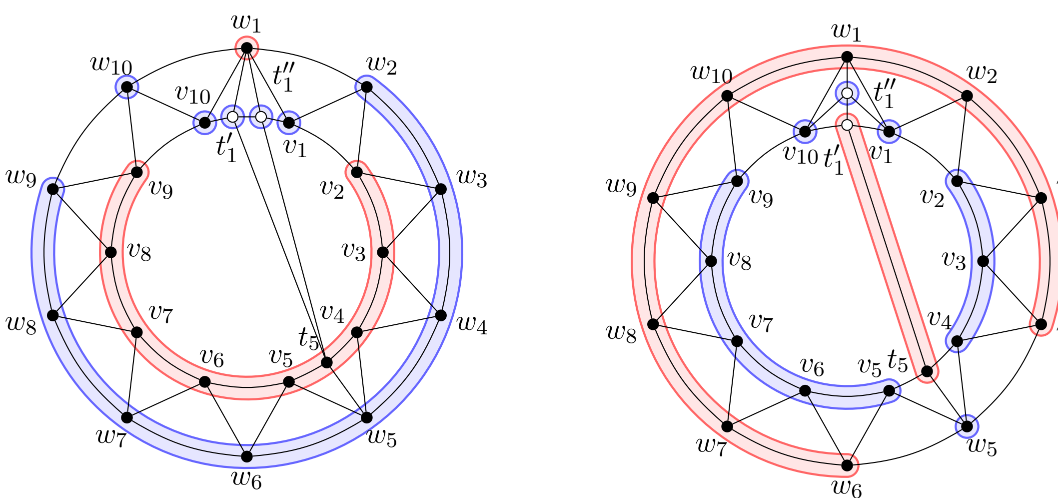
Figure 5.15: Splitting w_j of $D^1_{2n+1,i}$, $j \neq 1$

We consider splits of vertices t_1 (symmetric to t_i), w_1 (symmetric to w_i), w_j (for $j \notin \{1, i\}$), and v_j. We will assume that $i \neq \{2, n\}$, because these graphs are isomorphic to $D^0_{2(n+1),3}$, and as such have been covered in a previous section. As such, this analysis covers $n \geq 4$, for which it makes sense to have $2 < i < n$.

Splitting t_1 (similarly t_i) of $D^2_{2n+2,i}$:

Case 1: Split t_1-w_1t_i gives a DW_6-minor (see Figure 5.16a).

Case 2: Split t_1-v_1v_n gives a DW_6-minor (see Figure 5.16b).



(a) Split t_1-w_1t_i has a DW_6-minor.
Hub: $\{w_1\}$ and $\{t_i, v_2, \ldots, v_{n-1}\}$.
Rim: $\{t_1'\}$, $\{t_1''\}$, $\{v_1\}$, $\{w_2, \ldots, w_{n-1}\}$, $\{w_n\}$, and $\{v_n\}$.

(b) Split t_1-v_1v_n has a DW_6-minor.
Hub: $\{w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n\}$ and $\{t_1', t_i\}$.
Rim: $\{t_1''\}$, $\{v_1\}$, $\{v_2, \ldots, v_{i-1}\}$, $\{w_i\}$, $\{v_i, \ldots, v_{n-1}\}$, and $\{v_n\}$.

Figure 5.16: Splitting t_1 (similarly t_i) of $D^2_{2n+2,i}$

Splitting w_1 (similarly w_i) of $D^2_{2n+2,i}$:

Case 1: Split w_1-v_1 gives a DW_6-minor (see Figure 5.17a).

Case 2: Split w_1-v_n is equivalent to the split w_1-v_1 in $D^2_{2n+2,n-i+2}$.

Case 3: Split w_1-w_2 gives a DW_6-minor (see Figure 5.17b).

Case 4: Split w_1-w_n is equivalent to the split w_1-w_2 in $D^2_{2n+2,n-i+2}$, so will also give a DW_6-minor.

Case 5: Split w_1-t_1 gives $D^1_{2(n+1)+1,i}$ (see Figure 5.17c). Adding a second edge to either w_2 or w_n will give DW_6-minors, by Cases 3 and 4 respectively.

Splitting v_j of $D^2_{2n+2,i}$:

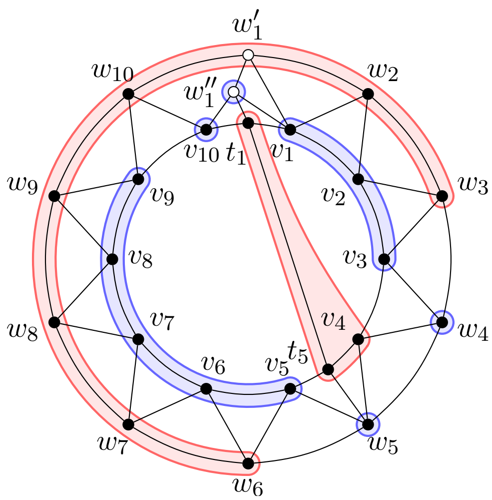
By symmetry, it suffices to assume that $j < i$.

Case 1: Split v_1-t_1w_2 gives a DW_6-minor (see Figure 5.18a).

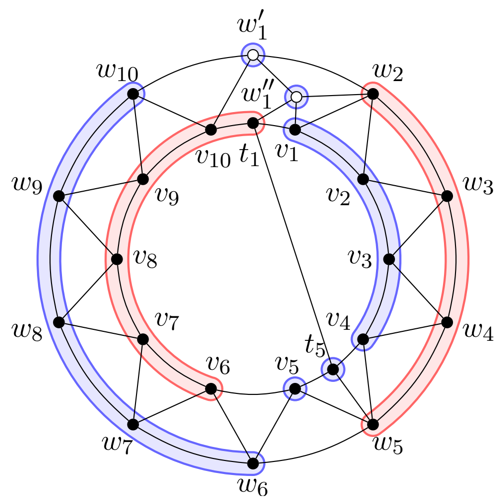
Case 2: Split v_j-$v_{j-1}w_{j+1}$, for $j \neq 1$, gives a DW_6-minor (see Figure 5.18b).

Case 3: Split v_{i-1}-t_iw_j is equivalent to split v_1-t_1w_2, by the automorphism swapping t_1 with t_i.

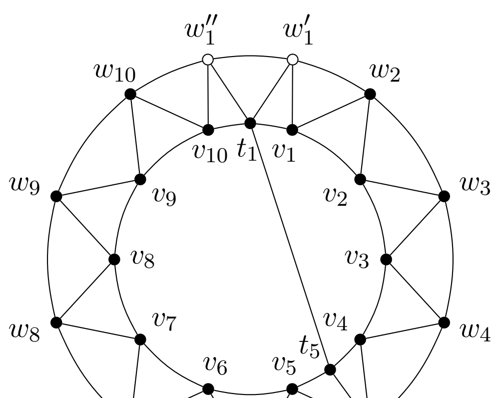
Case 4: Split v_j-$v_{j+1}w_j$, for $j \neq i-1$, is equivalent to the split $v_{j'}$-$v_{j'-1}w_{j'+1}$ where $j' = i - j$, by the automorphism swapping t_1 with t_i.



(a) Split w_1-v_1 has a DW_6-minor.
Hub: $\{w_1', \ldots, w_{i-2}, w_{i+1}, \ldots, w_n\}$ and $\{t_1, t_i, v_{i-1}\}$.
Rim: $\{w_1''\}$, $\{v_1, \ldots, v_{i-2}\}$, $\{w_{i-1}\}$, $\{w_i\}$, $\{v_i, \ldots, v_{n-1}\}$, and $\{v_n\}$.

