Learner-Driven Engagement in

Out-of-School Mathematics Spaces

By

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# CHAPTER 1

# GROUNDING MATHEMATICS IN LEARNERS' AESTHETICS TO BROADEN ACCESS


## Abstract

Mathematics is a polarizing discipline: it is often either loved or hated. Unfortunately, many learners fall into the latter category. This paper proposes that attending to the *aesthetic* nature of mathematical experiences might help mitigate such negative dispositions, and indeed, may even create new entry-points to participation in ways that support disciplinary enjoyment. Through synthesizing literature about mathematical aesthetics in the work of mathematicians, teachers, and learners, this paper offers design principles for creating learning environments that foster (a) learning authentic to the discipline and (b) positive relationships between learners and mathematics. While learners' aesthetics may be quite different from the aesthetics of mathematicians (people who are well indoctrinated into the world and social norms of mathematics), failing to intentionally leverage and support learners' aesthetics for mathematics teaching and learning is a mistake if the goal of mathematics education reform is to develop more authentic and humane learning environments.



***Keywords.*** mathematics learning; informal learning; mathematical aesthetics; problem-posing; disciplinary dispositions; learning environment design principles

## Introduction

Almost thirty years ago, Jean Lave declared, “All is not well with math learning in our system of school education” (1992, p. 76). Unfortunately, these words still ring true. Even after years of reform efforts in the intervening decades seeking to make mathematics learning more authentic to disciplinary ways of knowing (National Council of Teachers of Mathematics [NCTM], 1989), most students are still taught mathematics more or less the way their parents were taught (Jacobs et al., 2006; Litke, 2015), what Stigler and Hiebert (1999) described as *learning definitions and practicing procedures*. Mostly, this kind of ritualized experience — frequently devoid of personal meaning or connection to students' other interests — leads learners to feel like mathematics is “not for them,” even into adulthood (Boaler & Greeno, 2000; Boaler & Selling, 2017).

In reform-oriented communities of mathematics teaching and learning, we commonly argue that mathematics can be learned more meaningfully — in ways that support the development of positive disciplinary dispositions — when learning includes the disciplinary practices of those who make mathematics (Lampert, 1990; Silver, 1994): professional mathematicians. Unlike most students, mathematicians feel intimately connected to the mathematics that they study (Sinclair, 2004). They describe their experiences of doing mathematics as fundamentally about seeking *beauty* and *elegance.* In their essence, the words mathematicians use to describe their mathematical experiences show us that they have developed an *aesthetic* for mathematical work. Their aesthetic preferences guide *what* work they do and *why*, not just how they write it up for public display and critique (Silver & Metzger, 1989; Sinclair, 2004; Wells, 1990). While beauty and elegance are fundamental drivers of mathematical practice, many teachers and students find aesthetics for mathematics a mystifying concept.

Understanding how mathematicians' aesthetics drive and guide their work may help us make mathematics learning more *meaningful* and *enjoyable* for children. Of course, as Lave reminds us, "Most children going through public school curricula will not become […] mathematicians" (1992, p. 74, 75). For that reason, mathematical practices transform in different contexts to serve different purposes. For example, mathematicians are charged with the creative task of creating new mathematics, but that is rarely the task of engineers. Although both disciplines are mathematics based, the purposes and practices of these two disciplines differ in important ways. I do not argue that we should use *only* mathematicians as a model for authentic disciplinary practices to reflect in classrooms. But in this paper, I privilege mathematicians' practices because of the creativity in their production of new mathematics (which requires a kind of *learning* of mathematics) and because of their love for the discipline for its own sake.

Following a host of prior work (e.g., Dreyfus & Eisenberg, 1986; Silver & Metzger, 1989; Sinclair, 2006a; Wells, 1990), I contend that centering aesthetic experiences in our conceptualization of what it means to know and do mathematics is paramount for creating learning environments that (a) make mathematics learning more authentic to the discipline and (b) foster positive relationships between learners and mathematics. Based on a review of literature on mathematicians' aesthetics and on developing and leveraging aesthetics for mathematics learning, I offer a provisional set of design principles to ground mathematics education in learners' aesthetic sensemaking. If we can create opportunities for students to use personal aesthetics to guide the math they do — as they are learning it — we might end up with more people who actually *enjoy* and feel *competent* in mathematics.

## Background

Because the word *aesthetic* can seem like a surprising description for mathematical activity to many people, I want to clarify my use of the word. To do so, I use the Merriam Webster dictionary’s examples of *aesthetic* (Aesthetic, 2018) and connect them to the three roles of mathematical aesthetics described by Sinclair (2004). Specifically, Sinclair describes mathematical aesthetics as serving *evaluative*, *motivational*, and *generative* roles for mathematicians.

Consider the following three sentences that operationalize the word *aesthetic* in non-mathematics contexts (emphasis added):

1. My generation has an annoying penchant for treating luxuries as necessities and turning guilty pleasures into *aesthetic* and even moral *touchstones*. —Terrence Rafferty, *GQ*, October 1997
2. I suppose that jazz listening and prizefight watching are my two most passionate avocations, and this is largely so because the origins of my *aesthetic urges* are in the black working class. —Gerald Early, "The Passing of Jazz's Old Guard: … ," in *The Best American Essays 1986*, Elizabeth Hardwick & Robert Atwan, editors, 1986
3. Whereas the essence of Proust's *aesthetic position* was contained in the deceptively simple yet momentous assertion that "a picture's beauty does not depend on the things portrayed in it." —Alain de Botton, *How Proust Can Change Your Life*, 1997

These examples use aesthetic in more complex ways than simply as a synonym for beauty and visual appeal. Here, aesthetics function as values or sensibilities in the form of *touchstones*, *urges,* and *positions.* As a *touchstone,* our aesthetics are the source of our ability to make comparisons and evaluate worth and thus serve an *evaluative* function in our activity. As an *urge*, our aesthetics drive us to pursue activities or objects in enduring and meaningful ways and thus serve a *motivational* role in our activity. As a *position*, our aesthetics orient us to what matters — structuring how we engage and what we produce — and thus serve a *generative* (or innovative) role in our activity. In this way, aesthetics are the framework for how we experience the world — for how we judge what counts as good and for why and how we choose to engage or disengage.

## Research Questions

In the remainder of this paper, I explore how the evaluative, motivational, and generative roles of mathematical aesthetics have been operationalized in educational research and conclude with synthesized design principles. Specifically, I ask the following questions:

1. *What is mathematical aesthetics, and what is its role in the work of mathematicians?*
2. *How can we leverage learners' aesthetics in service of meaningful mathematics learning?*

Importantly, the cultural activities of *aesthetically-driven mathematics* and *school math* are dramatically different in scale and motivation. Aesthetically-driven mathematics is engaged by a tiny percentage of the population, largely by professionals who have voluntarily decided to participate in such mathematics. School mathematics, on the other hand, is engaged by nearly the entire population; it is mostly a compulsory activity by conscripted students. A reasonable reader might ask, how can better understanding the few (i.e., mathematicians) help us improve learning experiences for the masses (i.e., students in schools)? Likewise, if aesthetically-driven mathematics is an inconceivable activity for many teachers (as I have claimed), who else is in a position to open opportunities for leveraging aesthetics for mathematics learning?

I argue that if we leverage aesthetics in our design of learning materials and instruction — and foreground them in our educational goals — then we might be able to broaden access for adolescents whose identities are otherwise repelled by school mathematics. In other words, although the path into academic math is historically one of extreme winnowing, attending to the authentic nature of mathematical work, which is inherently aesthetic, can help combat the macro-*culture-of-exclusion* in mathematics education (Louie, 2017) that allows only the few to be successful. Aesthetics also provide a way to *humanize* mathematical activity (Buenrostro, 2016) by making it personal. By reshaping the cultural and material tools for mathematics learning both locally in classrooms and at scale across the nation, we may be able to transform an educational system that routinely leads too many students to dis-identify with the discipline (Funk & Parker, 2018; Jacobs, Lanza, Osgood, Eccles, & Wigfield, 2002).

## Methods: Literature Synthesis

To derive design principles for learning environments that broaden access to mathematics by grounding learning in aesthetics, I reviewed literature on the role of aesthetics in the production of the mathematical canon and in the production of mathematics learning. To do so, I first turned first to the work of a leading scholar in the field of mathematical aesthetics in education, Nathalie Sinclair — and her seminal piece from 2004, *The roles of the aesthetic in mathematical inquiry*. Through her work, I view mathematical aesthetics to be a personal and disciplined *taste* for what counts as interesting and worthwhile throughout the process of problem finding, solving, and evaluating. In this way, mathematical aesthetics are both a preference and a value system.

In order to understand how mathematical aesthetics have been operationalized in educational research, I searched educational databases (ERIC ProQuest and PsychInfo) for literature on mathematical aesthetics. After sorting out[1] articles that were not about mathematicians' practices, mathematics teachers, students of mathematics, or mathematics curriculum, I was left with 44 articles to review. As I examined these 44 articles, I began to see that they that fell into categories such as (a) historical analyses of mathematicians' work, (b) empirical and conceptual work that takes an aesthetic lens to understanding mathematical activity, (c) empirical work comparing the aesthetics of students and mathematicians, (d) empirical and conceptual work about developing mathematical aesthetics in teachers and students, and (e) a few outlier studies that uniquely operationalized aesthetics. These categories structure the way findings of this paper are organized within each role of mathematical aesthetics, with (b) and (c) combined and (e) interwoven where appropriate. In addition to these papers, I also include studies frequently cited within my literature search and studies recommended by colleagues engaged with me

[1] For example, I excluded articles that explored why mathematics is such a powerful tool for explaining the natural world (e.g., Gelfert, 2014) as well as studies that examined human aesthetic preferences in nature and in art in relation to mathematical constructs such as the golden ratio (e.g., Green, 1995).

in the early writing process.

Again, Sinclair's (2004) analysis of the evaluative, motivational and generative roles of mathematical aesthetics provided an especially compelling framework for my inquiry. Based on my theoretical framework, I selected empirical research that I concluded best demonstrated these three roles of mathematical aesthetics in education. This method for engaging the literature allowed me to identify gaps in the field's understanding of how aesthetics support and hinder mathematics learning, and to synthesize design principles that help us operationalize what we do know. Synthesizing design principles from the literature allows us to see how attention to specific features of learning environments can be leveraged in different ways across contexts (Ito et al., 2013), and thus is the beginning of an iterative process of reforming approaches to curriculum and its enactment in ways that are theoretically grounded yet responsive to local realities.

## Theoretical Framework: Aesthetics and Learning

By examining both the role of mathematical aesthetics in mathematicians' work and how we can leverage learners' aesthetics in service of meaningful mathematics learning, I seek to understand how young people can engage in mathematics in rich and meaningful ways that support enjoyment in mathematics learning. Both students who already identify as "math people" and those who do not should be able to participate in the cultural activity of mathematics in ways that bolster their competence and motivation.

To frame my review of research on mathematical aesthetics and learning, I draw on *situated perspectives* to understand what it means to engage in doing mathematics, specifically to understand the role of aesthetics in mathematics learning. In a situative perspective (Lave & Wenger, 1991), knowledge — the *what* in learning — is inextricably linked to learners' context and activity — the *where, who, why* and *how* in learning. In this way, knowledge is always tentative, partial, and linked to context. Because mathematical aesthetics are means of knowing and doing — as they are touchstones, urges, and positions — mathematical aesthetics are also tentative, partial, and linked in relation to the *where, who, why,* and *how* of learning. Taking this perspective of knowledge as inextricable from context and activity, mathematical aesthetics are *disciplined* (Stevens & Hall, 1998) and *negotiated* within mathematical *communities* (Lakatos, 1976; Lave & Wenger, 1991), but are also individual and emergent from the way our everyday worlds shape our *personal interests* (Nolen, Horn, & Ward, 2015).

Following Stevens and Hall (1998), I consider mathematical aesthetics to be socially constructed through a process of *disciplining perception*. Disciplining perception has value-driven and disciplinary dimensions: When an individual's actions are invalidated and corrected, that individual's activity becomes disciplined to conform to what counts as authentic participation. In this way, mathematical aesthetics come to exist because communities of individuals negotiate them during participation in a particular context for a particular purpose (Lave & Wenger, 1991; see also Lakatos, 1976).

In particular, I link learners' everyday aesthetics to mathematicians' mathematical aesthetics. Following Jean Lave (1992), who distinguishes between *everydaying mathematics* and *mathematizing the everyday*, I consider learners' everyday aesthetics to be transformed as they move through various contexts for various (and sometimes multiple) purposes. Because school is an everyday activity for K-12 children — "not a privileged site where universal knowledge is transmitted" (p. 81) — we know that learners likely already have everyday mathematical aesthetics from school mathematics. It seems reasonable to argue that, for many students, these everyday mathematical aesthetics lead them to prefer quick-and-correct solutions and thus to negatively evaluate mathematics as a whole.

This makes sense if we consider aesthetics to emerge through a process of disciplined perception, as the *logics of participation* (John, Torralba, & Hall, 1999) of school math are too often logics of credit. Many consider this type of engagement to be a game of "doing school" rather than of participating in meaningful learning, meaning that learners are oriented to achieving grades rather than to intellectual curiosity, insight, or mathematical beauty. By leveraging learners' aesthetic, we might be able to flip the script of what it means to learn math.

When I promote *leveraging* everyday aesthetics for mathematics learning *and grounding* mathematics learning in children's everyday aesthetics, I refer to the process of *mathematizing* the aesthetics that emerge in the context of mathematics activity. This is in addition to a *funds of knowledge* approach (Gonzalez, González, Moll, & Amanti, 2005) that might promote recruiting aesthetics from non-mathematical activities such as art (e.g., by hybridizing a mathematical task with an art task). In sum, I promote mathematizing by explicitly eliciting, attuning to, and valuing learners' aesthetics as resources for mathematical ways of knowing. I describe means for doing so in the design principles section of this paper.

The design principles, in accordance with situative perspectives, assume that aesthetics co-develop with mathematics learning, as what is (de)valued in particular communities comes to shape what it means to know and do (e.g., sociomathematical norms; Yackel & Cobb, 1996). In the contexts of mathematics communities (e.g., mathematicians who subscribe to a particular academic journal; a mathematics classroom), aesthetics become shared as they are passed down, (re)negotiated, and transformed through disciplining perception during participation in local cultural practices (Lave & Wenger, 1991). In addition, personal interests shape why and how individuals take-up or reject particular practices, such as practices that shape how individuals evaluate (touchstones), why individuals are motivated (urges), and what individuals generate (shaped by positions). These intrinsic personal interests emerge from *identities* developed through participation in everyday worlds, including but not limited to the world of mathematics (Nolen, Horn, & Ward, 2015).

In relation to Lave's (1992) notion of mathematizing the everyday, I take this to mean that learners' identities and interests from outside-of-school contexts can also be mathematized. In this way, learners' histories of participation across multiple communities of practice can come to have implications how they perceive themselves (identity) and how they want to be perceived (values) in relation to mathematics (Nolen, Horn, & Ward, 2015), and thus aesthetics are somewhat personal as well as shared within a community. This means that aesthetics are something that everyone possesses as an identity resource in relation to the three roles of mathematical aesthetics (touchstones/evaluation, urges/motivation, and positions/generation).

Because all students come to the table with mathematical aesthetics, these aesthetics can be developed through instruction and out-of-school experiences in productive and unproductive ways. By explicitly attending to the aesthetic nature of learners' experiences, we can begin to leverage aesthetics towards positive disciplinary dispositions and expansive mathematical competence. In sum, developing mathematical aesthetics can be better understood through attending to the organization and assembly of (a) individuals' socialization to mathematics through the process of disciplining perception and (b) individuals' personal interests as shaped by their histories of participation.

Attuning to aesthetics, and in particular, to the cultural and personal dimensions of aesthetics, can help us begin to make mathematics more meaningful and enjoyable for mathematics learners. Throughout this paper, I use this dual lens to interrogate research findings on the evaluative, motivational, and generative roles of aesthetics in the work of mathematical experts and novices with an eye towards how activity is or could be organized to support the development of mathematical aesthetics that support meaning making. I conclude with a synthesis in the form of design principles and implications for theory and practice.

## Evaluative Role of Mathematical Aesthetics: Aesthetic Preferences Guide Judgement

Earlier, I described the evaluative role of mathematical aesthetics is akin to a touchstone. In this section, I build-up a richer understanding of the evaluative role of aesthetics by connecting it directly to mathematics. To do so, I first summarize the evaluative role of aesthetics in mathematician's work. I then examine the similarities and differences between mathematicians' evaluative aesthetics and students' evaluative aesthetics, as described by prior research. Finally, I review key ways researchers have attempted to understand and support the role of aesthetics in learning mathematics.

### The Evaluative Role of Aesthetics in Mathematician's Work

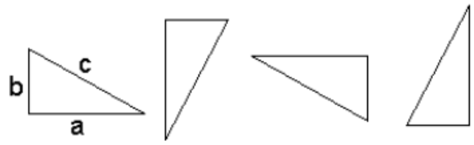
Mathematicians' aesthetic evaluations determine what mathematics gets produced, retained, and verified (Sinclair, 2004). Although aesthetic evaluations vary by subfields of mathematics (Wells, 1990), mathematicians tend to prefer write-ups that make solutions seem obvious or simple, spur surprise, or make novel connections between concepts (Pimm & Sinclair, 2009; Sinclair, 2004). For example, mathematician Philip Davis (1997) prefers his mathematical write ups to make the conclusion appear "as plain as the nose on your face" (p. 17, as cited in Sinclair, 2006b, p. 92), while mathematicians Henderson and Taimina (2006) prefer write ups that support inquiry and meaning making. Creating proofs is typically an iterative process: The first proof a mathematician creates, even if correct, is rarely the proof that gets published (e.g., Gauss, 1863).

The Pythagorean Theorem is a well-known example of this phenomenon. A ubiquitous part of school curricula all over the world, the Pythagorean Theorem states that the sum of the areas of the squares on the legs of a right triangle (call these legs $a$ and $b$, respectively) equals the area of the square on the hypotenuse (call this leg $c$). Symbolically, then, $a^2 + b^2 = c^2$.

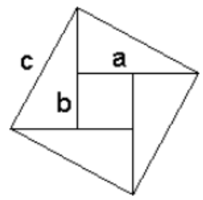
To illustrate different tastes in mathematical proofs, I provide two examples of Pythagorean Theorem proofs to contrast Davis' (1997) preference for self-evident proofs and Henderson and Taimina's (2006) preference for provocative proofs. I propose that this first proof, from ancient China, aligns with a self-evident proof aesthetic (Bogomolny, n.d.; see Table 1-1), while the second was a proof, written by Albert Einstein in his boyhood, aligns with the provocative proof aesthetic (Strogatz, 2015; see Table 2-1).

Table 2-1 *Ancient Chinese proof of the Pythagorean Theorem*

**Proof 1 of the Pythagorean Theorem**




Now we start with four copies of the same triangle. Three of these have been rotated 90 degrees, 180 degrees, and 270 degrees, respectively. Each has an area of $\frac{ab}{2}$. Put them together without additional rotations so that they form a square with side $c$.



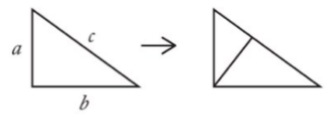

The square has a square hole with the side $(a - b)$. Summing the area of this square $(a - b)^2$ and the area of the four triangles $\left(4 \cdot \frac{ab}{2} \text{ or } 2ab\right)$, we get:

$$\begin{aligned} c^2 &= (a - b)^2 + 2ab \\ &= a^2 - 2ab + b^2 + 2ab \\ &= a^2 + b^2 \end{aligned}$$

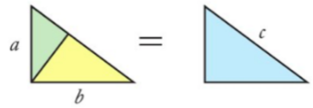
Table 2-2 *Albert Einstein's boyhood proof of the Pythagorean Theorem*

**Proof 2 of the Pythagorean Theorem**

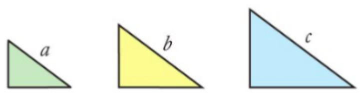
Step 1: Draw a perpendicular line from the hypotenuse to the right angle. This partitions the original right triangle into two smaller right triangles.



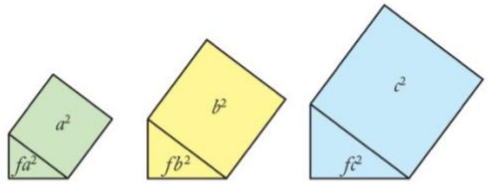

Step 2: Note that the area of the little triangle plus the area of the medium triangle equals the area of the big triangle.




Step 3: The big, medium, and little triangles are similar in the technical sense: their corresponding angles are equal and their corresponding sides are in proportion. Their similarity becomes clear if you imagine picking them up, rotating them, and arranging them like so, with their hypotenuses on the top and their right angles on the lower left:




Step 4: Because the triangles are similar, each occupies the same fraction $f$ of the area of the square on its hypotenuse. Restated symbolically, this observation says that the triangles have areas $fa^2$, $fb^2$, and $fc^2$, as indicated in the diagram.




Step 5: Remember, from Step 2, that the little and medium triangles add up to the original big one. Hence, from Step 4, $fa^2 + fb^2 = fc^2$.

Step 6: Divide both sides of the equation above by $f$. You will obtain $a^2 + b^2 = c^2$, which says that the areas of the squares add up. That's the Pythagorean Theorem.

These two proofs appeal to different aesthetic preferences. The first proof aligns with the self-evident proof aesthetic, since the combination of geometry and algebra leaves the knowledgeable reader with no doubt the theorem is veritable with very little effort. The second proof, in contrast, aligns with the provocative proof aesthetic. The explanation is a little longer, although more intuitive in relation the theorem's statement: it has clear links to the theorem's "areas of the squares" on the legs of a triangle, while the first does not. However, the second proof also requires a bit of inquiry on the part of the reader. It makes the reader ask, *why are the two smaller triangles congruent to the larger triangle and to each other?* (Steps 1 and 3; Strogatz, 2015) and *why do the smaller squares add up to the larger square?* (Step 6; Strogatz, 2015). These contrasting examples illustrate how the evaluative role of mathematical aesthetics can lead mathematicians to very different preferences in proving the same mathematical theorem. As evident in both Proof 1 and Proof 2 (Table 1-1), mathematicians' aesthetics often lead to solutions that omit many details necessary for understanding: They both assume an audience with particular prior knowledge.

This, of course, foreshadows that mathematical novices will likely have quite different aesthetics than expert mathematicians. However, from a situative stance, the potential for developing a mathematical aesthetic is accessible to all participants in the cultural activity of mathematics — not only the few who manage to persist in mathematics as a career — if they are provided the appropriate socialization experiences. To explore this further, I next discuss how the role of aesthetics has been operationalized in learning mathematics.

**The Relationship between Students' Aesthetics and Mathematicians' Aesthetics**

Based on what we know about mathematicians' evaluative aesthetics, what would it look like for students to have aesthetic values that allow them to perceive some solutions as more beautiful than others? While students' mathematical aesthetics likely are and should be different than those of mathematical experts, students who develop a sophisticated mathematical aesthetic might value creative solutions appealing to big ideas in mathematics beyond computation. They might also distinguish between more and less interesting (personally meaningful) justifications of mathematical claims. Such students would have personal tastes and be able to defend them. In classrooms that intentionally cultivate such mathematical aesthetics, we would expect students to have heterogenous tastes for mathematical problems and solutions and to be able to explain their preferences in meaningful ways while understanding the tastes of others. They might explore problems beyond arriving at an answer to find appealing ways to communicate the meaningfulness they find in the problem and solution. Such a classroom would provide an inroad for affect: students could be passionate about their mathematical work and experience feelings of surprise, curiosity, and pride in their work. But what does research tell us about students' aesthetic evaluations in mathematics?

Taste is perceptual in nature. Perception is based on both personal histories of experience as well as on participation in communities engaged in a common endeavor with shared norms — think of what counts as "sweet" in different cultures, from red bean ice cream in Japan to milk chocolate candy bars in the U.S. Similarly, because of their social origins, evaluative aesthetics can be considered both personal and shared. The socialization into a mathematical aesthetic may very well be part of what it means to become a mathematician. For example, in a study of children's mathematical aesthetics in relation to tessellations and tilings, Eberle (2014) found that children and mathematicians tended to describe the visual aesthetic appeal of tessellations very differently from each other: Mathematicians' evaluations were more like their peers' evaluations than the students' evaluations, and children's evaluations were more like their peers' evaluations than the mathematicians' evaluations.

However, even within these similarity clusters, there was much individual variation on what counted as aesthetically pleasing. Interestingly, while there was significant variation between how mathematicians and students described their aesthetic evaluations, both mathematicians and students preferred complexity in tessellations that was neither too simple (*boring, trivial*) nor too complex (*confusing*). However, their criteria for what counted as too simple and too complex differed between the two groups, likely due to differing levels of mathematical expertise and socialization that partially shape what was interpreted as boring or novel.

In another study of students' evaluative aesthetics, Tjoe (2015) found that "gifted students"[2] and mathematicians had nearly opposite aesthetic preferences for the write-up of solutions. The mathematicians interviewed for the study preferred write-ups that showed connections between multiple branches of mathematics, write-ups that made the solution seem simple or obvious, or write-ups that communicated a novel means of solving the problem. In contrast, the students preferred "brute force" computational solutions to the same problems. Contrary to Eberle's (2014) study, these students were informed of mathematician's aesthetic evaluations of the same problem solutions. When confronted with mathematicians' judgements about solutions, the students responded that the mathematicians were "trying too hard to be unique and clever, when they are totally unnecessary" (p. 172). This study reconfirmed the findings of Dreyfus and Eisenberg's (1989) pioneer study of college mathematics students' and mathematicians' aesthetic tastes towards problem solutions. While Tjoe (2015) seemed to have the goal to prove that gifted students had sophisticated aesthetics that mirrored mathematicians, from a situative perspective, this study largely demonstrated that students who are taught to prioritize finding quick-and-correct answers are socialized to prefer brute-force, computational solutions.

These dissimilarities between students' and mathematicians' mathematical aesthetics are not only a product of expertise (prior knowledge), but also of disciplined perception (Dreyfus & Eisenberg, 1986; Stevens & Hall, 1998). The shared values *within*

[2] I use quotes around the term "gifted students" to signal that this term is problematic, as the social function of labelling students as more or less gifted actually creates socially constructed categories such as gifted and disabled, and thus privileges some and oppresses others. For more detail, see McDermott and Varenne's (1995) paper on culture as disability.

school math communities and academic math communities — not *between* them — discipline participants to see their worlds similarly enough that they can, by and large, communicate within it unproblematically. However, individual differences in perception are always present. Because pupils engage in different communities (school math community) than do mathematicians (academic math community), we can assume that children's and mathematicians' evaluative aesthetics about mathematics will be different. In fact, some scholars argue that mathematical aesthetics of school-age children and youth should be different than the aesthetics of mathematicians: Children's aesthetics may be uniquely suited for children learning mathematics, just as mathematicians' aesthetics are uniquely suited to them and their endeavor (Sinclair, 2006a).

**Developing Mathematical Aesthetics**

One way that researchers and teachers have attempted to help students develop mathematical aesthetics that allow them to perceive some solutions as beautiful and others as not is by following the lead of Harré (1958, as cited in Sinclair & Pimm, 2006) and providing students with access to multiple solutions to the same mathematical problem so that they can begin to compare. This approach has endured across generations. In 1968, Edwin Rosenberg published an article in *The Arithmetic Teacher* about providing students with rich problems that can be solved in a multitude of ways with increasing sophistication. In many ways, this resonates with today's reform mathematics that encourages students to engage with multiple strategies and solutions by making their ideas public and open for debate (Horn, 2008, 2017; Lampert, 1990).

For example, Deborah Ball's (1993) well known episode of classroom discourse that led to the emergence of "Sean Numbers" involves students excitedly debating about whether the number six is even or odd. In the video data, there is evidence that Sean and his classmates are negotiating their taste for what counts as mathematically correct, and perhaps even mathematically significant. While Sean Numbers are not included in the Common Core Standards (Common Core State Standards Initiative [CCSSI], 2010), they are part of the mathematical canon dating back to *Euclid's Elements*, where they were specifically described as special subset of *even times odd numbers: two times an odd number* (Mathematics Teaching and Learning to Teach, 2010). More importantly for this argument, the students created and communicated about Sean Numbers using big mathematical ideas of grouping and classification. Most importantly, Sean Numbers were mathematically meaningful for the students' learning in that classroom at that time. While there is not much research about whether students taught with ambitious teaching methods develop aesthetic judgements about what counts as a beautiful mathematical solution or a worthwhile mathematical problem, it is clear from this well-known example that even elementary students express evaluative aesthetic responses such as surprise and disgust when engaged in meaningful mathematics learning.

The particulars of students' mathematical contexts seem to shape their mathematical aesthetics. In an experimental study on stimulating evaluative aesthetics, Koichu, Katz, and Berman (2017) found that most participants labeled some problems and solutions as more beautiful and surprising than others, depending on different conditions. The researchers exposed two groups of ninth-grade students to a sequence of two mathematical problems and solutions, with each group encountering the two problems in opposite orders. The problems were designed such that the problems looked similar, but the solutions were deceptively different. In both groups, the problems encountered last were considered more beautiful and their solutions more surprising. In other words, both problems in the study were evaluated as aesthetically appealing, but each of them was considered best when it was presented last. This suggests that the affective experiences of unexpectedness and surprise can lead students to evaluate some mathematical solutions as more appealing than others. While the design of the mathematical problems to be unexpectedly dissimilar facilitated this surprise, it was the affective experience of surprise — rather than anything intrinsic to either problem individually — that facilitated students' aesthetic evaluations.

In contrast to Tjoe's (2015) and Dreyfus and Eisenberg's (1989) experimental designs that required learners to evaluate "beautiful" solutions compared to their own solutions, Koichu, Katz, and Berman (2017) did not have students solve the problems themselves. They concluded that both their design of problem sequence as well as the emotionally safe nature of the study facilitated students' positive aesthetic evaluation of mathematical problems and solutions. Thus, emotional safety — the mitigation of social and academic risk (Horn, 2008) — along with socialization into what counts as mathematics can impact students' evaluative aesthetics. Importantly, studies of ambitious instruction have identified means of reducing social and academic risk while supporting students' mathematical competence (Horn, 2008). Still, Koichu, Katz, and Berman's (2017) study contributes an existence proof that students — as well as mathematicians — can experience problems as more or less aesthetically appealing.

But how does the evaluative role of mathematical aesthetics develop in relation to the motivational and generative roles? In a study of college mathematics majors' ability to extend their understanding to unconventional uses of mathematical tools, Mamolo and Zazkis (2012) theorized that the evaluative role of mathematical aesthetics was the gateway into motivational and generative roles. Specifically, they designed a teaching experiment in which they first introduced students to the unique ability of the derivative function to transform the formula for the area of a circle into the formula for the perimeter of a circle. They then sought to create surprise and pleasure for students by asking them whether or not this could generalize to other regular shapes such as a square — an unconventional use of the derivative. However, when they showed students that the derivative could indeed transform the area of a square into its perimeter, students were not motivated to pursue this pattern in a more general context. The researchers postulated that students' histories of participation in traditional mathematics courses led them to lack the particular evaluative aesthetics they sought to evoke. While this is likely, the study design did little to disrupt these histories of participation, giving students no opportunities to agentically explore problem spaces and develop a personal familiarity and a disciplined perception of what might be interesting (non-trivial) and doable (not too open-ended).

Of course, for learners to develop aesthetic preferences for meaningful solutions, they have to be *interested* in the mathematical question itself. Otherwise, they will likely prefer solution write-ups that show quick and correct answers rather than solutions that communicate understanding. In the next section, I discuss interest and persistence in mathematical inquiry by appealing to the motivational role of mathematical aesthetics.

## Motivational Role of Mathematical Aesthetics: Aesthetic Preferences Support Voluntary Sustained Engagement

My first description of the motivational role of mathematical aesthetics appealed to the non-academic language of "urges" to describe how aesthetics drive mathematical inquiry. In this section, I build on this description by attending explicitly to the role of aesthetics in doing mathematics. I first summarize the motivational role of aesthetics in mathematician's work. I then extrapolate what it might look like for students to be motivated by aesthetics in their learning of mathematics. Finally, I review key ways we have attempted to understand and support the role of aesthetics in learning mathematics.

### The Motivational Role of Aesthetics in Mathematicians' Work

Mathematical activity is driven by making connections to gain insight. In other words, *doing mathematics* requires taking one's current understanding of a mathematical object and looking for fruitful (mutually illuminating) relationships between that mathematical object and other mathematical objects in a way that produces a bigger picture understanding of particular mathematical phenomena (Sinclair, 2004). Thus, this notion of *meaningful connections* is disciplinarily specific: Mathematicians see some connections in mathematics as trivial and thus uninteresting, but others as important, surprising, and exciting. Often, this relates to issues of generalization, suggesting that making connections makes a group of things easier to understand by putting them in conversation with each other, either by shedding light on prior work or by opening up new possibilities for future inquiry (Silver & Metzger, 1989 p. 70).

