A PROPOSAL FOR A SELECTION CRITERION IN A CLASS OF DYNAMIC RATIONAL EXPECTATIONS MODELS WITH MULTIPLE EQILIBRIA

by

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A Proposal for a Selection Criterion in a Class of Dynamic Rational Expectations Models with Multiple Equilibria

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Abstract

The paper argues that multiple equilibria-whether non-stationary or stationary- are a generic property of dynamic rational expectations models. In light of this, this paper proposes a selection criterion for choosing between these multiple equilibria in an important class of dynamic rational expectations models. The criterion is based on the idea that agents can be assumed to coordinate their beliefs around the limit of a finite-horizon equilibrium. For three examples examined, all of which can have multiple stationary, i.e., non-explosive, rational expectations equilibria, there is, among the multiple equilibria of an infinite-horizon model, only one that is the limit of a finite-horizon model.

1 Introduction

Multiple equilibria in dynamic rational expectations models are both problematic and endemic. When multiple equilibria in these models exist, completion of the model as an equilibrium model requires some specification about how people coordinate their beliefs . If not, people may differ in their expectations, and not all of them can be right. Hence, any equilibrium that arises will not necessarily be consistent with rational expectations by all agents.

Within the dynamic rational expectations class of models, multiple equilibria appear in many forms and are a possibility in a wide variety of models. Some models have a continuum of equilibria, such as in Taylor (1977), Farmer (1993), or Karp (1996a), while others have a finite number, such as in McCafferty and Driskill (1980). Some models are stochastic, while others are deterministic. Some have ad hoc elements, while others derive behavioral relationships from an explicit optimization problem. Some have no strategic interactions among agents, while others are explicitly game-theoretic.

Researchers have adopted a variety of perspectives in response to this feature of multiple equilibria in dynamic rational expectations models. Some researchers have viewed the potential existence of multiple equilibria as a reflection of reality and a possible explanation of phenomena ranging from speculative booms to the Great Depression.[1] Many other researchers, though, have viewed multiple equilibria as an undesirable feature of these models. Consequently, they have made various attempts to modify models so as to eliminate the possibility of multiple equilibria. To this end, some add features such as imperfect capital markets (Shell and Stiglitz, 1967), heterogeneity (Herrandorf, Valentinyi, and Waldmann, 2000) or exogenous stochastic variability (Frankel and Pauzner 2000) to a basic model in an attempt to eliminate the possibility of multiple equilibria in a particular setting. Others have suggested that some equilibria are more "natural" than others and satisfy some ad hoc criterion such as having a minimum variance solution (Taylor, 1997) or having a "minimum state space representation" (McCallum (1983)[2].

Another literature considers whether some form of learning or recursive updating of expectations converges on a particular equilibrium, or whether only one of several equilibria is "robust" in the sense that if expectations are "wrong", recursive updating of these expectations converges to the rational expectation.[3]

In this paper we build on an observation of Karp (1996a) and propose a different selection criterion. Karp's rational expectations model of a durable-goods monopolist has an infinite number of equilibria. Karp noted this was a natural state of events in the absence of a "natural" boundary condition.[4] He also noted that Driskill (1997) had shown that for a special linear case of his model, only one solution to the infinite-horizon model was the limit of the associated finite-horizon model as the horizon approached infinity. Following on these results, we propose that in the presence of multiple equilibria, one should think of the most "natural" equilibrium as the limit of the backward-induction equilibrium of a finite-horizon model as the horizon tends to infinity. This selection criterion has several desirable attributes. First, because finite-horizon models that have behavioral relationships derived from optimizing behavior generally have a natural boundary condition that is grounded in economic theory, then equilibria that are the limit of such solutions share this feature. Thus, this

[1] See Hahn (1966) for an early example in which existence of both a non-stationary and stationary solution was viewed as a potential explanation of speculative booms. This approach is also found in Blanchard (1979), in which short-lived probabalistic bubbles are a result of existence of two solutions. See Bryant (1981, 1983) for an example in which existence of both a degenerate and non-degenerate solution is viewed as a potential explanation of "good times" and "bad times." See Howitt and McAfee (1988) and Diamond (1982) for examples in which non-degenerate multiple equilibria are viewed as potential explanations of "good times" and "bad times."