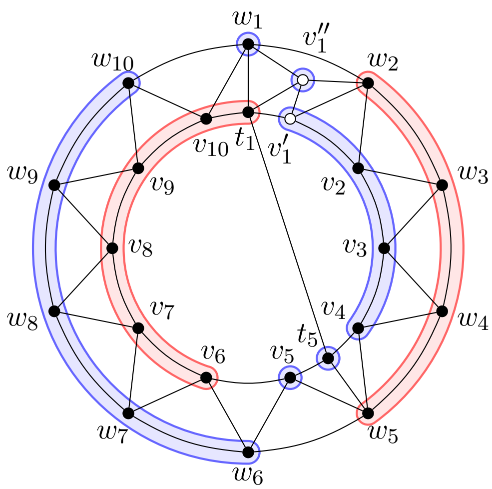


(b) Split w_1-w_2 has a DW_6-minor.
Hub: $\{v_{i+1}, \ldots, v_n, t_1\}$ and $\{w_2, \ldots, w_i\}$.
Rim: $\{w_1'\}$, $\{w_1''\}$, $\{v_1, \ldots, v_{i-1}\}$, $\{t_i\}$, $\{v_i\}$, and $\{w_{i+1}, \ldots, w_n\}$.

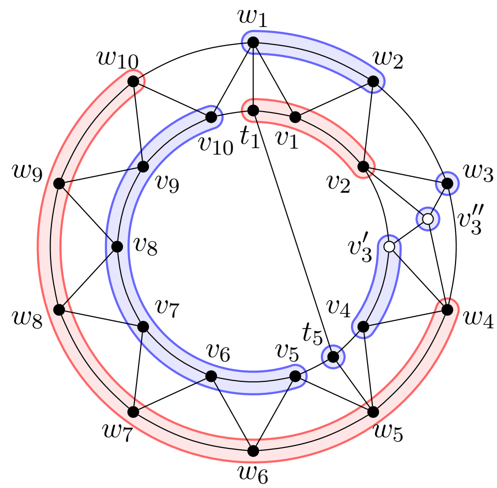


(c) Split w_1-t_1 gives $D^1_{2(n+1)+1,i}$.

Figure 5.17: Splitting w_1 (similarly w_i) of $D^2_{2n+2,i}$



(a) Split v_1-t_1w_2 has a DW_6-minor.
Hub: $\{v_{i+1}, \ldots, v_n, t_1\}$ and $\{w_2, \ldots, w_i\}$.
Rim: $\{w_1\}$, $\{v_1''\}$, $\{v_1', \ldots, v_{i-1}\}$, $\{t_i\}$, $\{v_i\}$, and $\{w_{i+1}, \ldots, w_n\}$.



(b) Split v_j-$v_{j-1}w_{j+1}$ for $j \neq 1$ has a DW_6-minor.
Hub: $\{t_1, v_1, \ldots, v_{j-1}\}$ and $\{w_{j+1}, \ldots, w_n\}$.
Rim: $\{v_j''\}$, $\{v_j', \ldots, v_{i-1}\}$, $\{t_i\}$, $\{v_i, \ldots, v_n\}$, $\{w_1, \ldots, w_{j-1}\}$, and $\{w_j\}$.

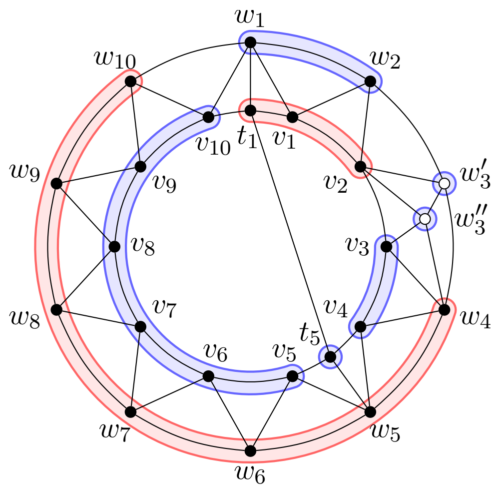
Figure 5.18: Splitting v_j of $D^2_{2n+2,i}$

Splitting w_j of $D^2_{2n+2,i}$, for $j \notin \{1, i\}$:

By symmetry, it suffices to assume that $j < i$.

Case 1: Split w_j-$v_{j-1}w_{j+1}$ gives a DW_6-minor (see Figure 5.19a).

Case 2: Split w_j-v_jw_{j-1} is equivalent to split $w_{j'}$-$v_{j'-1}w_{j'+1}$ with $j' = i - j + 1$, by the automorphism swapping t_1 with t_i.



(a) Split w_j-$v_{j-1}w_{j+1}$ has a DW_6-minor.
Hub: $\{t_1, v_1, \ldots, v_{j-1}\}$ and $\{w_{j+1}, \ldots, w_n\}$.
Rim: $\{w_1, \ldots, w_{j-1}\}$, $\{w_j'\}$, $\{w_j''\}$, $\{v_j, \ldots, v_{i-1}\}$, $\{t_i\}$, and $\{v_i, \ldots, v_n\}$.

Figure 5.19: Splitting w_j of $D^2_{2n+2,i}$, for $j \notin \{1, i\}$

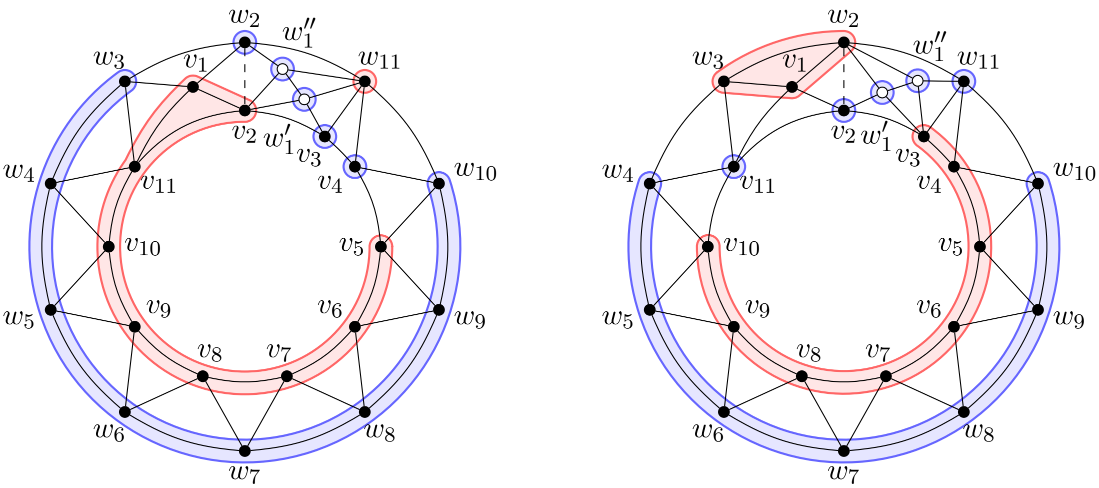
Splits of $X^{\pm}_{2n}$

Most of the following analysis applies to either type of graph in this family, so to make clear that we are allowing, but not requiring v_2w_2, we will draw pictures with this edge dashed, unless in a case in which we can assume its presence or absence. Note that the map exchanging w_i and v_i for each i is an automorphism of $X^{\pm}_{2n}$. We will break into separate cases for splits of the vertices w_1, w_2, w_3, w_j (for $j \notin \{1, 2, 3, n\}$), and w_n. We will assume that $n \geq 5$; for $n = 4$, see the section at the beginning of the proof.

Splitting w_1 (similarly v_1) of $X^{\pm}_{2n}$:

Case 1: Split w_1-v_2w_n gives a DW_6-minor (see Figure 5.20a).

Case 2: Split w_1-v_3w_2 gives a DW_6-minor (see Figure 5.20b).



(a) Split w_1-v_2w_n has a DW_6-minor.
Hub: $\{w_n\}$ and $\{v_5, \ldots, v_n, v_1, v_2\}$.
Rim: $\{w_2\}$, $\{w_1''\}$, $\{w_1'\}$, $\{v_3\}$, $\{v_4\}$, and $\{w_3, \ldots, w_{n-1}\}$.

(b) Split w_1-v_3w_2 has a DW_6-minor.
Hub: $\{v_1, w_2, w_3\}$ and $\{v_3, v_4, \ldots, v_{n-1}\}$.
Rim: $\{v_n\}$, $\{v_2\}$, $\{w_1'\}$, $\{w_1''\}$, $\{w_n\}$, and $\{w_4, \ldots, w_{n-1}\}$.