This relates back to the evaluative role of mathematical aesthetics. Mathematicians are motivated to pursue certain problems because they seek insight, and they evaluate the work of others based on the insight their proofs provide. For example, geometer David Henderson published a research paper (1973) that contained a concise and simple half-page proof that provoked more questions than any of his other papers. While his peer mathematicians accepted that the proof was logically correct, they asked *why is this true? where did it come from?* and *how did you see it?* (Henderson & Taimina, 2006, p. 67). Questions such as these make some mathematicians hesitant to use computers for proofs. For example, some mathematicians dislike the Apple-Haken proof of the Four-Color Theorem because it relies on computer computation that obscures why the theorem is true (Hersh, 1997). In other words, although the result was irrefutable from the brute-force proof that generated all possible cases, it did not satisfy the aesthetic criteria of insight. Hence, mathematicians are motivated not only to find out *whether* or not something is true, but also (and perhaps more importantly), they are motivated to understand *why* things are true. In sum, mathematicians' aesthetic preferences for certain problems are motivated by their personal histories of interest and by a desire to create fruitful understandings that connect the mathematical objects (or systems) they are interested in to more global mathematical phenomena. In fact, if the mathematical problem at hand does not seem fruitful for making connections (i.e., it is not interesting), mathematicians will often cease their pursuit of the problem once they *feel* they know how to achieve the solution (Dreyfus & Eisenberg, 1986).

When we consider designing for classrooms, the motivational ground shifts drastically. For most children, their motivation to learn math comes from extrinsic factors — grades, parent or teacher approval, future academic aspirations, or even social status. In the next section, I extrapolate what it might look like for students the to experience the motivational role of aesthetics in their learning of mathematics.

### The Relationship Between Students' Aesthetics and Mathematicians' Aesthetics

What would it look like for students to have a mathematical aesthetic that directs them to select and pursue mathematically interesting questions? While holding that mathematicians' aesthetics likely are (and should be) different than students' (since they are engaged in different endeavors in different contexts), I build on the previous section by suggesting several learning goals for mathematical aesthetics. I posit that students who have developed a sophisticated mathematical aesthetic would constantly be asking themselves why something is true rather than just whether something is true. They would seek connections to things they already know about, linking not only shapes and symbols but also other big ideas, such as *equivalence* or *symmetry*. Students would prefer some mathematical problems over others, and would want to pursue their mathematical questions for more than just the span of a lesson.

### Developing Mathematical Aesthetics

Mathematics educators have attempted to get students intrinsically interested in mathematics by making instruction interdisciplinary — by, for example, linking mathematics to domains like music and art. Such interventions seek to make mathematics less intimidating for students (Van der Veen, 2012), to illustrate mathematics' utility (Bush, Karp, Nadler, & Gibbons, 2016; Wilders & VanOyen, 2011), or to provide concrete experiences to make symbolization more meaningful (Bush, Karp, Bennett, Popelka, &

Nadler, 2013; Cipoletti, B., & Wilson, 2004). For example, Cipoletti and Wilson (2004) describe a task in which students use origami directions — modified to use geometric language — to produce an aesthetically pleasing object while also learning to use that language to communicate in concrete ways before using it symbolically. Such approaches to motivating students through aesthetics in art are most commonly found in teacher journals rather than research journals. Clearly, teachers are sharing resources for mathematical motivation that other teachers are surely searching for. Some might critique these interdisciplinary methods as locating the source of interest outside of mathematics, thus bolstering the belief that mathematics itself is aesthetically sterile (Sinclair, 2001). While this may be true, making mathematics more tangible and connected to other disciplines can also create new opportunities for mathematical engagement, offering ways for students to productively engage their aesthetics for mathematics learning.

Of course, motivation to pursue mathematical problems cannot be achieved without foregrounding meaning making, which requires starting with what students already know. Educators have sought to do this through project-based mathematics. In well-designed project-based mathematics, students focus on a broad question and learn new methods in a just-in-time approach (Boaler, 1998; Schwartz & Bransford, 1998; Gutstein, 2003). This learning arrangement allows students to orient towards the big picture and thus develop a heuristic understanding of content.

However, project-based mathematics does not address the motivational role of mathematical aesthetics in driving the *selection* and *pursuit* of mathematical problems based on personal interest (e.g., Sinclair & Watson, 2001). To incite students' intrinsic interest in particular mathematical problems — to make them excited to pursue that problem and *want* to persist in the face of trouble and frustration — requires creating instructional activities that intentionally elicit and leverage students' emotions. For example, Dietiker (2015) outlined a convincing argument that curriculum should be designed to tell a mathematical story, where "plot twists" challenge what students expect to happen in a mathematical activity. She gives an example of a mathematical game where students expect that the "fairness" of the game results from mathematical mechanisms such as equal area, equal distance, or averages, but the actual underlying mechanism is one of probability. In her experiences with students, this disruption to students' expectations evokes emotions such as surprise and curiosity, prompting students to generate questions such as *how is that possible?* and *why does that happen?* (p. 8). Questions such as these indicate intrinsic interest in the mathematics itself. Importantly, the enactment of the instruction, not only the design of the mathematical task, has implications for whether or not students will ask and pursue such questions (Ball & Cohen, 1996).

However, many teachers themselves have never experienced feelings of curiosity or surprise in mathematics, creating a significant barrier for them to design these kinds of experiences for students. In a study of preservice teachers' ability to pose and select mathematically interesting problems, Crespo and Sinclair (2008) found that teachers were better able to select mathematically interesting problems when they focused on problems that made them curious rather than on pedagogy. In order to help the preservice teachers grasp the notion of a *mathematically interesting* problem, they used the metaphors of *tasty* and *nutritious* as descriptors for mathematical problems. This food analogy was meant to highlight how their future students might not pursue nutritious problems if they were not also tasty, just as how people in general will not regularly seek out and consume foods that are good for them if they do not also enjoy their flavor and texture. In learning to pose problems that were both nutritious and tasty, these teachers considered the ways mathematical constraints in problems affected the problem's aesthetic value. The researchers found that when the preservice elementary teachers were given time to explore materials before posing problems, they were not only able to pose richer problems but were also able select problems that were mathematically interesting.

Just as Crespo and Sinclair's (2008) study of preservice teachers focused on teachers' selection of mathematically interesting problems resulting from the exploration of mathematical objects, my review of the literature did not unearth any empirical studies in which children developed *a taste* for interesting mathematical problems *without* first having opportunities to explore problem spaces. It appears that developing an interest that motivates selection and sustained pursuit of mathematical problems requires that students gain familiarity with a problem space and develop a taste for what is trivial (uninteresting) or unsolvable (too open-ended). Thus, rather than delving into the myriad ways mathematics researchers and educators have attempted to foster student interest and engagement in mathematics meaning making, I next explore how mathematicians develop the tastes that shape what they count as mathematically interesting — how they come to want to pursue and create new mathematics.

## The Generative Role of Mathematical Aesthetics: Aesthetic Preferences Support Innovation

Thus far, I have discussed how mathematicians' aesthetic preferences shape the value judgments they make on finished work and motivate them to pursue particular problems. However, I have not yet discussed how mathematicians' aesthetic preferences shape how they actually do mathematics — how aesthetic preferences shape their inquiry, their process of innovatively solving mathematical problems, and finding new ones. In this section, I elaborate the description of the generative role as a "position" or orientation towards mathematical inquiry by attending explicitly to aesthetics in mathematics. I first summarize the generative role of aesthetics in mathematicians' work. I then extrapolate what it might look like for students to experience the generative role of aesthetics in their learning of mathematics. As with the motivational role of mathematical aesthetics, my description of aesthetics as generative for students' learning is based on my reading of the mathematical literature, because comparative studies between students and mathematicians do not exist for the generative role of the aesthetic. Finally, I review an important study on how researchers have attempted to understand and support the role of aesthetics in learning mathematics.

**The Generative Role of Aesthetics in Mathematicians' Work**

Mathematicians engage the generative role of their mathematical aesthetics in three ways. First, mathematicians engross themselves in a state of mind where they can *playfully explore* a problem space (Sinclair, 2004). For example, many geometers use visualization software like *Geometer's Sketchpad*® to playfully explore mathematical objects in both concrete and abstract ways; it gives them a sense of pleasure and insight to see their theoretical mathematical objects come to life. For example, Hofstadter (1997) gives a powerful account of his playful exploratory activity in the software:

> One further key factor that mustn't be overlooked is the fortuitous existence and

> tremendous power of Geometer's Sketchpad. Somehow, this program precisely filled an inner need, a craving, that I had, to be able to see my beloved special points doing their intricate, complex dances inside and outside the triangle as it changed. (p. 13, as cited in Sinclair, 2004)

This playful exploration is not goal oriented in the sense of wanting to produce particular problem solutions; rather, it is playful exploration with the purpose of getting familiar, of seeing what exists, what is possible, and what stands out. In this way, playful exploration supports a conceptualization of mathematical insights as much or more than deductive reasoning. In today's technologically rich social-scape, examples of rich, playful mathematical exploration can be found in even informal contexts (see, for example, the Twitter thread https://twitter.com/j_lanier/status/1042889005220732929). Yet, this kind of mathematical engagement is often absent from classrooms.

Second, mathematicians experience the generative role of mathematical aesthetics by developing a sense of *intimacy* with mathematical objects (Sinclair, 2004). This intimacy is a personal connection with, and interest in, a mathematical object that makes them look far and wide for connections. Mathematicians do this to understand their mathematical objects better, but also to extend their interaction and connection with that particular mathematical object. In Hofstader's quote above, he expressed his intimate connection with his work in the way he described his mathematical objects as his "beloved special points."

Finally, mathematicians also experience the generative role of mathematical aesthetics when they appeal to their *intuition* in problem solving, with no initial deductive reasoning, just a feeling guiding them (Sinclair, 2004). According to Reuben Hersh (1997), “a realistic analysis of mathematical intuition should be a central goal of the philosophy of mathematics” (p. 62). In a pioneer study of mathematicians’ aesthetics, Silver and Metzger (1989) described aesthetic intuition as the link between metacognition and emotion in mathematical work: Mathematicians develop affective responses to mathematics that allow them to use intuition to generate interesting questions and ideas for solutions. Indeed, intuition is a primary generator of mathematics. For example, mathematicians worked for centuries to understand the shape of the hyperbolic plane. While multiple symbolic descriptions of the plane existed, mathematicians wanted a tangible way to explore the space, a way that would let them use their intuitions. Around 1880, Henri Poincaré developed the first planar model of hyperbolic space to accurately represent angles, and although accurate, this model did not fully support the exploration that mathematicians sought (Henderson & Taimina, 2006). Then, in 1997, Daina Taimina crocheted a hyperbolic plane that allowed mathematicians to garner a deeper understanding for how the hyperbolic plane is shaped, allowing for a link between intrinsic (local) and extrinsic (global) explorations of the space, thus allowing mathematicians to develop *intuitive* insights into the nature of lines and shapes in the hyperbolic plane (Henderson & Taimina, 2006).

In sum, mathematicians’ aesthetic preferences lead to the generation of new mathematics: They pose new mathematical questions and pursue insight through their playful explorations, personal connection and commitment to certain mathematical objects, and reliance on intuition. Next, I extrapolate what it might look like for students to experience the generative role of aesthetics in their learning of mathematics.

### The Relationship between Students' Aesthetics and Mathematicians' Aesthetics

What would it look like for students to playfully explore mathematics, to have intimate personal connections with mathematical objects, and to evoke intuition in their learning of mathematics? Extrapolating from the prior section, I posit that students engaged in the generative role of mathematical aesthetics would continuously be developing and refining opinions about what mathematics is interesting and meaningful. While the motivational role of the aesthetic involves selecting and pursuing interesting mathematical problems, the generative role of the aesthetic involves developing tastes for what is interesting. Because what counts as interesting changes as familiarity with the mathematical object changes (Wells, 1990), students would engage with mathematics through noticing, wondering, and posing new-to-them mathematical questions. They would seek out concrete encounters with resources such as mathematical software for visualization and computation, the internet, and physical objects to build on their existing visual, spatial, and physical sensitivities and holistically pursue their mathematical questions (Sinclair, 2001).

### Developing Mathematical Aesthetics

We can help students form opinions about what mathematics is interesting and meaningful in ways that leads them to wonder about and pose new-to-them mathematical questions by giving them rich mathematical problem spaces, rather than pre-determined mathematical problems. These problems spaces should be open-ended and yet structured in a way allows them to explore and develop mathematical tastes and questions. This puts the emphasis of their mathematical activity on making meaning, making connections, and developing interest, rather than on primarily knowing how to find answers to particular types of problems.

While there is research on how to get students to pose mathematical problems, the strategies most researchers have used are prescriptive — for example, by getting students to change some of the constraints in a math problem by asking *what if-not?* questions (Brown & Walter, 1983). While such prescriptive strategies *are* productive for expanding students' mathematical sensemaking, they do not necessarily address the key issue of concern here — developing a taste for what counts as an interesting or worthwhile problem (Crespo & Sinclair, 2008). I suggest we need to make space both inside and outside of school for learners to engage in playful exploration of rich mathematical problem spaces so that they can have the opportunity to notice, wonder, and explore questions that arise.

While these kinds of spaces are rare, Fiori and Selling (2016) created a summer school course where they supported students in posing their own mathematical questions based on their interactions with physical objects. Each of these objects was located at a different station in the classroom, and students could go to any table they pleased, stay as long as they were interested, and leave whenever they lost interest. Throughout the summer school class, the students developed their own mathematical questions.

In fact, through playful exploration, the development of personal connections to mathematical objects, and reliance on intuition, these students developed a taste for mathematically interesting questions. As the students explored the objects at the stations, they began to develop personal connections with their in-progress ideas. For example, one student, Marco, became very interested in creating different mazes with square tiles, eventually posing the mathematical question of, "Given a number of tiles, how many different mazes can be made with them?" and defended to his classmates what he counted as (or defined as) a maze (p. 223). Two other students became interested in discerning between configurations of snap cubes that could not "wiggle." While the physical snap cubes could always wiggle in some way, the two students developed a more abstract understanding of their structure, treating them as idealized objects (as mathematical models) as they intuitively investigated their mathematical definition of wiggling, which did not include physically possible motions such as twisting and stretching (p. 224). Throughout the course, students posed problems that balanced simplicity and complexity: they iteratively modified the constraints of their problems to make solutions possible (solvable) but also interesting (non-trivial). In this study, students' exploration of physical objects — rich with mathematical potential — led them to generate not only new-to-them mathematical problems, but also generated personal interest in some problems over others.

While Fiori and Selling's (2016) study provides an existence proof that students can generate rigorous (nutritious) and interesting (tasty) mathematical questions, they provide little insight into the mechanisms by which exploration shaped the aesthetic value systems that guided the students' activity. In fact, the instructors' facilitation moves are absent from the study's write-up, and it is not certain that the materials alone would generate such a range of interesting questions. While studies of students' evaluative aesthetics show us how schools' value systems shape what students count as good mathematical solutions, we know little about how students' aesthetic value systems might come into play during mathematical exploration, problem posing, and problem selection.

However, a study by Lehrer, Kobiela, and Weinberg (2013) offers some insight about how to facilitate mathematics learning to support students' mathematical problem generation in ways that are personally interesting and mathematically significant (non-trivial). Drawing on multiple veins of mathematics education research, they enacted facilitation moves including (a) labelling or mathematizing students' language during re-voicing moves (O'Connor & Michaels, 1993), (b) supporting student authorship of ideas (e.g., by labelling a mathematical guess as "Vern's conjecture"), (c) modelling and eliciting acts of noticing and wondering with statements such as

> ''What kinds of questions could we now ask, having made this thing? Cause in math, we don't make things unless we want to ask questions about them. So what kinds of questions could we ask about this thing that we've just made? … What else might we want to know? About this or anything related to it.'' (p. 372),

(d) drawing on and mathematizing bodily experiences (e.g., walking and turning as resources for understanding shapes and angles), and (e) providing new tangible experiences (e.g., a dynamic quadrilateral made of four strips of paper and brad fasteners). They found that when students participated in learning that was driven by students' own "investigate-able" questions (i.e., questions that are rich enough to support inquiry), students' reflections on their mathematical experiences featured discussion about agency, positive attitudes, practices for generating knowledge, and peer collaboration. These outcomes are considered highly desirable but are notoriously difficult to achieve. Even more, they also found that students had mathematically rigorous aesthetics for *what counts* as an interesting mathematical question (Table 1-3).

Table 2-3. *Student-generated criteria for qualities of good questions/conjectures from Lehrer et al. (2013, p. 372).*

| | Student generated criteria for qualities of good questions/conjectures |
|---|---|
| 1. | Help you learn something that you do not already know. |
| 2. | Something you are eager to know. |
| 3. | Shows good curiosity and thinking. |
| 4. | Has something you know about and something you want to find. |
| 5. | Has questions for what we know. |
| 6. | Topic of the discussion is inside the question. The question helps you keep thinking about the topic. |
| 7. | Good conjecture follows from the question. It links to what you know. |
| 8. | Questions are posed with good evidence. You can see why the question is relevant. |
| 9. | Leads to other conjectures. |
| 10. | Clear in wording and makes sense. |
| 11. | Good detail, specific focus |
| 12. | Good math wording—good math vocabulary helps you know what the question (or conjecture) is about |
| 13. | Question helps you understand what people are saying and why |

This study (Lehrer, Kobiela, Weinberg; 2013) exemplifies how aesthetically grounded mathematical learning — learning that begins with asking personally meaningful questions — can support the development of mathematical competence and positive disciplinary dispositions. Importantly, this aesthetically grounded learning was possible because of the epistemic culture (Knorr Centina, 1999) of the classroom (fostered through the design decisions above) that led students to prefer and thus generate questions that shed light on mathematical ideas they had worked on in the past and that opened up directions for future inquiry. Aesthetically grounded mathematics activity leads to more authentic, meaningful learning and also supports students to develop positive disciplinary dispositions.

In sum, if we wish to create mathematics learning environments that foster genuine interest in mathematics, we need to ground mathematics learners' aesthetics. In the next section, I synthesize the previous three sections on the evaluative, motivational, and generative roles of aesthetics in mathematical inquiry by presenting design principles. These design principles are meant to direct the readers' attention to aspects of learning environments that can support the emergence of mathematical problems as well as their meaningful pursuit and evaluation.

## Design Principles: Grounding Mathematics Learning in Children's Aesthetic Sensemaking

Across the three roles of mathematical aesthetics, my synthesis of research suggests design principles for leveraging mathematical aesthetics. In this section, I discuss those design principles and implications for mathematics learning.

My synthesis of research suggests that we can support students in developing evaluative mathematical aesthetics through design principles such as (a) emphasize insightful solutions over quick and correct solutions by supporting multiple modes of communicating findings (Dreyfus & Eisenberg, 1986; Tjoe, 2015), (b) create emotional safety and mitigate academic risk (Koichu, Katz, & Berman, 2017), and (c) disrupt conventional tool use (Mamolo & Zazkis, 2012). Of course, in order to achieve any of these design principles, we must identify conditions under which leaners and their teachers can escape the *didactical contract* (Brousseau, 2006) of schooling, where *logics of participation* (John, Torralba, & Hall, 1999) can too often add up to doing school rather than engaging in inquiry. This is especially challenging with the school subject of mathematics, as mathematics beyond simple computation is (too often) only considered to exist within the walls of classrooms, universities, and professional STEM careers — not in life outside of these more formal spaces. Indeed, many readers understand how difficult it can be to help even their own children pursue mathematics out of intellectual curiosity. Notably, ambitious teaching literature has been addressing all three of these design principles for decades, as facilitating students to make their ideas public and engage each other's ideas through *kind* debate (see Horn, 2008 for a description of such a participation structure) inherently foregrounds insight, emotional safety, and disrupts traditional means of learning mathematics by replacing direct instruction with student-to-student discourse.

My synthesis of research also suggests that we can support mathematical aesthetics as a resource for student engagement and motivation through design principles such as (d) provide just-in-time instruction in personally meaningful contexts (Boaler, Munson, & Williams, N.D.) (e) make tasks untidy (Dietiker, 2015) and tangible (e.g., Cipoletti & Wilson, 2004), (f) support exploration (Mamolo & Zazkis, 2012) that leads to problem posing (Fiori & Selling, 2016; Lehrer et al., 2013), and (g) mathematize concrete experiences (Lehrer et al., 2013). Of course, the phrase "personally meaningful" in Design Principle D connotes, at first blush, a different design for each child. However, research in the learning sciences has theorized that designing "disruptive" environments — by removing standard tools for problem solving — can facilitate students to recruit their own knowledge and practices (Ma, 2016), thereby bypassing the sometimes individualistic problems of *funds of knowledge* (Gonzalez et al., 2005) approaches. Designs such as these typically also account for the second and third design principles as well, as designing for disruptions often involves untidy and tangible tasks that can only be resolved by beginning with exploration.

Importantly, all of the design principles synthesized from this literature review have been contemplated in detail by literatures that centralize other phenomena. However, few research designs closely attend to all principles. From this review, the Lehrer et al. (2013) study is an exception, as this study incorporates all design principles, and adds nuance to them. For example, Lehrer and colleagues (2013) add nuance to Principle E (create untidy and tangible experiences) by distinguishing between experiences at the scale of gesture and object manipulation from the scale of whole bodies in motion. Their study also adds clarity to Principle F (supporting exploration that leads to problem-posing) by describing how to facilitate acts of noticing and wondering, as well as by providing descriptions of students' understandings of what constitutes a meaningful mathematical question. Thus, acknowledging that there are always tradeoffs in design, I argue that designing environments that encapsulate these principles is quite possible. Nonetheless, high-risk standardized measures of mathematics learning often result in research that is geared towards producing student learning that satisfies those measures (for an exception, see Sengupta-Irving and Enyedy's (2015) study on designing for disciplinary enjoyment by creating open means of participating in mathematical practices). Of course, this is due to a widespread cultural understanding of mathematics as *learning definitions and practicing procedures* (Stigler and Hiebert, 1999) rather than as fundamentally defined and driven by a value system. Regardless of this cultural conception, mathematics *is* driven by a value system — a value system that is aesthetic in nature.

## Future Directions: Examining the Role of Aesthetics for Mathematics Sensemaking in Designed Spaces

The idea that mathematics is a value driven discipline is radical yet not new. Back in 1996, Yackel and Cobb wrote about value systems for mathematics learning. They described these value systems by pointing to their manifestations in classrooms: sociomathematical norms.[3] This construct changed the way researchers and many educators understood what it means to participate in mathematics learning. In this paper, I have presented mathematicians' aesthetics in ways that support viewing them as a kind of value system for sociomathematical norms that centralize beauty in both *mathematics* and in the *mathematical experience*.

In fact, Yackel and Cobb's first publication on the construct of sociomathematical norms explicitly identified the notion of mathematical elegance as driven by such norms (p. 461). Yet value systems around mathematical beauty have not been strongly taken in up in subsequent research in the same ways that values around mathematical justification and other mathematical practices have been. This paper is built around the notion that this neglect of aesthetics is due to problematic cultural scripts that position mathematics as an emotionally and aesthetically "cold" discipline. This has limited our vision of what is possible, and thus have constrained our ability to design in ways that explicitly centralize goals of positive dispositions and competence. I have argued that mathematics can be taught in a way that allows learners to seek-out and experience mathematical aesthetics in relation to their prior knowledge just as mathematicians' aesthetically pose, choose to

[3] Sociomathematical norms are reflexively related to a community's values and goals, and they can be inferred from attending to a community's mathematical practices (Yackely & Cobb, 1996, p. 460). In particular, sociomathematical norms are constrained by both taken-as-shared classroom values and individuals' values, yet those values are also influenced by what is already normative (legitimized as acceptable activity). Sociomathematical norms and the values that constitute (and are constituted by) them are reflected in local practices, and thus can be inferred by attending to patterns of social interaction.

pursue, and evaluate based on their expertise. The design principles in the prior section are a promising start to support students in engaging in such meaningful — and potentially enjoyable — aesthetically driven mathematics learning.

Still, although mathematical aesthetics are core to disciplinary engagement for mathematicians, we know almost nothing about what the role of productive (both positive and negative) aesthetic values might be for learners. We know that as mathematicians are inducted into the discipline of mathematics, their perception of what counts as beautiful or elegant, both in problems and solutions, becomes disciplined to conform to what counts as valuable mathematics in particular communities of expertise (e.g., Dreyfus & Eisenberg, 1989). This disciplining of perception is how the discipline of mathematics remains a human practice (else mathematics would cease to exist, Stevens & Hall, 1998) — and it is akin to becoming attuned to sociomathematical norms. Importantly, the sociomathematical norms that shape the way mathematicians innovate — through playful exploration, developing intimacy, and leveraging intuition — is driven by their aesthetics. This contrasts sharply with how school versions of mathematical innovation (problem solving rather than problem posing) are conceptualized and operationalized in schools throughout the world.

What would a learning environment that did support playful exploration, developing intimacy, and leveraging intuition in mathematics look like? What kinds of spaces could support learners to explore rich mathematical problem spaces and begin to pose their own mathematical questions? Such a space would need to provide alternative logics of participation (John, Torralba, & Hall, 1999) to what is typically available in mathematics classrooms. What if we created aesthetically appealing mathematical objects (visual and tactile aesthetics), such as irregular tiling pentagons (e.g., see Figure 1-1), and allowed learners to playfully explore them (open opportunities for aesthetic touchstones, urges, and positions to emerge)? What kinds of mathematical content and practices do we think they would be engaging with? What kinds of questions might they ask? In order to answer these questions, we will need to find or design such spaces. If the spaces are designed, they will likely need to be improved over iterative studies of use, much like mathematics instruction. If they are found, we will need rich ethnographic work that describes what it means to participate in those spaces.

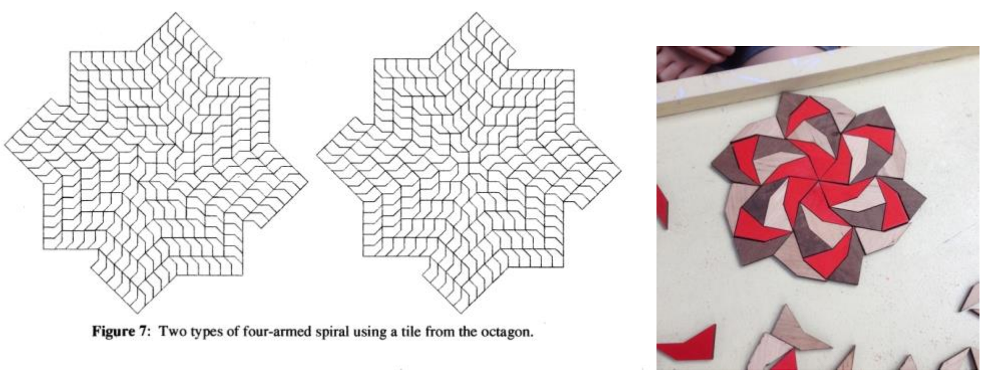



*Figure 2-1.* Mathematicians' irregular tiling pentagons (left, Gailiunas, 2000, p. 136), and playful aesthetic irregular tiling pentagons (right, Math On-A-Stick, Minnesota State Fair, 2016).

> When mathematicians playfully explore problem spaces, they ask questions such as *What if we could divide by 0, what if we were to throw away the parallel postulate, what if the irrational numbers were our counting blocks, what if we really could have staircases like those drawn by Escher, what if we could redefine differentiability to cope with some kinds of discontinuity?* (Sinclair & Watson, 2001, p. 40).

It is not far-fetched that our students might ask similar types of questions. Just as Penrose (1974) began much of his work because of a personal affection for the beauty of irregular tilings, we can expect that some students will notice and wonder about important mathematical questions as they explore rich problem spaces. For professional mathematicians, a *what if?* question might be, *what if there is another kind of tiling pentagon?* (Lord, 2016). For an eighth grader, it might be, *what if some shapes tile and some shapes do not tile?* (Civil, 2002). For an eight-year-old child on a mathematical playground, it might be, *what if these pentagons fit together in a way that covers the whole table or in a way that never ends?* or *what if I try to make this other kind of pentagon tile?* In this scenario, the child at play might be engaging with mathematical concepts of space-filling, tiling, and infinite planes.

Until we know how students engage in sensemaking in high-agency mathematical problem spaces in ways that productively exercise their aesthetics, we will not know what to change in mathematics classrooms. Even more, children today often do not have opportunities to playfully explore mathematical problem spaces in order to pose mathematical questions — either inside or outside of school — and so their personal connection to mathematical objects and their intuition are not adequately fostered. In relation to developing interest and positive disciplinary identities, learners need access to mathematical contexts with *conditions of practice* (Azevedo, 2011) that support multiple modes of aesthetic engagement with the discipline. Azevedo (2011) argues that developing and sustaining persistent engagement in an interest-based practice is supported by many interweaving threads of multiple practices (*lines of practice*). In this vein, I argue that providing learners with agentic play spaces designed for mathematical affordances can provide conditions of practice that make a wide array of learners' personal interests and aesthetics relevant. Studying engagement in such contexts can give us insight into how aesthetics are recruited as resources for interest driven engagement in mathematics.

In future studies, we need to follow holistic ethnographic approaches to examining how learners' aesthetics shape what they notice as problematic (generative role of mathematical aesthetics, positions), whether or not and how they decide to pursue the problem (motivational role of mathematical aesthetics, urges), and what counts as a good solution (evaluative roles of mathematical aesthetics, touchstones), as well as the relationship between these aspects of mathematical sesnsemaking. In addition to ethnographic studies that ask *what happens here?*, we also need microgentic studies that ask *how and why does activity change here?* in spaces designed for playful, aesthetic mathematical sensemaking. By designing and studying spaces for open participation through playful exploration with mathematically rich materials, we may learn how to better support the engagement of individuals with varying histories of participation in mathematics and other communities. Again, I contrast this approach to the "everydaying of mathematics" approaches (e.g., make word problems "reflect" real world scenarios), and instead propose creating contexts for mathematizing what we often think of as "everyday" experiences (Lave, 1992, p. 87).

Notably, because all forms of participation involve mind *and body*,[4] future research

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[4] According to Hutchins (2010), even "offline" cognition involves the body: "the premise that the particular bodies we have influence how we think... cognition is situated in the interaction of body and world, dynamic bodily processes such as motor activity can be part of reasoning processes, and offline cognition is body-

into learners' aesthetics should attempt to more fully describe the embodied practices that encompass physical, material, and discursive resources for shared meaning-making. In truth, mathematical aesthetics can be more fully theoretically described as an embodied performance genre of developing and exercising a morally implicated disciplined perception (Stevens & Hall, 1998). However, due to the backgrounding of the embodied nature of knowing in the literature on mathematical aesthetics, I have omitted this theoretical framework to make my analysis more concise.

## Conclusion

At the beginning of this article, I conjectured that centering aesthetic experiences as foundational to mathematics learning may facilitate the development of learning environments that (a) make mathematics learning more authentic to the discipline and (b) foster positive relationships between students and mathematics. To investigate this conjecture, I have connected the roles of mathematical aesthetics in mathematicians' inquiry with what we know as a field about the role of aesthetics in mathematics learning. This process led me to develop design principles for supporting authentic engagement and positive dispositions by grounding mathematics in learners' aesthetics. In addition, I proposed inquiry into new kinds of supplemental spaces for engaging with mathematics.

At this time, our understanding of how aesthetics can be leveraged to make mathematics more meaningful and enjoyable are limited. In fact, research that connects

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based too... cognition evolved for action [and thus] perception and action are not separate systems, but are inextricably linked to each other and cognition" (p. 428).

aesthetics in mathematics learning to other areas of education research, such as the role of discourse and scaffolding in learning, are scarce. This is a significant problem, as classroom discourse is a primary source for learners' socialization into mathematics (Yackel & Cobb, 1996). I have argued that mathematics learners would benefit from opportunities to exercise their aesthetics as a valued resource for mathematics learning. Importantly, children's' aesthetics in their mathematics classes may be quite different from mathematicians' aesthetics, although children's aesthetics may be more appropriate for mathematics learning (Sinclair, 2006a). By mathematizing "everyday" aesthetics (in designed, out-of-school, high-agency, mathematics contexts) towards developing new everyday mathematical aesthetics in schools, mathematics can be experienced as a connected set of big ideas, where intrinsic interest and meaning making are foregrounded.

Another significant gap in the literature is on the generative role of mathematical aesthetics — the role of the aesthetic that facilitates mathematical insight and innovation — in relation to mathematics learning. Because insight and innovation are the stuff of collective learning, it is pivotal that this gap narrows. Mathematicians leverage their mathematical aesthetic for insight and innovation by playfully exploring mathematical systems, developing intimacy with mathematical objects, and leveraging their intuition (Sinclair, 2004). However, many students and teachers have little opportunity to engage in this kind of activity (Jacobs et al., 2006; Litke, 2015). This forecloses opportunities to engage in one of the most important mathematical practices for participation in the discipline: the practice of posing interesting (non-trivial), meaningful (personally appealing), and valuable (significant to the community) mathematical problems. While studies have shown that teachers (Crespo & Sinclair, 2008) and students (Fiori & Selling,

2016) can come to have tastes for and pose mathematically interesting problems, our knowledge about the mechanisms for posing these questions authentically (not prescriptively) are limited.

Finally, this paper has provided a provisional outline to design principles for supporting students in developing sophisticated mathematical aesthetics. These design principles were linked not only to research on aesthetics but to multiple branches of research on mathematics teaching and learning. Namely, research in the veins of ambitious instruction (e.g., Lampert, 1990) and novel teaching experiments (e.g., Ma, 2016) have theoretical commitments that lead them to design for learning in ways purported by aesthetics literature. However, these literatures do not attend to learners' aesthetics, which I have argued is a residue of the history of mathematics education. Nonetheless, analyses of learners' aesthetics in these two veins of research might open new ways to conceptualize and thus support meaningful mathematics learning.