[2] See Turnovsky (1995) for an overview of these issues.

[3] See, for example, Evans (1986, 1989), and Guesnerie (2002). Again, Turnovsky (1995) has an overview of these issues.

[4] Karp also cites Tatsuo and Mino (1990), who ascribed multiple equilibria in a variant of Fershtman and Kamien's model of dynamic duopoly to the infinite time horizon.

selection criterion is not subject to the critiques made of such as hoc criteria as the minimum variance or minimal state-space representation criteria. Second, because finite-horizon models provide people (and economists) with much of their intuition about the workings of dynamic models, solutions to such models seem a natural "focal point" about which agents can be assumed to coordinate their "belief functions"[5]. For example, even in infinite-horizon models, finite-horizon ideas seem important in how economists formulate such things as no-ponzi-game restrictions on infinite-horizon budget constraints.[6] Finally, this criterion applies both to deterministic and to stochastic models, both to models with a finite and with an infinite number of equilibria, and both to models where the type of multiple equilibria are bubbles and where they are non-explosive. In a broad sense, what we suggest here is that in order to eliminate multiple potential solutions all of these models need one more boundary condition. We propose a particular one, based on the solution to a finite-horizon model.

The idea that infinite-horizon dynamic rational expectations models always are short one boundary condition is quite general. A heuristic way of thinking about the idea of rational expectations is to consider a model in which the equilibrium value of some endogenous variable, perhaps price, for example, is determined at some time t by the equilibrium condition that supply equals demand. For example, in a simple Cagan model of inflation with an exogenous, fixed nominal money supply, the conditional equilibrium price level at some time t is determined by the equilibrium of supply and demand of real money balances. Denoting the nominal money supply at time t by M_t and the price level by P_t, this equilibrium condition would be expressed as

$$\frac{M_t}{P_t} = L(\cdot)$$

where $L(\cdot)$ denotes the demand for real balances. The modifier "conditional" is used to emphasize that this price is really only one of many endogenous price variables: there is one for each time period, and the equilibrium values for these prices in general can not be solved independently of one another. In particular, for many models, either supply or demand depends behaviorally on the current-period expectation of next-period's equilibrium value. In the Cagan model, for instance, demand for today's real money balances depends not only on the current price level but also on today's expectation of next-period's price level:

$$L(\cdot) = L(\frac{P_t^{ex} - P_t}{P_t})$$

[5] Karp (1996b) made a similar argument about the desireability of using Markov strategies in a dynamic game specification. See Farmer (1993) and Matheny (1999) for a discussion and definition of belief functions. Such functions seem closely related to the learning updating rules used by Evans (1986) in his concept of expectational stability and to the common knowledge assumption used by Guesnerie (2002).

[6] A no-ponzi-game restriction does not allow an infinitely-lived agent to borrow more that the present value of his or her lifetime resources, i.e., does not allow chain-letter schemes. See Shell (1971) for a discussion of the arbitrariness of such a restriction in an infinite-horizon model.

The rational expectations assumption is that agents "know the model." Hence, a natural way to model how agents form their expectations of next-period's price is to assume they know that next-period's price will be that price that equilibrates next-period demand and supply.

Of course, next-period's demand in turn depends on the subsequent-period price, which depends on the price after that, and so on for as far into the future as the model permits. In an infinite-horizon model, this means that there is always one more price to be determined-the next-period price- than there are equilibrium conditions. That is, for any arbitrary number of periods n, there are n equations-the equilibrium condition that demand equals supply-but $n+1$ endogenous variables-the n prices plus the $(n+1)^{th}$ expected price.

This problem is reminiscent of the parable recounted by Shell (1971) of the hotelier with a hotel with an infinite number of rooms. A traveler approached and asked for a room. The hotelier responded that he was all booked up, but could still make room for him. He would simply move the guest in room #1 to room #2, the guest in room #2 to room #3, and so forth.

The equilibrium price at some initial time is analogous to the traveler. In an infinite-horizon model, room can be made for a variety of possible equilibrium values of this first price by changing all subsequent prices in an appropriate fashion.

Seen in this light, multiple equilibria in dynamic rational expectations models are a fundamental feature of infinite-horizon models. For such models, any selection criteria for choosing among the multiple equilibria must in some way provide an additional condition that makes the model determined.