Figure 5.20: Splitting w_1 (similarly v_1) of $X_{2n}^{\pm}$

Splitting w_2 (similarly v_2) of $X_{2n}^{\pm}$:

In Cases 1 and 2, we consider the graph formed from splitting w_2 in X_{2n}^{-} as shown in Figures 5.21a and 5.21b. The resulting graphs are not 4-connected, as in each case w_2'' only has degree three, but each already has a DW_6-minor. Each of the splits w_2-v_1w_n and w_2-w_1w_3 in X_{2n}^{-} and w_2-v_1 and w_2-w_1 in X_{2n}^{+} will contain one of these 3-connected splits as a subgraph, and hence will also have a DW_6-minor.

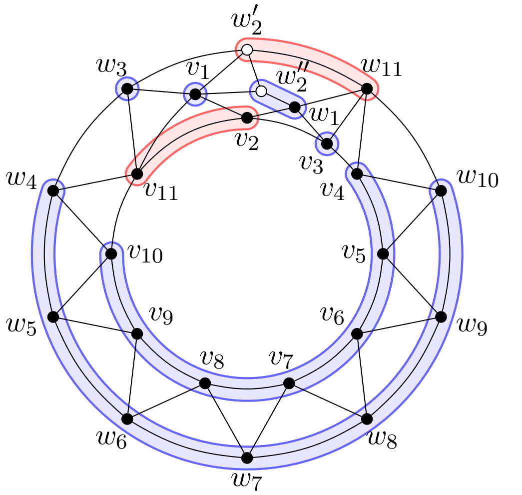
Case 1: Split w_2-v_1 gives a DW_6-minor (see Figure 5.21a).

Case 2: Split w_2-w_1 gives a DW_6-minor (see Figure 5.21b).

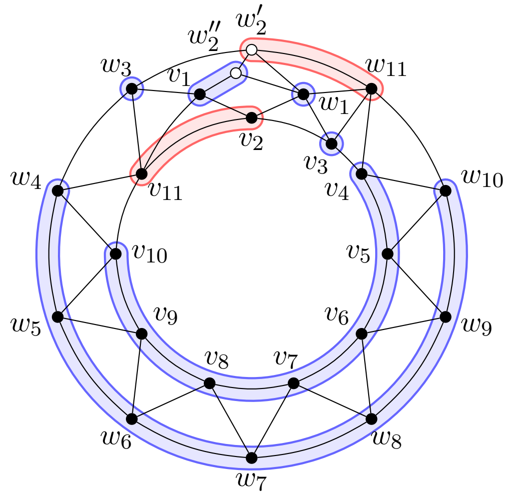
Case 3: Split w_2-w_3 in X_{2n}^{+} gives a DW_6-minor (see Figure 5.21c). The 4-connected splits of w_2 in X_{2n}^{-} are covered in Cases 1 and 2.

Case 4: Split w_2-w_n in X_{2n}^{+} gives a DW_6-minor (see Figure 5.21d). The 4-connected splits of w_2 in X_{2n}^{-} are covered in Cases 1 and 2.

Case 5: Split w_2-v_2 gives a DW_6-minor (see Figure 5.21e). Note that this split only makes sense in X^+_{2n}, because the edge v_2w_2 is not present in X^-_{2n}.

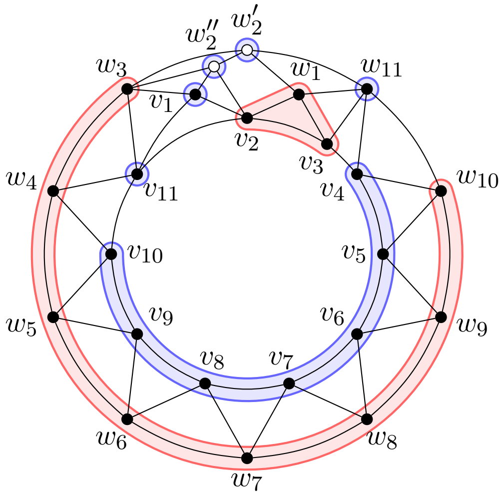


(a) Split w_2-v_1 has a DW_6-minor.
Hub: $\{v_2, v_n\}$ and $\{w_2', w_n\}$.
Rim: $\{v_1\}$, $\{w_1, w_2''\}$, $\{v_3\}$, $\{v_4, \ldots, v_{n-1}\}$, $\{w_4, \ldots, w_{n-1}\}$, and $\{w_3\}$.

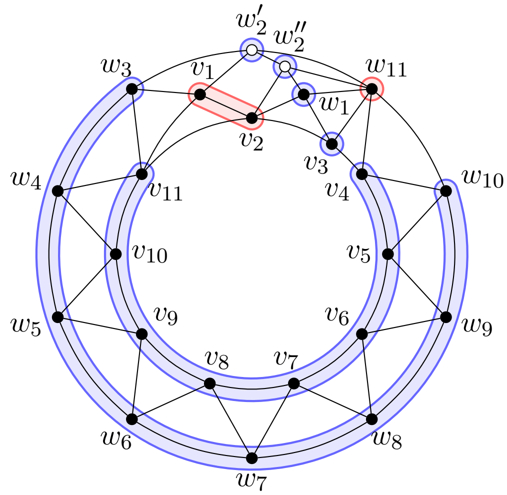


(b) Split w_2-w_1 has a DW_6-minor.
Hub: $\{v_2, v_n\}$ and $\{w_2', w_n\}$.
Rim: $\{v_1, w_2''\}$, $\{w_1\}$, $\{v_3\}$, $\{v_4, \ldots, v_{n-1}\}$, $\{w_4, \ldots, w_{n-1}\}$, and $\{w_3\}$.

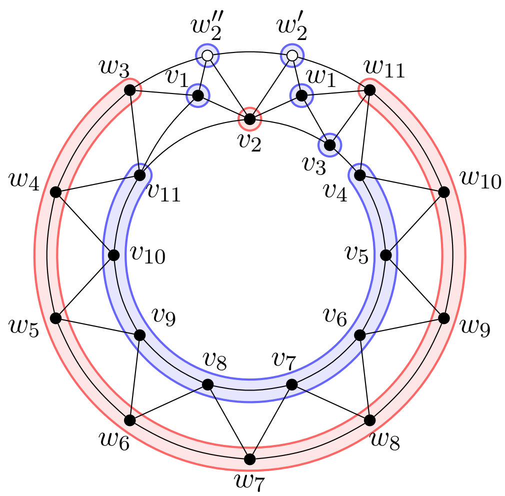
Figure 5.21: Splitting w_2 (similarly v_2) of $X^{\pm}_{2n}$



(c) Split w_2-w_3 has a DW_6-minor.
Hub: $\{w_1, v_2, v_3\}$ and $\{w_3, \ldots, w_{n-1}\}$.
Rim: $\{v_n\}$, $\{v_1\}$, $\{w_2''\}$, $\{w_2'\}$, $\{w_n\}$, and $\{v_4, \ldots, v_{n-1}\}$.



(d) Split w_2-w_n has a DW_6-minor.
Hub: $\{v_1, v_2\}$ and $\{w_n\}$.
Rim: $\{w_3, \ldots, w_{n-1}\}$, $\{w_2'\}$, $\{w_2''\}$, $\{w_1\}$, $\{v_3\}$, and $\{v_4, \ldots, v_n\}$.