Importantly, no matter the theoretical commitment, such studies must overcome the "doing school" logics of participation (John, Torralba, & Hall, 1999) that can emerge in classrooms, which no doubt influence learners' emergent aesthetic values. For this reason, I have further suggested that studies that examine how learners' aesthetics shape what they notice as problematic (generative role of mathematical aesthetics), whether or not and how they decide to pursue the problem (motivational role of mathematical aesthetics), and what counts as a good solution (evaluative roles of mathematical aesthetics) would be best carried out where children can agentically explore in mathematically rich spaces.

This leaves us with many unanswered questions. What would it look like for children to engage their aesthetics while playfully exploring mathematically rich problem spaces? What might we learn about supporting engagement Common Core mathematics practices (CCSSI, 2010) — such as making sense of problems, persevering in solving them, making use of structure, and using appropriate tools strategically — by centering aesthetics in both our analyses and our learning environment designs? What new mathematical practices might we see, and to what ends? How might we need to reorganize mathematics learning to support the development of mathematical aesthetics, specifically with an eye towards the motivational and generative roles, which are understudied in education literature?

While children's aesthetics may be quite different from the aesthetics of mathematicians (people who are well indoctrinated into the world and social norms of mathematics), failing to intentionally leverage and support learners' aesthetics for the learning of mathematics is a mistake if the goal of mathematics education reform is to develop more authentic and more humane learning environments. We need to better understand how learners' aesthetic preferences can be leveraged for mathematical sensemaking, in relation to developing interest and intuition. I believe we will find that centering aesthetics in both our designs for and analyses of mathematics learning will allow us to better support learners in developing a more perfect relationship with and understanding of mathematics.

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## References


Aesthetic. (2018). In *Merriam-Webster.com*. Retrieved May 15, 2018 from https://www.merriam-webster.com/dictionary/aesthetic

Au, W. (2007). High-stakes testing and curricular control: A qualitative metasynthesis. *Educational researcher*, *36*(5), 258-267.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, *93*(4), 373-397.

Ball, D. L., & Cohen, D. K. (1996). Reform by the book: What is—or might be—the role of curriculum materials in teacher learning and instructional reform?. *Educational researcher*, *25*(9), 6-14.

Boaler, J. (1998). Open and closed mathematics: Student experiences and understandings. *Journal for Research in Mathematics Education*, *29*(1), 41-62.

Boaler, J. (2016) *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching.* Jossey-Bass/Wiley: Chappaqua, NY.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex Publishing.

Boaler, J., Munson, J., & Williams, C. (N.D.). What is mathematical beauty? Teaching through big ideas and connections. Short impact paper. Retrieved on June 4, 2018 from https://bhi61nm2cr3mkdgk1dtaov18-wpengine.netdna-ssl.com/wp-content/uploads/2017/11/Big-Ideas-paper-12.17.pdf

Boaler, J., & Selling, S. K. (2017). Psychological Imprisonment or Intellectual Freedom? A

Longitudinal Study of Contrasting School Mathematics Approaches and Their Impact on Adults' Lives. *Journal for Research in Mathematics Education, 48*(1), 78-105. doi:10.5951/jresematheduc.48.1.0078

Bogomolny, A. (n.d.). Pythagorean Theorm and its many proofs. *Interactive Mathematics Miscellany and Puzzles.* Retrieved from https://www.cut-the-knot.org/pythagoras/index.shtml

Borwein, J. M. (2006). Aesthetics for the working mathematician. In N. Sinclair & D. Pimm & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 21-40). New York, NY: Springer.

Brousseau, G. (2006). *Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990* (Vol. 19). Springer Science & Business Media.

Brown, S. I., & Walter, M. I. (1983). *The art of problem posing.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Buenrostro, P. M. (2016). *Humanizing Mathematics: Students' Perspectives on Learning Math for Social Justice* (doctoral dissertation). University of Illinois at Chicago, Chicago, IL.

Burton, L. (1995). Moving towards a feminist epistemology of mathematics. *Educational Studies in Mathematics*, *28*(3), 275-291.

Bush, S. B., Karp, K. S., Bennett, V. M., Popelka, L., & Nadler, J. (2013). Framing measurement: An art gallery installation. *Mathematics Teaching in the Middle School, 18*(8), 474-483.

Bush, S. B., Karp, K. S., Nadler, J., & Gibbons, K. (2016). Using artwork to explore proportional reasoning. *Mathematics Teaching in the Middle School, 22*(4), 216-

223.

Cipoletti, B., & Wilson, N. (2004). Turning Origami into the Language of Mathematics. *Mathematics Teaching in the Middle School, 10*(1), 26-31.

Civil, M. (2002). Chapter 4: Everyday Mathematics, Mathematicians' Mathematics, and School Mathematics: Can We Bring Them Together? In M. E. Brenner & J. N. Moschkovich (Eds.), *Everyday and Academic Mathematics in the Classroom* (pp. 40-62). Journal of Research in Mathematics Education monograph series. Resoton, VA: National Council of Teachers of Mathematics.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*(5), 395-415.

Davis, P. (1997) Mathematical Encounters of the Second Kind,  Boston, MA, Birkhäuser.

De Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics, 83*(3), 453-470.

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom.* New York, NY: Cambridge University Press.

Dewey, J. (1929). *The quest for certainty: A study of the relation of knowledge and action.* New York: Minton, Balch and Company.

Dietiker, L. (2015). What mathematics education can learn from art: The assumptions, values, and vision of mathematics education. *Journal of Education, 195*(1), 1-10.

Eberle, R. S. (2014). The role of children's mathematical aesthetics: The case of tessellations. *The Journal of Mathematical Behavior, 35*, 129-143.

Fiori, N., & Selling, S. K. (2016). Truth isn't everything: Promoting aesthetically guided choice in mathematical work. *The Journal of Mathematical Behavior*, *41*, 219-234.

Freedman, K. (2003). *Teaching visual culture: Curriculum, aesthetics, and the social life of art*. Cambridge, MA**:** Teachers College Press.

Funk, C. & Parker, K. (2018). Women and men in STEM often at odds, Section 6: Many Americans say they liked math and science in school, thought about a STEM carreer. Washington D. C.: Pew Research Center. Retrieved from http://www.pewsocialtrends.org/2018/01/09/many-americans-say-they-liked-math-and-science-in-school-thought-about-a-stem-career/

Gailiunas, P. (2000). Spiral Tilings. *Bridges*, 133-140.

Gauss, C. (1863) Werke (Schering, E., ed.), II, Leipzig-Berlin, Königliche Gesellschaft der Wissenschaften zu Göttingen.

Gelfert, A. (2014). Applicability, indispensability, and underdetermination: Puzzling over Wigner's 'Unreasonable effectiveness of mathematics'. *Science & Education*, *23*(5), 997-1009.

Gonzalez, N., González, N., Moll, L., & Amanti, C. (2005). *Funds of knowledge theorizing practice in households, communities, and classrooms* . Mahwah, N.J.: L. Erlbaum Associates.

Green, C. D. (1995). All that glitters: A review of psychological research on the aesthetics of the golden section. *Perception*, *24*(8), 937-968.

Greeno, J. G., & Middle School Mathematics Applications Project [MMAP]. (1998). The situativity of knowing, learning, and research. *American Psychologist*, *53*, 5-26.

Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education*, *34*(1), 37-73. doi:10.2307/30034699

Hadamard, J. (1945). *The mathematician's mind: The psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Hardy, G. H. (1940). *A mathematician's apology.* Cambridge, UK: Cambridge University Press.

Henderson, D. (1973) 'A simplicial complex whose product with any ANR is a simplicial complex', General Topology and Its Applications 3 (1), 81-83.

Henderson, D, & Taimina, D. (2006). Experiencing meanings in geometry. In N. Sinclair & D. Pimm & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 58-83). New York, NY: Springer.

Harré, R. (1958) 'Quasi-aesthetic appraisals'. *Philosophy, 33*(125), 132-137.

Hersh, R. (1997). *What is mathematics Really?.* New York, NY: Oxford University Press.

Hofstadter, D. (1992). From Euler to Ulam: Discovery and dissection ofa geometric gem. (Preprint version available from author. An abbreviated version of this paper was published in Geometry turned on: Dynamic software in learning, teaching, and research, pp. 3–14, by J. King &D. Schattschneider, Eds., 1997, Washington, DC: MAA)

Horn, I. S. (2008). Accountable argumentation as a participation structure to support learning through disagreement. In A. Schoenfeld (Ed.) *A study of teaching: Multiple lenses, multiple views.* Journal of Research in Mathematics Education monograph series. Reston, VA: National Council of Teachers of Mathematics.

Horn, I.S. (2017). *Motivated: Designing Math Classrooms where Students Want to Join in.* Portsmouth, NH: Heinemann.

Hutchins, E. (2010). Enaction, imagination, and insight. In J. Stewart, O. Gapenne, & E. A. Di Paolo (Eds.), *Enaction: Toward a new paradigm for cognitive science* (pp. 423-450). Cambridge, MA: MIT Press.

Ito, M., Gutiérrez, K., Livingstone, S., Penuel, B., Rhodes, J., Salen, K., Schor, J., Sefton-Green, J. & Watkins, S. C. (2013). *Connected learning: An agenda for research and design*. Irvine, CA: Digital Media and Learning Research Hub. Retrieved online from http://eprints.lse.ac.uk/48114/1/__lse.ac.uk_storage_LIBRARY_Secondary_libfile_shared_repository_Content_Livingstone%2C%20S_Livingstone_Connected_learning_agenda_2010_Livingstone_Connected_learning_agenda_2013.pdf

Jacobs, J. K., Hiebert, J., Givvin, K. B., Hollingsworth, H., Garnier, H., & Wearne, D. (2006). Does eighth-grade mathematics teaching in the United States align with the NCTM Standards? Results from the TIMSS 1995 and 1999 video studies. *Journal for Research in Mathematics Education*, *37*(1), 5–32. doi:10.2307/30035050

Jacobs, J. E., Lanza, S., Osgood, D. W., Eccles, J. S., & Wigfield, A. (2002). Changes in children's self-competence and values: Gender and domain differences across grades one through twelve. *Child development*, *73*(2), 509-527.

John, A.S., Torralba, J.A., & Hall, R. (1999, April). *Comparing logics of participation: Working at the boundary between errors and findings in classroom and professional ecology*. Paper presented in M. Roth & R. Hall (Organizers), *Of* *lizards, ants, and arthropods: Scientists' and middle school students' representations of nature*. Panel at the annual meeting of the American Educational Research Association, Montreal, Canada.

Knorr Cetina, K. (1999). Epistemic cultures. Cambridge: Harvard University Press.

Koichu, B., Katz, E., & Berman, A. (2017). Stimulating student aesthetic response to mathematical problems by means of manipulating the extent of surprise. *The Journal of Mathematical Behavior*, *46*, 42-57.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29-63.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, UK: Cambridge University Press.

Lave, J. (1988). *Cognition in practice: Mind, mathematics and culture in everyday life*. Cambridge University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York, NY: Cambridge University Press.

Lehrer, R., Kobiela, M., & Weinberg, P. J. (2013). Cultivating inquiry about space in a middle school mathematics classroom. *ZDM*, *45*(3), 365-376.

Litke, E. G. (2015). *The state of the gate: A description of instructional practice in algebra in five urban districts* (Doctoral dissertation). Retrieved from http://nrs.harvard.edu/urn-3:HUL.InstRepos:16461050

Lord, N. (2016). Constructing the 15th pentagon that tiles the plane. *The Mathematical Gazette*, *100*(547), 154-158.

Louie, N. L. (2017). The Culture of Exclusion in Mathematics Education and Its Persistence in Equity-Oriented Teaching. *Journal for Research in Mathematics Education*, *48*(5), 488-519.

Ma, J. Y. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. *Journal of the Learning Sciences*, *25*(3), 335-371.

Mamolo, A., & Zazkis, R. (2012). Stuck on convention: a story of derivative relationships. *Educational Studies in Mathematics, 81*(2), 161-177.

Mathematics Teaching and Learning to Teach (2010). Introduction to Sean Numbers and Ofala Video. Retrieved from https://deepblue.lib.umich.edu/bitstream/handle/2027.42/65013/seannumbers-ofala_background.pdf?sequence=5&isAllowed=y on June 3, 2018.

McDermott, R., & Varenne, H. (1995). Culture as disability. *Anthropology & Education Quarterly*, *26*(3), 324-348.

Moschkovich, J. N. (2002). An introduction to examining everyday and academic mathematical practices. In M. E. Brenner & J. N. Moschkovich (Eds.), *Everyday and Academic Mathematics in the Classroom* (pp. 1-11). Journal of Research in Mathematics Education monograph series. Resoton, VA: National Council of Teachers of Mathematics.

No Child Left Behind (NCLB) Act of 2001, Pub. L. No. 107-110, § 115, Stat. 1425 (2002).

National Council of Teachers of Mathematics. (1989*). Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Nolen, S. B., Horn, I. S., & Ward, C. J. (2015). Situating motivation. *Educational Psychologist*, *50*(3), 234-247.

O'Connor & Michaels (1993). Aligning academic task and participation status through revoicing: Analysis of a classroom discourse strategy. *Anthropology and Education Quarterly, 24*(4), 318-335.

Pea, R., & Martin, L. (2010). Values that occasion and guide mathematics in the family. In W. R. Penuel & K. O'Connor (Eds.), *Research on learning as a human science: National Society for the Study of Education Yearbook*, 109(1). New York: Teachers College.

Penrose, R. (1974). The role of aesthetic in pure and applied mathematical research. *The Institute of Mathematics and its Applications, 7/8*(10), 266–271.

Pimm, D., & Sinclair, N. (2009). Audience, style and criticism. *For the Learning of Mathematics*, *29*(2), 23-27.

Pimm, D. (2006). Drawing on the image in mathematics and art. In N. Sinclair & D. Pimm & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 160-190). New York, NY: Springer.

Poincaré, H. (1956). Mathematical creation. In J. Newman (Ed.), *The world of mathematics* (pp. 2041–2050). New York: Simon & Schuster. (Original work published in 1908)

Rosenberg, E. A. (1968). Aesthetics in elementary mathematics. *The Arithmetic Teacher*, *15*(4), 333-336.

Schattschneider, D. (2006). Beauty and truth in mathematics. In N. Sinclair & D. Pimm & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 41-57). New York, NY: Springer.

Schwartz, D., & Bransford, J. (1998). A Time for Telling. *Cognition and Instruction*, 16(4), 475-522.

Sengupta-Irving, T., & Enyedy, N. (2015). Why engaging in mathematical practices may explain stronger outcomes in affect and engagement: Comparing student-driven with highly guided inquiry. *Journal of the Learning Sciences*, *24*(4), 550-592.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19-28.

Silver, E., & Metzger, W. (1989). Aesthetic influences on expert mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving* (pp. 59–74). New York: Springer-Verlag.

Sinclair, N. (2001). The Aesthetic" Is" Relevant. *For the Learning of Mathematics,* 21(1), 25-32.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, *6*(3), pp. 261-284.

Sinclair, N. (2006a). *Mathematics and beauty: Aesthetic approaches to teaching children.* Cambridge, MA: Teachers College Press.

Sinclair, N. (2006b). The aesthetic sensibilities of mathematicians. In N. Sinclair & D. Pimm & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 87-104). New York, NY: Springer.

Sinclair, N., De Freitas, E., & Ferrara, F. (2013). Virtual encounters: The murky and furtive world of mathematical inventiveness. *ZDM*, *45*(2), 239-252.

Sinclair, N., & Pimm, D. (2006). A historical gaze at the mathematical aesthetic. In N. Sinclair & D. Pimm & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 1-17). New York, NY: Springer.

Sinclair, N., & Watson, A. (2001). Wonder, the Rainbow and the Aesthetics of Rare

Experiences. *For the Learning of Mathematics, 21*(3), 39-42.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. Simon and Schuster.

Stahl, G. & Hohenwarter, M. (n. d.). The Euler Segment of a triangle. *Geogebra Online*, GeoGebra. Retrieved on June 12, 2018, from https://www.geogebra.org/m/zNH3A9MN

Stevens, R. & Hall, R. (1998). Disciplined perception: learning to see in technoscience. In M. Lampert and M. Blunk (Eds.), *Talking mathematics in school: Studies of teaching and learning* (pp. 107-149). Cambridge, UK: Cambridge University Press.

Strogatz, S. (2015, November 19). Einstein's first proof. *The New Yorker*. Retrieved from https://www.newyorker.com/tech/elements/einsteins-first-proof-pythagorean-theorem

Sugimoto, T., & Ogawa, T. (2000). Tiling problem of convex pentagon. *FORMA-TOKYO-*, *15*(1), 75-79.

Sullivan, J. (1925/1956) 'Mathematics as an art', in Newman, J. (ed.), *The World of Mathematics* (Vol. 3, pp. 2015-2021). New York, NY, Simon and Schuster.

Tjoe, H. (2015). Giftedness and aesthetics: Perspectives of expert mathematicians and mathematically gifted students. *Gifted Child Quarterly, 59*(3), 165-176.

Van der Veen, J. (2012). Draw Your Physics Homework? Art as a Path to Understanding in Physics Teaching. *American Educational Research Journal, 49*(2), 356-407.

Weisstein, E. W. (n.d). Euler line. *MathWorld—A Wolfram Web Resource.* Retrieved from http://mathworld.wolfram.com/EulerLine.html

Wells, D. (1990). Are these the most beautiful?. *The Mathematical Intelligencer*, *12*(3), 37-

41.

Wilders, R., & VanOyen, L. (2011). Turning students into symmetry detectives. *Mathematics Teaching in the Middle School, 17*(2), 103-107.

Yackel, E., & Cobb, P. (1996). Sociomathematical Norms, Argumentation, and Autonomy in Mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477. doi:10.2307/749877

Chapter 2

# FIXING THE CROOKED HEART: CHILDREN'S AESTHETIC AND MATHEMATICAL PRACTICES IN PLAY

Lara Jasien & Ilana Horn

**Abstract**

Supporting learners' meaningful engagement in mathematical practices is often challenging. To understand how learners' might engage in this way, we draw on mathematicians' descriptions of their work and conceptualize doing mathematics as an aesthetic endeavor. To help see learners' mathematical practices as aesthetic, we situate our study in an informal context that features design-centered play with mathematical objects. Drawing from video data that supports inferences on children's perspectives, we present a case-study of one child's aesthetic judgements about her designs, highlighting the emergence of aesthetic problems whose resolution required engagement in mathematical practices. As mathematics educators seek to expand participation, our findings have implications for how we view children's mathematical competencies and, relatedly, instructional design that supports their emergence and development.

## Introduction

In educational contexts, mathematical practices are often named as things people *do* — such as making sense of structure, attending to precision, and using appropriate tools strategically (Common Core State Standards Initiative [CCSSI], 2010). Indeed, lists of such practices can be found on many classroom walls to support teachers and students in using them. Certainly, these lists represent one way of thinking about developing practices: name them, then do them. We refer to this use as *the everyday meaning of practice.*

From an anthropological view, however, descriptions of activities do not quite suffice. To count as a practice, we must account not only for *what* people do, but also the underlying cultural logics of *why* they do it — the meanings they derive from their activity in context (Bourdieu, 1977). With this in mind, we argue here that, to make mathematical practices more authentically connected to learning and sensemaking, mathematics educators need to attend not only to supporting children in the *doing* of mathematical practices, but also to the *meanings* they derive while engaging with them. We refer to this use as the *anthropological meaning of practice*.

To make mathematical practices anchored in meaningful sensemaking, then, we must understand their meanings in the broader mathematical enterprise. Looking at professional mathematicians, we ask: Why do they make use of structure, attend to precision, and use appropriate tools strategically? What meaning do they derive from these activities? When we pursue these questions, we find frequent references to mathematical beauty. That is, mathematical aesthetics underlie much of their engagement (see Sinclair, Pimm, & Higginson, 2006). *Aesthetic practices*, according to contemporary theory (Rancière, 2004; Sinclair, 2018), involve a double meaning of sensemaking: Aesthetic practices exist as a relation between *sensory ways of knowing* (e.g., embodied[5] ways of knowing; what is perceived by the eyes, ears, nose, skin, emotions, etc.) and *making sens*e (e.g., what is taken as sensible and desirable versus insensible and undesirable). Invoking the anthropological meaning of practice, *mathematical* aesthetic practices shape human sensemaking of and appreciation (or distaste) for particular manifestations of mathematics. Mathematical practices support sensemaking and engagement.

Using this insight, we wonder: How can we leverage aesthetics to support children's engagement in mathematical practices, such as those valued by the US's Common Core Standards? How would aesthetic designs reorient them to mathematical practices in ways that are personally meaningful? As Sinclair (2004, 2006) has described, learners' aesthetic practices can help both learners and mathematicians construct meaning as they generate, pursue, and evaluate new-to-them mathematical questions.

If aesthetic practices are a promising way to make mathematical practices meaningful, it is worth understanding how children might use them. This question is significant: Although awareness of the importance of mathematical aesthetics in mathematicians' work and in students' learning is increasing, we have yet to empirically examine the relationship between children's aesthetic practices, mathematical practices, and mathematical sensemaking. To this end, we found a naturalistic environment to examine this phenomenon, at an out-of-school mathematics playground called Math On-A-

[5] In this study, we do not take up the post-humanist and new materialist ontologies associated with this definition of aesthetic practices that conceptualize the body as inclusive of human and more-than-human elements. As will be discussed in the theory section, we examine interaction between people and their social and material environment, including mathematical tools.

Stick (MOAS). MOAS was a space designed for learners to playfully engage with mathematical objects. We illustrate how playing with mathematical objects brought out aesthetic practices and facilitated meaningful engagement in mathematical practices, such as those valued by the Common Core. This study deepens the field's understanding of meaningful engagement in mathematical practices, both inside and outside of the classroom. Furthermore, our aesthetic lens on engagement in mathematical practices contributes towards a holistic understanding of what it means for children to do mathematics, offering one avenue for broadening participation in the field.

## Research Questions

In this exploratory study, we use a case study of a child in a mathematical playground to ask: *How can learners' aesthetic practices support meaningful engagement in mathematical practices?* Situating our study at MOAS provided ample opportunities to see children making choices about their mathematical activities. These, of course, given the nature of the materials, often included aesthetic choices about designs, games, and patterns. By delving into one illustrative case, we provide a rich description of how mathematical practices emerge in a learner's playful exploration of mathematical objects, and how the child's use of aesthetic practices — along with the objects' affordances — influence meanings behind those mathematical practices.

## Prior Work: Mathematical Sensemaking and Aesthetic Practices

### Mathematical Sensemaking

For almost a century, calls have come from many corners to foreground meaning in mathematics learning, what might be called *mathematical sensemaking.* From mathematician Alfred North Whitehead's (1929/1961) plea to "resc[ue] the subject from [...] being a mechanical discipline" (p. 89) to William Brownell's (1946) exhortation to teach arithmetic in meaningful ways, educators have long sought to help children make sense of the mathematics they are asked to learn. Indeed, current efforts to shift mathematics education towards mathematical practices rather than procedures stem from the insight that children learn better when they have sensemaking opportunities (National Council of Teachers of Mathematics [NCTM], 1989; Carpenter, Fennema, 1992; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Erlwanger, 1973; Schoenfeld, 1985, 1992). This perspective continues to be echoed by mathematicians (Cheng, 2015) and education researchers (Boaler, 2002; Doyle, 1988; NCTM, 2014; Schoenfeld, 1988; Stigler & Hiebert, 1999) who are invested in redressing typical U.S. mathematics instruction's tendency to bypass sensemaking and its related consequences for students' learning, instead focusing on memorization and procedures.

### Aesthetic Practices in Mathematics Engagement

One way to invite children's mathematical sensemaking is through *mathematical aesthetics*, a foundational influence in in shaping the discipline of mathematics (Sinclair, 2004). According to Sinclair (2004), mathematical aesthetics involve aesthetic practices that fulfill three roles in driving mathematical work: an evaluative role, a motivational role, and a generative role. As the labels suggest, these respectively refer to: how we determine what counts as quality or "good enough" mathematical work; why and how we choose to engage or disengage in particular problem spaces; and how we gain insight into mathematical problems. As we describe more fully in the methods section, we operationalize these roles in our analysis by looking for tacit questions that indicate each role of the aesthetic (Table 2-1). We explain each role in more detail in the following subsections, with an interest in how they might emerge in children's mathematical play.

Table 2-1. *Operationalizing questions for each role of mathematical aesthetics*

| Role of Aesthetic Practices | Indicating Questions |
|---|---|
| Generative Role:<br>*The emergence of mathematical problems* | 1. What is possible here?<br>2. What stands out as interesting? |
| Motivational Role:<br>*The selection and pursuit of problems* | 1. Is this worth pursuing?<br>2. What action-steps should I take? |
| Evaluative Role:<br>*The judgement of inquiry* | 1. Am I on the right track?<br>2. Is this finished problem good enough, or the best that I can make it? |

### The generative role of aesthetic practices

The generative role of aesthetic practices in mathematics emerges when mathematicians experiment and *play* with the elements of a mathematical situation (Sinclair, 2004). Sinclair and Watson (2001) argue that mathematicians engage in exploration ways that spark feelings of wonder and curiosity, which we consider to be quite playful. These feelings of wonder and curiosity signal to mathematicians that they have discovered a personally interesting or culturally significant mathematical problem or idea (Sinclair, 2004). Thus, we see the generative role of mathematical aesthetics to have a clear connection to the mathematical practices of problem-posing and problem-solving, or perhaps more appropriately, mathematical inquiry. By mathematical inquiry, we refer the back-and-forth process by which mathematical problems and solution strategies can co-evolve throughout problem-solving activity: In the words of mathematician Philip Davis, "problem formulation and problem solution go hand in hand, each eliciting the other as the investigation progresses" (Davis, 1985, p. 23). While mathematical inquiry in classrooms often begins with problems posed by teachers and textbooks, the mathematical inquiry of mathematicians often begins with wonder and curiosity. The problems of mathematicians are not always well defined at their first emergence, but rather are elicited and refined as mathematicians explore the problem and potential solutions.

Children are also capable of having pleasant and powerful intellectual experiences with emergent mathematical inquiry. Dewey (1938) articulated inquiry — which begins with problem-*finding*, or what he calls *snags* in activity — as the process of bringing structure to an ambiguous situation. He described such child-driven inquiry as the kind of activity that is both satisfying and satisfactory, and thus also enjoyable (Dewey, 1929, p. 259). Likewise, Duckworth (1972) described children's acts of noticing and wondering as "the having of wonderful ideas," and argued that such ideas are the essence of intellectual development (p. 218). In this way, mathematical inquiry can be conceptualized as a practice that arises out of exploration when children begin to notice, wonder, and agentically re-structure their environment to pursue their own curiosities.

Mathematical engagement that begins with children identifying their own problems to solve can be quite playful. While a growing body of research examines how to engage students in mathematical problem-*posing* (Singer, Ellerton, & Cai, 2013), the strategies most researchers have used are prescriptive (for an exception, see Fiori & Selling, 2016) — for example, by getting students to change constraints in a math problem by asking *what if-not?* questions (Brown & Walter, 1983). These prescriptive strategies *are* productive for expanding students' engagement in mathematical sensemaking, yet they do not address the key issue of developing a personal taste for what counts as an interesting or worthwhile problem (Crespo & Sinclair, 2008).

Consider the following mathematical questions (Sinclair & Watson, 2001, p. 40):

*What if we could divide by 0?*

*What if we were to throw away the parallel postulate?*

*What if the irrational numbers were our counting blocks?*

*What if we really could have staircases like those drawn by Escher?*

*What if we could redefine differentiability to cope with some kinds of discontinuity?*

Historically, mathematicians have considered these questions to be interesting and worthwhile. We also consider these questions to be quite playful, and we conjecture that learners can ask similarly playful mathematical questions. For professional mathematicians, a *what if?* question might be, *What if there is another kind of tiling pentagon?* (e.g., Lord, 2016). For an eighth grader, it might be, *What if some shapes tile and some shapes do not tile?* (e.g., Civil, 2002). For an 8-year-old child on a mathematical playground, it might be, *What if these pentagons fit together in a way that never ends?* or *What if I try to make another kind of pentagon tile?*

Keifert and Stevens (2018) describe such insight-driving questions as the pillars of inquiry from a member's perspective. In their longitudinal ethnographic study of young children across home and school contexts, they articulated and illustrated the beginnings, middles, and ends of inquiry as being defined by when people jointly orient to an unknown situation, make progress on understanding the situation by drawing on sense making resources, and then orient to a satisfactory end to inquiry. In this sense, a mathematical problem is not necessarily the result of a dilemma (as in the sense of Lampert, 1990; Hall, 1996; Hiebert et al., 1996; Lave, Murtaugh, de la Rocha, 1984; Pea & Martin, 2010) but can also emerge from a sense of puzzlement or curiosity. This *felt* sense of puzzlement then also partially constitutes the task to be done.

Thus, the generative role of mathematical aesthetics helps mathematicians answer tacit questions such as, *What is possible here?* and *What stands out as interesting?* In our analysis of children's activity, we look for these tacit questions as indicators of aesthetic practices playing a generative role.

**The motivational role of aesthetic practices**

In addition to the generative role, aesthetic practices play a motivational role in both problem selection (a *selective function*, Sinclair, 2004, p. 277) and strategy selection (a *heuristic function*, Sinclair, 2004, p. 289). Thus, the motivational role of mathematical aesthetic practices involves the development of personal interests that support motivation to engage in mathematics in particular ways. For example, Penrose (1974) suggested visual appeal attracted him to inquire into the strange symmetries in irregular tilings, a problem he pursued for the majority of his career. Unfortunately, students rarely have the opportunity to select their own problems to solve, and, even after decades of reform-efforts, are often still provided with rote means of solving assigned problems (Stigler & Hiebert, 1999; Jacobs, Hiebert, Givvin, Hollingsworth, Garnier, & Wearne, 2006; Litke, 2015). This forecloses opportunities for students to develop a taste for problems they enjoy pursuing and strategies they prefer for problem-solving.

One way mathematics educators have attempted to get students interested in mathematics is through interdisciplinary instruction — by, for example, linking mathematics to domains like music and art, as evidenced by STEAM initiatives that have gained popularity. Such interventions seek to make mathematics less intimidating for students (Van der Veen, 2012), to illustrate mathematics' utility (Bush, Karp, Nadler, & Gibbons, 2016; Wilders & Van Oyen, 2011), or to provide concrete experiences to make symbolization more meaningful (Bush, Karp, Bennett, Popelka, & Nadler, 2013; Cipoletti & Wilson, 2004). For example, Cipoletti and Wilson (2004) describe a task in which students use origami directions — modified to feature geometric language — to produce an aesthetically pleasing object while also learning to use mathematical vocabulary to communicate concretely before using it symbolically. These approaches to motivating students through aesthetics in art are most commonly found in teacher journals rather than research journals. Clearly, within classrooms, teachers find aesthetics motivate forms of mathematics enough to write and share about them.

Some critique these interdisciplinary methods as moving away from mathematics to foster student motivation, thus bolstering the belief that mathematics is aesthetically sterile (Sinclair, 2001). Although this is a legitimate concern, research has shown that disrupting longstanding mathematics education norms and practices while simultaneously opening up opportunities for students to bring in unconventional sensemaking resources can create expanded opportunities for mathematical engagement (de Freitas & Sinclair, 2014; Ma, 2016), especially when multimodal ways of engaging are leveraged (Kelton & Ma, 2018; Nemirovsky, Kelton, & Rhodehamel, 2013). While connections between such kinesthetic engagement and aesthetic practices are not often made, we argue that kinesthetics may provide their own aesthetic avenue for some learners. The general claim is that multimodal ways of knowing offer access for students to productively engage aesthetic practices for mathematics sensemaking.

In sum, the motivational role of mathematical aesthetics helps mathematicians engage tacit questions such as, *Is this worth pursuing?* and, *What action-steps should I take?* Aesthetic responses (such as wonder, curiosity, and surprise) help mathematicians answer these questions. To pursue our analysis of children's playful activity, we look for these tacit questions as indicators of aesthetic practices motivating activity.

**The evaluative role of aesthetic practices**

Finally, the evaluative role of mathematical aesthetics guides mathematicians both as they engage in formative assessment during inquiry and as they assess what mathematics counts as good enough to be produced, retained, and verified (Sinclair, 2004). For example, mathematicians evaluate of the meaningfulness of their current strategies in relationship to the ideas they personally value, which can lead to strategy shifting (Silver & Metzger, 1989). In fact, mathematicians tend to prefer solutions that *make them feel something,* such as surprise, wonder, or pleasure in simplicity (Wells, 1990). A mathematicians' work is often not deemed finished until they have achieved aesthetic appeal in their solution.

Following the lead of Harré (1958, as cited in Sinclair & Pimm, 2006), some researchers and teachers have attempted to help students develop mathematical aesthetics by providing them with access to multiple solutions to the same problem for comparison. This approach has endured across generations. In 1968, Rosenberg published an article in

*The Arithmetic Teacher* about providing students with rich problems that can be solved in a multitude of ways with increasing sophistication. This resonates with today's reform-oriented mathematics instruction that encourages students to engage with multiple strategies and solutions by making their ideas public and open for debate (e.g., Horn, 2008, 2017; Lampert, 1990; Smith & Stein; 2011).

