TWO PROBLEMS IN ASYMPTOTIC ANALYSIS PADÉ-ORTHOGONAL APPROXIMATION AND RIESZ POLARIZATION CONSTANTS AND CONFIGURATIONS

By

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To my parents,

Suwapan and Suchela.

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CHAPTER I

INTRODUCTION

In this dissertation, we investigate two subjects in asymptotic analysis: Padé-orthogonal approximants and Riesz polarization constants. The first focuses on a class of rational functions called *Padé-orthogonal approximants*. The second concerns the max-min and min-max quantities called *Riesz polarization constants* and associated *optimal Riesz polarization configurations*. We give a detailed description of these subjects in what follows.

I.1 Padé-orthogonal approximants

The history of Padé approximation is one of the longest among those of approximation theory. Padé approximants were named after H. Padé who developed them in a table and gave a connection of these rational functions to continued fractions in his thesis [44] in 1892. However, the subject had been introduced several times before Padé by J.L. Lagrange [36] in 1776, C.G.J. Jacobi [35] in 1845, and F.G. Frobenius [22] in 1881. In the last several decades, these Padé approximants were generalized in various forms such as multipoint Padé approximants, Padé-Faber approximants, Padé-Laurent approximants, etc. Padé approximants and their generalizations have been used in diverse areas such as numerical analysis, number theory, integral equations, the spectral theory of operators, random matrix theory, quantum mechanics, and quantum field theory. They also can be used as a tool to detect zeros or singularities of functions and study analytic continuation of power series or Fourier series.

The first part of this dissertation is devoted to a generalization of the classical construction of Padé approximants, namely Padé-orthogonal approximants. These rational functions are based on orthogonal polynomial expansions on some compact set in the complex plane $\mathbb{C}$. In order to define Padé-orthogonal approximants rigorously, we need to introduce some notation. Let E be a compact subset of the complex plane $\mathbb{C}$. Let μ be a finite positive Borel measure with infinite support contained in E and define the associated inner product

$$\langle g, h\rangle_\mu := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta).$$

We denote by

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, \dots,$$

the unique sequence of polynomials of respective degrees n with positive leading coefficients that are orthonormal with respect to $d\mu$; that is, $\langle p_n, p_m\rangle_\mu = \delta_{n,m}$. Padé-orthogonal approximants corresponding to μ are defined as follows:

Definition I.1.1. Let F be a holomorphic function in a neighborhood of E. A rational function $[n/m]_F^\mu := P_{n,m}^\mu/Q_{n,m}^\mu$ is a *(linear) Padé-orthogonal approximant of type* (n,m) *corresponding to* μ for the function F if $P_{n,m}^\mu$ and $Q_{n,m}^\mu$ are polynomials satisfying

$$\deg(P_{n,m}^\mu) \le n, \quad \deg(Q_{n,m}^\mu) \le m, \quad Q_{n,m}^\mu \not\equiv 0,$$

$$\langle Q_{n,m}^\mu F - P_{n,m}^\mu, p_j\rangle_\mu = 0, \quad \text{for } j = 0, 1, \dots, n+m.$$

It is not difficult to see that if $E = \{z \in \mathbb{C} : |z| \le 1\}$ and $d\mu = d\theta$, then these Padé-orthogonal approximants are exactly the classical Padé approximants (see the Frobenius definition [22]). The concept of linear Padé-orthogonal approximants was first introduced by H.J. Maehly [39] in 1960. In fact, he considered linear Padé-orthogonal approximants only for the case when $d\mu = dx/\sqrt{1-x^2}$ on $[-1,1]$. These rational functions are called Padé-Chebyshev approximants (see [6]) or sometimes cross-multiplied approximants (see [21]). Later, E.W. Cheney defined linear Padé-orthogonal approximants for a general setting (E is not just a finite interval) in his book [15]. The study of Padé-orthogonal approximants has mainly concentrated on the case when μ is supported in a finite interval (see e.g. [55, 56, 54, 29, 38, 28, 59, 12, 13]). S.P. Suetin [55, 54, 56] was the first to prove the convergence of row sequences of both linear and nonlinear Padé-orthogonal approximants on $[-1,1]$ (see the definition of nonlinear Padé-orthogonal approximants[1] in his paper [55]).

[1]S.P. Suetin called this rational functions “nonlinear Padé approximants of orthogonal expansions”.

Some problems on the convergence of *diagonal* sequences of these Padé-orthogonal approximants were considered in [29, 38, 28, 59]. For the case that μ is supported on the unit circle, there are a few articles [47, 46, 14, 1] studying (or using) these rational functions. However, the study of linear (or nonlinear) Padé-orthogonal approximants corresponding to μ supported on a general compact set has not yet been thoroughly explored. A rational function $[n/m]_F^\mu$ always exists, but may not be unique as we show in Example II.1.4. We also would like to emphasize that unlike the classical case, the definitions of linear and nonlinear Padé-orthogonal approximants may lead to distinct rational functions (see [60]). Since we consider only linear Padé-orthogonal approximants in this dissertation, we will omit the word "linear" when we refer to linear Padé-orthogonal approximants.

In this work, we focus on the study of a sequence $\{[n/m]_F^\mu\}_{n\in\mathbb{N}}$ when $m \in \mathbb{N}$ is fixed, which is called a *row sequence* of Padé-orthogonal approximants. Our goal is to investigate the relation of the convergence of poles of row sequences of Padé-orthogonal approximants corresponding to a measure supported on a general compact set and the singularities of the approximated function F. Under suitable restrictions on the set E, our main contributions are as follows.

- We prove convergence of row sequences of these rational functions (see Theorem II.2.1), namely an analogue of the theorem of Montessus de Ballore, under a ratio asymptotic condition on $\{p_k\}_{k\in\mathbb{N}}$. This result generalizes work of S.P. Suetin [56] who studied the case for measures supported on $[-1, 1]$.

- Under Szegő asymptotic conditions on $\{p_k\}_{k\in\mathbb{N}}$, we prove a direct analogue of the Fabry ratio theorem (see Theorem II.2.3) concerning the detection of the "nearest" singularity of a function F by using the limit of the ratio F_n/F_{n+1}, where $F_n := \langle F, p_n\rangle_\mu$. As a consequence of this, we provide a limit formula for the "nearest" singularity of the reciprocal of the interior Szegő function $1/S_{\text{int}}$ in terms of the Verblunsky coefficients (see Corollary II.2.4).

- We prove in Theorem II.2.8 that the row sequences of Padé-orthogonal approximants satisfy a Fabry ratio theorem when the measure supported on E satisfies Szegő asymptotic conditions. This result generalizes part of the result of V.I. Buslaev [13].

I.2 Riesz polarization constants and configurations

The second subject focuses on the study of Riesz polarization constants and optimal Riesz polarization configurations of infinite compact subsets of the m-dimensional Euclidean space $\mathbb{R}^m$ which are defined as follows.

Definition I.2.1. Let A be an infinite compact subset of the m-dimensional Euclidean space $\mathbb{R}^m$, $\omega_N = \{x_1, \dots, x_N\}$ denote a configuration of N (not necessarily distinct) points in A, and $\#\omega_N$ denote the cardinality of the multiset ω_N. For $s \in \mathbb{R}$, the *maximal N-point Riesz s-polarization constant of A* is given by

$$M_N^s(A) := \max_{\substack{\omega_N \subset A \\ \#\omega_N = N}} \min_{y \in A} \sum_{i=1}^{N} \frac{1}{|y - x_i|^s}, \quad \text{and} \quad M_N^0(A) := N, \tag{I.1}$$

and for $s \leq 0$, the *minimal N-point Riesz s-polarization constant of A* is given by

$$m_N^s(A) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} \max_{y \in A} \sum_{i=1}^{N} \frac{1}{|y - x_i|^s}, \quad \text{and} \quad m_N^0(A) := N. \tag{I.2}$$

We say that an N-point configuration ω_N is *optimal for $M_N^s(A)$ (optimal for $m_N^s(A)$)* if it attains the maximum in (I.1) (the minimum in (I.2)).

Such max-min quantities (I.1) were first introduced by M. Ohtsuka [43] who showed that the following limit exists as an extended real number:

$$\mathcal{M}^s(A) := \lim_{N \to \infty} \frac{M_N^s(A)}{N}$$

and that the limit $\mathcal{M}^s(A)$ is not less than the *Wiener constant* $W^s(A)$ corresponding to the same value of s. This constant is defined as

$$W^s(A) := \inf \int\int \frac{1}{|x-y|^s} d\mu(x) d\mu(y), \tag{I.3}$$

the infimum being taken over all Borel probability measures μ supported on A. The constants $M_N^s(A)/N$, $m_N^s(A)/N$, $\mathcal{M}^s(A)$, and $\lim_{N\to\infty} m_N^s(A)/N$ which were later called the N^{th}

Chebyshev constant of A, the N^{th} *dual Chebyshev constant of A,* the *Chebyshev constant of A*, and the *dual Chebyshev constant of A* were studied in [18, 20, 45, 19]. In [18], it was proved that the Chebyshev constant $\mathcal{M}^s(A)$ is the same as the Wiener constant $W^s(A)$ whenever the maximum principle is satisfied on A for the Riesz s-potential. The (abstract) Chebyshev constants in [20, 19] were used to study the so-called *rendezvous* or *average numbers.*

Another occurrence of such max-min and min-max constants is in the study of the N^{th} *linear polarization constant* and *polarization inequalities* in the theory of an infinite dimensional Banach space (see [45, Proposition 19-20], [4, Proposition 5 and Theorem 6], and [2, Theorem 1.12]). A very special case (see [2, Theorem 1.12]) of those polarization inequalities is equivalent to the following equality

$$M_N^2(\mathbb{S}^1) = \frac{N^2}{4}, \tag{I.4}$$

where $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$. Of course, if we know an optimal N-point configuration for $M_N^2(\mathbb{S}^1)$ which is intuitively a configuration of N distinct equally spaced points on $\mathbb{S}^1$, we can compute $M_N^2(\mathbb{S}^1)$. However, proving that a configuration of N distinct equally spaced points on $\mathbb{S}^1$ is optimal for $M_N^2(\mathbb{S}^1)$ (and $M_N^s(\mathbb{S}^1)$, for $s \in \mathbb{R} \setminus \{0\}$ and $m_N^s(\mathbb{S}^1)$, for $s \in (-\infty, 0)$) is a nontrivial problem. The optimality for $M_N^2(\mathbb{S}^1)$ of N distinct equally spaced points on $\mathbb{S}^1$ was proved by G. Ambrus in [2] and by G. Ambrus, K. Ball, and T. Erdélyi in [3]. T. Erdélyi and E.B. Saff [17] established this for $M_N^4(\mathbb{S}^1)$. The cases $M_N^s(\mathbb{S}^1)$ for $s > 0$ and $m_N^s(\mathbb{S}^1)$ for $-1 \leq s < 0$ were proved by D.P. Hardin, A.P. Kendall, and E.B. Saff in [32]. For some negative even integers s, we will see in Corollary III.2.8 that N distinct equally spaced points on $\mathbb{S}^1$ are not the only optimal configurations for $M_N^s(\mathbb{S}^1)$ and $m_N^s(\mathbb{S}^1)$.

The study of the dominant term of $M_N^s(A)$ for $s > 0$ as $N \to \infty$ is suggested by T. Erdélyi and E.B. Saff. In [17], they provide upper estimates and lower estimates for $M_N^s(A)$ on infinite compact sets A in $\mathbb{R}^m$ and focus on finding the dominant terms of $M_N^s(\mathbb{S}^m)$ and $M_N^s(\mathbb{B}^m)$, where $\mathbb{S}^m$ is the unit sphere in $\mathbb{R}^{m+1}$ and $\mathbb{B}^m$ is the closed unit ball in $\mathbb{R}^m$. In particular, they show that for an infinite compact set A in $\mathbb{R}^m$ of positive d-dimensional Hausdorff measure, one has $M_N^d(A) = O(N \ln N)$, $N \to \infty$, and $M_N^s(A) = O(N^{s/d})$, $N \to \infty$, for every $s > d$. Concerning the lower estimate of $M_N^s(A)$, it is not difficult to show that (see [17, 18, 20])

$$M_N^s(A) \geq \frac{1}{N-1}\mathcal{E}_s(A;N), \quad N \geq 2,$$

where $\mathcal{E}_s(A;N)$ is the minimal N-point Riesz s-energy of A as defined in the next paragraph. Combining this lower estimate and the so-called Poppy-seed Bagel Theorems (see [34, 10, 17]), T. Erdélyi and E.B. Saff showed that the order estimate for $s = d$ is sharp when A is contained in a d-dimensional C^1-manifold and the order estimate for $s > d$ is sharp when A is d-rectifiable.

The *minimal N-point Riesz s-energy* of an infinite compact set $A \subset \mathbb{R}^m$ is defined as

$$\mathcal{E}_s(A,N) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} \sum_{1 \leq j \neq k \leq N} \frac{1}{|x_j - x_k|^s}, \qquad \text{where} \qquad N \geq 2, \quad s > 0, \tag{I.5}$$

and an N-point configuration ω_N is called a *minimal N-point Riesz s-energy configuration of A* if it attains the minimum in (I.5). The basic asymptotic properties for N fixed and s varying of the minimal N-point Riesz s-energy are considered in [9, Chapter 2; Section 2.1-2.2]. It is known that as s gets large, the minimal Riesz s-energy problem tends to the *best-packing* problem and as s goes to 0, the minimal Riesz s-energy problem tends to the *minimal log-energy* problem (see [9, Proposition 2.9 and 2.14]). Similar basic properties for maximal and minimal Riesz polarization constants and configurations are considered in this dissertation in Section III.2.1. We will see in Theorems III.2.1-III.2.2 that as s goes to ∞, the maximal Riesz s-polarization problem tends to the *best-covering* problem and as s approaches 0 from the right, the maximal Riesz s-polarization problem tends to the *maximal log-polarization* problem.

The asymptotic behaviors for s fixed and $N \to \infty$ of the dominant term of $\mathcal{E}_s(A,N)$ and the limiting distribution of minimal N-point Riesz s-energy configurations have been investigated in [37, 41, 10, 34]. It appears that these asymptotic behaviors depend on the value of s. For an arbitrary compact set $A \subset \mathbb{R}^m$ with Hausdorff dimension d and $0 < s < d$, classical potential theory provides the relation of the continuous and discrete Riesz energies (see [9, Theorem 3.7]). The asymptotic behaviors for $s \geq d$ were proved by S.V. Borodachov, D.P. Hardin, and E.B. Saff in [10, 34] for a large class of sets. These results are known as Poppy-seed Bagel Theorems which we state in the following theorem.

Theorem A. *Let $d \in \mathbb{N}$, $A \subset \mathbb{R}^m$ be a compact d-rectifiable set, and $s \geq d$. If $s = d$, we further suppose that A is a subset of a d-dimensional C^1-manifold. Then*

$$\lim_{N \to \infty} \frac{\mathcal{E}_d(A, N)}{N^2 \log N} = \frac{\mathcal{H}_d(\mathbb{B}^d)}{\mathcal{H}_d(A)} \tag{I.6}$$

and

$$\lim_{N \to \infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}},$$

where $C_{s,d}$ is a finite positive constant independent of A and $\mathcal{H}_d$ denotes d-dimensional Hausdorff measure in $\mathbb{R}^m$ normalized so that the copy of the d-dimensional unit cube embedded in $\mathbb{R}^m$ has measure 1. Furthermore, under an additional assumption that $\mathcal{H}_d(A) > 0$, if $\{x_{k,N}^s\}_{k=1}^N$, $N \in \mathbb{N}$, is a sequence of minimal N-point Riesz s-energy configurations of A, we have in the weak* topology of measures

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_{i,N}^s} \xrightarrow{\ *\ } \frac{\mathcal{H}_d(\cdot)|_A}{\mathcal{H}_d(A)}, \quad N \to \infty, \tag{I.7}$$

where δ_x denotes the unit point mass at the point x.

Results analogous to (I.6) and (I.7) for the maximal Riesz d-polarization constant and configurations on an infinite compact subset of a d-dimensional C^1-manifold embedded in $\mathbb{R}^m$ (or a finite union of such sets provided that their pairwise intersections have d-dimensional Hausdorff measure zero) are proved in Section III.2.2. As a consequence of this, we show that maximal N-point Riesz d-polarization configurations are "good points" for discretizing such subsets of $\mathbb{R}^m$ in the uniformly distributed sense.

The main results obtained in this dissertation on the properties of the Riesz polarization constants and configurations can be summarized as follows:

- We study basic properties of $M_N^s(A)$ and $m_N^s(A)$ as functions of s in Section III.2.1.

We prove that for an infinite compact set A and for a fixed positive integer N, the function $f(s) := M_N^s(A)$ is continuous on $\mathbb{R}\setminus\{0\}$ and is not continuous at 0, and the function $g(s) := m_N^s(A)$ is continuous on $(-\infty, 0]$. More precisely, we prove that $f(s)$ is right-continuous at 0 but not left-continuous at 0. Moreover, we show that

$$\lim_{s\to\infty} M_N^s(A)^{1/s} = \frac{1}{\rho_N(A)},$$

where $\rho_N(A)$ is the N-point mesh norm (or N-point best-covering distance) of A. Additionally, we show that

$$\lim_{s\to 0^+} \frac{M_N^s(A) - N}{s} = M_N^{\log}(A),$$

where $M_N^{\log}(A)$ is the maximal N-point *log*-polarization constant of A.

- We determine the optimal configurations for $M_N^s(\mathbb{S}^m)$ and $m_N^s(\mathbb{S}^m)$ for those values of s for which there exists an N-point configuration whose Riesz s-potential function is constant on $\mathbb{S}^m$ in Section III.2.3.

- We prove Conjecture 2 of T. Erdélyi and E.B. Saff in [17] concerning the dominant term of $M_N^d(A)$ as $N \to \infty$ when A is an infinite compact subset of a d-dimensional C^1-manifold embedded in $\mathbb{R}^m$. Moreover, if we assume further that the d-dimensional Hausdorff measure of A is positive, we show that any sequence of optimal N-point configurations for $M_N^d(A)$ is asymptotically uniformly distributed with respect to the d-dimensional Hausdorff measure on the set A. These results also hold for finite unions of such sets A provided that their pairwise intersections have d-dimensional Hausdorff measure zero (see Section III.2.2).

CHAPTER II

PADÉ-ORTHOGONAL APPROXIMANTS

II.1 Introduction, background results, and notation

We begin this section by recalling the definition of classical Padé approximants.

Definition II.1.1. Let a pair of nonnegative integers (n, m) be given. The rational function $[n/m]_F := P_{n,m}/Q_{n,m}$ is called a *classical Padé approximant* of type (n, m) to a power series $F(z) = \sum_{k=0}^{\infty} f_k z^k$ if $P_{n,m}$ and $Q_{n,m}$ are polynomials satisfying

$$\deg(P_{n,m}) \leq n, \quad \deg(Q_{n,m}) \leq m, \quad Q_{n,m} \not\equiv 0,$$

$$(Q_{n,m}F - P_{n,m})(z) = O(z^{n+m+1}), \quad \text{as } z \to 0. \tag{II.1}$$

It is clear from (II.1) that once $Q_{n,m}$ is determined, then $P_{n,m}$ is simply the nth truncation of the power series for $Q_{n,m}F$. Finding $Q_{n,m}$ is equivalent to solving a system of m homogeneous linear equations on $m+1$ unknowns. However, the ratio of any pair $(P_{n,m}, Q_{n,m})$ defines the same rational function $[n/m]_F$, although the polynomials $Q_{n,m}$ are not unique. We will review shortly in this section only some properties of row sequences of classical Padé approximants which we will consider for Padé-orthogonal approximants. We refer the reader to the book of G.A. Baker and P. Graves-Morris [6] and survey papers [60, 61, 5] for more details and recent progresses in the subject of classical Padé approximants.

In this dissertation, we will restrict our consideration to the sets E as described below. Let E be a compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}}\setminus E$ is simply connected. Then, there exists a unique exterior conformal bijection Φ sending $\overline{\mathbb{C}}\setminus E$ onto $\overline{\mathbb{C}}\setminus\{w \in \mathbb{C} : |w| \leq 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. We assume that E is such that the inverse function Ψ of Φ can be extended continuously to $\overline{\mathbb{C}} \setminus \{w \in \mathbb{C} : |w| < 1\}$. Note that the closure of a Jordan region and a finite interval fall in our consideration.

Let μ be a finite positive Borel measure with infinite support $\text{supp}(\mu)$ contained in E.

We write $\mu \in \mathcal{M}(E)$ and denote the associated inner product

$$\langle g,\, h\rangle_\mu := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \quad g, h \in L_2(\mu).$$

We denote by

$$p_n(z) := \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, \ldots,$$

the unique sequence of polynomials of respective degrees n with positive leading coefficients that are orthonormal with respect to $d\mu$; that is, $\langle p_n,\, p_m\rangle_\mu = \delta_{n,m}$. Denote by $H(E)$ the space of all functions holomorphic in some neighborhood of E. Padé-orthogonal approximants corresponding to μ are defined as follows:

Definition II.1.2. Let $F \in H(E)$, $\mu \in \mathcal{M}(E)$, and a pair of nonnegative integers (n, m) be given. A rational function $[n/m]_F^\mu := P_{n,m}^\mu / Q_{n,m}^\mu$ is a *Padé-orthogonal approximant corresponding to* μ of type (n, m) to F if $P_{n,m}^\mu$ and $Q_{n,m}^\mu$ are polynomials satisfying

$$\deg(P_{n,m}^\mu) \leq n, \quad \deg(Q_{n,m}^\mu) \leq m, \quad Q_{n,m}^\mu \not\equiv 0, \tag{II.2}$$

$$\langle Q_{n,m}^\mu F - P_{n,m}^\mu,\, p_j\rangle_\mu = 0, \quad \text{for } j = 0, 1, \ldots, n+m. \tag{II.3}$$

Since $Q_{n,m}^\mu \not\equiv 0$, we will normalize it by requiring that its leading coefficient equals 1.

Note that if $E = \{z \in \mathbb{C} : |z| \leq 1\}$ and $d\mu = d\theta$, then these Padé-orthogonal approximants are exactly the classical Padé approximants.

Hereafter, we consider only $F \in H(E)$. For any $\rho > 1$, we set

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\}, \qquad \text{and} \qquad \gamma_\rho := \{w \in \mathbb{C} : |w| = \rho\}.$$

Denote by D_ρ the interior of Γ_ρ and by $\mathbb{B}(z, \rho)$ the open disk centered at z of radius ρ. We will call Γ_ρ and D_ρ a *level curve of index* ρ and a *canonical domain of index* ρ (with respect to E), respectively. For convenience, we let $\mathbb{B} := \mathbb{B}(0, 1)$ be the open unit ball and $\mathbb{T} := \partial\mathbb{B}$ be the unit circle. We denote by $\rho_0(F)$ the maximal index $\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a holomorphic function and by $\rho_m(F)$ the maximal index

$\rho > 1$ of the largest canonical domain D_ρ to which F can be extended as a meromorphic function whose number of poles does not exceed m (counting their multiplicities). Define the Fourier coefficient of F corresponding to p_n :

$$F_n := \langle F,\, p_n \rangle = \int F(z)\overline{p_n(z)}d\mu(z). \tag{II.4}$$

In all that follows, the phrase "uniform convergence inside a domain" means "uniform convergence on each compact subset of the domain".

One can easily determine the domain of holomorphy $D_{\rho_0(F)}$ of F by the following analogue of Cauchy's theorem for power series (see e.g. [53, Theorem 6.6.1] for the proof):

Lemma II.1.3. *Let $F \in H(E)$. Assume that*

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \tag{II.5}$$

uniformly inside $\mathbb{C} \setminus E$. Then,

$$\rho_0(F) = \left(\limsup_{n\to\infty} |F_n|^{1/n}\right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} F_n p_n(z)$ converges to $F(z)$ uniformly inside $D_{\rho_0(F)}$ and the series $\sum_{n=0}^{\infty} F_n p_n(z)$ diverges pointwise for all $z \in \mathbb{C} \setminus \overline{D_{\rho_0(F)}}$.

Therefore, if μ satisfies the condition (II.5), the polynomials $P^\mu_{n,m}$ and $Q^\mu_{n,m}$ verify

$$Q^\mu_{n,m}(z)F(z) - P^\mu_{n,m}(z) = \sum_{k=n+m+1}^{\infty} \langle Q^\mu_{n,m}F,\, p_k\rangle_\mu p_k(z)$$

for all $z \in D_{\rho_0(F)}$ and $P^\mu_{n,m} = \sum_{k=0}^{n} \langle Q^\mu_{n,m}F,\, p_k\rangle_\mu p_k$ is uniquely determined by $Q^\mu_{n,m}$.

In contrast with classical Padé approximants, the rational function $[n/m]^\mu_F$ may not be unique as the following example shows.

Example II.1.4. Consider $E = [-1, 1]$, $d\mu = dx/\sqrt{1-x^2}$ and

$$F(x) = \frac{37}{x-3} + \sum_{k=0}^{4} c_k p_k(x),$$

where the p_k are normalized Chebyshev polynomials, and

$$c_0 := 37, \quad c_1 := 6(-271\sqrt{\pi} + 192\sqrt{2\pi}), \quad c_2 := -\sqrt{2} + 315\sqrt{\pi} - 222\sqrt{2\pi},$$

$$c_3 := 3513\sqrt{\pi} - 2484\sqrt{2\pi}, \quad c_4 := \sqrt{2} + 10674\sqrt{\pi} - 7548\sqrt{2\pi}.$$

Using the program Mathematica it is easy to check that both $Q^{\mu}_{1,2}(x) = x$ and $Q^{\mu}_{1,2}(x) = (x-3)^2$ satisfy

$$\langle Q^{\mu}_{1,2} F,\, p_k \rangle_{\mu} = 0, \quad k = 2, 3.$$

These denominators $Q^{\mu}_{1,2}$ give us

$$[1/2]^{\mu}_{F}(x) = \frac{4756\sqrt{\pi} - 3363\sqrt{2\pi} - 36\sqrt{2\pi}x + 144x}{4\sqrt{\pi}x},$$

and

$$[1/2]^{\mu}_{F}(x) = \frac{1404 - 28536\sqrt{\pi} + 19827\sqrt{2\pi} - 864x + 90364\sqrt{\pi}x - 63681\sqrt{2\pi}x}{4\sqrt{\pi}(x-3)^2},$$

respectively, which are clearly distinct.

