EQUIVARIANT INDEX THEORY AND NON-POSITIVELY CURVED MANIFOLDS

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To my parents, and my wife Kai

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CHAPTER I

INTRODUCTION

An elliptic differential operator D on a compact manifold M is a Fredholm operator. The only topological invariant for a Fredholm operator is the Fredholm index [Dou72], which is defined to be $\dim(\ker D) - \dim(\operatorname{coker} D)$. Fredholm index is a homotopy invariant. The Atiyah-Singer index theorem calculates the Fredholm index of D in terms of its symbol $\sigma(D)$ and M. This theorem establishes a bridge between analysis, geometry and topology [AS1, AS3]. The Fredholm index is often related to the geometry of the manifold. An example of this is that the nonnegativity of the scalar curvature implies the vanishing of the Fredholm index for the Dirac operator.

Index theorems have been generalized to noncompact manifolds of various sorts. Elliptic operators on noncompact manifolds are no longer Fredholm in the classical sense, but are Fredholm in a generalized sense with respect to certain operator algebras. An important topological invariant for an elliptic operator is the generalized Fredholm index, which lives in the K-theory of an operator algebra. An early example of this was the index theorem for almost periodic Toeplitz operators, which computes partially a generalized Fredholm index [CDSS]. Some other examples are the index theorem for coverings [A76, MS, CM], for foliations [CS], for homogeneous spaces of Lie groups [CM82], and for complete manifolds of bounded geometry with regular exhaustions [R88]. in the case of a complete manifold M, Dirac type operators on M are generalized Fredholm operators in the sense that they are invertible modulo the Roe-algebra. Hence the indices of Dirac operators live in the K-theory of the Roe algebra.

In this thesis we define the equivariant index map for proper group actions and prove that this equivariant index map is injective for certain manifolds and groups. We also prove that the index map [Y95, Y97] is injective for spaces which admit a coarse embedding into a simply-connected complete Riemannian manifold with nonpositive sectional curvature, which is the joint work with Qin Wang.

CHAPTER II

K-THEORY FOR C*-ALGEBRAS

In this chapter, we will review the K-theory for C^*-algebras over $\mathbb{C}$. All material in this chapter is standard and can be found in most K-theory books, such as [Mur, T, W].

II.1 The group $K_0(A)$.

Let A be a unital C^*-algebra, and set

$$\mathcal{P}(A) = \bigcup_{n\in\mathbb{N}} \left\{p \in M_n(A) : p^* = p,\ p^2 = p\right\}.$$

Definition 1. *Let $p, q \in \mathcal{P}(A)$.*

- *We say p and q are* **equivalent**, *denoted by $p \sim q$, if $p = uu^*$ and $q = u^*u$ for some partial isometry $u \in M_m(A)$.*
- *We say p and q are* **unitarily equivalent**, *denoted by $p \sim_u q$, if $p = u^*qu$ for some unitary u in $M_m(A)$.*
- *We say p and q are* **homotopic**, *denoted by $p \sim_h q$, if p and q are connected by a norm continuous path of projections in $\mathcal{P}_m(A)$.*

Proposition 1. *If p and q are in $\mathcal{P}(A)$, then*

$$p \sim_h q \Rightarrow p \sim_u q \Rightarrow p \sim q.$$

And

$$p \sim q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \text{ and } p \sim_u q \Rightarrow \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

This proposition tells us that those three equivalence relations are equivalent in $\mathcal{P}(A)$. So let the equivalent classes be denoted by $[\cdot]$. Let $V(A) = \mathcal{P}(A)/\sim$ be the set of equivalent classes of all projections in $\mathcal{P}(A)$. Define the addition in $V(A)$ by

$$[p] + [q] \overset{def}{=} \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right].$$

Lemma 1. *$V(A)$ is an abelian semi-group with additive identity $0 = [0]$.*

If A, B are unital C^*-algebras and if $\phi : A \to B$ is a $*$-homomorphism, the induced map $\phi_* : V(A) \to V(B)$ given by

$$\phi_*([(a_{ij})]) = [(\phi(a_{ij}))]$$

is a well-defined homomorphism of semigroups.

Example 1. $V(\mathbb{C}) = \mathbb{N} \cup \{0\}$.

Definition 2. *Let A be a unital C^*-algebra. $K_0(A)$ is defined to be the Grothendieck group of $V(A)$.*

Example 2. $K_0(\mathbb{C}) = \mathbb{Z}$.

By the universal property of Grothendieck groups, the homomorphism $\phi_* : V(A) \to V(B)$ induced by some homomorphism $\phi : A \to B$ extends to a group homomorphism $\phi_* : K_0(A) \to K_0(B)$.

Definition 3. *Let A be a non-unital C^*-algebra and let A^+ be a unitization of A such that $A^+/A = \mathbb{C}$. Let $\pi : A^+ \to \mathbb{C}$ be the projection. Define*

$$K_0(A) \overset{def}{=} \ker \left\{ \pi_* : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z} \right\}.$$

Example 3. $K_0(C_0(\mathbb{R}^2)) = \mathbb{Z}$. *This can be computed by the Bott periodicity Lemma 12.*

Remark. When A is unital, we have

$$\ker \left\{ \pi_* : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z} \right\} \cong K_0(A),$$

where $K_0(A)$ is defined in Definition 2. Therefore we use $\ker \{\pi_* : K_0(A^+) \to K_0(\mathbb{C}) \cong \mathbb{Z}\}$ to be the definition of K_0-groups of all C^*-algebras.

Theorem 1. *Let A be a C^*-algebra.*

1. $K_0(A)$ *is an abelian group.*

2. *The element of $K_0(A)$ can be written as $[p] - [1_n]$, where $p, 1_n \in M_m(A^+)$ for some m and 1_n is the matrix with n 1's in the diagonal and 0 elsewhere and $p - 1_n \in M_m(A)$. If A is unital, we can choose $p, 1_n \in M_m(A)$.*

Theorem 2. *The homomorphism $A \to A \otimes \mathbb{K}$ by sending $a \to a \otimes e_1$, where e_1 is a rank 1 projection in $\mathbb{K}$, induces an isomorphism $K_0(A) \cong K_0(A \otimes \mathbb{K})$.*

Theorem 3. *Let J be an ideal in A. Then the exact sequence $0 \longrightarrow J \xrightarrow{i} A \xrightarrow{\pi} A/J \longrightarrow 0$ induces a short exact sequence of K_0-groups:*

$$K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J).$$

Definition 4. *Let A and B be C^*-algebras.*

1. *Two homomorphisms $\phi, \psi : A \to B$ are homotopic, denoted by $\phi \sim_h \psi$, if there is a path $\{\gamma_t\}_{[0,1]}$ of homomorphisms $\gamma_t : A \to B$, such that $t \to \gamma_t(a)$ is a norm continuous path in B for every fixed a in A and such that $\gamma_0 = \phi, \gamma_1 = \psi$.*

2. *A homomorphism $\phi : A \to B$ is equivalence if there is another homomorphism $\psi : B \to A$ such that $\phi \circ \psi$ and $\psi \circ \phi$ both are homotopic to the identity.*

Theorem 4. *When $\phi_0, \phi_1 : A \to B$ are homotopic, then $\phi_{0*} = \phi_{1*}$ for the induced homomorphisms $K_0(A) \to K_0(B)$.*

II.2 The group $K_1(A)$ and the index map.

Let A be a unital C^*-algebra, and let

$$\mathcal{U}(A) = \bigcup_{n \in \mathbb{N}} \{u \in U_n(A) : u \text{ is a unitary}\}.$$

Definition 5. *Let u and v be in $\mathcal{U}(A)$. We say that u and v are* **homotopic***, denoted by $u \sim_h v$, if they are connected by a norm continuous path of unitaries in $\mathcal{U}_m(A)$.*

Let the equivalent classes be denoted by $[\cdot]$. Define the addition in $\mathcal{U}(A)$ by

$$[u]+[v] \stackrel{def}{=} \left[\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right].$$

Definition 6. *Define $K_1(A) = \mathcal{U}(A)/\sim_h$ to be the set of equivalent classes of unitaries in $\mathcal{U}(A)$.*

Theorem 5. *$K_1(A)$ is an abelian group.*

Example 4. $K_1(\mathbb{C}) = 0$.

Definition 7. *Let A a non-unital C^*-algebra and let A^+ be a unitization of A such that $A^+/A = \mathbb{C}$. Let $\pi : A^+ \to \mathbb{C}$ be the projection. Define*

$$K_1(A) \stackrel{def}{=} \ker\left\{\pi_* : K_1(A^+) \to K_1(\mathbb{C}) \cong 0\right\}.$$

Therefore, $K_1(A) = K_1(A^+)$.

Example 5. $K_1(C_0(\mathbb{R})) = K_1(C(\mathbb{T})) = \mathbb{Z}$.

Theorem 6. *Let A be a C^*-algebra, then $K_1(A) \cong K_1(A \otimes \mathcal{K})$.*

Theorem 7. *Let J be an ideal in A. Then the exact sequence $0 \longrightarrow J \stackrel{i}{\longrightarrow} A \stackrel{\pi}{\longrightarrow} A/J \longrightarrow 0$ induces a short exact sequence of K_1-groups:*

$$K_1(J) \stackrel{i_*}{\longrightarrow} K_1(A) \stackrel{\pi_*}{\longrightarrow} K_1(A/J).$$

Definition 8. *The suspension of a C^*-algebra A is the C^*-algebra*

$$SA \stackrel{def}{=} C_0(\mathbb{R} \to A) \cong C_0(\mathbb{R}) \otimes A \cong C_0(0,1) \otimes A.$$

Theorem 8. *For every C^*-algebra A, there is an isomorphism*

$$\theta : K_1(A) \longrightarrow K_0(SA).$$

Definition 9. *For a C^*-algebra A, define the $K_n(A)$ by*

$$K_n(A) = K_0(S^n A), \qquad n \in \mathbb{N}.$$

Theorem 9. *When $\phi_0, \phi_1 : A \longrightarrow B$ are homotopic, then $\phi_{0*} = \phi_{1*}$ for the induced homomorphisms $K_1(A) \longrightarrow K_1(B)$.*

Lemma 2. *Let $u \in \mathcal{U}(A)$. There exists $v \in \mathcal{U}(A)$ such that $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ is homotopic to $1_m(A)$.*

Lemma 3. *If A and B are C^*-algebras and $\phi : A \longrightarrow B$ is a surjective morphism, then ϕ extends to a unital surjective morphism $\phi^+ : A^+ \longrightarrow B^+$ that can lift unitaries in the connected component of 1 in B^+ to unitaries in the connected component of 1 in A^+.*

Definition 10 (The Index Map)**.** *Let J be an ideal in a C^*-algebra A and $u \in \mathcal{U}((A/J)^+)$. Find a $v \in \mathcal{U}((A/J)^+)$ for which $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ is homotopic to $1_m \in \mathcal{U}((A/J)^+)$. Let $w \in \mathcal{U}(A^+)$ be a unitary lift of $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$.*

The **index map** $\mathrm{Ind} : K_1(A/J) \longrightarrow K_0(J)$ *is defined by*

$$\mathrm{Ind}(x) \stackrel{def}{=} [wp_n w^*] - [p_n],$$

where $x = [u] \in K_1(A/J)$.

We remark here that $[wp_n w^*] - [p_n] \in K_0(J)$. The reason is the following. Let $\pi_J : A \longrightarrow A/J$ be the projection. Then $\pi_J(wp_n w^*) = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} p_n \begin{pmatrix} u^* & 0 \\ 0 & v^* \end{pmatrix} = p_n$. Hence $wp_n w^* \in \mathcal{P}(J^+)$. And let $\pi : J^+ \longrightarrow \mathbb{C}$. We have $\pi_*([wp_n w^*] - [p_n]) = [p_n] - [p_n] = 0$ because $\pi(w)p_n\pi(w)^* \sim_u p_n$.

Theorem 10 (The Long Exact Sequence)**.** *Let J be an ideal in a C^*-algebra A. The* *following sequence is exact everywhere:*

$$\cdots \longrightarrow K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{\pi_*} K_1(A/J) \xrightarrow{\text{Ind}} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J).$$

II.3 The Bott periodicity and the six-term exact sequence.

Let A be a unital C^*-algebra. First, we can get the following characterization:

$$K_1(SA) \cong \{[f] | f \in C(\mathbb{T} \longrightarrow \mathcal{U}(A)),\ f(1) \sim_h 1_n \text{ for some n.}\}$$

This description tells that the elements in $K_1(SA)$ are loops in $\mathcal{U}(A)$. Let $p \in \mathcal{P}(A)$, we define

$$f_p(t) = e^{2\pi i t p} = 1 + p(e^{2\pi i t} - 1), \quad \forall\ 0 \leq t \leq 1.$$

f_p has the following properties.

- $f_p(t)^* = (e^{2\pi i t p})^* = e^{-2\pi i t p^*} = e^{-2\pi i t p} = f_p(-t)$,
- $f_p(t) f_p(-t) = 1$,
- $f(0) = f(1) = 1$.

Therefore f_p gives an element in $K_1(SA)$.

Lemma 4. *Let A be a unital C^*-algebra. Then the map*

$$\begin{array}{rccc} \beta_A : & K_0(A) & \longrightarrow & K_1(SA) \\ & [p] - [q] & \longrightarrow & [f_p f_q^*]. \end{array}$$

defines a group homomorphism.

Theorem 11. *β_A is an isomorphism.*

Combine Theorem 8 and Theorem 11, we have the following famous theorem.

Theorem 12 (Bott Periodicity)**.** $K_2(A) \cong K_0(A)$.

Theorem 13 (Six-term Exact Sequence)**.** *Let A be a C^*-algebra and let J be an ideal of A. The following loop is exact every where*

$$\begin{array}{ccccc} K_0(J) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/J) \\ \uparrow & & & & \downarrow \\ K_1(A/J) & \longleftarrow & K_1(A) & \longleftarrow & K_1(J). \end{array}$$

CHAPTER III

EQUIVARIANT K-HOMOLOGY

In this chapter, we will review the equivariant K-homology. Since K-homology has its root in differential operators, we first review Sobolev spaces and the first-order partial differential operators. This chapter is based on [KAS88, HR00].

III.1 Sobolev spaces and first-order partial differential operators

Let $\mathcal{S}$ be the Schwartz class of $\mathbb{R}^n$ which is the set of all smooth complex-valued functions on $\mathbb{R}^n$ such that for all α, β, there exists a constant $C_{\alpha,\beta}$ such that

$$|x^\alpha D_x^\beta f| \leq C_{\alpha,\beta},$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $D_x^\beta = (-i)^{\beta_1 + \cdots + \beta_n} \left(\frac{\partial}{\partial x_1}\right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\beta_n}$. The extra factors of $(-i)$ defining D_x^β are present to simplify later formulas. The functions in $\mathcal{S}$ have their derivatives decreasing faster at infinity than the inverse of any polynomials. Let $C_0^\infty(\mathbb{R}^n)$ denote the set of smooth functions with compact support on $\mathbb{R}^n$, then it is a dense subset of $\mathcal{S}$.

Definition 11. *Let u be a smooth, compactly supported function on $\mathbb{R}^n$. Let s be a non-negative real number. The Sobolev s-norm of u is the quantity $\|u\|_s$ defined by*

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where $\hat{u}$ is the Fourier transform

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} u(x) dx.$$

If U is an open subset of $\mathbb{R}^n$ then the Sobolev space $H^s(U)$ is the completion in the Sobolev s-norm of the space of smooth functions on $\mathbb{R}^n$ which are compactly supported in U.

The Plancherel formula from Fourier theory asserts that

$$\int_{\mathbb{R}^n} |u(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 d\xi.$$

Thus, up to a multiplicative constant, the Sobolev 0-norm is the same thing as the ordinary L^2-norm. If $s_1 > s_2$ then $\|u\|_{s_1} > \|u\|_{s_2}$. It follows that $H^{s_1}(U)$ may be regarded as a (dense) subspace of $H^{s_2}(U)$. In particular all of the Sobolev spaces $H^s(U)$ can be regarded as subspaces of the Hilbert space $L^2(U)$.

If u is a smooth, compactly supported function on $\mathbb{R}^n$, then the Fourier transform of the function $D^\alpha u$ is the function $\xi^\alpha \hat{u}(\xi)$.

Theorem 14. *If $s \geq 0$ and $s \in \mathbb{N}$, then the Sobolev s-norm is equivalent to the norm*

$$\sqrt{\sum_{\alpha \leq s} \|D^\alpha u\|^2_{L^2(\mathbb{R}^n)}}.$$

This theorem follows from Plancherel's theorem that

$$\sum_{\alpha \leq s} \|D^\alpha u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \sum_{\alpha \leq s} \int_{\mathbb{R}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2 d\xi$$

and the fact that the function $\sum_{\alpha \leq s} \xi^{2\alpha}$ and $(1+|\xi|^2)^s$ are bounded multiples of one another.

Roughly speaking the Sobolev space $H^s(U)$ consists of functions supported in U all of whose derivatives of order s or less belong to $L^2(U)$.

In order to globalize the Sobolev norms to manifolds we shall need the following lemma:

Lemma 5. *If σ is a smooth function on an open set $U \subset \mathbb{R}^n$ whose derivatives of all orders are bounded functions on U, then pointwise multiplication by σ extends to a bounded linear operator on $H^s(U)$, for every s. In addition, if $\Phi : U' \longrightarrow U$ is a diffeomorphism from one open set in $\mathbb{R}^n$ to another whose derivatives of all orders are bounded functions, then the operation of composition with Φ extends to a bounded linear operator from $H^s(U')$ to $H^s(U)$.*

Suppose now that M is a compact smooth manifold. Choose a finite coordinate cover $\{U_j\}$ for M and a partition of unity $\{\sigma_j\}$ subordinate to this cover. Using this structure any function on M can be broken up into a list of compactly supported functions on $\mathbb{R}^n$; we construct a *Sobolev s-norm* of the function on u by combining the s-norm of the constituent pieces $\sigma_j u$, which we regard as compactly supported function on $\mathbb{R}^n$. Thus:

$$\|u\|_s^2 = \sum_j \|\sigma_j u\|_s^2 .$$

This norm depends on the choice we made, but the different sets of choices give equivalent norms.

Definition 12. *Let M be a compact smooth manifold. The Sobolev space $H^s(M)$ is the completion of $C^\infty(M)$ in the above Sobolev s-norm.*

Theorem 15 (Rellich Lemma)**.** *Let $\{f_m\} \in \mathcal{S}$ be a sequence of functions with support in a fixed compact set $K \subset \mathbb{R}^n$. We suppose there is a constant C so $\|f_m\|_s \leq C$ for all m. Let $s > t$. There exists a subsequence f_{m_k} which converges in H_t.*

Definition 13. *A first-order partial differential operator on $\mathbb{R}^n$ is*

$$\mathcal{D} = \sum_{r \leq m} A_j(x) D_j + B(x)$$

where $D_j = -i\dfrac{\partial}{\partial x_j}$ for all $1 \leq j \leq n$ and $A_j(x)$ and $B(x)$ are in $M_p(C^\infty(\mathbb{R}^n))$ for some $p \in \mathbb{N}$.

Example 6. *On the plane $\mathbb{R}^2$, let*

$$\mathcal{D} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x_1} + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x_2}.$$

Then $\mathcal{D}$ is a first-order partial differential operator on $\mathbb{R}^2$.

Definition 14. *Let M be a smooth manifold. A first-order partial differential operator on M is a linear map P on $\overbrace{C^\infty(M) \oplus \cdots \oplus C^\infty(M)}^{p}$ such that on each local coordinate $\{x_1, \cdots, x_n\}$,*

$$D = \sum_{j \leq n} A_j(x) D_j + B$$

where $D_j = -i\frac{\partial}{\partial x_j}$ *and* $A_j(x)$ *and* B *are in* $M_p(C^\infty(\mathbb{R}^n))$.

Remark. In fact, we can define first-order partial differential operators on vector bundles on M. In this case, we use the smooth sections to replace $C^\infty(M)$. Recall that an *m-dimensional complex vector bundle* is a triple $\eta = (p, E, X)$ such that

1. E and X are topological spaces,
2. $p : E \longrightarrow X$ is a continuous map such that for each $x \in X$, $p^{-1}(x)$ is an m-dimensional vector space.
3. for each $x \in X$, there is a neighborhood $U \subset X$ such that $p^{-1}(U)$ is isomorphic to $U \times \mathbb{C}^m$.