However, much prior work has backgrounded affective feelings such as surprise that are key to the mathematical *aesthetics* of evaluation. Such affective feelings are shaped by the situations in which solutions are encountered. For instance, in an experimental study on stimulating evaluative aesthetics, Koichu, Katz, and Berman (2017) found that most participants labeled some problems and solutions as more beautiful and surprising than others, depending on different conditions. In fact, while students evaluated both problems in their study as aesthetically appealing, only the problem presented *second* was identified as the most appealing. This suggests that aesthetic appeal may not be inherent to the solutions themselves, but rather that affective experiences of unexpectedness and surprise can lead students to evaluate some mathematical solutions as more appealing than others. It is likely that other situational conditions — including prior knowledge, culture, histories, and mathematical encounters —also shape people's affective experiences and thus their aesthetic evaluations.

To summarize, mathematical aesthetic practices help mathematicians answer the evaluative questions: *Am I on the right track?* or *Is this finished problem good enough?* or *Is this argument the best that I can make it?* They often *feel* the answers to these questions, appealing to intuition and affective responses. In our analysis, we look for these tacit questions and affective responses as indicators of aesthetic practices playing an evaluative role in activity.

## Children's Engagement with Mathematical Aesthetic Practices

Despite its central role in mathematicians' activity, most students never get to engage in formulating, exploring, and answering *what if?* questions that support aesthetically driven mathematical inquiry. Instead of asking children why only some shapes tile, direct instruction tells students which shapes tile, sometimes (but not always) why they tile, and then requires them to store these facts as part of their accumulated math knowledge. In inquiry-oriented classrooms, we might ask students if all shapes tile and then structure their investigation into which do and do not by directing them to use angle measurements in their quest for a correct answer (e.g., Civil, 2002). In typical classrooms, children get few opportunities to experience the emergence of inquiry from exploration. Beginning to understand the role of aesthetic practices in both noticing and resolving these curiosity-sparking experiences, and connecting aesthetic practices to mathematical practices, motivates the present study.

## Theoretical Framework: Aesthetic Practices Support of Children's Mathematical Sensemaking

To examine how engagement in aesthetic practices can support meaningful engagement in mathematical practices, we take a situative and interactionist perspective on learning in activity (Jordan & Henderson, 1995; Lave & Wenger, 1991; Schön, 1992; Wertsch, 1998). From a situative perspective, engagement in aesthetic practices involves participating in practices around particular "'ways of doing and making' that have been chosen among many others" (Sinclair, 2018, p.4). Although these "ways of doing and making" are *a product of* participating in a community engaged in a particular endeavor for a particular purpose, they also function *to produce* particular forms of visibility and sensibility. Relatedly, they also obscure some things as *in*visible and *in*sensible.

For instance, aesthetic practices can influence sensibilities around what it means to be mathematically competent and how it is acceptable to express or communicate mathematical competence. Notably, displaying mathematical competence in school math looks very different than displaying it in an engineering firm (Stevens & Hall, 1998). Since MOAS engages learners in ways of doing and making that are quite different than those of school math, it is reasonable to expect that the aesthetic practices that emerge at MOAS will transform what mathematics looks like.

Our tight theoretical link between ways of doing and making, aesthetic and mathematical practices, and sensemaking follows Dewey's problematization of dualisms between thought and action, science and common sense, and the academy and out-of-school life (Dewey, 1938). In this vein, we look for moments of uncertainty that trigger learners to restructure their environment in ways that remove ambiguity and create determinacy. Schön (1992) described this process as *reflective conversation with the situation*, a term meant to emphasize the activity mode of *designing* as a primary means of engaging in inquiry through "getting in touch with the understandings we form spontaneously in the midst of action" (p. 126).

From an interactionist perspective (e.g., Wertsch, 1998), this points us to look closely at *trouble and repair* episodes (Jordan & Henderson, 1995), where participants encounter uncertainties (previously described as problematic experiences, snags, or dilemmas; e.g., Lave, Murtaugh, de la Rocha, 1984) in their activity and seek ways of moving activity forward. This process inherently involves sensemaking, as learners revise activities and develop new sensitivities and ways of moving forward. As mentioned in the conceptual framework, the source of inquiry — the *trouble* — is often experienced not only as a felt problem, but also as a sense of puzzlement or curiosity (Keifert & Stevens, 2018).

In addition, the mathematical objects at MOAS were designed to facilitate encounters with mathematical concepts. We take the perspective that body movements are part of mathematical thinking, not just actions that give rise to it. In more theoretical terms, this is an embodied, non-dualist approach to approach to understanding tool use, meaning that we take all mathematical sensemaking to be multimodal, both overtly and covertly constituted and expressed in bodily activity (Kelton & Ma, 2018; Nemirovsky, Kelton, and Rhodehamel, 2013). In this way, although the mathematical concepts were not always explicitly named as they are in typical school instruction, they were afforded in the objects, supporting an analysis of children's emergent mathematical sensemaking. For this reason, we conceptualize the mathematical objects at MOAS as cultural tools (Wertsch, 1998), with an eye toward their role in mediating activity and sensemaking. To understand participants' meanings, and the mathematics participants encountered in their play, we examine the designed affordances of materials as well as the realized affordances of tools-in-use — alongside the more ephemeral tools of language and interaction (Ahearn, 2011; Goodwin, 2006). However, while we analyze our participants' activity for mathematical sensemaking, we do not imply that participants view their activity as mathematics.

Across the data from MOAS, we see many instances where children's goals have aesthetic dimensions — ones that invite representation, pattern making, or design. Like professional mathematicians, these aesthetic goals made mathematical practices meaningful. When children pursued aesthetic goals and encountered uncertainty, we often saw that the pursuit of aesthetic goals supported persistence in ways that supported mathematical sensemaking. These moments gave us a window on the unique role of aesthetic practices in children's mathematical sensemaking.

## Research Design

### Setting and Participants

Our study took place at a mathematical playground called Math On-A-Stick at the Minnesota State Fair in 2016. In the US, state fairs are pop-up leisure gatherings that happen once a year, typically around harvest time; they often feature carnival rides, fried food, games, and other forms of entertainment, appealing to families and people from all walks of life. We collected data over the 10 days of the fair. We noted that families frequently used MOAS as a retreat from other crowded and busy attractions. A white picket fence set the space of the mathematical playground apart, providing a boundary to let children roam, as well as nine *exhibits* — picnic tables with colorful umbrellas, decked out with an array of mathematical tools for children to use and explore as they pleased. Volunteers in orange aprons staffed the tables, facilitating children's play or simply organizing and securing the materials. MOAS is described on the fair's website as:

> …a welcoming space where kids and grown-ups can explore fun math concepts at the fair. Play with geometric and reptile-shaped tiles to create designs and patterns.

> Sort, count and look for what's the same and what's different in groups of colored eggs on captivating cards. Take a break from the hustle and bustle of the fair to enjoy a shapes or numbers book. (Minnesota State Fair, n.d.)

We consider MOAS to be a mathematical playground within the setting of the fair because of the open, playful participation structure supported by the organization of the space and the mathematical nature of the materials within its exhibits. The nine exhibit tables were labeled with the following signs: *Cones & Spheres, Eggs & Crates, Lizards & Turtles, Pattern Machine, Pentagons, Spiral Machine, Stepping Stones*, and *Tiles & Patterns*. In addition, the ninth table, marked as the *Visiting Mathematician,* changed daily with a specific activity supervised by a volunteer mathematician who invited children to investigate a topic. In exploring the materials at the different tables, children had opportunities to encounter many mathematical ideas, including symmetry, tessellations, arrays, angles, along with many types of patterns. Children's voluntary participation, their freedom in how (and how long) to engage with the exhibits, and the mathematical richness of the designed materials support our investigation of how children's aesthetic practices might support their engagement in mathematical practices**.**

Over the 10 days of data collection, we recruited 345 children between the ages of 4 and 17 to participate, with their age distribution represented in Figure 2-1.

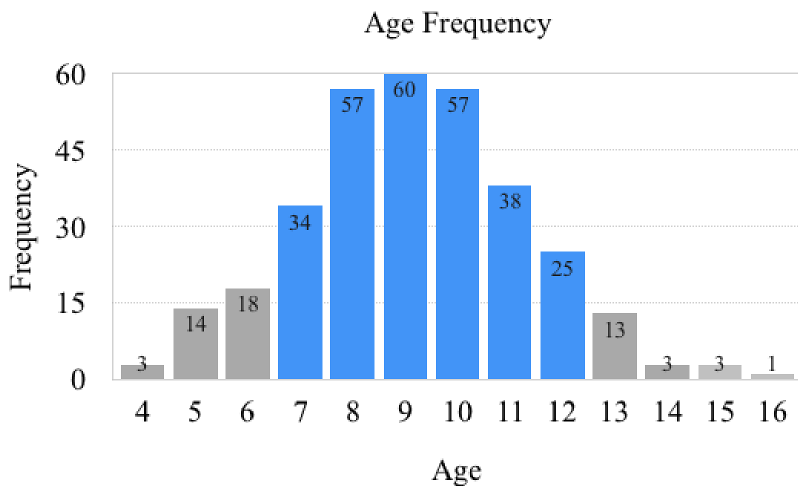



*Figure 2-1*. Distribution of participants' age in years.

**Data Collection**

Our primary interest in this study was to understand how learners' aesthetic practices might support meaningful engagement in mathematical practices. Thus, our data speak to what happens during play rather than on participants' recollections of their play. Following sociocultural studies of families' interaction in naturalistic settings (Goodwin, 2006), our primary data source is the 345 video recordings of children at MOAS. To offer insight into participants' social locations and experiences, we also collected intake and exit surveys, and an exit interview, but these sources are not used in this analysis.

After completing the intake survey, participants roamed the playground. Researchers mounted cameras on hats for the children to wear so they aimed downwards and slightly forward to capture the children's and adults' talk, gestures, and object manipulation (Figure 2-2). The head-mounted cameras aimed to capture children's perspectives and activities in the MOAS exhibits. This view captures the locus of the children's attention when they are playing with objects: When the child looks up, the camera shows what they are looking at. Even slight glances at other children playing are captured. This video record supports inferences about children's attention and interest. In the end, we collected a total of 127 hours of video, with the average visit lasting 27 minutes (*sd* = 16.46 min).

Figure 2-2. *Child with video capturing device.*

**Data Analysis**

**Case selection**

Our focal case was purposively selected (Yin, 2017) from the data corpus for several analytical reasons. Olivia, an 8 year-old-girl, played at the *Eggs and Crate* exhibit for over 12 minutes, much longer than the average stay time at an exhibit that primarily held the attention of children younger than our age band. The table featured a plentiful supply of colored plastic eggs along with cardboard egg-crates with slots arranged in a 6 x 5 array. During her stay at the exhibit, Olivia engaged in three episodes of goal-based activities. We focus on Olivia's last one, where she worked to make a heart design using pink plastic eggs in the egg crate over approximately seven minutes. This single design took her longer than most participants remained at this exhibit (*mdn* = 2.79 min), making her an atypical case, but one that offered a rich record of children's engagement in aesthetic practices in support of mathematical practices.

We devote our analysis to her based on multimodal richness in *mathematical sensemaking* during *persistent, aesthetic, goal-driven play*. Furthermore, because Olivia's mother was attentive during this case of play, their interaction was analytically useful, since their dialogue gave us increased access to Olivia's thinking. In this way, our goal is not representativeness but enriching the reader's understanding of aesthetic and mathematical practices through thick description (Geertz, 1973) of complex episodes. By immersing the reader in the details of Olivia's activity, we aim to show how her goal-driven engagement with the *Eggs and Crate* exhibit created opportunities for her to develop thicker understandings of mathematical ideas such as symmetry.

**Mathematical affordances of the exhibit**

The materiality of the *Eggs and Crate* exhibit is significant in this analysis. The exhibit's design included:

- Multi-colored (rather than monotone) eggs to encourage design
- A plethora of eggs, making it rare to run out of any particular color
- Eggs in a size graspable by children with varying degrees of sensorimotor coordination
- Rectangular dimensions of the crate, including the even and odd number of slots on the two orthogonal dimensions (a 6 x 5 crate), making different conditions for

symmetry

- The overall scale of the crate to be approximately the width of a child's body
- The overall tactile and visual aesthetic appeal of the materials.

These features all worked together in children's designs and, in turn, in their mathematical sensemaking. For example, the multi-colored eggs afforded children the opportunity to pattern in ways that attended to and made use of color, and the grid-like nature of the crate afforded children the opportunity to pattern in ways that utilized the crate as an array of slots where eggs could be placed as points. Thus, the exhibit afforded engagement with mathematical concepts involving structure, patterning, and spatial reasoning — concepts that we know are very important to students' success in mathematics, but that are often downplayed in mathematics curriculums (Sinclair & Bruce, 2014). Similarly, the odd- and even- crate dimensions offered different kinds of symmetry as children build patterns, depending on the rotation of the crate and children's goals, while the crate's scale — approximately the width of children's torsos — allowed for easy bilateral coordination (i.e., mirroring gestures in both hands) in the pursuit of symmetry. Notably, even the aesthetic appeal of the exhibit is significant for mathematical sensemaking, as this feature invites both initial exploration and sustained engagement across multiple episodes of making.

**Unit of analysis: Episodes trouble-and-repair**

Using interaction analysis (Jordan & Henderson, 1995) — a method centered on foregrounding participants' meaning-making in activity — we identified when Olivia experienced trouble and worked to repair it. Because *trouble* occurs when participants' expectations are broken (Jordan & Henderson, 1995, p. 69), this method allows us to distinguish between play that produces mathematical ideas that participants *might* have noticed and play that produces mathematical ideas that we can empirically argue that they noticed. In other words, while we can certainly infer mathematical ideas from children's play activities, we seek to identify the mathematics that participants *attend to* as they work to achieve their aesthetic goals, even if the mathematics is not formally named. Because common goals at MOAS involved making things, trouble occurred when children's patterns and designs did not emerge as they intended.

This analytic approach allowed us to attend to both verbal and nonverbal features of interaction. Verbal indicators of trouble include when children explicitly asked for help ("How do you make a circle?"), sought formative feedback ("Dad, can you tell what I'm making?"), expressed frustration ("Ughhhh"). Nonverbal cues included inquisitive gestures, pauses accompanied by gazing at the trouble spots, or trial-and-error revisions (moving pieces to make a pattern "look right"). We transcribed the focal case for a close analysis of discourse and interaction, using transcript conventions to capture some of the rhythms and intonations of participants' speech. The extra punctuation denotes speech characteristics like breaks -, elo::::ngation, and *intensity*.

This emic approach (Harris, 1976) allows us to centralize Olivia's perspective as she experienced (a) the emergence of problems (generative role of aesthetic practices), (b) the selection and pursuit of those problems (motivational role of aesthetic practices), and (c) judgement during and after inquiry (evaluative role of aesthetic practices). To identify when Olivia's aesthetic practices surface, we look for the tacit questions that signal each role of the aesthetic practices, as identified in the conceptual framework (see Table 2-1). In our analysis, the *generative role* was signaled by the emergence of a mathematical problem

— or in interaction analysis terms, trouble. Specifically, we look for Olivia's articulation of (and thus insight into) what is possible or interesting in relation to resolving the trouble. The *motivational role* was signaled as she repaired the trouble that she had identified, as well as by her interactional moves to extend her goal-oriented activity. Finally, the *evaluative role* came up when Olivia described the extent to which her design was "on the right track" or "good enough" — in other words, the extent to which repair had been achieved. Consistent with our conceptualization of the roles of aesthetic practices in children's mathematical activity, when children persist in resolving trouble by (re)formulating their understanding of the trouble and restructuring their attempts to repair it, we view this as indication of sensemaking supported by their goals and the mathematical tools.

## Findings

In this section, we illustrate how aesthetic practices can support engagement in mathematical practices by narrating cycles of trouble-and-repair as aesthetic cycles that encompass the three roles of mathematical aesthetic practices: Throughout Olivia's play, the generative, motivational, and evaluative roles of her aesthetic practices mutually elicit and inform each other to facilitate her mathematical sensemaking. By attending to Olivia's sensemaking, we identify mathematical practices in her play. To do so, we focus primarily on the last episode of trouble-and-repair, a long instance of goal-based play where Olivia attempted to build a pink heart in the egg crate. We start with an analytic description of Olivia and her mother's activities at the *Egg and Crate* exhibit, then we interpret this in light of our question about how learners' aesthetic practices support meaningful engagement in mathematical practices.

### Prior to Focal Case of Goal-Based Play: Orientation and Preliminary Designs

Upon arriving at the *Egg and Crate* exhibit, Olivia asked the volunteer, “How do you do it?” The volunteer told Olivia that she could make her own designs, and Olivia’s mother agreed, telling Olivia, “You can do anything you want.” This invitation underscores a widespread norm at MOAS, inculcated by the designer in the volunteers’ training materials: Children can make whatever they want, which differs not only from school activity but also from more structured mathematics exhibits at science museums that often come with models or directions. In other words, MOAS’s material and social design deliberately invited exploration and presented children with freedom in their activity.

In her first two EOMs, Olivia made what she described as a “flower” and a “rainbow.” In the flower design, Olivia struggled with the symmetry of her design. She made several attempts to repair the problem, intermittently asking aloud evaluative aesthetic questions such as, “Or, would it look better this way?” Eventually, Olivia settled on the first way she had made her flower as “good enough,” leaving the symmetry unresolved, proudly showing it to her mother (Figure 2-3a).

In the rainbow design, Olivia attended to both the color and shape of rainbows (Figure 2-3b). Most children who made rainbows at the *Egg and Crate* exhibit did so by striping the columns of the crate with solid colors in a way that attended to color but not shape (e.g., one row or column all blue, the next all green, etc.). Olivia’s fidelity to archetypes (the color *and* the arch of the rainbow) was characteristic of her later work — and became a source of trouble in her next design.

(a) (b)

*Figure 2-3.* (a) Olivia's flower. (b) Olivia's rainbow.

**Focal Episode of Goal-Based Play: Making a Heart**

After showing her rainbow to her mother, Olivia deconstructed it and asked aloud, "What else can I do?" We take this as evidence that Olivia was attempting to create a design that was interesting to her — and perhaps more challenging. This demonstrates the generative role of Olivia's aesthetic practices, as she attempted to understand what was possible and interesting at the *Egg and Crate* exhibit. She then excitedly exclaimed, "Oh! I can do a hea*:::*rt!", generating her next goal. While Olivia was talking, her mother stood up and asked if she wanted to go to the nearby *Pattern Machine* exhibit. Olivia declined with a determined, "*Noooo*. I want to keep doing th*i::*s." Her mother acquiesced, and Olivia began building her heart.

### First cycle of aesthetic practices: Emergence of trouble

Olivia's first attempt at making a heart lasted approximately two and half minutes and included several revisions: Olivia hesitantly and repeatedly placed eggs in the crate, picked them back up, and moved them to other slots in the crate (Figure 2-4a). During this two and a half minutes, her mother offered suggestions, saying, "Make it the same on either side" (Figure 2-4b).


(a) (b) (c)

*Figure 2-4.* (a) Olivia struggling to make her heart. (b) Mother attempting scaffold Olivia (c) Olivia maintaining control of her heart by asserting aesthetic evaluations.

#### *Trouble*

Once the heart was constructed (Figure 2-4c), Olivia verbalized an aesthetic evaluation of the heart, saying, "I feel like there's something not right." This aesthetic evaluation was the first time Olivia generated a description of trouble, although this formulation of the trouble did not include any direction for repair. Her mother responded, asking Olivia, “Can I try?” (Figure 2-4c). Olivia agreed, and her mother started to manipulate the eggs. However, as soon her mother moved an egg, Olivia changed her mind. In an urgent voice, she said, “I want to do it *thou::gh*, that doesn't look right,” pushing her mother’s hand away and returning the egg to its prior position (Figur e2- 4c). In this move, Olivia asserted both her aesthetic judgement and agency, maintaining ownership of her play and the design of her heart. We argue that this authoritative assertion of aesthetic judgement reflected Olivia's motivation to create a heart that was meaningful to her.

***Summary of first cycle***

In this episode of trouble-and-repair, Olivia’s aesthetic practices allowed her to pursue new-to-her challenges of creating a heart with symmetry (Table 2-2). Although she had not yet made a symmetric heart, her aesthetic practices supported her persistence in resolving a problematic situation: Olivia generated a goal of creating a heart, she was motivated to pursue that goal, and she evaluated the heart as not yet good enough. When her mother tried to take over, Olivia maintained ownership of the heart’s construction (“No, I want to do it”), again demonstrating the motivational role of aesthetic practices. Thus, in this first episode of trouble-and-repair, Olivia’s aesthetic practices played generative, motivational, and evaluative roles.

*Table 2-2.* Summary of the first cycle of aesthetic practices.

| Roles of the Aesthetic* | Analytic Indicating Questions | Observed Evidence | Nature of Practices |
|---|---|---|---|
| G | What is possible here? | Generated Goal | |
| M | Is this worth pursuing? | Pursues symmetric heart | |
| E | Am I on the right track? | Critiques the heart (“I feel like there’s something not right”) | Aesthetic practice supports *attending to precision* |
| M | Is this worth pursuing? | Olivia maintains ownership of repair (“No, I want to do it though”) | Aesthetic practice supports *making sense of problems and persevering in solving them* |

*The abbreviations G, M, and E respectively stand for the generative role, motivational role, and evaluative role.

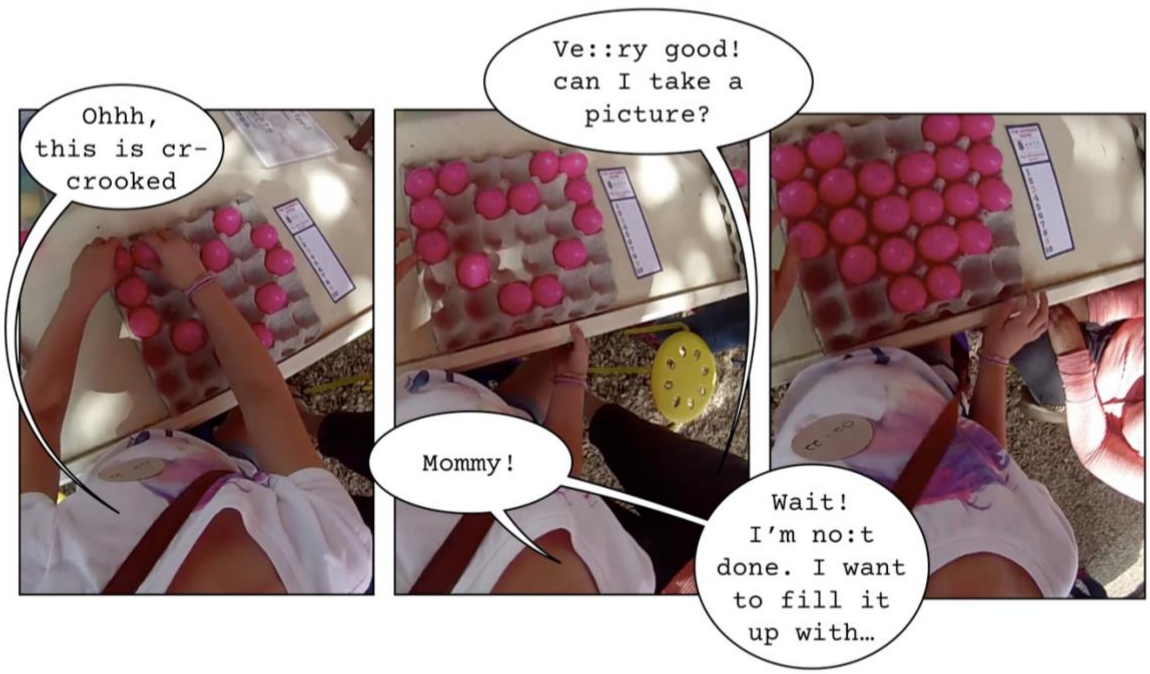
Although we do not claim Olivia had yet engaged in mathematical sensemaking, we do argue that her aesthetic practices facilitated her engagement in activities that resemble mathematical practices. For example, her evaluative aesthetic practices led her to *attend to precision* and her motivational aesthetic practices led her to *make sense of problems and persevere in solving them.* Furthermore, we conceptualize Olivia’s goal-setting of making a heart to be the aesthetic generation of an out-of-school problem, as she did not yet know how she would make a heart. In this sense, her goal-setting was an out-of-school version of mathematical problem-posing, which, as we will see, does in fact create opportunities for her to engage in mathematical practices that facilitate mathematical sensemaking.

### Second cycle of aesthetic practices: Repairing trouble

After a little conversation with her mother about where eggs should and should not be placed (Mother: “I think you need to have one there” Olivia: “Or take that one away”), Olivia rearticulated the trouble of what was wrong with the heart. While her initial evaluation was, “I feel like there’s something not right,” Olivia refined her formulation of the trouble, saying, “Ohhh, this is cr-crooked” (Figure 2-5a). This fine tuning of the problem supported Olivia’s mathematical sensemaking, as she homed in on the central mathematical issue — namely, symmetry. In other words, Olivia’s aesthetic exploration of the problem space became mathematically generative in this reformulation of the trouble (“It’s crooked”), pointing her towards a resolution — make the heart symmetrical. This aesthetic goal supported her motivation to pursue repair.

While Olivia’s statement of “it’s crooked” could be analytically interpreted as evaluative, we take the surrounding utterances to point to another interpretation. In particular, her intonation and her exclamation (“Ohhh”) before stating “it’s crooked” indicated that Olivia was not asking herself whether or not she was on the right track or whether the heart was good enough — in fact she already expressed that something was not right. Instead, her identification of the heart’s crookedness was a tacit answer to the question, *What is possible here?* Although she may have apprehended that something was amiss prior to stating the issue, identifying crookedness as the source of her trouble guided her work going forward.

Over the next minute and a half, Olivia created a heart with midline symmetry (Figure 2-5b). Although her mother interacted with Olivia throughout this time, Olivia still called for her mother’s attention to show her the heart (“Mommy!”). We take this an indication that Olivia had evaluated the heart as good enough to share. In response, her mother praised Olivia and attempted to take a picture of the heart — a commonplace signal that a making activity had ended at the MOAS playground — but Olivia interjected that she was not yet finished because she wanted to "fill up" the heart (Figure 2-5c). Just as mathematicians are motivated to pursue mathematical problems because of their personal intimacy with ideas, Olivia was motivated to continue work on her heart because it was meaningful to her. Here, we see Olivia's aesthetic practices as motivating her continued activity.




(a) (b) (c)

*Figure 2-5*. Olivia reformulates her understanding of trouble. (b) Olivia achieves a symmetric heart. (c) Olivia extends her activity by filling up the heart.

Once the heart was filled up, Olivia backed away from the table to get a better view of it, positioning her body for evaluation, asking, “Does that look like a heart?” (Figure 2-6a). She then exclaimed in a frustrated tone, “It still doesn’t look like a heart!” (Figure 2-6b). Here, Olivia’s aesthetic practices led her to re-evaluate the heart, and she found it wanting. She then began removing the pink eggs that filled up the heart, starting in the middle and removing two eggs at a time by using both her hands (Figure 2-6c), using her body’s bilateral symmetry to operate on the symmetry of the heart. Once again, rather than giving up, Olivia’s personal identification with creating a heart suiting her archetype motivated her to persist.


(a) (b) (c)

*Figure 2-6.* (a) Olivia aesthetically evaluating and asking for her mother’s aesthetic evaluation. (b) Olivia negatively evaluating after filling in the heart’s background. (c) Olivia begins to deconstruct the heart using bilateral symmetry of her body.

### ***Summary of second cycle***

In this episode of trouble-and-repair, Olivia's aesthetic practices continued to support her engagement and facilitated her emergent mathematical sensemaking (Table 2-3). First, Olivia generated a productive articulation of trouble ("It's crooked"), leading her to pursue a solution oriented towards the mathematical property of symmetry. This mathematical sensemaking emerged as Olivia's aesthetic practices facilitated her to *look for and making use of structure* in relation to the symmetry of the crate. Thus, her attention to the structure of the heart (and also the beginning of her attention to the structure of the crate as a grid) connects to mathematical sensemaking around the properties of crookedness and symmetry.

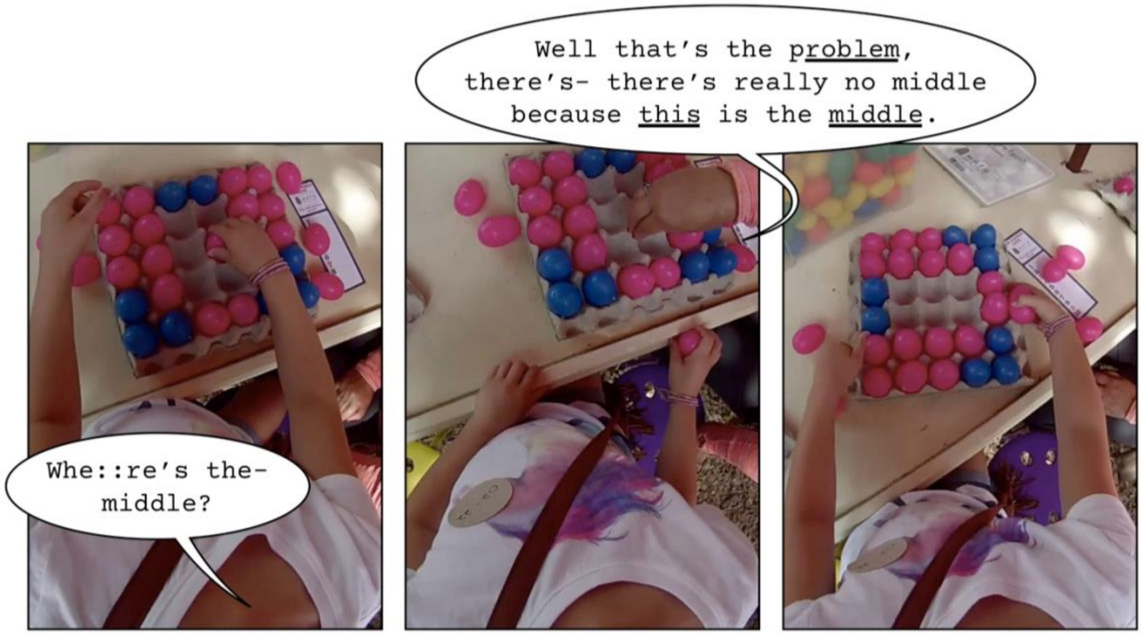
*Table 2-3.* Summary of the second cycle of aesthetic practices.

| Roles of the Aesthetic | Analytic Indicating Questions | Observed Evidence | Nature of Practices |
|---|---|---|---|
| G | What is possible here? | Articulated trouble ("It's crooked") | Increasingly mathematical practice of *looking for and making use of structur*e as she attends to symmetry; beginning to treat crate as a grid |
| M | What action-steps should I take? | Pursues symmetric heart | Increasingly mathematical practice of *make sense of problems and persevere in solving them* |
| E | Is this finished problem good enough, or the best that I can make it? | Shows pride in work (calls for her mother's attention, "mommy!") | |
| M | Is this worth pursuing? | Extends work by filling up the heart | |
| E | Is this finished problem good enough, or the best that I can make it? | Evaluates the heart (Does that still look like a heart?) | Aesthetic practice supports *attending to precision* |
| M | What action-steps should I take? | Pursues heart by filling in a blue background | Aesthetic practice supports *making sense of problems and persevering in solving them* |
| E | Is this finished problem good enough, or the best that I can make it? | Critiques the heart ("It still doesn't look like a heart") | Increasingly mathematical practice of *attending to precision* as she treats the crate as grid as she works towards a heart with a point |
| M | What action-steps should I take? Is this worth pursuing? | Begins to retry to repair the heart as she removes eggs using bilateral coordination | Increasingly mathematical practice of *make sense of problems and persevere in solving them* |

At first, when she made a symmetric heart, Olivia evaluated it as good enough, proudly showing it to her mother. However, Olivia's aesthetic practices then motivated her to fill up the heart, putting a pink interior into the heart and blue background of the crate. This altered the look of the heart, leading Olivia to re-evaluate its design as unsatisfactory, perhaps because the blue background highlighted the heart's lack of a single *point* (the heart ended with two eggs comprising its vertex, instead of just one). We take this to be another instance of *attending to precision* — this time with a stronger mathematical orientation. Rather than give up on making a heart that matched her archetype, Olivia persisted in further attempts to create a *heart with a point*, with her aesthetic goals once again motivating her continued play. We consider her *pursuit of precision* and her persistence in *making sense of problems and persevering in solving them* to be increasingly reflective of desired mathematical practices: She treated the crate as a grid when she began to investigate its symmetry more systematically by removing eggs bilaterally.

**Third cycle of aesthetic practices: Reformulation of trouble**

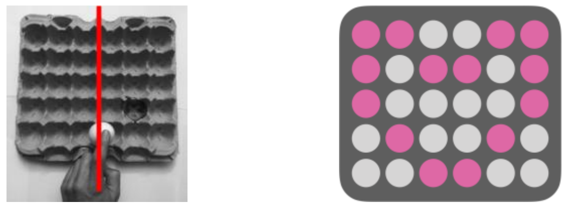
After Olivia removed a few eggs using her body's bilateral symmetry, she took one egg and hesitantly moved it back and forth between the two middle slots in the egg crate, eventually touching an egg to the high ridge on the egg crate's midline of symmetry and asked aloud, "Where's the middle?" (Figure 2-7a). This question — "Where's the middle?" — is Olivia's third reformulation of her trouble. Her understanding of the problem shifted from not looking like a heart (first cycle), to being crooked (second cycle), to missing a middle (third cycle). In the end, this reformulation structured Olivia's investigation in a way that allowed her to eventually resolve the trouble.