However, if

$$\Delta_{n,m}(F,\mu) := \begin{vmatrix} \langle F, p_{n+1}\rangle_{\mu} & \langle zF, p_{n+1}\rangle_{\mu} & \cdots & \langle z^{m-1}F, p_{n+1}\rangle_{\mu} \\ \vdots & \vdots & \vdots & \vdots \\ \langle F, p_{n+m}\rangle_{\mu} & \langle zF, p_{n+m}\rangle_{\mu} & \cdots & \langle z^{m-1}F, p_{n+m}\rangle_{\mu} \end{vmatrix} \neq 0 \qquad \text{(II.6)}$$

or for every solution of (II.2)-(II.3), the polynomial $Q^{\mu}_{n,m}$ is of degree m, then $Q^{\mu}_{n,m}$ is unique and $[n/m]^{\mu}_{F}$ is also unique. One can easily show that $\Delta_{n,m}(F,\mu) \neq 0$ and the condition that for every solution of (II.2)-(II.3), the polynomial $Q^{\mu}_{n,m}$ is of degree m are equivalent.

For the case when $E = \overline{\mathbb{B}}$ and the support of μ is $\mathbb{T}$, D. Barrios Rolanía, G. López Lagomasino, and E.B. Saff (see [47]) use the determinants $\Delta_{n,m}(F,\mu)$ to determine the radii of meromorphy of F, namely they show that

$$\rho_m(F) = \frac{l_m}{l_{m+1}}$$

(by convention $0/0 = \infty$), where

$$l_m := \limsup_{n\to\infty} |\Delta_{n,m}(F,\mu)|^{1/n}, \quad l_0 := 1$$

under the two assumptions that μ satisfies Szegő's condition and the reciprocal of the interior Szegő function $1/S_{\text{int}}(z)$ has an analytic continuation to $\mathbb{B}(0,r)$ for some $r > 1$.

In this dissertation, we focus on row sequences of the Padé table. We use $\{[n/m]_F\}_{n,m=0,1,\ldots}$ ($\{[n/m]_F^\mu\}_{n,m=0,1,\ldots}$) to denote the *classical Padé (Padé-orthogonal) table* for the function F and $\{[n/m]_F\}_{n=0,1,\ldots}$ ($\{[n/m]_F^\mu\}_{n=0,1,\ldots}$) to represent the *mth row of the classical Padé (Padé-orthogonal) table* for the function F. Before proceeding with this study we recall some basic results for power series and classical Padé approximants.

For a meromorphic function F with exactly m poles within an open disk centered at the origin, Montessus de Ballore's theorem (cf. e.g. [6]) asserts the convergence of the mth row sequence $\{[n/m]_F\}_{n=0,1,\ldots}$ inside the region obtained removing the poles of F from the open disk.

Theorem (Montessus de Ballore). *Let F be a function that is meromorphic in the disk $\mathbb{B}(0,R)$, with poles in the distinct points $z_1,\ldots,z_p$, where*

$$0 < |z_1| \le |z_2| \le \cdots \le |z_p| < R.$$

Let m_k be the multiplicity of z_k and $\sum_{k=1}^p m_k =: m$. Then

$$F(z) = \lim_{n\to\infty} [n/m]_F(z)$$

uniformly inside $\mathbb{B}(0,R) \setminus \{z_1,\dots,z_p\}$, *and*

$$\lim_{n\to\infty} Q_{n,m}(z) = \prod_{k=1}^{p}(z-z_k)^{m_k}.$$

Various generalizations of this theorem were given by e.g. E.B. Saff [49, 31, 30], A.A. Gonchar [26], and S.P. Suetin [55, 56]. In particular, in [55] a Montessus de Ballore type result is given for Padé-orthogonal approximants corresponding to a measure supported on the interval $[-1,1]$.

In the converse direction, a natural question is: What conclusions can be drawn concerning the singularities of F if we know the asymptotic behavior of the poles of its approximants? Such problems are called of *inverse type* in Padé approximation theory. In this direction, an interest classical result is due to E. Fabry (see [16, p. 377]):

Theorem (Fabry). *Suppose that the coefficients of a power series* $\sum_{n=0}^{\infty} f_n z^n$ *are such that*

$$\lim_{n\to\infty} \frac{f_n}{f_{n+1}} = \lambda \neq 0 \tag{II.7}$$

exists. Then the series $\sum_{n=0}^{\infty} f_n z^n$ *converges uniformly inside the disk* $\mathbb{B}(0,|\lambda|)$ *and* λ *is a singularity of the function* $F(z) = \sum_{n=0}^{\infty} f_n z^n$.

The boundary of $\mathbb{B}(0,|\lambda|)$ may contain more than one singularity. For example, if

$$F(z) = \frac{1}{z+1} + \frac{1}{(z-1)^2} = \sum_{n=0}^{\infty}((n+1)+(-1)^n)z^n$$

then $\lim_{n\to\infty} f_n/f_{n+1} = 1$. However, the function F has poles at ± 1.

The first conclusion of Fabry's theorem concerning the convergence of the series is a consequence of the Cauchy-Hadamard formula. The second conclusion, concerning the singularity, is far from trivial. L. Bieberbach mentioned in [7] that "it requires much effort to penetrate Fabry's works so as to get pleasure from them and fully understand the elegance and simplicity of the arguments of this master".

It is easy to check that if $f_n \neq 0$ and $f_{n+1} \neq 0$, then f_n/f_{n+1} in (II.7) is the pole of the classical Padé approximant $[n/1]_F$. In [27], A.A. Gonchar conjectured that Fabry's theorem can be generalized to the mth row of the Padé table. In its general form, the conjecture was proved by S.P. Suetin in [58] (see also [57]).

Theorem (Suetin). *Suppose that the coefficients of a power series $F(z) := \sum_{n=0}^{\infty} f_n z^n$ are such that, for any fixed $m \in \mathbb{N}$ and any sufficiently large $n \in \mathbb{N}$, the classical Padé approximants $[n/m]_F$ have precisely m finite poles $\lambda_{n,1}, \ldots, \lambda_{n,m}$ that are convergent:*

$$\lim_{n\to\infty} \lambda_{n,i} = \lambda_i \neq 0, \quad i = 1, \ldots, m,$$

and set $R_{\min} := \min_{1\leq i\leq m} |\lambda_i|$, $R_{\max} := \max_{1\leq i\leq m} |\lambda_i|$. Then,

(i) the series $\sum_{n=0}^{\infty} f_n z^n$ converges uniformly inside the disk $\mathbb{B}(0, R_{\min})$;

(ii) the function F admits a meromorphic continuation to the disk $\mathbb{B}(0, R_{\max})$;

(iii) $\lambda_1, \ldots, \lambda_m$ are singularities of F; those lying in $\mathbb{B}(0, R_{\max})$ are poles, and F has no other poles in $\mathbb{B}(0, R_{\max})$.

Similar inverse type results for row sequences of multipoint Padé approximants, Padé-Faber approximants, and Padé-orthogonal approximants corresponding to a measure supported on $[-1, 1]$ were proved by V.I. Buslaev in [13]. In this work, we will use the methods employed in [12] and [13] to prove analogues of the Fabry and Suetin theorems for Padé-orthogonal approximants corresponding to a measure supported on a general compact set $E \subset \mathbb{C}$ as described above.

II.2 Main results

We will make the following assumptions on the asymptotic behavior of the sequence of orthonormal polynomials with respect to a given measure $\mu \in \mathcal{M}(E)$. We write $\mu \in \mathcal{R}(E)$ when the corresponding sequence of orthonormal polynomials has ratio asymptotics; that is,

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+1}(z)} = \frac{1}{\Phi(z)}. \tag{II.8}$$

We say that Szegő or strong asymptotics takes place, and write $\mu \in \mathcal{S}(E)$, if

$$\lim_{n\to\infty} \frac{p_n(z)}{c_n \Phi^n(z)} = S(z) \quad \text{and} \quad \lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 1, \tag{II.9}$$

The first limit in (II.9) and the one in (II.8) are assumed to hold uniformly inside $\overline{\mathbb{C}} \setminus E$, the c_n's are positive constants, and $S(z)$ is some holomorphic and non-vanishing function on $\overline{\mathbb{C}} \setminus E$. Obviously, (II.9) $\Rightarrow$ (II.8) $\Rightarrow$ (II.5).

An analogue of Montessus de Ballore's theorem for Padé-orthogonal approximants is the following:

Theorem II.2.1. *Suppose $F \in H(E)$ has poles of total multiplicity exactly m in $D_{\rho_m(F)}$ at the (not necessarily distinct) points $\lambda_1, \dots, \lambda_m$ and let $\mu \in \mathcal{R}(E)$. Then, $[n/m]_F^\mu$ is uniquely determined for all sufficiently large n and the sequence $[n/m]_F^\mu$ converges uniformly to F inside $D_{\rho_m(F)} \setminus \{\lambda_1, \dots, \lambda_m\}$ as $n \to \infty$. Moreover, for any compact subset K of $D_{\rho_m(F)} \setminus \{\lambda_1, \dots, \lambda_m\}$,*

$$\limsup_{n\to\infty} \|F - [n/m]_F^\mu\|_K^{1/n} \le \frac{\max\{|\Phi(z)| : z \in K\}}{\rho_m(F)}, \tag{II.10}$$

where $\|\cdot\|_K$ denotes the sup-norm on K and if $K \subset E$, then $\max\{|\Phi(z)| : z \in K\}$ is replaced by 1. Additionally,

$$\limsup_{n\to\infty} \|Q_{n,m}^\mu - Q_m\|^{1/n} \le \frac{\max\{|\Phi(\lambda_j)| : j = 1, \dots, m\}}{\rho_m(F)} < 1, \tag{II.11}$$

where $\|\cdot\|$ denotes (for example) the coefficient norm in the space of polynomials of degree m and $Q_m(z) = \prod_{k=1}^m (z - \lambda_k)$.

Remark II.2.2. When $K = E$, the rate of convergence in (II.10) cannot be improved; that is,

$$\limsup_{n\to\infty} \|F - [n/m]_F^\mu\|_E^{1/n} = \limsup_{n\to\infty} \sigma_{n,m}^{1/n} = \frac{1}{\rho_m(F)}, \tag{II.12}$$

where

$$\sigma_{n,m} := \inf_r \|F - r\|_E,$$

and the infimum is taken over the class of all rational functions of type (n, m)

$$r(z) = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}.$$

We refer the reader to [24, 48] for more information on the second equality in (II.12).

In [54, Theorem 1], S.P. Suetin proves this result for measures supported on a bounded interval of the real line and states without proof that a similar theorem may be obtained for measures supported on a continuum of the complex plane whose sequence of orthonormal polynomials and their associated second type functions have strong asymptotic behavior. The assumptions of our Theorem II.2.1 are substantially weaker.

The natural analogue of Fabry's theorem is the following:

Theorem II.2.3. *Let $F \in H(E)$ and $\mu \in \mathcal{S}(E)$. If*

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \tau,$$

then $\Psi(\tau)$ is a singularity of F and $\rho_0(F) = |\tau|$.

If $E = \overline{\mathbb{B}}$ and the measure μ supported on $\mathbb{T}$ satisfies the Szegő condition,

$$\int_0^{2\pi} \log w(\theta) d\theta > -\infty, \tag{II.13}$$

(where $d\mu(\theta) = w(\theta)d\theta/2\pi + d\mu_s(\theta)$ is the Radon-Nikodym decomposition of μ), it is well known that the orthonormal polynomials φ_n satisfy the Szegő asymptotics (II.9) (with $c_n = 1$), the leading coefficients of the orthonormal polynomials φ_n satisfy

$$\lim_{n\to\infty} \kappa_n = \kappa := \exp\left\{-\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) d\theta\right\},$$

and

$$\frac{1}{S_{\text{int}}(z)} = \frac{1}{\kappa}\sum_{k=0}^{\infty} \overline{\varphi_k(0)}\varphi_k(z), \quad \text{uniformly inside } \mathbb{B},$$

where

$$S_{\text{int}}(z) := \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log w(\theta) \frac{e^{i\theta}+z}{e^{i\theta}-z} d\theta\right), \quad z \in \mathbb{B},$$

denotes the interior Szegő function (see [23, p. 19-20] for the proof). Therefore, Theorem II.2.3 can be applied to locate the nearest singularity of the reciprocal of the interior Szegő function in terms of the *Verblunsky (or Schur)* coefficients α_n ($\alpha_n := -\overline{\varphi_n(0)}/\kappa_n$).

Corollary II.2.4. *Let μ satisfy the Szegő condition* (II.13) *and assume that $1/S_{\text{int}} \in H(\overline{\mathbb{B}})$. Suppose that the Verblunsky coefficients α_n corresponding to μ verify*

$$\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = \lambda.$$

Then λ is a singularity of $1/S_{\text{int}}$ and $1/S_{\text{int}}$ is holomorphic on $\mathbb{B}(0, |\lambda|)$.

This result complements [46, Theorem 2] where, under stronger assumptions, it is shown that λ is a simple pole and $1/S_{\text{int}}$ has no other singularity in a neighborhood of $\overline{\mathbb{B}(0, |\lambda|)}$.

Using the definition of $Q_{n,1}^{\mu}$, it is easy to verify that whenever $F_{n+1} \neq 0$, we have

$$Q_{n,1}^{\mu}(z) = z - \frac{\langle zF, p_{n+1}\rangle_{\mu}}{F_{n+1}}.$$

The next result enables one to locate the singularity of F nearest E using the zeros of $Q_{n,1}^{\mu}$.

Theorem II.2.5. *Let $F \in H(E)$ and $\mu \in \mathcal{S}(E)$. If*

$$\lim_{n\to\infty} \frac{\langle zF, p_n\rangle_{\mu}}{F_n} = \lambda,$$

then λ is a singularity of F and $\rho_0(F) = |\Phi(\lambda)|$.

The proofs of Theorems II.2.3 and II.2.5 are reduced to Fabry's theorem by using the following result.

Theorem II.2.6. *Let $F \in H(E)$ and $\mu \in \mathcal{S}(E)$. Define $f(w) := F(\Psi(w))$ and denote the Laurent series of f about 0 by $\sum_{k=-\infty}^{\infty} f_k w^k$. Then, the following limits are equivalent:*

(*a*) $\lim_{n\to\infty} F_n/F_{n+1} = \tau$, (*b*) $\lim_{n\to\infty}\langle zF, p_n\rangle_\mu / F_n = \lambda$,

(*c*) $\lim_{n\to\infty} f_n/f_{n+1} = \tau$,

where τ *and* λ *are finite and related by the formula* $\Phi(\lambda) = \tau$.

An analogue of Suetin's theorem (on the inverse problem) for Padé-orthogonal approximants is the following:

Theorem II.2.7. *Let* $F \in H(E)$ *and* $\mu \in \mathcal{S}(E)$. *If for all* n *sufficiently large,* $[n/m]_F^\mu$ *has precisely* m *finite poles* $\lambda_{n,1}, \ldots, \lambda_{n,m}$, *and*

$$\lim_{n\to\infty} \lambda_{n,j} = \lambda_j, \quad j = 1, 2, \ldots, m,$$

($\lambda_1, \ldots, \lambda_m$ *are not necessarily distinct), then*

(*i*) F *is holomorphic in* $D_{\rho_{\min}}$ *where* $\rho_{\min} := \min_{1\le j\le m} |\Phi(\lambda_j)|$;

(*ii*) $\rho_{m-1}(F) = \max_{1\le j\le m} |\Phi(\lambda_j)|$;

(*iii*) $\lambda_1, \ldots, \lambda_m$ *are singularities of* F; *those lying in* $D_{\rho_{m-1}(F)}$ *are poles, and* F *has no other poles in* $D_{\rho_{m-1}(F)}$.

Theorem II.2.7 is an immediate consequence of the following result and Suetin's theorem (on the inverse problem).

Theorem II.2.8. *Let* $F \in H(E)$ *and* $\mu \in \mathcal{S}(E)$. *Define* $f(w) := F(\Psi(w))$ *and denote the Laurent series of* f *about* 0 *by* $\sum_{k=-\infty}^{\infty} f_k w^k$ *and the regular part of* f *by* $\hat{f}(w) := \sum_{k=0}^{\infty} f_k w^k$. *For each fixed* $m \ge 1$, *the following conditions are equivalent:*

(*a*) *The poles of* $[n/m]_{\hat{f}}$ *have finite limits* $\tau_1, \ldots, \tau_m$, *as* $n \to \infty$.

(*b*) *The poles of* $[n/m]_F^\mu$ *have finite limits* $\lambda_1, \ldots, \lambda_m$, *as* $n \to \infty$.

Under appropriate enumeration of the sub-indices, the values λ_j *and* τ_j, $j = 1, \ldots, m$, *are related by the formula* $\Phi(\lambda_j) = \tau_j$ *for all* $j = 1, \ldots, m$.

II.3 Proofs

II.3.1 Auxiliary lemmas

We collect all lemmas used to prove the main results in this section.

For convenience of the reader, we begin stating two lemmas due to V.I. Buslaev (see [13, Theorem 5-6]). These results constitute basic tools for proving our inverse type results. We make use of the following notation. Let $f(w) = \sum_{k=-\infty}^{\infty} f_k w^k$ be a Laurent series. We denote the regular part of $f(w)$ by $\hat{f}(w) := \sum_{k=0}^{\infty} f_k w^k$. If $\hat{f}(w)$ is holomorphic at 0, we denote by $R_m(\hat{f})$ the radius of the largest disk centered at the origin to which $\hat{f}(w)$ can be extended as a meromorphic function with at most m poles (counting their multiplicities). Define the annulus

$$T_{\delta,m}(f) := \{w \in \mathbb{C} : e^{-\delta} R_0(\hat{f}) \leq |w| \leq e^{\delta} R_{m-1}(\hat{f})\},$$

where $m \in \mathbb{N}$ and $\delta \geq 0$. We will use $[\cdot]_n$ to denote the coefficient of w^n in the Laurent series expansion around 0 of the function in the square brackets. Set

$$U := \overline{\mathbb{C}} \setminus \overline{\mathbb{B}}.$$

Lemma II.3.1 (Buslaev [13])**.** *Let $m \in \mathbb{N}, \delta > 0$, and let $f(w) = \sum_{n=-\infty}^{\infty} f_n w^n$ be a Laurent series such that*

$$0 < R_0(\hat{f}) \leq R_{m-1}(\hat{f}) < \infty, \quad \overline{\lim_{n\to\infty}} |f_{-n}|^{1/n} \leq R_0(\hat{f}). \tag{II.14}$$

Assume further that

$$\lim_{n\to\infty} [f \alpha_n \eta_{n,j}]_n R_{m-1}^n(\hat{f}) e^{\delta n} = 0, \quad j = 0, \ldots, m-1, \tag{II.15}$$

where the functions $\alpha_n, \eta_{n,j} \in H(T_{\delta,m}(f))$ have the limits

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) \not\equiv 0, \quad \eta_j(w) := \lim_{n\to\infty} \eta_{n,j}(w) = \eta^j(w), \quad j = 0, \ldots, m-1,$$

uniformly on $T_{\delta,m}(f)$*,* $\eta(w)$ *is a univalent function on* $T_{\delta,m}(f)$*, and* $\alpha(w)$ *has at most* m *zeros in the annulus* $T_{0,m}(f)$*. Then the function* $\alpha(w)$ *has precisely* m *zeros* $\tau_1, \ldots, \tau_m$ *in* $T_{0,m}(f)$ *and* $\lim_{n\to\infty} \tau_{n,j} = \tau_j$*, where the* $\tau_{n,j}\,(j = 1, \ldots, m)$ *are poles of the classical Padé approximants* $[n/m]_{\hat{f}}(w)$*. Moreover, for any functions* $K_{n,1}, \ldots, K_{n,m}, L_{n,1}, \ldots, L_{n,m} \in H(T_{\nu,m}(f))$*,* $\nu > 0$*, that converge to* $K_1, \ldots, K_m$*,* $L_1, \ldots, L_m$ *uniformly on* $T_{\nu,m}(f)$*,*

$$\lim_{n\to\infty} \frac{\det([fK_{n,i}L_{n,j}]_n)_{i,j=1,\ldots,m}}{\det(f_{n-i-j})_{i,j=0,\ldots,m-1}} = \frac{\det(K_r(\tau_s))_{s,r=1,\ldots,m} \det(L_r(\tau_s))_{s,r=1,\ldots,m}}{W^2(\tau_1, \ldots, \tau_m)}, \qquad \text{(II.16)}$$

where $W(\tau_1, \ldots, \tau_m) = \det(\tau_s^{r-1})_{s,r=1,\ldots,m}$ *is the Vandermonde determinant of the numbers* $\tau_1, \ldots, \tau_m$ *(for multiple zeros the right-hand side of* (II.16) *is defined by continuity). In particular, for any* $k_1, \ldots, k_m, q_1, \ldots, q_m \in \mathbb{Z}$*, the limits*

$$\lim_{n\to\infty} \frac{\det(f_{n-k_i-q_j})_{i,j=1,\ldots,m}}{\det(f_{n-i-j})_{i,j=0,\ldots,m-1}} = \frac{\det(\tau_s^{k_r})_{s,r=1,\ldots,m} \det(\tau_s^{q_r})_{s,r=1,\ldots,m}}{W^2(\tau_1, \ldots, \tau_m)}$$

exist.

The assumptions $R_{m-1}(\hat{f}) < \infty$ and (II.15) in Lemma II.3.1 can be replaced by the following: the functions $\alpha_n(w)$ and $w^{-j}\eta_{n,j}(w)$ are holomorphic in the set $\overline{\mathbb{C}} \setminus \mathbb{B}(0, e^{-\delta}R_0(\hat{f}))$, and

$$[f\alpha_n\eta_{n,j}]_n = 0, \quad j = 0, \ldots, m-1, \quad n \geq n_0.$$

Hence, we also have.

Lemma II.3.2 (Buslaev [13])**.** *Let* $m \in \mathbb{N}$*,* $\sigma > 1$*, let* $f(w) = \sum_{n=-\infty}^{\infty} f_n w^n$ *be a holomorphic function in the annulus* $\{1 < |w| < \sigma\}$*. Assume further that*

$$[f\alpha_n\eta_{n,j}]_n = 0, \quad j = 0, \ldots, m-1, \quad n \geq n_0, \qquad \text{(II.17)}$$

hold, where $\alpha_n(w)$ *and* $w^{-j}\eta_{n,j}(w)$ *are holomorphic functions in* U*, the limits*

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) \not\equiv 0, \quad \eta_j(w) := \lim_{n\to\infty} \eta_{n,j}(w) = \eta^j(w), \quad j = 0, \ldots, m-1,$$

exist uniformly inside $U \setminus \{\infty\}$*, the function* $\alpha(w)$ *has at most* m *zeros in* $U \setminus \{\infty\}$*, and*

$\eta(w)$ is a univalent function in U such that $\eta(\infty) = \infty$. Then, only one of the following assertions takes place:

(i) *$\hat{f}(w)$ is a rational function with at most $m-1$ poles;*

(ii) *$\alpha(w)$ has precisely m zeros $\tau_1, \dots, \tau_m$ in $U \setminus \{\infty\}$, these zeros are singularities of $f(w)$, with an appropriate ordering $|\tau_1| = R_0(\hat{f}), \dots, |\tau_m| = R_{m-1}(\hat{f})$, and the limits $\lim_{n\to\infty} \tau_{n,j} = \tau_j$ exist, where the $\tau_{n,j}, j = 1, \dots, m$, are the poles of the classical Padé approximants $[n/m]_{\hat{f}}(w)$.*

The second type functions $s_n(z)$ (corresponding to the orthonormal polynomials $p_n(z)$) defined by

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z-\zeta} d\mu(\zeta), \quad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu),$$

where $\operatorname{supp}(\mu)$ is the support of μ, have been extensively used in applications of orthogonal polynomials to rational approximation (see e.g. [12], [13], and [54]). Since we use these functions in our proofs, we list some of their properties in the next lemma.

Lemma II.3.3. *If $\mu \in \mathcal{R}(E)$, then*

$$\lim_{n\to\infty} p_n(z) s_n(z) = \frac{\Phi'(z)}{\Phi(z)},$$

uniformly inside $\overline{\mathbb{C}} \setminus E$. Consequently, for any compact set $K \subset \mathbb{C} \setminus E$, there exists n_0 (n_0 depends on K) such that $s_n(z) \neq 0$ for all $z \in K$ and $n \geq n_0$.