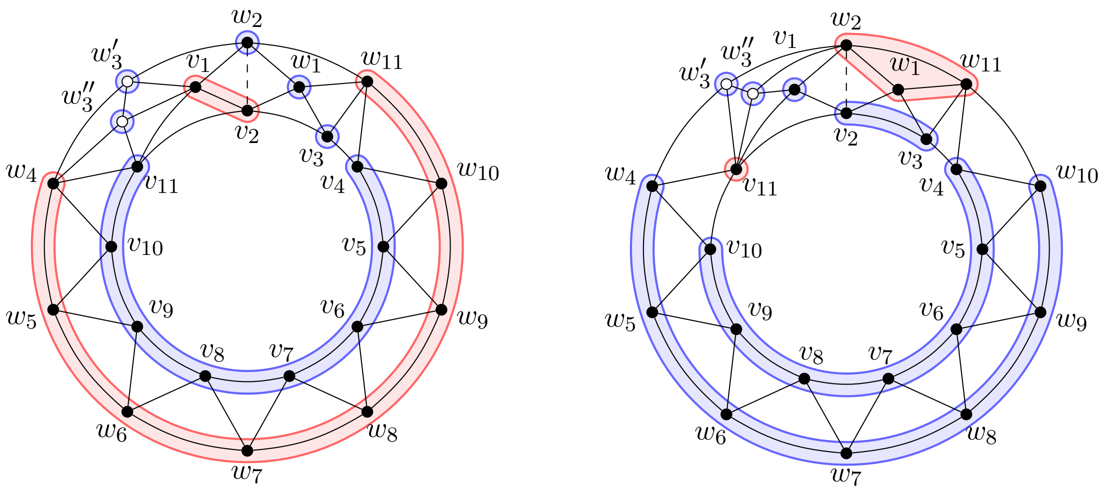
(e) Split w_2-v_2 has a DW_6-minor.
Hub: $\{v_2\}$ and $\{w_3, \ldots, w_n\}$.
Rim: $\{v_1\}$, $\{w_2''\}$, $\{w_2'\}$, $\{w_1\}$, $\{v_3\}$, and $\{v_4, \ldots, v_n\}$.

Figure 5.21: Splitting w_2 (similarly v_2) of $X_{2n}^{\pm}$

Splitting w_3 (similarly v_3) of $X_{2n}^{\pm}$:

Case 1: Split w_3-v_1w_4 gives a DW_6-minor (see Figure 5.22a).

Case 2: Split w_3-v_nw_2 gives a DW_6-minor (see Figure 5.22b).



(a) Split w_3-v_1w_4 has a DW_6-minor.
Hub: $\{v_1, v_2\}$ and $\{w_4, \ldots, w_n\}$.
Rim: $\{w_3''\}$, $\{w_3'\}$, $\{w_2\}$, $\{w_1\}$, $\{v_3\}$, and $\{v_4, \ldots, v_n\}$.

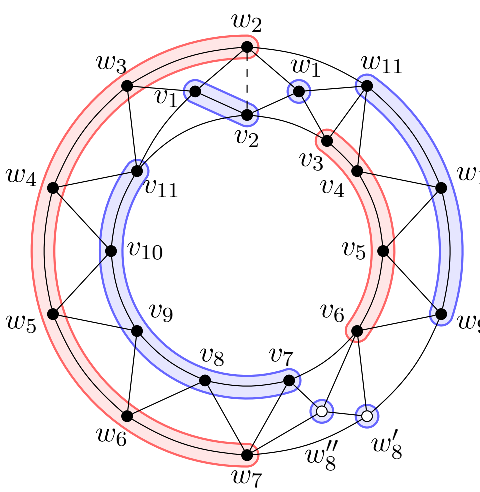
(b) Split w_3-v_nw_2 has a DW_6-minor.
Hub: $\{w_1, w_2, w_n\}$ and $\{v_n\}$.
Rim: $\{w_3''\}$, $\{v_1\}$, $\{v_2, v_3\}$, $\{v_4, \ldots, v_{n-1}\}$, $\{w_4, \ldots, w_{n-1}\}$, and $\{w_3'\}$.

Figure 5.22: Splitting w_3 (similarly v_3) of $X_{2n}^{\pm}$

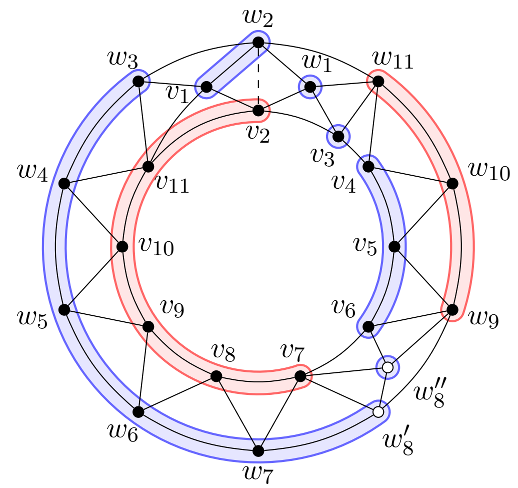
Splitting w_j (similarly v_j) of $X_{2n}^{\pm}$, for $4 \leq j \leq n-1$:

Case 1: Split w_j-$w_{j-1}v_{n-j+3}$ gives a DW_6-minor (see Figure 5.23a).

Case 2: Split w_j-$w_{j+1}v_{n-j+4}$ gives a DW_6-minor (see Figure 5.23b).



(a) Split w_j-$w_{j-1}v_{n-j+3}$ has a DW_6-minor. Hub: $\{v_3, v_4, \ldots, v_{n-j+3}\}$ and $\{w_2, \ldots, w_{j-1}\}$.
Rim: $\{v_{n-j+4}, \ldots, v_n\}$, $\{v_1, v_2\}$, $\{w_1\}$, $\{w_{j+1}, \ldots, w_n\}$, $\{w_j'\}$, and $\{w_j''\}$.



(b) Split w_j-$w_{j+1}v_{n-j+4}$ has a DW_6-minor. Hub: $\{v_{n-j+4}, \ldots, v_n, v_2\}$ and $\{w_{j+1}, \ldots, w_n\}$.
Rim: $\{w_3, \ldots, w_j'\}$, $\{v_1, w_2\}$, $\{w_1\}$, $\{v_3\}$, $\{v_4, \ldots, v_{n-j+3}\}$, and $\{w_j''\}$.

Figure 5.23: Splitting w_j (similarly v_j) of $X_{2n}^{\pm}$, for $4 \leq j \leq n-1$

Splitting w_n (similarly v_n) of $X_{2n}^{\pm}$:

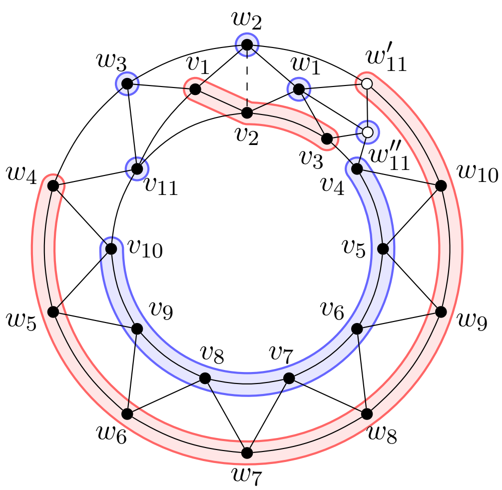
Case 1: Split w_n-w_1 gives a DW_6-minor (see Figure 5.24a).

Case 2: Split w_n-w_2 gives a DW_6-minor (see Figure 5.24b).

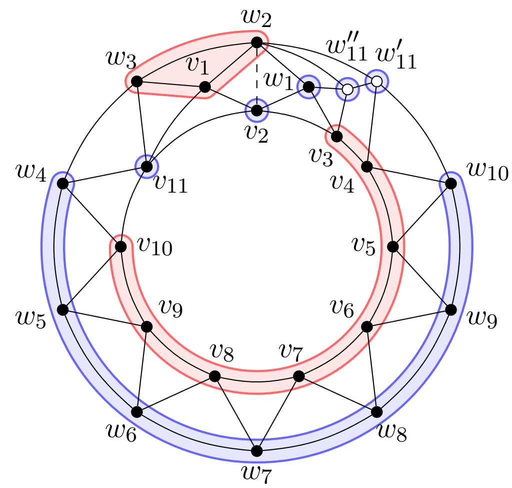
Case 3: Split w_n-v_4 gives a DW_6-minor (see Figure 5.24c).

Case 4: Split w_n-w_{n-1} gives a DW_6-minor (see Figure 5.24d).