(a) (b) (c)

*Figure 2-7*. (a) Olivia reformulates the problem for a third time. (b) Mother affirms Olivia's evaluation. (c) Olivia uses her whole body to explore the egg crate.

Importantly, when Olivia asked, "Where's the middle," she moved the egg to slots on either side of the actual middle in the egg crate (midline of symmetry) and then rested the egg on the actual middle (Figure 2-7a). While doing this, Olivia and her mother looked at the egg crate together. We take this joint attention and *Olivia's placement of the egg on the actual middle while asking "Where's the middle?"* to mean that Olivia was explicitly asking her mother for help finding the *practical middle* in a 6 x 5 egg crate (Figure 2-8). We distinguish between *actual* and *practical* middles because, by moving the egg, Olivia drew her mother's attention to the absence of a practical midline — a slot for the egg in her hand — in her current use of the crate. In other words, with the six slots parallel to her body, the egg could not rest in the center.



(a) (b)

*Figure 2-8*. (a) No practical midline symmetry and (b) a schematic diagram of Olivia's heart centered on practical middle from Figure 5b.

Her mother then pointed out the missing midline symmetry for the entire carton, saying, "Well that's the problem, there's- there's really no middle because *this* is the middle" while running her fingers along the same midline ridge her daughter had just rested the egg on (Figure 2-7b). Her mother then broke their joint attention to the egg crate by turning away to talk with another adult, providing Olivia with the opportunity to investigate more on her own. Olivia, motivated by her aesthetic desire to create the heart she envisioned, continued exploring the problem.

***Summary of third cycle***

In this episode of trouble-and-repair, Olivia generated a new articulation of trouble ("Where's the middle") and pursued her quest to make the heart more aesthetically pleasing with the help of her mother's scaffolding (Table 2-4). By taking the eggs out two at a time, moving one egg back and forth across the actual midline of symmetry and eventually resting the egg (or point) on it, and *then* inquiring about a middle, Olivia used multimodal resources to make sense of the egg crate as a grid. We take this as evidence of meaningful engagement in mathematical practices — such as *attending to precision, looking for and making use of structure,* and *using appropriate tools strategically* — that facilitated her sensemaking around a concrete mathematical problem of finding the practical middle of six.

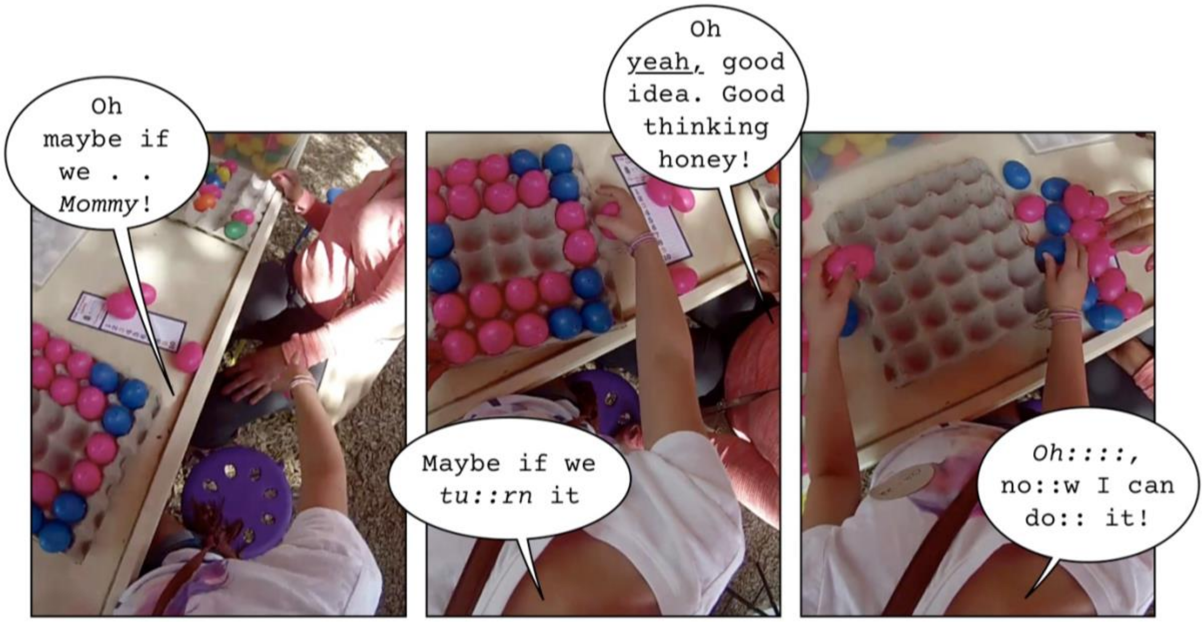
Table 2-4. *Summary of the third cycle of aesthetic practices.*

| Roles of the Aesthetic | Analytic Indicating Questions | Observed Evidence | Nature of Practices |
|---|---|---|---|
| G | What is possible here? | Articulated trouble ("Where's the middle") | Mathematical practices of *attending to precision, looking for and making use of structure,* and *using appropriate tools strategically* as she uses the egg crate as a grid to investigate the relationship between *actual* and *practical* middles of six |
| M | What action-steps should I take? Is this worth pursuing? | Her mother scaffolded and Olivia continued to pursue repair | |

**Fourth cycle of aesthetic practices: Final articulation of trouble**

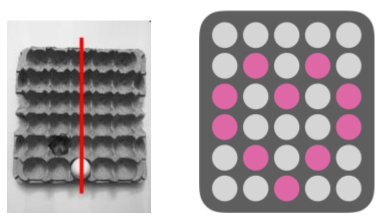
After her mother's scaffolding, Olivia began to investigate the crate, eventually moving her body to align with the side of the egg crate with five slots, grabbing the crate with both hands, and then rotating her body back to its original position, thus rotating the crate 90 degrees. Now, with the five slots parallel to her body, the egg could rest in the center. This rotation resulted in the crate being oriented such that its midline of symmetry and the practical middle for making a *heart with a point* became aligned.

Olivia made an urgent bid for her mother's attention, saying "Oh maybe if we // *Mommy::*" while grabbing her mother's arm to pull her body back into a position of joint attention (Figure 2-9a). While doing this, Olivia simultaneously rotated the crate back to its original position (six slots parallel to her body) so that her mother could witness the rotation and thus the change in the problem space (five slots parallel to her body, finding the practical middle of five instead of six). Once she had her mother's attention, Olivia demonstrated her discovery to her mother by re-enacting the rotation of the crate, saying "Maybe if we *tu::rn* it" (Figure 2- 9b). We count this as Olivia's final articulation of trouble, and we take her demonstration of the crate rotation to her mother as evidence that she now knew how to resolve her trouble: by aligning the actual and practical middles (Figure 2-10). While her previous articulations of trouble were expressed in the form of a statement ("It's crooked") and a question ("Where's the middle"), this articulation of trouble was in the form of a suggestion for action ("Maybe if we turn it"). The progression of Olivia's narration of trouble indicates a narrowing between what Olivia identified as problematic and her strategies for repair.




(a) (b) (c)

*Figure 2-9.* (a) Olivia works to regain her mother's attention. (b) Olivia shows her mother her discovery. (c) Olivia states that she no knows how to repair the trouble and make a symmetric heart with a point.



(a) (b)

*Figure 2-10.* (a) Practical midline symmetry and (b) ) a schematic diagram of Olivia's heart centered on both the actual and practical middles upcoming in Figure 11b.

Her mother then excitedly praised Olivia's ingenuity with a term of endearment while also recognizing Olivia's solution to the problem of a missing midline, saying "Oh *yeah*, good idea. Good thinking, honey!" (Figure 2-9b). Olivia articulated that the trouble had been repaired, saying "*Oh::::*, no::w I can do:: it!" as she took all of the eggs out of the carton, clearing the crate (Figure 2-9c). Here, before she had put even a single egg in the crate, Olivia again demonstrated that she knew she had generated a solution that would repair her trouble with the heart. In this moment, we see Olivia's aesthetic practices playing evaluative, motivational, and generative roles: She generated an articulation of trouble that points her to a solution worth pursuing and that would result in a satisfying and satisfactory heart with-a-point. Olivia proceeded to create her heart with a point unproblematically (Figure 2-11).

(a) (b) (c)

*Figure 2-11.* (a) Olivia places the point of her heart on the practical and actual midline of symmetry of the egg crate. (b) Olivia creates a symmetric heart with a point. (c) Olivia's finished heart.

***Summary of fourth cycle***

In this last episode of trouble-and-repair, Olivia fixed her crooked heart (Table 2-5). Through exploring what was possible with the crate, Olivia generated her final articulation of trouble ("Maybe if we turn it"). Upon receiving praise from her mother, Olivia was aesthetically motivated to make her heart with a point. Importantly, her statement of "*Oh::::,* no::w I can do:: it" before ever placing an egg in the crate indicates that Olivia was engaged in the mathematical practice of *reasoning abstractly and quantitatively*: Her multimodal interaction with the crate constituted an emergence of "simple" mathematical ideas in novel contexts, offering a deeper understanding about the relationship between the symmetry of her heart and the slots in the grid.

Table 2-5. *Summary of the fourth cycle of aesthetic practices.*

| Roles of the Aesthetic | Analytic Indicating Questions | Observed Evidence | Nature of Practices |
|---|---|---|---|
| G | What is possible here? What stands out as interesting? | Articulated trouble ("Maybe if we turn it") | Mathematical practices of *attending to precision, looking for and making use of structure,* and *using appropriate tools strategically* as she uses the egg crate as a grid to investigate the relationship between *actual* and *practical* middles of six |
| M | What action-steps should I take? Is this worth pursuing? | Olivia pursues remake of heart with a middle ("Ohhh, now I can do it!) | Mathematical practice of *reasoning abstractly and quantitatively* as her multimodal interaction with the create facilitated a strong intuition about the relationship between slots of the grid and the symmetry of her heart |
| E | Is this finished problem good enough, or the best that I can make it? | Evaluates the heart as good enough ("Done, mommy") | |

Upon completing her heart, Olivia proudly showed it to her mother. In fact, when her mother asked to take a picture of her with the heart, Olivia asked for another picture with only the heart. We find this request especially compelling, as we see it as evidence that she wanted to preserve the heart on its own, much like mathematicians remove their own voices from proofs to give their work an aesthetic of permanence and import.

## Summary: The Role of Aesthetic Practices in Olivia's Mathematical Engagement

Across these four cycles, Olivia engaged not only in making a heart during play at a mathematical playground but also meaningfully engaged in mathematical practices. These practices became personally meaningful to Olivia in relationship to her aesthetic practices. Specifically, Olivia's aesthetic practices (1) supported the *generation* of problem spaces and insights into the source of troubles, which, in turn, led her to engage in mathematical sense-making, (2) *motivated* her to pursue the problem and persist in investigating repair strategies informed by her articulations of trouble, and (3) grounded the way she *evaluated* the trouble and what counted as a good enough heart (see Tables 2-2 through 2-5).

Olivia's aesthetic, goal-centered play was mediated by the mathematics afforded by the egg crate. The crate's 6 x 5 design created trouble for Olivia as she attempted to make a heart with a point, which would be represented with a single egg. As she worked to repair this trouble, Olivia engaged in mathematical practices that facilitated sensemaking around relationships between symmetry and the middles of even and odd sides of the crate.

### Aesthetic Practices Support Engagement in Mathematical Practices

Olivia's aesthetic practices supported increasingly authentic engagement in mathematical practices. For example, in the second cycle, Olivia *looked for and made use of structure*, resulting in the generation of her first trouble-and-repair statement ("it's crooked”). In that same cycle, Olivia *attended to precision* as she evaluated her symmetric heart to be not precise enough and *made sense of problems and persevered in solving them* as she pursued her heart through further exploration of the egg crate. Throughout the four cycles, Olivia’s aesthetic practices recruited these practices in increasingly mathematical (and *not* decreasingly aesthetic) ways, as indicated by her repair strategies and the mathematical sensemaking they evidence (see Tables 2-2 through 2-5).

Importantly, the *generative* role of Olivia’s aesthetic practices emerged as she engaged in the mutually informative aesthetic and mathematical practices of exploration and articulation of trouble as she tacitly asked herself, *What if?* Her aesthetic practices guided each of her (re)formulations of the trouble in ways that generated increasingly productive potential solution paths. As indicated in Tables 2-2 through 2-5, the generative role of Olivia’s aesthetic practices was pivotal for continued activity, as the exploration and insight that define the generative role served as the entry point for further cycles of her aesthetic practices. Indeed, with each reformulation of the trouble, Olivia’s personal investment and interest in the aesthetics of the heart *motivated* her to pursue the solution paths suggested by her articulation of trouble, making mathematics sensemaking possible. Tables 2-2 through 2-5 indicate that as Olivia pursued repair, she also generated new articulations of trouble: She *noticed* new things about her heart (it’s crooked), *wondered* how to fix them (where’s the middle), and *experimented* (*what if* I turn the crate). We take this as an indication that opportunities to “try out” repair strategies that emerge from articulations of trouble during exploration in the generative role are pivotal to making sense of activity in a way that leads to motivation, evaluation, and then again to generating new ideas.

## Engagement in Mathematical Practices Facilitates Mathematical Sensemaking

Conceptually, Olivia's aesthetic provided a heuristic that facilitated her attention to mathematical properties of the egg crate — such as the presence or absence of a true middle — and relations between them. In other words, Olivia structured a new problem with new affordances for symmetry through her visible attention to aligning *actual middles* (i.e., midline symmetry is at the middle of six, Figure 2-8) with *practical middles* (i.e., midline symmetry is at the middle of five, Figure 2-10).

## Social and Material Scaffolding

Because Olivia persisted in solving her problem of making a heart, we were able to see how both her mother and the design of the exhibit supported Olivia's solution. The strategic choice of a 6 x 5 egg crate as a mathematical tool at MOAS provided both the opportunity for Olivia to notice the mathematical affordances and resolve her problem as she continued to explore the egg crate. After creating a symmetric heart that did not have one egg as a point, she eventually jointly rotated the crate and her body such that the odd side of the crate was parallel to her torso when standing in her original position. This rotation allowed her to align the actual midline of the crate as a grid with the practical midline of the crate as an array of slots for eggs. Olivia's gestures and other body movements were not just actions that gave rise to mathematical sensemaking; rather, her corporeal engagement constituted an important dimension of her mathematical sensemaking. Thus, the design of the materials — in conjunction with Olivia's aesthetic goal to create a heart that closely matched her prototype of a symmetric heart with a point — was a key element to the richness of this episode of making.

Furthermore, we note that her mother effectively scaffolded Olivia, taking up her questions while also allowing Olivia to maintain control of the decision-making process, including what counted as an aesthetically acceptable heart ("Done, Mommy!"). We posit that Olivia's agency in decision-making allowed her aesthetic practices to drive her engagement, creating opportunities for mathematical sensemaking through increasingly mathematical practices.

While we do not claim that Olivia's aesthetic understanding of the relationship between symmetry and the grid of the crate will transfer to other contexts, her sustained play at the *Egg and Crate* exhibit demonstrates how playful experiences with such mathematical tools can support meaningful engagement in mathematical practices and afford embodied experiences of core mathematical ideas.

## Conclusion

This study contributes to our understanding of how aesthetic practices can support children's meaningful engagement in mathematical practices during play. We illustrated how one child, Olivia, satisfied an aesthetic goal through the use of mathematical practices. In this way, Olivia's case offers an example of aesthetic practices driving mathematical inquiry. Specifically, as she worked to find the practical middle of her heart, Olivia persevered, made use of structure, attended to precision, and reasoned abstractly (CCSSI, 2010). Olivia's aesthetic practices were pivotal in this process as they influenced what she noticed as problematic (generative role), motivated her to pursue and persist in the problem (motivational role), and organized how she evaluated what counted as a satisfying and satisfactory heart (evaluative role).

By operationalizing the roles of mathematical aesthetics in this way, we have provided an illustration of how child-driven mathematical inquiry can emerge organically from activity. More specifically, our study adds to literature on both mathematical aesthetics and problem-posing by providing thick description of how exploration can lead to the posing of and persistence in increasingly sophisticated problems based on and driven by children's own interests and tastes. Building off of synthetic analyses of the aesthetic practices of mathematicians (Sinclair, 2004), research on children's mathematical aesthetics (Koichu, Katz, & Berman, 2017; Cipoletti & Wilson, 2004), ethnographic studies of children's inquiry across formal and informal contexts (Keifert & Stevens, 2018), and teaching experiments that support new modalities for mathematical sensemaking (Fiori & Selling, 2016; Ma, 2016), we argue that Olivia's activity provided her with experiences that could be leveraged for future mathematical learning opportunities, as evidenced by her sustained effort to repair the symmetry trouble she encountered in her design.

A key motivation for this study was to better understand the role of aesthetic practices in noticing and resolving inquiry-sparking experiences. We selected Olivia's case for close analysis as it offered a sustained and clear instance of aesthetic practices guiding sensemaking. While existing research has documented the richness of out-of-school inquiry (e.g., Keifert & Stevens, 2018; Kelton & Ma, 2018; Nemirovsky, Kelton, & Rhodehamel, 2013), our findings suggest that *aesthetic practices* are key in sparking and sustaining engagement in mathematical practices. In particular, our study suggests that informal mathematical environments that support design activity may provide meaningful contexts for children's mathematical practices to emerge in personally meaningful ways.

Even more, it suggests that creating opportunities for learners to engage in exploration may lead to the emergence of trouble in activity, and thus to the generation of mathematical dilemmas and sensemaking: As children make judgements about their own aesthetic goals, problems arise that require engaging in mathematical inquiry to resolve. In this way, aesthetic practices in play led to engagement in a mathematical practice in ways reflective of the anthropological meaning of practice — particularly the practice of problem-finding that spurs mathematical inquiry.

Notably, we do not claim that all aesthetic engagement or all trouble led to mathematical sensemaking. Indeed, in the larger corpus, we have examples that look more and less like Olivia's. A question arises, then, about how to design and foster the episodes that lead to mathematical sensemaking. Through our thick description of how aesthetic practices in children's play can lead to the emergence of mathematical practices during pursuit of aesthetic goals during trouble-and-repair sequences, we invite other investigators to explore these issues. By tracing the particulars of Olivia's activity and the particular role of aesthetic practices within it, we sought to advance the field's understanding of what it means for learners to engage their aesthetic practices for mathematical sensemaking and how this might provide a means to make mathematical practices personally meaningful.

This analysis also highlights how children's mathematical competencies might look in an out-of-school settings. By identifying the mathematical practices that emerged from in activity, we saw how critical Olivia's mathematical competencies — such as trying new approaches and reasoning systematically — were to her design problem and its resolution. Although our research did not allow us to follow up with Olivia or the other children at MOAS, we suspect that their informal mathematical encounters could support future formal concept development, perhaps giving them different experiential intuitions about issues such as symmetry and spatial reasoning.

From an instructional design perspective, Olivia's spontaneous and meaningful use of mathematical practices, like making use of structure and persevering in solving problems, is decidedly non-trivial: These are practices many educators hope to inculcate in their students. It is the hope behind the lists of practices posted on classroom walls. The emergence of mathematical practices in children's play at MOAS suggests that carefully designed mathematical playgrounds offer organic opportunities for mathematical practices to emerge. The challenge of formal instruction, of course, is figuring out how to support multiple children to identify and engage in these practices. But perhaps instead of incorporating play solely to serve curricular goals, play can contribute to students' positive mathematical identities and help teachers see broader forms of mathematical competence. Indeed, some teachers, inspired by MOAS, have incorporated Play Tables in their math classrooms. For example, Van Der Werf (2017) found that she "noticed brilliance in students that was hidden under fear of being wrong and/or lack of quick fact fluency. I noticed students working together I had no idea ever talked to one another."

Future research can help us better understand the relationship between informal child-driven mathematical play and formal learning. Certainly, as a field, we have a lot of work to do just to legitimize aesthetic engagement and the mathematical practices it supports as authentically mathematical. Indeed, at the end of her daughter's remarkable work on fixing the crooked heart, Olivia's mother asked a nearby MOAS volunteer if he could "give [Olivia] a math problem to do," suggesting that she had not viewed Olivia's design work as "real" mathematics. While our perspective pushes against deeply ingrained cultural scripts of what it means to learn and do mathematics (Stigler & Hiebert, 1999), our analysis illustrates the rich resources learners bring with them into the classroom that too often rendered invisible, because commonplace lenses on mathematics learning obscure people's ability to see them.

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## References


Ahearn, L. 2012. Living Language: An Introduction to Linguistic Anthropology. Malden MA: Wiley-Blackwell.

Boaler, J. (2002). Learning from Teaching: Exploring the Relationship between Reform Curriculum and Equity. Journal for Research in Mathematics Education, 33(4), 239–58.

Boaler, J., & Staples, M. E. (2008, March). Transforming students' lives through an equitable mathematics approach: The case of Railside School. *Teachers College Record, 110*, 608–645.

Bourdieu, P. (1977). The economics of linguistic exchanges. *Information (International Social Science Council), 16*(6), 645-668.

Brown, S. I., & Walter, M. I. (1983). The art of problem posing. Hillsdale, NJ: Lawrence Erlbaum Associates.

Brownell, W. A. (1946): The Measurement of Understanding. *45th Yearbook National Social Studies Education.* Chicago, IL: Chicago University Press.

Bush, S. B., Karp, K. S., Nadler, J., & Gibbons, K. (2016). Using artwork to explore proportional reasoning. *Mathematics Teaching in the Middle School, 22*(4), 216-223.

Bush, S. B., Karp, K. S., Bennett, V. M., Popelka, L., & Nadler, J. (2013). Framing measurement: An art gallery installation. *Mathematics Teaching in the Middle School, 18*(8), 474-483.

Carpenter, T. P., Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. *International Journal of Educational*

*Research,17*(5), pp. 457-470. ISSN 0883-0355, https://doi.org/10.1016/S0883-0355(05)80005-9.
(http://www.sciencedirect.com/science/article/pii/S0883035505800059)

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American educational research journal*, *26*(4), 499-531.

Cheng, E. (2015). *How to Bake Pi: An Edible Exploration of the Mathematics of Mathematics*. New York, NY: Basic Books.

Cipoletti, B., & Wilson, N. (2004). Turning Origami into the Language of Mathematics. *Mathematics Teaching in the Middle School, 10*(1), 26-31.

Civil, M. (2002). Chapter 4: Everyday Mathematics, Mathematicians' Mathematics, and School Mathematics: Can We Bring Them Together? In M. E. Brenner & J. N. Moschkovich (Eds.), *Everyday and Academic Mathematics in the Classroom*(pp. 40-62). Journal of Research in Mathematics Education monograph series. Reston, VA: National Council of Teachers of Mathematics.

Crespo, S., & Sinclair, N. (2008). What makes a problem mathematically interesting? Inviting prospective teachers to pose better problems. *Journal of Mathematics Teacher Education, 11*(5), 395-415.

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Davis, P. J. (1985). What do I know? A study of mathematical self-awareness. *The College Mathematics Journal*, *16*(1), 22-41.

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge, MA: Cambridge University Press.

Dewey, J. (1929). *The quest for certainty: A study of the relation of knowledge and action*. New York: Minton, Balch and Company.

Dewey, J. (1938). Experience and education. New York: Macmillan.

Doyle, W. (1988). Work in Mathematics Classes: The Context of Students' Thinking During Instruction. *Educational psychologist., 23*(2), 167–180. doi:10.1207/s15326985ep2302_6

Duckworth, E. (1972). The having of wonderful ideas. *Harvard Educational Review*, *42*(2), 217-231.

Erlwanger, S. H. (1973). Benny's conception of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior*, *1*(2), 7-26.

Fiori, N., & Selling, S. K. (2016). Truth isn't everything: Promoting aesthetically guided choice in mathematical work. *The Journal of Mathematical Behavior*, *41*, 219-234.

Geertz, C. (1973). *The Interpretation of Cultures*. *The interpretation of cultures*. New York, NY: Basic Books.

Goodwin, M. H. (2006). Participation, affect, and trajectory in family directive/response sequences. *Text & Talk-An Interdisciplinary Journal of Language, Discourse Communication Studies*, *26*(4-5), 515-543.

Hall, R. (1996). Representation as shared activity: Situated cognition and Dewey's cartography of experience. *The Journal of the Learning Sciences*, *5*(3), 209-238.

Harris, M. (1976). History and significance of the emic/etic distinction. *Annual Review of Anthropology*, *5*(1), 329-350.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., Olivier, A., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational Researcher, 25*(12), pp. 12-21.

Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *The Journal of the Learning Sciences*, *16*(1), 37-79.

Horn, I.S. (2008). Turnaround students in high school mathematics: Constructing identities of competence through mathematical worlds. *Mathematical Thinking and Learning, 10*(3), 201-239.

Horn, I.S. (2012, March). Teacher discourse and the construction of school mathematics. *For the Learning of Mathematics,* 4-6.

Horn, I.S. (2017). *Motivated: Designing Math Classrooms where Students Want to Join in.* Portsmouth, NH: Heinemann.

Jacobs, J. K., Hiebert, J., Givvin, K. B., Hollingsworth, H., Garnier, H., & Wearne, D. (2006). Does eighth-grade mathematics teaching in the United States align with the NCTM Standards? Results from the TIMSS 1995 and 1999 video studies. *Journal for Research in Mathematics Education, 37*(1), 5–32. doi:10.2307/30035050

Jordan, B., & Henderson, A. (1995) Interaction analysis: Foundations and practice. *Journal of the Learning Sciences, 4*(1), 39-103.

Keifert, D., & Stevens, R. (2018). Inquiry as a Members' Phenomenon: Young Children as Competent Inquirers. *Journal of the Learning Sciences*, 1-39.

Kelton, M. L., & Ma, J. Y. (2018). Reconfiguring mathematical settings and activity through multi-party, whole-body collaboration. *Educational Studies in*

*Mathematics, 98*(2), 177-196.

Koichu, B., Katz, E., & Berman, A. (2017). Stimulating student aesthetic response to mathematical problems by means of manipulating the extent of surprise. *The Journal of Mathematical Behavior, 46*, 42-57.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29-63.

Lave, J. E., Murtaugh, M, de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B.E. Rogoff & J. E. Lave (Eds.), *Everyday cognition: Its development in social context*. Boston, MA: Harvard University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation.* New York, NY: Cambridge University Press.

Litke, E. G. (2015). *The state of the gate: A description of instructional practice in algebra in five urban districts* (Doctoral dissertation). Retrieved from http://nrs.harvard.edu/urn-3:HUL.InstRepos:16461050

Lord, N. (2016). Constructing the 15th pentagon that tiles the plane. *The Mathematical Gazette, 100*(547), 154-158.

Ma, J. Y. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. *Journal of the Learning Sciences, 25*(3), 335-371.

Mamolo, A., & Zazkis, R. (2012). Stuck on convention: a story of derivative relationships. *Educational Studies in Mathematics, 81*(2), 161-177.

Minnesota State Fair. (n.d.). Math On-A-Stick. Retrieved from https://www.mnstatefair.org/location/math-on-a-stick/

National Council of Teachers of Mathematics. (1989*). Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education*, *44*(2), 372-415.

Pea, R., & Martin, L. (2010). Values that occasion and guide mathematics in the family. *Research on Learning as a Human Science. New York: Teachers College*.

Penrose, R. (1974). The role of aesthetic in pure and applied mathematical research. *The Institute of Mathematics and its Applications, 7/8*(10), 266–271.

Rancière, J. (2004). (G. Rockhill, Trans.). *The politics of aesthetics: The distribution of the sensible*. New York, NY: Continuum.

Rosenberg, E. A. (1968). Aesthetics in elementary mathematics. *The Arithmetic Teacher*, *15*(4), 333-336.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational psychologist.*, *23*(2), 145-166.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

Schön, D. A. (1992). The theory of inquiry: Dewey's legacy to education. *Curriculum*

*Inquiry*, *22*(2), 119-139.

Silver, E., & Metzger, W. (1989). Aesthetic influences on expert mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving*(pp. 59–74). New York: Springer-Verlag.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, *6*(3), pp. 261-284.

Sinclair, N. (2006). *Mathematics and beauty: Aesthetic approaches to teaching children.* Cambridge, MA: Teachers College Press.

Sinclair, N., & Bruce, C. D. (2014). Spatial reasoning for young learners. In *Proceedings of the Joint Meeting of PME* (Vol. 38, pp. 15-20).

Sinclair, N., & Watson, A. (2001). Wonder, the Rainbow and the Aesthetic practices of Rare Experiences. *For the Learning of Mathematics, 21*(3), 39-42.

Sinclair, N., Pimm D., & Higginson, W. (2006). *Mathematics and the aesthetic: New approaches to an ancient affinity*. New York, NY: Springer.

Singer, F. M., Ellerton, N., & Cai, J. (2013). Problem-posing research in mathematics education: New questions and directions. *Educational Studies in Mathematics*, *83*(1), 1-7.

Smith, M., & Stein, M. K. (2011). *Five practices for orchestrating productive mathematical discourse*. Reston, VA: National Council of Teachers of Mathematics.

Stevens, R., & Hall, R. (1998). Disciplined perception: Learning to see in technoscience. *Talking mathematics in school: Studies of teaching and learning*, 107-149.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York, NY: Simon and Schuster.

Van der Veen, J. (2012). Draw Your Physics Homework? Art as a Path to Understanding in Physics Teaching. *American Educational Research Journal*, *49*(2), 356-407.

Van Der Werf, S. (2017). Why you need a play table in your mathematics classroom asap. Accessed October 1, 2018 from https://saravanderwerf.com/2017/05/29/you-need-a-play-table-in-your-math-classroom/

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*(4), 375-400.

Wells, D. (1990). Are these the most beautiful?. *The Mathematical Intelligencer*, *12*(3), 37-41.

Wertsch, J. V. (1998). *Mind as action*. New York, NY: Oxford University Press.

Whitehead, A. N. (1929/1961). The Aims of Education. *New York, NY: New American Library*.

Yin, R. K. (2017). *Case study research and applications: Design and methods*. Thousand Oaks: Sage Publications.

Chapter 3

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# PLAYING WITH MATH: HYBRIDITY BETWEEN IN- AND OUT-OF-SCHOOL ACTIVITIES

Lara Jasien & Melissa Gresalfi

**Abstract**

This study examines how children hybridize out-of-school mathematics activities with school-mathematics. By situating our study at an out-of-school mathematics space designed to support children's playful engagement in patterning, we illustrate how children can invoke school mathematics frames in ways that allow traditional school-mathematics norms and practices to hybridize with open play. Specifically, taking a phenomenological lens on children's activity frames, we offer a multilayered analysis of participants' orientations to (a) authority during task determination and (b) emergent evaluations of their activity. We closely attend to two purposively sampled cases with multiple features that both overlap and contrast along the two lines of analysis. We share two cases of 12-year-old girls, Dia and Aimee, during mathematics play. Dia's case illustrates how a *traditional school mathematics frame*, with its norms and practices around authority and evaluation, hybridized with play, leading her to express uncertainty about and dissatisfaction with her activity. In contrast, Aimee's case illustrates how a *play frame* led to engagement in expansive exploration, feeling pleasure in activity, and satisfaction with what was produced. We suggest implications for designs of out-of-school mathematics

environments, offering conjectures for future research into how children's different mathematical identities might influence engagement in such spaces.

## Introduction

Over the last 20 years, research in education — and in STEM education in particular — has increasingly tried to foster meaningful engagement for all students by creating opportunities for learners to bring knowledge and practices from outside of school into school learning activities (e.g., Banks et al., 2007). Such designs for learning have been posited as important for supporting children to see school mathematics as relevant to their lives (Abreu & Cline, 2003) and to see their out-of-school practices and identities as relevant to their school learning (Gutiérrez, Baquedano-López & Tejeda, 1999). Thus, out-of-school experiences are potentially rich resources for supporting interest and learning in schools. However, while we know that out-of-school experiences can be valuable for in-school learning, we know much less about the role of in-school learning for out-of-school activities.

In this study, we are interested in understanding how the expectations children develop through experiences with *school mathematics* might influence their experiences in *out-of-school mathematics.* For example, do negative experiences with school mathematics lead children to engage in out-of-school mathematics activities differently than those who have positive experiences with mathematics? This connection is important to understand as there are significant current efforts towards creating opportunities for students to have non-school, informal, experiences with mathematics — for example in museums, through the Maker Movement, and through intentionally designed play activities like the *National Math Festival.* Although there are many possible justifications for the development of these, one dominant argument is that these informal, experiential, and enjoyable activities could potentially transform students' relationship with mathematics in enduring ways

(Banks et al., 2007; Petrich, Wilkinson, & Bevan, 2013; Quinn, & Bell, 2013). Indeed, there is reason to invest in out-of-school activities, as there is evidence that such experiences strongly influence self-selection into STEM fields. A retrospective study surveying STEM professionals found that 94 percent of their sample reported out-of-school experiences as important for their decisions to pursue a STEM career (Jones, Taylor, & Forrester, 2011).

However, we know little about the experiences people have in those spaces, especially in relation to their experiences in school. The causal direction is unclear: In the survey, did those who had positive experiences in out-of-school activities also enjoy school mathematics? Were those out-of-school experiences transformative, or simply broadening? Phenomenologically, we assume experiences connect across contexts in important and significant ways. For example, one study demonstrated that when children invoke school-based expectations of learning in museums, they often do not see their activity as meaningful for learning — unless they are given a worksheet to fill out during the museum visit (Griffin, 1994, 2004). This suggests not only that the practices of schooling are easily invoked in out-of-school spaces, but also, that those practices can be offered as a benchmark to make sense of content, engagement, and learning.