Proof of Lemma II.3.3. From orthogonality, we get

$$p_n(z) s_n(z) = \int \frac{|p_n(\zeta)|^2}{z-\zeta} d\mu(\zeta), \quad z \notin \operatorname{supp}(\mu).$$

Since p_n is of norm 1 in $L_2(\mu)$, it readily follows that $\{\int |p_n(\zeta)|^2/(z-\zeta) d\mu(\zeta)\}_{n\in\mathbb{N}}$ forms a normal family in $\overline{\mathbb{C}} \setminus E$. Consequently, the limit stated follows from pointwise convergence in a neighborhood of infinity. Now, for all z sufficiently large, since $\mu \in \mathcal{R}(E)$ from [51, Theorem 1.8] it follows that[1]

$$\lim_{n\to\infty}\int\frac{|p_n(\zeta)|^2}{z-\zeta}d\mu(\zeta)=\lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\int\zeta^k|p_n(\zeta)|^2d\mu(\zeta)=\sum_{k=0}^{\infty}\frac{1}{z^{k+1}}\frac{1}{2\pi}\int_{\mathbb{T}}\Psi(w)^k\frac{dw}{wi}$$

$$=\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{1}{w(z-\Psi(w))}dw=\frac{1}{2\pi i}\int_{\Psi(\mathbb{T})}\frac{\Phi'(\zeta)}{\Phi(\zeta)(z-\zeta)}d\zeta=\frac{\Phi'(z)}{\Phi(z)}.$$

Since the function on the right-hand side never vanishes in $\mathbb{C}\setminus E$, the rest of the statements follow at once. □

Define

$$h_n(w):=c_nw^{n+1}s_n(\Psi(w))\Psi'(w),\quad w\in U:=\overline{\mathbb{C}}\setminus\overline{\mathbb{B}}.$$

This sequence of functions is needed to define the $\alpha_n(w)$ and $\eta_{n,j}(w)$ in Buslaev's lemmas. So, we will list some of their properties.

Lemma II.3.4. *Let $F\in H(E)$. Define $f(w):=F(\Psi(w))$. The functions $h_n(w)$ are holomorphic in U, $F_n=[fh_n]_n/c_n$ and $\langle zF,p_n\rangle_\mu=[\Psi fh_n]_n/c_n$. If $\mu\in\mathcal{S}(E)$, then the sequence $h_n(w)$ converges to some non-vanishing function $h(w)$ uniformly inside U.*

Proof of Lemma II.3.4. Clearly, $h_n(w)$ is holomorphic in U. Let $\epsilon>0$ be a small number so that $\Gamma_{1+\varepsilon}$ is in the domain of holomorphy of $F(z)$. By Fubini's theorem and Cauchy's integral formula, we have

$$F_n=\int F(z)\overline{p_n(z)}d\mu(z)=\int\left(\frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}}\frac{F(\zeta)}{\zeta-z}d\zeta\right)\overline{p_n(z)}d\mu(z)$$

$$=\frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}}F(\zeta)\int\frac{\overline{p_n(z)}}{\zeta-z}d\mu(z)d\zeta=\frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}}F(\zeta)s_n(\zeta)d\zeta$$

$$=\frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}}f(w)s_n(\Psi(w))\Psi'(w)dw=\frac{1}{c_n}\frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}}\frac{f(w)h_n(w)}{w^{n+1}}dw=\frac{1}{c_n}[fh_n]_n.$$

[1]We note that in [51, Theorem 1.8] it is assumed that E is a compact set bounded by a Jordan curve. However, as pointed out to us by the author, the result remains valid if E verifies the conditions imposed in this paper.

and

$$\langle zF, p_n\rangle_\mu = \int zF(z)\overline{p_n(z)}d\mu(z) = \int \left(\frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \frac{\zeta F(\zeta)}{\zeta - z}d\zeta\right)\overline{p_n(z)}d\mu(z)$$

$$= \frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \zeta F(\zeta)\int \frac{\overline{p_n(z)}}{\zeta - z}d\mu(z)d\zeta = \frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \zeta F(\zeta)s_n(\zeta)d\zeta$$

$$= \frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}} \Psi(w)f(w)s_n(\Psi(w))\Psi'(w)dw = \frac{1}{c_n}\frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}} \frac{\Psi(w)f(w)h_n(w)}{w^{n+1}}dw = \frac{1}{c_n}[\Psi f h_n]_n.$$

If $\mu \in \mathcal{S}(E)$, then $\mu \in \mathcal{R}(E)$ and using Lemma II.3.3, we have

$$\lim_{n\to\infty} h_n(w) = \lim_{n\to\infty} c_n w^{n+1} s_n(\Psi(w))\Psi'(w)$$

$$= w\Psi'(w)\lim_{n\to\infty}\frac{c_n w^n}{p_n(\Psi(w))}\lim_{n\to\infty} p_n(\Psi(w))s_n(\Psi(w)) = \frac{1}{S(\Psi(w))} =: h(w),$$

uniformly inside U. □

Finally, we state a lemma due to A.A. Gonchar which is quite useful in the theory of rational approximation. We recall the definition of the logarithmic capacity of A :

$$\operatorname{cap}(A) := e^{-\gamma(A)},$$

where

$$\gamma(A) := \inf\{\int\int \log\frac{1}{|z-t|}d\mu(z)d\mu(t) \ : \ \mu \geq 0, \quad \operatorname{supp}(\mu) \subset A, \quad \|\mu\| = 1\}.^2 \qquad \text{(II.18)}$$

Definition II.3.5. Let $W(z)$ and $W_n(z)$, $n \in \mathbb{N}$, be functions defined on an open region Ω. We say that the sequence $W_n(z)$ *converges to* $W(z)$ *in capacity inside* Ω, if for any $\varepsilon > 0$ and for any compact subset K of Ω,

$$\lim_{n\to\infty} \operatorname{cap}(\{z \in K : |W_n(z) - W(z)| \geq \varepsilon\}) = 0.$$

Lemma II.3.6 (Gonchar [25])**.** *Suppose that the sequence* $W_n(z)$ *converges to* $W(z)$ *in*

[2] $\gamma(A)$ is known as the Ronin constant of A.

capacity inside an open region Ω. If the $W_n(z)$ are meromorphic and have no more than $m < \infty$ poles in Ω, and $W(z)$ is meromorphic and has exactly m poles $z_1, \ldots, z_m$ in Ω, then the sequence $W_n(z)$ converges to $W(z)$ uniformly inside $\Omega \setminus \{z_1, \ldots, z_m\}$, for all sufficiently large n, $W_n(z)$ has exactly m poles in Ω and the poles of the $W_n(z)$ converge to the poles of $W(z)$ according to their order.

II.3.2 Proofs of main results

Proof of Theorem II.2.1. From (II.8), it follows that

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n+l}(z)} = \frac{1}{\Phi(z)^l}, \quad l = 0, 1, \ldots, \tag{II.19}$$

uniformly inside $\overline{\mathbb{C}} \setminus E$. By (II.19) and Lemma II.3.3, for any $l = 0, 1, \ldots,$ we have

$$\lim_{n\to\infty} \frac{s_{n+l}(z)}{s_n(z)} = \lim_{n\to\infty} \frac{p_n(z)}{p_{n+l}(z)} \frac{p_{n+l}(z)s_{n+l}(z)}{p_n(z)s_n(z)} = \frac{1}{\Phi(z)^l} \frac{\Phi'(z)/\Phi(z)}{\Phi'(z)/\Phi(z)} = \frac{1}{\Phi(z)^l}, \tag{II.20}$$

uniformly inside $\overline{\mathbb{C}} \setminus E$. Furthermore,

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \tag{II.21}$$

and

$$\lim_{n\to\infty} |s_n(z)|^{1/n} = \frac{1}{|\Phi(z)|}, \tag{II.22}$$

uniformly inside $\mathbb{C} \setminus E$, are trivial consequences of (II.19) and (II.20), respectively.

By the definition of Padé-orthogonal approximants and the condition (II.21), we have

$$Q^{\mu}_{n,m}(z)F(z) - P^{\mu}_{n,m}(z) = \sum_{k=n+m+1}^{\infty} a_{k,n} p_k(z), \quad z \in D_{\rho_0(F)}, \tag{II.23}$$

where

$$a_{k,n} := \langle Q^{\mu}_{n,m} F,\, p_k \rangle_{\mu}, \quad k = 0, 1, \ldots,$$

$$a_{k,n} = 0, \quad k = n+1, \dots, n+m.$$

Using Cauchy's integral formula and Fubini's theorem, we obtain, for $k = 0, 1, \dots,$

$$\begin{aligned} a_{k,n} &:= \langle Q^{\mu}_{n,m} F,\, p_k \rangle_{\mu} = \int \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{Q^{\mu}_{n,m}(t) F(t)}{t-z} dt \overline{p_k(z)} d\mu(z) \\ &= \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} Q^{\mu}_{n,m}(t) F(t) \int \frac{\overline{p_k(z)}}{t-z} d\mu(z) dt = \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} Q^{\mu}_{n,m}(t) F(t) s_k(t) dt, \end{aligned} \tag{II.24}$$

where $1 < \rho_1 < \rho_0(F)$. Let $\{\alpha_1, \dots, \alpha_\gamma\}$ be the set of the distinct poles of F in $D_{\rho_m(F)}$ and m_k be the multiplicity of α_k so that

$$Q(z) := \prod_{j=1}^{m} (z - \lambda_j) = \prod_{k=1}^{\gamma} (z - \alpha_k)^{m_k}, \quad \sum_{k=1}^{\gamma} m_k =: m.$$

Multiplying the equation (II.23) by Q and expanding $\sum_{k=n+m+1}^{\infty} a_{k,n} Q p_k (= QQ^{\mu}_{n,m}F - QP^{\mu}_{n,m} \in H(D_{\rho_m(F)}))$ in terms of the Fourier series corresponding to the orthonormal system $\{p_\nu\}_{\nu=0}^{\infty}$, we obtain that for $z \in D_{\rho_m(F)}$,

$$Q(z) Q^{\mu}_{n,m}(z) F(z) - Q(z) P^{\mu}_{n,m}(z) = \sum_{k=n+m+1}^{\infty} a_{k,n} Q(z) p_k(z) = \sum_{\nu=0}^{\infty} b_{\nu,n} p_\nu(z), \tag{II.25}$$

where

$$b_{\nu,n} := \sum_{k=n+m+1}^{\infty} a_{k,n} \langle Q p_k,\, p_\nu \rangle_{\mu}, \quad \nu = 0, 1, \dots.$$

First of all, we will estimate $|a_{k,n}|$ in terms of $|\tau_{k,n}|$ where

$$\tau_{k,n} := \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} Q^{\mu}_{n,m}(t) F(t) s_k(t) dt, \quad \rho_{m-1}(F) < \rho_2 < \rho_m(F), \quad k = 0, 1 \dots. \tag{II.26}$$

Note that the only difference between the integral in (II.26) and the last integral in (II.24) is the domains of the integrals. The greater number ρ of Γ_ρ will allow to have a better bound on $|s_k|$. For each $k \geq 0$, the function $Q^{\mu}_{n,m} F s_k$ is meromorphic on $\overline{D_{\rho_2}} \setminus D_{\rho_1} = \{z \in \mathbb{C} : \rho_1 \leq |\Phi(z)| \leq \rho_2\}$ and has poles at $\alpha_1, \dots, \alpha_\gamma$ with multiplicities at most $m_1, \dots, m_\gamma$, respectively.

Applying Cauchy's residue theorem to the functions $Q^{\mu}_{n,m}Fs_k$, we have

$$\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^{\mu}_{n,m}(t)F(t)s_k(t)dt - \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} Q^{\mu}_{n,m}(t)F(t)s_k(t)dt = \sum_{j=1}^{\gamma}\text{res}(Q^{\mu}_{n,m}Fs_k,\, \alpha_j), \quad \text{(II.27)}$$

for $k \geq 0$. Recall that the limit formula for residue is

$$\text{res}(Q^{\mu}_{n,m}Fs_k,\, \alpha_j) = \frac{1}{(m_j-1)!}\lim_{z\to\alpha_j}((z-\alpha_j)^{m_j}Q^{\mu}_{n,m}(z)F(z)s_k(z))^{(m_j-1)}. \quad \text{(II.28)}$$

By the Leibniz formula and the fact that for n sufficiently large, $s_n(z) \neq 0$ for $z \in \mathbb{C}\setminus E$ (see Lemma II.3.3), we can transform the expression under the limit sign as follows

$$((z-\alpha_j)^{m_j}Q^{\mu}_{n,m}(z)F(z)s_k(z))^{(m_j-1)} = \left((z-\alpha_j)^{m_j}Q^{\mu}_{n,m}(z)F(z)s_n(z)\frac{s_k(z)}{s_n(z)}\right)^{(m_j-1)}$$

$$= \sum_{p=0}^{m_j-1}\binom{m_j-1}{p}((z-\alpha_j)^{m_j}Q^{\mu}_{n,m}(z)F(z)s_n(z))^{(m_j-1-p)}\left(\frac{s_k(z)}{s_n(z)}\right)^{(p)}.$$

To avoid long expressions, let us introduce the following notation:

$$\beta_n(j,p) := \frac{1}{(m_j-1)!}\begin{pmatrix} m_j-1 \\ p \end{pmatrix}\lim_{z\to\alpha_j}((z-\alpha_j)^{m_j}Q^{\mu}_{n,m}(z)F(z)s_n(z))^{(m_j-1-p)},$$

for $j = 1, \ldots, \gamma$ and $p = 0, \ldots, m_j - 1$ (notice that the $\beta_n(j,p)$ do not depend on k), so we can rewrite the equality (II.27) as

$$a_{k,n} = \tau_{k,n} - \sum_{j=1}^{\gamma}\left(\sum_{p=0}^{m_j-1}\beta_n(j,p)\left(\frac{s_k}{s_n}\right)^{(p)}(\alpha_j)\right), \quad n \geq n_0 \quad \text{and} \quad k = 0, 1, \ldots. \quad \text{(II.29)}$$

Since $a_{k,n} = 0$, for $k = n+1, n+2, \ldots, n+m$, it follows from (II.29) that

$$\sum_{j=1}^{\gamma}\sum_{p=0}^{m_j-1}\beta_n(j,p)\left(\frac{s_k}{s_n}\right)^{(p)}(\alpha_j) = \tau_{k,n}, \quad k = n+1, \ldots, n+m. \quad \text{(II.30)}$$

We will view this as a system of m equations with m unknowns $\beta_n(j,p)$. If we can show that

$$\Lambda_n := \begin{vmatrix} \left(\frac{s_{n+1}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+1}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+1}}{s_n}\right)^{(m_j-1)}(\alpha_j) \\ \left(\frac{s_{n+2}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+2}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+2}}{s_n}\right)^{(m_j-1)}(\alpha_j) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{s_{n+m}}{s_n}\right)(\alpha_j) & \left(\frac{s_{n+m}}{s_n}\right)'(\alpha_j) & \cdots & \left(\frac{s_{n+m}}{s_n}\right)^{(m_j-1)}(\alpha_j) \end{vmatrix}_{j=1,\dots,\gamma} \neq 0, \qquad \text{(II.31)}$$

(this expression represents the determinant of order m in which the indicated groups of columns are successively written out for $j=1,\dots,\gamma$), then we can express $\beta_n(j,p)$ in terms of $(s_k/s_n)^{(p)}(\alpha_j)$ and $\tau_{k,n}$, for $k=n+1,\dots n+m$. However, since

$$\lim_{n\to\infty}\Lambda_n = \Lambda := \begin{vmatrix} R(\alpha_j) & R'(\alpha_j) & \cdots & R^{(m_j-1)}(\alpha_j) \\ R^2(\alpha_j) & (R^2)'(\alpha_j) & \cdots & (R^2)^{(m_j-1)}(\alpha_j) \\ \vdots & \vdots & \vdots & \vdots \\ R^m(\alpha_j) & (R^m)'(\alpha_j) & \cdots & (R^m)^{(m_j-1)}(\alpha_j) \end{vmatrix}_{j=1,\dots,\gamma}$$

$$= \prod_{j=1}^{\gamma}(m_j-1)!! \prod_{j=1}^{\gamma}(-\Phi'(\alpha_j))^{m_j(m_j-1)/2} \prod_{j=1}^{\gamma}\Phi(\alpha_j)^{-m_j^2} \prod_{1\le i<j\le\gamma}\left(\frac{1}{\Phi(\alpha_j)}-\frac{1}{\Phi(\alpha_i)}\right)^{m_i m_j},$$

where $R(z)=1/\Phi(z)$ and $n!! = 0!1!\cdots$n! (using for example [52, Theorem 1] for proving this), for sufficiently large n, $\Lambda_n \neq 0$. In fact, for sufficiently large n, $|\Lambda_n| \geq c_1 > 0$ where the number c_1 does not depend on n (from now on, we will denote some constants that do not depend on n by $c_2, c_3, \dots$ and we will consider only n large enough so that $|\Lambda_n| \geq c_1 > 0$).

Applying Cramer's rule to (II.30), we have

$$\beta_n(j,p) = \frac{\Lambda_n(j,p)}{\Lambda_n} = \frac{1}{\Lambda_n}\sum_{s=1}^{m}\tau_{n+s,n}C_n(s,q), \qquad \text{(II.32)}$$

where $\Lambda_n(j,p)$ is the determinant obtained from Λ_n replacing the column with index $q = (\sum_{l=0}^{j-1} m_l)+p+1$ (where we define $m_0 := 0$) with the column $[\tau_{n+1,n}\ \dots\ \tau_{n+m,n}]^T$ and $C_n(s,q)$ is the determinant of the $(s,q)^{\text{th}}$ cofactor matrix of $\Lambda_n(j,p)$. Substituting $\beta_n(j,p)$ in the formula (II.29) with the expression in (II.32), we obtain

$$a_{k,n} = \tau_{k,n} - \frac{1}{\Lambda_n} \sum_{j=1}^{\gamma} \sum_{p=0}^{m_j-1} \sum_{s=1}^{m} \tau_{n+s,n} C_n(s,q) \left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j), \quad k \geq n+m+1. \tag{II.33}$$

Let $\delta > 0$ be sufficiently small so that $\rho_0(F) - 2\delta > 1$ and $\epsilon > 0$ be sufficiently small so that

$$\{z \in \mathbb{C} : |z - \alpha_j| = \epsilon\} \subset \{z \in \mathbb{C} : |\Phi(z)| \geq \rho_0(F) - \delta\}$$

and

$$\left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j) = \frac{p!}{2\pi i} \int_{|z-\alpha_j|=\epsilon} \frac{s_k(z)}{s_n(z)(z-\alpha_j)^{p+1}} dz, \quad k = 0, 1 \ldots, \quad p = 0, \ldots, m_j - 1. \tag{II.34}$$

Applying (II.20) and (II.34), we can easily check that

$$\left| \left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j) \right| \leq c_2, \quad p = 0, \ldots, m_j - 1, \quad j = 1, \ldots, \gamma, \quad k = n+1, \ldots, n+m, \tag{II.35}$$

for $n \geq n_1$, and

$$\left| \left(\frac{s_k}{s_n}\right)^{(p)} (\alpha_j) \right| \leq \frac{c_3}{(\rho_0(F) - 2\delta)^{k-n}}, \quad p = 0, \ldots, m_j - 1, \quad j = 1, \ldots, \gamma, \quad k \geq n+m+1, \tag{II.36}$$

for $n \geq n_2$. The equation (II.35) implies that

$$|C_n(s,q)| \leq (m-1)! c_2^{m-1} = c_4, \quad s, q = 1, \ldots, m, \tag{II.37}$$

for $n \geq n_3$. Combining the estimates (II.35), (II.36), (II.37), and $|\Lambda_n| \geq c_1 > 0$, we see from (II.33) that

$$\begin{aligned} |a_{k,n}| &\leq |\tau_{k,n}| + \frac{m c_4 c_3}{c_1} \frac{1}{(\rho_0(F) - 2\delta)^{k-n}} \sum_{s=1}^{m} |\tau_{n+s,n}| \\ &\leq |\tau_{k,n}| + \frac{c_5}{(\rho_0(F) - 2\delta)^{k-n}} \sum_{s=1}^{m} |\tau_{n+s,n}|, \qquad k \geq n+m+1, \end{aligned} \tag{II.38}$$

for $n \ge n_4$.

Secondly, we will give an estimate of $|b_{\nu,n}|$ in terms of $|\tau_{k,n}|$. By the Cauchy-Schwarz inequality and the orthonormality of p_ν, we have

$$|\langle Qp_k,\, p_\nu\rangle_\mu|^2 \le \langle Qp_k,\, Qp_k\rangle_\mu \langle p_\nu,\, p_\nu\rangle_\mu \le \max_{z\in E}|Q(z)|^2 = c_6, \quad k,\nu = 0,1,\ldots. \tag{II.39}$$

By (II.38), (II.39), and the fact that $\sum_{k=n+m+1}^{\infty}(\rho_0(F)-2\delta)^{n-k} < \infty$, we obtain, for n sufficiently large and for all $\nu \ge 0$,

$$\begin{aligned} |b_{\nu,n}| &\le \sum_{k=n+m+1}^{\infty} |a_{k,n}||\langle Qp_k,\, p_\nu\rangle| \le \sqrt{c_6}\sum_{k=n+m+1}^{\infty}|a_{k,n}| \\ &\le \sqrt{c_6}\left(\sum_{k=n+m+1}^{\infty}|\tau_{k,n}| + c_5 \sum_{k=n+m+1}^{\infty}\frac{1}{(\rho_0(F)-2\delta)^{k-n}}\sum_{s=1}^{m}|\tau_{n+s,n}|\right) \\ &\le c_7 \sum_{k=n+1}^{\infty}|\tau_{k,n}|. \end{aligned} \tag{II.40}$$

Thirdly, we show that $P^\mu_{n,m}/Q^\mu_{n,m}$ converges in capacity to F inside $D_{\rho_m(F)}$, as $n\to\infty$. Let K be a compact subset of $D_{\rho_m(F)}$ and σ be the smallest positive number such that $\sigma \ge 1$ and $K \subset \overline{D_\sigma} \subset D_{\rho_m(F)}$. Choose $\delta > 0$ so small that

$$\rho_2 := \rho_m(F) - \delta > \rho_{m-1}(F), \quad \rho_0(F) - 2\delta > 1, \quad \text{and} \quad \frac{\sigma+\delta}{\rho_2-\delta} < 1. \tag{II.41}$$

We write (II.25) in the form

$$|Q(z)Q^\mu_{n,m}(z)F(z) - Q(z)P^\mu_{n,m}(z)| \le \sum_{\nu=0}^{n+m}|b_{\nu,n}||p_\nu(z)| + \sum_{\nu=n+m+1}^{\infty}|b_{\nu,n}||p_\nu(z)|. \tag{II.42}$$

Define

$$A_n^1(z) := \frac{\sum_{\nu=0}^{n+m}|b_{\nu,n}||p_\nu(z)|}{|Q(z)Q^\mu_{n,m}(z)|} \quad \text{and} \quad A_n^2(z) := \frac{\sum_{\nu=n+m+1}^{\infty}|b_{\nu,n}||p_\nu(z)|}{|Q(z)Q^\mu_{n,m}(z)|},$$

and let $Q^\mu_{n,m}(z) := \prod_{j=1}^{m_n}(z-\lambda_{n,j})$. Therefore, the relation (II.42) implies

$$\left|F(z) - \frac{P^\mu_{n,m}(z)}{Q^\mu_{n,m}(z)}\right| \le A_n^1(z) + A_n^2(z),$$

for all $z \in \hat{D}_\sigma := \overline{D}_\sigma \setminus (\cup_{n=0}^\infty \{\lambda_{n,1}, \dots, \lambda_{n,m_n}\} \cup \{\lambda_1, \dots, \lambda_m\})$.