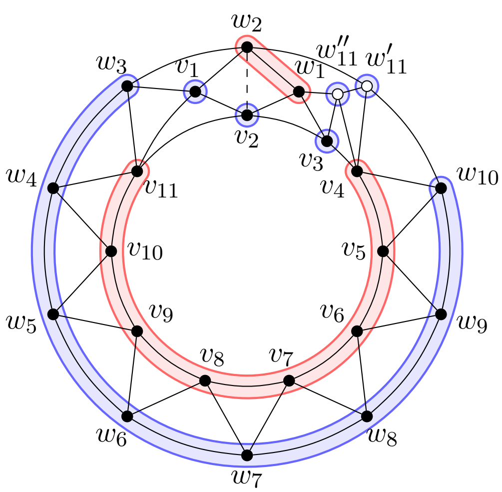
Case 5: Split w_n-v_3 gives $X_{2n+1}^{\pm}$ (see Figure 5.24e), but adding a second edge to either w_2 or w_{n-1} will introduce a DW_6-minor, as is shown in Cases 2 and 4 respectively.



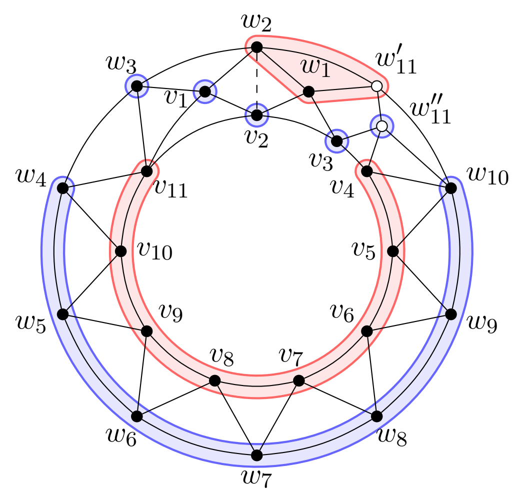
(a) Split w_n-w_1 has a DW_6-minor.
Hub: $\{v_1, v_2, v_3\}$ and $\{w_4, \ldots, w_{n-1}, w'_n\}$.
Rim: $\{v_n\}$, $\{w_3\}$, $\{w_2\}$, $\{w_1\}$, $\{w''_n\}$, and $\{v_4, \ldots, v_{n-1}\}$.



(b) Split w_n-w_2 has a DW_6-minor.
Hub: $\{v_1, w_2, w_3\}$ and $\{v_3, v_4, \ldots, v_{n-1}\}$.
Rim: $\{v_n\}$, $\{v_2\}$, $\{w_1\}$, $\{w''_n\}$, $\{w'_n\}$, and $\{w_4, \ldots, w_{n-1}\}$.

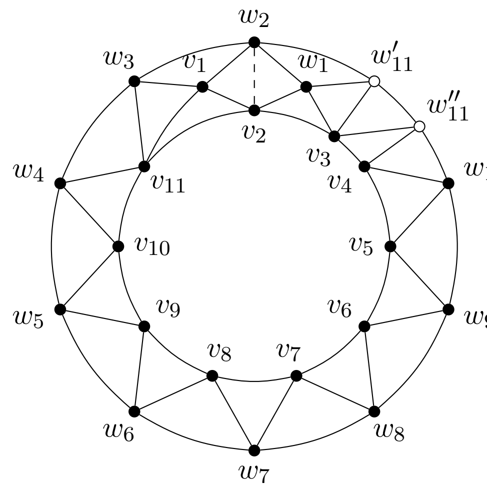


(c) Split w_n-v_4 has a DW_6-minor.
Hub: $\{v_4, \ldots, v_n\}$ and $\{w_1, w_2\}$.
Rim: $\{w_3, \ldots, w_{n-1}\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{w''_n\}$, and $\{w'_n\}$.



(d) Split w_n-w_{n-1} has a DW_6-minor.
Hub: $\{v_4, \ldots, v_n\}$ and $\{w_1, w_2, w'_n\}$.
Rim: $\{w_4, \ldots, w_{n-1}\}$, $\{w_3\}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, and $\{w''_n\}$.

Figure 5.24: Splitting w_n (similarly v_n) of $X^{\pm}_{2n}$



(e) Split w_n-v_3 gives $X^{\pm}_{2n+1}$.

Figure 5.24: Splitting w_n (similarly v_n) of $X^{\pm}_{2n}$

Splits of $X^{\pm}_{2n+1}$

Most of the following analysis applies to either type of graph in this family, so to make clear that we are allowing, but not requiring tw_1, we will draw pictures with this edge dashed, unless in a case in which we can assume its presence or absence. Note that the map fixing w_1 and t and taking v_i to v_{n-i+1} (for all i) and w_i to w_{n-i+2} (for $i \neq 1$) is an automorphism. We will break into separate cases for splits of the vertices t, w_1, v_1, v_2, w_2, v_j (for $3 \leq j \leq n-2$), and w_j (for $3 \leq j \leq n-1$). We will assume that $n \geq 5$; for $n = 4$, see the section at the beginning of the proof.

Splitting t of $X^{\pm}_{2n+1}$:

In Case 2, we consider the graph formed by splitting t in X^{-}_{2n+1} as shown in Figure 5.25b. The resulting graph is not 4-connected, as t'' only has degree three, but it does already have a DW_6-minor. Each of the splits t-v_1v_{n-1} in X^{-}_{2n+1} and t-v_1 in X^{+}_{2n+1} will contain this 3-connected split as a subgraph, and hence will also have a DW_6-minor.

Case 1: Split t-w_1 gives a DW_6-minor (see Figure 5.25a). Note that this split only makes sense in X^{+}_{2n+1}, because the edge tw_1 is not present in X^{-}_{2n+1}

Case 2: Split t-v_1 gives a DW_6-minor (see Figure 5.25b).

Case 3: Split t-v_n is equivalent to split t-v_1, so will also introduce a DW_6-minor.

Case 4: Split t-v_2 in X^+_{2n+1} gives a DW_6-minor (see Figure 5.25c). The 4-connected splits of t in X^-_{2n+1} are covered in Cases 2 and 3.

Case 5: Split t-v_{n-1} is equivalent to split t-v_2, so will also introduce a DW_6-minor.

Splitting w_1 of $X^{\pm}_{2n+1}$:

In Case 2, we consider the graph formed by splitting w_1 in X^-_{2n+1} as shown in Figure 5.26b. The resulting graph is not 4-connected, as w_1'' only has degree three, but it does already have a DW_6-minor. Each of the splits w_1-v_1w_n in X^-_{2n+1} and w_1-v_1 in X^+_{2n+1} will contain this 3-connected split as a subgraph, and hence will also have a DW_6-minor.

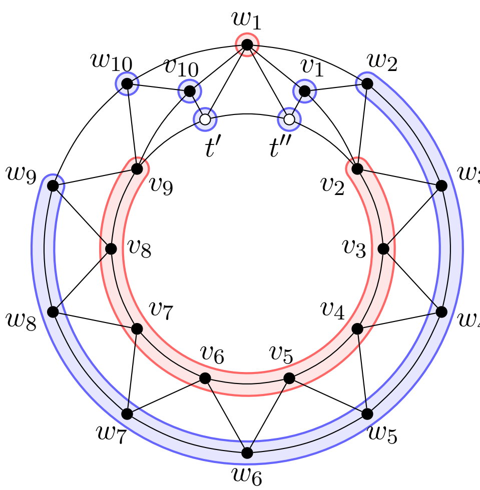
Case 1: Split w_1-t gives a DW_6-minor (see Figure 5.26a). Note that this split only makes sense in X^+_{2n+1}, because the edge w_1t is not present in X^-_{2n+1}

Case 2: Split w_1-v_1 gives a DW_6-minor (see Figure 5.26b).