This research highlights the need to better understand how participants connect experiences across contexts. This paper contributes by building on the idea of *hybridizing* (Bakhtin, 1981), the process in which people integrate two or more cultural activities in a way that is more than the sum of their parts, resulting in the creation of new activities (Gutiérrez , Baquedano-López, & Tejeda, 1999). Research in schools has shown how hybridizing with out-of-school activities can be a mechanism for opening up learning activities (e.g., Calabrese-Barton & Tan, 2009; Ma, 2016), drawing on ways that children's lives are rich with meaning in ways that school often lacks (Taylor, 2009; Walkerdine, 1990). Such hybridizing fundamentally transforms the mathematics activity of a classroom, resulting in the creation of activity that is neither children's typical school mathematics activity nor their out-of-school mathematics activity, but rather is a new activity in service of mathematics learning (Ma, 2016). While we know that creating hybridity with out-of-school can support meaningful engagement in school mathematics, we know quite little about how children's activity in out-of-school spaces might benefit or suffer from hybridization with school mathematics.

We can imagine hybridization with school mathematics in out-of-school activities being either productive or unproductive based on children's prior experiences with mathematics. For example, if a child integrates school-mathematics expectations around *looking for and making use of structure* (Common Core State Standards Initiative [CCSSI], 2010) into their patterning work at a math exhibit outside of school, this could enrich the child's experience at the exhibit (e.g., Jasien & Horn, under review). In contrast, if a child integrates school-based mathematics expectations around producing quick-and-correct answers, this could lead to shallow engagement or even lead the child to struggle to enjoy participating in activities designed for open engagement. Decades of research demonstrate how such expectations are shaped through participation in the norms and practices of school mathematics (Boaler, 1999; Schoenfeld, 1988; Stigler, Hiebert, & Manaster, 1999). With this in mind, this paper considers the following question: *How do children hybridize out-of-school mathematics activities with school-mathematics norms and practices?*

## Conceptual Framework: Norms and Practices across In- and Out-of-School Mathematics Activities

Because we are interested in how children hybridize out-of-school mathematics activities with school-mathematics norms and practices, we aim to uncover relationships between typical mathematics experiences inside and outside of school. More specifically, we explore how the norms and practices of school might shape children's expectations for and engagement in out-of-school mathematics when children make school mathematics relevant to their activity. In this section, we first articulate how norms and practices organize expectations for activity. We then discuss what we know about the norms and practices of school mathematics and out-of-school mathematics.

### Norms and Practices Organize Expectations Within and Across Cultural Activities

Mathematics education research has documented the wide range of norms that can organize classroom mathematics and their consequences for student engagement. Some of that scholarship has distinguished between "reform" versus "traditional" mathematics practices, with the former emphasizing collaboration and communication — for example, through normalizing mistakes or investigating different approaches to solving problems — and the latter emphasizing attentive listening and rote practice of known procedures. Reform mathematics aims to disrupt the authoritative discourse of traditional school mathematics by endowing students with the authority to make sense of intellectual problems (Boaler & Staples, 2008). What is more, scholarship on learning and identity argues that these differences in norms shape the relationships students are likely to develop with school mathematics (Boaler & Greeno, 2000; Boaler & Staples, 2008; Cobb, Gresalfi,

& Hodge, 2009). For example, students in traditional mathematics classrooms frequently view mathematics as steps and procedures to follow and often disaffiliate with mathematics; students in reform classrooms tend to view mathematics as an endeavor of creative sensemaking and often affiliate with mathematics (Boaler & Greeno, 2000), thus developing more positive and agentic mathematical identities.

Research has also argued that mathematical practices are locally constituted and, indeed, can dramatically transform the nature of mathematical activity (Yackel & Cobb, 1996). Building on early ethnographic work conducted by anthropologists and cultural psychologists (Saxe, 1988; Lave, Murtaugh, & de la Rocha, 1984), we understand mathematical activity as a socially mediated process shaped by history, tools, and participation. This view implies that school mathematics is a fairly peculiar enterprise reflecting the norms and organizational needs of school as an institution (e.g., crowded classrooms, graded work, decisions for promotion and retention; see Schoenfeld, 1988). Nonetheless, this organization influences what people ultimately think about mathematics (Boaler & Greeno, 2000; Nasir, 2002; Schoenfeld, 1989), a human endeavor much older than the institution of schooling. If mathematical practices are locally constituted, then it follows that out-of-school mathematics practices may also influence the nature of students' mathematical knowing. Yet because schools, as institutions, have the power to certify and sanction certain forms of knowledge, people can demonstrate knowing in one context but be evaluated as not knowing in another (Nasir, 2002; Noss, Hoyles, & Pozzi, 2002; Saxe, 1988). Typically, schools' judgments are what is taken up in people's mathematical identities (Abreu & Cline, 2003; Gargoetzi, Horn, Chavez & Byun, under review).

This situation stems, in part, from the different social value put on in-school versus out-of-school learning. Additionally, productive connections are not always made between the two social contexts. For example, Carraher, Carraher, and Schliemann (1985) demonstrated how child street vendors could flawlessly carry out arithmetic computations during street vending but could not do so on schoolish written assessments. The children's enculturation into the norms and practices of school mathematics led them to (incorrectly) use school mathematics algorithms rather than their street vending arithmetic practices, likely cued by its mediating form, suggesting that the children expected their street vending arithmetic practices to be irrelevant in the context of the school mathematics problem. To date, we know little about the converse of this ethnographic research, namely how in-school mathematics practices — such as those around authority and evaluation — influence engagement in out-of-school mathematics activities.

However, prior work from museum studies offers insight into how the norms and practices of school can shape engagement in other contexts. First, designers of these learning environments often borrow school-structures to cue participants to the learning goals. Museum-based designs for organized learning can end up looking a lot like traditional classrooms, with learners primarily sitting in rows and listening to more knowledgeable others (Russell, Knutson, & Crowley 2013). This arrangement encourages children to function within traditional school-based norms of authority and evaluation. Second, participants' expectations often carry over school expectations to out-of-school settings. To expand on an earlier example, Griffin (1994, 2004) found that both teachers and students on science museum fieldtrips claimed that, to learn from a museum, students needed to fill out worksheets while there. Students claimed that the worksheets made the fieldtrip less fun because it constrained what they were able to look at, but they also stated that “just looking around” would not count as learning. In other words, students did not evaluate their activity as learning when there was not an authoritative tool guiding their engagement.

We have evidence, however, that these judgments about learning may be heavily cued by worksheets as an artifact. When Falk and Dierking (1992) asked students on a fieldtrip what they remembered, rather than what they learned, students responded about specific exhibits and displays in the museum that had interested them, even when they did not use a worksheet. Thus, as common sense would suggest, open engagement often leads to learning, but students’ school-based expectations around evaluation and authority preclude them from viewing their activity as such.

**Key Norms and Practices that Travel Across Contexts: Authority and Evaluation**

A theme across the research presented here is that school mathematics norms and practices around *authority* and *evaluation* are especially key for understanding how expectations rooted in school mathematics might influence children’s engagement in other, non-school mathematics contexts. Although we discussed norms and practices of both reform and traditional mathematics, most children are still taught the way their parents were taught (Jacobs, Lanza, Osgood, Eccles, & Wigfield, 2002; Litke, 2015), by learning definitions and practicing procedures. We focus our investigation into how the norms and practices of school mathematics can be hybridized in out-of-school mathematics activities by narrowing our attention to the norms and practices of *traditional* school mathematics, particularly those around authority and evaluation. Thus, we present a refinement of our research question: How do children hybridize out-of-school mathematics activities with *traditional* school-mathematics norms and practices around *authority to determine mathematical tasks* and *evaluating correctness*?

As previously discussed, traditional school mathematics norms and practices are organized around an authoritative discourse. As a cultural activity, traditional classrooms have a familiar organization that supports (and is supported by) these norms and practices. Teachers stand at the front of the room, asking known-answer questions (Mehan, 1979) to funnel students towards learning definitions and practicing procedures (Stigler & Hiebert, 1999). This discourse pattern values student compliance over sensemaking (Cazden, 2001), partially constituting a hidden curriculum (Jackson, 1968) that teaches children that they do not have the authority to pose or make sense of mathematical problems independently but must instead rely on the authority of teachers and textbooks (Lampert, 1990; Herbel-Eisenmann, 2007). Traditional school mathematics' organization also teaches children that demonstrating mathematical competence involves producing correct answers quickly and that evaluation of correctness is the primary goal of mathematics learning (Boaler, 1997). In the rest of this section, we present how authority and evaluation are constituted in school mathematics, comparing it with their constitution in out-of-school mathematics.

**Authority and Evaluation in Schools: Predetermined Tasks and Correctness**

First, to examine *authority*, we follow Engle and Conant (2002), who define student authority as the idea that

> tasks, teachers, and other members of the learning community generally encourage students to be authors and producers of knowledge, with ownership over it, rather than mere consumers of it (p. 405).

Notably, traditional mathematics classrooms grant students little authority to determine how to engage and determine what is right. Instead, authority is often distributed between school and teacher policies that prescribe what must be learned, restrict how to learn it, and constrain student behavior and bodies (Jackson, 1968; Louie, 2017). In other words, students are positioned as receivers and consumers of knowledge rather than as authors with ownership. Such over-ritualized, authority-lacking experience of mathematics can leave students feeling unsure of what to do when faced with a legitimate problem to solve (Schoenfeld, 1988). Even more, this issue is amplified when it comes to students' ability to pose their own mathematical problems, as school mathematics largely involves solving problems predetermined by the distant authorial voices of textbooks (Herbel-Eisenmann, 2007).

This version of authority contrasts starkly with authority in professional mathematics, as professional mathematics involves exploration, problem-posing, and constructing aesthetically appealing solution strategies (Lakatos, 1978; Pickering, 1995; Sinclair, 2003; Sinclair, Pimm, & Higginson, 2006). Indeed, part of Engle and Conant's (2002) call is to redistribute authority to students; this is a critical mode of facilitating productive disciplinary engagement, as well as disciplinary enjoyment (Sengupta-Irving & Enyedy, 2015).

We know that efforts to redistribute mathematical authority in line with this vision are non-trivial. After nearly thirty years of reform efforts (NCTM, 1989), the continued ubiquity of traditional norms of mathematical authority (Jacobs, Lanza, Osgood, Eccles, & Wigfield, 2002; Litke, 2015; Louie, 2017) has strengthened widespread narratives that more open learning formats are unfeasible (Stigler & Hiebert, 1998). Even teachers outwardly committed to equitable teaching sometimes inadvertently frame mathematics as a fixed body of knowledge and position students as deficient (Louie, 2017), invoking traditional norms of mathematical authority. This framing unintentionally precludes students' meaningful engagement with mathematics and their development of positive mathematical identities.

Second, to examine *evaluation* in schools, we look at positive and negative labelling of what students produce as well as interactional positioning of students as either failing or succeeding. This kind of labelling, which is a subset of but not limited to assessment practices, includes informal labels that students integrate into their own identities (Anderson, 2009; Gargoetzi et al, under review; Horn, 2007; Horn 2008) as well as more formal, institutionalized labels that lead others to view them as particular kinds of people (McDermott, 2001). For many students — including high achieving students — traditional evaluation practices can lead to cheating or even to disaffiliation with mathematics (Boaler & Greeno, 2000; Pope, 2001). If people make the evaluation norms and practices of *traditional* school mathematics relevant in out-of-school mathematics environments, engagement may become less productive than if norms and practices around open exploration are invoked, especially for those who already dislike mathematics.

**Authority and Evaluation in Out-of-School Mathematics: Open Engagement and Expansive Outcomes**

Most prior research on engagement in out-of-school in STEM emphasizes science, technology, and engineering rather than explicitly mathematics engagement. The lack of research on out-of-school mathematics is likely due to the scarcity of such spaces. We find this problematic: Because school mathematics is a widely disliked and a gatekeeping subject, providing students opportunities to engage in mathematics in ways that are personally meaningful seems like a potentially useful way to broaden field-level participation. While this study does not involve designing such spaces, our investigation into children's mathematical activity therein and its relationship to school mathematics seems an important first step toward this goal.

From prior work on out-of-school STEM spaces, we see that norms and practices around authority and evaluation differ from those in schools and, in many ways, support meaningful engagement. Such spaces are largely designed to be self-directed, engaging, and fun. Indeed, out-of-school STEM spaces such as interactive museums are commonly referred to as *free-choice learning environments* (e.g., Ballantyne, & Packer, 2005; Bamberger & Tal, 2007; Falk, 2004; Falk & Dierking, 2002; Falk, Storksdieck, & Dierking, 2007), a label that starkly contrasts the presumed norms and practices for learning in school.

In relation to *authority,* free-choice environments, as their name suggests, often involve participants' open-engagement where they determine the goals of their activity and how to pursue them. Some out-of-school STEM learning research has attempted to deepen inquiry by semi-structuring participants' goals, but this can lead to "killing the playfulness" that is thought to be one of the primary learning benefits of these spaces (Honey & Hilton, 2011; Falk & Dierking, 1992, 2013; Little, Wimer & Weiss, 2008; National Research Council [NRC], 2009). For example, Gutwill and Allen (2012) designed a "juicy question" participation structure in which students on a fieldtrip first explored museum exhibits, individually posed questions about the exhibit, and then collaboratively selected a question to jointly pursue. However, they found that students had difficulty posing questions because, from the teacher's perspective, students' desire to play was an obstacle to question posing. While Gutwill and Allen's interventions succeeded in supporting a number of desirable learning behaviors — students posed more questions, interpreted the results of their inquiry, and produced more collaborative explanations — some students and chaperones complained that the interventions killed the interest-driven nature of activity that trips to the museum usually supported. Thus, limiting children's authority, even in ways that only partially structure activity to be more like school, can dramatically shape their experiences.

In relation to norms and practices around *evaluation,* free-choice environments support many learning outcomes, like enjoyment of and aesthetic appreciation for the discipline (Schauble, Gleason, Lehrer, Bartlett, Petrosino, Allen, & Street, 2002). Activity in these spaces does not involve any kind of external evaluations: Visitors self-evaluate based on whether or not they are satisfied with their activity and whether or not they met their own goals. This can lead visitors to engage shallowly with the museum without exploring the full potential of any exhibits. This very phenomenon prompted Gutwill and Allen's (2012) study on deepening inquiry in museums. However, as stated in the introduction, out-of-school experiences can support long-term interest and the pursuit of STEM careers (Jones, Taylor, & Forrester, 2011). Thus, evaluation norms and practices in free-choice environments partially support the interest development that fodders the pursuit of STEM careers.

Implied in the above descriptions of authority and evaluation in free-choice environments is a description of engagement in such places as quite *playful.* In play, children tacitly orient to the questions “What does this object do?” and “What can I do with this object” (Hutt, 1966; Pelligrini, 2009). Scholarship on play points to the importance of *exploration* (indicated by an orientation to *what materials can do*) as a precursor to *play* (indicated by an orientation to *what they can do with materials*). Within the diverse literature on play, generally accepted features include active rather than passive engagement, intrinsic motivation, self-selection and choice in process, enjoyment, intense focus or a state of flow (e.g., Nakamura, J., & Csikszentmihalyi, 2014), self-perpetuation in activity, and empowerment (Jerret, 2015, citing Klugman & Fasoli, 1995).

A common theme in all these play descriptors is high-levels of child *authority* and a prioritization of process over product (i.e., little emphasis on *evaluation*). Indeed, first and foremost, play is a child-defined activity. Wing (1995) reported how a child described herself as *playing* with sand in her classroom when she was allowed to determine her own goals for interacting with the sand, but described herself as *working* when the teacher positioned her activity with the sand to be about estimating, even though her physical manipulation of the sand looked the same to an observer in both situations. Thus, the norms and practices of play are often quite the opposite from the norms and practices of traditional school math, in particular around lines of authority and evaluation.

In sum, authority and evaluation norms and practices in free-choice environments are inherently more expansive than those of traditional school mathematics. It thus seems reasonable to conjecture that when participants in free-choice environments hybridize their play with *traditional* school mathematics norms around *authority* and *evaluation,* the outcomes may be less positive than when hybridity is created in the other direction.

## Theoretical Framework: Hybridizing Activity through Framing

Given the potential of out-of-school mathematics learning to sustain learners' persistence in the subject, we seek to investigate the relationship between in- and out-of-school mathematics learning more closely. Because norms and practices around authority and evaluation highlight the potential conflicts in these activities, we hone in on how children make sense of these in out-of-school spaces. In particular, to understand how children hybridize these norms and practices from in- and out-of-school, we draw on two separate but related ideas: sociocultural theories of hybridity, and frame theory. As we briefly explained in the introduction and will explain in more detail here, *hybridizing* is the act of individuals or groups integrating norms and practices (including language practices, etc.) from one cultural activity into another cultural activity, resulting in a new activity that is different from either (Bakhtin, 1981; Calabrese-Barton & Tan, 2009; Gutiérrez et al., 1999; Ma, 2016). *Framing* is the way people make cultural activities relevant in local interactions (Goffman, 1974), and thus is a way to trace the construction of hybridity. When multiple frames are evoked in a way that leads to a changed, new activity, then hybridity has been created (Figure 3-1). In what follows, we review the concept of frames, and then use it to clarify the idea of hybridity as we use it in this piece.

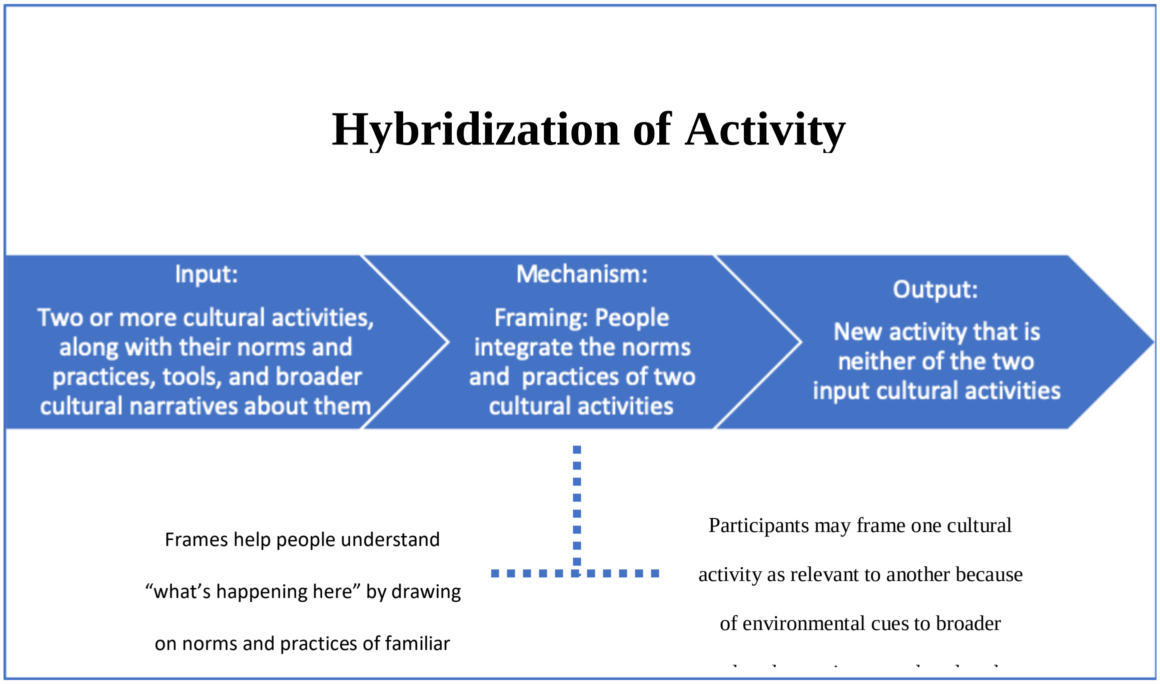


*Figure 3-1*. Relationship between hybridizing and framing.

Frames help people make sense of their activity by cuing the norms and practices of familiar cultural activities and thus help people organize activity in ways that make it recognizable to themselves and others (Goffman, 1974). In other words, frames help us understand when it is "desirable, appropriate, or at least socially acceptable" to make something from one cultural activity relevant in another (Engle, 2006, p. 455); it centers questions about what participants recognize themselves to be doing (Goffman, 1974) and has implications for what participants see as relevant to their activity. Importantly, framing is not necessarily a conscious activity, but rather is emergent in interaction between people and their social and material environment.

Indeed, frames are almost never stated explicitly in interaction, but we can still see them analytically. Frames are most analytically visible when they are called into question or when participants' expectations are broken. For example, Hammer, Elby, Scherr, and Redish (2005) describe a group of students questioning their joint endeavor by making bids for reasonable but opposing ways of engaging with a physics problem (kinesthetic sensemaking versus quantitative analysis). This example involves explicit questioning of the problem-solving frame: The statement "*Do we* even *need to* do all that calculation?" (***emphasis*** added) indicates that the bid for quantitative analysis might be excessive and *uncalled for* (p. 12). Participants engaged in a joint endeavor can also need to re-negotiate frames when expectations are unexpectedly broken. For example, if two cohabiting adults go to the grocery store, with one of them thinking it is a quick trip to grab a few things for dinner and the other thinking it is a trip to spend quality time exploring new foods and finding good deals, they will likely realize at some point — cued by differences in pace, path, and discussion — that their frames are misaligned and need renegotiation. In this way, frame negotiation happens when participants' intentionally or unintentionally challenge each other's understanding of "what's happening here." In sum, although frames are rarely stated, they can be seen in interaction at points of renegotiation and confusion, which are particularly visible as "breaks" in the fairly stable interactions involved in widespread cultural activities.

We propose analyzing frames helps us see how hybridization takes place, with the caveat that a multiplicity of frames does not necessarily result in hybridization. For example, when watching a film such as *Waiting for Superman,* people may be engaged in leisure frames while simultaneously drawing on frames about what happens in schools using the culture-of-poverty thesis (Bulman, 2002). In this example, multiple frames are evoked, but the film viewers are not making sense of their current movie watching activity (their leisure frame) by drawing on other frames. Rather, the film viewers are drawing on frames about schooling as a cultural activity to make sense of the movie, not of their own leisure activity. Although multiple frames are invoked,[6] they are not in conflict with each other. Thus, this is a non-example of hybridity, because the norms and practices of multiple cultural activities are not integrated in a way that leads to new activity.

In contrast, hybridizing takes place when multiple frames of cultural activities are integrated in ways that lead to novel patterns of talk and interaction. For example, when children in Gutiérrez and colleagues' study (1999) engaged in a lesson about human reproduction, they leveraged both their school-science discourse as well as their home discourse to make sense of new ideas, which expanded their patterns of talk in ways that include not just school discourse plus home discourse, but a merged discourse geared towards learning about teacher sanctioned content (Gutiérrez et al., 1999). This merged, hybrid discourse included informal anatomy terms as part of the official discourse of the classroom. Most adults do not expect to walk into a science classroom and hear informal language about human anatomy, especially from the teacher. Students' initial anxious laughter and hesitancy in participation suggests that their expectations were also broken, indicating that new norms were being negotiated — and thus a new activity was being created — as the two discourses were integrated.

The complexity of looking at multiple frames as indicators of hybridity continues.

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[6] We are using hybridize as a verb that children do, so we have chosen to use the verb "invoke" in relation to frames and norms and practices. However, it this is somewhat misleading, because framing it is often not conscious / intentional and is often cued by the environment. However, framing does not happen without people as the actors, and so we have chosen to use this term to put the emphasis on the role of people.

Just as multiple frames do not always yield hybridizing, not all hybridizing leads to productive new activity. For this reason, scholars have introduced terms such *third space* (Gutiérrez et al., 1999) and *productive hybridity* (Ma, 2016) to distinguish hybridity that supports desired learning from other hybridity. For example, using a worksheet to invite students to hybridize their activity in a museum with school science may not be desirable if it forecloses opportunities for students' interest to be sparked. Because we are specifically interested in how children hybridize out-of-school activities with *traditional* school mathematics norms and practices around *authority* and *evaluation*, it is unlikely that we will analytically uncover such productive hybridization.

Furthermore, it is easier to see hybridization of traditional school activities by out-of-school activities than vice versa (i.e., it is easier to see hybridity in school than outside of school). This is because of the relative stability of classroom norms and practices. When these fairly stable norms and practices are disrupted, we can often see hybridity being constructed as teachers and students recruit resources from other contexts in an attempt to continue on with activity (Ma, 2016). However, out-of-school activities often have a broader norms and practices — so broad in fact that it can be difficult to determine when they are disrupted. This is magnified in novel out-of-school activities.

In relation to mathematics, novel activities that disrupt both typical processes of engaging with mathematics and broader cultural narratives around what counts as mathematics invite hybridity (Ma, 2016). This invitation for hybridity is caused by experiences of uncertainty as participants' understanding of activity is disrupted. In considering how uncertainty functions in the invocation of framing, Goffman (1974) stated that:

> It is perfectly possible for individuals, especially one at a time and briefly, to be in doubt about what it is that is going on […] And insofar as the individual is moved to engage in action of some kind — a very usual possibility — the ambiguity will be translated into felt uncertainty and hesitancy. Note, ambiguity as here defined is itself of two kinds: one, where there is question as to *what could possibly be going on here*; the other as to *which one of two or more clearly possible things is going on* (pp. 302, 303, emphasis added).

In this way, novel activities that have less established — and thus, more ambiguous — norms and practices may facilitate hybridity since participants must recruit their resources from familiar cultural activities to resolve their uncertainty in how to engage. We refer to such activities as *hybridity-inviting*, and we consider this recruiting of resources to be a process *framing.*

How do we see framing in hybridity-inviting activities — activities in which participants may still be searching for what expectations are reasonable? In this case, we must look for evidence of frames that we think participants are likely to find relevant to their activity based on features of the social and material environment. As we are interested in framing around *traditional* school math, and particularly in norms and practices around *authority* and *evaluation*, we next describe how we might see these frames during playful activity in novel, non-school mathematics activities.

**Authority**

One way to see framing around traditional school mathematics' norms of authority in novel, non-school activities is through verbal orienting work. For example, participants' explicit talk or questioning about what they are *supposed* to do can indicate a schoolish orientation towards tasks as necessarily (almost definitionally) determined by authoritative others. Lack of such questioning or answering such questions is an indication of self-determining tasks and thus also indicates an internal locus of authority (an exercise of personal agency), which does not correspond with traditional school mathematics norms around authority. Orienting work can also be seen by attending to physical interaction with materials. For example, when participants explore materials before beginning to use them, this can be seen as an exercise of agency that does not well align with traditional school math authority norms, as those norms would suggest that participants should be able to quickly identify and solve problems.

**Evaluations**

Participants' verbal evaluations of their activity also support an analysis of framing, as evaluations provide clues as to what participants perceive themselves to be up to and how successful they viewed their engagement to be. In particular, attending to the nature of evaluations, such as whether they are positive or negative can give insight into whether participants have met their own expectations for activity. If participants repeatedly evaluate their work, especially negatively, this may indicate a schoolish emphasis on correctness. This, in conjunction with orienting to authority, may indicate that a participant is invoking a frame around traditional school mathematics.

## Research Design

### Setting

In the summer of 2016, we went to a mathematical playground called Math On-A-Stick (MOAS), located at the Minnesota State Fair (Figure 3-2). In the United States, state fairs happen once a year in each state for approximately 1–2 weeks. They are places of leisure and entertainment, with thrilling roller coaster rides, indulgent foods, and games with prizes. The Minnesota State Fair is the second biggest fair in the United States, and it is a very busy place. MOAS was tucked away in a quiet corner of the fairgrounds near educational booths and buildings. The soundscape at MOAS contributed to its pleasant, child-friendly atmosphere: sounds of birds chirping and frogs croaking floated in the air from the nearby EcoExperience building (visible in Figure 3-2), and cheerful children's songs punctuated the soundscape from shows performed at the nearby stage (visible in the same block as MOAS in Figure 3-2). Finally, MOAS offered a rare respite at the state fair: trees for shade and plentiful seating. Thus, MOAS was an oasis for families that needed a break from the hustle and bustle of the fairgrounds.


*Figure 3-2.* Minnesota State Fair Map (left) with a zoom in of the MOAS area via Google Maps.

Despite its mathematical label, the materials at MOAS look quite different than typical, school-based mathematics materials (e.g., paper, pencil, ruler, calculator). Some examples of the exhibits included:

- *Tiles and Patterns* exhibit (Figure 3-3 a, b) and *Pentagons* exhibit (Figure 3-3 c, d): shapes that tile the plane without necessarily requiring or producing symmetry and that tile plane in many ways, including ways that pattern.

- *Pattern Machine* exhibit (Figure 3-4 a, b) and *Egg and Crate* exhibit (Figure 3-4 c, d): grid-like materials that embody arrays (with different affordances for mathematical ideas based on whether the array is square or rectangle, with even or odd number of grid spaces).

(a) (b) (c) (d)

*Figure 3-3.* (a) A "pleasing-to-the-eye" pattern (b) and mono-color pattern at the *Tiles and Patterns* exhibit. (c) A mosasaur representation (d) and a multi-color design with radial symmetry at the *Pentagons* exhibit.

(a) (b) (c) (d)

*Figure 3-4.* (a) Design with rotational symmetry or snowflake representation (b) and heart representation at the *Pattern Machine* exhibit. (c) A heart representation and (d) a diagonal-stripes at the *Eggs and Crate* exhibit.

MOAS offered visitors little structure in terms of instructions. Materials were out on tables, and volunteers were often (but not always) stationed to offer ideas. Sometimes visitors left their patterns or designs out when they departed, and thus sometimes when a new visitor approached an exhibit they would see a design that someone else made. Other times tools were scattered randomly or neatly stored in containers. Approaching a table, visitors usually decided for themselves what they wanted to do. Typically, visitors might spend their entire time at MOAS designing and playing as they wished; mostly, the volunteers helped if asked or offered suggestions if pressed. With such an open participation structure, visitors framed their activities in many different ways.

Notably, even though MOAS was designed to encourage play, not all fairgoers found it equally inviting. Our fieldnotes capture this poignantly. Sitting on the sidewalk of MOAS watching passersby, we heard them rejecting MOAS with statements like, "Mathematics at the fair! No thank you!" or "Math? Ugh, keep walking." Of course, many fairgoers loved the idea and came in to explore. Despite this clear indication of some significant self-selection based on (dis)affiliation with mathematics, our study includes participants who claimed that mathematics was their least favorite school subject.

Most importantly for our analysis, MOAS is a hybridity-inviting space. As its name suggests, it is a place to play with mathematics. Thus, it possible for multiple — sometimes contradictory — frames to be relevant. The openness of the exhibits, the availability of materials, and the dominance of child activity over adult activity all make it possible that a play frame will be invoked into a hanging-out-at-the-fair or leisure frame.

Based on our description of free-choice environments in the conceptual framework, we would expect children at MOAS engaged in a play frame to rely on their own authority around what to do and how to do it, and we would expect them to focus on the process of making rather than on evaluating either their activity or their finished products. At the same time, the name of the space (*Math* On-A-Stick) makes it possible that a school mathematics frame will be invoked. Based on the description of traditional school math that we provided in the conceptual framework, we would expect children invoking a traditional school mathematics frame at MOAS to seek out the authority of others around what to do and how to do it, and we would expect them to struggle with self-evaluating the quality of their own work.

Thus, it is not surprising that fairgoers often need some assistance to engage: Because the novelty of the activity at MOAS makes important norms and practices unclear and potentially unshared by participants, it is not always clear which frames (e.g., school mathematics or play) to invoke. Based on personal histories and local interactions, MOAS visitors have to decide what cultural activities — and their associated norms and practices — are and are not relevant.

**Data Collection**

For this analysis, our primary interest is to understand how traditional school-mathematics norms and practices around authority and evaluation hybridize in free-choice mathematics environments. To recruit participants, we asked children and their parents if they would like to participate in a study of how children engage with mathematics through play. Because we are interested in participants' perspectives, our primary data source is line-of-sight video. We also collected an intake survey to capture demographic information, an exit survey to capture experiences with school mathematics, and a brief semi-structured exit interview to capture participants' immediate reflection on their experiences at MOAS.

Each participant was given a record locator number that identified the day of the month and the child's place in our queue of participants. For example, 05-25 would indicate the child was our 25th participant on the fifth of the month. This preserved participants' anonymity, but also allowed us to retrospectively identify children who participated with friends and siblings since their record locators were sequential and they often appeared in each other's recordings.

**Video data**

For the line-of-sight video, we mounted GoPro™ cameras on baseball caps aimed downwards and slightly forward to capture their talk, gestures, and object manipulation (Figure 3-5). This view captures the locus of the children's visual attention when they are playing with objects and supports inferences about their gaze and interest in relation to talk. In the end, we collected a total of 127 hours of video from 345 participants, with the average visit lasting 26 minutes per child (*sd* = 16.46 min).

*Figure 3-5.* A participant wearing the GoPro™ camera.

**Intake survey**

As one researcher finished the consent process and set the GoPro™ camera on the child's head, another collected an "About me" survey on an iPad (see Appendix A), asking about their favorite and least favorite school subjects, how they like to spend their leisure time, along with basic demographic information. The majority of the children in our sample were between 7 and 12 years old ($n$ = 274) and our analyses focus on this age band.

**Exit survey and interview**

As participants exited MOAS, we conducted and recorded brief exit interviews (Appendix B) before we removed the GoPro™ cameras from their heads. We asked about their experiences at MOAS, such as why they came to MOAS and which exhibits they liked most and why. They also completed a second brief iPad survey (Appendix C) with twelve statements about school and mathematics, with 5-point forced-choice Likert scale responses.