Let us bound $A_n^1(z)$ from above. We will first estimate $|\tau_{k,n}/Q_{n,m}^\mu(z)|$ for $z \in \hat{D}_\sigma$ and for $k \geq n+1$. By definition of $\tau_{k,n}$,

$$\frac{\tau_{k,n}}{Q_{n,m}^\mu(z)} = \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} s_k(t)F(t)\frac{Q_{n,m}^\mu(t)}{Q_{n,m}^\mu(z)}dt, \quad k \geq n+1, \tag{II.43}$$

so we shall approximate the factors multiplying $F(t)$ in the integral signs separately. For n sufficiently large,

$$|s_k(t)| \leq \frac{c_8}{(\rho_2 - \delta)^k}, \quad k \geq n+1.$$

Define

$$Q_{n,m,\rho_2}^\mu(t) := \prod_{\lambda_{n,j} \in D_{\rho_2}} (t - \lambda_{n,j}).$$

It is easy to see that

$$\left|\frac{t-\zeta}{z-\zeta}\right| \leq c_9,$$

for all $t \in \Gamma_{\rho_2}, z \in \hat{D}_\sigma$, and $\zeta \in \mathbb{C} \setminus D_{\rho_2}$ (notice that the last condition in (II.41) implies $\rho_2 > \sigma$). Then,

$$\left|\frac{Q_{n,m}^\mu(t)}{Q_{n,m}^\mu(z)}\right| \leq {c_9}^m \left|\frac{Q_{n,m,\rho_2}^\mu(t)}{Q_{n,m,\rho_2}^\mu(z)}\right| \leq \frac{c_{10}}{|Q_{n,m,\rho_2}^\mu(z)|}, \quad z \in \hat{D}_\sigma, \quad t \in \Gamma_{\rho_2}. \tag{II.44}$$

By (II.43), we obtain

$$\left|\frac{\tau_{k,n}}{Q_{n,m}^\mu(z)}\right| \leq \frac{c_{11}}{|Q_{n,m,\rho_2}^\mu(z)|(\rho_2-\delta)^k}, \quad z \in \hat{D}_\sigma, \quad k \geq n+1, \quad n \geq n_5,$$

which implies

$$\left|\frac{b_{\nu,n}}{Q_{n,m}^\mu(z)}\right| \leq \frac{c_{12}}{|Q_{n,m,\rho_2}^\mu(z)|(\rho_2-\delta)^n}, \quad z \in \hat{D}_\sigma, \quad n \geq n_6. \tag{II.45}$$

Applying (II.21) and the maximum modulus principle, we have

$$|p_\nu(z)| \leq c_{13}(\sigma+\delta)^\nu, \quad z \in \overline{D_\sigma}, \quad \nu \geq 0. \tag{II.46}$$

Using (II.45) and (II.46), we obtain the estimate:

$$A_n^1(z) = \frac{1}{|Q(z)|}\sum_{\nu=0}^{n+m}\frac{|b_{\nu,n}||p_\nu(z)|}{|Q_{n,m}^\mu(z)|} \le \frac{(n+m+1)c_{12}c_{13}(\sigma+\delta)^{n+m}}{|Q(z)Q_{n,m,\rho_2}^\mu(z)|(\rho_2-\delta)^n}, \quad z\in \hat{D}_\sigma.$$

We choose $\theta>0$ such that $(\sigma+\delta)/(\rho_2-\delta)<\theta<1$. Therefore, for n sufficiently large,

$$A_n^1(z) \le \frac{c_{14}\theta^n}{|Q(z)Q_{n,m,\rho_2}^\mu(z)|}, \quad z\in \hat{D}_\sigma. \tag{II.47}$$

Next, let us approximate $A_n^2(z)$. Since $\deg(QP_{n,m}^\mu)\le n+m$, by a computation similar to (II.24), we obtain

$$b_{\nu,n} = \langle QQ_{n,m}^\mu F,\, p_\nu\rangle_\mu = \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q(t)Q_{n,m}^\mu(t)F(t)s_\nu(t)dt, \quad \nu\ge n+m+1. \tag{II.48}$$

As before, from (II.22) and (II.48), we have

$$\frac{|b_{\nu,n}|}{|Q(z)Q_{n,m}^\mu(z)|} \le \frac{c_{15}}{|Q(z)Q_{n,m,\rho_2}^\mu(z)|(\rho_2-\delta)^\nu}, \quad z\in\hat{D}_\sigma, \quad \nu\ge n+m+1, \tag{II.49}$$

for $n\ge n_7$. Then, using (II.46) and (II.49), for n sufficiently large, we obtain

$$A_n^2(z) \le \frac{c_{16}(\sigma+\delta)^n}{|Q(z)Q_{n,m,\rho_2}^\mu(z)|(\rho_2-\delta)^n} < \frac{c_{17}\theta^n}{|Q(z)Q_{n,m,\rho_2}^\mu(z)|}, \quad z\in\hat{D}_\sigma. \tag{II.50}$$

Combining (II.47) and (II.50), we have, for n sufficiently large,

$$\left|F(z)-\frac{P_{n,m}^\mu(z)}{Q_{n,m}^\mu(z)}\right| \le \frac{c_{18}\theta^n}{|Q(z)Q_{n,m,\rho_2}^\mu(z)|}, \quad z\in\hat{D}_\sigma. \tag{II.51}$$

Let $T_n(z):=Q(z)Q_{n,m,\rho_2}^\mu(z)$. Then, $T_n(z)$ is a monic polynomial of degree at most $2m$. Let $\varepsilon>0$. Clearly,

$$e_n := \left\{z\in\hat{D}_\sigma : \left|F(z)-\frac{P_{n,m}^\mu(z)}{Q_{n,m}^\mu(z)}\right|\ge\varepsilon\right\} \subset \left\{z\in\hat{D}_\sigma : \left|Q(z)Q_{n,m,\rho_2}^\mu(z)\right| \le \frac{c_{18}\theta^n}{\varepsilon}\right\} =: E_n.$$

The capacity function is monotonic and has the well-known property,

$$\operatorname{cap}\{z \in \mathbb{C} : |z^n + a_{n-1}z^{n-1} + \ldots + a_0| \le \rho^n\} = \rho, \quad \rho > 0.$$

Hence, we find that for n sufficiently large

$$\operatorname{cap} e_n \le \operatorname{cap} E_n \le \left(\frac{1}{\varepsilon} c_{18}\theta^n\right)^{1/\deg T_n} \le \left(\frac{1}{\varepsilon} c_{18}\theta^n\right)^{1/2m} \le c_{19}\theta^{n/2m}.$$

This means that $\operatorname{cap}\{z \in \overline{D}_\sigma : \left|F(z) - \frac{P^\mu_{n,m}(z)}{Q^\mu_{n,m}(z)}\right| \ge \varepsilon\} = \operatorname{cap} e_n \to 0$, as $n \to \infty$. This proves that $[n/m]^\mu_F$ converges in capacity to F inside $D_{\rho_m(F)}$, as $n \to \infty$. Applying Gonchar's lemma, we have $[n/m]^\mu_F$ converges to F uniformly inside $D_{\rho_m(F)} \setminus \{\lambda_1, \ldots, \lambda_m\}$, as $n \to \infty$. In addition we get that each pole of F in $D_{\rho_m(F)}$ attracts as many zeros of $Q^\mu_{n,m}$ as its order. Therefore, $\deg Q^\mu_{n,m} = m$ for all sufficiently large n which in turn implies that $[n/m]^\mu_F$ is uniquely determined for such n. We have obtained (II.10) and (II.11) except for the rate of convergence exhibited in those relations.

To show (II.10), we let K be a compact subset of $D_{\rho_m(F)} \setminus \{\lambda_1, \ldots \lambda_m\}$, σ be the smallest positive number such that $\sigma \ge 1$ and $K \subset \overline{D_\sigma} \subset D_{\rho_m(F)}$, and choose an arbitrarily small number $\delta > 0$ such that ρ_2 satisfies (II.41). Note that what we just proved implies

$$\max_{z \in D_{\rho_m(F)}} |Q^\mu_{n,m}(z)| \le c_{20}.$$

From (II.22) and (II.48), for $n \ge n_8$,

$$|b_{\nu,n}| = \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q(t)Q^\mu_{n,m}(t)F(t)s_\nu(t)dt\right| \le \frac{c_{21}}{(\rho_2 - \delta)^\nu}, \quad \nu \ge n + m + 1,$$

$$|\tau_{k,n}| = \left|\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^\mu_{n,m}(t)F(t)s_k(t)dt\right| \le \frac{c_{22}}{(\rho_2 - \delta)^k}, \quad k \ge n + 1. \tag{II.52}$$

Then, by (II.40), for $n \ge n_9$,

$$|b_{\nu,n}| \le c_7 \sum_{k=n+1}^{\infty} |\tau_{k,n}| \le \frac{c_{23}}{(\rho_2 - \delta)^n}, \quad 0 \le \nu \le n + m.$$

Using (II.46), we can prove that for $z \in \overline{D_\sigma}$ and for $n \geq n_{10}$,

$$|Q(z)Q^{\mu}_{n,m}(z)F(z) - Q(z)P^{\mu}_{n,m}(z)| \leq \sum_{\nu=0}^{\infty} |b_{\nu,n}||p_\nu(z)| \leq c_{24}\left(\left(\frac{\sigma+\delta}{\rho_2-\delta}\right)+\delta\right)^n. \qquad \text{(II.53)}$$

Dividing the previous inequality by $|QQ^{\mu}_{n,m}|$, we have, for $n \geq n_{10}$,

$$\left|F(z) - \frac{P^{\mu}_{n,m}(z)}{Q^{\mu}_{n,m}(z)}\right| \leq \frac{c_{25}}{|Q(z)Q^{\mu}_{n,m}(z)|}\left(\left(\frac{\sigma+\delta}{\rho_2-\delta}\right)+\delta\right)^n, \quad z \in K.$$

Since for n sufficiently large, the zeros of $Q^{\mu}_{n,m}(z)$ are distant from K, it follows that

$$\limsup_{n\to\infty} \|F - [n/m]^{\mu}_{F}\|_K^{1/n} \leq \left(\frac{\sigma+\delta}{\rho_2-\delta}\right)+\delta.$$

Taking $\delta \to 0^+$ and $\rho_2 \to \rho_m(F)$, we obtain (II.10). Moreover, if K is any compact set contained in $D_{\rho_m(F)}$, we can use similar arguments to show that (II.53) implies

$$\limsup_{n\to\infty} \|QFQ^{\mu}_{n,m} - QP^{\mu}_{n,m}\|_K^{1/n} \leq \frac{\|\Phi\|_K}{\rho_m(F)}. \qquad \text{(II.54)}$$

Finally, we prove (II.11). We first need to show that for $k = 1, \ldots, \gamma$,

$$\limsup_{n\to\infty} |(Q^{\mu}_{n,m})^{(j)}(\alpha_k)|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}, \quad j = 0, \ldots, m_k - 1. \qquad \text{(II.55)}$$

Let $\varepsilon > 0$ be arbitrarily small so that $\overline{\mathbb{B}(\alpha_k, \varepsilon)} \subset D_{\rho_m(F)}$ for all $k = 1, \ldots, \gamma$ and the disks $\overline{\mathbb{B}(\alpha_k, \varepsilon)}$, $k = 1, \ldots, \gamma$, are pairwise disjoint. As a consequence of (II.54), we have

$$\limsup_{n\to\infty} \|(z-\alpha_k)^{m_k}FQ^{\mu}_{n,m} - (z-\alpha_k)^{m_k}P^{\mu}_{n,m}\|_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}}^{1/n} \leq \frac{\|\Phi\|_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}}}{\rho_m(F)}, \qquad \text{(II.56)}$$

so by Cauchy's integral formula for the derivative, we obtain

$$\limsup_{n\to\infty} \left\| \left[(z-\alpha_k)^{m_k}FQ^{\mu}_{n,m} - (z-\alpha_k)^{m_k}P^{\mu}_{n,m}\right]^{(j)} \right\|_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}}^{1/n} \leq \frac{\|\Phi\|_{\overline{\mathbb{B}(\alpha_k,\varepsilon)}}}{\rho_m(F)}, \qquad \text{(II.57)}$$

for all $j \geq 0$. Since $\varepsilon > 0$ can be taken arbitrarily small, the inequality (II.56) implies that

$$\limsup_{n\to\infty} |L_k Q^{\mu}_{n,m}(\alpha_k)|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)},$$

where $L_k := \lim_{z\to\alpha_k}(z-\alpha_k)^{m_k}F(z) \neq 0$ (because F has a pole of order m_k at α_k). Therefore,

$$\limsup_{n\to\infty} |Q^{\mu}_{n,m}(\alpha_k)|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}.$$

Proceeding the proof by induction, we let $r \leq m_k - 1$ and assume that

$$\limsup_{n\to\infty} |(Q^{\mu}_{n,m})^{(j)}(\alpha_k)|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}, \quad j = 0, \ldots, r-1. \tag{II.58}$$

We want to show that the above inequality also holds for $j = r$. Using (II.57), since $r < m_k$, we obtain

$$\limsup_{n\to\infty} |[(z-\alpha_k)^{m_k} F Q^{\mu}_{n,m}]^{(r)}(\alpha_k)|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}. \tag{II.59}$$

By the Leibniz formula, we have

$$[(z-\alpha_k)^{m_k} F Q^{\mu}_{n,m}]^{(r)}(\alpha_k) = \sum_{l=0}^{r} \binom{r}{l} [(z-\alpha_k)^{m_k} F]^{(l)}(\alpha_k)(Q^{\mu}_{n,m})^{(r-l)}(\alpha_k).$$

Therefore, by (II.58), (II.59), and the fact that $L_k \neq 0$, we have

$$\limsup_{n\to\infty} |(Q^{\mu}_{n,m})^{(r)}(\alpha_k)|^{1/n} \leq \frac{|\Phi(\alpha_k)|}{\rho_m(F)}$$

which completes the induction and the proof of (II.55).

Let $\{q_{k,s}\}_{k=1,\ldots,\gamma,\, s=0,\ldots,m_k-1}$ be a system of polynomials such that $\deg q_{k,s} \leq m-1$ for all k, s and

$$q^{(i)}_{k,s}(\alpha_j) = \delta_{j,k}\delta_{i,s}, \quad 1 \leq j \leq \gamma, \quad 0 \leq i \leq m_j - 1.$$

It is not difficult to check that $q_{k,s}$ exist (using for example [52, Theorem 1]). Then,

$$Q_{n,m}^{\mu}(z) = \sum_{k=1}^{\gamma} \sum_{s=0}^{m_k-1} (Q_{n,m}^{\mu})^{(s)}(\alpha_k) q_{k,s}(z) + Q_m(z).$$

This formula combined with (II.55) imply

$$\limsup_{n\to\infty} \|Q_{n,m}^{\mu} - Q_m\|^{1/n} \leq \frac{\max_{k=1,\ldots,\gamma} |\Phi(\alpha_k)|}{\rho_m(F)}.$$

□

Proof of Theorem II.2.6. First of all, we will prove that (a) implies (c) and (b) implies (c) at the same time by using Lemma II.3.2 for $m = 1$. Assume that (a) or (b) is satisfied, that is,

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \tau, \quad \text{or} \quad \lim_{n\to\infty} \frac{\langle zF, p_n\rangle_\mu}{F_n} = \lambda. \tag{II.60}$$

Let

$$\tau_n := \frac{F_n}{F_{n+1}}, \quad \lambda_n := \frac{\langle zF, p_n\rangle_\mu}{F_n}.$$

Define

$$\alpha_n^1(w) := \frac{c_n}{c_{n+1}} \frac{\tau_n h_{n+1}(w)}{w} - h_n(w), \quad \alpha_n^2(w) := \frac{h_{n+1}(w)(\lambda_{n+1} - \Psi(w))}{w}, \quad w \in U$$

and

$$\eta_{n,0}(w) := 1, \quad w \in U.$$

The functions $\alpha_n^1(w)$ and $\alpha_n^2(w)$ are holomorphic on U. By Lemma II.3.4, for $\varepsilon > 0$ sufficiently small so that $f(w)$ is holomorphic in a neighborhood of $\gamma_{1+\varepsilon}$,

$$[f\alpha_n^1 \eta_{n,0}]_n = \frac{c_n}{c_{n+1}} \frac{\tau_n}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{f(w)h_{n+1}(w)}{w^{n+2}} dw - \frac{1}{2\pi i} \int_{\gamma_{1+\varepsilon}} \frac{f(w)h_n(w)}{w^{n+1}} dw$$

$$= \frac{c_n}{c_{n+1}} \frac{F_n}{F_{n+1}} [fh_{n+1}]_{n+1} - [fh_n]_n = 0$$

and

$$[f\alpha_n^2\eta_{n,0}]_n = \frac{\lambda_{n+1}}{2\pi i}\int_{\gamma_{1+\varepsilon}} \frac{f(w)h_{n+1}(w)}{w^{n+2}}dw - \frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}} \frac{\Psi(w)f(w)h_{n+1}(w)}{w^{n+2}}dw$$

$$= \frac{c_{n+1}}{c_{n+1}}\frac{\langle zF, p_{n+1}\rangle_\mu}{F_{n+1}}[fh_{n+1}]_{n+1} - [\Psi f h_{n+1}]_{n+1} = 0.$$

If (a) holds, then

$$\alpha^1(w) := \lim_{n\to\infty} \alpha_n^1(w) = h(w)\left(\frac{\tau}{w} - 1\right), \quad \text{uniformly inside } U,$$

and if (b) holds, then

$$\alpha^2(w) := \lim_{n\to\infty} \alpha_n^2(w) = \frac{h(w)(\lambda - \Psi(w))}{w}, \quad \text{uniformly inside } U.$$

Since $h(w)$ is never zero on U, each function $\alpha^j(w)$, $j = 1, 2$, has at most one zero in U (which is τ). It is also easy to check that $\eta_{n,0}(w)$ satisfies the rest of the required conditions in Lemma II.3.2. Moreover, if $f_n = 0$ for $n \geq n_0$, then $F_n = [fh_n]_n = 0$ (recall that $h_n(w)$ is analytic at ∞). Therefore, by (ii) in Lemma II.3.2, $\tau \in U \setminus \{\infty\}$ and $\lim_{n\to} f_n/f_{n+1} = \tau$.

Now, we prove that (c) implies (a) and (b) by Lemma II.3.1 for $m = 1$. Assume that $\lim_{n\to\infty} f_n/f_{n+1} = \tau$. By Fabry's theorem, we have (II.14). Set

$$\tau_n := \frac{f_n}{f_{n+1}}, \quad \alpha_n(w) := \frac{\tau_n}{w} - 1, \quad \text{and} \quad \eta_{n,0}(w) = 1, \quad w \in U.$$

Therefore,

$$[f\alpha_n\eta_{n,0}]_n = \tau_n f_{n+1} - f_n = 0,$$

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) = \frac{\tau}{w} - 1, \quad \text{uniformly inside } U,$$

$$\eta_0(w) := \lim_{n\to\infty} \eta_{n,0}(w) = 1 = w^0, \quad \text{uniformly on } \mathbb{C},$$

and $\alpha(w)$ has at most one zero in U. Applying (II.16) in Lemma II.3.1, if we select $K_{n,1}(w) =$

$h_n(w)$ and $L_{n,1}(w) = 1$, we have

$$\lim_{n\to\infty} \frac{[fh_n]_n}{f_n} = h(\tau),$$

and if we select $K_{n,1}(w) = \Psi(w)h_n(w)$ and $L_{n,1}(w) = 1$, we have

$$\lim_{n\to\infty} \frac{[\Psi fh_n]_n}{f_n} = \Psi(\tau)h(\tau).$$

Since $h(w)$ vanishes nowhere on the domain U,

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{c_n}{c_{n+1}} \frac{[fh_n]_n}{[fh_{n+1}]_{n+1}} = \lim_{n\to\infty} \frac{c_n}{c_{n+1}} \frac{[fh_n]_n}{f_n} \frac{f_n}{f_{n+1}} \frac{f_{n+1}}{[fh_{n+1}]_{n+1}} = \tau,$$

and

$$\lim_{n\to\infty} \frac{\langle zF, p_n\rangle_\mu}{F_n} = \lim_{n\to\infty} \frac{c_n}{c_n} \frac{[\Psi fh_n]_n}{[fh_n]_n} = \lim_{n\to\infty} \frac{[\Psi fh_n]_n}{f_n} \frac{f_n}{[fh_n]_n} = \Psi(\tau) = \lambda.$$

Then, the proof is complete. □

Proof of Theorem II.2.8. First of all, we prove (b) implies (a) by using Lemma II.3.2. We assume that the zeros of $Q^\mu_{n,m}(z)$ have limits $\lambda_1, \dots, \lambda_m$, as $n \to \infty$. For $w \in U$, we define

$$\alpha_n(w) := w^{-m}h(w)Q^\mu_{n,m}(\Psi(w)),$$

$$\eta_{n,j}(w) := \frac{c_{n+m-j}w^{n+m+1}s_{n+m-j}(\Psi(w))\Psi'(w)}{h(w)}, \quad j = 0, \dots, m-1.$$

The functions $\alpha_n(w)$ and $w^{-j}\eta_{n,j}(w) = h_{n+m-j}(w)/h(w)$, $j = 1, \dots, m-1$, are holomorphic on U, and

$$\alpha(w) := \lim_{n\to\infty} \alpha_n(w) = w^{-m}h(w)\prod_{j=1}^{m}(\Psi(w) - \lambda_j),$$

$$\eta_j(w) := \lim_{n\to\infty} \eta_{n,j}(w) = w^j, \quad j = 0, 1, \dots, m-1,$$

uniformly inside $U \setminus \{\infty\}$. Since $h(w)$ is never zero on U, $\alpha(w)$ has at most m zeros in $U \setminus \{\infty\}$. By Cauchy's integral formula, Fubini's theorem, and the definition of $Q^\mu_{n,m}$, we have, for $\epsilon > 0$ sufficiently small so that $F(z)$ is analytic on $D_{1+\varepsilon}$, and for $j = 0, \ldots, m-1$,

$$[f\alpha_n\eta_{n,j}]_n = \frac{c_n}{2\pi i}\int_{\gamma_{1+\varepsilon}} F(\Psi(w))Q^{\mu}_{n,m}(\Psi(w))s_{n+m-j}(\Psi(w))\Psi'(w)dw$$

$$= \frac{c_n}{2\pi i}\int_{\Gamma_{1+\varepsilon}} F(t)Q^{\mu}_{n,m}(t)s_{n+m-j}(t)dt = \frac{c_n}{2\pi i}\int_{\Gamma_{1+\varepsilon}} F(t)Q^{\mu}_{n,m}(t)\int \frac{\overline{p_{n+m-j}(z)}}{t-z}d\mu(z)dt$$

$$= c_n\int\frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}}\frac{F(t)Q^{\mu}_{n,m}(t)}{t-z}dt\overline{p_{n+m-j}(z)}d\mu(z) = c_n\int F(z)Q^{\mu}_{n,m}(z)\overline{p_{n+m-j}(z)}d\mu(z) = 0.$$

Therefore, we prove the required conditions for Lemma II.3.2. If the regular part of $f(w)$ is a rational function with at most $m-1$ poles, then $F(z)$ is a rational function with at most $m-1$ poles which implies that $\Delta_{n,m}(F,\mu) = 0$ for n sufficiently large. This is impossible, because $\deg(Q^{\mu}_{n,m}) = m$, for n sufficiently large. Therefore, by Lemma II.3.2, $\alpha(w)$ has precisely m zeros $\tau_1, \ldots, \tau_m$ in $U \setminus \{\infty\}$ and the limits of the poles of the classical Padé approximants $[n/m]_{\hat{f}}(w)$ are $\tau_1, \ldots, \tau_m$, as $n \to \infty$.

Now, we prove (a) implies (b) by using Lemma II.3.1. Assume that the poles of $[n/m]_{\hat{f}}(w)$ have limits $\tau_1, \ldots, \tau_m$, as $n \to \infty$. We assume further that $Q_{n,m}(w)$ is monic. By Suetin's theorem, we have (II.14).

Define, for $w \in U$,

$$\tilde{\alpha}_n(w) := w^{-m}Q_{n,m}(w),$$

$$\tilde{\eta}_{n,\nu}(w) := w^{\nu}, \quad \nu = 0, \ldots, m-1.$$

Then,

$$\tilde{\alpha}(w) := \lim_{n\to\infty}\tilde{\alpha}_n(z) = w^{-m}\prod_{j=1}^{m}(w-\tau_j),$$

$$\tilde{\eta}_{\nu}(w) = w^{\nu}, \quad \nu = 0, \ldots, m-1,$$

uniformly inside $U \setminus \{\infty\}$. By the definition of $Q_{n,m}(z)$, it follows that, for $\epsilon > 0$ sufficiently small so that $f(w)$ is holomorphic on $\gamma_{1+\varepsilon}$ and for n sufficiently large,

$$[f\tilde{\alpha}_n\tilde{\eta}_{n,\nu}]_n = [\hat{f}\tilde{\alpha}_n\tilde{\eta}_{n,\nu}]_n = \frac{1}{2\pi i}\int_{\gamma_{1+\varepsilon}}\frac{\hat{f}(w)Q_{n,m}(w)}{w^{m-\nu+n+1}}dw = 0, \quad \nu = 0, \ldots, m-1.$$

We can easily check the rest of all required conditions in Lemma II.3.1 for $\tilde{\alpha}_n(w)$ and $\tilde{\eta}_{n,\nu}(w)$, so we can apply the equality (II.16) in Lemma II.3.1.