A section of η is a map $s : X \longrightarrow E$ such that $p \circ s = \mathrm{id} : X \longrightarrow X$. Let (p, E, X) be a vector bundle over X and X and E be smooth manifolds, a section $s : X \longrightarrow E$ is smooth if it is smooth as a map from X to E.

Example 7. *On the unit circle* $\mathbb{T}$, $\mathcal{D} = -i\dfrac{d}{d\theta}$ *is a first-order partial differential operator on* $\mathbb{T}$.

Definition 15. *Let* $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ *and* $p(x, \xi) = \sum_{i=1}^{n} A_i(x)\xi_i + B(x)$. *Then the partial differential operator defined in definition 13 can be written as* $P = p(x, D)$. *Let* $\sigma(x, \xi) = \sum_{i=1}^{n} A_i(x)\xi_i$. $\sigma(x, \xi)$ *is called the symbol of* P.

Definition 16. *Let* P *be a partial differential operator on* $\mathbb{R}^n$ *and let* $\sigma(x, \xi)$ *be the symbol of* P. *If* $\sigma(x, \xi)$ *is invertible for all* $x \in \mathbb{R}^n$ *and* $\xi(\neq 0) \in \mathbb{R}^n$, *we call* P *elliptic.*

Theorem 16 (Garding's Inequality)**.** *Let* D *be a first-order partial differential operator on* M *and let* K *be a compact subset of* M. *If* D *is elliptic over a neighborhood of* K *then there is a constant* $c > 0$ *such that*

$$\|u\|_{L^2(M)} + \|Du\|_{L^2(M)} \geq c\,\|u\|_{H^1(K)},$$

for all $u \in H^1(K)$.

Example 8. *$H^1(\mathbb{T})$ is the completion of* $\left\{ \sum_{n\in\mathbb{Z}} a_n e^{int} \in C(\mathbb{T}) \middle| \; \sum_{n\in\mathbb{Z}} n^2 |a_n|^2 < \infty \right\}$ *in the Sobolev 1-norm, i.e.* $\|u\|_1 \approx \|f\|_{L^2(\mathbb{T})} + \|Df\|_{L^2(\mathbb{T})}$.

Definition 17. *An unbounded Hilbert space operator T is closable if the norm-closure of its graph is the graph of another unbounded operator, called the closure of T and denoted $\bar{T}$.*

Lemma 6. *Every differential operator D is closable.*

Definition 18. *An operator which has a unique self-adjoint extension is said to be essentially self-adjoint.*

Theorem 17 (Sobolev Embedding Theorem)**.** *If $s > \frac{n}{2} + k$, then $H^s(\mathbb{R}^n)$ is included within $C_0^k(\mathbb{R}^n)$ the Banach space of k-times continuously differentiable functions on $\mathbb{R}^n$, whose derivatives up to order k vanish at infinity.*

Proof. We need to show that the C^k-norm of a smooth, compactly supported function is bounded by a multiple of the Sobolev s-norm, whenever $s > \frac{n}{2} + k$. This will imply that the identity map on $C_c^\infty(\mathbb{R}^n)$ extends to a continuous map of $H^s(\mathbb{R}^n)$ into $C^k(\mathbb{R}^n)$, as required. If $|\alpha| \leq k$, we compute, using the Fourier inversion formula, that

$$D^\alpha u(x) = \int e^{i\xi\cdot x} \xi^\alpha \hat{u}(\xi) d\xi.$$

Therefore, by the Cauchy-Schwarz inequality,

$$|D^\alpha u(x)|^2 \leq \int (1+\xi^2)^{-s} \xi^{2\alpha} d\xi \cdot \int (1+\xi^2)^s |\hat{u}(\xi)|^2 d\xi.$$

If $s > \frac{n}{2} + k$ and $k \geq |\alpha|$ then the first integral is finite. Taking the square roots we get the required estimate

$$\sup_x |D^\alpha u(x)| \leq C \|u\|_s$$

for some constant C. □

Theorem 18 (Elliptic Regularity Principle)**.** *Let M be a smooth manifold and $U \subset M$. If D is elliptic over U and if u is a distribution such that Du is smooth over U, then in fact* *u itself is smooth over U.*

To prove this theorem, we need the Sobolev Embedding Theorem and Garding Inequality. The Garding Inequality implies that the eigenvectors of elliptic operators on compact manifolds belong to every Sobolev space $H^k(M)$. The Sobolev Embedding Theorem asserts that $\cap_k H^k(M)$ is made up entirely of smooth functions on M.

It is immediate from the definition of the Sobolev space that every first-order differential operator D is *bounded* when considered as an operator from $H^1(K)$ to $L^2(M)$.

Theorem 19. *Let M be a manifold and let D be an essentially selfajoint differential operator on M. If D is elliptic over an open subset $U \subset M$ then for every $\phi \in C_0(\mathbb{R})$ and every $g \in C_0(U)$ the operator $\rho(g)\phi(D) : L^2(M) \to L^2(M)$ is compact.*

To prove this theorem, we need the Rellich Lemma and Garding Inequality. First we factor the map as follows:

$$\rho(g)\phi(D) : L^2(M) \longrightarrow H^1(K) \longrightarrow L^2(M)$$

where ϕ is a compactly supported function and K is the compact support of ϕ. The first map is bounded via the Garding Inequality, the second map is compact via Rellich Lemma.

Theorem 20. *Let M be a compact smooth manifold and let P be an elliptic differential operator on M. Then P has closed range,* $\ker P$ *and* $\operatorname{coker} P$ *are finite dimensional.*

Example 9. *For the differential operator $\mathcal{D}$ in Example 7,* $\ker \mathcal{D}$ *is the set of all constant functions and of dimension 1.* $\operatorname{coker} \mathcal{D}$ *is the same.*

Definition 19. *Let M be a smooth manifold. An ungraded Fredholm module is a triple (H, ϕ, F) such that*

1. *H is a Hilbert space;*

2. *$\phi : C_0(M) \longrightarrow \mathcal{L}(H)$ is a homomorphism;*

3. *$F \in \mathcal{L}(H)$ such that $\phi(f)(F^*F - I)$, $\phi(f)(FF^* - I)$ and $[\phi(f), F]$ are in $\mathcal{K}(H)$.*

Example 10. *We show that the differential operator $\mathcal{D}$ in Example 7 induces an ungraded Fredholm module. Let $H = L^2(\mathbb{T})$, $\phi : C(\mathbb{T}) \longrightarrow \mathcal{L}(H)$ is the multiplication operator, i.e. $\phi(f)(g) = fg$ for all $f \in C(\mathbb{T})$ and $g \in H$. $\mathcal{D}$ is unbounded on H, $F = \dfrac{\mathcal{D}}{\sqrt{I+\mathcal{D}^2}}$ is a bounded operator on H. Let us check the conditions in the definition 19.*

(1) $\phi(f)(FF^* - I)$ and $\phi(f)(F^*F - I)$ are in $\mathcal{K}(H)$.

Since $\mathbb{T}$ is compact, we only need to show that $FF^* - I, F^*F - I$ are compact.

$$\begin{aligned}\langle \mathcal{D}f, g\rangle &= \int_{\mathbb{T}} -i\frac{d}{dt}f(t)\overline{g(t)}dt \\ &= -(-i)\int_{\mathbb{T}} f(t)\frac{d}{dt}\overline{g(t)}dt \qquad \textit{(integration by parts)} \\ &= \int_{\mathbb{T}} f(t)\overline{-i\frac{d}{dt}g(t)}dt \\ &= \langle f, \mathcal{D}g\rangle.\end{aligned}$$

So $\mathcal{D}$ is symmetric, then $F^* = F$ and $FF^* - I = F^*F - I = (I + \mathcal{D}^2)^{-1}$. Since the spectrum of $\mathcal{D}$ is $\{0, \pm 1, \pm 2, \cdots, \pm n, \cdots\}$, $FF^* - I = F^*F - I = (I + \mathcal{D}^2)^{-1}$ are in $\mathcal{K}(H)$.

(2) $[\phi(f), F] \in \mathcal{K}(H)$.

On the standard basis $\{e^{int}\}_{n\in\mathbb{Z}}$, $\mathcal{D}(e^{int}) = ne^{int}$ and $F(e^{int}) = \dfrac{n}{\sqrt{1+n^2}}e^{int}$. Let $f = e^{im_0t} \in C(\mathbb{T})$,

$$(Ff - fF)(e^{int}) = \left(\frac{n+m_0}{\sqrt{1+(n+m_0)^2}} - \frac{n}{\sqrt{1+n^2}}\right)e^{i(n+m_0)t}.$$

Hence $[F, f]$ is a shift operator and $\dfrac{n+m_0}{\sqrt{1+(n+m_0)^2}} - \dfrac{n}{\sqrt{1+n^2}}$ approaches 0 when n approaches infinity. So $[F, f]$ can be approximated by finite rank operators. This means that $[F, f]$ is a compact operator on H. For any $f \in C(\mathbb{T})$ and $\epsilon > 0$, we can find a finite sum $g(t) = \sum_{j=-l}^{l} a_j e^{ijt}$ such that $\|f - g\| < \epsilon$. Since g is finite sum of compact operators, f is a compact operator.

Therefore (H, ϕ, F) is an ungraded Fredholm operator on $\mathbb{T}$.

Definition 20. *Let M be a smooth manifold. A graded Fredholm module is a triple (H, ϕ, F) such that*

1. *H is a $\mathbb{Z}_2$-graded Hilbert space;*

2. $\phi : C_0(M) \longrightarrow \mathcal{L}(H)$ *is a homomorphism of degree 0;*

3. $F \in \mathcal{L}(H)$ *is of degree 1 and* $\phi(f)(F - F^*)$, $\phi(f)(F^2 - I)$ *and* $[\phi(f), F]$ *are in* $\mathcal{K}(H)$.

Example 11. *We show that the differential operator* $\mathcal{D}$ *in Example 6 induces a graded Fredholm module. Let* $H = L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$, $\phi : C_0(\mathbb{R}^2) \longrightarrow \mathcal{L}(H)$ *is the multiplication operator, i.e.* $\phi(f)(g_1 \oplus g_2) = (fg_1 \oplus fg_2)$ *for all* $f \in C_0(\mathbb{R}^2)$ *and* $(g_1, g_2) \in H$. *Let* $\partial_z = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}$ *and* $\partial_{\bar{z}} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$, *then* $\mathcal{D} = \begin{pmatrix} 0 & -\partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}$ *is unbounded on* H, $F = \frac{\mathcal{D}}{\sqrt{I + \mathcal{D}^2}}$ *is a bounded operator on* H. *Let us check the conditions in the definition 20.*

(1) $\phi(f)(F - F^*) \in \mathcal{K}(H)$.

First for $f_1, f_2 \in C^\infty(\mathbb{R}^2)$,

$$\langle -\partial_z f_1, f_2 \rangle = \langle f_1, \partial_{\bar{z}} f_2 \rangle.$$

Then for $f \oplus g, h \oplus l \in C^\infty(\mathbb{C}) \oplus C^\infty(\mathbb{C})$,

$$\begin{aligned} \langle \mathcal{D}(f \oplus g), h \oplus l \rangle &= \langle -\partial_z g \oplus \partial_{\bar{z}} f, h \oplus l \rangle \\ &= \langle -\partial_z g, h \rangle + \langle \partial_{\bar{z}} f, l \rangle \\ &= \langle g, \partial_{\bar{z}} h \rangle + \langle f, -\partial_z l \rangle \\ &= \langle f \oplus g, -\partial_z l \oplus \partial_{\bar{z}} h \rangle \\ &= \langle f \oplus g, \mathcal{D}(h \oplus l) \rangle. \end{aligned}$$

So $\mathcal{D}$ *is symmetric, then* $F^* = F$ *and* $\phi(f)(F - F^*) = 0$ *for all* $f \in C_0(\mathbb{R}^2)$.

(2) $\phi(f)(F^2 - I) \in \mathcal{K}(H)$.

Let $f \in C_c^\infty(\mathbb{R}^2)$ *be a compactly supported smooth function on* $\mathbb{R}^2$. *That* $\phi(f)(F^2 - I)$ *is a compact operator follows from Rellich Lemma.*

(3) $[F, f] \in \mathcal{K}(H)$.

Here we have a formula

$$F = \frac{2}{\pi} \int_0^\infty \frac{\mathcal{D}}{1 + \lambda^2 + \mathcal{D}^2} d\lambda.$$

Since the $\mathcal{D}$ is symmetric, the above formula comes from the integration

$$\frac{x}{\sqrt{1+x^2}} = \frac{2}{\pi}\int_0^\infty \frac{x}{1+\lambda^2+x^2}d\lambda, \qquad (\int \frac{1}{1+\lambda^2}d\lambda = \tan^{-1}(\lambda)).$$

Then for any $f \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}[F,f] &= \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2+\mathcal{D}^2}\left((1+\lambda^2+\mathcal{D}^2)\mathcal{D}f - f\mathcal{D}(1+\lambda^2+\mathcal{D}^2)\right)\frac{1}{1+\lambda^2+\mathcal{D}^2}d\lambda \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2+\mathcal{D}^2}\left((1+\lambda^2)[\mathcal{D},f] + \mathcal{D}[f,\mathcal{D}]\mathcal{D}\right)\frac{1}{1+\lambda^2+\mathcal{D}^2}d\lambda.\end{aligned}$$

Since $\frac{1+\lambda^2}{1+\lambda^2+\mathcal{D}^2}$ is uniformly bounded and $[\mathcal{D},f]$ is compactly supported, $\frac{1+\lambda^2}{1+\lambda^2+\mathcal{D}^2}[\mathcal{D},f]\frac{1}{1+\lambda^2+\mathcal{D}^2}$ is compact by Rellich Lemma. Similarly, $\frac{\mathcal{D}}{1+\lambda^2+\mathcal{D}^2}$ is uniformly bounded and $[f,\mathcal{D}]$ is compactly supported, $\frac{\mathcal{D}}{1+\lambda^2+\mathcal{D}^2}[f,\mathcal{D}]\frac{\mathcal{D}}{1+\lambda^2+\mathcal{D}^2}$ is compact by Rellich Lemma. Both $\left\|\frac{1}{1+\lambda^2+\mathcal{D}^2}\right\|$ and $\left\|\frac{\mathcal{D}}{1+\lambda^2+\mathcal{D}^2}\right\|$ approach 0 when λ approaches infinity. Then for any bounded interval $[0,a]$ in $[0,\infty)$, the integral

$$F_a = \frac{2}{\pi}\int_{[0,a]} \frac{1}{1+\lambda^2+\mathcal{D}^2}\left((1+\lambda^2)[\mathcal{D},f] + \mathcal{D}[f,\mathcal{D}]\mathcal{D}\right)\frac{1}{1+\lambda^2+\mathcal{D}^2}d\lambda$$

is compact. When a is big enough, $\|F_a - [F,f]\|$ can be smaller than any given positive number, therefore $[F,f]$ is compact.

III.2 Equivariant K-homology

Let M be a Riemannian manifold, and let Γ act on M properly and isometrically. Recall that Γ acts on M properly in the sense that the map

$$\begin{array}{ccc} X\times\Gamma & \to & X\times X \\ (x,\gamma) & & (\gamma x, x) \end{array}$$

is proper, i.e. the preimage of compact sets are compact. In this case we call X a *proper Γ-space.*

Definition 21. *Let H be a separable graded Hilbert space with Γ action. For $\gamma \in \Gamma$ and*

$F \in \mathcal{L}(H)$, we define $\gamma(F) \in \mathcal{L}(H)$ as

$$\gamma(F)(x) = \gamma(F(\gamma^{-1}(x)))$$

for all $x \in H$.

Definition 22. *Let X be a proper Γ-space. A graded equivariant Kasparov Γ-module over X is a triple (H, ϕ, F) satisfying*

(1) H is a separable graded Hilbert space with Γ action;

*(2) $\phi : C_0(X) \to \mathcal{L}(H)$ is a *-homomorphism of degree 0;*

(3) $F \in \mathcal{L}(H)$ of degree 1 such that the graded commutator $[F, \phi(f)]$, $(F^2 - 1)\phi(f)$, $(F - F^*)\phi(f)$ and $(\gamma(F) - F)\phi(f)$ are all in $\mathcal{K}(H)$ for all $f \in C_0(X)$ and $\gamma \in \Gamma$.

If $[F, \phi(f)] = (F^2 - 1)\phi(f) = (F - F^*)\phi(f) = 0$ for all $f \in C_0(X)$, we call (H, ϕ, F) degenerate.

In the next definition, we give three equivalence relations between graded equivariant Kasparov Γ-modules.

Definition 23. *1. Let (H, ϕ, F_t) be a graded equivariant Kasparov Γ-module over X for $t \in [0, 1]$. If the map $t \longrightarrow F_t$ is norm continuous, we call that (H, ϕ, F_0) is operator homotopic to (H, ϕ, F_1);*

2. Let (H, ϕ, F) be a graded equivariant Kasparov Γ-module over X, H' be a Hilbert space and $U : H' \longrightarrow H$ be a unitary isomorphism preserving the grading. Then $(H', U^*\phi U, U^*FU)$ is a graded equivariant Kasparov Γ-module and we call it is unitarily equivalent to (H, ϕ, F);

3. Let (H, ϕ, F) and (H, ϕ, F') be graded equivariant Kasparov Γ-modules over X and the $(F - F')\phi(f)$ is compact for all $f \in C_0(X)$. We call that (H, ϕ, F') is a compact perturbation of (H, ϕ, F).

Operator homotopy implies compact perturbation. The linear path will give the homotopy. And now we can normalize F such that F is self-adjoint. In fact we can use $\frac{F+F^*}{2}$ to replace F and they are the compact perturbation to each other.

Definition 24. *We denote by $E_0^\Gamma(X)$ the set of all equivalence classes of graded equivariant Kasparov Γ-modules and denote by $D_0^\Gamma(X)$ the set of equivalence classes containing degenerate elements in E_0^Γ. We set $K_0^\Gamma(X) = E_0^\Gamma(X)/D_0^\Gamma(X)$.*

Lemma 7. *$K_0^\Gamma(X)$ is an abelian group with addition given by direct sum $(H_1, \phi_1, F_1) \oplus (H_2, \phi_2, F_2) = (H_1 \oplus H_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2)$. Any degenerate module will give the zero element and $-[(H, \phi, F)] = [(H^{op}, \phi, -F)]$ where $H^{op} = H$ as a Hilbert space but with the reverse grading.*

Similarly we can define the $K_1(X)$.

Definition 25. *Let X be a proper Γ-space. An ungraded equivariant Kasparov Γ-module over X is a triple (H, ϕ, F) satisfying*

(1) H is a separable Hilbert space with Γ action;

*(2) $\phi : C_0(X) \to \mathcal{L}(H)$ is a *-homomorphism;*

(3) $F \in \mathcal{L}(H)$ such that the commutator $[F, \phi(f)]$, $(F^2 - 1)\phi(f)$, $(F - F^*)\phi(f)$ and $(\gamma(F) - F)\phi(f)$ are all in $\mathcal{K}(H)$ for all $f \in C_0(X)$ and $\gamma \in \Gamma$.

If $[F, \phi(f)] = (F^2 - 1)\phi(f) = (F - F^*)\phi(f) = 0$ for all $f \in C_0(X)$, we call (H, ϕ, F) degenerate.

Similarly, we give three equivalence relations between ungraded equivariant Kasparov Γ-modules.

Definition 26.

1. *Let (H, ϕ, F_t) be an ungraded equivariant Fredholm Γ-module over X for $t \in [0, 1]$. If the map $t \longrightarrow F_t$ is norm continuous, we call that (H, ϕ, F_0) is operator homotopic to (H, ϕ, F_1);*

2. *Let (H, ϕ, F) be an ungraded equivariant Fredholm Γ-module over X, H' be a Hilbert space and $U : H' \longrightarrow H$ be a unitary isomorphism. Then $(H', U^*\phi U, U^*FU)$ is an ungraded equivariant Kasparov Γ-module and we call it is unitarily equivalent to (H, ϕ, F);*

3. *Let* (H, ϕ, F) *and* (H, ϕ, F') *be ungraded equivariant Fredholm* Γ*-modules over* X *and the* $(F - F')\phi(f)$ *is compact for all* $f \in C_0(X)$. *We call that* (H, ϕ, F') *is a compact perturbation of* (H, ϕ, F).