In the remainder of the paper, we illustrate use of the criteria espoused here in three examples. All three examples share a common feature in that the solutions to all three are characterized by a fundamental second-order dynamic equation. These dynamic equations all arise in part from an equilibrium condition much like that discussed for the Cagan example: at any moment in time, the value of the equilibrating variable at that moment depends upon the future adjacent-moment's value of that equilibrating variable.

They also share the feature that they permit the possibility of multiple *stable* equilibria. While the proposed criterion applies to selection between both stable and unstable equilibria, the case of multiple stable equilibria has always been viewed as more problematic.[7]

They all differ, though, in ways that illustrate both the scope of the proposed selection criterion and the common unifying element that makes it appropriate. The first example is the classic Muth (1961) model of inventory speculation as amended by McCafferty and Driskill (1980). We choose this model for several reasons. First, this model readily permits existence of two stable equilibria, and

[7] As noted earlier, people differ in their views about the importance of multiple equilibria. For "bubble" solutions in particular, though, these differences tend to be more pronounced. In fact, Muth (1961) recognized that his original inventory speculation model had a "bubble" solution, but he dismissed this as uninteresting. In contrast, people who saw Hahn present his 1996 paper report that he claimed that existence of this same fundamental feature of his model to be "the last nail in the coffin of capitalism."

thus distinguishes it from models like the Cagan model where only one equilibria is stable. Second, it has behavioral relationships grounded in optimizing behavior. Hence, the finite-horizon version of this model has a boundary condition that arises naturally out of economic theory. Third, it is stochastic, which the other two examples are not. In addition, there are no strategic elements in it, distinguishing it from the work on the durable goods monopoly of Karp (1996a) and Driskill (1997). Finally, it has been used by Evans (1989) in his work on expectational stability, and thus allows an explicit comparison with this other proposed selection criterion.

The second example is a dynamic duopoly model related to both Fershtman and Kamien (1987, 1990) and Driskill and McCafferty (2001). This model posits Nash-Cournot duopolists who compete in a market in which demand depends on both current and cumulated consumption. These duopolists attempt to maximize the present discounted value of profits. Hence, by construction, any stable, non-explosive equilibria satisfy transversality conditions, and the Obstfeld and Rogoff (1986) procedure of eliminating equilibria that do not satisfy transversality conditions is not relevant. For some parameter values, two stable Perfect Markov Equilibria exist for the infinite-horizon model. Only one of these equilibria, though, is the limit of the associated finite-horizon game.

The third example is a two-sector overlapping-generations model in which an individual's productivity in one sector depends on the total number of people at work in that sector. Such a model is similar to those used by Matsuyama (1991) and Krugman (1991) in their attempts to describe the process of industrialization.[8]. In the model used in this paper, newborn members of each generation make an irreversible choice to work either in a constant-returns sector or a sector in which returns increase with the size of the sector. The decision of which sector to choose thus depends on a newborn's expectations of what members of future generations will choose. Again, for some parameter values of an infinite-horizon model, multiple equilibria occur, only one of which can possibly be the limit of the associated finite-horizon model. One feature that is interesting about this model is that the overlapping-generations feature coupled with a constant probability of death for members of the economy make it impossible to eliminate "explosive" solutions, i.e., solutions associated with positive roots to characteristic equations, via appeal to transversality conditions or unreasonableness of such a solution. This result also sheds some light on the "expectations versus history" debate addressed by this literature.[9]

The other interesting feature of this model is that some parameter values lead to cyclical equilibrium paths. We use such equilibria to clarify those conditions under which the selection criterion proposed here does not select a unique equilibrium. For the constant-amplitude cycles that arise from this specification of the model, there is not a unique limit of a finite-horizon model.

[8] Another multiple-equilibria OLG model with a similar flavor is Howitt and McAfee (1992). Lucas (2002) also has a two-sector model based on a specification of scale economies used by Eaton and Eckstein (1997).

[9] See Krugman (1991) and Matsuyama (1991).