Next, we set

$$\tilde{Q}_{n,m}(z) := \begin{vmatrix} c_{n+1}\langle F, p_{n+1}\rangle_\mu & c_{n+1}\langle zF, p_{n+1}\rangle_\mu & \cdots & c_{n+1}\langle z^m F, p_{n+1}\rangle_\mu \\ \vdots & \vdots & \cdots & \vdots \\ c_{n+m}\langle F, p_{n+m}\rangle_\mu & c_{n+m}\langle zF, p_{n+m}\rangle_\mu & \cdots & c_{n+m}\langle z^m F, p_{n+m}\rangle_\mu \\ 1 & z & \cdots & z^m \end{vmatrix}. \tag{II.61}$$

Note that the polynomials $\tilde{Q}_{n,m}(z)$ satisfy

$$\langle \tilde{Q}_{n,m} F, p_\nu \rangle_\mu = 0, \quad \nu = n+1, \ldots, n+m, \tag{II.62}$$

and if we can show that $\Delta_{n,m}(F,\mu) \neq 0$ (the coefficient of $\tilde{Q}_{n,m}(z)/\prod_{j=1}^m c_{n+j}$) which we will show at the end of this proof, then $Q^\mu_{n,m}(z)$ is unique and

$$Q^\mu_{n,m}(z) = \frac{\tilde{Q}_{n,m}(z)}{\Delta_{n,m}(F,\mu)\prod_{j=1}^m c_{n+j}}.$$

Using Cauchy's integral formula and Fubini's theorem, for $\varepsilon > 0$ sufficiently small so that $F(z)$ is holomorphic on $D_{1+\varepsilon}$, for $j = 1, \ldots, m+1$, and $\nu = 1, \ldots, m$,

$$\begin{aligned} & c_{n+\nu}\langle z^{j-1}F, p_{n+\nu}\rangle_\mu = c_{n+\nu}\int \frac{1}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \frac{\zeta^{j-1}F(\zeta)}{\zeta - z} d\zeta \overline{p_{n+\nu}(z)} d\mu(z) \\ &= \frac{c_{n+\nu}}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \zeta^{j-1}F(\zeta)\int \frac{\overline{p_{n+\nu}(z)}}{\zeta - z} d\mu(z) d\zeta = \frac{c_{n+\nu}}{2\pi i}\int_{\Gamma_{1+\varepsilon}} \zeta^{j-1}F(\zeta)s_{n+\nu}(\zeta)d\zeta \\ &= \frac{c_{n+\nu}}{2\pi i}\int_{\gamma_{1+\varepsilon}} \Psi^{j-1}(w)f(w)s_{n+\nu}(\Psi(w))\Psi'(w)dw = [f(w)w^{-\nu}h_{n+\nu}(w)\Psi^{j-1}(w)]_n. \end{aligned} \tag{II.63}$$

Computing the determinant in (II.61) along the last row and applying (II.63), we obtain

$$\tilde{Q}_{n,m}(z) = \sum_{k=0}^m (-1)^{m+k} z^k \det([fK_{n,t}L_{n,r}]_n)_{t=1,\ldots,m,\, r=1,\ldots,k,k+2,\ldots,m+1}, \tag{II.64}$$

where

$$K_{n,t}(w) := w^{-t}h_{n+t}(w), \quad t=1,\ldots,m,$$

$$L_{n,r}(w) := \Psi^{r-1}(w), \quad r=1,\ldots,m+1.$$

Moreover, all functions $K_{n,t}(w)$ and $L_{n,r}(w)$, are holomorphic on $U \setminus \{\infty\}$, and

$$K_t(w) := \lim_{n\to\infty} K_{n,t}(w) = w^{-t}h(w), \quad t=1,\ldots,m,$$

$$L_r(w) := \Psi^{r-1}(w), \quad r=1,\ldots,m+1,$$

uniformly inside $U \setminus \{\infty\}$. By Lemma II.3.1 and (II.64), we have $\tau_1,\ldots,\tau_m \in U$ and

$$\lim_{n\to\infty} \frac{\tilde{Q}_{n,m}(z)}{\det(f_{n-i-j})_{i,j=0,1,\ldots,m-1}}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{m} (-1)^{m+k} z^k \frac{\det([fK_{n,t}L_{n,r}]_n)_{t=1,\ldots,m,\ r=1,\ldots,k,k+2,\ldots,m+1}}{\det(f_{n-i-j})_{i,j=0,1,\ldots,m-1}}$$

$$= \sum_{k=0}^{m} (-1)^{m+k} z^k \frac{\det(K_r(\tau_t))_{t,r=1,\ldots,m} \det(L_r(\tau_t))_{t=1,\ldots,m,\ r=1,\ldots,k,k+2,\ldots,m+1}}{W^2(\tau_1,\tau_2,\ldots,\tau_m)}$$

$$= \frac{\det(K_r(\tau_t))_{r,t=1,2,\ldots,m}}{W^2(\tau_1,\tau_2,\ldots,\tau_m)} \begin{vmatrix} 1 & \Psi(\tau_1) & \cdots & \Psi^m(\tau_1) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \Psi(\tau_m) & \cdots & \Psi^m(\tau_m) \\ 1 & z & \cdots & z^m \end{vmatrix} \tag{II.65}$$

$$= (-1)^{(m)(m-1)/2} \frac{\prod_{i=1}^m h(\tau_i)}{\prod_{i=1}^m \tau_i^m} \prod_{1\le i<j\le m} \left(\frac{\Psi(\tau_j)-\Psi(\tau_i)}{\tau_j-\tau_i} \right) z^m + \ldots, \tag{II.66}$$

where $W(\tau_1,\tau_2,\ldots,\tau_m) = \det(\tau_t^{r-1})_{t,r=1,\ldots,m}$ is the Vandermonde determinant of the numbers $\tau_1,\ldots,\tau_m$. Since the degree of the polynomial in the last expression is m, the degree of $\tilde{Q}_{n,m}(z)$ is m for all n sufficiently large. This implies that $\Delta_{n,m}(F,\mu) \neq 0$ and $Q^{\mu}_{n,m}(z) = \tilde{Q}_{n,m}(z)/(\Delta_{n,m}(F,\mu)\prod_{j=1}^m c_{n+j})$. Moreover, the zeros of the polynomial in (II.65) are $\lambda_1,\ldots,\lambda_m$, so the zeros of $\tilde{Q}_{n,m}(z)$ (and $Q^{\mu}_{n,m}(z)$) converge to $\lambda_1,\ldots,\lambda_m$, as $n\to\infty$. $\square$

CHAPTER III

RIESZ POLARIZATION CONSTANTS AND CONFIGURATIONS

III.1 Introduction, background results, and notation

Throughout this chapter, A will denote an infinite compact subset of the m-dimensional Euclidean space $\mathbb{R}^m$. Let $\omega_N = \{x_1, \ldots, x_N\}$ denote a configuration of N (not necessarily distinct) points in $\mathbb{R}^m$ (such configurations are known as multisets, however, we will still use the word *configurations*). The class of *Riesz s-potential* functions and *log-potential* function corresponding to a fixed configuration $\omega_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^m$ is defined by

$$U^s(\omega_N; y) := \begin{cases} \sum_{i=1}^{N} |y - x_i|^{-s}, & \text{if } s \in \mathbb{R} \setminus \{0\}, \\ N, & \text{if } s = 0, \\ \sum_{i=1}^{N} \log |y - x_i|^{-1}, & \text{if } s = \log, \end{cases}$$

where $y \in \mathbb{R}^m$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$.

For a configuration $\omega_N = \{x_1, \ldots, x_N\} \subset A$, we define the following quantities

$$M^s(\omega_N; A) := \min_{y \in A} U^s(\omega_N; y), \quad s \in \mathbb{R}, \tag{III.1}$$

$$m^s(\omega_N; A) := \max_{y \in A} U^s(\omega_N; y), \quad s \leq 0, \tag{III.2}$$

$$M^{\log}(\omega_N; A) := \min_{y \in A} U^{\log}(\omega_N; y). \tag{III.3}$$

For a fixed configuration $\omega_N \subset A$, since the potential functions $f(y) := U^s(\omega_N; y), s > 0$ and $g(y) := U^{\log}(\omega_N; y)$ are lower semi-continuous in y on A and A is an infinite compact set, the functions $f(y)$ and $g(y)$ attain their minimums on A. Moreover, for a fixed configuration $\omega_N \subset A$, the potential function $h(y) := U^s(\omega_N; y), s \leq 0$ is continuous in y on A, so by the compactness of A, the function $h(y)$ attains its maximum and minimum on A. Therefore, the maximum and the minimums in (III.1), (III.2), and (III.3) are well-defined.

Let $\#W$ denote the cardinality of the multiset W. The definitions of the maximal and minimal N-point Riesz s-polarization constants and configurations of A and the maximal N-point *log*-polarization constant and configurations of A are the following.

Definition III.1.1. Let A be an infinite compact subset of the m-dimensional Euclidean space $\mathbb{R}^m$. For $s \in \mathbb{R}$, the *maximal N-point Riesz s-polarization constant of A* is given by

$$M_N^s(A) := \max_{\substack{\omega_N \subset A \\ \#\omega_N = N}} M^s(\omega_N; A), \quad \text{and} \quad M_N^0(A) := N, \tag{III.4}$$

for $s \leq 0$, the *minimal N-point Riesz s-polarization constant of A* is given by

$$m_N^s(A) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} m^s(\omega_N; A), \quad \text{and} \quad m_N^0(A) := N, \tag{III.5}$$

and the *maximal N-point log-polarization constant of A* is given by

$$M_N^{\log}(A) := \max_{\substack{\omega_N \subset A \\ \#\omega_N = N}} M^{\log}(\omega_N; A). \tag{III.6}$$

We say that a configuration ω_N is a *maximal N-point Riesz s-polarization configuration of A*, a *minimal N-point Riesz s-polarization configuration of A*, or a *maximal N-point log-polarization configuration of A*, if it attains the maximum in (III.4), the minimum in (III.5), or the maximum in (III.6), respectively. For short, sometimes, those configurations are simply called *optimal for $M_N^s(A)$, $m_N^s(A)$*, or $M_N^{\log}(A)$, respectively.

The existences of optimal configurations in (III.4), (III.5), and (III.6) follow from the continuities of the functions $f(\mathbf{x}_N) := M^s(\mathbf{x}_N; A), s \leq 0$ and $g(\mathbf{x}_N) := m^s(\mathbf{x}_N; A), s \leq 0$ in $\mathbf{x}_N$ on A^N and the upper semi-continuities of the functions $h(\mathbf{x}_N) := M^s(\mathbf{x}_N; A), s > 0$, and $k(\mathbf{x}_N) := M^{\log}(\mathbf{x}_N; A)$ in $\mathbf{x}_N$ on A^N (see Lemma III.3.1).

For a configuration $\omega_N = \{x_1, \ldots, x_N\}$, its covering distance relative to A is defined by

$$\rho(\omega_N; A) := \max_{y \in A} \min_{1 \leq j \leq N} |y - x_j|.$$

The N*-point best-covering distance (or* N*-point mesh norm) of* A is defined by

$$\rho_N(A) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} \rho(\omega_N; A). \tag{III.7}$$

We call an N-point configuration ω_N an N*-point best-covering configuration of* A if it attains the minimum in (III.7).

In order to state our results in Section III.2.1 about asymptotic properties for N fixed and s varying of optimal Riesz polarization configurations, we need to introduce the following definition of cluster point.

Definition III.1.2. We say that a configuration $\omega_N \subset A$ with $\#\omega_N = N$ is a *cluster point as* $s \to t$ *of maximal (minimal)* N*-point Riesz* s*-polarization configurations of* A if there is a sequence $\{\omega_N^{s_k}\}_{k=1}^{\infty}$ of maximal (minimal) N-point Riesz s_k-polarization configurations on A such that $\lim_{k\to\infty} s_k = t$, and $\lim_{k\to\infty} \omega_N^{s_k} = \omega_N$ in the product topology on A^N.

Our main result in Section III.2.2 about the dominant term of $M_N^d(A)$ and the limiting distribution of maximal N-point Riesz d-polarization configurations will be stated on subsets of the following d-dimensional C^1-manifolds in $\mathbb{R}^m$.

Definition III.1.3. A set $W \subset \mathbb{R}^m$ is called a d*-dimensional* C^1*-manifold embedded in* $\mathbb{R}^m$, $d \leq m$, if every point $y \in W$ has an open neighborhood V relative to W such that V is homeomorphic to an open set $U \subset \mathbb{R}^d$ with the homeomorphism $f : U \to V$ being a C^1-continuous mapping and the Jacobian matrix

$$J_x^f := \begin{bmatrix} \nabla f_1(x) \\ \dots \\ \nabla f_m(x) \end{bmatrix}$$

of the function f having rank d at any point $x \in U$ (here $f_1, \dots, f_m$ denote the coordinate mappings of f).

Hausdorff measures defined as follows will play a significant role in the study of these asymptotic behaviors. For a given set $A \subset \mathbb{R}^m$ and $0 \le \alpha \le m$, define the quantity

$$\tilde{\mathcal{H}}_\alpha^\delta(A) := \inf\{\sum_{i=1}^{\infty}(\mathrm{diam}(E_i))^\alpha : \ A \subset \cup_{i=1}^{\infty} E_i, \ \mathrm{diam}(E_i) \le \delta\}, \quad \delta > 0,$$

where E_i are arbitrary non-empty subsets of $\mathbb{R}^m$. Define a set function $\tilde{\mathcal{H}}_\alpha : \{A : A \subset \mathbb{R}^m\} \to [0, \infty]$ by

$$\tilde{\mathcal{H}}_\alpha(A) := \lim_{\delta \to 0^+} \tilde{\mathcal{H}}_\alpha^\delta(A) = \sup_{\delta > 0} \tilde{\mathcal{H}}_\alpha^\delta(A).$$

We call $\tilde{\mathcal{H}}_\alpha$ the *α-dimensional Hausdorff measure* in $\mathbb{R}^m$. For $d \in \mathbb{N}$, we denote by $\mathcal{H}_d$ the d-dimensional Hausdorff measure in $\mathbb{R}^m$ normalized so that the copy of the d-dimensional unit cube embedded in $\mathbb{R}^m$ has measure 1. Moreover, we will denote by $\beta_d = \mathcal{H}_d(\mathbb{B}^d)$ the volume of the d-dimensional unit ball.

For a subset $K \subset A$, we will denote by $\partial_A K$ the boundary of K relative to A.

We say that a sequence $\{\mu_n\}_{n=1}^\infty$ of Borel probability measures in $\mathbb{R}^m$ converges to a Borel probability measure μ in the weak* topology of measures (and write $\mu_n \xrightarrow{*} \mu$, $n \to \infty$) if for every continuous function $f : \mathbb{R}^m \to \mathbb{R}$,

$$\int f \, d\mu_n \to \int f \, d\mu, \quad n \to \infty. \tag{III.8}$$

Remark III.1.4. It is well known that to prove (III.8) when μ and all the measures μ_n are supported on a compact set $A \subset \mathbb{R}^m$, it is sufficient to show that

$$\mu_n(K) \to \mu(K), \quad n \to \infty,$$

for every closed subset K of A with $\mu(\partial_A K) = 0$.

We call a sequence $\{\omega_N\}_{N=1}^\infty$ of N-point configurations on A *asymptotically maximal for the N-point d-polarization problem on A* if

$$\lim_{N \to \infty} \frac{M^d(\omega_N; A)}{M_N^d(A)} = 1.$$

The polarization problem is related to the minimal Riesz energy problem described below.

For a collection $\omega_N = \{x_1, \ldots, x_N\}$ of $N \geq 2$ pairwise distinct points in $\mathbb{R}^m$ and $s > 0$ we let

$$E_s(\omega_N) := \sum_{1 \leq i \neq j \leq N} \frac{1}{|x_i - x_j|^s}.$$

The *minimal N-point Riesz s-energy* of an infinite compact set $A \subset \mathbb{R}^m$ is defined as

$$\mathcal{E}_s(A, N) := \min_{\substack{\omega_N \subset A \\ \#\omega_N = N}} E_s(\omega_N).$$

D.P. Hardin and E.B. Saff proved in [34] (see also [33]) that if A is an infinite compact subset of a d-dimensional C^1-manifold embedded in $\mathbb{R}^m$, then

$$\lim_{N \to \infty} \frac{\mathcal{E}_d(A, N)}{N^2 \ln N} = \frac{\beta_d}{\mathcal{H}_d(A)}. \tag{III.9}$$

Furthermore, if A is as in above condition and $\mathcal{H}_d(A) > 0$, then for any sequence $\omega_N = \{x_{k,N}\}_{k=1}^N$, $N \in \mathbb{N}$, of asymptotically d-energy minimizing N-point configurations in A in the sense that

$$\lim_{N \to \infty} \frac{E_d(\omega_N)}{\mathcal{E}_d(A, N)} = 1,$$

we have

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{x_{i,N}} \xrightarrow{\ *\ } \frac{\mathcal{H}_d(\cdot)|_A}{\mathcal{H}_d(A)}, \quad N \to \infty, \tag{III.10}$$

in the weak* topology of measures. Here δ_x denotes the unit point mass at the point x.

Relations (III.9) and (III.10) have recently been extended by D.P. Hardin, E.B. Saff, and J.T. Whitehouse to the case of A being a finite union of compact subsets of $\mathbb{R}^m$ where each compact set is contained in some d-dimensional C^1-manifold in $\mathbb{R}^m$ and the pairwise intersections of such compact sets have $\mathcal{H}_d$-measure zero. These authors observed that the methods of [40] could be applied (see [9]). For convenience of the reader, part of this statement and its proof are reproduced in this dissertation in Proposition III.3.13.

III.2 Main results

III.2.1 Basic properties of maximal and minimal Riesz polarization constants and configurations

Theorem III.2.1. *Let $N \in \mathbb{N}$ be fixed and A be an infinite compact subset of $\mathbb{R}^m$. We have*

$$\lim_{s\to\infty} M_N^s(A)^{1/s} = \frac{1}{\rho_N(A)},$$

where $\rho_N(A)$ is the N-point best-covering distance of A. Furthermore, every cluster point as $s \to \infty$ of maximal N-point Riesz s-polarization configurations of A is an N-point best-covering configuration of A.

Theorem III.2.2. *Let $N \in \mathbb{N}$ be fixed and A be an infinite compact subset of $\mathbb{R}^m$. We have*

$$M_N^{\log}(A) = \lim_{s\to 0^+} \frac{M_N^s(A) - N}{s}. \tag{III.11}$$

Furthermore, every cluster point as $s \to 0^+$ of maximal N-point Riesz s-polarization configurations of A is a maximal N-point log-polarization configuration of A.

Remark III.2.3. The equality (III.11) shows that $f(s) := M_N^s(A)$ is right differentiable at 0 and its right derivative is $M_N^{\log}(A)$.

Theorem III.2.4. *Let $N \in \mathbb{N}$ be fixed and A be an infinite compact subset of $\mathbb{R}^m$. The function $f(s) := M_N^s(A)$ is continuous for all $s \in (-\infty, 0) \cup (0, \infty)$. More precisely, the function $f(s)$ is right-continuous but not left-continuous at 0. The function $g(s) := m_N^s(A)$ is continuous for all $s \in (-\infty, 0]$. Furthermore, for $t \in (-\infty, \infty)$, every cluster point as $s \to t$ of maximal N-point Riesz s-polarization configurations of A is a maximal N-point Riesz t-polarization configuration of A, and for $t \in (-\infty, 0]$, every cluster point as $s \to t$ of minimal N-point Riesz s-polarization configurations of A is a minimal N-point Riesz t-polarization configuration of A.*

III.2.2 Asymptotics of maximal Riesz d-polarization on subsets of d-dimensional manifolds

Our main result about the dominant term of $M_N^d(A)$ and the limiting distribution of maximal N-point Riesz d-polarization configurations as $N \to \infty$ is the following.

Theorem III.2.5. *Let $A = \cup_{i=1}^{l} A_i$ be an infinite subset of $\mathbb{R}^m$, where each set A_i is a compact subset contained in some d-dimensional C^1-manifold in $\mathbb{R}^m$, $d \le m$, and $\mathcal{H}_d(A_i \cap A_j) = 0$, $1 \le i < j \le l$. Then*

$$\lim_{N\to\infty} \frac{M_N^d(A)}{N \ln N} = \frac{\beta_d}{\mathcal{H}_d(A)}. \tag{III.12}$$

Furthermore, under an additional assumption that $\mathcal{H}_d(A) > 0$, if $\omega_N = \{x_{i,N}\}_{i=1}^N$, $N \in \mathbb{N}$, is a sequence of asymptotically maximal configurations for the N-point d-polarization problem on A, then in the weak topology of measures we have*

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_{i,N}} \xrightarrow{\ *\ } \frac{\mathcal{H}_d(\cdot)|_A}{\mathcal{H}_d(A)}, \quad N \to \infty. \tag{III.13}$$

Remark III.2.6. Note that the conditions imposed on the set A imply $\mathcal{H}_d(A) < \infty$. Moreover, if $\mathcal{H}_d(A) = 0$, then the limit in (III.12) is understood to be ∞.

III.2.3 Maximal and minimal N-point Riesz s-polarization configurations of the m-dimensional sphere

Two 1-dimensional circles in different planes

Let $\omega_N = \{x_1, \ldots, x_N\}$ denote N (not necessarily distinct) points in $\mathbb{R}^2$. Let $R > 0$. Denote by

$$\mathbb{S}_R^1 := \{x \in \mathbb{R}^2 : |x| = R\}$$

the circle centered at 0 of radius R in $\mathbb{R}^2$. When $R = 1$, we simply use the notation $\mathbb{S}^1$. We consider the generalization of Riesz polarization constants and configurations of two concentric circles in the following way. For $s \in \mathbb{R}$ and $h \geq 0$,

$$M_N^{s,h}(\mathbb{S}^1;\mathbb{S}_R^1) := \max_{\substack{\omega_N \subset \mathbb{S}^1 \\ \#\omega_N = N}} \min_{y \in \mathbb{S}_R^1} \sum_{j=1}^{N} \left(\sqrt{|y-x_j|^2+h}\right)^{-s}, \quad \text{and} \quad M_N^{0,h}(\mathbb{S}^1;\mathbb{S}_R^1) := N, \tag{III.14}$$

and

$$m_N^{s,h}(\mathbb{S}^1;\mathbb{S}_R^1) := \min_{\substack{\omega_N \subset \mathbb{S}^1 \\ \#\omega_N = N}} \max_{y \in \mathbb{S}_R^1} \sum_{j=1}^{N} \left(\sqrt{|y-x_j|^2+h}\right)^{-s}, \quad \text{and} \quad m_N^{0,h}(\mathbb{S}^1;\mathbb{S}_R^1) := N. \tag{III.15}$$

We will call ω_N a *maximal (minimal) N-point Riesz (s,h)-polarization configuration of* $(\mathbb{S}^1;\mathbb{S}_R^1)$ if ω_N attains the maximum in (III.14) (minimum in (III.15)). Clearly, if $R=1$ and $h=0$, then $M_N^{s,h}(\mathbb{S}^1;\mathbb{S}_R^1) = M_N^s(\mathbb{S}^1)$ and $m_N^{s,h}(\mathbb{S}^1;\mathbb{S}_R^1) = m_N^s(\mathbb{S}^1)$. The term h of the potential function $\sum_{j=1}^N \left(\sqrt{|y-x_j|^2+h}\right)^{-s}$ in (III.14) and (III.15) can be interpreted as follows. Let us consider two circles in 3D: one is $\mathbb{S}^1 \times \{0\}$ and the other is $\mathbb{S}_R^1 \times \{\sqrt{h}\}$. The potential function $f(y) := \sum_{j=1}^N \left(\sqrt{|y-x_j|^2+h}\right)^{-s}$ is actually the Riesz s-potential function on $\mathbb{S}_R^1 \times \{\sqrt{h}\}$ when $\omega_N := \{x_1,\ldots,x_N\}$ is fixed on $\mathbb{S}^1 \times \{0\}$.

Because the Euclidean space $\mathbb{R}^2$ and the complex space $\mathbb{C}$ have the same dimension and the same norm, we will embed $\mathbb{S}^1$ and $\mathbb{S}_R^1$ into $\mathbb{C}$ and adopt the notation x^j, where $j \in \mathbb{N}$ and $1/x$ from complex numbers.

A complete characterization of all maximal and minimal N-point Riesz (s,h)-polarization configurations of $(\mathbb{S}^1;\mathbb{S}_R^1)$ when $s=-2,\ldots,-2N+2$ is the following.

Theorem III.2.7. *Let $N \in \mathbb{N}$, $p \in \{1,2,\ldots,N-1\}$, $R>0$, and $h \geq 0$. We have*

$$M_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}_R^1) = m_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}_R^1) = \frac{N}{2^p}\sum_{j=0}^{p}\binom{p}{j}^2 b^{2j}\left(a \pm \sqrt{a^2-b^2}\right)^{p-2j},$$

where $a := R^2+h+1$ and $b := 2R$. Furthermore, an N-point configuration $\omega_N = \{x_1,\ldots,x_N\}$ is a maximal or minimal N-point Riesz $(-2p,h)$-polarization configuration of $(\mathbb{S}^1;\mathbb{S}_R^1)$ if and only if $\sum_{i=1}^N x_i = \sum_{i=1}^N x_i^2 = \cdots = \sum_{i=1}^N x_i^p = 0$.

Letting $R=1$ and $h=0$, we have the following corollary.

Corollary III.2.8. *Let $N \in \mathbb{N}$ and $p \in \{1, 2, \ldots, N-1\}$. We have*

$$M_N^{-2p}(\mathbb{S}^1) = m_N^{-2p}(\mathbb{S}^1) = N \sum_{j=0}^{p} \binom{p}{j}^2 R^{2p-2j}.$$

Furthermore, an N-point configuration $\omega_N = \{x_1, \ldots, x_N\}$ is a maximal or minimal N-point Riesz $-2p$-polarization configuration of $\mathbb{S}^1$ if and only if $\sum_{i=1}^N x_i = \sum_{i=1}^N x_i^2 = \cdots = \sum_{i=1}^N x_i^p = 0$.

The m-dimensional spheres

We will examine optimal configurations for $M_N^s(\mathbb{S}^m)$ and $m_N^s(\mathbb{S}^m)$, where

$$\mathbb{S}^m := \{x \in \mathbb{R}^{m+1} : |x| = 1\},$$

for some negative even integers.

The simplest case is when $s = -2$.

Theorem III.2.9. *Let $N \in \mathbb{N}$. Then*

$$M_N^{-2}(\mathbb{S}^m) = m_N^{-2}(\mathbb{S}^m) = 2N.$$

Moreover, an N-point configuration $\omega_N = \{x_1, \ldots, x_N\}$ is optimal for $M_N^{-2}(\mathbb{S}^m)$ or $m_N^{-2}(\mathbb{S}^m)$ if and only if $\sum_{j=1}^N x_j = 0$.

The next result shows that if ω_N is an N-point configuration on $\mathbb{S}^m$ such that its associated Riesz s-potential function is constant on $\mathbb{S}^m$, then ω_N is optimal for $M_N^s(\mathbb{S}^m)$ and $m_N^s(\mathbb{S}^m)$.

Theorem III.2.10. *Let $s \leq 0$. Assume that there exists an N-point configuration $\omega_N := \{x_1, \ldots, x_N\}$ on $\mathbb{S}^m$ such that its Riesz potential function $f(y) := U^s(\omega_N; y) = \sum_{i=1}^N |x_i - y|^{-s}$ is constant on $\mathbb{S}^m$. Then, such ω_N is optimal for $M_N^s(\mathbb{S}^m)$ and $m_N^s(\mathbb{S}^m)$.*

As a consequence of this theorem and the results in [42, Theorem 3-5], we show that many natural configurations on $\mathbb{S}^m$ are optimal for $M_N^s(\mathbb{S}^m)$ and $m_N^s(\mathbb{S}^m)$ for some certain negative even integers s and certain positive integer m.

Corollary III.2.11. *Let $N \in \mathbb{N}$. We have the following:*

(i) Any $N+2$-point regular simplex configuration on $\mathbb{S}^N$ is optimal for $M^{-4}_{N+2}(\mathbb{S}^N)$ and $m^{-4}_{N+2}(\mathbb{S}^N)$.