Operator homotopy implies compact perturbation. The linear path will give the homotopy. We can normalize F such that F is self-adjoint.

Definition 27. *We denote by* $E_1^\Gamma(X)$ *the set of all equivalence classes of ungraded equivariant Fredholm* Γ*-modules and denote by* $D_1^\Gamma(X)$ *the set of equivalence classes containing degenerate elements in* E_0^Γ. *We set* $K_1^\Gamma(X) = E_1^\Gamma(X)/D_1^\Gamma(X)$.

Lemma 8. $K_1^\Gamma(X)$ *is an abelian group with addition given by direct sum* $(H_1, \phi_1, F_1) \oplus (H_2, \phi_2, F_2) = (H_1 \oplus H_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2)$. *Any degenerate module will give the zero element and* $-[(H, \phi, F)] = [(H, \phi, -F)]$.

CHAPTER IV

COARSE GEOMETRY

In this chapter, we study the coarse structure for metric spaces and compute the K-groups of the C^*-algebras associated to several examples. This chapter is based on [HRY, HR00, Y95, Y97].

Definition 28. *Let X, Y be metric spaces. The map $f : X \to Y$ is called coarse if*

- *for any $s > 0$, there exists $r > 0$ such that for any $x_1, x_2 \in X$ and $d_X(x_1, x_2) < s$, $d_Y(f(x_1), f(x_2)) < r$,*
- *for any $R > 0$, there exists $S > 0$ such that for any $x_1, x_2 \in X$ and $d_Y(f(x_1), f(x_2)) < R$, $d_X(x_1, x_2) < S$.*

Definition 29. *Let X be a metric space and let S be any set. Two maps $\phi_1, \phi_2 : S \to X$ are close if*

$$\sup_{s \in S} d(\phi_1(s), \phi_2(s)) < \infty.$$

Definition 30. *Let X, Y be metric spaces. The maps $f, g : X \to Y$ are called (coarsely) equivalent if f, g are close.*

Definition 31. *Let X, Y be metric spaces. X, Y are called (coarsely) equivalent if there exist $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is close to id_Y and $g \circ f$ is close id_X.*

Let X be a proper metric space (a metric space is called *proper* if every closed ball is compact). An X*-module* is a separable Hilbert space equipped with a faithful and non-degenerate $*$-representation of $C_0(X)$ whose range contains no nonzero compact operators, where $C_0(X)$ is the algebra of all complex-valued continuous functions on X which vanish at infinity.

Definition 32. *Let X and Y be proper metric spaces, and let H_X and H_Y be X-module and Y-module, respectively.*

1. *The* support $\text{Supp}(T)$ *of a bounded linear operator* T *from* H_X *to* H_Y *is defined to be the complement (in* $X \times Y$*) of the set of points* $(x, y) \in X \times Y$ *for which there exist functions* $\phi \in C_0(X)$, $\psi \in C_0(Y)$ *such that* $\psi T \phi = 0$ *and* $\phi(x) \neq 0$, $\psi(y) \neq 0$.

2. *The* propagation *of a bounded linear operator* $T : H_X \to H_X$ *is defined to be*

$$\sup \Big\{ d(x, y) : \ (x, y) \in \text{Supp}(T) \subset X \times X \Big\}.$$

3. *A bounded linear operator* $T : H_X \to H_X$ *is said to be* locally compact *if the operators* ϕT *and* $T \phi$ *are compact for all* $\phi \in C_0(X)$.

Definition 33. *Let* H_X *be an* X*-module. The Roe algebra* $C^*(X, H_X)$ *is the operator norm closure of the* $*$*-algebra of all locally compact, finite propagation operators acting on* H_X.

Example 12. *Let* X *be a bounded metric space and let* H_X *be an* X*-module. Then* $C^*(X, H_X) = \mathcal{K}$ *because local compactness implies compactness and every operators in* $\mathcal{L}(H_X)$ *have finite propagation.*

Lemma 9. *Let* X, Y *be metric spaces and let* H_X, H_Y *be* X *and* Y*-modules, respectively. Let* $f : X \to Y$ *be a coarse map. There is an isometry* $V_f : H_X \to H_Y$ *such that for some* $R > 0$,

$$\text{Supp}(V) \subset \{(x, y) \in X \times Y | d(f(x), y) < R\}.$$

We call that V_f *covers* f.

With V_f as above, $Ad(V_f)(u) = V_f u (V_f)^*$ for $u \in C^*(X, H_X)$ maps $C^*(X, H_X)$ to $C^*(Y, H_Y)$. Clearly this V_f is not unique.

Lemma 10. *Let* V_1, V_2 *be two isometries satisfying the above lemma. The induced maps on K-theory are equal:*

$$Ad(V_1)_* = Ad(V_2)_* : K_*(C^*(X, H_X)) \to K_*(C^*(Y, H_Y)).$$

Let X be a metric space and let H_1 and H_2 be two X-modules. The identity map $id : X \to X$ induces an isomorphism $id_* : K_*(C^*(X, H_1)) \to K_*(C^*(X, H_2))$. So up to isomorphism, $K_*(C^*(X, H_1))$ does not depend on the choice of X-modules. If $f, g : X \to Y$ are close and V_f covers f, then V_f covers g too. We have

Lemma 11. *Let X, Y be two coarsely equivalent metric spaces, then $K_*(C^*(X)) \cong K_*(C^*(Y))$.*

Example 13. $K_*(C^*(\mathbb{Z})) \cong K_*(C^*(\mathbb{R}))$.

CHAPTER V

EQUIVARIANT INDEX THEORY AND NONPOSITIVELY-CURVED MANIFOLDS

In this chapter, we will define the equivariant higher index map and prove the following theorem.

Theorem 21. *If X is a simply-connected complete Riemannian manifold with nonpositive sectional curvature and Γ is a torsion-free discrete group acting on X properly and isometrically, then the equivariant higher index map*

$$\text{Ind} : K_*^{\Gamma}(X) \to K_*(C^*(X)^{\Gamma})$$

is injective.

V.1 The equivariant higher index map.

In this section, we will define the equivariant higher index map. Let X be a complete Riemannian manifold and let Γ be a torsion-free discrete group acting on X properly and isometrically, where the Γ action on X is *proper* in the sense that the map

$$\begin{array}{ccc} X \times \Gamma & \to & X \times X \\ (x, \gamma) & & (\gamma x, x) \end{array}$$

is proper, i.e. the preimage of a compact set is compact. Γ acts on X isometrically if $d(\gamma x, \gamma y) = d(x, y)$ for all $\gamma \in \Gamma$ and $x, y \in X$. In this case, we call X a *proper* Γ*-space*. If Γ is torsion-free, the properness of the Γ action implies that the Γ action is free.

Definition 34. *An X-module is a Hilbert space H equipped with a $*$-homomorphism $\phi : C_0(X) \to \mathcal{L}(H)$ of C^*-algebras. If H is $\mathbb{Z}/2\mathbb{Z}$-graded, we require that $\phi(f)$ is of degree 0 for all $f \in C_0(X)$. An X-module H is called adequate if $\overline{\phi(C_0(X))H} = H$ and there is no non-zero element in $C_0(X)$ acting on H as a compact operator. In this paper, we denote $\phi(f)v$ by fv for all $v \in H$.*

Definition 35. *Let X be a proper Γ-space. For $\gamma \in \Gamma$ and $f \in C_0(X)$, we define $\gamma(f) \in C_0(X)$ as*

$$\gamma(f)(x) = f(\gamma^{-1}(x))$$

for $x \in X$.

Definition 36. *Let H be an X-module. We say that H is a covariant X-module if it is equipped with a unitary action ρ of Γ, i.e. $\rho : \Gamma \to \mathcal{U}(H)$ is a group homomorphism from Γ to the set of all unitary elements in $\mathcal{L}(H)$, compatible with the action of Γ on X, in the sense that for all $v \in H$, $f \in C_0(X)$ and $\gamma \in \Gamma$*

$$(\gamma(f))(v) = \rho(\gamma)\left(f(\rho(\gamma)^*(v))\right).$$

For $\gamma \in \Gamma$ and $T \in \mathcal{L}(H)$, we define $\gamma(T) \in \mathcal{L}(H)$ as

$$\gamma(T)(v) = \rho(\gamma)T\rho(\gamma)^*(v)$$

for $v \in \mathcal{L}(H)$. In this paper, we assume that all X-modules are adequate and covariant.

Definition 37. *Let H_X be an X-module. A bounded operator $T : H_X \to H_X$ is Γ-invariant if $\gamma(T) = T$ for all $\gamma \in \Gamma$.*

Definition 38. *The equivariant Roe algebra $C^*(X)^\Gamma$ is the operator norm closure of the $*$-algebra of all locally compact and Γ-invariant operators with finite propagation in $\mathcal{L}(H_X)$.*

Lemma 12. *The Roe algebra $C^*(X)^\Gamma$ does not depend on the choice of the X-module H_X.*

The proof is similar to Lemma 6.2 in [Y95].

To define the equivariant higher index map, we need to make locally almost Γ-invariant operators in $K_0^\Gamma(X)$ to be Γ-invariant by a Γ-averaging process.

Lemma 13. *Let X be a metric space and Γ be a torsion-free discrete group acting on X properly. If (H_X, ϕ, F) is a cycle for $K_1^\Gamma(X)$, then there exists another operator $F' \in \mathcal{L}(H_X)$ such that*

$$[(H_X, \phi, F)] = [(H_X, \phi, F')] \in K_1^\Gamma(X)$$

and

$$\gamma(F') = F' \text{ for all } \gamma \in \Gamma.$$

The same result is also true for cycles for $K_0^\Gamma(X)$.

Proof. We only prove it for cycles for $K_1^\Gamma(X)$. The proof for cycles for $K_0^\Gamma(X)$ is similar. Since the Γ action on X is proper and Γ is torsion-free, we can find a cover $\{U_i\}_{i\in I}$ of X such that U_i is Γ-equivariantly diffeomorphic to $\Gamma \times O_i$, where O_i is an open subset of some vector space and $(\{\gamma_1\} \times O_i) \cap (\{\gamma_2\} \times O_i) = \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma$ and $\gamma_1 \neq \gamma_2$. To simplify notations, we identify U_i with $\Gamma \times O_i$. Let $\{\phi_i\}_{i\in I}$ be the partition of unity subordinate to the cover $\{U_i\}_{i\in I}$. Let

$$F'' = \sum_{i\in I} \phi_i^{\frac{1}{2}} F \phi_i^{\frac{1}{2}} = \sum_{i\in I} F_i'',$$

where $F_i'' = \phi_i^{\frac{1}{2}} F \phi_i^{\frac{1}{2}}$ for $i \in I$ and $\sum_{i\in I} F_i''$ converges in strong operator topology. For all $f \in C_c(X)$,

$$f(F - F'') = f\left(\sum_{i\in I} \phi_i^{\frac{1}{2}} \left(\phi_i^{\frac{1}{2}} F - F\phi_i^{\frac{1}{2}}\right)\right)$$

is a finite sum and therefore compact. Hence (H_X, ϕ, F) and (H_X, ϕ, F'') represent the same element in $K_1^\Gamma(X)$.

Choose $\chi_{i,\gamma} \in C_0(X)$ such that $\chi_{i,\gamma}(x) = 1$ when $x \in \{\gamma\} \times O_i$ and $\chi_{i,\gamma}(x) = 0$ when $x \notin \{\gamma\} \times O_i$ for $\gamma \in \Gamma$ and $i \in I$. Let

$$G_{i,\gamma} = \chi_{i,\gamma} F_i'' \chi_{i,\gamma}$$

for $\gamma \in \Gamma$ and $i \in I$. Then for all $f \in C_0(X)$

$$f(F_i'' - \sum_{\gamma\in\Gamma} G_{i,\gamma})$$

is compact, where the infinite sum converges in strong topology. For any $f \in C_c(X)$,

$$f\left(\sum_{\gamma\in\Gamma}(G_{i,\gamma}-\gamma(G_{i,e}))\right) = f\left(\sum_{\gamma\in\Gamma}\chi_{i,\gamma}F_i''\chi_{i,\gamma}-\gamma\chi_{i,e}F_i''\chi_{i,e}\gamma^{-1}\right)$$
$$= \sum_{\gamma\in\Gamma}\chi_{i,\gamma}\left(f(F_i''-\gamma(F_i''))\right)\chi_{i,\gamma}$$

is a finite sum and therefore compact. Let $F_i' = \sum_{\gamma\in\Gamma}\gamma(G_{i,e})$ and $F' = \sum_{i\in I}F_i'$, then $f(F'-F'')$ is compact for all $f \in C_0(X)$, therefore $[(H_X,\phi,F')] = [(H_X,\phi,F'')] \in K_1^\Gamma(X)$ and $\gamma(F') = F'$ for all $\gamma \in \Gamma$.

□

Now let us define the equivariant higher index map. Based on Lemma 13, let (H_X,ϕ,F) be a cycle for $K_0^\Gamma(X)$ such that F is Γ-invariant and has finite propagation. If we have the decomposition $F = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}$, then U is a multiplier of $C^*(X)^\Gamma$ in the multiplier algebra $M(C^*(X)^\Gamma)$ and U is a unitary modulo $C^*(X)^\Gamma$ and $\partial(U) \in K_0(C^*(X)^\Gamma)$, where ∂ is the boundary map: $K_1(M(C^*(X)^\Gamma)/C^*(X)^\Gamma) \longrightarrow K_0(C^*(X)^\Gamma)$. $\partial(U)$ depends only on the class $[(H_X,\phi,F)]$ in $K_0^\Gamma(X)$. Similarly, if (H_X,ϕ,F) is a cycle for $K_1^\Gamma(X)$ such that F is Γ-invariant and has finite propagation. Then F is a multiplier of $C^*(X)^\Gamma$ and $\frac{I+F}{2}$ is a projection modulo $C^*(X)^\Gamma$. Hence $\partial[\frac{I+F}{2}] \in K_1(C^*(X)^\Gamma)$, where ∂ is the boundary map: $K_0(M(C^*(X)^\Gamma)/C^*(X)^\Gamma) \longrightarrow K_1(C^*(X)^\Gamma)$. $\partial[\frac{I+F}{2}]$ depends only on the class $[(H_X,\phi,F)]$ in $K_1^\Gamma(X)$.

Definition 39.

1. *Let $[(H_X,\phi,F)] \in K_0^\Gamma(X)$ such that F is Γ-invariant, self-adjoint and has finite propagation. If $F = \begin{pmatrix} 0 & V \\ U & 0 \end{pmatrix}$, define*

$$\mathrm{Ind} : K_0^\Gamma(X) \to K_0(C^*(X)^\Gamma)$$

by $\mathrm{Ind}([(H_X,\phi,F)]) = \partial(U)$, called the Γ-index of $[(H_X,\phi,F)]$ in $K_0(C^*(X)^\Gamma)$;

2. *Let* $[(H_X, \phi, F)] \in K_1^\Gamma(X)$ *such that* F *is* Γ*-invariant, self-adjoint and has finite propagation. Define*

$$\mathrm{Ind} : K_1^\Gamma(X) \to K_1(C^*(X)^\Gamma)$$

by $\mathrm{Ind}([(H_X, \phi, F)]) = \partial[\frac{I+F}{2}]$*, called the* Γ*-index of* $[(H_X, \phi, F)]$ *in* $K_1(C^*(X)^\Gamma)$*.*

V.2 Local index theorem.

In this section, we construct the associated localization algebra $C_L^*(X)^\Gamma$, define the local Γ-index map

$$\mathrm{Ind_L} : K_*^\Gamma(X) \to K_*(C_L^*(X)^\Gamma)$$

and prove the local index theorem.

Definition 40. *The localization algebra* $C_L^*(X)^\Gamma$ *is the norm-closure of the algebra of all uniformly bounded and uniformly norm-continuous functions* $f : [0, \infty) \to C^*(X)^\Gamma$ *such that*

$$\sup\{d(m, m') : (m, m') \in supp(f(t))\} \to 0$$

as $t \to \infty$*.*

Let's define the associated local Γ-index map. For each positive integer n, let $\{U_{n,i}\}_i$ be a locally finite and Γ-invariant open cover for X such that diameter$(U_{n,i}) < \frac{1}{n}$ for all i. Let $\{\phi_{n,i}\}_i$ be a continuous partition of unity subordinate to $\{U_{n,i}\}_i$. Let $[(H_X, \phi, F)] \in K_0^\Gamma(X)$. Define a family of operators $F(t)$ $(t \in [0, \infty))$ acting on H_X by

$$F(t) = \sum_i ((1-(t-n))\phi_{n,i}^{\frac{1}{2}} F \phi_{n,i}^{\frac{1}{2}} + (t-n)\phi_{n+1,i}^{\frac{1}{2}} F \phi_{n+1,i}^{\frac{1}{2}})$$

for all $t \in [n, n+1]$, where the infinite sum converges in strong topology. Notice that the propagation of $(F(t)) \to 0$ as $t \to \infty$.

Definition 41.

1. *Let* (H_X, ϕ, F) *be a cycle for* $K_0^\Gamma(X)$ *such that* F *is* Γ*-invariant, self-adjoint and has finite propagation. Define* $F(t)$ *as above and assume that* $F(t) = \begin{pmatrix} 0 & V(t) \\ U(t) & 0 \end{pmatrix}$.

The local index map

$$\mathrm{Ind_L} : K_0^\Gamma(X) \to K_0(C_L^*(X)^\Gamma)$$

is defined by $\mathrm{Ind_L}([(H_X, \phi, F)]) = [\partial(U(t))]$*, called the local* Γ*-index of* $[(H_X, \phi, F)]$ *in* $K_0(C_L^*(X)^\Gamma)$*;*

2. *Let* (H_X, ϕ, F) *be a cycle for* $K_1^\Gamma(X)$ *such that* F *is* Γ*-invariant, self-adjoint and has finite propagation. Define* $F(t)$ *as above. The local index map*

$$\mathrm{Ind_L} : K_1^\Gamma(X) \to K_1(C_L^*(X)^\Gamma)$$

is defined by $\mathrm{Ind_L}([(H_X, \phi, F)]) = [\partial(\frac{I+F(t)}{2})]$*, called the local* Γ*-index of* $[(H_X, \phi, F)]$ *in* $K_1(C_L^*(X)^\Gamma)$*.*

Next we will use Mayer-Vietoris argument to show that the local Γ-index is an isomorphism. Since Γ acts on X properly, X is covered by sets of the form $\Gamma \times_F U$, where U is F-equivariantly contractible and F is a finite subgroup of Γ ([BCH]). The existence of Mayer-Vietoris sequence for localization algebras without group action has been proved in [Y97] and [HRY]. Then the existence of Mayer-Vietoris sequence for localization algebras with a proper group action follows from those two results.

Theorem 22. *Let* X *be a metric space and let* Γ *be a torsion-free discrete group acting on* X *properly and isometrically. If* X_1 *and* X_2 *are two* Γ*-invariant open subspaces of* X *endowed with the subspace metric and are proper* Γ*-spaces such that* $X = X_1 \cup X_2$ *and* $A = X_1 \cap X_2$*, then we have the following six term exact sequence*

$$\begin{array}{ccccc} K_0(C_L^*(A)^\Gamma) & \longrightarrow & K_0(C_L^*(X_1)^\Gamma) \oplus K_0(C_L^*(X_2)^\Gamma) & \longrightarrow & K_0(C_L^*(X)^\Gamma) \\ \uparrow & & & & \downarrow \\ K_1(C_L^*(X)^\Gamma) & \longleftarrow & K_1(C_L^*(X_1)^\Gamma) \oplus K_1(C_L^*(X_2)^\Gamma) & \longleftarrow & K_1(C_L^*(A)^\Gamma)\ . \end{array}$$

The proof of this requires a couple of lemmas.

Definition 42 ([HRY]). *Let* U *be a* Γ*-invariant open subspace of a* Γ*-space* X *and let* H_M *be an* X*-module. Denote by* $C^*(U; X)^\Gamma$ *the norm closure of the set of all locally compact, finite propagation and* Γ*-invariant operators* T *on* H_M *whose support is contained* *in* $\{x \in X | d(x, U) < R\} \times \{x \in X | d(x, U) < R\}$, *for some* $R > 0$ *(depending on* T*). Denote by* $C_L^*(U; X)^\Gamma$ *the norm closure of the algebra of all uniformly bounded and uniformly continuous functions* $f : [0, \infty) \longrightarrow C^*(U; X)^\Gamma$ *such that*

$$\sup \left\{ d(m, m') : (m, m') \in supp(f(t)) \right\} \longrightarrow 0$$

as $t \longrightarrow \infty$.