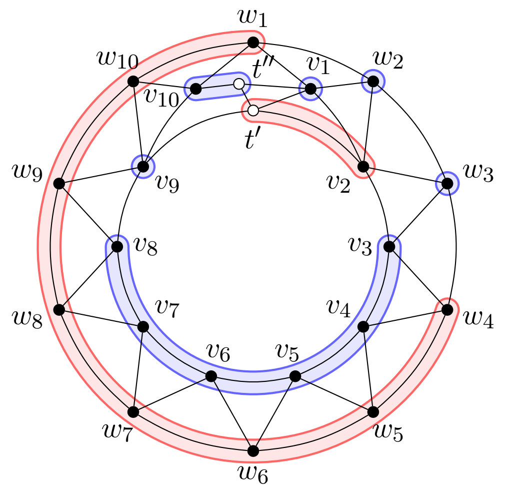
Case 3: Split w_1-v_n is equivalent to the split w_1-v_1, so will also introduce a DW_6-minor.

Case 4: Split w_1-w_2 in X^+_{2n+1} gives a DW_6-minor (see Figure 5.26c). The 4-connected splits of w_1 in X^-_{2n+1} are covered in Cases 2 and 3, and so both will give DW_6-minors.

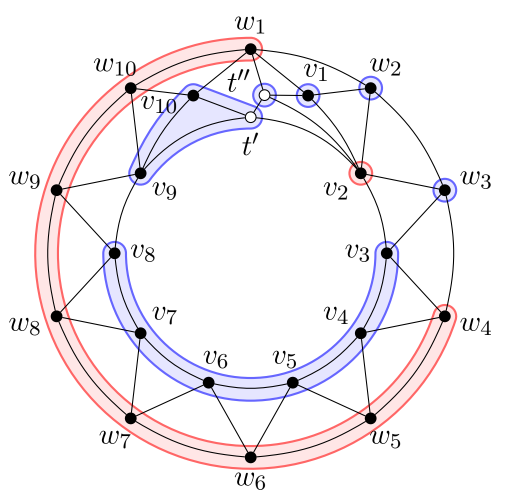
Case 5: Split w_1-w_n is equivalent to split w_1-w_2, so will also introduce a DW_6-minor.



(a) Split t-w_1 has a DW_6-minor.
Hub: $\{w_1\}$ and $\{v_2, \ldots, v_{n-1}\}$.
Rim: $\{w_n\}$, $\{v_n\}$, $\{t'\}$, $\{t''\}$, $\{v_1\}$, and $\{w_2, \ldots, w_{n-1}\}$.

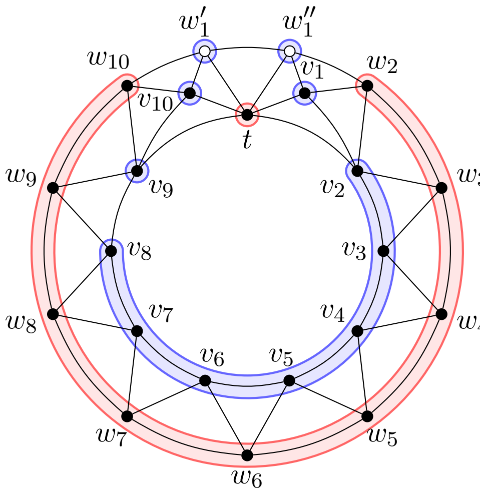


(b) Split t-v_1 has a DW_6-minor.
Hub: $\{t', v_2\}$ and $\{w_1, w_4, \ldots, w_n\}$.
Rim: $\{v_{n-1}\}$, $\{t'', v_n\}$, $\{v_1\}$, $\{w_2\}$, $\{w_3\}$, and $\{v_3, \ldots, v_{n-2}\}$.

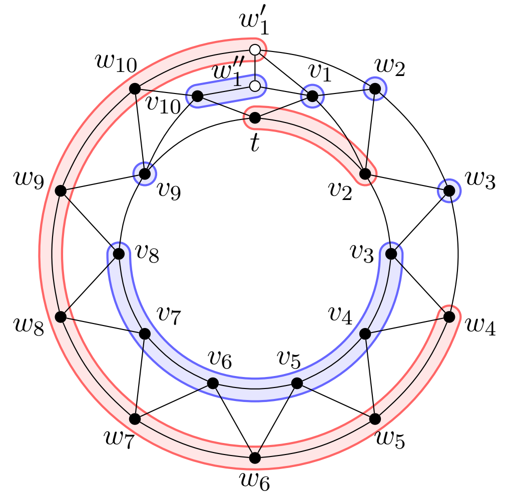


(c) Split t-v_2 has a DW_6-minor.
Hub: $\{w_1, w_4, \ldots, w_n\}$ and $\{v_2\}$.
Rim: $\{v_{n-1}, v_n, t'\}$, $\{t''\}$, $\{v_1\}$, $\{w_2\}$, $\{w_3\}$, and $\{v_3, \ldots, v_{n-2}\}$.

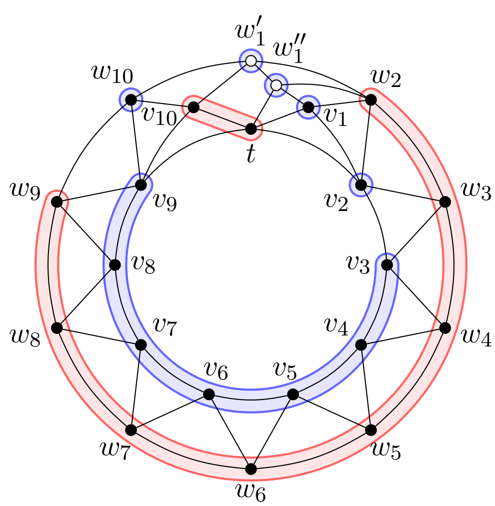
Figure 5.25: Splitting t of $X^{\pm}_{2n+1}$



(a) Split w_1-t has a DW_6-minor.
Hub: $\{t\}$ and $\{w_2, \ldots, w_n\}$.
Rim: $\{w_1'\}$, $\{w_1''\}$, $\{v_1\}$, $\{v_2, \ldots, v_{n-2}\}$, $\{v_{n-1}\}$, and $\{v_n\}$.



(b) Split w_1-v_1 has a DW_6-minor.
Hub: $\{t, v_2\}$ and $\{w_4, \ldots, w_n, w_1'\}$.
Rim: $\{v_{n-1}\}$, $\{v_n, w_1''\}$, $\{v_1\}$, $\{w_2\}$, $\{w_3\}$, and $\{v_3, \ldots, v_{n-2}\}$.



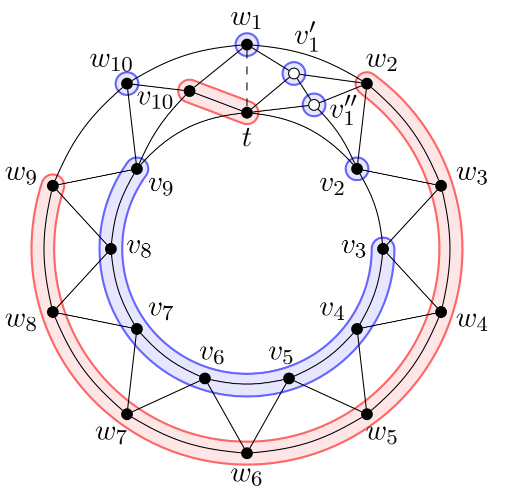
(c) Split w_1-w_2 has a DW_6-minor.
Hub: $\{v_n, t\}$ and $\{w_2, \ldots, w_{n-1}\}$.
Rim: $\{w_n\}$, $\{w_1'\}$, $\{w_1''\}$, $\{v_1\}$, $\{v_2\}$, and $\{v_3, \ldots, v_{n-1}\}$.