(ii) Any $2N+2$-point cross-polytope configuration on $\mathbb{S}^N$ is optimal for $M^{-4}_{2N+2}(\mathbb{S}^N)$, $M^{-6}_{2N+2}(\mathbb{S}^N)$, $m^{-4}_{2N+2}(\mathbb{S}^N)$, and $m^{-6}_{2N+2}(\mathbb{S}^N)$.

(iii) Any 2^{N+1}-point hypercube configuration on $\mathbb{S}^N$ is optimal for $M^{-4}_{2^{N+1}}(\mathbb{S}^N)$, $M^{-6}_{2^{N+1}}(\mathbb{S}^N)$, $m^{-4}_{2^{N+1}}(\mathbb{S}^N)$, and $m^{-6}_{2^{N+1}}(\mathbb{S}^N)$.

Remark III.2.12. For those values s and those configurations, their associated Riesz s-potential functions are constant on $\mathbb{S}^N$.

III.3 Proofs

III.3.1 Proofs of III.2.1

Lemma III.3.1. *For a fixed vector $\boldsymbol{x}_N \in A^N$, the functions $f(y) := U^s(\boldsymbol{x}_N; y)$, $s > 0$ and $g(y) := U^{\log}(\boldsymbol{x}_N; y)$ are lower semi-continuous on A. Moreover, the functions $h(\boldsymbol{x}_N) := M^s(\boldsymbol{x}_N; A)$, $s > 0$ and $k(\boldsymbol{x}_N) := M^{\log}(\boldsymbol{x}_N; A)$ are upper semi-continuous on A^N.*

Proof of Lemma IV.1.1. For a fixed vector $\mathbf{x}_N \in A^N$, the lower semi-continuities of $U^s(\mathbf{x}_N; y)$, $s > 0$ and $U^{\log}(\mathbf{x}_N; y)$ as functions of y on A are well-known. We prove only the second statement. Let $s > 0$ and $\mathbf{x}'_N := (x'_1, \ldots, x'_N) \in A^N$ and let $\{\mathbf{x}^k_N\}_{k \in \mathbb{N}}$ be a sequence in A^N such that $\mathbf{x}^k_N \to \mathbf{x}'_N$, as $k \to \infty$. Because $f(y) := U^s(\mathbf{x}'_N; y)$ is lower semi-continuous on A, there exists $y^0 \in A$ such that

$$U^s(\mathbf{x}'_N; y^0) = M^s(\mathbf{x}'_N; A).$$

Notice that $|y^0 - x'_i| > 0$ for all i since the cardinality of A is infinity. Therefore, the function $h(\mathbf{x}_N) := U^s(\mathbf{x}_N; y^0)$ is continuous in a small neighborhood of $\mathbf{x}'_N$. Hence,

$$\limsup_{k\to\infty} M^s(\mathbf{x}^k_N; A) \le \limsup_{k\to\infty} U^s(\mathbf{x}^k_N; y^0) = \lim_{k\to\infty} U^s(\mathbf{x}^k_N; y^0) = M^s(\mathbf{x}'_N; A).$$

The same argument can be applied to the case $s = \log$. □

Lemma III.3.2. *For a fixed vector* $(x_1, \dots, x_N) \in A^N$, *the function* $f(y) := \min_{1\le j\le N} |y-x_j|$ *is continuous on* A. *Moreover, the function* $g(\boldsymbol{x}_N) := \rho(\boldsymbol{x}_N; A)$ *is continuous in* $\boldsymbol{x}_N$ *on* A^N.

Proof. The proof of this lemma is trivial. □

Proof of Theorem III.2.1. Let $s > 0$. Clearly, for $y \notin \{x_1, \dots, x_N\}$,

$$\frac{1}{\min_{1\le j\le N} |y-x_j|} \le \left(\sum_{j=1}^{N} \frac{1}{|y-x_j|^s} \right)^{1/s} \le \frac{N^{1/s}}{\min_{1\le j\le N} |y-x_j|}. \tag{III.16}$$

By Lemma III.3.1 and Lemma III.3.2, since the function $h(t) := t^{1/s}$ is increasing and continuous on $[0, \infty)$, we have

$$\frac{1}{\rho_N(A)} \le M_N^s(A)^{1/s} = \max_{\substack{\omega_N \subset A \\ \#\omega_N = N}} \min_{y\in A} \left(\sum_{j=1}^{N} \frac{1}{|y-x_j|^s} \right)^{1/s} \le \frac{N^{1/s}}{\rho_N(A)}. \tag{III.17}$$

This implies that

$$\lim_{s\to\infty} M_N^s(A)^{1/s} = \frac{1}{\rho_N(A)}.$$

Let $\{\omega_N^{s_k}\}_{k\in\mathbb{N}} := \{\{x_1^{s_k}, \dots, x_N^{s_k}\}\}_{k\in\mathbb{N}}$ be a sequence of maximal N-point Riesz s_k-polarization configurations of A such that $s_k \to \infty$ and $\omega_N^{s_k} \to \omega_N^* := \{x_1^*, \dots, x_N^*\}$, as $k \to \infty$. Let $\omega_N = \{x_1, \dots, x_N\}$ be any $N-$point configuration on A. Using (III.16) and the fact that $h(t) := t^{1/s}$ is continuous and increasing on $[0, \infty)$, we obtain

$$\frac{1}{\rho(\omega_N; A)} \le \left(\min_{y\in A} \sum_{j=1}^{N} \frac{1}{|y-x_j|^{s_k}} \right)^{1/s_k} \le \left(\min_{y\in A} \sum_{j=1}^{N} \frac{1}{|y-x_j^{s_k}|^{s_k}} \right)^{1/s_k} \le \frac{N^{1/s_k}}{\rho(\omega_N^{s_k}; A)}.$$

Now, let $k \to \infty$, it follows from the continuity of the function $\rho(\omega_N, A)$ that

$$\frac{1}{\rho(\omega_N; A)} \le \frac{1}{\rho(\omega_N^*; A)}, \quad \text{for all } \omega_N \subset A.$$

Therefore, $\rho_N(A) = \rho(\omega_N^*; A)$. □

Before we prove Theorem III.2.2, we need to state some lemmas. For $s > 0$, we define

$$U^{s,\log}(\omega_N; y) := \sum_{i=1}^{N} \frac{1}{|y - x_i|^s} \log \frac{1}{|y - x_i|},$$

where $\omega_N := \{x_1, \ldots, x_N\}$.

Lemma III.3.3. *Let $\omega_N := \{x_1, \ldots, x_N\}$ be an $N-$point configuration on A and y be a point in A such that* $\min_{1\le i\le N} |y - x_i| > 0$. *Then, for $s > t \geq 0$,*

$$U^{t,\log}(\omega_N; y) \leq \frac{U^s(\omega_N; y) - U^t(\omega_N; y)}{s - t} \leq U^{s,\log}(\omega_N; y).$$

Proof of Lemma III.3.3. The lemma follows from the inequalities

$$a^t \log a \leq \frac{a^s - a^t}{s - t} \leq a^s \log a, \quad s > t \geq 0, \quad a > 0. \tag{III.18}$$

□

Lemma III.3.4. *Let $\omega_N := \{x_1, \ldots, x_N\}$ be an $N-$point configuration on A and y be a point in A such that* $\min_{1\le i\le N} |y - x_i| > 0$. *Then,*

$$U^{\log}(\omega_N; y) = \lim_{s\to 0^+} \frac{U^s(\omega_N; y) - N}{s}.$$

Proof of Lemma III.3.4. This immediately follows from Lemma III.3.3. □

Proof of Theorem III.2.2. Let $\omega_N^{\log}$ be a maximal N-point *log*-polarization configuration of A and y^s be a point in A such that

$$U^s(\omega_N^{\log}; y^s) = M^s(\omega_N^{\log}; A).$$

Using Lemma III.3.3 for $s > t = 0$, we have

$$M^{\log}(\omega_N^{\log}; A) \leq U^{\log}(\omega_N^{\log}; y^s) \leq \frac{U^s(\omega_N^{\log}; y^s) - N}{s} = \frac{M^s(\omega_N^{\log}; A) - N}{s} \leq \frac{M_N^s(A) - N}{s},$$

which implies

$$\liminf_{s\to 0^+} \frac{M_N^s(A)-N}{s} \geq M_N^{\log}(A). \tag{III.19}$$

Let $\omega_N' := \{x_1', \ldots, x_N'\}$ be a cluster point as $s \to 0^+$ of maximal N-point Riesz s-polarization configurations $\omega_N^s := \{x_1^s, \ldots, x_N^s\}$ of A and y^0 be a point in A such that

$$U^{\log}(\omega_N'; y^0) = M^{\log}(\omega_N'; A).$$

Then there is a sequence $\{\omega_N^{s_k}\}_{k=1}^{\infty}$ such that $s_k \to 0^+$ and $\omega_N^{s_k} \to \omega_N'$, as $k \to \infty$. So, we have

$$\begin{aligned} M^{\log}(\omega_N'; A) = U^{\log}(\omega_N'; y^0) &= \lim_{k\to\infty} \frac{U^{s_k}(\omega_N'; y^0) - N}{s_k} \\ &= \lim_{k\to\infty} \left(\frac{U^{s_k}(\omega_N^{s_k}; y^0) - N}{s_k} + \frac{U^{s_k}(\omega_N'; y^0) - U^{s_k}(\omega_N^{s_k}; y^0)}{s_k} \right) \\ &\geq \limsup_{k\to\infty} \left(\frac{M^{s_k}(\omega_N^{s_k}; A) - N}{s_k} + \frac{U^{s_k}(\omega_N'; y^0) - U^{s_k}(\omega_N^{s_k}; y^0)}{s_k} \right), \end{aligned} \tag{III.20}$$

which the second equality follows from Lemma III.3.4. If we can show that

$$\lim_{k\to\infty} \frac{U^{s_k}(\omega_N'; y^0) - U^{s_k}(\omega_N^{s_k}; y^0)}{s_k} = 0, \tag{III.21}$$

then it follows from (III.20) that

$$M^{\log}(\omega_N'; A) \geq \limsup_{k\to\infty} \frac{M_N^{s_k}(A) - N}{s_k}. \tag{III.22}$$

Combining (III.19) and (III.22), we have

$$M^{\log}(\omega_N'; A) \geq M_N^{\log}(A).$$

This implies

$$\lim_{k\to\infty} \frac{M_N^{s_k}(A) - N}{s_k} = M^{\log}(\omega_N'; A) = M_N^{\log}(A)$$

and ω_N' is optimal for $M_N^{\log}(A)$.

Now, we prove (III.21). We consider

$$\left|\frac{U^{s_k}(\omega_N^{s_k};y^0)-U^{s_k}(\omega_N';y^0)}{s_k}\right| = \left|\frac{\sum_{i=1}^N |y^0-x_i^{s_k}|^{-s_k}-\sum_{i=1}^N |y^0-x_i'|^{-s_k}}{s_k}\right|$$
$$=\left|\sum_{i=1}^N \frac{|y^0-x_i^{s_k}|^{-s_k}-|y^0-x_i'|^{-s_k}}{s_k}\right| \leq \sum_{i=1}^N \frac{1}{|y^0-x_i'|^{s_k}}\left|\frac{(|y^0-x_i'|/|y^0-x_i^{s_k}|)^{s_k}-1}{s_k}\right|. \tag{III.23}$$

By (III.18), for $i=1,\dots,N$,

$$\left|\frac{(|y^0-x_i'|/|y^0-x_i^{s_k}|)^{s_k}-1}{s_k}\right| \leq \max\left\{1,\left|\frac{y^0-x_i'}{y^0-x_i^{s_k}}\right|^{s_k}\right\}\left|\log\left|\frac{y^0-x_i'}{y^0-x_i^{s_k}}\right|\right| \leq 2^{s_k}\left|\log\left|\frac{y^0-x_i'}{y^0-x_i^{s_k}}\right|\right| \to 0,$$

as $k\to\infty$. By this and (III.23), we obtain

$$\lim_{k\to\infty}\frac{U^{s_k}(\omega_N^{s_k};y^0)-U^{s_k}(\omega_N';y^0)}{s_k}=0.$$

Next, we will show (III.11). Let $\{s_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $s_k\to 0$, as $k\to\infty$. Denote by $\omega_N^{s_k}$ a corresponding maximal N-point Riesz s_k-polarization configuration of A. By compactness of A^N, there exists a convergent subsequence $\{\omega_N^{s_{k_l}}\}_{l\in\mathbb{N}}\subset\{\omega_N^{s_k}\}_{k\in\mathbb{N}}$, say $\omega_N^{s_{k_l}}\to\omega_N'$, as $l\to\infty$. Using above argument, we obtain

$$\lim_{l\to\infty}\frac{M^{s_{k_l}}(\omega_N^{s_{k_l}};A)-N}{s_{k_l}}=M^{\log}(\omega_N';A)=M_N^{\log}(A).$$

This means for every sequence $\{(M_N^{s_k}(A)-N)/s_k\}_{k\in\mathbb{N}}$ such that $s_k\to 0^+$, as $k\to\infty$, there exists a subsequence

$$\left\{\frac{M_N^{s_{k_l}}(A)-N}{s_{k_l}}\right\}_{l\in\mathbb{N}}\subset\left\{\frac{M_N^{s_k}(A)-N}{s_k}\right\}_{k\in\mathbb{N}}$$

such that

$$\lim_{l\to\infty}\frac{M_N^{s_{k_l}}(A)-N}{s_{k_l}}=M_N^{\log}(A).$$

Hence, we prove (III.11).

$\square$

To show Theorem III.2.4, we need the following three lemmas.

Lemma III.3.5. *Let $t > 0$. Assume that*

$$s_k \to t, \qquad \text{as } k \to \infty,$$

$$\omega_N^k := \{x_1^k, \dots, x_N^k\} \to \omega_N := \{x_1, \dots, x_N\}, \qquad \text{as } k \to \infty,$$

$$y^k \to y, \qquad \text{as } k \to \infty,$$

$$U^t(\omega_N; y) < \infty,$$

and $|U^{s_k,\log}(\omega_N^k; y^k)|$ and $|U^{t,\log}(\omega_N^k; y^k)|$ are uniformly bounded for k sufficiently large. Then,

$$\lim_{k\to\infty} U^{s_k}(\omega_N^k; y^k) = U^t(\omega_N; y).$$

Proof of Lemma III.3.5. Assume that

$$\max\{|U^{s_k,\log}(\omega_N^k; y^k)|, |U^{t,\log}(\omega_N^k; y^k)|\} \leq M < \infty, \qquad \text{for all } k \geq n_0.$$

By Lemma III.3.3,

$$|U^{s_k}(\omega_N^k; y^k) - U^t(\omega_N; y)| \leq |U^{s_k}(\omega_N^k; y^k) - U^t(\omega_N^k; y^k)| + |U^t(\omega_N^k; y^k) - U^t(\omega_N; y)|$$

$$\leq |s_k - t| \max\{|U^{s_k,\log}(\omega_N^k; y^k)|, |U^{t,\log}(\omega_N^k; y^k)|\} + |U^t(\omega_N^k; y^k) - U^t(\omega_N; y)|$$

$$\leq |s_k - t| M + |U^t(\omega_N^k; y^k) - U^t(\omega_N; y)|.$$

Then,

$$\lim_{k\to\infty} U^{s_k}(\omega_N^k; y^k) = U^t(\omega_N; y).$$

$\square$

Lemma III.3.6. *Let $t > 0$. Every cluster point as $s \to t$ of maximal N-point Riesz s-polarization configurations is a maximal N-point Riesz t-polarization configuration.*

Proof of Lemma III.3.6. Let $\omega_N^* := \{x_1^*, \dots, x_N^*\}$ be a cluster point as $s \to t$ of maximal N-point Riesz s-polarization configurations of A. This implies that there exists a sequence $\{\omega_N^{s_k}\}_{k\in\mathbb{N}}$ of maximal N-point Riesz s_k-polarization configurations of A such that $s_k \to t$ and $\omega_N^{s_k} \to \omega_N^*$, as $k \to \infty$. Let $\omega_N := \{x_1, \dots, x_N\}$ be any configuration on A. Denote by $y^{s_k} \in A$ a point such that

$$U^{s_k}(\omega_N^{s_k}; y^{s_k}) = M_N^{s_k}(A)$$

and by $y'^{s_k} \in A$ a point such that

$$U^{s_k}(\omega_N; y'^{s_k}) = M^{s_k}(\omega_N; A).$$

By compactness of A, we can find convergent subsequences of $\{y^{s_k}\}_{k\in\mathbb{N}}$ and $\{y'^{s_k}\}_{k\in\mathbb{N}}$. To avoid complicated indexes, we will assume that

$$y^{s_k} \to \hat{y} \quad \text{and} \quad y'^{s_k} \to \hat{\hat{y}}, \quad \text{as } k \to \infty.$$

We claim that

$$\lim_{k\to\infty} M^{s_k}(\omega_N; A) = U^t(\omega_N; \hat{\hat{y}}) = M^t(\omega_N; A) \tag{III.24}$$

and

$$\lim_{k\to\infty} M^{s_k}(\omega_N^{s_k}; A) = U^t(\omega_N^*; \hat{y}) = M^t(\omega_N^*; A). \tag{III.25}$$

If we prove (III.24) for all N-point configurations $\omega_N \subset A$ and (III.25), then we will have for all N-point configurations $\omega_N \subset A$,

$$M^t(\omega_N^*; A) = \lim_{k\to\infty} M^{s_k}(\omega_N^{s_k}; A) \geq \lim_{k\to\infty} M^{s_k}(\omega_N; A) = M^t(\omega_N; A),$$

which implies $M^t(\omega_N^*; A) = M_N^t(A)$ and the proof will be complete.

Now, we show the claims. Note that we will show only (III.25). The same proof can be applied to (III.24). To show the first equality in (III.25), by Lemma III.3.5, we need to show that $|U^{s_k,\log}(\omega_N^{s_k}; y^{s_k})|$ and $|U^{t,\log}(\omega_N^{s_k}; y^{s_k})|$ are uniformly bounded for k sufficiently large.

Let $\varepsilon > 0$ be a small number that $t-\varepsilon > 0$. For $0 < t-\varepsilon \leq s_k < t+1$ and for $i = 1, \ldots, N$, we have

$$\min\left\{\frac{1}{|y^{s_k} - x_i^{s_k}|^{t-\varepsilon}}, \frac{1}{|y^{s_k} - x_i^{s_k}|^{t+1}}\right\} \leq \frac{1}{|y^{s_k} - x_i^{s_k}|^{s_k}} \leq M^{s_k}(\omega_N^{s_k}; A)$$
$$\leq \frac{N}{\rho_N(A)^{s_k}} \leq N \max\left\{\frac{1}{\rho_N(A)^{t-\varepsilon}}, \frac{1}{\rho_N(A)^{t+1}}\right\} < \infty, \tag{III.26}$$

where the third inequality follows from (III.17). Since A is infinite, $\rho_N(A) > 0$. So, there exists $M > 0$ such that

$$\operatorname{diam}(A) \geq |y^{s_k} - x_i^{s_k}| \geq M > 0, \quad k \in \mathbb{N}, \qquad i = 1, \ldots, N.$$

Therefore, $|U^{s_k,\log}(\omega_N^{s_k}; y^{s_k})|$ and $|U^{t,\log}(\omega_N^{s_k}; y^{s_k})|$ are uniformly bounded for all k. Hence, we prove

$$\lim_{k\to\infty} M^{s_k}(\omega_N^{s_k}; A) = U^t(\omega_N^*; \hat{y}). \tag{III.27}$$

Now, we prove the second equality in (III.25), i.e.

$$U^t(\omega_N^*; \hat{y}) \leq U^t(\omega_N^*; y), \quad \text{for all } y \in A.$$

Let $y \in A$ be such that $U^t(\omega_N^*; y) < \infty$ (otherwise the inequality is clear). Since $\omega_N^{s_k} \to \omega_N^*$ as $k \to \infty$, we will consider only large k such that $|x_i^{s_k} - x_i^*| \leq |y - x_i^*|/2$ for all i. Therefore,

$$\operatorname{diam}(A) \geq |y - x_i^{s_k}| \geq |y - x_i^*| - |x_i^{s_k} - x_i^*| \geq \frac{|y - x_i^*|}{2} \geq \min_{1\leq i\leq N} \frac{|y - x_i^*|}{2} = M_y > 0,$$

and $|U^{s_k,\log}(\omega_N^{s_k}; y)|$ and $|U^{t,\log}(\omega_N^{s_k}; y)|$ are uniformly bounded for k sufficiently large. Using Lemma III.3.3 and the equality (III.27), we have

$$U^t(\omega_N^*; \hat{y}) = \lim_{k\to\infty} U^{s_k}(\omega_N^{s_k}; y^{s_k}) \leq \lim_{k\to\infty} U^{s_k}(\omega_N^{s_k}; y) = U^t(\omega_N^*; y).$$

This proves the second equality in (III.25). Then the proof is complete. $\square$

Lemma III.3.7. *Let $t \leq 0$. Every cluster point as $s \to t$ of maximal (minimal) N-point Riesz s-polarization configurations is a maximal (minimal) N-point Riesz t-polarization configuration.*

Proof of Lemma III.3.7. In this case, the Riesz t-potential function $f(y) := U^t(\omega_N; y)$ is nicely continuous on A. It is easy to show Lemma III.3.5 for $t \leq 0$ without the uniform boundedness conditions on $|U^{s_k,\log}(\omega_N^k; y^k)|$ and $|U^{t,\log}(\omega_N^k; y^k)|$ for k sufficiently large. Moreover, the proof of this case can be processed by the same argument as the proof of Lemma III.3.6. So, we leave the details for the reader. $\square$

Proof of Theorem III.2.4. We will omit the proofs of the continuity of $M_N^t(A)$ and $m_N^t(A)$ for $t \in (-\infty, 0)$, because their proofs are exactly the same as the proof of the continuity of $M_N^t(A)$ for $t \in (0, \infty)$.

Now, we will show that $M_N^t(A)$ is continuous as a function of t for all $t \in [0, \infty)$. Recall that Theorem III.2.2 shows that $M_N^t(A)$ is right differentiable at 0, so $M_N^t(A)$ is right-continuous at 0. Let $t > 0$. We want to show that for every sequence $\{M_N^{s_k}(A)\}_{k\in\mathbb{N}}$ such that $s_k \to t$, as $k \to \infty$, there exists a subsequence of $\{M_N^{s_k}(A)\}_{k\in\mathbb{N}}$ that converges to $M_N^t(A)$. Let $\{\omega_N^{s_k}\}_{k\in\mathbb{N}}$ be a sequence of maximal N-point Riesz s_k-polarization configurations of A. By the compactness of A^N, there exists a subsequence $\{\omega_N^{s_{k'}}\}_{k'} \subset \{\omega_N^{s_k}\}_k$ such that $\omega_N^{s_{k'}} \to \omega_N^*$ as $k' \to \infty$ for some $\omega_N^* \subset A$. The proof of Lemma III.3.6 actually shows that we can extract a subsequence $\{\omega_N^{s_{k''}}\}_{k''} \subset \{\omega_N^{s_{k'}}\}_{k'}$ such that

$$\lim_{k''\to\infty} M_N^{s_{k''}}(A) = M_N^t(A).$$

This proves the continuity of $M_N^t(A)$ for $t > 0$.

Next, we prove that $M_N^t(A)$ is not left-continuous at 0. Let $\omega_N^s := \{x_1^s, \ldots, x_N^s\}$ denote a maximal N-point Riesz s-polarization configuration of A and let $y^s := x_1^s$. Then,

$$\limsup_{s\to0^-} M_N^s(A) \leq \limsup_{s\to0^-} \sum_{i=1}^{N} |x_i^s - y^s|^{-s} \leq \lim_{s\to0^-} \sum_{i=1}^{N-1} \operatorname{diam}(A)^{-s} = N - 1 < N. \tag{III.28}$$

Therefore, $M_N^t(A)$ is not left-continuous at 0.

Finally, we prove that $m_N^t(A)$ is left-continuous at 0. Since the function $\rho(\mathbf{x}_N; A)$ is continuous on A^N, A^N is compact, and $\rho(\mathbf{x}_N; A) > 0$ for all $\mathbf{x}_N \in A^N$, there exists a constant $C > 0$ such that

$$\rho(\omega_N; A) \geq C > 0,$$

for all N-point configurations $\omega_N \subset A$. Let $\omega_N^s := \{x_1^s, \ldots, x_N^s\}$ be a minimal N-point Riesz s-polarization configuration of A and y'^s be a point in A such that

$$\min_{1\leq j\leq N} |y'^s - x_j^s| = \rho(\omega_N^s; A).$$

Then,

$$\begin{aligned} N = \lim_{s\to 0^-} NC^{-s} &\leq \liminf_{s\to 0^-} \sum_{i=1}^{N} |y'^s - x_i^s|^{-s} \leq \liminf_{s\to 0^-} m_N^s(A) \\ &\leq \limsup_{s\to 0^-} m_N^s(A) \leq \lim_{s\to 0^-} \sum_{i=1}^{N} \operatorname{diam}(A)^{-s} = N. \end{aligned}$$

Hence, the function $m_N^t(A)$ is left-continuous at 0.

□

III.3.2 Proofs of III.2.2

Upper estimate

For a compact set $A \subset \mathbb{R}^m$, define the quantity

$$\overline{\alpha}_d(A; \varepsilon) := \sup_{0<r\leq\varepsilon} \sup_{x\in A} \frac{\mathcal{H}_d(B(x,r)\cap A)}{\beta_d r^d}. \tag{III.29}$$

Let also

$$\underline{h}_d(A) := \liminf_{N\to\infty} \frac{M_N^d(A)}{N\ln N} \quad \text{and} \quad \overline{h}_d(A) := \limsup_{N\to\infty} \frac{M_N^d(A)}{N\ln N}.$$

The main lemma of this section is given below.