Lemma 14 ([HRY]). *Let* H_{Y_1} *and* H_{Y_2} *be adequate* Y_1 *and* Y_2*-modules and let* $F : Y_1 \longrightarrow Y_2$ *be a* Γ*-invariant coarse map. The exists a* Γ*-invariant isometry* $V : H_{Y_1} \longrightarrow H_{Y_2}$ *such that for some* $R > 0$

$$supp(V) \subset \{(y_1, y_2) \in Y_1 \times Y_2 | d(F(y_1), y_2) \leq R\}.$$

Clearly $C_L^*(U; X)^\Gamma$ is a closed two-sided ideal in $C_L^*(X)^\Gamma$. If $V : H_U \longrightarrow H_X$ is a Γ-invariant isometry associated to the inclusion morphism $U \longrightarrow X$ as in the previous lemma, then the range of the map $Ad(V) : C_L^*(U)^\Gamma \longrightarrow C_L^*(X)^\Gamma$ lies within $C_L^*(U; X)^\Gamma$.

Lemma 15 ([HRY]). *The induced map*

$$Ad(V) : K_*(C_L^*(U)^\Gamma) \longrightarrow K_*(C_L^*(U; X)^\Gamma)$$

is an isomorphism.

The Proof of Theorem 22. Observe that $C_L^*(X_1; X)^\Gamma$ and $C_L^*(X_2; X)^\Gamma$ are ideals of $C_L^*(X)^\Gamma$ and $C_L^*(X_1; X)^\Gamma + C_L^*(X_2; X)^\Gamma = C_L^*(X)^\Gamma$. The Theorem follows from Lemma 15 and the Mayer-Vietoris sequence on page 90 in [HRY].

Theorem 23. $\mathrm{Ind_L} : K_*^\Gamma(X) \to K_*(C_L^*(X)^\Gamma)$ *is an isomorphism.*

Proof. The required isomorphism may be shown by Mayer-Vietoris argument, based on the fact that X is covered by sets of the form $\Gamma \times U$, where U is contractible. Let P be a one-point set. Note that P is homotopic to U. By Mayer-Vietoris sequence and the five lemma it suffices to show that for a single such space $\Gamma \times U$ the map

$$\mathrm{Ind_L} : K_*^\Gamma(\Gamma \times U) \to K_*(C_L^*(\Gamma \times U)^\Gamma)$$

is an isomorphism. For this, it suffices to show that

$$\mathrm{Ind_L}: K_*^{\Gamma}(\Gamma \times P) \to K_*(C_L^*(\Gamma \times P)^{\Gamma})$$

is an isomorphism. Both sides are isomorphic to the K-theory of compact operators, therefore it is an isomorphism.

□

V.3 Twisted Roe algebras and twisted localization algebras.

In this section, we define certain twisted Roe algebras and twisted localization algebras. In the case of coarse embedding into Hilbert space, these algebras are introduced by Yu in [Y00].

Assume that X is a proper Γ-space. Let $X \times \mathbb{R}$ be the metric space with the product metric. Define the Γ action by

$$\gamma(x, r) = (\gamma x, r)$$

for $\gamma \in \Gamma$ and $(x, r) \in X \times \mathbb{R}$. Note that this Γ action is proper and isometric.

Lemma 16.

$$K_i(C^*(X \times \mathbb{R})^{\Gamma}) \cong K_{i+1}(C^*(X)^{\Gamma}))$$

for $i = 0, 1$.

By the six-term exact sequence, we only need to show that $K_i(C^*(X \times \mathbb{R}^+)^{\Gamma}) = 0$ for $i = 0, 1$. This proof is the same as the proof when Γ is trivial.

Lemma 17. *The following diagram commutes*

$$\begin{array}{ccc} K_i^{\Gamma}(X \times \mathbb{R}) & \xrightarrow{Ind} & K_i(C^*(X \times \mathbb{R})^{\Gamma}) \\ \cong\downarrow & & \downarrow\cong \\ K_{i+1}^{\Gamma}(X) & \xrightarrow{Ind} & K_{i+1}(C^*(X)^{\Gamma}) \end{array}$$

for $i = 0, 1$.

This is the naturality of the index map.

Based on these two lemmas we only prove Theorem 21 for even-dimensional manifold X. Let $\mathcal{A} = C_0(X, \mathrm{Cliff}(TX))$. Choose a countable dense subset X' of X such that X' is Γ-invariant.

Let $C^*_{alg}(X, \mathcal{A})^{\Gamma}_{X'}$ be the set of all functions T on $X' \times X'$ such that

(1) $T(x, y) \in \mathcal{A} \otimes \mathcal{K}$ for all $x, y \in X'$, where $\mathcal{K}$ is the algebra of compact operators;

(2) $\exists\, L > 0$ such that $T(x, y) = 0$ if $d(x, y) > L$ for all $x, y \in X$;

(3) $\exists\, r > 0$ such that $\mathrm{support}(T(x, y)) \subset B(x, r)$ for all $x, y \in X$;

(4) $\gamma(T) = T$ for all $\gamma \in \Gamma$, where $\gamma(T)(x, y) = \gamma(T(\gamma^{-1}(x), \gamma^{-1}(y)))$;

(5) $\exists\, M > 0$ and $N > 0$ such that $\|T(x, y)\| \leq M$ for all $x, y \in X'$, and for each $y \in X'$, $\#\{x : T(x, y) \neq 0\} \leq N$, $\#\{x : T(y, x) \neq 0\} \leq N$.

We define a product structure on $C^*_{alg}(X, \mathcal{A})^{\Gamma}_{X'}$ by:

$$(T_1 T_2)(x, y) = \sum_{z \in X'} T_1(x, z) T_2(z, y).$$

Let

$$E = \left\{ \sum_{x \in X'} a_x[x] : a_x \in \mathcal{A} \otimes \mathcal{K}, \sum_{x \in X'} a_x^* a_x < \infty \right\}.$$

E is a Hilbert module over $\mathcal{A} \otimes \mathcal{K}$:

$$\left\langle \sum_{x \in X'} a_x[x], \sum_{x \in X'} b_x[x] \right\rangle = \sum_{x \in X'} a_x^* b_x,$$

$$\left(\sum_{x \in X'} a_x[x] \right) a = \sum_{x \in X'} a_x a[x]$$

for all $a \in \mathcal{A} \otimes \mathcal{K}$ and $\sum_{x \in X'} a_x[x] \in E$.

$C^*_{alg}(X, \mathcal{A})^{\Gamma}_{X'}$ acts on E by:

$$T\left(\sum_{x \in X'} a_x[x] \right) = \sum_{y \in X'} \left(\sum_{x \in X'} T(y, x) a_x \right) [y],$$

where $T \in C^*_{alg}(X, \mathcal{A})^\Gamma_{X'}$ and $\sum_{x \in X'} a_x[x] \in E$. T is a module homomorphism which has an adjoint module homomorphism.

Definition 43. *$C^*(X, \mathcal{A})^\Gamma$ is the operator norm closure of $C^*_{alg}(X, \mathcal{A})^\Gamma_{X'}$ in $B(E)$, the C^*-algebra of all module homomorphisms from E to E for which there is an adjoint module homomorphism.*

Remark ([Y00]). The twisted equivariant Roe algebra $C^*(X, \mathcal{A})^\Gamma$ does not depend on the choice of X'.

Let $C^*_{L,alg}(X, \mathcal{A})^\Gamma_{X'}$ be the set of all uniformly continuous and uniformly bounded functions $g : [0, \infty) \to C^*_{alg}(X, \mathcal{A})^\Gamma_{X'}$ such that

(1) $\exists$ a bounded function $r(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{t \to \infty} r(t) = 0$ and if $d(x, y) > r(t)$, $g(t)(x, y) = 0$ for all $x, y \in X$;

(2) $\exists\ R > 0$ such that $supp(g(t)(x, y)) \subset B(x, R)$ for all $x, y \in X$ and $t \in [0, \infty)$.

Definition 44. *$C^*_L(X, \mathcal{A})^\Gamma$ is the operator norm closure of $C^*_{L,alg}(X, \mathcal{A})^\Gamma_{X'}$, where $C^*_{L,alg}(X, \mathcal{A})^\Gamma_{X'}$ is endowed with the norm:*

$$\|g\| = \sup_{t \in [0, \infty)} \|g(t)\|_{B(E)}.$$

V.4 K-Theory of twisted Roe algebras and twisted localization algebras.

In this section, we compute the K-theory of twisted Roe algebras and twisted localization algebras.

Definition 45.

1. *The support of an element T in $C^*_{alg}(X, \mathcal{A})^\Gamma_{X'}$ is defined to be*

$$\left\{(x, y, u) \in X' \times X' \times X : u \in supp(T(x, y))\right\};$$

2. *The support of an element g in $C^*_{L,alg}(X, \mathcal{A})^\Gamma_{X'}$ is defined to be*

$$\bigcup_{t \in [0, \infty)} supp(g(t)).$$

Let O be a subset of X. Define $C^*_{alg}(X,\mathcal{A})^\Gamma_O$ to be the subalgebra of $C^*_{alg}(X,\mathcal{A})^\Gamma_{X'}$ consisting of all elements whose supports are contained in $X' \times X' \times O$. Define $C^*(X,\mathcal{A})^\Gamma_O$ to be the norm closure of $C^*_{alg}(X,\mathcal{A})^\Gamma_O$. We can similarly define $C^*_L(X,\mathcal{A})^\Gamma_O$. Let $C^*_{L,0}(X,\mathcal{A})^\Gamma_O$ be the C^*-subalgebra of $C^*_L(X,\mathcal{A})^\Gamma_O$ consisting of elements g satisfying $g(0) = 0$.

Lemma 18. *If O is Γ invariant and cocompact, then*

$$e_* : K_*(C^*_L(X,\mathcal{A})^\Gamma_O) \to K_*(C^*(X,\mathcal{A})^\Gamma_O)$$

is an isomorphism.

Proof. This proof is similar to the proofs of Lemma 6.4 and Lemma 6.7 in [Y00]. We have a short exact sequence:

$$0 \to C^*_{L,0}(X,\mathcal{A})^\Gamma_O \to C^*_L(X,\mathcal{A})^\Gamma_O \to C^*(X,\mathcal{A})^\Gamma_O \to 0.$$

Hence it is enough to show

$$K_*(C^*_{L,0}(X,\mathcal{A})^\Gamma_O) = 0.$$

Let $\mathcal{A}_O$ be the subset of $\mathcal{A}$ consisting of elements f satisfying that $supp(f) \subseteq O$ and $supp(f)$ is compact. For all $R > 0$, let

$$O(R) = \{x \in X : \exists z \in O \text{ such that } d(z,x) < R\}.$$

Since O is cocompact, we have

$$C^*(X,\mathcal{A})^\Gamma_O = \lim_{R\to\infty} \mathcal{A}_O \otimes C^*(O(R))^\Gamma.$$

Clearly $O(R)$ is cocompact, then

$$C^*(X,\mathcal{A})^\Gamma_O = \lim_{R\to\infty} \mathcal{A}_O \otimes C^*_r(\Gamma) \otimes \mathcal{K} = \mathcal{A}_O \otimes C^*_r(\Gamma) \otimes \mathcal{K}.$$

Hence $C^*_{L,0}(X,\mathcal{A})^\Gamma_O$ is the set of $g : [0,\infty) \to \mathcal{A}_O \otimes C^*_r(\Gamma) \otimes \mathcal{K}$ satisfying that $g(0) = 0$, and

g is uniformly continuous and uniformly bounded.

We prove that

$$K_1(C^*_{L,0}(X,\mathcal{A})^\Gamma_O) = 0.$$

Clearly $C^*_{L,0}(X,\mathcal{A})^\Gamma_O$ is stable. Therefore any element in $K_1(C^*_{L,0}(X,\mathcal{A})^\Gamma_O)$ can be represented by a unitary u in $(C^*_{L,0}(X,\mathcal{A})^\Gamma_O)^+$.

For each $s \in [0,\infty)$, we define

$$u_s(t) = \begin{cases} I & \text{if } 0 \le t \le s; \\ u(t-s) & \text{if } s \le t < \infty. \end{cases}$$

Consider

$$w(s) = (\oplus_{k=0}^\infty u_k \oplus I)(I \oplus (\oplus_{k=1}^\infty u^{-1}_{k-s}) \oplus I),$$

where $s \in [0,\infty)$. Notice that $w(s)$ is an element in $(C^*_{L,0}(X,\mathcal{A})^\Gamma_O)^+$ for each $s \in [0,1]$. We have

$$w(0) = u \oplus (\oplus_{k=1}^\infty I) \oplus I,$$

$$w(1) = (\oplus_{k=0}^\infty u_k \oplus I)(I \oplus (\oplus_{k=1}^\infty u^{-1}_{k-1}) \oplus I).$$

$w(1)$ is equivalent to $\oplus_{k=0}^\infty I \oplus I$ in $K_1(C^*_{L,0}(X,\mathcal{A})^\Gamma_O)$ by a rotation. Therefore u is equivalent to the zero element in $K_1(C^*_{L,0}(X,\mathcal{A})^\Gamma_O)$.

Using suspension, we can also prove that $K_0^\Gamma(C^*_{L,0}(X,\mathcal{A})^\Gamma_O) = 0.$

□

Theorem 24. $e_* : K_*(C^*_L(X,\mathcal{A})^\Gamma) \to K_*(C^*(X,\mathcal{A})^\Gamma)$ *is an isomorphism.*

Proof. We have the exact sequence

$$0 \to C^*_{L,0}(X,\mathcal{A})^\Gamma \to C^*_L(X,\mathcal{A})^\Gamma \to C^*(X,\mathcal{A})^\Gamma \to 0.$$

Therefore it is enough to show that $K_*(C^*_{L,0}(X,\mathcal{A})^\Gamma) = 0$. Fix $x \in X$ and let B_r be the ball with radius r and centered at x. Let O_r the smallest Γ-invariant subset of X containing B_r. Then O_r is cocompact and

$$C^*_{L,0}(X,\mathcal{A})^\Gamma = \lim_{r\to\infty} C^*_{L,0}(X,\mathcal{A})^\Gamma_{O_r}.$$

From the previous lemma, we know that

$$K_*(C^*_{L,0}(X,\mathcal{A})^\Gamma_{O_r}) = 0.$$

Hence

$$K_*(C^*_{L,0}(X,\mathcal{A})^\Gamma) = 0.$$

□

V.5 The Bott elements and Bott maps.

In this section we define the Bott elements and Bott maps. We will define the Bott element and Bott map for K_0, using suspension we can extend our definition onto K_1. Let X be a simply-connected complete Riemannian manifold with nonpositive sectional curvature and Γ be a torsion-free discrete group acting on X properly and isometrically. First we define the Bott element for each $x \in X$. Let $x, z \in X$ and $\sigma(t)$ be the unique geodesic connecting x and z such that $\sigma(0) = x$ and $\sigma(1) = z$. Let $v_x(z) = \frac{\sigma'(1)}{\|\sigma'(1)\|} \in T_zX$. Define $f_x(z) = \phi(z)v_x(z)$ where $\phi(z)$ is a continuous function such that

$$\phi(z) = \begin{cases} 1 & \text{if} \quad d(x,z) > c\ , \\ 0 & \text{if} \quad d(x,z) < \frac{99}{100}c \end{cases}$$

for some constant c.

Let $\mathcal{B} = C_b(X, \mathrm{Cliff}(TX))$ and $\mathcal{A} = C_0(X, \mathrm{Cliff}(TX))$.

Lemma 19. *f_x is an invertible element in $\mathcal{B}/\mathcal{A}$ with an inverse $-f_x$.*

We have a short exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A} \to 0,$$

then we have a boundary map

$$\partial : K_1(\mathcal{B}/\mathcal{A}) \to K_0(\mathcal{A}).$$

Let $u = [f_x] \in K_1(\mathcal{B}/\mathcal{A})$ and $g_x = -f_x \in \mathcal{B}$ be a representative of $u^{-1} \in K_1(\mathcal{B}/\mathcal{A})$. Define

$$\omega = \begin{pmatrix} 1 & f_x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g_x & 1 \end{pmatrix} \begin{pmatrix} 1 & f_x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\beta_1^x = \omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \omega^{-1}$ and $\beta_0^x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then β_1^x, β_0^x are both projections in $M_2(\mathcal{A}^+)$ and $\beta_1^x - \beta_0^x \in M_2(\mathcal{A})$. We define

$$\partial([f_x]) = [\beta_1^x] - [\beta_0^x] \in K_0(\mathcal{A}).$$

With the constant c in the definition of f_x, we can construct "almost flat" Bott element in the following sense.

Lemma 20. *Let X be a simply-connected complete Riemannian manifold with nonpositive sectional curvature and $x, y \in X$. Let $f_x, \beta_1^x, \beta_0^x$ and $f_y, \beta_1^y, \beta_0^y$ defined as above. Then for all $r, \epsilon > 0$, there exists $c > 0$ such that for all $z \in X$,*

$$\|\beta_1^x(z) - \beta_1^y(z)\| < \epsilon \qquad \textit{if} \ \ d(x,y) < r,$$

where c depends only on r and ϵ.

Proof. First we prove that it is true for Euclidean spaces. Let $x, y, z \in \mathbb{R}^n$, $z \neq x, y$ and $d(x, y) = r$. Let $v_x(z)$ and $v_y(z)$ be the unit vectors defined as above and let θ denote the angle formed by $v_x(z)$ and $v_y(z)$. Let $c = \frac{100r}{\epsilon}$. When $d(x, z) > c$ and $d(y, z) > c$,

$$\begin{aligned} \|v_x(z) - v_y(z)\| &= \sqrt{\|v_x(z)\|^2 + \|v_y(z)\|^2 - 2\|v_x(z)\|\,\|v_y(z)\|\cos\theta} \\ &= \sqrt{2 - 2\cos\theta} < \epsilon. \end{aligned}$$

This is true since θ will decrease to zero when $d(x,z)$ and $d(y,z)$ approach infinity. We just choose this c to define f_x and f_y. Clearly we have

$$\|\beta_1^x(z) - \beta_1^y(z)\| < \epsilon.$$

Now let X be a simply-connected complete Riemannian manifold with nonpositive sectional curvature and $x, y, z \in X$. Assume that $z \neq x, y$ and $d_X(x,y) = r$ where d_X is the Riemannian metric on X. Let $v_x(z), v_y(z) \in T_zX$ defined as above. Let $\exp : T_zX \longrightarrow X$ be the exponential map which is a diffeomorphism between T_zX and X. Since X is nonpositively curved, by the comparison theorem in Riemannian geometry $d_{T_zX}(\exp^{-1}(x), \exp^{-1}(y)) \leq d_X(x,y) = r$. And $\left\|\exp^{-1}(x)\right\| = d_X(x,z)$ and $\left\|\exp^{-1}(y)\right\| = d_X(y,z)$ since $\exp^{-1}(z) = 0$. Since T_zX is a Euclidean space, from the first part we know that on T_zX we can find c such that when $\left\|\exp^{-1}(x)\right\| > c$ and $\left\|\exp^{-1}(y)\right\| > c$,

$$\|v_x(z) - v_y(z)\| < \epsilon.$$

We just use this c to define f_x and f_y. Similarly we have

$$\|\beta_1^x(z) - \beta_1^y(z)\| < \epsilon.$$

□

To define the Bott maps, we need the following difference construction introduced by Kasparov and Yu in [KY]. Let A be a unital C^*-algebra and J be a two-sided ideal of A. Let $p, q \in M_k(A)$ be two idempotents and $p - q \in M_k(J)$. Let

$$Z(q) = \begin{pmatrix} q & 0 & 1-q & 0 \\ 1-q & 0 & 0 & q \\ 0 & 0 & q & 1-q \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

then $Z(q)$ is invertible and

$$Z(q)^{-1} = \begin{pmatrix} q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-q & 0 & q & 0 \\ 0 & q & 1-q & 0 \end{pmatrix}.$$

Let

$$D_0(p,q) = Z(q)^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(q),$$

then

$$D_0(p,q) \in J^+ \otimes M_4(\mathbb{C})$$

and

$$D_0(p,q) - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in J \otimes M_4(\mathbb{C}).$$

Lemma 21 ([KY]). *Let $A, J, p, q, Z(q)$ and $D_0(p,q)$ defined as above, then*

$$D(p,q) = [D_0(p,q)] - \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \in K_0(J)$$

and we call $D(p,q)$ the difference of p, q in $K_0(J)$.