## Analysis

### Phase 1: Coding the corpus

To get a handle on our large data corpus, we viewed the entirety of all 345 videos and used Studiocode™ software to code for when participants were at each exhibit. Of the 127 hours of video we collected, 101 hours were of participants within our age band ($n$ = 279 children) and 21 hours were of participants outside of our age band ($n$ = 66 children) at exhibits. The remaining 5 hours of video contained records of participants engaged in transitional activities, such as checking in with their parents or eating snacks. This initial viewing allowed us to gain familiarity with the data and garner a sense of typical activity at each exhibit.

### Phase 2: Data reduction

Following Gutwill and Allen (2012) in their study of inquiry in museums, we initially characterized engagement by looking to the duration of stay times at each exhibit. We used the median length of stay time for the corpus as an indication of typical engagement (Table 1), and we then compared the duration of each child's stay time at each exhibit to typical stay time to categorize as a measure their engagement as more or less engaged. We chose the median rather than the mean as a measure of *typical engagement* because we had multiple occurrences of outliers with significantly longer times that skewed the data.

Our initial viewing of the data suggested that analyzing children who stayed longer at exhibits would offer a richer understanding of the potential for *mathematical* engagement. Thus, we selected participants with long stay times, which resulted in re-watching 311 video clips (43 hours) of child level occurrences of play at particular exhibits (Table 3-1).[7]

Table 3-1. *Statistical summary comparing all 7–12 yr old participants in sample age band*

| | **Stay times (in minutes) per station for all 7 to 12 year-old participants** | | | | **Stay times (in minutes) at selected exhibits of 7-12 year-olds who stayed longer than typical** | | | |
|---|---|---|---|---|---|---|---|---|
| Exhibit | *n* | *M* | *Sd* | *Mdn* | *n** | *Min* | *Max* | *Mdn* |
| Tiles and Patterns | 180 | 3.49 | 3.82 | 2.15 | 69 | 3.01 | 22.67 | 4.91 |
| Pentagons | 156 | 6.38 | 7.67 | 4.07 | 70 | 5.14 | 51.72 | 7.71 |
| Pattern Machine | 237 | 5.35 | 4.58 | 4.32 | 104 | 5.01 | 34.77 | 7.63 |
| Eggs | 156 | 3.38 | 3.19 | 2.76 | 68 | 3.00 | 26.61 | 4.58 |
| Lizards and Turtles | 183 | 5.89 | 4.45 | 4.95 | — | — | — | — |
| Spiral Machine | 90 | 4.16 | 3.78 | 3.26 | — | — | — | — |

*Tiles and Patterns (rejected 34 participants whose duration was between 2.0 and 2.99 because *mdn* = 2.15); Pentagons (rejected 9 whose duration was between 4.0 and 4.99 because *mdn*= 4.07); Pattern Machine (rejected 24 whose duration was between 4.0 and 4.99 because *mdn*= 4.32); Eggs (rejected 37 whose duration was between 2.0 and 2.99 because *mdn* = 2.76)

[7] We term these exhibit visits *child level occurrences* to distinguish that some participants attended stations more than once or stayed longer than typical at multiple exhibits, and thus are counted multiple times in our 311 video clips (63 participants stayed longer than typical at more than one exhibit). In our parsing of the data, we have each separate occurrence of participants' time at each exhibit counted as a unique occurrence. Thus, the count for "child level occurrences" does not reflect the number of children but rather the number of times the station was visited for longer than the roof of the median of exhibit stay time for 7-12 year olds.

### Phase 3: Analyzing patterns in engagement

Centering children's perspectives in video records lends itself to interaction analysis methods, since this approach seeks to understand participants meaning in activity (Hall & Stevens, 2016; Jordan & Henderson, 1995). To analyze participants' meaning making, interaction analysis methods begin by determining appropriate analytic chunks of the video. In this analysis, our analytic chunks, or units of analysis, are *Episodes of Making* (EOMs), which center participants' interaction with the materials. An EOM, as we define it, is an instance of activity that is oriented towards a particular goal *or* end-state; the making of "something." To emically define EOMs, we identified beginnings and endings of activity from participants' perspectives. According to Jordan and Henderson (1995), these "starting up" and "winding down" segments of interaction tend to be the temporal location of significant events. Delineating the boundaries of EOMs can be difficult due to fuzzy boundaries of goal shifting in play, so ultimately, we specified the endpoint of each EOM by attending to when participants "erased" their prior work (i.e., making themselves a "blank slate" by destroying what they built) or made major revisions that resulted in a new object. We then rewound the video to determine when that activity had begun (often right after the end of the previous EOM). If the child made a blank slate and re-attempted their previous goal, we rewound the video and coded the entire instance as one goal.

### Phase 4: Selecting focal participants for contrasting engagement

While coding for EOMs, we noticed that two of our participants were navigating MOAS together. Because their record locators were sequential (i.e., they were friends consented into the study at the same time), we coded their videos consecutively and noticed that while they were moving through MOAS *together*, they were also "doing it" very *differently*. In other words, they seemed to engage different norms and practices — particularly around authority and evaluation — in their activity even as their activity was constantly visible to each other. We call our two participants Aimee and Dia, 12-year-old girls who worked as friends. Looking into their cases more deeply, we found that they were our only two participants who stayed at all four selected exhibits for longer than was typical, making them our two "most engaged" participants (following Gutwill & Allen, 2012). Although they were our two "most engaged" participants by this measure, by other measures their engagement was also fairly representative. For example, their stay times at each were longer than the median stay time for all 7-to-12 year-olds, yet their stay times were usually closer to the median of all 7-to-12 year-olds than to the median of 7-to-12 year-olds who stayed longer than typical (Table 3-2). As a note, 62 participants between the ages of 7 and 12 stayed longer than typical at more than one exhibit, with 13 of those 62 participants staying longer than typical at three exhibits, 45 participants staying longer than typical at 2 exhibits, and two participants staying longer than typical at only one exhibit. Their extended engagement, friendship, and co-navigation of MOAS offered a rich narrative record of their activity, fodder for interaction analysis.

Table 3-2. *Aimee and Dia's stay times at selected exhibits.*

| | Stay times (in minutes) per station for all 7 to 12 year-old participants | | Stay times (in minutes) at selected exhibits for 7-12 year-olds who stayed longer than typical | | Aimee and Dia's stay times | |
|---|---|---|---|---|---|---|
| Exhibit | *n* | *Mdn* | *n** | *Mdn** | *Aimee* | *Dia* |
| Tiles and Patterns | 180 | 2.15 | 69 | 4.91 | 6.54 | 6.45 |
| Pentagons | 156 | 4.07 | 70 | 7.71 | 5.34 | 5.86 |
| Pattern Machine | 237 | 4.32 | 104 | 7.63 | 6.42 | 6.51 |
| Eggs | 156 | 2.76 | 68 | 4.58 | 5.03 | 3.53 |
| Lizards and Turtles | 183 | 4.95 | — | — | — | — |
| Spiral Machine | 90 | 3.26 | — | — | — | — |

Adding to our interest, Aimee and Dia professed contrasting experiences of mathematics in their entrance surveys, with Aimee claiming mathematics as her favorite subject, and Dia claiming mathematics as her least favorite subject. In fact, Dia seemed to think that her dislike of mathematics was either funny or awkward in the context of our study of *mathematics* activity at *Math* On-A-Stick, as she laughed uncomfortably when answering the intake survey question (asked aloud by a researcher) about her least favorite subject.

However, Dia and Aimee had somewhat similar opinions about mathematics as they experienced it in school, as indicated by our short Likert scale exit survey.[8] They both

[8] Notably, our video records cut off before the girls begin their exit surveys so it is possible that they copied one another's answers, but we find this unlikely as other questions on the survey have non-matching responses, as well as because our presence collecting the data leads us to believe that such copying of responses was uncommon.

thought that mathematics is moderately challenging (*sometimes*), that it consists of steps (*usually*) and facts to remember (*sometimes*), and that they needed their teacher to tell them whether their answers are correct or not (*sometimes*). They both also responded that they had the agency to figure out whether or not their answers are correct on their own. Interestingly, Dia — whose least favorite subject is mathematics — provided a slightly higher agency response (*strongly agree*) than Aimee (*agree*).

Their survey responses differed in relation to whether they found mathematics interesting or boring, with Dia finding mathematics very boring (*strongly agree*) and Aimee finding it very interesting (*strongly agree*). In this way, we interpret Dia's dislike for mathematics not to be related to her feelings of competence or agency, but rather to her feelings about whether or not mathematics is interesting. Yet, as we will show in the findings, Dia positioned herself as failing to produce interesting EOMs at MOAS — that is, as failing to find interesting problems — and thus also evaluated herself as not fully competent in this out-of-school context. Analyzing these two participants allows us to add nuance to what it means to be engaged in free-choice environments that invite hybridity.

**Phase 5: Frame analysis of hybridity between school mathematics and play**

Based on Aimee's and Dia's extended engagement, their contrasting behaviors at MOAS, and our curiosity about the girls' expressed differences in their liking of mathematics, we drew on frame theory as a means of orienting our analysis into whether and how Aimee and Dia hybridized their activity in the math-play space of MOAS with school-mathematics norms and practices around authority and evaluation. Frames are almost never stated explicitly in interaction, and thus we began with conjectures about aspects of interaction likely to give insight into Aimee's and Dia's notions of "what is happening here."

We primarily looked for indicators of the girls' frames by attending to the content of their talk, the majority of which involved making comments about their work. We first stitched their independent GoPro™ videos together such so that we could see what each girl was doing at the same time as the other (Figure 3-6) and make sense of their engagement as a joint endeavor, as this gave us the ability to analyze points of renegotiation and confusion during activity framing. We then transcribed the entirety of their stay at MOAS using a joint transcript that attended to both girls' talk and manipulation of exhibit materials. Next, we coded this transcript to identify talk and interaction around authority and evaluation.


*Figure 3-6.* A screenshot of Aimee's and Dia's videos stitched together such that their activity was synced in time. Aimee's arms can be seen reaching across Dia's body to grab eggs in both Aimee's video (right) and Dia's video (left).

***Authority in task determination***

We looked for *authority* by attending to whether or not the girls oriented to the exhibits in self-determined ways. For example, we coded statements such as "So I just play the games?", "What are you supposed to do?", and "So now we just make more designs?" as evidence for an external locus of authority (i.e., searching for what others determined the task to be). We coded the entire transcript for authority, finding nine such utterances in the first 13 minutes of activity and none thereafter. In our findings, we discuss the significance of this for framing, especially in relation to who initiated these orienting statements and questions. By attending to who asked orienting questions and who answered them, we are able to understand felt distributions of authority, with those asking orienting questions appealing to the authority of others rather than engaging in agentic, self-determined activity. We also carefully attended to the time lapse between when participants first began manipulating materials at each exhibit and when they began making their first EOMs at each exhibit. We consider this span of time to be exploratory activity, with exploratory activity as an indication that the girls expected to exercise personal authority to make sense of the materials.

***Evaluation of correctness***

We looked for *evaluation* by attending to how the girls positioned the products of their EOMs and their work during EOMs. Following Allen's (2003) study of learning through talk in museums, we coded our video data for metaperformance talk, or participants' evaluations (or requests for evaluations) of their own performance, actions, or abilities. We then considered these comments in relation to Allen's category of affective talk: Because comments such as "beautiful," "cool," "I like that," "ugly," "whoa!," and "ooooh" can be considered evaluations of the things produced by participants in the context of MOAS, we collapse both meta-performance talk codes and affective talk codes into an evaluation code. We then classified the girls' evaluations as positive (expressing satisfaction), negative (expressing dissatisfaction), or neutral (expressing neither satisfaction nor dissatisfaction). We counted statements such as "I like it!" and "There!" as *positive evaluations*; statements such as "I can't make it," "No that's ugly," and sarcastically saying positive words like "Great" with a downward intonation as *negative evaluations;* and statements such as "I don't know what I'm making" and "It's hard to figure out" as *neutral evaluations.* Upon going back to the video data to validate our coding of the transcripts with our interpretations of the video, we added some gestures to our account of evaluations. The gestures we coded as evaluations were all coded as positive evaluations, for example, tracing designs or rubbing hands together with pleasure after completing designs. In the interest of not over-interpreting the data, we left uncoded any gestures that may have been neutral and negative evaluations as these were more difficult to interpret. For example, although we did notice that the quickness with which EOMs were destroyed after completion seemed to be somewhat indicative of negative evaluations, we did not include this in our count of negative evaluations as we felt that creating a boundary on what counts as a quick and negative evaluation would be somewhat arbitrary. As a consequence, it is possible that we may be underreporting negative evaluations.[9]

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[9] Our viewing of the data leads us to infer that, if we are under-reporting neutral and negative evaluations, we are primarily doing so for Dia which does not affect our findings.

We coded the entire transcript and found evidence for authority and evaluation throughout EOMs. Because we conjecture that frames of school mathematics and play are likely to be evoked at MOAS, we then engaged in interpretive work around evidence for school mathematics and play frames in the analyses of authority and evaluation described above. While we acknowledge that neither a school mathematics nor a play frame may be consistently invoked by either girl, we sought to understand whether a school-mathematics frame or a play frame could be considered the leading frame for each girl.

***School-mathematics frames***

To identify school-mathematics frames at MOAS, we looked for talk that explicitly aligned activity at MOAS with school mathematics, as well as talk that tacitly bid to match activity at MOAS to activity in school mathematics. We see such tacit bids when the girls engaged in activities that mimicked expectations of school, namely expectations of fulfilling tasks predetermined by higher authorities (e.g., a mathematics teacher; the designer of MOAS, see Authority Section above) and an orientation towards doing so correctly (see Evaluation Section above). To be specific, in our data, we see bids for predetermined tasks when the girls expressed hesitancy in self-determining their own goals — for example, by appealing to the *authority* of others about *what they are supposed to do* — and we see bids for correctness when the girls seek *evaluations* from others or self-evaluate their own activity.

We attend to both the quantity and quality of evaluations, as more evaluations in conjunction with many of those evaluations being negative has quite different connotations than if many of those evaluations were positive. While we do not equate frequent negative evaluations with school mathematics in general, in this context, frequent negative evaluations indicate that participants frame their activity at MOAS as unsuccessful. If this co-occurs with explicit talk aligning MOAS with school mathematics and other talk that tacitly maps to school mathematics through the practice of looking to others to determine the task, then we take frequent negative evaluations as a further indication that participants were evoking a traditional school-mathematics frame.

***Play frames***

To identify play frames at MOAS, we looked for talk and gestures that explicitly conveyed comfort and confidence in establishing self-determined goals (authority), a prioritization of process over product (evaluation), and enjoyment. For example, when the girls engaged in self-determining their own EOMs (see Authority Section above), we inferred that they were taking an authoritative stance and interpreted that as an indicator that a play frame was leading activity. We also interpreted fewer evaluations (see Evaluation Section above) as an indication that the process of engaging was satisfying and satisfactory and so also considered this to be an indication of a play frame. Finally, we took the occurrence of more positive than negative evaluations, including affective expressions of pleasure (see Evaluation Section above) to be an indicator of a play frame. Significantly, the evidence we take as an indication of play frames is not incompatible with or mutually exclusive to school mathematics in general, but it is productive for us to look for them as separate frames in this analysis of how children hybridize their engagement in out-of-school mathematics activities with *traditional* school mathematics norms and practices around authority and evaluation.

## Limitations

While we did not conduct a negative case analysis to systematically look for disconfirming evidence in our entire corpus of data for how school-mathematics frames can influence activity, our extensive engagement with the video data of all 345 participants in multiple iterations of viewing and coding, purposive sampling of Aimee and Dia, and our systematic data analysis procedures lend trustworthiness to our findings. The generalizability of how traditional school-mathematics norms and practices can shape engagement in out-of-school mathematics activities remains an open question. In addition, we note our thin data on our participants' prior histories with mathematics, relying primarily on a short, self-report, Likert-scale questionnaire. Follow-up studies should include denser child-level data, ideally with re-occurring qualitative observations of classroom and out-of-school experiences, as well as iterative in-depth interviews to achieve a nuanced understanding of salient experiences inside and outside of school mathematics.

## Findings

Looking across Aimee and Dia's videos from their visit to Math-On-A-Stick, the two girls framed their activity in this hybridity-inviting math-play space differently. How the girls determined how to engage at each exhibit (i.e., by appealing to external authority or to themselves) and how they evaluated their engagement serve as the primary evidence of this claim. More specifically, Dia framed her activity using a traditional version of school-mathematics norms and practices around authority and evaluation, while Aimee framed her activity as play. Dia's hybridizing of MOAS with traditional school-mathematics norms and practices lead to hesitancy and dissatisfaction, foreclosing engagement in expansive exploration. Dia's framing prompted her to attune to the ambiguity of the situation as troublesome and unpleasant rather than playful. In contrast, Aimee framed the activity as more like play, and so she experienced the same ambiguity and openness as unproblematic and even enjoyable opportunities for free exploration.

To illustrate our central claim, we organize our findings by first presenting Dia's framing, characterizing her activity as *uncertain* and *dissatisfied*. We then present Aimee's framing activity and characterize it as *expansive exploration* and *pleasure in making* before interpreting both girls' frames in relation to how they hybridized the cultural activities of school mathematics and play.

**Dia's Framing: Uncertainty and Dissatisfaction**

As stated, the first approach to determining how the girls framed this activity was to look at the ways they oriented to new exhibits, with the assumption that they would be actively making sense of the purpose for their activity. Dia explicitly mapped the purpose of her activity onto school mathematics multiple times. For example, at minute seven when approaching the *Pattern Machine* (her second exhibit), she said, "This is the kind of mathematics I like," suggesting that she was attuned to looking for school mathematics as she contrasted the mathematics at MOAS ("this kind" and "mathematics I like") with other mathematics (mathematics she does not like: school math). Indeed, just a few minutes later, Dia asked if she was supposed to count at the *Pattern Machine,* providing more evidence that she was expecting school-like mathematics content and practices and thus was framing school-mathematics as relevant to her activity.

### Uncertainty in self-determination

Dia also explicitly questioned what she was supposed to be doing, suggesting that she was attempting to match the authority practices of schooling around completing predetermined tasks with her activity at MOAS. Within the first 13 minutes of her 35-minute visit, Dia offered nine utterances that focused on the purpose or intention of her participation, saying things like, “So I just play the games?” or, “We have to do all of them, I think.” Thus, Dia was searching for instructions or rules, indicating that, for her, the openness of the activity and the lack of instructions was uncomfortable, a problem or ambiguity that needed to be resolved.

Dia’s physical interaction with the materials at MOAS further indicated an orientation towards an expectation for predetermined, well-structured tasks rather than towards using the materials to explore the parameters of what was possible and interesting at each exhibit. In fact, Dia rarely explored the materials of the exhibits before using them in an EOM; her explorations generally lasted only 2–16 seconds. Instead, she almost always integrated the first materials she touched into her first EOM. This suggests that Dia was not exercising her personal agency to make sense of the materials, which corroborates our interpretation of her orientation to activity as searching for pre-defined rules and expectations. It seemed that, in the absence of finding such clear rules or expectations, Dia resorted to using materials as an act of compliance with the expectation that she do something, rather than an act of intentional design. This lack of exploration also aligns with traditional school-mathematics norms around authority, where tasks and appropriate solution strategies are often predetermined, and so exploration is unnecessary or even deviant.

Dia's discursive moves of looking for tasks predetermined by external authorities is consistent with a traditional school-mathematics frame in which a new activity typically begins with instructions, and students are tasked with listening to explanations and then complying. Dia's repeated search for predetermined tasks suggests that she was attempting to frame the activity in such a way as to match it with what she might otherwise expect mathematics to be, yet, because such tasks do not exist at MOAS, Dia experienced uncertainty in her activity.

**Dissatisfaction in evaluations**

We also looked for evidence of Dia's frames in the ways that she evaluated her activity. Dia evaluated her activity quite frequently (Table 3): While working on her EOMs, Dia produced a total of 27 evaluations of her activity over 14 distinct EOMS (up to 8 evaluations per EOM). These frequent evaluations tell us that Dia framed MOAS as a space in which some EOMs were more valuable or more correct than others.

Table 3-3. *Summary of Dia's evaluations during each EOM**

| Exhibit | EOM | Positive | Neutral | Negative | Total |
|---|---|---|---|---|---|
| Eggs (Exhibit 1) | EOM 1.1 | 5 | 2 | 1 | **8** |
| Eggs (Exhibit 1) | EOM 1.2 | 0 | 0 | 0 | **0** |
| Pattern Machine (Exhibit 2) | EOM 2.1 | 0 | 1 | 1 | **2** |
| Pattern Machine (Exhibit 2) | EOM 2.2 | 0 | 0 | 1 | **2** |
| Pattern Machine (Exhibit 2) | EOM 2.3 | 0 | 0 | 0 | **0** |
| Pattern Machine (Exhibit 2) | EOM 2.3 | 0 | 1 | 0 | **2** |
| Tiles and Patterns (Exhibit 3) | EOM 3.1 | 0 | 0 | 1 | **2** |
| Tiles and Patterns (Exhibit 3) | EOM 3.2 | 0 | 1 | 0 | **2** |
| Tiles and Patterns (Exhibit 3) | EOM 3.3 | 0 | 2 | 1 | **3** |
| Tiles and Patterns (Exhibit 3) | EOM 3.4 | 0 | 2 | 0 | **2** |
| Pentagons (Exhibit 8) | EOM 8.1 | 0 | 0 | 3 | **3** |
| Pentagons (Exhibit 8) | EOM 8.2 | 0 | 0 | 0 | **0** |
| Pentagons (Exhibit 8) | EOM 8.3 | 0 | 0 | 0 | **0** |
| Lizards and Turtles (Exhibit 9) | EOM 9.1 | 3 | 2 | 0 | **5** |
| **Total** | | **8** | **11** | **8** | **27** |

*Exhibits 4 (Visiting Mathematician), 5 (Stepping Stones), 6 (return to Visiting Mathematician), and 7 (Book Table) are not included in this table because they did not involve EOMs.

Notably, one might interpret Dia's frequent self-evaluations as evidence of Dia positioning herself as the authority for determining what was right, but a closer look at the nature of her evaluations suggests something quite different. In her comments about her work, Dia appeared to demonstrate a concern for doing things correctly. When she paused to evaluate her own work, there was little evidence that she was considering whether it met with her own standards or goals. As Table 3 shows, Dia offered a negative self-assessment in six of 14 EOMs, and offered positive evaluations during only two EOMs. In addition, she made neutral statements like "I don't know what I'm making" in seven out of 14 EOMs, and these statements functioned discursively to devalue her work. For example, at the *Tiles and Patterns* exhibit, she repeatedly made neutral comments about not knowing ("I don't know what I'm making;" "What is this, nobody knows") and negative comments about being unsuccessful ("Dang it I can't do it;" "Great" in a sarcastic tone while destroying her design). Figure 3-7 provides a photograph of Dia's EOMs at the *Tiles and Patterns* exhibit in order to visually illustrate how these evaluations played out in relation to the products of her EOMs. We note that the patterns she created do not look like the kinds of patterns with visually repeating units that we typically value in schools.

*Figure 3-7.* The product of Dia's four EOMs at the Tiles and Patterns exhibit.

Thus, even as Dia appeared to eventually determine what her task was in each exhibit (although she initially sought predetermined tasks), she continued to check with herself and with others that she was on-track or doing the right thing. Her emphasis on evaluation is reminiscent of traditional school mathematics and is not indicative of an enjoyment of process over product (an indicator of a play frame). While it may be that Dia was simply a talkative 12-year-old girl, the frequency and content of her evaluations indicates that she was attuned to, and concerned with, correctness. While her evaluations in themselves are not a strong indication of a school-mathematics frame, these evaluations in conjunction with her explicit references to school mathematics and explicit search for tasks predetermined by external authorities suggest that a school-mathematics frame was shaping her activity.

**Illustration of uncertainty and dissatisfaction**

We further illustrate the way Dia invoked authority and evaluation with the transcript in Table 4. In this transcript excerpt from the *Pattern Machine*, Dia repeatedly questioned what she was supposed to be doing (see times 06:37, 07:04, and 07:23) and negatively evaluated her work. Although Dia showed interest when suggesting the *Pattern Machine* as their next exhibit ("Let's do this, this look interesting"), she immediately followed up her suggestion with a request for guidance in understanding the task. Aimee answered Dia, telling her that the point of the exhibit was to make patterns. Later, Dia asked if they were supposed to count at the *Pattern Machine*, to which Aimee firmly responded "No." While noting that they had been doing a lot of pattern making at MOAS and that she liked pattern making (in contrast to other kinds of mathematics, as noted earlier), Dia also denigrated her pattern making twice, once while making her pattern, saying, "I don't know what this is supposed to be" and once at the end of making her pattern, saying, "Tah-dah," with a downward intonation that communicated the opposite of excitement or pride in work (i.e., sarcasm). Thus, this analysis of Dia's activity suggests framing around authority and evaluation that lead to uncertainty in and dissatisfaction with her activity.

Table 3-4. *An excerpt of transcript from the Pattern Machine exhibit*.*

| Time | Actor | Talk | Action |
|---|---|---|---|
| 06:34 | Dia | Let's do this, this looks interesting | |
| 06:37 | | What are you supposed to do? | *((beginning to touch Pattern Machine))* |
| 06:41 | Aimee | You make a pattern | *((beginning to touch Pattern Machine))* |
| 06:43 | Dia | There's a lot of just making patterns | |
| 06:44 | | This is the kind of mathematics I like | |
| 06:57 | | I'm tired of standing | *((grabs Pattern Machine and moves to opposite side of table, such that she and Aimee are face-to-face across the table from each other))* |
| 07:04 | | I don't know what this is supposed to be | *((sits down))* |
| 07:23 | Dia | Ohhh are we supposed to count? | *((beginning to push down design))* |
| 07:26 | Aimee | No | *((beginning to make heart from the bottom))* |
| 07:37 | | | *((finishes up the last bit of a heart, and after clicking down the last buttons moves her hands away quickly as if to draw attention to the fact that she has finished))* |
| 07:56 | Dia | Tah-dah | *((dull tone, picks up her Pattern Machine to show what she has made))* |
| 07:59 | Aimee | Can I see!? | |
| 08:00 | Dia | | *((turns her Pattern Machine so Aimee can see it))* |
| 08:13 | Aimee | Oh. I want to pop up- | *((begins popping all of the buttons up quickly, meanwhile Dia is pushing her design down))* |

*This transcript has been modified to include talk that center's Dia, and so talk between Aimee and their chaperone has been omitted for the sake of clarity and page limits.

### Summary

Dia's uncertainty as she repeatedly searched for pre-determined tasks and dissatisfaction as she negatively evaluated her activity suggest that this mathematical space that offered very little structure with which to comply was confusing for Dia and did not match her expectations for what she anticipated she might be doing. In other words, Dia's interactional self-positioning of "not knowing," when taken with her evaluations of her in-progress and finished work, suggest that she largely framed her activity as one in which she did not (a) have authority to determine what she should do or (b) feel confident in evaluating her activity as good or correct.

Indeed, even when Dia did make positive evaluations, they often served as a transition into modifying her design rather than as accepting it as satisfying and satisfactory. For example, at the *Eggs and Crate* exhibit, Dia repeatedly evaluated her "cool dude" (i.e., a smiley face) positively as she worked to make her design more interesting ("Now it's even better, it's got a green face! No green lips and a pink face, no a blue face and pink eyes!"; "Aimee, do you like my smiley face?"). Thus, even her positive evaluations do not give a clear indication of satisfaction with her activity, although they might point to enjoyment of process over product, an indicator of a play frame. Overall, Dia's explicit talk aligning MOAS with school mathematics, her explicit appeal to external authorities for task determination, and evaluation of her activity as incorrect or unsuccessful together indicate that Dia evoked a traditional school-mathematics frame, although not to the exclusion of any indicators of a play frame.

Importantly, Dia also made comments indicating that, while her confusion about the goal of her activity was unclear, she was nevertheless enjoying herself ("I have no idea what I'm actually supposed to be doing, but I'm having fun doing it," and, "This is the kind of mathematics I like."). Thus, while Dia's self-evaluations often found her work lacking, she also appeared to sometimes or in some way also be playing. Although enjoyment is an indication of a play frame, school mathematics is the leading frame of Dia's hybrid activity as her comments suggest a reliance on the familiar norms and practices of authority and evaluation in tractional school mathematics to make sense of her activity. Because Dia's activity at MOAS involved an integration of play and traditional school mathematics norms and practices in a way that led both cultural activities to be only partially recognizable, we identify Dia's activity as hybrid.

### Aimee's Framing: Expansive Exploration and Pleasure in Making

In contrast to Dia, Aimee did not explicitly talk about school mathematics at all during her time at MOAS. Instead, all of Aimee's talk directly referred to the activity she was engaging in rather than the activity she "should" be engaging in. In general, Aimee's frame was one of play as she engaged in self-determining her goals, actively displayed pleasure in the process of making her EOMs, and did not harshly evaluate her own activity.

#### Expansive exploration towards self-determining tasks

In contrast to Dia, Aimee did not ask a single orienting question or make any comments that indicated her confusion or uncertainty about her activity. Indeed, occasionally when Dia posed questions about their purpose, Aimee's responses were expansive. For example, when Dia and Aimee approached the *Pattern Machine,* Dia asked "What are you supposed to do, just press things?"; Aimee answered by offering an alternative to Dia's proposal, saying, "Or you make a pattern." At the *Tiles and Patterns* exhibit, Dia again asked aloud what they were supposed to do, and Aimee again answered that they could make patterns. The *Tiles and Patterns* was their third exhibit (after *Eggs* *and Crates* and *Pattern Machine*) and all previous exhibits also involved patterning. Dia noted this as unexpected, saying, "That's all we've been doing lately. That's cool I guess." However, Aimee treated the open, unstructured exhibits as an invitation and did not struggle to consider whether her activity was aligned with any external goals.

This is not to say that Aimee immediately knew what to do when she moved to a new exhibit. Whereas Dia approached the first three exhibits asking some form of "What am I supposed to do," Aimee approached each of these exhibits by picking up the materials and exploring their properties, then leveraged the structure of the materials towards deliberate designs. For example, Aimee took almost twice as long as Dia to get started with her first EOMs at all exhibits (with generous coding on what counted as Dia's exploration). While the first materials Dia touched often remained part of her first EOM, Aimee's rarely did, with Aimee frequently taking 20 – 34 seconds to explore and Dia taking between 2 – 16 seconds to explore. This suggests more intentionality in Aimee's work than in Dia's. For example, at the *Tiles and Patterns* exhibit, Dia used the first two tiles she put together in her first EOM in exactly the way she first put them together. Aimee, on the other hand, explored for approximately 33 seconds, which was enough time for her to select only green colored tiles and to notice that there were two kinds of green colored tiles. It took Dia until her third EOM at this exhibit to notice this design feature (as indicated by her surprise, selection and rejection of tiles, and talk). The products of their EOMs at this exhibit strongly reflect this difference (Figures3- 7 and 3-8). At the *Pentagons* exhibit, Dia explored for 14 seconds and then tried to make a pinwheel with red and dark brown pentagons but soon gave up, saying "Dang it I can't do it." She then went on to make a pinwheel without the red pentagons and evaluated it as a "boring circle." Aimee, on the other hand, explored for 22 seconds, much of which was spent examining the red pentagons (which required special attention to orientation due to being red on one side and light brown on the other), and then successfully completed a red and dark brown pinwheel (Figure 3-9).

*Figure 3-8.* The product of Aimee's four EOMs at the Tiles and Patterns exhibit.

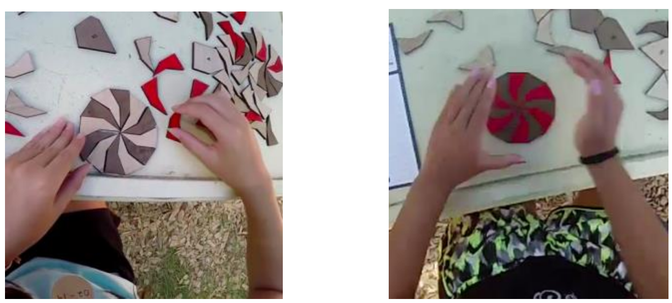

*Figure 3-9.* (a) Dia's "boring" pinwheel and (b) Aimee's red and brown pinwheel.

Thus, Aimee's discursive moves and interactions with the materials convey a play frame in which Aimee was confident in self-determining her tasks. At each exhibit, the she determined her tasks after initially exploring the materials. This exploration was followed by intentional pattern construction. In this way, Aimee was exercising her authority in a way that does not match or mimic the traditional norms and practices around authority in school mathematics, yet does match norms and practices of play.

**Pleasure in making: Satisfying activity and satisfactory evaluations**

Aimee produced only one third the number of evaluations as Dia, with 15 evaluations total — less than one evaluation per EOM (Table 5). These evaluations took place at the end of her activity, when she stepped back to comment on what she had made. Most (six out of nine) of these evaluations were positive. However, Aimee did have one negative evaluation. At the *Pattern Machine*, she attempted to make an “A” for Aimee but was unable to make it the way she wanted to. Dia showed Aimee how to make an A, but Aimee discounted Dia’s suggestion by saying that she wanted to make her A fill-up the board and her A to have a point at the top (i.e., rather than an “A” with a flat top), which is indeed impossible on the nine-by-nine grid of the *Pattern Machine.* In this way, Aimee’s negative evaluation was quite different than Dia’s evaluations, as Aimee’s was related to the constraints of the tool in relation to her goals, whereas Dia’s were based on her affective orientation towards her own design work. This suggests Aimee felt capable and confident in activity at MOAS.