Lemma III.3.8. *Let $d, m \in \mathbb{N}$, $d \le m$, and $A \subset \mathbb{R}^m$ be a compact set with $0 < \mathcal{H}_d(A) < \infty$, containing a closed subset B of zero $\mathcal{H}_d$-measure such that every compact subset $K \subset A \setminus B$ satisfies*

$$\lim_{\epsilon \to 0^+} \overline{\alpha}_d(K;\epsilon) \le 1. \tag{III.30}$$

Then

$$\overline{h}_d(A) \le \frac{\beta_d}{\mathcal{H}_d(A)}. \tag{III.31}$$

If an equality holds in (III.31)*, then any infinite sequence $\omega_N = \{x_{k,N}\}_{k=1}^N$, $N \in \mathcal{N} \subset \mathbb{N}$, of configurations on A such that*

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{M^d(\omega_N; A)}{N \ln N} = \frac{\beta_d}{\mathcal{H}_d(A)} \tag{III.32}$$

satisfies

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_{i,N}} \xrightarrow{\;*\;} \frac{\mathcal{H}_d(\cdot)|_A}{\mathcal{H}_d(A)}, \quad \mathcal{N} \ni N \to \infty. \tag{III.33}$$

We precede the proof of Lemma III.3.8 with the following auxiliary statements.

Lemma III.3.9. *Let $0 < R \le r$, $D \subset \mathbb{R}^m$ be a compact set with $\mathcal{H}_d(D) < \infty$, $d \in \mathbb{N}$, $d \le m$, and $y \in D$. Then*

$$\int\limits_{D\setminus B(y,R)} \frac{d\mathcal{H}_d(x)}{|x-y|^d} \le r^{-d}\mathcal{H}_d(D) + \beta_d \overline{\alpha}_d(D;r) \ln\left(\frac{r}{R}\right)^d.$$

Proof of Lemma III.3.9. We have

$$\begin{aligned}
\int\limits_{D\setminus B(y,R)} \frac{d\mathcal{H}_d(x)}{|x-y|^d} &= \int_0^\infty \mathcal{H}_d\{x \in D \setminus B(y,R) : |x-y|^{-d} > t\} dt \\
&= \int_0^\infty \mathcal{H}_d\{x \in D \setminus B(y,R) : t^{-1/d} > |x-y|\} dt \\
&\le \int_0^{R^{-d}} \mathcal{H}_d(B(y, t^{-1/d}) \cap D) dt \\
&\le r^{-d}\mathcal{H}_d(D) + \int_{r^{-d}}^{R^{-d}} \mathcal{H}_d(B(y, t^{-1/d}) \cap D) dt
\end{aligned}$$

$$\le r^{-d}\mathcal{H}_d(D) + \beta_d \int_{r^{-d}}^{R^{-d}} \overline{\alpha}_d(D;r)t^{-1}dt$$
$$= r^{-d}\mathcal{H}_d(D) + \beta_d\overline{\alpha}_d(D;r)\ln\left(\frac{r}{R}\right)^d,$$

which completes the proof. □

Lemma III.3.10. *Let $d, m \in \mathbb{N}$, $d \le m$, and $A \subset \mathbb{R}^m$ be a compact set with $0 < \mathcal{H}_d(A) < \infty$, containing a closed subset B of zero $\mathcal{H}_d$-measure such that every compact subset of the set $A \setminus B$ satisfies* (III.30)*. Then for any infinite sequence $\{\omega_N\}_{N\in\mathcal{N}}$, $\mathcal{N} \subset \mathbb{N}$, of N-point configurations on the set A, the inequality*

$$\frac{\mathcal{H}_d(K)}{\beta_d} \cdot \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{M^d(\omega_N;A)}{N\ln N} \le \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap K)}{N} \tag{III.34}$$

holds for any compact subset $K \subset A$ with $\mathcal{H}_d(K) > 0$ and $\mathcal{H}_d(\partial_A K) = 0$.

Proof of Lemma III.3.10. Without loss of generality, we can assume that $B \neq \emptyset$ since in the case $B = \emptyset$ we can also use as B any non-empty compact subset of A with $\mathcal{H}_d(B) = 0$.

Let $x_{1,N}, \ldots, x_{N,N}$ be the points in the configuration ω_N, $N \in \mathcal{N}$, and let $K \subset A$ be any compact subset of positive $\mathcal{H}_d$-measure such that $\mathcal{H}_d(\partial_A K) = 0$. Denote

$$K_\rho := \{x \in K : \mathrm{dist}(x, B \cup \partial_A K) \ge \rho\}, \quad \rho > 0.$$

Choose an arbitrary number $\rho > 0$ such that $\mathcal{H}_d(K_{2\rho}) > 0$. Let $r > 0$ be any number such that $2\beta_d r^d < \mathcal{H}_d(K_{2\rho})$. For each $j = 1, \ldots, N$, define the set

$$\mathcal{D}_{j,N} := K_{2\rho} \setminus B(x_{j,N}, rN^{-1/d}) \quad \text{and let} \quad \mathcal{D}_N := \bigcap_{j=1}^{N} \mathcal{D}_{j,N}.$$

Notice that $\mathrm{dist}(K_{2\rho}, K \setminus K_\rho) \ge \rho > 0$. Furthermore, $\mathrm{dist}(K_{2\rho}, A \setminus K) > 0$. Indeed, if there were sequences $\{x_n\}$ in $K_{2\rho}$ and $\{y_n\}$ in $A \setminus K$ such that $|x_n - y_n| \to 0$, $n \to \infty$, then by compactness of $K_{2\rho}$ and A there would exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ having the same limit $z \in K_{2\rho}$. Since $\{y_{n_k}\} \subset A \setminus K$ the point z must belong to $\partial_A K$, which contradicts to the definition of the set $K_{2\rho}$. Thus, we have

$$h := \operatorname{dist}(K_{2\rho}, A \setminus K_\rho) = \min\{\operatorname{dist}(K_{2\rho}, K \setminus K_\rho), \operatorname{dist}(K_{2\rho}, A \setminus K)\} > 0.$$

Choose $N \in \mathcal{N}$ to be such that $rN^{-1/d} < h$ and $\overline{\alpha}_d(K_\rho; rN^{-1/d}) \le 2$ (such N exists since K_ρ is a compact subset of $A \setminus B$, and by assumption, satisfies $\lim_{N\to\infty} \overline{\alpha}_d(K_\rho; rN^{-1/d}) \le 1$). Then

$$\mathcal{H}_d(\mathcal{D}_N) = \mathcal{H}_d\left(K_{2\rho} \setminus \bigcup_{j=1}^{N} B(x_{j,N}, rN^{-1/d})\right)$$

$$= \mathcal{H}_d\left(K_{2\rho} \setminus \bigcup_{x_{j,N}\in K_\rho} B(x_{j,N}, rN^{-1/d})\right)$$

$$\ge \mathcal{H}_d(K_{2\rho}) - \sum_{x_{j,N}\in K_\rho} \mathcal{H}_d\left(K_\rho \cap B(x_{j,N}, rN^{-1/d})\right)$$

$$\ge \mathcal{H}_d(K_{2\rho}) - \beta_d r^d \frac{\#(\omega_N \cap K_\rho)}{N} \cdot \overline{\alpha}_d(K_\rho; rN^{-1/d})$$

$$\ge \mathcal{H}_d(K_{2\rho}) - \beta_d r^d \overline{\alpha}_d(K_\rho; rN^{-1/d}) \ge \mathcal{H}_d(K_{2\rho}) - 2\beta_d r^d =: \gamma_{r,\rho} > 0.$$

Let $\widetilde{\mathcal{D}}_{j,N} := K_\rho \setminus B(x_{j,N}, rN^{-1/d})$. Then

$$M^d(\omega_N; A) = \min_{x\in A} \sum_{j=1}^{N} \frac{1}{|x - x_{j,N}|^d}$$

$$\le \frac{1}{\mathcal{H}_d(\mathcal{D}_N)} \sum_{j=1}^{N} \int_{\mathcal{D}_N} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d} \le \frac{1}{\gamma_{r,\rho}} \sum_{j=1}^{N} \int_{\mathcal{D}_{j,N}} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d}$$

$$\le \frac{1}{\gamma_{r,\rho}} \left(\sum_{x_{j,N}\in K_\rho} \int_{\widetilde{\mathcal{D}}_{j,N}} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d} + \sum_{x_{j,N}\in A\setminus K_\rho} \int_{\mathcal{D}_{j,N}} \frac{d\mathcal{H}_d(x)}{|x - x_{j,N}|^d} \right).$$

Taking into account Lemma III.3.9 with $R = rN^{-1/d}$ and $D = K_\rho$ and the fact that

$\operatorname{dist}(\mathcal{D}_{j,N}, A \setminus K_\rho) \geq \operatorname{dist}(K_{2\rho}, A \setminus K_\rho) = h > 0$, we will have

$$M^d(\omega_N; A) \leq \frac{1}{\gamma_{r,\rho}} \Bigg(\#(\omega_N \cap K_\rho) \left(\frac{\mathcal{H}_d(K_\rho)}{r^d} + \beta_d \overline{\alpha}_d(K_\rho; r) \ln N \right)$$

$$+ \sum_{x_{j,N} \in A \setminus K_\rho} \frac{\mathcal{H}_d(\mathcal{D}_{j,N})}{h^d} \Bigg).$$

Consequently,

$$\frac{M^d(\omega_N; A)}{N \ln N} \leq \frac{1}{\gamma_{r,\rho}} \left(\frac{\#(\omega_N \cap K_\rho)}{N} \left(\frac{\mathcal{H}_d(K_\rho)}{r^d \ln N} + \beta_d \overline{\alpha}_d(K_\rho; r) \right) + \frac{\mathcal{H}_d(A)}{h^d \ln N} \right). \tag{III.35}$$

Passing to the lower limit in (III.35) we will have

$$\tau := \liminf_{\substack{N \to \infty \\ N \in \mathcal{N}}} \frac{M^d(\omega_N; A)}{N \ln N} \leq \frac{\beta_d \overline{\alpha}_d(K_\rho; r)}{\mathcal{H}_d(K_{2\rho}) - 2\beta_d r^d} \liminf_{\substack{N \to \infty \\ N \in \mathcal{N}}} \frac{\#(\omega_N \cap K_\rho)}{N}.$$

Letting $r \to 0$ and taking into account (III.30) and the fact that $K_\rho \subset K$, we will have

$$\tau \leq \frac{\beta_d}{\mathcal{H}_d(K_{2\rho})} \liminf_{\substack{N \to \infty \\ N \in \mathcal{N}}} \frac{\#(\omega_N \cap K_\rho)}{N} \leq \frac{\beta_d}{\mathcal{H}_d(K_{2\rho})} \liminf_{\substack{N \to \infty \\ N \in \mathcal{N}}} \frac{\#(\omega_N \cap K)}{N}.$$

Since $\lim_{\rho \to 0^+} \mathcal{H}_d(K_{2\rho}) = \mathcal{H}_d(K \setminus (B \cup \partial_A K)) = \mathcal{H}_d(K)$, we finally have

$$\tau \leq \frac{\beta_d}{\mathcal{H}_d(K)} \liminf_{\substack{N \to \infty \\ N \in \mathcal{N}}} \frac{\#(\omega_N \cap K)}{N},$$

which implies (III.34). □

Proof of Lemma III.3.8. Let $\mathcal{N}_0 \subset \mathbb{N}$ be an infinite subset such that

$$\overline{h}_d(A) = \lim_{\substack{N \to \infty \\ N \in \mathcal{N}_0}} \frac{M_N^d(A)}{N \ln N}.$$

Let $\{\overline{\omega}_N\}_{N \in \mathcal{N}_0}$ be a sequence of N-point configurations on A such that $M_N^d(A) = M^d(\overline{\omega}_N; A)$,

$N \in \mathcal{N}_0$. Then applying Lemma III.3.10 with $K = A$, we will have

$$\overline{h}_d(A) = \lim_{\substack{N\to\infty \\ N\in\mathcal{N}_0}} \frac{M^d(\overline{\omega}_N; A)}{N \ln N} \le \frac{\beta_d}{\mathcal{H}_d(A)} \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}_0}} \frac{\#(\overline{\omega}_N \cap A)}{N} = \frac{\beta_d}{\mathcal{H}_d(A)}$$

and inequality (III.31) follows.

Assume now that $\overline{h}_d(A) = \beta_d \mathcal{H}_d(A)^{-1}$ and let $\{\omega_N\}_{N\in\mathcal{N}}$, $\mathcal{N} \subset \mathbb{N}$, be any infinite sequence of N-point configurations on A satisfying (III.32). For any closed subset $D \subset A$ with $\mathcal{H}_d(D) > 0$ and $\mathcal{H}_d(\partial_A D) = 0$, by Lemma III.3.10 we have

$$\liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap D)}{N} \ge \frac{\mathcal{H}_d(D)}{\beta_d} \lim_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{M^d(\omega_N; A)}{N \ln N} = \frac{\mathcal{H}_d(D)}{\mathcal{H}_d(A)}. \tag{III.36}$$

Let now $P \subset A$ be any closed subset of zero $\mathcal{H}_d$-measure. Show that

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap P)}{N} = 0. \tag{III.37}$$

If $P = \emptyset$, then (III.37) holds trivially. Let $P \neq \emptyset$. Since $\mathcal{H}_d(A) < \infty$, for every $\epsilon > 0$, there are at most finitely many numbers $\delta > 0$ such that the set $P[\delta] := \{x \in A : \mathrm{dist}(x, P) = \delta\}$ has $\mathcal{H}_d$-measure at least ϵ. This implies that there are at most countably many numbers $\delta > 0$ such that $\mathcal{H}_d(P[\delta]) > 0$. Denote also $P_\delta = \{x \in A : \mathrm{dist}(x, P) \ge \delta\}$, $\delta > 0$. Then there exists a positive sequence $\{\delta_n\}$ monotonically decreasing to 0 such that every set $\partial_A P_{\delta_n} \subset P[\delta_n]$ has $\mathcal{H}_d$-measure zero. Since P_{δ_n} is closed and $\mathcal{H}_d(P_{\delta_n}) > 0$ for every n greater than some n_1, in view of (III.36), we have

$$\liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap (A \setminus P))}{N} \ge \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap P_{\delta_n})}{N} \ge \frac{\mathcal{H}_d(P_{\delta_n})}{\mathcal{H}_d(A)}, \quad n > n_1.$$

Since $\mathcal{H}_d(P_{\delta_n}) \to \mathcal{H}_d(A \setminus P) = \mathcal{H}_d(A)$, $n \to \infty$, we have

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap (A \setminus P))}{N} = 1,$$

which implies (III.37).

Since the set $\overline{A \setminus D}$ is also a closed subset of A and $\mathcal{H}_d(\partial_A(A \setminus D)) = \mathcal{H}_d(\partial_A D) = 0$, by (III.36) and (III.37) (with $P = \partial_A D$) we have

$$\limsup_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap D)}{N} = 1 - \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap (A \setminus D))}{N}$$

$$= 1 - \liminf_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap \overline{A \setminus D})}{N} \leq 1 - \frac{\mathcal{H}_d(\overline{A \setminus D})}{\mathcal{H}_d(A)} = \frac{\mathcal{H}_d(D)}{\mathcal{H}_d(A)}.$$

Thus,

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{\#(\omega_N \cap D)}{N} = \frac{\mathcal{H}_d(D)}{\mathcal{H}_d(A)} \tag{III.38}$$

for any closed subset $D \subset A$ with $\mathcal{H}_d(D) > 0$ and $\mathcal{H}_d(\partial_A D) = 0$. In view of (III.37) relation (III.38) also holds when $D \subset A$ is closed and $\mathcal{H}_d(D) = 0$. Then in view of Remark III.1.4 we have (III.33). □

Auxiliary statements

We will show in this section that for every set A satisfying the assumptions of Theorem III.2.5, the assumptions of Lemma III.3.8 necessarily hold.

Proposition III.3.11. *Let A be a compact subset of a d-dimensional C^1-manifold embedded in $\mathbb{R}^m$, $d \leq m$. Then for such a set A,*

$$\lim_{\varepsilon\to 0^+} \overline{\alpha}_d(A;\varepsilon) \leq 1. \tag{III.39}$$

The proof of this statement is given in the Chapter IV.

Lemma III.3.12. *Let $A = \cup_{i=1}^{l} A_i$, where each set A_i is a compact set contained in some d-dimensional C^1-manifold in $\mathbb{R}^m$, $d \leq m$, and $\mathcal{H}_d(A_i \cap A_j) = 0$, $1 \leq i < j \leq l$. Then there is a compact subset $B \subset A$ with $\mathcal{H}_d(B) = 0$ such that every compact subset $K \subset A \setminus B$ satisfies $\lim\limits_{\epsilon\to 0^+} \overline{\alpha}_d(K;\epsilon) \leq 1$.*

Proof of Lemma III.3.12. Denote $B := \bigcup\limits_{1\leq i<j\leq l} A_i \cap A_j$. Let $K \subset A \setminus B$ be a compact subset.

Then

$$\delta_0 := \min_{1\le i<j\le l} \operatorname{dist}(A_i \cap K, A_j \cap K) > 0.$$

Choose any $\epsilon \in (0, \delta_0)$. Choose also arbitrary $r \in (0, \epsilon]$ and $x \in K$. We have $x \in A_i$ for some $1 \le i \le l$ and $x \notin A_j$ for every $j \ne i$. Since $r < \delta_0$, we have $B(x,r) \cap K \subset B(x,r) \cap A_i$ and consequently,

$$\frac{\mathcal{H}_d(B(x,r)\cap K)}{\beta_d r^d} \le \frac{\mathcal{H}_d(B(x,r)\cap A_i)}{\beta_d r^d}$$
$$\le \sup_{t\in(0,\epsilon]} \sup_{y\in A_i} \frac{\mathcal{H}_d(B(y,t)\cap A_i)}{\beta_d t^d} = \overline{\alpha}_d(A_i;\epsilon) \le \max_{1\le j\le l} \overline{\alpha}_d(A_j;\epsilon).$$

Consequently,

$$\overline{\alpha}_d(K;\epsilon) = \sup_{r\in(0,\epsilon]} \sup_{x\in K} \frac{\mathcal{H}_d(B(x,r)\cap K)}{\beta_d r^d} \le \max_{1\le j\le l} \overline{\alpha}_d(A_j;\epsilon). \tag{III.40}$$

Since each A_i is a compact subset of a d-dimensional C^1-manifold, by Proposition III.3.11, we have $\lim_{\epsilon\to 0^+} \overline{\alpha}_d(A_i;\epsilon) \le 1$, $i = 1, \ldots, l$. Then in view of (III.40) we have $\lim_{\epsilon\to 0^+} \overline{\alpha}_d(K;\epsilon) \le 1$. □

The following proposition is a part of the result by D.P. Hardin, E.B. Saff, and J.T. Whitehouse mentioned at the end of Section III.1. For completeness, we will reproduce its proof.

Proposition III.3.13. *Let $A = \cup_{i=1}^{l} A_i$, where each A_i is a compact set contained in some d-dimensional C^1-manifold in $\mathbb{R}^m$ and $\mathcal{H}_d(A_i \cap A_j) = 0$, $1 \le i < j \le l$. Then*

$$\underline{g}_d(A) := \liminf_{N\to\infty} \frac{\mathcal{E}_d(A,N)}{N^2 \ln N} \ge \frac{\beta_d}{\mathcal{H}_d(A)}.$$

Proof of Proposition III.3.13. Since every set A_i is a compact subset of a d-dimensional C^1-manifold, in view of Theorem 2.4 in [34], there holds $\underline{g}_d(A_i) \ge \beta_d \mathcal{H}_d(A_i)^{-1}$, $i = 1, \ldots, l$. In view of inequality (34) from Lemma 3.2 in [34], we then have

$$\underline{g}_d(A) = \underline{g}_d\left(\bigcup_{i=1}^{l} A_i\right) \ge \left(\sum_{i=1}^{l} \underline{g}_d(A_i)^{-1}\right)^{-1} \ge \left(\frac{1}{\beta_d}\sum_{i=1}^{l} \mathcal{H}_d(A_i)\right)^{-1} = \frac{\beta_d}{\mathcal{H}_d(A)},$$

which yields the desired inequality. ☐

Proof of Theorem III.2.5

Here is the proof of the main theorem in Section III.2.2.

Proof of Theorem III.2.5. The proof of the lower estimate in (III.12) will repeat the proof of inequality (2.9) in [17]. It is known that (see [17], [18], or [20]) for any infinite compact set $A \subset \mathbb{R}^m$,

$$M_N^s(A) \geq \frac{1}{N-1}\mathcal{E}_s(A,N), \quad N \geq 2, \quad s > 0. \tag{III.41}$$

Then Proposition III.3.13 and inequality (III.41) give the lower estimate for $M_N^d(A)$:

$$\liminf_{N\to\infty} \frac{M_N^d(A)}{N\ln N} \geq \liminf_{N\to\infty} \frac{\mathcal{E}_d(A,N)}{(N-1)N\ln N} \geq \frac{\beta_d}{\mathcal{H}_d(A)}.$$

Note that if $\mathcal{H}_d(A) = 0$, then $\lim_{N\to\infty} M_N^d(A)/(N\ln N) = \infty$.

Now, assume that $\mathcal{H}_d(A) > 0$. In view of Lemma III.3.12 and Remark III.2.6, the set A satisfies the assumptions of Lemma III.3.8. Consequently

$$\limsup_{N\to\infty} \frac{M_N^d(A)}{N\ln N} \leq \frac{\beta_d}{\mathcal{H}_d(A)}.$$

This implies (III.12).

Every sequence $\{\omega_N\}_{N=1}^\infty$ of N-point configurations, which is asymptotically maximal for the N-point d-polarization problem on A must satisfy (III.32) with $\mathcal{N} = \mathbb{N}$. Since $\overline{h}_d(A) = \beta_d \mathcal{H}_d(A)^{-1}$, by Lemma III.3.8 we obtain (III.13). ☐

III.3.3 Proofs of III.2.3

Proof of Two one-dimensional circles in different planes

The N-roots of unity, i.e. the solution of $z^N = 1,\ z \in \mathbb{C}$, have the following basic property.

Lemma III.3.14. *Let $\{x_1^*, \ldots, x_N^*\}$ be the set of the N-roots of unity. Then $\sum_{j=1}^N x_j^{*k} = 0$ for all $k \in \{1, \ldots, N-1\}$.*

Proof of Lemma III.3.14. For $k \in \mathbb{N}$, we define the functions $F_k : \mathbb{C}^N \to \mathbb{C}$ and the functions $e_k : \mathbb{C}^N \to \mathbb{C}$ by

$$F_k(x_1, \dots, x_N) := \sum_{i=1}^{N} x_i^k, \quad \text{and} \quad e_k(x_1, \dots, x_N) := \sum_{1 \le j_1 < j_2 \dots < j_k \le N} x_{j_1} \dots x_{j_k},$$

where $(x_1, \dots, x_N) \in \mathbb{C}^N$. Let $\{x_1^*, \dots, x_N^*\}$ be the set of the $N-$roots of unity. Since

$$\prod_{i=1}^{N} (x - x_i^*) = x^N - e_1(x_1^*, \dots, x_N^*)x^{N-1} + e_2(x_1^*, \dots, x_N^*)x^{N-2} - \dots + (-1)^N e_N(x_1^*, \dots, x_N^*),$$

$$e_1(x_1^*, \dots, x_N^*) = e_2(x_1^*, \dots, x_N^*) = \dots = e_{N-1}(x_1^*, \dots, x_N^*) = 0.$$

Using the Newton's identities, we have for $1 \le k \le N-1$,

$$F_k(x_1^*, \dots, x_N^*) = \left(\sum_{j=1}^{k-1} (-1)^{j+1} e_j(x_1^*, \dots, x_N^*) F_{k-j}(x_1^*, \dots, x_N^*) \right) + (-1)^{k+1} k e_k(x_1^*, \dots, x_N^*) = 0.$$

This completes the proof. □

Next, we show that a configuration of N distinct equally spaced points on $\mathbb{S}^1$ is a maximal and minimal N-point Riesz (s, h)-polarization configuration of $(\mathbb{S}^1; \mathbb{S}^1_R)$.

Lemma III.3.15. *Let $N \in \mathbb{N}$, $p \in \{1, 2, \dots, N-1\}$, $R > 0$, and $h \ge 0$. Then, any configuration of N distinct equally spaced points on $\mathbb{S}^1$ is a maximal and minimal N-point Riesz $(-2p, h)$-polarization configuration of $(\mathbb{S}^1; \mathbb{S}^1_R)$.*

Proof of Lemma III.3.15. Let $\omega_N := \{x_1, \dots, x_N\}$ be a configuration of N distinct equally spaced points on $\mathbb{S}^1$, $p \in \{1, \dots, N-1\}$ be fixed, and $h \ge 0$ be fixed. By [42, Theorem 1], we know that $f(x) := \sum_{j=1}^{N} (|x - x_j|^2 + h)^p$ is constant as a function of x on $\mathbb{S}^1_R$, say $f(x) \equiv C$ for all $x \in \mathbb{S}^1_R$.