Figure 5.26: Splitting w_1 of $X_{2n+1}^{\pm}$

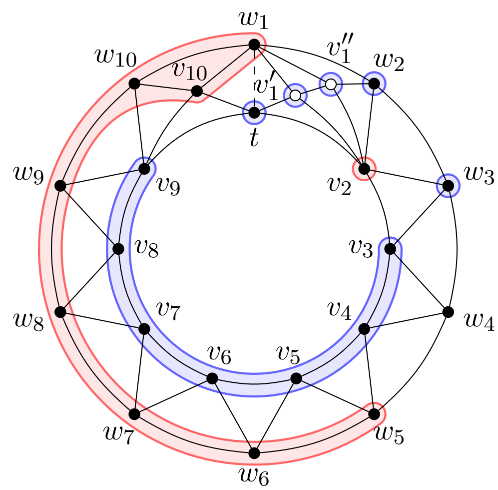
Splitting v_1 (similarly v_n) of $X^{\pm}_{2n+1}$:

Case 1: Split v_1-tw_2 gives a DW_6-minor (see Figure 5.27a).

Case 2: Split v_1-v_2w_1 gives a DW_6-minor (see Figure 5.27b).



(a) Split v_1-tw_2 has a DW_6-minor.
Hub: $\{v_n, t\}$ and $\{w_2, \ldots, w_{n-1}\}$.
Rim: $\{w_n\}$, $\{w_1\}$, $\{v_1'\}$, $\{v_1''\}$, $\{v_2\}$, and $\{v_3, \ldots, v_{n-1}\}$.



(b) Split v_1-v_2w_1 has a DW_6-minor.
Hub: $\{v_2\}$ and $\{w_4, \ldots, w_n, w_1, v_n\}$.
Rim: $\{t\}$, $\{v_1'\}$, $\{v_1''\}$, $\{w_2\}$, $\{w_3\}$, and $\{v_3, \ldots, v_{n-1}\}$.

Figure 5.27: Splitting v_1 (similarly v_n) of $X^{\pm}_{2n+1}$

Splitting v_2 (similarly v_{n-1}) of $X^{\pm}_{2n+1}$:

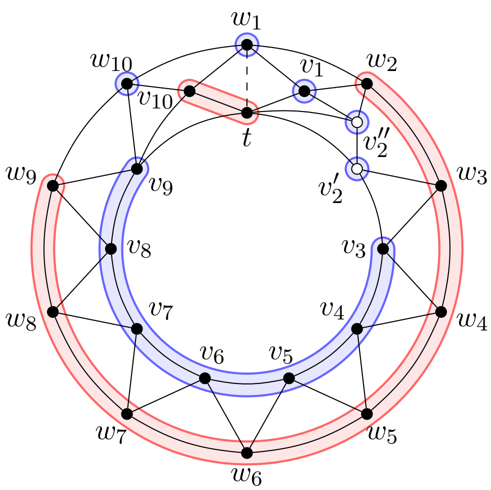
Case 1: Split v_2-t gives a DW_6-minor (see Figure 5.28a).

Case 2: Split v_2-v_1 gives a DW_6-minor (see Figure 5.28b).

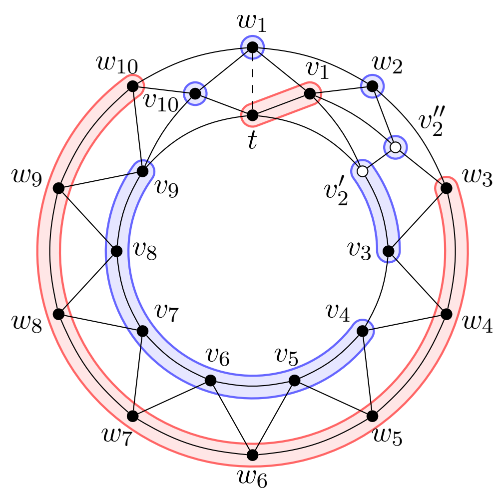
Case 3: Split v_2-w_3 gives a DW_6-minor (see Figure 5.28c).

Case 4: Split v_2-v_3 gives a DW_6-minor (see Figure 5.28d).

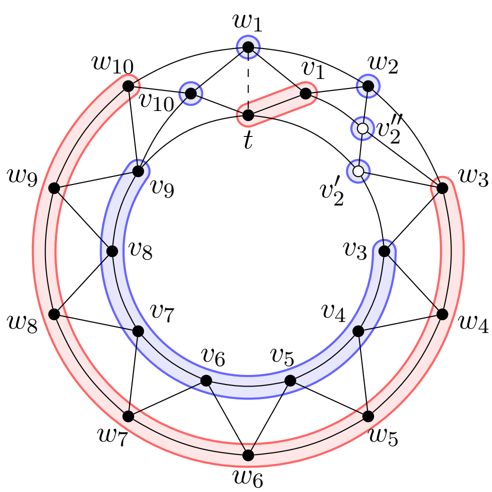
Case 5: Split v_2-w_2 gives $X^{\pm}_{2(n+1)}$ (see Figure 5.28e). Adding a second edge to either t or v_3 will give a DW_6-minor, as is covered in Cases 1 and 4 respectively.



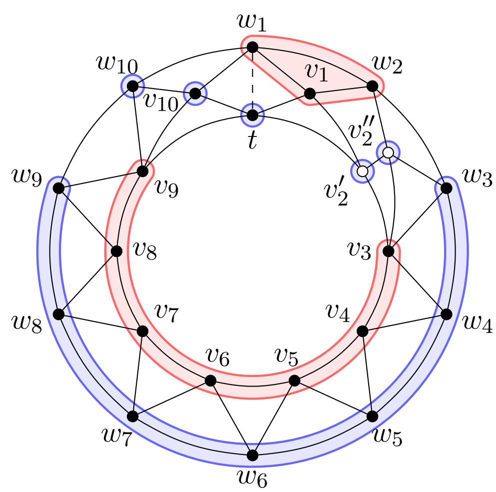
(a) Split v_2-t has a DW_6-minor.
Hub: $\{v_n, t\}$ and $\{w_2, \ldots, w_{n-1}\}$.
Rim: $\{w_n\}$, $\{w_1\}$, $\{v_1\}$, $\{v_2''\}$, $\{v_2'\}$, and $\{v_3, \ldots, v_{n-1}\}$.



(b) Split v_2-v_1 has a DW_6-minor.
Hub: $\{t, v_1\}$ and $\{w_3, \ldots, w_n\}$.
Rim: $\{v_n\}$, $\{w_1\}$, $\{w_2\}$, $\{v_2''\}$, $\{v_2', v_3\}$, and $\{v_4, \ldots, v_{n-1}\}$.

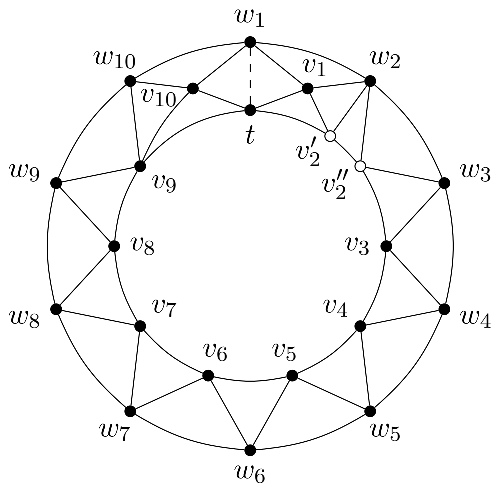


(c) Split v_2-w_3 has a DW_6-minor.
Hub: $\{t, v_1\}$ and $\{w_3, \ldots, w_n\}$.
Rim: $\{v_n\}$, $\{w_1\}$, $\{w_2\}$, $\{v_2''\}$, $\{v_2'\}$, and $\{v_3, \ldots, v_{n-1}\}$.



(d) Split v_2-v_3 has a DW_6-minor.
Hub: $\{v_1, w_1, w_2\}$ and $\{v_3, \ldots, v_{n-1}\}$.
Rim: $\{w_n\}$, $\{v_n\}$, $\{t\}$, $\{v_2'\}$, $\{v_2''\}$, and $\{w_3, \ldots, w_{n-1}\}$.

Figure 5.28: Splitting v_2 (similarly v_{n-1}) of $X^{\pm}_{2n+1}$



(e) Split v_2-w_2 gives $X^{\pm}_{2(n+1)}$.