Now, let us construct the Bott maps. For any $[(k(x,y))_{x,y\in X'}] - [p_k] \in K_0(C^*(X)^\Gamma)$, choose $(k_n(x,y))_{x,y\in X'} \in (C^*_{alg}(X)^\Gamma)^+$ such that $\left\|(k(x,y))_{x,y\in X'} - (k_n(x,y))_{x,y\in X'}\right\| < \frac{1}{n}$ and $\left\|\left((k_n(x,y))_{x,y\in X'}\right)^2 - (k_n(x,y))_{x,y\in X'}\right\| < \frac{1}{n}$ and the propagation of $(k_n(x,y))_{x,y\in X'}$ is r. Here $(k(x,y))_{x,y\in X'}$ is considered as an infinite dimensional matrix and the product is the matrix product. Then when n is big enough, there is a constant c in the definition of $f_x, \beta_0^x, \beta_1^x$ depending on r and ϵ such that

$$\left\| \left((k_n(x,y) \otimes \beta_i^x)_{x,y\in X'} \right)^2 - (k_n(x,y) \otimes \beta_i^x)_{x,y\in X'} \right\| < \epsilon \ \text{ for } \ i = 0, 1.$$

For $x, z \in X$ and $\gamma \in \Gamma$, we have

$$\gamma(\beta_1^{\gamma^{-1}x})(z) = \gamma(\beta_1^{\gamma^{-1}x}(\gamma^{-1}z)) = \beta_1^x(z).$$

This implies that $(k_n(x,y) \otimes \beta_1^x)_{x,y\in X'}$ is Γ-invariant and

$$(k_n(x,y) \otimes \beta_1^x)_{x,y\in X'} - (k_n(x,y) \otimes \beta_0^x)_{x,y\in X'} \in C^*(X, \mathcal{A})^\Gamma.$$

When ϵ is small enough, $(k_n(x,y) \otimes \beta_1^x)_{x,y\in X'}$ and $(k_n(x,y) \otimes \beta_0^x)_{x,y\in X'}$ define two idempotents p_1 and p_0 by functional calculus. Then we construct the Bott map as follows.

Definition 46. *The Bott map*

$$\beta : K_0(C^*(X)^\Gamma) \longrightarrow K_0(C^*(X, \mathcal{A})^\Gamma)$$

is defined by

$$\beta([(k(x,y))_{x,y\in X'}] - [p_k]) = D(p_1, p_0)$$

for all $[(k(x,y))_{x,y\in X'}] - [p_k] \in K_0(C^*(X)^\Gamma)$*, where* p_1 *and* p_0 *are defined above and* $D(p_1, p_0)$ *is the difference of* p_1 *and* p_0 *defined in Lemma 21.*

Using suspension, we have the Bott map

$$\beta : K_1(C^*(X)^\Gamma) \longrightarrow K_1(C^*(X, \mathcal{A})^\Gamma).$$

Similarly we define the Bott map for K-groups of the localization algebras

$$(\beta_L)_* : K_i(C_L^*(X)^\Gamma) \to K_i(C_L^*(X, \mathcal{A})^\Gamma) \qquad i = 0, 1.$$

In the next lemma we will show that $(\beta_L)_*$ is an isomorphism.

By the proof of Theorem 22, we can easily see that $K_*(C_L^*(X, \mathcal{A})^\Gamma)$ also has the Mayer-Vietoris sequence.

Lemma 22. $(\beta_L)_* : K_*(C_L^*(X)^\Gamma) \to K_*(C_L^*(X, \mathcal{A})^\Gamma)$ *is an isomorphism.*

Proof. This proof is the composition of Mayer-Vietoris argument, the five lemma and Bott periodicity. X is covered by sets of the form $\Gamma \times U$, where U is contractible. Let P be the one-point set. Note that P is homotopic to U. By Mayer-Vietoris argument and the five lemma it suffices to show that for a single such space $\Gamma \times U$ the map

$$(\beta_L)_* : K_*(C_L^*(\Gamma \times U)^\Gamma) \to K_*(C_L^*(\Gamma \times U, \mathcal{A})^\Gamma)$$

is an isomorphism. For this, it suffices to show that

$$(\beta_L)_* : K_*(C_L^*(\Gamma \times P)^\Gamma) \to K_*(C_L^*(\Gamma \times P, \mathcal{A})^\Gamma)$$

is an isomorphism. The last map is an isomorphism by the Bott periodicity.

□

V.6 The proof of the main theorem.

Theorem 25. *Let X be a simply-connected Riemannian manifold with nonpositive sectional curvature and Γ be a torsion-free discrete group acting on X properly and isometrically. Then the equivariant index map*

$$\mathrm{Ind} : K_*^\Gamma(X) \to K_*(C^*(X)^\Gamma)$$

is injective.

Proof. We have the commuting diagram

$$\begin{array}{ccccc} & & K_*(C_L^*(X)^\Gamma) & \xrightarrow{(\beta_L)_*} & K_*(C_L^*(X, \mathcal{A})^\Gamma) \\ & \nearrow^{\mathrm{Ind_L}} & \downarrow{e_*} & & \downarrow{e_*} \\ K_*^\Gamma(X) & \xrightarrow{\mathrm{Ind}} & K_*(C^*(X)^\Gamma) & \xrightarrow{\beta_*} & K_*(C^*(X, \mathcal{A})^\Gamma). \end{array}$$

From above, we have the isomorphism

$$e_* \circ (\beta_L)_* \circ \mathrm{Ind_L} : K_*^\Gamma(X) \to K_*(C^*(X, \mathcal{A})^\Gamma)) \ .$$

Therefore,

$$\mathrm{Ind} : K_*^\Gamma(X) \to K_*(C^*(X)^\Gamma)$$

is injective.

□

CHAPTER VI

SUBSPACES OF A SIMPLY CONNECTED COMPLETE RIEMANNIAN MANIFOLD OF NONPOSITIVE SECTIONAL CURVATURE

This chapter is the joint work with Qin Wang. Let X and Y be two metric spaces. A map $f : X \to Y$ is said to be a *coarse embedding* or *uniform embedding* [G] if there exist non-decreasing functions ρ_1 and ρ_2 from $\mathbb{R}_+ = [0, \infty)$ to $\mathbb{R}_+$ such that

1. $\rho_1(d(x, y)) \leq d(f(x), f(y)) \leq \rho_2(d(x, y))$;

2. $\lim_{r\to\infty} \rho_i(r) = \infty$ for $i = 1, 2$.

The main purpose of this chapter is to prove the following result:

Theorem 26. *Let Γ be a discrete metric space with bounded geometry. If Γ admits a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature, then the coarse geometric Novikov conjecture holds for Γ, i.e., the index map from $\lim_{d\to\infty} K_*(P_d(\Gamma))$ to $K_*(C^*(\Gamma))$ is injective, where $K_*(P_d(\Gamma)) = KK_*(C_0(P_d(\Gamma)), \mathbb{C})$ is the locally finite K-homology group of the Rips complex $P_d(\Gamma)$, and $K_*(C^*(\Gamma))$ is the K-theory group of the Roe algebra $C^*(\Gamma)$ associated to Γ.*

Recall that a discrete metric space X is said to have bounded geometry if for any $r > 0$ there is $N > 0$ such that any ball of radius r in X contains at most N elements. The coarse geometric Novikov conjecture provides an algorithm of determining non-vanishing of the higher index for elliptic differential operators on noncompact complete Riemannian manifolds. It implies Gromov's conjecture stating that a uniformly contractible Riemmanian manifold with bounded geometry can not have uniformly positive scalar curvature, and the zero-in-the-spectrum conjecture stating that the Laplacian operator acting on the space of all L^2-forms of a uniformly contractible Riemannian manifold has zero in its spectrum.

The coarse geometric Novikov conjecture holds for bounded geometry metric spaces which are coarsely embeddable into Hilbert space [Y00]. More generally, Kasparov and Yu proved the coarse geometric Novikov conjecture for bounded geometry metric spaces which are coarsely embeddable into uniformly convex Banach space [KY]. The coarse geometric Novikov conjecture holds for a simply connected complete Riemannian manifold of nonpositive sectional curvature [HR, HRY, Y97]. Yet it is an open problem if any simply connected complete Riemannian manifold with nonpositive sectional curvature admits a coarse embedding into a Hilbert space or a uniformly convex Banach space. It is also an interesting problem if a bounded geometry metric space which admits a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature is coarsely embeddable into a Hilbert space or a uniformly convex Banach space. We remark here that Dranishnikov proved that a metric space with a finite asymptotic dimension is coarsely embeddable into a non-positively curved manifold [D].

VI.1 The coarse geometric Novikov conjecture

In this section, we shall recall the concept of the Roe algebra [R93], the coarse geometric Novikov conjecture and Yu's localization algebras [Y97].

Let X be a proper metric space (a metric space is called *proper* if every closed ball is compact). and let (H_X, F) be a cycle for $K_0(X)$. Let $\{U_j\}_j$ be a locally finite and uniformly bounded open cover of X and $\{\phi_j\}_j$ be a continuous partition of unity subordinate to the open cover $\{U_j\}_j$. Define

$$F' = \sum_j \phi_j^{\frac{1}{2}} F \phi_j^{\frac{1}{2}}$$

where the infinite sum converges in strong topology. Then it is not difficult to verify that (H_X, F') is equivalent to (H_X, F) in $K_0(X)$. Note that F' has finite propagation so that F' is a multiplier of $C^*(X)$. It is easy to see that F' is a unitary modulo $C^*(X)$. Hence F' gives rise to an element, denoted by $\partial[F']$, in $K_0(C^*(X))$. We define the index of the K-homology class of (H_X, F) to be $\partial[F']$. Similarly, we can define the index map from $K_1(X)$ to $K_1(C^*(X))$.

Recall that a discrete metric space is said to be *locally finite* if every ball contains finitely many elements.

Definition 47. *Let Γ be a locally finite discrete metric space. For each $d \geq 0$, the* Rips complex $P_d(\Gamma)$ *is defined to be the simplicial polyhedron in which the set of vertices is Γ, and a finite subset $\{\gamma_0, \gamma_1, \cdots, \gamma_n\} \subset \Gamma$ spans a simplex if and only if $d(\gamma_i, \gamma_j) \leq d$ for all $0 \leq i, j \leq n$.*

Endow $P_d(\Gamma)$ with the spherical metric. Recall that on each path connected component of $P_d(\Gamma)$, the spherical metric is the maximal metric whose restriction to each simplex $\{\sum_{i=0}^n t_i\gamma_i \mid \ t_i \geq 0, \ \sum_{i=0}^n t_i = 1\}$ is the metric obtained by identifying the simplex with $S_+^n := \{(s_0, s_1, \cdots, s_n) \in \mathbb{R}^{n+1} : s_i \geq 0, \sum_{i=0}^n s_i^2 = 1\}$ via the map

$$\sum_{i=0}^n t_i\gamma_i \mapsto \left(\frac{t_0}{\sqrt{\sum_{i=0}^n t_i^2}}, \frac{t_1}{\sqrt{\sum_{i=0}^n t_i^2}}, \cdots, \frac{t_n}{\sqrt{\sum_{i=0}^n t_i^2}} \right)$$

where S_+^n is endowed with the standard Riemannian metric. The distance of a pair of points in different connected components of $P_d(\Gamma)$ is defined to be infinity.
The following conjecture is called the coarse geometric Novikov conjecture:

Conjecture 1. *If Γ is a discrete metric space with bounded geometry, then the index map*

$$ind: \lim_{d\to\infty} K_*(P_d(\Gamma)) \to \lim_{d\to\infty} K_*(C^*(P_d(\Gamma))) \cong K_*(C^*(\Gamma))$$

is injective.

The coarse geometric Novikov conjecture is false if the bounded geometry condition is dropped [Y98]. In the rest of this section, we shall recall the localization algebra [Y97] and its relation with K-homology. Let X be a proper metric space.

Definition 48 ([Y97])**.** *The* localization algebra $C_L^*(X)$ *is the norm-closure of the algebra of all bounded and uniformly norm-continuous functions $g: \ [0, \infty) \to C^*(X)$ such that*

$$propagation(g(t)) \to 0 \quad as \quad t \to \infty.$$

The evaluation homomorphism e from $C_L^*(X)$ to $C^*(X)$ is defined by $e(g) = g(0)$ for

$g \in C_L^*(X)$. There exists a local index map [Y97]

$$ind_L: \ K_*(X) \to K_*(C_L^*(X)).$$

Theorem 27 ([Y97]). *For every finite dimensional simplicial complex X endowed with the spherical metric, the local index map $ind_L: \ K_*(X) \to K_*(C_L^*(X))$ is an isomorphism. Consequently, if Γ is a discrete metric space with bounded geometry, we have the following commuting diagram:*

$$\begin{array}{ccc} & & \lim\limits_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) \\ & \overset{ind_L}{\underset{\cong}{\nearrow}} & \big\downarrow e_* \\ \lim\limits_{d\to\infty} K_*(P_d(\Gamma)) & \xrightarrow{ind} & \lim\limits_{d\to\infty} K_*(C^*(P_d(\Gamma))). \end{array}$$

VI.2 Twisted Roe algebras and twisted localization algebras

In this section, we shall define the twisted Roe algebras and the twisted localization algebras for bounded geometry spaces which admit a coarse embedding into a simply connected complete Riemannian manifold of nonpositive sectional curvature. The construction of these twisted algebras is similar to those twisted algebras introduced in [Y00].

Let M be a simply connected complete Riemannian manifold of nonpositive sectional curvature. In the following, we shall assume that the dimension of M is even. If $\dim(M)$ is odd, we can replace M by $M \times \mathbb{R}$. Indeed, the product manifold $M \times \mathbb{R}$ is also a simply connected complete Riemannian manifold with nonpositive sectional curvature. And if $f: \Gamma \to M$ is a coarse embedding, then the induced map $f': \Gamma \to M \times \mathbb{R}$ defined by $f'(\gamma) = (f(\gamma), 0)$ is also a coarse embedding so that we can replace f by f'. Thus, without loss of generality, we assume $\dim M = 2n$ for some integer $n > 0$.

Let $\mathcal{A} = C_0(M, \mathrm{Cliff}(TM))$ be the C^*-algebra of continuous functions a on M which have value $a(x) \in \mathrm{Cliff}(T_xM)$ at each point $x \in M$ and vanish at infinity, where $\mathrm{Cliff}(T_xM)$ is the complexified Clifford algebra of the tangent space T_xM at $x \in M$ with respect to the inner product on T_xM given by the Riemannian structure of M. Note that the exponential map at any point $x \in M$

$$\exp_x : T_x M \longrightarrow M$$

is a homeomorphism, which induces a $*$-isomorphism:

$$\mathcal{A} \cong C_0(\mathbb{R}^{2n}) \otimes \mathcal{M}_{2^n}(\mathbb{C}),$$

where by $\mathcal{M}_k(\mathbb{C})$ we denote the algebra of $k \times k$ complex matrices.

Let Γ be a discrete metric space with bounded geometry. Let $f : \Gamma \to M$ be a coarse embedding.

For each $d > 0$, we shall extend the map f to the Rips complex $P_d(\Gamma)$ in the following way. Note that f is a coarse map, i.e., there exists $R > 0$ such that for all $\gamma_1, \gamma_2 \in \Gamma$,

$$d(\gamma_1, \gamma_2) \leq d \implies d_M(f(\gamma_1), f(\gamma_2)) \leq R.$$

For any point $x = \sum_{\gamma \in \Gamma} c_\gamma \gamma \in P_d(\Gamma)$, where $c_\gamma \geq 0$ and $\sum_{\gamma \in \Gamma} c_\gamma = 1$, we choose a point $f_x \in M$ such that

$$d(f_x, f(\gamma)) \leq R$$

for all $\gamma \in \Gamma$ with $c_\gamma \neq 0$. The correspondence $x \mapsto f_x$ gives a coarse embedding $P_d(\Gamma) \to M$, also denoted by f.

Choose a countable dense subset Γ_d of $P_d(\Gamma)$ for each $d > 0$ in such a way that $\Gamma_d \subset \Gamma_{d'}$ when $d < d'$. Let $\mathcal{K}$ be the algebra of all compact operators on a separable Hilbert space H_0.

Let $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ be the set of all functions

$$T : \Gamma_d \times \Gamma_d \to \mathcal{A} \otimes \mathcal{K}$$

such that

1. there exists $C > 0$ such that $\|T(x, y)\| \leq C$ for all $x, y \in \Gamma_d$;

2. there exists $R > 0$ such that $T(x, y) = 0$ if $d(x, y) > R$;

3. there exists $L > 0$ such that for every $z \in P_d(\Gamma)$, the number of elements in the following set

$$\{(x, y) \in \Gamma_d \times \Gamma_d : \ d(x, z) \leq 3R, \ d(y, z) \leq 3R, \ T(x, y) \neq 0\}$$

is less than L;

4. there exists $r > 0$ such that

$$\text{Supp}(T(x, y)) \subset B(f(x), r)$$

for all $x, y \in \Gamma_d$, where $B(f(x), r) = \{p \in M : \ d(p, f(x)) < r\}$ and, for all $x, y \in \Gamma_d$, the entry $T(x, y) \in \mathcal{A} \otimes \mathcal{K}$ is a function on M with $T(x, y)(p) \in \text{Cliff}(T_pM) \otimes \mathcal{K}$ for each $p \in M$ so that the *support* of $T(x, y)$ is defined by

$$\text{Supp}(T(x, y)) := \{p \in M : \ T(x, y)(p) \neq 0\}.$$

Remark. For any $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$, there is $r > 0$ such that

$$\text{Supp}(T(x, y)) \subset B(f(x), r)$$

for all $x, y \in \Gamma_d$. Since $f : \ P_d(\Gamma) \to M$ is a coarse embedding, there exists $S > 0$ such that $d(f(x), f(y)) < S$ whenever $d(x, y) < R$. It follows that

$$\text{Supp}(T(x, y)) \subset B(f(y), S + r)$$

for all $x, y \in \Gamma_d$. Hence, the adjoint T^* of T defined by

$$T^*(x, y) = (T(y, x))^* \in \mathcal{A} \otimes \mathcal{K} \qquad (\forall x, y \in \Gamma_d)$$

is also an element of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$. Therefore, $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ is a $*$-algebra.

A product structure on $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ can be defined by

$$(T_1T_2)(x,y) = \sum_{z\in\Gamma_d} T_1(x,z)T_2(z,y).$$

Let

$$E = \left\{ \sum_{x\in\Gamma_d} a_x[x] : \; a_x \in \mathcal{A}\otimes\mathcal{K}, \; \sum_{x\in\Gamma_d} a_x^* a_x \;\text{ converges in norm} \right\}.$$

Then E is a Hilbert module over $\mathcal{A}\otimes\mathcal{K}$:

$$\left\langle \sum_{x\in\Gamma_d} a_x[x], \; \sum_{x\in\Gamma_d} b_x[x] \right\rangle = \sum_{x\in\Gamma_d} a_x^* b_x,$$

$$\left(\sum_{x\in\Gamma_d} a_x[x] \right) a = \sum_{x\in\Gamma_d} a_x a[x]$$

for all $a \in \mathcal{A}\otimes\mathcal{K}$. The $*$-algebra $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ acts on E by

$$T\left(\sum_{x\in\Gamma_d} a_x[x] \right) = \sum_{y\in\Gamma_d} \left(\sum_{x\in\Gamma_d} T(y,x)a_x \right)[y],$$

where $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})$ and $\sum_{x\in\Gamma_d} a_x[x] \in E$. Note that T is a module homomorphism which has an adjoint module homomorphism.

Definition 49. *The twisted Roe algebra $C^*(P_d(\Gamma), \mathcal{A})$ is defined to be the operator norm closure of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ in $\mathcal{B}(E)$, the C^*-algebra of all module homomorphisms from E to E for which there is an adjoint module homomorphism.*

The above definition of the twisted Roe algebra is similar to that in [Y00].

Let $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$ be the set of all bounded, uniformly norm-continuous functions

$$g : \mathbb{R}_+ \to C^*_{alg}(P_d(\Gamma), \mathcal{A})$$

such that

1. there exists a bounded function $R(t) : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} R(t) = 0$ such that $(g(t))(x,y) = 0$ whenever $d(x,y) > R(t)$;

2. there exists $L > 0$ such that for every $z \in P_d(\Gamma)$, the number of elements in the following set

$$\{(x, y) \in \Gamma_d \times \Gamma_d : \ d(x, z) \leq 3R, \ d(y, z) \leq 3R, \ g(t)(x, y) \neq 0\}$$

is less than L for every $t \in \mathbb{R}_+$.