Let $\{y_1, \dots, y_N\}$ be any N-point configuration on $\mathbb{S}^1$. Then,

$$CN = \sum_{j=1}^{N} \sum_{i=1}^{N} \left(\left| x_i - \frac{R}{y_j} \right|^2 + h \right)^p = \sum_{j=1}^{N} \sum_{i=1}^{N} \left(\left| y_j - \frac{R}{x_i} \right|^2 + h \right)^p = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\left| y_j - \frac{R}{x_i} \right|^2 + h \right)^p. \tag{III.42}$$

Therefore, there exists $i_0, i_0' \in \{1, \dots, N\}$ such that

$$\sum_{j=1}^{N} \left(\left| y_j - \frac{R}{x_{i_0}} \right|^2 + h \right)^p \geq C \quad \text{and} \quad \sum_{j=1}^{N} \left(\left| y_j - \frac{R}{x_{i_0'}} \right|^2 + h \right)^p \leq C.$$

Then, we have

$$\max_{x \in \mathbb{S}^1_R} \sum_{j=1}^{N} \left(|y_j - x|^2 + h \right)^p \geq C = \max_{x \in \mathbb{S}^1_R} \sum_{j=1}^{N} \left(|x_j - x|^2 + h \right)^p$$

and

$$\min_{x \in \mathbb{S}^1_R} \sum_{j=1}^{N} \left(|y_j - x|^2 + h \right)^p \leq C = \min_{x \in \mathbb{S}^1_R} \sum_{j=1}^{N} \left(|x_j - x|^2 + h \right)^p,$$

which imply

$$\max_{x \in \mathbb{S}^1_R} \sum_{j=1}^{N} \left(|x_j - x|^2 + h \right)^p = m_N^{-2p,h}(\mathbb{S}^1; \mathbb{S}^1_R)$$

and

$$\min_{x \in \mathbb{S}^1_R} \sum_{j=1}^{N} \left(|x_j - x|^2 + h \right)^p = M_N^{-2p,h}(\mathbb{S}^1; \mathbb{S}^1_R).$$

□

We recall that the usual dot product in $\mathbb{C}$ is defined by

$$a \cdot b := a_1 b_1 + a_2 b_2$$

where $a := a_1 + a_2 i$, $b := b_1 + b_2 i \in \mathbb{C}$.

Proof of Theorem III.2.7. A simple calculation shows that for $y \in \mathbb{S}^1_R$ and $x_j \in \mathbb{S}^1$,

$$(|y - x_j|^2 + h)^p = (R^2 + h + 1 - 2y \cdot x_j)^p.$$

Let $y := R\cos t + iR\sin t$ and $x_j := \cos t_j + i \sin t_j$. Then

$$f_j(t) := (|y - x_j|^2 + h)^p = (R^2 + h + 1 - 2R\cos(t - t_j))^p.$$

We know that

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(t-t_j)}{\sqrt{\pi}}, \dots, \frac{\cos(p(t-t_j))}{\sqrt{\pi}}\right\}$$

forms orthonormal system with respect to the inner product

$$\langle f, g\rangle := \int_0^{2\pi} f(t)g(t)dt,$$

and

$$\text{span}\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(t-t_j)}{\sqrt{\pi}}, \dots, \frac{\cos(p(t-t_j))}{\sqrt{\pi}}\right\} = \text{span}\{1, \cos(t-t_j), \cos^2(t-t_j), \dots, \cos^p(t-t_j)\}.$$

Therefore,

$$\begin{aligned} f_j(t) &= \langle f_j(t), \frac{1}{\sqrt{2\pi}}\rangle \frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{p} \langle f_j(t), \frac{\cos(k(t-t_j))}{\sqrt{\pi}}\rangle \frac{\cos(k(t-t_j))}{\sqrt{\pi}} \\ &= \frac{1}{2\pi}\langle f_j(t), 1\rangle + \frac{1}{\pi}\sum_{k=1}^{p}\langle f_j(t), \cos(k(t-t_j))\rangle \frac{(y^k \cdot x_j^k)}{R^k}. \end{aligned}$$

Let

$$C_{k,j} := \langle f_j(t), \cos(k(t-t_j))\rangle, \quad k \in \{0, 1, \dots, p\}, \quad j \in \{1, \dots, N\}.$$

Notice that $C_{k,j}$ do not depend on j, so we will let $\tilde{C}_k := C_{k,j}$. Therefore,

$$\sum_{j=1}^{N}(|y-x_j|^2+h)^p = \frac{N\tilde{C}_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{p}\frac{\tilde{C}_k}{R^k}\left(y^k \cdot \sum_{j=1}^{N} x_j^k\right) \tag{III.43}$$

and

$$\begin{aligned} \tilde{C}_k &= \int_0^{2\pi}(R^2+h+1-2R\cos(t))^p\cos(kt)dt \\ &= \frac{(-1)^k\pi}{2^{p-1}}\sum_{j=0}^{p-k}\binom{p}{j}\binom{p}{k+j} b^{2j+k}\left(a+\sqrt{a^2-b^2}\right)^{p-k-2j}, \end{aligned} \tag{III.44}$$

where

$$a := R^2+h+1, \quad \text{and} \quad b := 2R,$$

and the square root function in (III.44) can be chosen to be both branches of the complex square root function (see the computation in Lemma IV.2.1). Moreover, we note that $(R^2 + h+1)^2 \geq 4R^2$ and if we choose the branch that the square root of positive number is positive number, then clearly, $\tilde{C}_k \neq 0$ for all k.

By Lemma III.3.15, we know that the set of the N-roots of unity is optimal for $M_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$ and $m_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$. Then, by Lemma III.3.14, we have

$$M_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R) = m_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R) = \frac{N\tilde{C}_0}{2\pi} = \frac{N}{2^p}\sum_{j=0}^{p}\binom{p}{j}^2 b^{2j}\left(a+\sqrt{a^2-b^2}\right)^{p-2j}.$$

Moreover, any configuration $\{x_1,\dots,x_N\}$ such that $\sum_{j=1}^N x_j = \sum_{j=1}^N x_j^2 = \cdots = \sum_{j=1}^N x_j^p = 0$ is optimal for $M_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$ and $m_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$.

Now, we show that any optimal configuration for $M_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$ satisfies

$$\sum_{j=1}^N x_j = \sum_{j=1}^N x_j^2 = \cdots = \sum_{j=1}^N x_j^p = 0.$$

The proof of the $m_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$ case is similar. So, we will prove only the $M_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$ case. Let $\omega_N := \{x_1,\dots,x_N\}$ be an optimal configuration for $M_N^{-2p,h}(\mathbb{S}^1;\mathbb{S}^1_R)$. Then, by (III.43), we have, for all $z \in \mathbb{S}^1_R$,

$$\frac{N\tilde{C}_0}{2\pi} = \min_{y\in\mathbb{S}^1_R}\sum_{j=1}^N \left(|y-x_j|^2+h\right)^p \leq \sum_{j=1}^N \left(|z-x_j|^2+h\right)^p = \frac{N\tilde{C}_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^p \frac{\tilde{C}_k}{R^k}\left(z^k\cdot\sum_{j=1}^N x_j^k\right),$$

which means

$$0 \leq \frac{1}{\pi}\sum_{k=1}^p \frac{\tilde{C}_k}{R^k}\left(z^k\cdot\sum_{j=1}^N x_j^k\right), \qquad \text{for all } z\in\mathbb{S}^1_R.$$

Let $z = R\cos(t) + iR\sin(t)$ and $\sum_{j=1}^N x_j^k = \cos(t'_k) + i\sin(t'_k)$. Then, for all $t \in [0,2\pi]$,

$$0 \leq \frac{1}{\pi}\sum_{k=1}^p \tilde{C}_k\left(\cos(kt)\cos(t'_k)+\sin(kt)\sin(t'_k)\right) = \sum_{k=1}^p \left(D_k\cos(kt)+D'_k\sin(kt)\right), \quad \text{(III.45)}$$

where

$$D_k = \frac{\tilde{C}_k \cos(t'_k)}{\pi} \qquad \text{and} \qquad D'_k = \frac{\tilde{C}_k \sin(t'_k)}{\pi}.$$

Because

$$\int_0^{2\pi} \sum_{k=1}^{p} (D_k \cos(kt) + D'_k \sin(kt))\, dt = 0$$

and

$$\sum_{k=1}^{p} D_k \cos(kt) + D'_k \sin(kt) \geq 0, \qquad t \in [0, 2\pi],$$

$\sum_{k=1}^{p} (D_k \cos(kt) + D'_k \sin(kt)) = 0$ for all $t \in [0, 2\pi]$. Then, for all $j \in \{1, \dots, p\}$,

$$\pi D_j = \langle \sum_{k=1}^{p} (D_k \cos(kt) + D'_k \sin(kt)), \cos(jt) \rangle = 0$$

and

$$\pi D'_j = \langle \sum_{k=1}^{p} (D_k \cos(kt) + D'_k \sin(kt)), \sin(jt) \rangle = 0.$$

Since $\tilde{C}_k \neq 0$ for all k, $\cos(t'_k) = \sin(t'_k) = 0$ for all k. Hence, $\sum_{j=1}^{N} x_j^k = 0$ for all $k \in \{1, \dots, p\}$. □

Proofs of the m-dimensional sphere case

Proof of Theorem III.2.9. We will prove only the $M_N^{-2}(\mathbb{S}^m)$ case. The proof of the $m_N^{-2}(\mathbb{S}^m)$ case is basically the same. For $y, x_j \in \mathbb{S}^m$,

$$\sum_{j=1}^{N} |y - x_j|^2 = 2N - 2(y \cdot \sum_{j=1}^{N} x_j).$$

Therefore, our maximization problem is equivalent to finding all N-point configurations on $\mathbb{S}^m$ minimizing the following quantity

$$\min_{\substack{\omega_N \subset \mathbb{S}^m \\ \#\omega_N = N}} \max_{y \in \mathbb{S}^m} \left(y \cdot \sum_{j=1}^{N} x_j \right).$$

Clearly, if $\sum_{j=1}^N x_j = 0$, then $\max_{y\in\mathbb{S}^m}\left(y\cdot\sum_{j=1}^N x_j\right) = 0$. If $\sum_{j=1}^N x_j \neq 0$, then

$$\max_{y\in\mathbb{S}^m}\left(y\cdot\sum_{j=1}^N x_j\right) \geq \left(\frac{\sum_{j=1}^N x_j}{|\sum_{j=1}^N x_j|}\right)\cdot\sum_{j=1}^N x_j = |\sum_{j=1}^N x_j| > 0.$$

Therefore, the function $f(\omega_N) := \max_{y\in\mathbb{S}^m}\left(y\cdot\sum_{j=1}^N x_j\right)$ attains its minimum if and only if $\sum_{j=1}^N x_j = 0$. Then, $\omega_N := \{x_1,\ldots,x_N\}$ is optimal for $M_N^{-2}(\mathbb{S}^m)$ if and only if $\sum_{j=1}^N x_j = 0$. Moreover, $M_N^{-2}(\mathbb{S}^m) = m_N^{-2}(\mathbb{S}^m) = 2N$.

□

Proof of Theorem III.2.10. Let $\omega_N^* := \{x_1^*,\ldots,x_N^*\}\subset\mathbb{S}^m$ be a configuration such that its Riesz potential function $f(y) := \sum_{i=1}^N |y-x_i^*|^{-s}$ is constant on $\mathbb{S}^m$, say

$$f(y)\equiv C,\quad y\in\mathbb{S}^m.$$

Let $\omega_N = \{x_1,\ldots,x_N\}$ be any $N-$point configuration in $\mathbb{S}^m$. Then,

$$CN = \sum_{i=1}^N\sum_{j=1}^N |x_i - x_j^*|^{-s} = \sum_{j=1}^N\sum_{i=1}^N |x_i - x_j^*|^{-s}.$$

Therefore, there exists $j_0, j_0' \in \{1,\ldots,N\}$ such that

$$\sum_{i=1}^N |x_i - x_{j_0}^*|^{-s} \geq C \quad\text{and}\quad \sum_{i=1}^N |x_i - x_{j_0'}^*|^{-s} \leq C.$$

Then, we have

$$\max_{y\in\mathbb{S}^m}\sum_{i=1}^N |x_i - y|^{-s} \geq C = \max_{y\in\mathbb{S}^m}\sum_{j=1}^N |x_j^* - y|^{-s},$$

and

$$\min_{y\in\mathbb{S}^m}\sum_{i=1}^N |x_i - y|^{-s} \leq C = \min_{y\in\mathbb{S}^m}\sum_{j=1}^N |x_j^* - y|^{-s},$$

which imply

$$\max_{y\in\mathbb{S}^m}\sum_{j=1}^N |x_j^* - y|^{-s} = m_N^s(\mathbb{S}^m),$$

and

$$\min_{y \in \mathbb{S}^m} \sum_{j=1}^{N} |x_j^* - y|^{-s} = M_N^s(\mathbb{S}^m),$$

respectively. Hence, ω_N^* is optimal for $M_N^s(\mathbb{S}^m)$ and $m_N^s(\mathbb{S}^m)$. $\square$

CHAPTER IV

AUXILIARY RESULTS

IV.1 Proof of Proposition III.3.11

We say that a set B in $\mathbb{R}^m$ is *bi-Lipschitz homeomorphic to a set* $D \subset \mathbb{R}^n$ *with a constant* $M \geq 1$, if there is a mapping $\varphi : B \to D$ such that $\varphi(B) = D$ and

$$M^{-1} |x - y| \leq |\varphi(x) - \varphi(y)| \leq M |x - y|, \quad x, y \in B.$$

Lemma IV.1.1. *Let $U \subset \mathbb{R}^d$ be a non-empty open set and $f : U \to \mathbb{R}^m$, $m \geq d$, be an injective C^1-continuous mapping such that its inverse $f^{-1} : f(U) \to U$ is continuous and the Jacobian matrix*

$$J_x^f := \begin{bmatrix} \nabla f_1(x) \\ \dots \\ \nabla f_m(x) \end{bmatrix} \tag{IV.1}$$

of f has rank d at any point $x \in U$. Then for every $\epsilon > 0$ and every point $y_0 \in f(U)$, there is a closed ball B centered at y_0 such that the set $B \cap f(U)$ is bi-Lipschitz homeomorphic to some compact set in $\mathbb{R}^d$ with a constant $1 + \epsilon$.

Proof of Lemma IV.1.1. Let $x_0 \in U$ be the point such that $f(x_0) = y_0$. Choose any $\epsilon > 0$ and let $\delta = \delta(x_0, \epsilon) > 0$ be such that $B[x_0, \delta] \subset U$ and

$$|\nabla f_i(x) - \nabla f_i(x_0)| < \epsilon, \quad x \in B[x_0, \delta], \ i = 1, \dots, m.$$

Let $x, y \in B[x_0, \delta]$ be two arbitrary points. Define the function $g_i(t) := f_i(x + t(y - x))$, $t \in [0, 1]$. Then there exists $\xi_i \in (0, 1)$ such that

$$f_i(y) - f_i(x) = g_i(1) - g_i(0) = g_i'(\xi_i) = \nabla f_i(z_i) \cdot (y - x)$$

$$= \nabla f_i(x_0) \cdot (y - x) + (\nabla f_i(z_i) - \nabla f_i(x_0)) \cdot (y - x),$$

where $z_i = x + \xi_i(y - x)$, $i = 1, \ldots, m$. Since $z_i \in B[x_0, \delta]$, we have

$$|f_i(y) - f_i(x) - \nabla f_i(x_0) \cdot (y - x)|$$

$$= |(\nabla f_i(z_i) - \nabla f_i(x_0)) \cdot (y - x)| \leq \epsilon \, |y - x| , \quad i = 1, \ldots, m,$$

and hence (we treat x and y as vector-columns below),

$$\left| f(y) - f(x) - J^f_{x_0}(y - x) \right| \leq \epsilon\sqrt{m} \, |y - x| , \quad x, y \in B[x_0, \delta]. \tag{IV.2}$$

Since the matrix $J^f_{x_0}$ has rank d, for every standard basis vector e_i from $\mathbb{R}^d$, there is a vector $v_i \in R^m$ such that $(J^f_{x_0})^T v_i = e_i$, $i = 1, \ldots, d$, where $(J^f_{x_0})^T$ denotes the transpose of the matrix $J^f_{x_0}$. Then the $d \times m$ matrix $Z := [v_1, \ldots, v_d]^T$ satisfies $Z J^f_{x_0} = I_d$, where I_d is the $d \times d$ identity matrix. Taking into account (IV.2) we have

$$\left| f(y) - f(x) - J^f_{x_0}(y - x) \right| \leq \epsilon\sqrt{m} \left| Z J^f_{x_0}(y - x) \right|$$

$$\leq \epsilon\sqrt{m} \|Z\| \left| J^f_{x_0}(y - x) \right| , \quad x, y \in B[x_0, \delta],$$

where $\|Z\| := \max\{|Zu| : u \in \mathbb{R}^m, \ |u| = 1\}$. Consequently,

$$\left(1 - \epsilon\sqrt{m}\|Z\|\right) \left| J^f_{x_0}(y - x) \right| \leq |f(y) - f(x)|$$

$$\leq \left(1 + \epsilon\sqrt{m}\|Z\|\right) \left| J^f_{x_0}(y - x) \right| , \quad x, y \in B[x_0, \delta].$$

Let $u_1, \ldots, u_d$ be an orthonormal basis in the subspace H of R^m spanned by the columns of the matrix $J^f_{x_0}$ and let $D := [u_1, \ldots, u_d]$ be the $m \times d$ matrix with columns $u_1, \ldots, u_d$. Since the columns of $J^f_{x_0}$ also form a basis in H, there exists an invertible $d \times d$ matrix Q such that $D = J^f_{x_0} Q$.

Let $V \subset \mathbb{R}^d$ be the open set such that $\Phi(V) = B(x_0, \delta)$, where $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is the linear mapping given by $\Phi(v) = Qv$. Since the columns of the matrix D are orthonormal, for every

$u, v \in \overline{V}$, we will have

$$|f \circ \Phi(u) - f \circ \Phi(v)| = |f(Qu) - f(Qv)|$$

$$\leq \left(1 + \epsilon\sqrt{m}\|Z\|\right) \left|J^f_{x_0} Q(u - v)\right| = \left(1 + \epsilon\sqrt{m}\|Z\|\right) |D(u - v)|$$

$$= \left(1 + \epsilon\sqrt{m}\|Z\|\right) |u - v| .$$

Similarly,

$$|f \circ \Phi(u) - f \circ \Phi(v)| \geq \left(1 - \epsilon\sqrt{m}\|Z\|\right) |u - v| , \quad u, v \in \overline{V},$$

which implies that for $0 < \epsilon < (\sqrt{m}\|Z\|)^{-1}$, the restriction of the mapping $\psi := f \circ \Phi$ to the set $\overline{V}$ is a bi-Lipschitz mapping onto the set $f(\Phi(\overline{V})) = f(B[x_0, \delta])$ with constant $M_\epsilon := \max\{1 + \epsilon\sqrt{m}\|Z\|, (1 - \epsilon\sqrt{m}\|Z\|)^{-1}\}$.

Since f is a homeomorphism of U onto $f(U)$, the set $f(B(x_0, \delta))$ is open relative to $f(U)$. Then there is a closed ball B in $\mathbb{R}^m$ centered at $y_0 = f(x_0)$ such that $B \cap f(U) \subset f(B(x_0, \delta))$. Then the set $B \cap f(U) = B \cap f(B[x_0, \delta])$ is bi-Lipschitz homeomorphic (with constant M_ϵ) to the set

$$V_1 := \psi^{-1}(B \cap f(U)) = \psi^{-1}(B \cap f(B[x_0, \delta])),$$

which is compact in $\mathbb{R}^d$. Since $M_\epsilon \to 1$ as $\epsilon \to 0^+$, the assertion of the lemma follows. □

Proof of Proposition III.3.11. Let W denote the d-dimensional C^1-manifold that contains A and let $\epsilon > 0$ be arbitrary. In view of Definition III.1.3, for every point $x \in W$, there is an open neighborhood V_x of x relative to W which is homeomorphic to an open set $U_x \subset \mathbb{R}^d$ such that the homeomorphism $f : U_x \to V_x$ is a C^1-continuous mapping and the Jacobian matrix J^f_u (see the definition J^f_u in (IV.1)) has rank d for every $u \in U_x$. There is also a number $\epsilon_x > 0$ such that $B(x, \epsilon_x) \cap W \subset V_x$. By Lemma IV.1.1, there is a number $0 < \delta(x) < \epsilon_x/2$ such that the set $B[x, 2\delta(x)] \cap W = B[x, 2\delta(x)] \cap f(U_x)$ is bi-Lipschitz homeomorphic to a compact set D_x from $\mathbb{R}^d$ with constant $1 + \epsilon$. Since A is compact, the open cover $\{B(x, \delta(x))\}_{x \in A}$ has a finite subcover $\{B(x_i, \delta(x_i))\}_{i=1}^p$.

Denote $\delta_\epsilon := \min\limits_{j=1,\ldots,p} \delta(x_j)$. Let x be any point in A and $r \in (0, \delta_\epsilon]$. There is an index

i such that $x \in B(x_i, \delta(x_i))$. Since $B(x,r) \cap A \subset B[x_i, 2\delta(x_i)] \cap W$, the set $B(x,r) \cap A$ is bi-Lipschitz homeomorphic to a set $D_i \subset D_{x_i}$ with constant $1+\epsilon$. If $\varphi : B(x,r) \cap A \to D_i$ denotes the corresponding bi-Lipschitz mapping, we have $D_i \subset B(\varphi(x), (1+\epsilon)r)$. Then

$$\mathcal{H}_d(B(x,r) \cap A) \leq (1+\epsilon)^d \mathcal{L}_d(D_i) \leq \beta_d r^d (1+\epsilon)^{2d}.$$

Consequently,

$$\overline{\alpha}_d(A; \delta_\epsilon) = \sup_{r \in (0, \delta_\epsilon]} \sup_{x \in A} \frac{\mathcal{H}_d(B(x,r) \cap A)}{\beta_d r^d} \leq (1+\epsilon)^{2d},$$

which implies that $\lim_{\delta \to 0^+} \overline{\alpha}_d(A; \delta) \leq 1$. □

IV.2 Integrals

We collect our computations of all integrals in this section.

Lemma IV.2.1. *Let $m \in \mathbb{N}$, $k \in \{1, \ldots, m\}$, $z \in \mathbb{C}$. Then,*

$$\int_0^{2\pi} (z^2 + 1 - 2z\cos(t))^m \cos(kt) dt = (-1)^k 2\pi \sum_{j=0}^{m-k} \binom{m}{j} \binom{m}{k+j} z^{2m-k-2j}. \tag{IV.3}$$

Proof of Lemma IV.2.1. Let $m \in \mathbb{N}$ and $k \in \{1, \ldots, m\}$. First, we prove the equality (IV.3) for $z \in \mathbb{R}$.

Let $z \in \mathbb{R}$. Then, for $\zeta = e^{it}$,

$$\begin{aligned}
\int_0^{2\pi} (z^2 + 1 - 2z\cos(t))^m \cos(kt) dt &= \int_0^{2\pi} (z^2 + 1 - z(e^{it} + e^{-it}))^m e^{ikt} dt \\
&= \int_0^{2\pi} (z - e^{it})^m (z - e^{-it})^m e^{ikt} dt \\
&= \frac{1}{i} \int_0^{2\pi} (z - \zeta)^m (z - 1/\zeta)^m \zeta^{k-1} d\zeta \\
&= 2\pi \cdot \operatorname{res}\left(\frac{(z-\zeta)^m (z\zeta - 1)^m}{\zeta^{m-k+1}}; 0 \right) \\
&= (-1)^k 2\pi \sum_{j=0}^{m-k} \binom{m}{j} \binom{m}{k+j} z^{2m-k-2j},
\end{aligned}$$

where the first equality follows from the fact that the last expression is real number. Notice that the left-hand side and the right-hand side of the equation (IV.3) are polynomials as functions of z. Then, both functions are analytic on $\mathbb{C}$ and we have the equation (IV.3) for all $z \in \mathbb{C}$. □

Lemma IV.2.2. *Let $m \in \mathbb{N}$ and $k \in \{1, \ldots, m\}$. For $a, b \in \mathbb{C}$,*

$$\int_0^{2\pi} (a - b\cos(t))^m \cos(kt)dt = \frac{(-1)^k \pi}{2^{m-1}} \sum_{j=0}^{m-k} \binom{m}{j} \binom{m}{k+j} b^{2j+k} \left(a \pm \sqrt{a^2 - b^2}\right)^{m-k-2j}, \tag{IV.4}$$

where the square root function in (IV.4) can be selected to be both branches of the complex square root function.

Proof of Lemma IV.2.2. Clearly, if $b = 0$, then the equation in (IV.4) is 0. Let $b \in \mathbb{C} \setminus \{0\}$ and $a \in \mathbb{C}$. To reduce the equation (IV.4) to the equation (IV.3), we need to consider

$$(\lambda a - \lambda b\cos(t))^m,$$

where λ is chosen to that

$$2z = b\lambda \qquad \text{and} \qquad z^2 + 1 = a\lambda,$$

for some $z \in \mathbb{C}$. From above equations,

$$z = \frac{a \pm \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \lambda = \frac{2a \pm 2\sqrt{a^2 - b^2}}{b^2}.$$

Moreover, $\lambda \neq 0$, because if $\lambda = 0$, then $b = 0$. Therefore, by Lemma IV.2.1,

$$\begin{aligned} \int_0^{2\pi} (a - b\cos(t))^m \cos(kt)dt &= \frac{1}{\lambda^m} \int_0^{2\pi} (\lambda a - \lambda b\cos(t))^m \cos(kt)dt \\ &= \frac{1}{\lambda^m} \int_0^{2\pi} (z^2 + 1 - 2z\cos(t))^m \cos(kt)dt \\ &= \frac{(-1)^k \pi}{2^{m-1}} \sum_{j=0}^{m-k} \binom{m}{j} \binom{m}{k+j} b^{2j+k} \left(a \pm \sqrt{a^2 - b^2}\right)^{m-k-2j}. \end{aligned}$$

□

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