Table 3-5. *Summary of Aimee's evaluations during each EOM**

| **Exhibit** | **EOM** | **Positive** | **Neutral** | **Negative** | **Total** |
|---|---|---|---|---|---|
| Eggs and Crate (Exhibit 1) | EOM 1.1 | 1 | 0 | 0 | **1** |
| Eggs and Crate (Exhibit 1) | EOM 1.2 | 1 | 0 | 0 | **1** |
| Eggs and Crate (Exhibit 1) | EOM 1.3 | 0 | 0 | 0 | **0** |
| Pattern Machine (Exhibit 2) | EOM 2.1 | 0 | 0 | 0 | **0** |
| Pattern Machine (Exhibit 2) | EOM 2.2 | 0 | 1 | 0 | **1** |
| Pattern Machine (Exhibit 2) | EOM 2.3 | 1 | 0 | 0 | **1** |
| Pattern Machine (Exhibit 2) | EOM 2.4 | 0 | 0 | 1 | **1** |
| Pattern Machine (Exhibit 2) | EOM 2.5 | 1 | 0 | 0 | **1** |
| Pattern Machine (Exhibit 2) | EOM 2.6 | 0 | 0 | 0 | **0** |
| Tiles and Patterns (Exhibit 3) | EOM 3.1 | 0 | 0 | 0 | **0** |
| Tiles and Patterns (Exhibit 3) | EOM 3.2 | 0 | 0 | 0 | **0** |
| Tiles and Patterns (Exhibit 3) | EOM 3.3 | 1 | 0 | 0 | **1** |
| Tiles and Patterns (Exhibit 3) | EOM 3.3 | 1 | 0 | 0 | **1** |
| Pentagons (Exhibit 8) | EOM 8.1 | 0 | 0 | 0 | **0** |
| Lizards and Turtles (Exhibit 8) | EOM 9.1 | 0 | 1 | 0 | **1** |
| **Total** | | **6** | **2** | **1** | **9** |

*Exhibits 4 (Visiting Mathematician), 5 (Stepping Stones), 6 (return to Visiting Mathematician), and 7 (Book Table) are not included in this table because they did not involve EOMs.

Aimee's enjoyment of her work carried through all her EOMs. For example, at the *Pattern Machine*, Aimee talked about engaging with the materials as "strangely fascinating" and engaged in tracing of her final designs. At the *Tiles and Patterns* exhibit, she gleefully exclaimed "I made something!," then rubbed her hands together with pleasure and continued to extend her design. Upon finishing this design, she again rubbed her hands together with pleasure and then traced units of her pattern (black and green diamonds) before carefully disassembling it (Figure 3-10). Thus, Aimee felt capable and confident in her self-assessments, and she verbally assessed her own work much less frequently than Dia.

*Figure 3-10.* Aimee's TWO evaluations in her third EOM at the Tiles and Patterns exhibit.

Interestingly, when we compare the products of Aimee's EOMs to the product of Dia's EOMs, we notice that Aimee's EOMs look much more like something we would value in school than Dia's because Aimee's make more visible use of the the structure and symmetry of the materials. Yet, Aimee does not invoke a mathematics frame as Dia does. Rather, Aimee engages in a play frame that supports exploration, self-determination, and enjoyment of process. This does not mean that Aimee did not ever invoke a mathematics frame, but rather that she did not invoke traditional norms and practices around authority and evaluation.

**Illustration of exploration and pleasure**

We further illustrate the way Aimee invoked a play frame with the transcript in Table 6. In this transcript excerpt, Aimee set a goal of making a heart at the *Eggs and Crate* exhibit. Before beginning, she attended to the midline of the crate, rotating the crate so that the side with five slots was parallel to her body and thus created a center row that would allow her to make a heart with one egg at the upper and lower points. When she finished making her heart, she rescued it from Dia's eager hands attempting to steal the blue eggs that provided a background for her heart (Figure 3-11). When Dia retreated and asked Aimee what she had made, Aimee raised the pitch of her voice so that it sounded quite sweet and said, "It's a heart" while tracing the mirrored sides of the heart at the same time using two hands. This preservation, tone change, and tracing suggest that Aimee enjoyed and took pride in her activity.

*Figure 3-11*. Aimee's heart with blue background at the Eggs and Crate exhibit.

*Table 3-6.* Aimee's third EOM at the Eggs and Crates exhibit.*

| Time | Actor | Talk | Action |
|---|---|---|---|
| 03:25 | Aimee | Ohh wait, I know what I wanna do | *((dumps crate and grabs 2 pink eggs, rotates crate so that side with 5 slots is parallel to her body instead of side with six slots, then places egg in middle slot on the second row from the top and begins to make a pink heart))* |
| 04:54 | Aimee | Can I have more blues? | *((filling up background of heart in egg crate))* |
| 04:55 | Dia | Yeah | |
| 04:55 | Aimee | | *((reaches over Dia's egg crate to grab more blues))* |
| 05:09 | Dia | Wait no I want the bluuuue | *((grabs to take blues out of Aimee's crate, which is now completed with it's blue background))* |
| 05:10 | Aimee | NO NO! Wait wait, Wa-wait wait wait | *((covers Dia's hands to prevent her from removing the blue eggs))* |
| 05:12 | Dia | What is it? | *((removes hands from Aimee's crate))* |
| 05:14 | Aimee | It's a heart | *((traces the heart with her fingers using both hands in a coordinated motion))* |
| 05:15 | Dia | Oh! Well can I- | *((reaches to grab the blue eggs out of Aimee's crate again))* |
| 05:15 | Aimee | | *((dumps out all the eggs from her crate before Dia grabs any out, many of them landing on Dia's crate))* |
| 05:17 | Dia | Not on mine! | *((removes the eggs that landed on her crate from Aimee dumping eggs))* |
| 05:26 | Aimee | | *((starts putting blue eggs back into her crate, then takes them out and replaces them with green eggs))* |

**Summary**

Aimee's confidence as she repeatedly engaged in exploration and expressed satisfaction with her EOMs suggest that this mathematical space that offered very little structure with which to comply was unproblematic and even inviting for Aimee. In other words, Aimee's interactional self-positioning as knowing what she was up to and what was acceptable at MOAS (note that she often answered Dia's questions about what they were supposed to do), when taken with her positive evaluations of her finished work, suggest that she was largely framing her activity as one in which she (a) had authority to determine what she should do and (b) felt confident in evaluating her activity as good or correct (or even did not feel the need to evaluate at all). Overall, Aimee's lack of explicit talk aligning MOAS with school mathematics, her explicit appeal to herself as an authority for task determination, and evaluation of her activity as meaningful and worthy of pride together indicate that Aimee evoked a play frame, although not to the exclusion of the possibility of a mathematics frame that was not captured by our coding for traditional norms and practices of authority and evaluation.

We have no data about whether Aimee or Dia attended traditional or reform-oriented mathematics classrooms. Thus, Aimee's playful approach to her design work at the exhibits does not preclude the possibility that she made connections to her activity in mathematics classrooms, especially if her prior experiences with school mathematics took place in reform-oriented classrooms. However, we do not see evidence of this interactionally. We characterize Aimee's activity as led by a play frame as she explicitly conveyed comfort and confidence in establishing self-determined goals, a prioritization of process over product, and enjoyment. We do not characterize Aimee's activity as hybridized with traditional school mathematics.

## The Hybridization of Play with Traditional School-Mathematics Frames

Dia's uncertainty and dissatisfaction are linked to her attempts to match activity at MOAS to school mathematics, while Aimee's expansive, exploratory approach to these exhibits is linked to her framing of MOAS activity as one of play. Aimee was not concerned with others' potential ideas about what she should do but was comfortable immediately structuring her activity on her own. Likewise, the openness of the activities immediately invited exploration for her, and then unproblematic movement into plans and designs of her own. Whereas Dia approached the exhibits with an eye towards the external rules with which she was expected to comply, Aimee approached the table with an eye towards her own interest and enjoyment.

We also found the quantity and nature of their evaluations to indicate a difference in the ways the two girls framed their activity. Overall, Dia engaged in significantly more evaluations of her work than Aimee, considering whether what she was doing was "right" or "made sense." These comments, and their frequency, suggest that Dia was aware of, or concerned about, the audience for her work. This is consistent with a traditional school-mathematics frame, which typically offers cycles of work that involve demonstration, practice, and evaluation. In MOAS, evaluations from "authorities" were rarely available, which contributed to Dia's uncertainty about her activity. In contrast, Aimee was comfortable with the exploratory and unstructured nature of the environment, making changes to her design without comment, and then indicating her satisfaction with her work upon its completion. We think the sheer difference in quantity of evaluations provides additional meaningful evidence that the girls framed MOAS quite differently, with Aimee orienting to the activity with playfulness and high personal authority, and Dia invoking traditional school-mathematics norms and practices around authority and evaluation. This evidence — in conjunction with Dia's explicit comments matching and contrasting the activity with school mathematics — suggest a contrast in the way the girls framed the math-play activity of MOAS, with Dia hybridizing her play with the norms and practices of school mathematics and Aimee not.

## Discussion

In this paper, we have explored how hybridity-inviting activities can be framed differently by different participants, and how that framing connects with their experience of the activity. In particular, we investigated the question: *How do children hybridize out-of-school mathematics activities with school-mathematics norms and practices?* By attending to children's framing of their activity, particularly around authority and evaluation norms and practices of traditional school mathematics, we found traditional school mathematics' norms around authority and evaluation often negatively shaped Dia's activity. Because Dia hybridized in- and out-of-school mathematics, Dia's experience at MOAS was quite different that Aimee's, even though they traveled through together. Notably, Aimee did not hybridize her play activity by invoking traditional school mathematics frames, instead engaging primarily in play frames. Thus, our description of Dia's activity illustrates how challenging it can be to provide meaningful out-of-school mathematical experiences when children invoke their in-school mathematics frames while our description of Aimee's activity illustrates that it is possible to facilitate out-of-school engagement that involves mathematical concepts and actions — such as manipulating and exploring mathematical objects in order to pose questions or make things — without necessarily invoking school mathematics norms and practices.

Importantly, we do not claim that Aimee did not hybridize her play with school mathematics at all, but rather that she did not hybridize her play with traditional school mathematics. In contrast to traditional school mathematics, reform-oriented mathematics invites engagement in activities such as exploration, conjecturing, reflecting, and justification (NCTM 2000; CCSM 2010). Based on this, we would expect children invoking reform mathematics frames to engage with a high level of authority and engage in self-evaluation geared towards conceptual understanding rather than only towards correctness. Thus, we expected that distinguishing between reform-oriented mathematics frames and play frames would be analytically blurry, and so we narrowed our inquiry to the invocation of traditional school mathematics frames. Thus again, Aimee may have invoked a mathematics frame that was not captured by our analysis. We discuss this more later in this section.

Our study contributes to research on out-of-school STEM activities by providing a case study of how the norms and practices of traditional school mathematics can shape engagement out-of-school. This is a contribution for two reasons. First, most studies of out-of-school STEM engagement neglect mathematics. Second, little is known about how school norms and practices shape engagement in out-of-school STEM activities. In our study, we explored how framing around school mathematics can hybridize out-of-school mathematics activities in ways that constrain opportunities to feel competent, even if such activities are designed for open, evaluation-free participation. In particular, we illustrated how Dia hybridized her activity at MOAS with traditional school-mathematics norms of authority and evaluation. Dia's search for school mathematics at MOAS corresponded with a devaluing of her own play. In contrast, Aimee engaged in expansive exploration.

Although we cannot know *why* the two girls framed their activity differently, it seems likely that their own conceptions of mathematics — and themselves in relation to mathematics — contributed to the way the they invoked different frames. In particular, we conjecture that something about the space, or participation in the *Playful Mathematics Learning* research project, seemed to make Dia's not-a-mathematics person identity relevant, such that she repeatedly engaged in negative evaluations of her own play, making comments like, “I can’t do it,” “[I made a] boring circle” (as opposed to the not-boring circle she originally tried to make), and “I don’t know what I’m making.” We conjecture the opposite is true for Aimee: Her self-identification as a mathematics person may have played out in her EOMs, as she repeatedly engaged in positive evaluations of her own play, making comments like “I made something!,” clapping and rubbing her hands together in pleasure during arrivals, and tracing the elements of her patterns or representations before carefully disassembling them.

Indeed, when asked why they came to MOAS in their interviews, Aimee claimed they came because she loves mathematics whereas Dia claimed they just saw it as they were on their way to the nearby EcoExperience building and decided to stop by. The EcoExperience building is a science center — Dia’s favorite subject. Because of this, we conjecture that the girls’ personal, affective histories with school mathematics may have been made relevant in their engagement at MOAS. This conjecture is supported by existing research on in school learning, which has shown that the disciplinary identities of individual students — in addition to local practices — affects the actual learning opportunities that are available to students more so than the design of the activity (Esmonde, 2009). It also supported by research on free-choice environments that documents how visitors’ motivations for engaging are a larger determinant of what they learn than the design itself (Falk, 2006).

This conjecture and our findings are quite interesting in relation to each girls’ responses to exit interview questions that probed whether or not their activity at MOAS reminded them of school mathematics. Dia told her interviewer that MOAS was nothing like school mathematics, while Aimee said it reminded her of school mathematics in some ways because, after finishing assigned mathematics work, she and her classmates were allowed to play with shapes and "try to make something out of them." This is an indication that Aimee's playful experiences in school mathematics, even though they came after "official" mathematics work, seemed to allow Aimee to connect MOAS to legitimate mathematical engagement unproblematically. Thus, as discussed earlier in this section, Aimee may have invoked a reform-oriented mathematics frame as she engaged in her play. While we cannot draw any firm conclusions about this, it also seems reasonable that Dia struggled to make sense of her activity at MOAS because she did not have prior experiences of mathematical play that allowed her to have an expansive understanding of mathematics.

Regardless of participants' prior experiences with mathematics, we find it significant that *traditional school-mathematics norms can shape engagement in free-choice environments in ways that lead participants to devalue their activity*. We posit that this happened with Dia because of the ambiguity of the norms of MOAS. In the future, we hope to better understand how to design activities that invite norms of playful exploration and curiosity rather than hesitancy and self-critique. Such future research would add nuance to studies that attend to how participant motivations for engaging impact what is learned (Falk, 2006) by helping us better understand how creating norms around playfulness can (re)shape motivations and engagement. Our study adds to the existing literature by offering some evidence about the influence of hybridity with traditional school-mathematics norms and practices in free-choice mathematics environments.

In addition, our research expands our knowledge about how engagement in free-choice learning environments can become structured to look much like school not only when the designers have a (personal or institutional) goal for all participants' learning to be uniform (Russell, Knutson, & Crowley, 2013), but also when participants expectations for the activity draw on traditional school frames. Indeed, although we present two cases here, we note that traditional, school-based conceptions of mathematics are frequently brought into MOAS throughout our corpus, especially by parents. This phenomenon is so common in our data that we have informally come to call this kind of framing "schoolitizing," meaning that parents attempt to turn patterning work into memorization of facts and procedures — for example, by asking children to calculate how many eggs fit into the egg crate using skip counting or multiplication. We have not conducted an exhaustive analysis of this trend, but our sense is that schoolitizing tends to cut off play (*not* hybridize with it), with children often choosing to leave exhibits after play is subsumed by the authority of adults invoking traditional school mathematics. This is something we may wish to attend to in future designs by making the play norms of the space more visible to both parents and children.

Our vision of out-of-school mathematics can best be described as a material-cultural activity that entails high levels of child-agency in exploring mathematical objects, determining their own goals during exploration, and having opportunities to notice and wonder, and even to pose and pursue their own questions. In this way, we do not envision out-of-school mathematics as subordinate to school mathematics, but rather we emphasize what is gained by engagement in out-of-school mathematics. Of course, we do not recommend doing away with school mathematics, as the kinds of out-of-school environments we envision also lose some of the benefits of school mathematics, such as the comfort of knowing when the official correct answer has been reached, a potential for generalization, and even certain instantiations of precision. Nor do we necessarily recommend a complete absence of adult intervention. Rather, we envision activity that is driven by children's ideas and goals, with mathematical sensemaking scaffolded by the design of materials and adult guidance instead of the other way around.

Finally, we conjecture that MOAS may have provided Aimee with a rare opportunity to engage in playful, out-of-school mathematics experiences in ways that buttressed her self-identification and interests as a mathematics person. While such out-of-school experiences are known to support interest development and the pursuit of STEM careers (Jones, Taylor, & Forrester, 2011), such activities designed explicitly for mathematics are quite rare. Even more, it seems unlikely that Dia benefited from her time at MOAS in the same way Aimee did, which prompts many questions for future research, discussed in the implications section below.

## Implications

This study leaves many open questions. If negative school-mathematics experiences can transform MOAS to a place where some children can evaluate themselves as failing, how can we design for this not to occur? This also points to the need for in-school studies that examine how teaching and learning change when we centralize developing positive mathematical identities as a core goal of mathematics learning, while still pressing for rigor. It also illustrates how equitable interactions do not just happen, even in spaces designed to foster play. In addition, opening up opportunities for children to engage in creative and self-driven activity in school mathematics — such as finding problems for rather than only solving predetermined problems — can help us begin to understand the complexities of producing positive mathematics experiences through interest-driven mathematics learning. Such opportunities and studies are also needed outside of school. In conclusion, while we do not claim that hybridity between school mathematics and play would play out identically in other contexts, we posit that examining how mathematical engagement emerges in relation (to personal histories with) norms and practices of school mathematics is a productive way to begin re-imagining engagement in the discipline. Future research will benefit from across context studies that can trace participants' practices and engagement across school and free-choice mathematics contexts.

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## References


Abreu, G. D., & Cline, T. (2003). Schooled mathematics and cultural knowledge. *Pedagogy, Culture and Society*, *11*(1), 11-30.

Allen, S. (2003). Looking for learning in visitor talk: A methodological exploration. In G. Leinhard, K. Crowley, & K Knutson (Eds.), *Learning conversations in museums* (pp. 268 – 303). Mahwah, NJ: Lawrence Erlbaum Associates.

Anderson, K. T. (2009). Applying positioning theory to the analysis of classroom interactions: Mediating micro-identities, macro-kinds, and ideologies of knowing. *Linguistics and Education*, *20*(4), 291-310.

Bakhtin, M. (1981). The dialogic imagination (M. Holquist & C. Emerson, Trans.). Austin: University of Texas Press.

Ballantyne, R., & Packer, J. (2005). Promoting environmentally sustainable attitudes and behaviour through free-choice learning experiences: what is the state of the game?. Environmental Education Research, 11(3), 281-295.

Bamberger, Y., & Tal, T. (2007). Learning in a personal context: Levels of choice in a free choice learning environment in science and natural history museums. *Science Education*, *91*(1), 75-95.

Banks, J., Au, K., Ball, A. F., Bell, P., Gordon, E., Gutiérrez, K., Heath, S. B, Lee, C. D., Lee, Y., Mahiri, J., Nasir, N. S. & Valdes, G. , Zhou, M. (2007). Learning in and out of school in diverse environments: Life-long, life-wide, life-deep. Retrieved from http://www.life-slc.org/docs/Banks_etal-LIFE-Diversity-Report.pdf

Boaler, J. (1997). Setting, social class and survival of the quickest. *British Educational Research Journal*, *23*(5), 575-595.

Boaler, J. (1999). Participation, knowledge and beliefs: A community perspective on mathematics learning. *Educational Studies in Mathematics*, *40*(3), 259-281.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. *Multiple perspectives on mathematics teaching and learning* (pp. 171-200). Westport, CT: Ablex Publishing.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of Railside School. Teachers College Record, 110(3), 608-645.

Bulman, R.C. (2002). Teachers in the 'hood: Hollywood's middle-class fantasy. *The Urban Review, 34* (3), 251-276.

Calabrese Barton, A., & Tan, E. (2009). Funds of knowledge and discourses and hybrid. *Journal of Research in Science Teaching*, *46*(1), 50-73.

Carraher, T. N., Carraher, D. W., & Schliemann, A. D. (1985). Mathematics in the streets and schools. *British Journal of Developmental Psychology, 3*(?), pp. 21-29.

Cazden, C. B. (2001). Classroom Discourse: The Language of Teaching and Learning. New York, NY: Teachers College Press.

Civil, M. (2002). Everyday Mathematics, Mathematicians' Mathematics, and School Mathematics: Can We Bring Them Together? *Journal for Research in Mathematics Education. Monograph,11*, 40-62. doi:10.2307/749964

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education,* 40-68.

Common Core State Standards Initiative. (2010). *Common core state standards for* *mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Dewey, J. (1938). Experience and education. New York: Macmillan.

Engle, R. A. (2006). Framing interactions to foster generative learning: A situative explanation of transfer in a community of learners classroom. *The Journal of the Learning Sciences, 15*(4), pp. 451-498.

Engle, R. A., & Conant, F. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a Community of Learners classroom. *Cognition and Instruction, 20*, 399-483.

Esmonde, I. (2009). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *Journal of the Learning Sciences*, *18*(2), 247-284.

Falk, J. H. (2004). The director's cut: Toward an improved understanding of learning from museums. *Science Education, 88*(S1), S83-S96.

Falk, J. H. (2006). An identity-centered approach to understanding museum learning. Curator: *The Museum Journal, 49*(2), 151-166.

Falk, J. H. ,& Dierking, L.D. (1992). The Museum Experience. Washington, DC: Whalesback Books.

Falk, J. H., & Dierking, L. D. (2002). Lessons without limit: How free-choice learning is transforming education. Rowman Altamira.

Falk, J. H., & Dierking, L. D. (2013). The museum experience revisited (Chapters 3, Exhibits, & 6, The Physical Context). Walnut Creek, CA: Left Coast Press.

Falk, J. H., Storksdieck, M., & Dierking, L. D. (2007). Investigating public science interest and understanding: Evidence for the importance of free-choice learning. Public

Understanding of Science, 16(4), 455-469.

Falk, J. H., Storksdieck, M., & Dierking, L. D. (2007). Investigating public science interest and understanding: Evidence for the importance of free-choice learning. *Public Understanding of Science*, *16*(4), 455-469.

Gargroetzi, E., Horn, I.S., Chavez, R.,  & Byun, S. (Under review). Institution-Identities in the Neoliberal Era: Challenging Differential Opportunities for Mathematics Learning. *Journal of Research in Mathematics Education.*

Goffman, E. (1974). *Frame analysis: An essay on the organization of experience*. Boston, MA: Harvard University Press.

Gonzales, P., Williams, T., Jocelyn, L., Roey, S., Kastberg, D., & Brenwald, S. (2008). Highlights from TIMSS 2007: Mathematics and Science Achievement of US Fourth-and Eighth-Grade Students in an International Context. NCES 2009-001. *National Center for Education Statistics*.

Griffin, J. (1994). Learning to learn in informal settings. *Research in Science Education, 24*, 121 – 128.

Griffin, J. (2004). Research on students and museums: Looking more closely at the students in school groups. *Science Education, 88*(S1), S59-S70.

Gutiérrez, K. D., Baquedano-López, P., & Tejeda, C. (1999). Rethinking diversity: Hybridity and hybrid language practices in the third space. *Mind, Culture, and Activity, 6*(4), 286-303.

Gutwill, P., & Allen, S. (2012). Deepening Students' Scientific Inquiry Skills During a Science Museum Fieldtrip. *Journal of the Learning Sciences, 21*(1), 130–181. https://doi.org/10.1080/10508406.2011.555938

Hall, R. & Stevens, R. (2016). Interaction analysis approaches to knowledge in use. In A. A. diSessa, M. Levin, & J. S. Brown (Eds.), *Knowledge and interaction: A synthetic agenda for the learning sciences* (pp. 72-108). New York, NY: Routledge.

Hammer, D., Elby, A., Scherr. R. E., & Redish, E. F. (2005). Resources, framing, and transfer. In J. Mestre (Ed.), *Transfer of learning: Research and perspective*s (pp. 89–119). Greenwich, CT: Information Age.

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the" voice" of a mathematics textbook. *Journal for Research in Mathematics Education*, 344-369.

Heyd-Metzuyanim, E., Munter, C., & Greeno, J. (2018). Conflicting frames: a case of misalignment between professional development efforts and a teacher's practice in a high school mathematics classroom. *Educational Studies in Mathematics*, *97*(1), 21-37.

Hiebert, J., Stigler, J. W., & Manaster, A. B. (1999). Mathematical features of lessons in the TIMSS Video Study. *ZDM*, *31*(6), 196-201.

Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers' conversations. *The Journal of the Learning Sciences*, *16*(1), 37-79.

Horn, I.S. (2008). Turnaround students in high school mathematics: Constructing identities of competence through mathematical worlds. *Mathematical Thinking and Learning, 10*(3), 201-239.

Honey, M. A., & Hilton, M. (2011). Learning science through computer games and simulations. *National Academies Press*.

Hutt, C. (1966). Exploration and play in children. In *Symposia of the Zoological Society of London* (Vol. 18, No. 1, pp. 61-81).

Jackson, P.W. (1968, 1990). The Daily Grind. *Life in Classrooms* (pp. 1-37). New York: Teachers College Press.

Jacobs, J. E., Lanza, S., Osgood, D. W., Eccles, J. S., & Wigfield, A. (2002). Changes in children's self-competence and values: Gender and domain differences across grades one through twelve. *Child development*, *73*(2), 509-527.

Jerret, O. S. (2015). Recess and Learning: Research on the Effects of Recess on Children and Teachers. In J. E. Johnson, S. G. Eberle, T. S. Henricks, & D. Kuschner (Eds.), *The handbook of the study of play, Volume 1* (299-318). New York: Rowman & Littlefield.

Jones, G., Taylor, A., & Forrester, J. H. (2011). Developing a scientist: A retrospective look. *International Journal of Science Education*, *33*(12), 1653-1673.

Jordan, B., & Henderson, A. (1995) Interaction analysis: Foundations and practice. *Journal of the Learning Sciences*, *4*(1), 39-103.

Klugman, E., & Fasoli, L. (1995). Taking the high road toward a definition of play. *Play, policy and practice*, 13-32.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge, UK: Cambridge University Press.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, *27*(1), 29-63.

Lave, J. E., Murtaugh, M, de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In B.E. Rogoff & J. E. Lave (Eds.), *Everyday cognition: Its development in social context*(pp. 67-289). Boston, MA: Harvard University Press.

Litke, E. G. (2015). *The state of the gate: A description of instructional practice in algebra in five urban districts* (Doctoral dissertation). Retrieved from https://dash.harvard.edu/handle/1/16461050

Little, P., Wimer, C., & Weiss, H. B. (2008). After school programs in the 21st century: Their potential and what it takes to achieve it. *Issues and Opportunities in Out-of-school Time Evaluation*, *10*(1-12).

Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education*, *48*(5), 488-519.

Ma, J. Y. (2016). Designing disruptions for productive hybridity: The case of walking scale geometry. *Journal of the Learning Sciences*, *25*(3), 335-371.

McDermott, R. (2001). The acquisition of a child by a learning disability. *Understanding learning: Influences and outcomes*, 60-70.

Mehan, H. (1979). "What time is it, Denise?": Asking known information questions in classroom discourse. Theory into practice, 18(4), 285–294.

Minnesota State Fair (2019). Mathematics On-A-Stick [website]. Retrieved from https://www.mnstatefair.org/location/math-on-a-stick/.

Nakamura, J., & Csikszentmihalyi, M. (2014). The concept of flow. In *Flow and the foundations of positive psychology* (pp. 239-263). Springer, Dordrecht.

Nasir, N. (2002). Identity, goals, and learning: Mathematics in cultural practice. In N. Nasir & P. Cobb (Eds.), *Mathematical thinking and learning. Vol. 4*(pp. 213–248). New

York: Teachers College Press.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Research Council. (2009). *Learning science in informal environments: People, places, and pursuits*. National Academies Press.

Noss, R., Hoyles, C, & Pozzi, S. (2002). Abstraction in expertise: A study of nurses' conceptions of concentration. *Journal for Research in Mathematics Education, 33*(3), pp. 204-229. http://www.jstor.org/stable/749725 .

Pellegrini, A. D. (2009). *The role of play in human development*. Oxford University Press, USA.

Petrich, M., Wilkinson, K., & Bevan, B. (2013) It looks like fun but are they learning? In M. Honey & D. E. Kanter (Eds.), *Design, make, play: Growing the next generation of STEM innovators* (pp. 50-70). New York, NY: Routledge.

Pickering, A. (1995). *The Mangle of Practice: Time, Agency, and Science*. Chicago, IL: University of Chicago Press.

Pope, D. (2001). Doing School: how we are creating a generation of stressed out, materialistic, and miseducated students. New Haven, CT: Yale University Press.

Quinn, H. & Bell, P. (2013). How designing, making, and playing relate to the learning goals of K-12 science education. In M. Honey & D. E. Kanter (Eds.), *Design, make, play: Growing the next generation of STEM innovators* (pp. 17-33). New York, NY: Routledge.

Russell, J. L., Knutson, K., & Crowley, K. (2013). Informal learning organizations as part of an educational ecology: Lessons from collaboration across the formal-informal divide. *Journal of Educational Change*, *14,*259-281.

Saxe, G. B. (1988). Candy selling and mathematics learning. *Educational researcher*, *17*(6), 14-21.

Schauble, L., Gleason, M., Lehrer, R., Bartlett, K., Petrosino, A., Allen, A., & Street, J. (2002). Supporting science learning in museums. *Learning Conversations in Museums*, *425*, 452.

Schegloff, E. A. (1992). On talk and its institutional occasions. In P. Drew & J. Heritage (Eds.)*,Talk at work: Interaction in institutional settings* (pp. 101–134). Cambridge, UK: Cambridge University Press.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational psychologist.*,*23*(2), 145-166.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics Education*, 338-355.

Sengupta-Irving, T., & Enyedy, N. (2015). Why engaging in mathematical practices may explain stronger outcomes in affect and engagement: Comparing student-driven with highly guided inquiry. *Journal of the Learning Sciences*, *24*(4), 550-592.

Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. Mathematical Thinking and Learning, 6(3), pp. 261-284.

Sinclair, N., Pimm, D, & Higginson, W. (2006). Mathematics and the aesthetic: New approaches to an ancient affinity. New York, NY: Springer.

Stigler, J.W. & Hiebert, J. (1998, Winter). Teaching is a cultural activity. *American Educator*,1-10.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers* *for improving education in the classroom*. New York, NY: Simon and Schuster.

Taylor, E. V. (2009). The purchasing practice of low-income students: The relationship to mathematical development. *The Journal of the Learning Sciences,* 18(3), 370-415.

Walkerdine, V. (1990). Difference, cognition, and mathematics education. *For the Learning of Mathematics*, *10*(3), 51-56.

Wing, L. A. (1995). Play is not the work of the child: Young children's perceptions of work and play. *Early Childhood Research Quarterly*, *10*(2), 223-247.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), pp. 458-477. http://www.jstor.org.proxy.library.vanderbilt.edu/stable/749877

Yin, R. K. (2014). Case study research: Design and methods (5th ed.). Los Angeles, CA: Sage Publishing.

# APPENDIX A: INTAKE SURVEY

1. What is your favorite school subject? ___________________________________

1. What is your least favorite school subject? _______________________________

1. What do you like to do in your free time? _________________________________

1. Did you come to Mathematics On-A-Stick last year?

   ______________________________

1. What language(s) do you speak at home? _______________________________

1. What is your gender? _________________________________________________

1. How old are you? ____________________________________________________

1. With which race or ethnicity do you identify? ______________________________

1. What kind of of school do you attend?

   1. Public

   2. Private

3. Charter
4. Homeschool
5. Other ______________________________________________

# APPENDIX B: EXIT INTERVIEW

1. Did you have fun?

○ *(Probes: affect)*

1. Why did you decide to come into Mathematics On-a-Stick?

○ *(Probes: family disposition towards mathematics, child disposition toward math)*

1. What was your favorite activity you did in Mathematics On-a-Stick? Why?

○ Why do you think that activity was at Mathematics On-a-Stick?

○ *(Probes: What each child finds most playful, hopefully what kind of interactions they found playful)*

1. Was anything frustrating? Why? If yes, did you do anything that helped you get less frustrated?

○ *(Probes: What students found challenging, whether children found frustration or confusion productive, whether children persisted)*

1. Did you think about something new during your time in Mathematics On-a-Stick?

○ *(Probes: What mathematics children think they learned, how excited children are about their efforts at Mathematics On-a-Stick)*

1. Did you do anything here that feels like things you do at school?

○ *(Probes: Do children think this is it mathematics at all, what they think about school mathematics and Mathematics On-a-Stick mathematics, how the volunteers interacted with students vv)*

# APPENDIX C: EXIT SURVEY

*I need my teacher to tell me whether my mathematics work is right or wrong.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*School mathematics is challenging.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*I get to talk about mathematics with my classmates in mathematics class.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*School mathematics is interesting.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*I do really well in mathematics without trying.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*School mathematics is a lot of steps that I have to remember.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*It is important that I stay quiet in class and listen to my teacher.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*School mathematics is a lot of facts that I have to remember.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*I think school mathematics is boring.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*I can figure out whether or not school mathematics makes sense on my own.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*I get to talk about mathematics with my classmates in mathematics class.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |

*I can investigate things and use my own own ideas in school mathematics.*

| Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|
| | | | |