Figure 5.28: Splitting v_2 (similarly v_{n-1}) of $X^{\pm}_{2n+1}$

Splitting w_2 (similarly w_n) of $X^{\pm}_{2n+1}$:

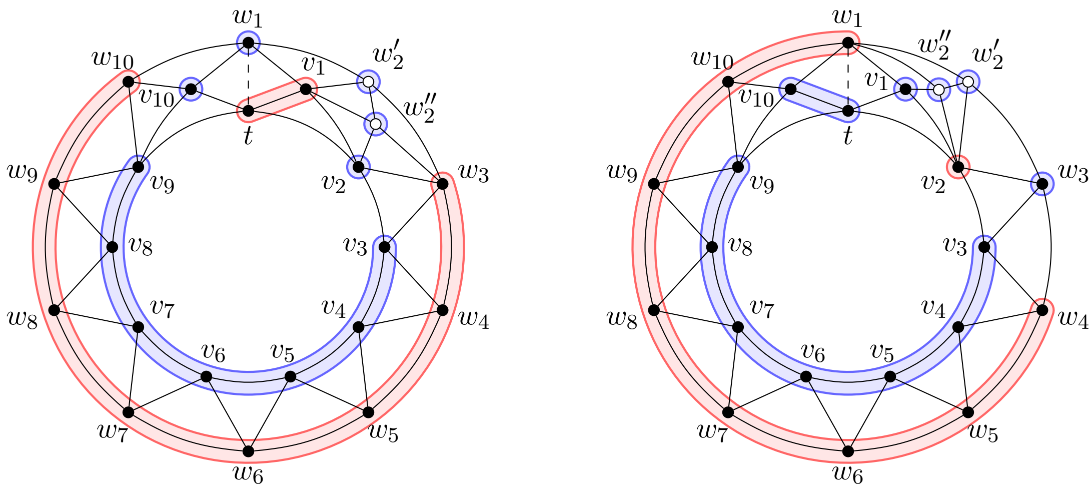
Case 1: Split w_2-v_1w_3 gives a DW_6-minor (see Figure 5.29a).

Case 2: Split w_2-v_2w_1 gives a DW_6-minor (see Figure 5.29b).

Splitting w_j of $X^{\pm}_{2n+1}$, for $3 \leq j \leq n-1$:

Case 1: Split w_j-$v_{j-1}w_{j+1}$ gives a DW_6-minor (see Figure 5.30a).

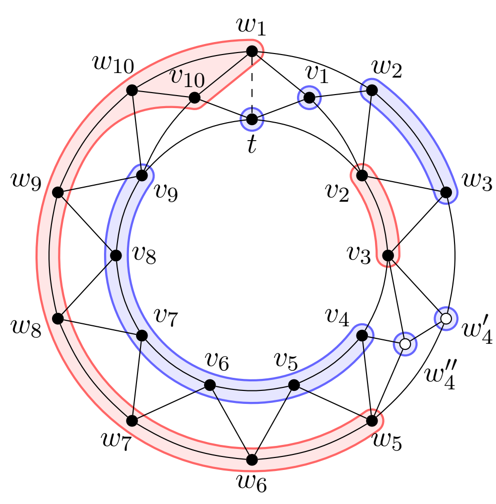
Case 2: Split w_j-v_jw_{j-1} is equivalent to the split $w_{j'}$-$v_{j'-1}w_{j'+1}$ where $j' = n - j + 2$, so will also introduce a DW_6-minor.



(a) Split w_2-v_1w_3 has a DW_6-minor.
Hub: $\{t, v_1\}$ and $\{w_3, \ldots, w_n\}$.
Rim: $\{v_n\}$, $\{w_1\}$, $\{w_2'\}$, $\{w_2''\}$, $\{v_2\}$, and $\{v_3, \ldots, v_{n-1}\}$.

(b) Split w_2-v_2w_1 has a DW_6-minor.
Hub: $\{v_2\}$ and $\{w_4, \ldots, w_n, w_1\}$.
Rim: $\{v_n, t\}$, $\{v_1\}$, $\{w_2''\}$, $\{w_2'\}$, $\{w_3\}$, and $\{v_3, \ldots, v_{n-1}\}$.

Figure 5.29: Splitting w_2 (similarly w_n) of $X^{\pm}_{2n+1}$



(a) Split w_j-$v_{j-1}w_{j+1}$ has a DW_6-minor.
Hub: $\{v_2, \ldots, v_{j-1}\}$ and $\{w_{j+1}, \ldots, w_n, w_1, v_n\}$.
Rim: $\{t\}$, $\{v_1\}$, $\{w_2, \ldots, w_{j-1}\}$, $\{w_j'\}$, $\{w_j''\}$, and $\{v_j, \ldots, v_{n-1}\}$.

Figure 5.30: Splitting w_j of $X^{\pm}_{2n+1}$, for $3 \le j \le n-1$

Splitting v_j of $X^{\pm}_{2n+1}$, for $3 \leq j \leq n-2$:

Case 1: Split v_j-$v_{j-1}w_{j+1}$ is isomorphic to split w_j-$v_{j-1}w_{j+1}$, and hence gives a DW_6-minor.

Case 2: Split v_j-$v_{j+1}w_j$ is equivalent to split $v_{j'}$-$v_{j'-1}w_{j'+1}$ where $j' = n - j + 1$, so will also introduce a DW_6-minor.

This completes the proof. $\square$

Chapter 6

Future Directions

Given our characterization of 4-connected $K_{2,5}$-minor-free graphs in Chapter 4, it might be natural to try to characterize 4-connected $K_{1,1,5}$- or $K_{2,6}$-minor-free graphs. We know from Theorem 2.29 that graphs in the latter family can all be formed by taking graphs of bounded size and possibly adding some number of strips, as described in Subsection 2.3.3. By considering which splits of 4-connected $K_{2,5}$-minor-free graphs would not introduce a $K_{2,6}$-minor, it seems likely that this family can in fact be formed by adding a single strip to a graph on a very small number of vertices, perhaps as few as seven or eight. This is supported by preliminary computer results. If true, it should be possible to describe exactly which small graphs are needed and where a strip might be attached to each. It would then also likely be possible to characterize the family of 4-connected $K_{1,1,5}$-minor-free graphs, as it is contained in the family of 4-connected $K_{2,6}$-minor-free graphs and contains the 4-connected $K_{2,5}$-minor-free graphs, thus sitting between two similar and well understood structural families.

It might also be possible to characterize 3-connected $K_{2,5}$-minor-free graphs. From Theorem 2.29, we know that these graphs can be formed by taking graphs of bounded size and possibly adding some number of strips and fans. We have generated all graphs in this family on up to 16 vertices, then restricted our attention to those that were edge maximal and did not contain a fan. The remaining graphs suggest that this family consists of fairly small graphs to which only fans can be added, along with 3-connected minors of $C_n[K_2]$.

It should also be possible to generalize the enumerative results of Section 4.3 to find the number of strips of on m vertices which could potentially be part of a 4-connected graph. We suspect that this number will asymptotically grow like $O(\alpha^m)$

for the same α defined in that section. This would imply, for any t, that the number of 4-connected $K_{2,t}$-minor-free graphs on n vertices would be $O(n^{c(t)}\alpha^m)$, where $c(t)$ would depend on the maximum number of strips that could be added to any graph without necessarily creating a $K_{2,t}$-minor. While it is not clear how to best bound this number, we know that it is finite for all t by Ding's Theorem 2.29. It may also be possible to similarly describe the asymptotic growth of 3-connected $K_{2,t}$-minor-free graphs.

Characterizing (nonplanar) 4-connected DW_6-minor-free graphs seems potentially difficult. Computer testing suggests that the only cubic cyclically-4-edge-connected graph (on at least 12 vertices) whose line graph does not contain a DW_6-minor is the Möbius ladder on $2n$ vertices. Even still, it is unclear whether or not the non-planar 4-connected DW_6-minor-free graphs have as nice a structure as the planar family described in Chapter 5. Because it grows very quickly, we have so far only been able to exhaustively generate this family on up to 11 vertices, and it is possible that more structure would reveal itself if one were able to continue.

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