2 The Muth Model

2.1 The Infinite-Horizon Model

Consider Muth's classic rational expectations model of inventory speculation (Muth 1961). The model consists of flow demand and supply functions, a speculative demand function for stocks, a market-clearing equation, the rational expectations assumption, and an initial condition:

$$
\begin{aligned}
C_t &= -\beta p_t;\ \beta > 0;\ \text{(Flow Demand)} && (2.1)\\
P_t &= \gamma E_{t-1} p_t + u_t;\ \gamma > 0;\ \text{(Flow Supply)} && (2.2)\\
I_t &= \alpha [E_t p_{t+1} - p_t];\ \text{(Inventory Speculation)} && (3.3)\\
I_t - I_{t-1} &= P_t - C_t;\ \text{(Market Equilibrium)} && (2.4)\\
& I_{-1} \text{ given; (Initial Condition)} && (2.5)
\end{aligned}
$$

where $t = 0, 1, 2, ...$ P_t represents production in the i^{th} period, C_t represents consumption in the i^{th} period, p_t represents market price, E_{t-j} is the conditional mathematical expectation operator, and u_t is a zero-mean, serially-uncorrelated random variable with finite variance σ_u^2. All variables are measured as deviations from equilibrium values.

Muth derives (2.3) from maximization of expected utility of profits, and shows that

$$\alpha = \frac{\widehat{K}}{\sigma_{t,1}^2} \tag{2.6}$$

where $\widehat{K}$ is a non-negative parameter that is a measure of risk aversion and $\sigma_{t,1}^2$ is the conditional variance of next period's price forecast. As shown in McCafferty and Driskill (1980), the equilibrium price path for all t is described by the following first-order stochastic difference equation, which we denote as the equilibrium price function

$$p_t = \lambda p_{t-1} - \frac{u_t}{\beta + \alpha(1-\lambda)};\ t = 0, 1, 2, ... \tag{2.7}$$

where λ and α satisfy the following two equations:

$$
\begin{aligned}
\frac{\widehat{K}}{\sigma_u^2}[\beta + \gamma\lambda]^2 &= (\gamma + \beta)\lambda; && (2.8)\\
\alpha &= \frac{(\gamma + \beta)\lambda}{(1-\lambda)^2}. && (2.9)
\end{aligned}
$$

For this model, there exist two real solutions to (2.8) if and only if

$$\frac{\beta + \gamma}{\beta\gamma} > 4K,\ K \equiv \frac{\widehat{K}}{\sigma_u^2}. \tag{2.10}$$

Throughout the remainder of this paper, we assume this restriction is satisfied. Denote the smaller root of (2.8) as λ_1 and the larger root as λ_2. As shown in McCafferty and Driskill (1980),

$$0 < \lambda_1 < \lambda_2. \tag{2.11}$$

Denote by $p_{i,t}$ the equilibrium price associated with the equilibrium price function (2.8) when $\lambda = \lambda_i, i = 1, 2..$ That is,

$$p_{i,t} = (\lambda_i)(p_{i,t-1}) - \frac{u_t}{\beta + \alpha_i(1-\lambda_i)};\ t = 0, 1, 2, ... \tag{2.12}$$

where $\alpha_i = \frac{(\gamma+\beta)\lambda_i}{(1-\lambda_i)^2}$

When $0 < \lambda_1 < 1$ and $\lambda_2 > 1$, the equilibrium price function associated with λ_2 is non-stationary. Existence of such explosive solutions is a well-known feature of many rational expectations models, and is frequently referred to as a "bubble" solution. As noted in the introduction, many researchers have simply ignored this possible solution as uninteresting because of the implied non-boundedness of the price process, while others have viewed this as a possible explanation of the real-world phenomena of price bubbles More problematic, though, has been the case emphasized by McCafferty and Driskill in which $0 < \lambda_1 < \lambda_2 < 1$.[10] In such a case, ad hoc selection criteria such as suggested by Taylor (1977) and McCallum (1983) and arguments such as put forth by Obstfeld and Rogoff (1986) and Roberts (1998) fail to work..[11]

2.2 The Finite-Horizon Model

Consider now a finite-horizon version of this model, where $t = 0, 1, 2, ..., T$. The only difference in the structural model associated with this finite-horizon assumption is that utility-maximization by inventory speculators implies that $I_T = 0$. This condition serves as a second boundary condition, and permits a solution by backwards induction. Let $\widetilde{p}_t$ denote the equilibrium price for this finite-horizon model. The changes in the solution to the infinite-horizon model generated by imposition of this new boundary condition are characterized by the following proposition:

Proposition 1 *The equilibrium price path for the above finite-horizon Muth inventory speculation model is described by the following first-order stochastic difference equation:*

$$\widetilde{p}_{T-k} = \lambda_{T-(k+1)}\widetilde{p}_{T-(k+1)} + \pi_{T-k}u_{T-k},\ k = 0, 1, 2, ..., T; \tag{2.13}$$

[10] An example given by Evans (1989) of parameter values for which this occurs is: $\beta = .1, \gamma = .4, K = 2.5$. In this case, $\lambda_1 = .1,\ \lambda_2 = .66$.