3. there exists $r > 0$ such that $\mathrm{Supp}((g(t))(x, y)) \subset B(f(x), r)$ for all $t \in \mathbb{R}_+$, $x, y \in \Gamma_d$, where $f : P_d(\Gamma) \to M$ is the extension of the coarse embedding $f : \Gamma \to M$ and $B(f(x), r) = \{p \in M : d(p, f(x)) < r\}$.

Definition 50. *The twisted localization algebra $C_L^*(P_d(\Gamma), \mathcal{A})$ is defined to be the norm completion of $C_{L,alg}^*(P_d(\Gamma), \mathcal{A})$, where $C_{L,alg}^*(P_d(\Gamma), \mathcal{A})$ is endowed with the norm*

$$\|g\|_\infty = \sup_{t \in \mathbb{R}_+} \|g(t)\|_{C^*(P_d(\Gamma), \mathcal{A})}.$$

The above definition of the twisted localization Roe algebra is similar to that in [Y00]. The evaluation homomorphism e from $C_L^*(P_d(\Gamma), \mathcal{A})$ to $C^*(P_d(\Gamma), \mathcal{A})$ defined by $e(g) = g(0)$ induces a homomorphism at K-theory level:

$$e_* : \lim_{d \to \infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})) \to \lim_{d \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A})).$$

Theorem 28. *Let Γ be a discrete metric space with bounded geometry which admits a coarse embedding $f : \Gamma \to M$ into a simply connected, complete Riemannian manifold M of non-positive sectional curvature. Then the homomorphism*

$$e_* : \lim_{d \to \infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})) \to \lim_{d \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A}))$$

is an isomorphism.

The proof of Theorem 28 is similar to the proof of Theorem 6.8 in [Y00]. To begin with, we need to discuss ideals of the twisted algebras associated to open subsets of the manifold M.

Definition 51.

1. *The support of an element T in $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ is defined to be*

$$\begin{aligned} \mathrm{Supp}(T) &= \Big\{(x, y, p) \in \Gamma_d \times \Gamma_d \times M : \ p \in \mathrm{Supp}(T(x, y))\Big\} \\ &= \Big\{(x, y, p) \in \Gamma_d \times \Gamma_d \times M : \ (T(x, y))(p) \neq 0\Big\}; \end{aligned}$$

2. *The support of an element g in $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})$ is defined to be*

$$\bigcup_{t \in \mathbb{R}_+} \mathrm{Supp}(g(t)).$$

Let $O \subset M$ be an open subset of M. Define $C^*_{alg}(P_d(\Gamma), \mathcal{A})_O$ to be the subalgebra of $C^*_{alg}(P_d(\Gamma), \mathcal{A})$ consisting of all elements whose supports are contained in $\Gamma_d \times \Gamma_d \times O$, i.e.,

$$C^*_{alg}(P_d(\Gamma), \mathcal{A})_O \ = \ \{T \in C^*_{alg}(P_d(\Gamma), \mathcal{A}) : \ \mathrm{Supp}(T(x, y)) \subset O, \ \forall \, x, y \in \Gamma_d\}.$$

Define $C^*(P_d(\Gamma), \mathcal{A})_O$ to be the norm closure of $C^*_{alg}(P_d(\Gamma), \mathcal{A})_O$. Similarly, let

$$C^*_{L,alg}(P_d(\Gamma), \mathcal{A})_O = \Big\{g \in C^*_{L,alg}(P_d(\Gamma), \mathcal{A}) : \ \mathrm{Supp}(g) \subset \Gamma_d \times \Gamma_d \times O\Big\}$$

and define $C^*_L(P_d(\Gamma), \mathcal{A})_O$ to be the norm closure of $C^*_{L,alg}(P_d(\Gamma), \mathcal{A})_O$ under the norm $\|g\|_\infty = \sup_{t \in \mathbb{R}_+} \|g(t)\|_{C^*(P_d(\Gamma), \mathcal{A})}$.

Note that $C^*(P_d(\Gamma), \mathcal{A})_O$ and $C^*_L(P_d(\Gamma), \mathcal{A})_O$ are closed two-sided ideals of $C^*(P_d(\Gamma), \mathcal{A})$ and $C^*_L(P_d(\Gamma), \mathcal{A})$, respectively. We also have an evaluation homomorphism $e : C^*_L(P_d(\Gamma), \mathcal{A})_O \to C^*(P_d(\Gamma), \mathcal{A})_O$ given by $e(g) = g(0)$.

Lemma 23. *For any two open subsets O_1, O_2 of M, we have*

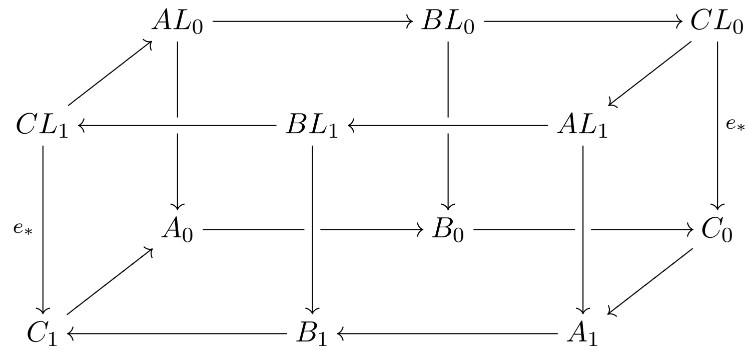
$$C^*(P_d(\Gamma), \mathcal{A})_{O_1} + C^*(P_d(\Gamma), \mathcal{A})_{O_2} = C^*(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2},$$

$$C^*(P_d(\Gamma), \mathcal{A})_{O_1} \cap C^*(P_d(\Gamma), \mathcal{A})_{O_2} = C^*(P_d(\Gamma), \mathcal{A})_{O_1 \cap O_2},$$

$$C^*_L(P_d(\Gamma), \mathcal{A})_{O_1} + C^*_L(P_d(\Gamma), \mathcal{A})_{O_2} = C^*_L(P_d(\Gamma), \mathcal{A})_{O_1 \cup O_2},$$

$$C_L^*(P_d(\Gamma),\mathcal{A})_{O_1} \cap C_L^*(P_d(\Gamma),\mathcal{A})_{O_2} = C_L^*(P_d(\Gamma),\mathcal{A})_{O_1\cap O_2}.$$

Consequently, we have the following commuting diagram connecting two Mayer-Vietoris sequences at K-Theory level:



where, for $* = 0, 1$,

$$AL_* = K_*\Big(C_L^*(P_d(\Gamma),\mathcal{A})_{O_1\cap O_2}\Big), \quad CL_* = K_*\Big(C_L^*(P_d(\Gamma),\mathcal{A})_{O_1\cup O_2}\Big),$$

$$A_* = K_*\Big(C^*(P_d(\Gamma),\mathcal{A})_{O_1\cap O_2}\Big), \qquad C_* = K_*\Big(C^*(P_d(\Gamma),\mathcal{A})_{O_1\cup O_2}\Big),$$

$$BL_* = K_*\Big(C_L^*(P_d(\Gamma),\mathcal{A})_{O_1}\Big)\bigoplus K_*\Big(C_L^*(P_d(\Gamma),\mathcal{A})_{O_2}\Big),$$

$$B_* = K_*\Big(C^*(P_d(\Gamma),\mathcal{A})_{O_1}\Big)\bigoplus K_*\Big(C^*(P_d(\Gamma),\mathcal{A})_{O_2}\Big).$$

Proof. We shall prove the first equality. Other equalities can be proved similarly. Then the two Mayer-Vietoris exact sequences follow from Lemma 2.4 of [HRY].

To prove the first equality, it suffices to show that

$$C_{alg}^*(P_d(\Gamma),\mathcal{A})_{O_1\cup O_2} \subseteq C_{alg}^*(P_d(\Gamma),\mathcal{A})_{O_1} + C_{alg}^*(P_d(\Gamma),\mathcal{A})_{O_2}.$$

Now suppose $T \in C_{alg}^*(P_d(\Gamma),\mathcal{A})_{O_1\cup O_2}$. Take a continuous partition of unity $\{\varphi_1,\varphi_2\}$ on $O_1\cup O_2$ subordinate to the open over $\{O_1,O_2\}$ of $O_1\cup O_2$. Define two functions

$$T_1,\ T_2 :\ \ \Gamma_d\times\Gamma_d \longrightarrow \mathcal{A}\otimes\mathcal{K}$$

by

$$T_1(x,y)(p) = \varphi_1(p)\Big(T(x,y)(p)\Big),$$

$$T_2(x,y)(p) = \varphi_2(p)\Big(T(x,y)(p)\Big)$$

for $x, y \in \Gamma_d$ and $p \in M$. Then $T_1 \in C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_1}$, $T_2 \in C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_2}$, and

$$T = T_1 + T_2 \in C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_1} + C^*_{alg}(P_d(\Gamma), \mathcal{A})_{O_2}$$

as desired. □

It would be convenient to introduce the following notion associated with the coarse embedding $f : \Gamma \to M$.

Definition 52. *Let $r > 0$. A family of open subsets $\{O_i\}_{i \in J}$ of M is said to be* (Γ, r)-separate *if*

1. *$O_i \cap O_j = \emptyset$ if $i \neq j$;*
2. *there exists $\gamma_i \in \Gamma$ such that $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each $i \in J$.*

Lemma 24. *If $\{O_i\}_{i \in J}$ is a family of (Γ, r)-separate open subsets of M, then*

$$e_* : \lim_{d \to \infty} K_*(C^*_L(P_d(\Gamma), \mathcal{A})_{\sqcup_{i \in J} O_i}) \to \lim_{d \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A})_{\sqcup_{i \in J} O_i})$$

is an isomorphism, where $\sqcup_{i \in J} O_i$ is the (disjoint) union of $\{O_i\}_{i \in J}$.

We will prove Lemma 24 in the next section. Granting Lemma 24 for the moment, we are able to prove Theorem 28.

Proof of Theorem 28. [Y00]. For any $r > 0$, we define $O_r \subset M$ by

$$O_r = \bigcup_{\gamma \in \Gamma} B(f(\gamma), r),$$

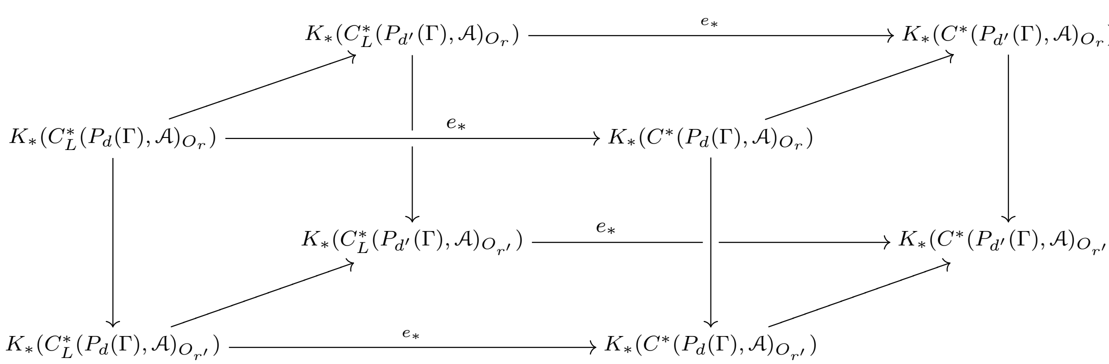
where $f : \Gamma \to M$ is the coarse embedding and $B(f(\gamma), r) = \{p \in M : d(p, f(\gamma)) < r\}$.

For any $d > 0$, if $r < r'$ then $C^*(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq C^*(P_d(\Gamma), \mathcal{A})_{O_{r'}}$ and $C_L^*(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq C_L^*(P_d(\Gamma), \mathcal{A})_{O_{r'}}$. By definition, we have

$$C^*(P_d(\Gamma), \mathcal{A}) = \lim_{r \to \infty} C^*(P_d(\Gamma), \mathcal{A})_{O_r},$$

$$C_L^*(P_d(\Gamma), \mathcal{A}) = \lim_{r \to \infty} C_L^*(P_d(\Gamma), \mathcal{A})_{O_r}.$$

On the other hand, for any $r > 0$, if $d < d'$ then $\Gamma_d \subseteq \Gamma_{d'}$ in $P_d(\Gamma) \subseteq P_{d'}(\Gamma)$ so that we have natural inclusions $C^*(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq C^*(P_{d'}(\Gamma), \mathcal{A})_{O_r}$ and $C_L^*(P_d(\Gamma), \mathcal{A})_{O_r} \subseteq C_L^*(P_{d'}(\Gamma), \mathcal{A})_{O_r}$. These inclusions induce the following commuting diagram



which allows us to change the order of limits from $\lim\limits_{d \to \infty} \lim\limits_{r \to \infty}$ to $\lim\limits_{r \to \infty} \lim\limits_{d \to \infty}$ in the second piece of the following commuting diagram

$$\begin{array}{ccc}
\lim\limits_{d \to \infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})) & \xrightarrow{e_*} & \lim\limits_{d \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A})) \\
\cong \downarrow & & \cong \downarrow \\
\lim\limits_{d \to \infty} \lim\limits_{r \to \infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})_{O_r}) & \xrightarrow{e_*} & \lim\limits_{d \to \infty} \lim\limits_{r \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A})_{O_r}) \\
\cong \downarrow & & \cong \downarrow \\
\lim\limits_{r \to \infty} \lim\limits_{d \to \infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})_{O_r}) & \xrightarrow{e_*} & \lim\limits_{r \to \infty} \lim\limits_{d \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A})_{O_r})
\end{array}$$

So, to prove Theorem 28, it suffices to show that, for any $r > 0$,

$$e_* : \lim_{d \to \infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})_{O_r}) \to \lim_{d \to \infty} K_*(C^*(P_d(\Gamma), \mathcal{A})_{O_r})$$

is an isomorphism.

Let $r > 0$. Since Γ has bounded geometry and $f : \Gamma \to M$ is a coarse embedding, there exist finitely many mutually disjoint subsets of Γ, say $\Gamma_k := \{\gamma_i : i \in J_k\}$ with some index set J_k for $k = 1, 2, \cdots, k_0$, such that $\Gamma = \bigsqcup_{k=1}^{k_0} \Gamma_k$ and, for each k, $d(f(\gamma_i), f(\gamma_j)) > 2r$ for distinct elements γ_i, γ_j in Γ_k.

For each $k = 1, 2, \cdots, k_0$, let

$$O_{r,k} = \bigcup_{i \in J_k} B(f(\gamma_i), r).$$

Then $O_r = \bigcup_{k=1}^{k_0} O_{r,k}$ and each $O_{r,k}$, or an intersection of several $O_{r,k}$, is the union of a family of (Γ, r)-separate (Definition 52) open subsets of M. Now Theorem 28 follows from Lemma 24 together with a Mayer-Vietoris sequence argument by using Lemma 23. □

VI.3 Strong Lipschitz homotopy invariance

In this section, we shall present Yu's arguments about strong Lipschitz homotopy invariance for K-theory of the twisted localization algebras [Y00], and prove Lemma 24 of the previous section.

Let $f : \Gamma \to M$ be a coarse embedding of a bounded geometry discrete metric space Γ into a simply connected complete Riemannian manifold M of nonpositive sectional curvature, and let $r > 0$. Let $\{O_i\}_{i \in J}$ be a family of (Γ, r)-separate open subsets of M, i.e., (1) $O_i \cap O_j = \emptyset$ if $i \neq j$; (2) there exists $\gamma_i \in \Gamma$ such that $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each $i \in J$. For $d > 0$, let X_i, $i \in J$, be a family of closed subsets of $P_d(\Gamma)$ such that $\gamma_i \in X_i$ for every $i \in J$ and $\{X_i\}_{i \in J}$ is uniformly bounded in the sense that there exists $r_0 > 0$ such that $diameter(X_i) \leq r_0$ for each $i \in J$. In particular, we will consider the following three cases of $\{X_i\}_{i \in J}$:

1. $X_i = B_{P_d(\Gamma)}(\gamma_i, R) := \{x \in P_d(\Gamma) : \ d(x, \gamma_i) \leq R\}$, for some common $R > 0$ for all $i \in J$;

2. $X_i = \Delta_i$, a simplex in $P_d(\Gamma)$ with $\gamma_i \in \Delta_i$ for each $i \in J$;

3. $X_i = \{\gamma_i\}$ for each $i \in J$.

For each $i \in J$, let $\mathcal{A}_{O_i}$ be the C^*-subalgebra of $\mathcal{A} = C_0(M, \mathrm{Cliff}(TM))$ generated by those functions whose supports are contained in O_i. We define

$$\begin{aligned} A^*(X_i : i \in J) &= \prod_{i\in J} C^*(X_i) \otimes \mathcal{A}_{O_i} \\ &= \left\{ \bigoplus_{i\in J} T_i \;\middle|\; T_i \in C^*(X_i) \otimes \mathcal{A}_{O_i},\ \sup_{i\in J} \|T_i\| < \infty \right\}. \end{aligned}$$

Similarly we define $A_L^*(X_i : i \in J)$ to be the C^*-subalgebra of

$$\left\{ \bigoplus_{i\in J} b_i \;\middle|\; b_i \in C_L^*(X_i) \otimes \mathcal{A}_{O_i},\ \sup_{i\in J} \|b_i\| < \infty \right\}$$

generated by elements $\bigoplus_{i\in J} b_i$ such that

1. the function
$$\bigoplus_{i\in J} b_i : \mathbb{R}_+ \to \prod_{i\in J} C^*(X_i) \otimes \mathcal{A}_{O_i}$$
is uniformly norm-continuous in $t \in \mathbb{R}_+$.

2. there exists a bounded function $c(t)$ on $\mathbb{R}_+$ with $\lim_{t\to\infty} c(t) = 0$ such that $(b_i(t))(x, y) = 0$ whenever $d(x, y) > c(t)$ for all $i \in J$, $x, y \in X_i$ and $t \in \mathbb{R}_+$.

For each natural number $s > 0$, let $\Delta_i(s)$ be the simplex with vertices $\{\gamma \in \Gamma : d(\gamma, \gamma_i) \leq s\}$ in $P_d(\Gamma)$ for $d > s$.

Lemma 25. *Let $O = \sqcup_{i\in J} O_i$ be the (disjoint) union of a family of (Γ, r)-separate open subsets $\{O_i\}_{i\in J}$ of M as above. Then*

1. $C^*(P_d(\Gamma), \mathcal{A})_O \cong \lim_{R\to\infty} A^*(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J)$;

2. $C_L^*(P_d(\Gamma), \mathcal{A})_O \cong \lim_{R\to\infty} A_L^*(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J)$;

3. $\lim_{d\to\infty} C^*(P_d(\Gamma), \mathcal{A})_O \cong \lim_{s\to\infty} A^*(\Delta_i(s) : i \in J)$;

4. $\lim_{d\to\infty} C_L^*(P_d(\Gamma), \mathcal{A})_O \cong \lim_{s\to\infty} A_L^*(\Delta_i(s) : i \in J)$.

Proof. [Y00]. Let $\mathcal{A}_O$ be the C^*-subalgebra of $\mathcal{A} = C_0(M, \mathrm{Cliff}(TM))$ generated by elements whose supports are contained in O. The support of an element $\sum_{x\in\Gamma_d} a_x[x]$ in the Hilbert module

$$E = \{ \sum_{x \in \Gamma_d} a_x[x] : \ a_x \in \mathcal{A} \otimes \mathcal{K}, \ \sum_{x \in \Gamma_d} a_x^* a_x \text{ converges in norm} \}$$

is defined to be

$$\{(x, p) \in \Gamma_d \times M : \ p \in \mathrm{Supp}(a_x)\}.$$

Let E_O be the closure of the set of all elements in E whose supports are contained in $\Gamma_d \times O$. Then E_O is a Hilbert module over $\mathcal{A}_O \otimes \mathcal{K}$ and $C^*(P_d(\Gamma), \mathcal{A})_O$ has a faithful representation on E_O. We have a decomposition

$$E_O = \bigoplus_{i \in J} E_{O_i}.$$

Each $T \in C^*_{alg}(P_d(\Gamma), \mathcal{A})_O$ has a corresponding decomposition

$$T = \bigoplus_{i \in J} T_i$$

such that there exists $R > 0$ for which each T_i is supported on

$$\{(x, y, p) : \ p \in O_i, \ x, y \in \Gamma_d, \ d(x, \gamma_i) \leq R, \ d(y, \gamma_i) \leq R\}.$$

On the other hand, the C^*-algebra $C^*(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\}) \otimes \mathcal{A}_{O_i}$ has a natural faithful representation on

$$\ell^2(\{x \in \Gamma_d : d(x, \gamma_i) \leq R\}) \otimes \mathcal{K} \otimes \mathcal{A}_{O_i}$$

so that on E_O, for each $R > 0$, the algebra $A^*(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J)$ can be represented as a subalgebra of $C^*(P_d(\Gamma), \mathcal{A})_O$. In this way, the decomposition $T = \oplus_{i \in J} T_i$ induces a $*$-isomorphism

$$C^*(P_d(\Gamma), \mathcal{A})_O \cong \lim_{R \to \infty} A^*(\{x \in P_d(\Gamma) : d(x, \gamma_i) \leq R\} : i \in J)$$

as desired in (1). Then (2),(3),(4) follows easily from (1). □

Now we turn to recall the notion of strong Lipschitz homotopy [Y97, Y98, Y00].

Let $\{Y_i\}_{i\in J}$ and $\{X_i\}_{i\in J}$ be two families of uniformly bounded closed subspaces of $P_d(\Gamma)$ for some $d > 0$ with $\gamma_i \in X_i$, $\gamma_i \in Y_i$ for every $i \in J$. A map $g : \bigsqcup_{i\in J} X_i \to \bigsqcup_{i\in J} Y_i$ is said to be *Lipschitz* if

1. $g(X_i) \subseteq Y_i$ for each $i \in J$;
2. there exists a constant c, independent of $i \in J$, such that

$$d(g(x), g(y)) \leq c\, d(x, y)$$

   for all $x, y \in X_i$, $i \in J$.

Let g_1, g_2 be two Lipschitz maps from $\bigsqcup_{i\in J} X_i$ to $\bigsqcup_{i\in J} Y_i$. We say g_1 is *strongly Lipschitz homotopy* equivalent to g_2 if there exists a continuous map

$$F : [0,1] \times (\sqcup_{i\in J} X_i) \to \sqcup_{i\in J} Y_i$$

such that

1. $F(0, x) = g_1(x)$, $F(1, x) = g_2(x)$ for all $x \in \sqcup_{i\in J} X_i$;
2. there exists a constant c for which $d(F(t,x), F(t,y)) \leq c\, d(x,y)$ for all $x, y \in X_i$, $t \in [0,1]$, where i is any element in J;
3. F is equicontinuous in t, i.e., for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(F(t_1, x), F(t_2, x)) < \varepsilon$ for all $x \in \sqcup_{i\in J} X_i$ if $|t_1 - t_2| < \delta$.

We say $\{X_i\}_{i\in J}$ is *strongly Lipschitz homotopy* equivalent to $\{Y_i\}_{i\in J}$ if there exist Lipschitz maps $g_1 : \sqcup_{i\in J} X_i \to \sqcup_{i\in J} Y_i$ and $g_2 : \sqcup_{i\in J} Y_i \to \sqcup_{i\in J} X_i$ such that $g_1 g_2$ and $g_2 g_1$ are respectively strongly Lipschitz homotopy equivalent to identity maps.

Define $A^*_{L,0}(X_i : i \in J)$ to be the C^*-subalgebra of $A^*_L(X_i : i \in J)$ consisting of elements $\oplus_{i\in J} b_i(t)$ satisfying $b_i(0) = 0$ for all $i \in J$.

Lemma 26 ([Y00]). *If $\{X_i\}_{i\in J}$ is strongly Lipschitz homotopy equivalent to $\{Y_i\}_{i\in J}$ then $K_*(A^*_{L,0}(X_i : i \in J)$ is isomorphic to $K_*(A^*_{L,0}(Y_i : i \in J))$.*

Let e be the evaluation homomorphism from $A^*_L(X_i : i \in J)$ to $A^*(X_i : i \in J)$ given by $\oplus_{i\in J}\, g_i(t) \mapsto \oplus_{i\in J} g_i(0)$.

Lemma 27 ([Y00]). *Let $\{\gamma_i\}_{i\in J}$ be as above, i.e., $O_i \subseteq B(f(\gamma_i), r) \subset M$ for each i. If $\{\Delta_i\}_{i\in J}$ is a family of simplices in $P_d(\Gamma)$ for some $d > 0$ such that $\gamma_i \in \Delta_i$ for all $i \in J$, then*

$$e_* : K_*(A^*_L(\Delta_i : i \in J)) \to K_*(A^*(\Delta_i : i \in J))$$

is an isomorphism.

Proof. ([Y00]) Note that $\{\Delta_i\}_{i\in J}$ is strongly Lipschitz homotopy equivalent to $\{\gamma_i\}_{i\in J}$. By an argument of Eilenberg swindle, we have $K_*(A^*_{L,0}(\{\gamma_i\} : i \in J)) = 0$. Consequently, Lemma 27 follows from Lemma 26 and the six term exact sequence of C^*-algebra K-theory. □

We are now ready to give a proof to Lemma 24 of the previous section.

Proof of Lemma 24. ([Y00]) By Lemma 25 we have the following commuting diagram

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} C^*_L(P_d(\Gamma), \mathcal{A})_{\sqcup_{i\in J} O_i} & \xrightarrow{e} & \lim\limits_{d\to\infty} C^*(P_d(\Gamma), \mathcal{A})_{\sqcup_{i\in J} O_i} \\
\cong \downarrow & & \downarrow \cong \\
\lim\limits_{s\to\infty} A^*_L(\Delta_i(s)_i :\ i \in J) & \xrightarrow{e} & \lim\limits_{s\to\infty} A^*(\Delta_i(s)_i :\ i \in J)
\end{array}$$

which induces the following commuting diagram at K-theory level

$$\begin{array}{ccc}
\lim\limits_{d\to\infty} K_*\Big(C^*_L(P_d(\Gamma), \mathcal{A})_{\sqcup_{i\in J} O_i}\Big) & \xrightarrow{e_*} & \lim\limits_{d\to\infty} K_*\Big(C^*(P_d(\Gamma), \mathcal{A})_{\sqcup_{i\in J} O_i}\Big) \\
\cong \downarrow & & \downarrow \cong \\
\lim\limits_{s\to\infty} K_*\Big(A^*_L(\Delta_i(s) :\ i \in J)\Big) & \xrightarrow{e_*} & \lim\limits_{s\to\infty} K_*\Big(A^*(\Delta_i(s) :\ i \in J)\Big).
\end{array}$$

Now Lemma 24 follows from Lemma 27. □

VI.4 Almost flat Bott elements and Bott maps

In this section, we shall construct uniformly almost flat Bott generators for a simply connected complete Riemannian manifold with nonpositive sectional curvature, and define a Bott map from the K-theory of the Roe algebra to the K-theory of the twisted Roe algebra and another Bott map between the K-theory of corresponding localization algebras. We show that the Bott map from the K-theory of the localization algebra to the K-theory of the twisted localization algebra is an isomorphism (Theorem 29).

Let M be a simply connected complete Riemannian manifold with nonpositive sectional curvature. As remarked at the beginning of Section VI.2, without loss of generality, we assume in the following $\dim(M) = 2n$ for some integer $n > 0$.

Recall that $\mathcal{A} := C_0(M, \mathrm{Cliff}(TM))$ is the C^*-algebra of all continuous functions a on M, with values $a(x) \in \mathrm{Cliff}(T_xM)$ for every $x \in M$, such that $\lim_{x\to\infty} a(x) = 0$, where $\mathrm{Cliff}(T_xM)$ denotes the complexified Clifford algebra of the tangent space T_xM with respect to the inner product on T_xM given by the Riemannian structure on M. Since $\dim M = 2n$, the exponential map

$$\exp_x : \ T_xM \cong \mathbb{R}^{2n} \to M$$

at any point $x \in M$ induces an isomorphism

$$C_0(M, \mathrm{Cliff}(TM)) \cong C_0(\mathbb{R}^{2n}) \otimes \mathcal{M}_{2^n}(\mathbb{C}).$$

Similarly, we define $\mathcal{B} := C_b(M, \mathrm{Cliff}(TM))$ to be the C^*-algebra of all bounded functions a on M with $a(x) \in \mathrm{Cliff}(T_xM)$ at all $x \in M$.

Let $x \in M$. For any $z \in M$, let $\sigma : [0, 1] \to M$ be the unique geodesic such that

$$\sigma(0) = x, \qquad \sigma(1) = z.$$

Let $v_x(z) := \frac{\sigma'(1)}{\|\sigma'(1)\|} \in T_zM$. For any $c > 0$, take a continuous function $\phi_{x,c} : M \to [0, 1]$ satisfying

$$\phi_{x,c}(z) = \begin{cases} 0, & \text{if } d(x, z) \le \frac{c}{2}; \\ 1, & \text{if } d(x, z) \ge c. \end{cases} \tag{1}$$

For any $z \in M$, let

$$f_{x,c}(z) := \phi_{x,c}(z) \cdot v_x(z) \in T_z M.$$

Then $f_{x,c} \in C_b(M, \mathrm{Cliff}(TM))$. The following result describes certain "uniform almost flatness" of the functions $f_{x,c}$ ($x \in M$, $c > 0$).

Lemma 28. *For any $R > 0$ and $\varepsilon > 0$, there exist a constant $c > 0$ and a family of continuous function $\{\phi_{x,c}\}_{x \in M}$ satisfying the above condition (1) such that, if $d(x, y) < R$, then*

$$\sup_{z \in M} \|f_{x,c}(z) - f_{y,c}(z)\|_{T_z M} < \varepsilon.$$

Proof. Let $c = \frac{2R}{\varepsilon}$. For any $x \in M$, define $\phi_{x,c} : M \to [0, 1]$ by

$$\phi_{x,c}(z) = \begin{cases} 0, & \text{if } d(x,z) \le \frac{R}{\varepsilon}; \\ \frac{\varepsilon}{R} d(x,z) - 1, & \text{if } \frac{R}{\varepsilon} \le d(x,z) \le \frac{2R}{\varepsilon}; \\ 1, & \text{if } d(x,z) \ge \frac{2R}{\varepsilon}. \end{cases}$$

Let $x, y \in M$ such that $d(x, y) < R$. Then we have several cases for the position of $z \in M$ with respect to x, y.

Consider the case where $d(x, z) > c = \frac{2R}{\varepsilon}$ and $d(y, z) > c = \frac{2R}{\varepsilon}$. Since $\phi_{x,c}(z) = \phi_{y,c}(z) = 1$, we have

$$f_{x,c}(z) - f_{y,c}(z) = v_x(z) - v_y(z).$$

Without loss of generality, assume $d(x, z) \le d(y, z)$. Then there exists a unique point y' on the unique geodesic connecting y and z such that $d(y', z) = d(x, z)$. Then $d(y', y) < R$ since $d(x, y) < R$, so that $d(x, y') < 2R$.

Let $\exp_z^{-1} : M \to T_z M$ denote the inverse of the exponential map

$$\exp_z : T_z M \to M$$

at $z \in M$. Then we have

1. $\|\exp_z^{-1}(x)\| = d(x, z) = d(y', z) = \|\exp_z^{-1}(y')\| > c = \frac{2R}{\varepsilon}$;

2. $\| \exp_z^{-1}(x) - \exp_z^{-1}(y') \| \leq d(x, y') < 2R$, since M has nonpositive sectional curvature;

3. $v_x(z) = -\frac{\exp_z^{-1}(x)}{\|\exp_z^{-1}(x)\|}$ and $v_y(z) = -\frac{\exp_z^{-1}(y')}{\|\exp_z^{-1}(y')\|}$.

Hence, for any $z \in M$, we have

$$\|f_{x,c}(z) - f_{y,c}(z)\| = \|v_x(z) - v_y(z)\| < 2R/(2R/\varepsilon) = \varepsilon$$

whenever $d(x, y) < R$. Similarly, we can check the inequality in other cases where $z \in M$ satisfies either $d(x, z) \leq c$ or $d(y, z) \leq c$. □

Now let's consider the short exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \xrightarrow{\pi} \mathcal{B}/\mathcal{A} \longrightarrow 0,$$

where $\mathcal{A} = C_0(M, \mathrm{Cliff}(TM))$ and $\mathcal{B} = C_b(M, \mathrm{Cliff}(TM))$. For any $f_{x,c}$ ($x \in M$, $c > 0$) constructed above, it is easy to see that $[f_{x,c}] := \pi(f_{x,c})$ is invertible in $\mathcal{B}/\mathcal{A}$ with its inverse $[-f_{x,c}]$. Thus $[f_{x,c}]$ defines an element in $K_1(\mathcal{B}/\mathcal{A})$. With the help of the index map

$$\partial : K_1(\mathcal{B}/\mathcal{A}) \to K_0(\mathcal{A}),$$

we obtain an element $\partial([f_{x,c}])$ in

$$K_0(\mathcal{A}) = K_0\Big(C_0(M, \mathrm{Cliff}(TM))\Big) \cong K_0\Big(C_0(\mathbb{R}^{2n}) \otimes \mathcal{M}_{2^n}(\mathbb{C})\Big) \cong \mathbb{Z}.$$

It follows from the construction of $f_{x,c}$ that, for every $x \in M$ and $c > 0$, $\partial([f_{x,c}])$ is just the Bott generator of $K_0(\mathcal{A})$.

The element $\partial([f_{x,c}])$ can be expressed explicitly as follows. Let

$$W_{x,c} = \begin{pmatrix} 1 & f_{x,c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f_{x,c} & 1 \end{pmatrix} \begin{pmatrix} 1 & f_{x,c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$b_{x,c} = W_{x,c} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W_{x,c}^{-1} ,$$

$$b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

Then both $b_{x,c}$ and b_0 are idempotents in $\mathcal{M}_2(\mathcal{A}^+)$, where $\mathcal{A}^+$ is the algebra jointing a unit to $\mathcal{A}$. It is easy to check that

$$b_{x,c} - b_0 \in C_c(M, \mathrm{Cliff}(TM)) \otimes \mathcal{M}_2(\mathbb{C}),$$

the algebra of 2×2 matrices of compactly supported continuous functions, with

$$\mathrm{Supp}(b_{x,c} - b_0) \subset B_M(x, c) := \{z \in M : d(x, z) \leq c\},$$

where for a matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ of functions on M we define the support of a by

$$\mathrm{Supp}(a) = \bigcup_{i,j=1}^{2} \mathrm{Supp}(a_{i,j}).$$

Now we have the explicit expression

$$\partial([f_{x,c}]) = [b_{x,c}] - [b_0] \in K_0(\mathcal{A}).$$

Lemma 29 (Uniform almost flatness of the Bott generators)**.** *The family of idempotents* $\{b_{x,c}\}_{x \in M, c>0}$ *in* $\mathcal{M}_2(\mathcal{A}^+) = C_0(M, \mathit{Cliff}(TM))^+ \otimes \mathcal{M}_2(\mathbb{C})$ *constructed above are uniformly almost flat in the following sense:*

for any $R > 0$ *and* $\varepsilon > 0$*, there exist* $c > 0$ *and a family of continuous functions* $\left\{\phi_{x,c} : \ M \to [0,1]\right\}_{x \in M}$ *such that, whenever* $d(x,y) < R$*, we have*

$$\sup_{z \in M} \|b_{x,c}(z) - b_{y,c}(z)\|_{\mathit{Cliff}(T_zM) \otimes \mathcal{M}_2(\mathbb{C})} < \varepsilon,$$

where $b_{x,c}$ *is defined via* $W_{x,c}$ *and* $f_{x,c} = \phi_{x,c} v_x$ *as above, and* $Cliff(T_zM)$ *is the complexified Clifford algebra of the tangent space* T_zM*.*

Proof. Straightforward from Lemma 28. ☐

It would be convenient to introduce the following notion:

Definition 53. *For* $R > 0, \varepsilon > 0, c > 0,$ *a family of idempotents* $\{b_x\}_{x\in M}$ *in* $\mathcal{M}_2(\mathcal{A}^+) = C_0(M, Cliff(TM))^+ \otimes \mathcal{M}_2(\mathbb{C})$ *is said to be* $(R, \varepsilon; c)$*-flat if*

1. *for any* $x, y \in M$ *with* $d(x, y) < R$ *we have*

$$\sup_{z\in M} \|b_x(z) - b_y(z)\|_{Cliff(T_zM)\otimes\mathcal{M}_2(\mathbb{C})} < \varepsilon.$$

2. $b_x - b_0 \in C_c(M, Cliff(TM)) \otimes \mathcal{M}_2(\mathbb{C})$ *and*

$$\mathrm{Supp}(b_x - b_0) \subset B_M(x, c) := \{z \in M : d(x, z) \le c\}.$$

Construction of the Bott map β_***:**

Now we shall use the above almost flat Bott generators for

$$K_0(\mathcal{A}) = K_0\Big(C_0(M, \mathrm{Cliff}(TM))\Big)$$

to construct a "Bott map"

$$\beta_* : \ K_*(C^*(P_d(\Gamma))) \to K_*(C^*(P_d(\Gamma), \mathcal{A})).$$

To begin with, we give a representation of $C^*(P_d(\Gamma))$ on $\ell^2(\Gamma_d) \otimes H_0$, where Γ_d is the countable dense subset of $P_d(\Gamma)$ and H_0 is the Hilbert space as in the definition of $C^*(P_d(\Gamma), \mathcal{A})$.

Let $C^*_{alg}(P_d(\Gamma))$ be the algebra of functions

$$Q : \ \Gamma_d \times \Gamma_d \to \mathcal{K}(H_0)$$

such that

1. there exists $C > 0$ such that $\|Q(x,y)\| \leq C$ for all $x, y \in \Gamma_d$;

2. there exists $R > 0$ such that $Q(x,y) = 0$ whenever $d(x,y) > R$;

3. there exists $L > 0$ such that for every $z \in P_d(\Gamma)$, the number of elements in the following set

$$\{(x,y) \in \Gamma_d \times \Gamma_d : \ d(x,z) \leq 3R, \ d(y,z) \leq 3R, \ Q(x,y) \neq 0\}$$

is less than L.

The product structure on $C^*_{alg}(P_d(\Gamma))$ is defined by

$$(Q_1 Q_2)(x,y) = \sum_{z \in \Gamma_d} Q_1(x,z) Q_2(z,y).$$

The algebra $C^*_{alg}(P_d(\Gamma))$ has a $*$-representation on $\ell^2(\Gamma_d) \otimes H_0$. The operator norm completion of $C^*_{alg}(P_d(\Gamma))$ with respect to this $*$-representation is $*$-isomorphic to $C^*(P_d(\Gamma))$ when Γ has bounded geometry.

Note that $C^*(P_d(\Gamma))$ is stable in the sense that $C^*(P_d(\Gamma)) \cong C^*(P_d(\Gamma)) \otimes \mathcal{M}_k(\mathbb{C})$ for all natural number k. Any element in $K_0(C^*(P_d(\Gamma)))$ can be expressed as the difference of the K_0-classes of two idempotents in $C^*(P_d(\Gamma))$. To define the Bott map $\beta_* : \ K_0(C^*(P_d(\Gamma))) \to K_0(C^*(P_d(\Gamma), \mathcal{A}))$, we need to specify the value $\beta_*([P])$ in $K_0(C^*(P_d(\Gamma), \mathcal{A}))$ for any idempotent $P \in C^*(P_d(\Gamma))$.

Now let $P \in C^*(P_d(\Gamma)) \subseteq \mathcal{B}(\ell^2(\Gamma_d) \otimes H_0)$ be an idempotent. For any $0 < \varepsilon_1 < 1/100$, take an element $Q \in C^*_{alg}(P_d(\Gamma))$ such that

$$\|P - Q\| < \varepsilon_1.$$

Then $\|Q - Q^2\| < 4\varepsilon_1$ and there is $R_{\varepsilon_1} > 0$ such that $Q(x,y) = 0$ whenever $d(x,y) > R_{\varepsilon_1}$. For any $\varepsilon_2 > 0$, take by Lemma 29 a family of $(R_{\varepsilon_1}, \varepsilon_2; c)$-flat idempotents $\{b_x\}_{x \in M}$ in

$\mathcal{M}_2(\mathcal{A}^+)$ for some $c > 0$. Define

$$\widetilde{Q},\ \widetilde{Q}_0 :\ \Gamma_d \times \Gamma_d \to \mathcal{A}^+ \otimes \mathcal{K} \otimes \mathcal{M}_2(\mathbb{C})$$

by

$$\widetilde{Q}(x, y) = Q(x, y) \otimes b_x$$

and

$$\widetilde{Q}_0(x, y) = Q(x, y) \otimes b_0,$$

respectively, for all $(x, y) \in \Gamma_d \times \Gamma_d$, where $b_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then

$$\widetilde{Q}, \widetilde{Q}_0 \in C^*_{alg}(P_d(\Gamma), \mathcal{A}^+ \otimes \mathcal{M}_2(\mathbb{C})) \cong C^*_{alg}(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$$

and

$$\widetilde{Q} - \widetilde{Q}_0 \in C^*_{alg}(P_d(\Gamma), \mathcal{A}) \otimes \mathcal{M}_2(\mathbb{C}).$$

Since Γ has bounded geometry, by the almost flatness of the Bott generators (Lemma 29), we can choose ε_1 and ε_2 small enough to obtain $\widetilde{Q}, \widetilde{Q}_0$ as constructed above such that $\|\widetilde{Q}^2 - \widetilde{Q}\| < 1/5$ and $\|\widetilde{Q}_0^2 - \widetilde{Q}_0\| < 1/5$.

It follows that the spectrum of either $\widetilde{Q}$ or $\widetilde{Q}_0$ is contained in disjoint neighborhoods S_0 of 0 and S_1 of 1 in the complex plane. Let $f : S_0 \sqcup S_1 \to \mathbb{C}$ be the function such that $f(S_0) = \{0\}, f(S_1) = \{1\}$. Let $\Theta = f(\widetilde{Q})$ and $\Theta_0 = f(\widetilde{Q}_0)$. Then Θ and Θ_0 are idempotents in $C^*(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$ with

$$\Theta - \Theta_0 \in C^*(P_d(\Gamma), \mathcal{A}) \otimes \mathcal{M}_2(\mathbb{C}).$$

Note that $C^*(P_d(\Gamma), \mathcal{A}) \otimes \mathcal{M}_2(\mathbb{C})$ is a closed two-sided ideal of $C^*(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$. At this point we need to recall the *difference construction* in K-theory of Banach algebras introduced by Kasparov-Yu [KY]. Let J be a closed two-sided ideal of a Banach algebra B. Let $p, q \in B^+$ be idempotents such that $p - q \in J$. Then a difference element

$D(p,q) \in K_0(J)$ associated to the pair p, q is defined as follows. Let

$$Z(p,q) = \begin{pmatrix} q & 0 & 1-q & 0 \\ 1-q & 0 & 0 & q \\ 0 & 0 & q & 1-q \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{M}_4(B^+).$$

We have

$$(Z(p,q))^{-1} = \begin{pmatrix} q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1-q & 0 & q & 0 \\ 0 & q & 1-q & 0 \end{pmatrix} \in \mathcal{M}_4(B^+).$$

Define

$$D_0(p,q) = (Z(p,q))^{-1} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Z(p,q).$$

Let

$$p_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $D_0(p,q) \in \mathcal{M}_4(J^+)$ and $D_0(p,q) = p_1$ modulo $\mathcal{M}_4(J)$. We define the difference element

$$D(p,q) := [D_0(p,q)] - [p_1]$$

in $K_0(J)$.

Finally, for any idempotent $P \in C^*(P_d(\Gamma))$ representing an element $[P]$ in $K_0(C^*(P_d(\Gamma)))$, we define

$$\beta_*([P]) = D(\Theta, \Theta_0) \in K_0(C^*(P_d(\Gamma), \mathcal{A})),$$

The correspondence $[P] \to \beta_*([P])$ extends to a homomorphism, the Bott map

$$\beta_* : \ K_0(C^*(P_d(\Gamma))) \to K_0(C^*(P_d(\Gamma), \mathcal{A})).$$

By using suspension, we similarly define the Bott map

$$\beta_* : \ K_1(C^*(P_d(\Gamma))) \to K_1(C^*(P_d(\Gamma), \mathcal{A})).$$

Construction of the Bott map $(\beta_L)_*$:

Next we shall construct a Bott map for K-theory of localization algebras:

$$(\beta_L)_* : \ K_*(C_L^*(P_d(\Gamma))) \to K_*(C_L^*(P_d(\Gamma), \mathcal{A})).$$

Let $C^*_{L,alg}(P_d(\Gamma))$ be the $*$-algebra of all bounded, uniformly continuous functions

$$g : \ \mathbb{R}_+ \to C^*_{alg}(P_d(\Gamma)) \subset \mathcal{B}(\ell^2(\Gamma_d) \otimes H_0)$$

with the following properties:

1. there exists a bounded function $R : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} R(t) = 0$ such that $g(t)(x,y) = 0$ whenever $d(x,y) > R(t)$ for every t;
2. there exists $L > 0$ such that for every $z \in P_d(\Gamma)$, the number of elements in the following set

$$\{(x,y) \in \Gamma_d \times \Gamma_d : \ d(x,z) \le 3R, \ d(y,z) \le 3R, \ g(t)(x,y) \neq 0\}$$

   is less than L for every $t \in \mathbb{R}_+$.

The localization algebra $C_L^*(P_d(\Gamma))$ is $*$-isomorphic to the norm completion of $C^*_{L,alg}(P_d(\Gamma))$ under the norm

$$\|g\|_\infty := \sup_{t\in\mathbb{R}_+} \|g(t)\|$$

when Γ has bounded geometry. Note that $C_L^*(P_d(\Gamma))$ is stable in the sense that

$C_L^*(P_d(\Gamma)) \cong C_L^*(P_d(\Gamma)) \otimes \mathcal{M}_k(\mathbb{C})$ for all natural number k. Hence, any element in $K_0(C_L^*(P_d(\Gamma)))$ can be expressed as the difference of the K_0-classes of two idempotents in $C_L^*(P_d(\Gamma))$. To define the Bott map $(\beta_L)_* :\ K_0(C_L^*(P_d(\Gamma))) \to K_0(C_L^*(P_d(\Gamma), \mathcal{A}))$, we need to specify the value $(\beta_L)_*([g])$ in $K_0(C_L^*(P_d(\Gamma), \mathcal{A}))$ for any idempotent $g \in C_L^*(P_d(\Gamma))$ representing an element $[g] \in K_0(C_L^*(P_d(\Gamma)))$.

Now let $g \in C_L^*(P_d(\Gamma))$ be an idempotent. For any $0 < \varepsilon_1 < 1/100$, take an element $h \in C_{L,alg}^*(P_d(\Gamma))$ such that

$$\|g - h\|_\infty < \varepsilon_1.$$

Then $\|h - h^2\| < 4\varepsilon_1$ and there is a bounded function $R_{\varepsilon_1}(t) > 0$ with $\lim_{t\to\infty} R_{\varepsilon_1}(t) = 0$ such that $h(t)(x, y) = 0$ whenever $d(x, y) > R_{\varepsilon_1}(t)$ for every t. Let $\widetilde{R}_{\varepsilon_1} = \sup_{t\in\mathbb{R}_+} R(t)$. For any $\varepsilon_2 > 0$, take by Lemma 29 a family of $(\widetilde{R}_{\varepsilon_1}, \varepsilon_2; c)$-flat idempotents $\{b_x\}_{x\in M}$ in $\mathcal{M}_2(\mathcal{A}^+)$ for some $c > 0$. Define

$$\widetilde{h},\ \widetilde{h}_0 :\ \mathbb{R}_+ \to C_{alg}^*(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$$

by

$$\Big(\widetilde{h}(t)\Big)(x, y) = \Big(h(t)(x, y)\Big) \otimes b_x \in \mathcal{A}^+ \otimes \mathcal{K} \otimes \mathcal{M}_2(\mathbb{C}),$$

$$\Big(\widetilde{h}_0(t)\Big)(x, y) = \Big(h(t)(x, y)\Big) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}^+ \otimes \mathcal{K} \otimes \mathcal{M}_2(\mathbb{C})$$

for each $t \in \mathbb{R}_+$. Then we have

$$\widetilde{h},\ \widetilde{h}_0\ \in\ C_{L,alg}^*(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$$

and

$$\widetilde{h} - \widetilde{h}_0\ \in\ C_{L,alg}^*(P_d(\Gamma), \mathcal{A}) \otimes \mathcal{M}_2(\mathbb{C}).$$

Since Γ has bounded geometry, by the almost flatness of the Bott generators, we can choose ε_1 and ε_2 small enough to obtain $\widetilde{h}, \widetilde{h}_0$, as constructed above, such that $\|\widetilde{h}^2 - \widetilde{h}\|_\infty < 1/5$ and $\|\widetilde{h}_0^2 - \widetilde{h}_0\| < 1/5$. The spectrum of either $\widetilde{h}$ or $\widetilde{h}_0$ is contained in disjoint neighborhoods S_0 of 0 and S_1 of 1 in the complex plane. Let $f : S_0 \sqcup S_1 \to \mathbb{C}$ be the function such that $f(S_0) = \{0\}, f(S_1) = \{1\}$. Let $\eta = f(\widetilde{h})$ and $\eta_0 = f(\widetilde{h}_0)$. Then η

and η_0 are idempotents in $C_L^*(P_d(\Gamma), \mathcal{A}^+) \otimes \mathcal{M}_2(\mathbb{C})$ with

$$\eta - \eta_0 \in C_L^*(P_d(\Gamma), \mathcal{A}) \otimes \mathcal{M}_2(\mathbb{C}).$$

Thanks to the difference construction, we define

$$(\beta_L)_*([g]) = D(\eta, \eta_0) \in K_0(C_L^*(P_d(\Gamma), \mathcal{A})).$$

This correspondence $[g] \mapsto (\beta_L)_*([g])$ extends to a homomorphism, the Bott map

$$(\beta_L)_* : \; K_0(C_L^*(P_d(\Gamma))) \to K_0(C_L^*(P_d(\Gamma), \mathcal{A})).$$

By suspension, we similarly define

$$(\beta_L)_* : \; K_1(C_L^*(P_d(\Gamma))) \to K_1(C_L^*(P_d(\Gamma), \mathcal{A})).$$

This completes the construction of the Bott map $(\beta_L)_*$.

It follows from the constructions of β_* and $(\beta_L)_*$, we have the following commuting diagram

$$\begin{array}{ccc} K_*(C_L^*(P_d(\Gamma))) & \xrightarrow{(\beta_L)_*} & K_*(C_L^*(P_d(\Gamma), \mathcal{A})) \\ \big\downarrow{\scriptstyle e_*} & & \big\downarrow{\scriptstyle e_*} \\ K_*(C^*(P_d(\Gamma))) & \xrightarrow{\beta_*} & K_*(C^*(P_d(\Gamma), \mathcal{A})) \end{array}$$

Theorem 29. *For any $d > 0$, the Bott map*

$$(\beta_L)_* : \; K_*(C_L^*(P_d(\Gamma))) \to K_*(C_L^*(P_d(\Gamma), \mathcal{A}))$$

is an isomorphism.

Proof. Note that Γ has bounded geometry, and both the localization algebra and the twisted localization algebra have strong Lipschitz homotopy invariance at the K-theory level. By a Mayer-Vietoris sequence argument and induction on the dimension of the skeletons [Y97, CW02], the general case can be reduced to the 0-dimensional case, i.e., if

$D \subset P_d(\Gamma)$ is a δ-separated subspace (meaning $d(x, y) \geq \delta$ if $x \neq y \in D$) for some $\delta > 0$, then

$$(\beta_L)_* : \ K_*(C_L^*(D)) \to K_*(C_L^*(D, \mathcal{A}))$$

is an isomorphism. But this follows from the facts that

$$K_*(C_L^*(D)) \cong \prod_{\gamma \in D} K_*(C_L^*(\{\gamma\})),$$

$$K_*(C_L^*(D, \mathcal{A})) \cong \prod_{\gamma \in D} K_*(C_L^*(\{\gamma\}, \mathcal{A}))$$

and that $(\beta_L)_*$ restricts to an isomorphism from $K_*(C_L^*(\{\gamma\})) \cong K_*(\mathcal{K})$ to

$$K_*(C_L^*(\{\gamma\}, \mathcal{A})) \cong K_*(\mathcal{K} \otimes \mathcal{A})$$

at each $\gamma \in D$ by the classic Bott periodicity. □

VI.5 Proof of the Main Theorem

Proof of Theorem 26. We have the commuting diagram

$$\begin{array}{ccccc} & & \lim\limits_{d\to\infty} K_*(C_L^*(P_d(\Gamma))) & \xrightarrow[\cong]{(\beta_L)_*} & \lim\limits_{d\to\infty} K_*(C_L^*(P_d(\Gamma), \mathcal{A})) \\ & \overset{ind_L}{\underset{\cong}{\nearrow}} & e_* \Big\downarrow & & \cong \Big\downarrow e_* \\ \lim\limits_{d\to\infty} K_*(P_d(\Gamma)) & \xrightarrow{ind} & \lim\limits_{d\to\infty} K_*(C^*(P_d(\Gamma))) & \xrightarrow{\beta_*} & \lim\limits_{d\to\infty} K_*(C^*(P_d(\Gamma), \mathcal{A})). \end{array}$$

Hence, $\beta_* \circ ind = e_* \circ (\beta_L)_* \circ ind_L$. It follows from Theorem 27, Theorem 28 and Theorem 29 that $\beta_* \circ ind$ is an isomorphism. Consequently, the index map

$$ind : \lim_{d\to\infty} K_*(P_d(\Gamma)) \to \lim_{d\to\infty} K_*(C^*(P_d(\Gamma))) \cong K_*(C^*(\Gamma))$$

is injective. □

BIBLIOGRAPHY

[A76] Atiyah, M. F., *Elliptic operators, discrete groups and von Neumann algebras*, Asterisque **32**(1976) 43–72.

[A] Atiyah, M. F., *Bott periodicity and the index of elliptic operators*, Q. J. Math., **19**(1968), 113-140

[ABS] Atiyah, M. F., Bott, R., and Shapiro, A., *Clifford modules*, Topology, **3**, Suppl. 1, (1964), 3–38.

[AS1] Atiyah, M. F. and Singer, I. M., *The index of elliptic operators. I*, Ann. of Math. (2) **87**(1968) 484–530.

[AS3] Atiyah, M. F. and Singer, I. M., *The index of elliptic operators. III* . Ann. of Math. (2) **87**(1968) 546–604.

[BC] Baum, P. and Connes, A., *K-theory for discrete groups*, in *Operator Algebras and Applications,* Evens and Takesaki, editors, Cambridge University Press (1989), 1-20.

[BCH] Baum, P., Connes, A. and Higson, N., *Classifying space for proper actions and K-theory of group C^*-algebras*, C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 240–291, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994.

[B] Blackadar, B., *K-Theory for Operator Algebras*, (2nd edition), Cambridge Univ. Press, 1998.

[CE] Cheeger, J. and Ebin, D. G., *Comparison theorems in Riemannian geometry*, North-Holland Publishing Company, Amsterdam, 1975.

[CW02] Chen, X. and Wang, Q., *Localization algebras and duality*, J. London Math. Soc., **66**(2)(2002) 227–239.

[CDSS] Coburn, L., Douglas, R., Schaeffer, D., Singer, I., *C^*-algebras of operators on a half plane, II: Index theory*, Publ. I.H.E.S. **40**(1971) 69–80.

[C86] Connes, A., *Cyclic cohomology and the transverse fundamental class of a foliation.* Geometric methods in operator algebras (Kyoto, 1983), 52–144, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.

[C94] Connes, A., *Noncommutative geometry*. Academic Press, Inc., San Diego, CA, 1994.

[CM82] Connes, A. and Moscovici, H., *The L^2-index theorem for homogeneous spaces of Lie groups*, Ann. Math. (2) **115**(1982) 291–330.

[CM] Connes, A. and Moscovici, H., *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology 29 (1990), 345–388.

[CS] Connes, A. and Skandalies, G., *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Soc. **20**(1984) 1139–1183.

[Dou72] Douglas, R., *Banach Algebra Techniques in Operator Theory*, Academic Press, New Yor, San Francisco, London, 1972.

[D] Dranishnikov, A. N., *On hypersphericity of manifolds with finite asymptotic dimension*, Trans. Amer. Math. Soc. **355**(1)(2003), 155–167.

[FRR] S. Ferry; A. Ranicki and J. Rosenberg, *A history and survey of the Novikov conjecture*. Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), 7–66, London Math. Soc. Lecture Note Ser., 226, Cambridge Univ. Press, Cambridge, 1995.

[G] Gromov, M., *Asymptotic invariants for infinite groups*, Vol 2, Proc. 1991 Sussex Conference on Geometry Group Theory, LMS Lecture Note Ser. 182, Academic Press, New York, 1993.

[HR] Higson, N. and Roe, J., *On the coarse Baum-Connes conjecture*, in: S. Ferry, A. Ranicki and J. Rosenberg (eds), Proc. 1993 Oberwolfach Conference in the Novikov Conjecture, London Math. Soc. Lecture Note Series 227, Cambridge University Press, 1995, PP. 227-254.

[HR00] Higson, N. and Roe, J., *Analytic K-homology*, Oxford Mathematical Monographs. Oxford Science Publications. Oxford University Press, Oxford, 2000.

[HRY] Higson, N., Roe, J. and Yu, G., *A coarse Mayer-Vietoris principle*, Math. Proc. Camb. Phil. Soc., **114**(1993) 85–97.

[KAS75] Kasparov, G. G., *Topological invariants of elliptic operators I: K-homology.* Math. USSR Izvestija, **9** (1975) 751–792.

[KAS88] Kasparov, G., *Equivariant KK-theory and the Novikov conjecture*, Invent. Math., **91** (1988)147–201.

[KY] Kasparov, G. and Yu, G., *The coarse geometric Novikov conjecture and uniform convexity*, to appear in Advances in Mathematics, 2006.

[LLR] Larsen, F., Laustsen, N. and Rordam, M., *An Introduction to K-Theory for C*-Algebras* London Mathematical Society Student Texts, 49. Cambridge University Press, Cambridge, 2000.

[LM] Lawson, H. B. and Michelsohn, M. L., *Spin Geometry*, Princeton, 1990.

[M] Mishchenko, A. S., *Infinite-dimensional representations of discrete groups, and higher signatures (Russian)*, Izv. Akad. Nauk SSSR Ser. Mat., **38**(1974), 81–106.

[MS] Mishchenko, A. S. and Fomenko, A. T., *The index of elliptic operators over C^*-algebras*, Izv. Akad. Nauk. SSSR, Ser. Mat. **43**(1979), 831–859.

[Mur] Murphy, G. J., *C^*-algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990.

[R88] Roe, J., *An index theorem on open manifold*, J. Differential Geom. **27**(1988) 87–113.

[R93] Roe, J., *Coarse cohomology and index theory on complete Riemannian manifolds*, Mem. Amer. Math. Soc. **104** (1993), no. 497, x+90 pp.

[R96] Roe, J., *Index theory, coarse geometry, and the topology of manifolds*, CBMS Conference Proceedings 90, American Mathematical Society, Providence, R.I., 1996.

[S] Shan, L., *An equivariant higher index theory and nonpositively curved manifolds*, Preprint, 2006.

[T] Taylor, J. L., *Banach algebras and topology*, Algebras in analysis (Proc. Instructional Conf. and NATO Advanced Study Inst., Birmingham, 1973), pp. 118–186. Academic Press, London, 1975.

[W] Wegge-Olsen, N. E., *K-Theory and C*-algebras*, A friendly approach. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[Y95] Yu, G., *Coarse Baum-Connes conjecture*, K-Theory, **9**(3)(1995) 199–221.

[Y97] Yu, G., *Localization algebras and the coarse Baum-Connes conjecture*, K-theory **11**(1997) 307–318.

[Y98] Yu, G., *The Novikov conjecture for groups with finite asymptotic dimension*, Annals of Mathematics, **147**(2) (1998), 325-355.

[Y00] Yu, G., *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math., **139**(2000) 201–240.

[Y06] Yu, G., *Higher index theory of elliptic operators and geometry of groups*, Proceedings of International Congress of Mathematicians, Madrid, 2006, vol. II, 1623–1639.