[11] Obstfeld and Rogoff (1986) showed that some bubbles may be ruled out by transversality conditions associated with fully-specified intertemporal optimizing models. Recently, Roberts (1998) has shown that risk-aversion rules out the possibility of rational probabilistic bubbles such as studied in Blanchard(1979) because in this case the rational expectations solution becomes process-inconsistent in the sense of Flood and Garber (1980). This result strengthens interest in the case studied below, where there are multiple stationary equilibria.

where λ_{T-k} , π_{T-k} , *and* α_{T-k} *obey the following recursive relationships:*

$$\lambda_{T-(k+1)} = \frac{\alpha_{T-(k+1)}[1-\lambda_{T-(k+1)}]-\gamma\lambda_{T-(k+1)}}{\beta+\alpha_{T-k}[1-\lambda_{T-k}]};\ k=1,2,...,T; \quad (2.14)$$

$$\pi_{T-k} = \frac{-1}{\beta+\alpha_{T-k}[1-\lambda_{T-k}]};\ k=1,2,...,T; \quad (2.15)$$

$$\alpha_{T-k} = (K)\{\beta+\alpha_{T-(k-1)}[1-\lambda_{T-(k-1)}]\}^2; K\equiv\frac{\widehat{K}}{\sigma_u^2};\ k=1,2,..., T; \quad (2.16)$$

and where the following boundary conditions apply:

$$\lambda_{T-1} = \frac{\alpha_{T-1}}{\beta+\gamma+\alpha_{T-1}}; \quad (2.17)$$

$$\alpha_{T-1} = K\beta^2; \quad (2.18)$$

$$\pi_T = \frac{-1}{\beta}. \quad (2.19)$$

Proof. First consider the equilibrium condition (2.4) at $t=T$:

$$\gamma E_{T-1}\widetilde{p}_T+u_T+\beta\widetilde{p}_T=-\alpha_{T-1}[E_{T-1}\widetilde{p}_T-\widetilde{p}_{T-1}]. \quad \text{(P.1.1)}$$

This can be solved for $\widetilde{p}_T$ as a function of $\widetilde{p}_{T-1}$, $E_{T-1}\widetilde{p}_T$, and u_T. Hence, the period-$(T-1)$ expectation of period-T price can be found as:

$$E_{T-1}\widetilde{p}_T=\frac{\alpha_{T-1}}{\beta+\gamma+\alpha_{T-1}}\widetilde{p}_{T-1}. \quad \text{(P.1.2)}$$

Substituting this expectation back into the equilibrium condition (P.1.1) yields the following equilibrium price function for $\widetilde{p}_T$:

$$\widetilde{p}_T=\lambda_{T-1}\widetilde{p}_{T-1}+\pi_T u_T;\ \lambda_{T-1}\equiv\frac{\alpha_{T-1}}{\beta+\gamma+\alpha_{T-1}};\ \pi_T\equiv\frac{-1}{\beta}. \quad \text{(P.1.3)}$$

Knowing this price function, the period-T forecast variance can be computed as

$$\sigma_{T,1}^2=\frac{\sigma_u^2}{\beta^2}. \quad \text{(P.1.4)}$$

Now, using this in the definition of α_{T-1} yields

$$\alpha_{T-1}=K\beta^2=K(\pi_T)^2. \quad \text{(P.1.5)}$$

(P.1.2) and (P.1.5) establish the boundary conditions (2.17) and (2.18).

Knowing the equilibrium price function $\widetilde{p}_T$, the equilibrium price function for $\widetilde{p}_{T-1}$can be constructed by the same steps as used in construction of the equilibrium price function $\widetilde{p}_T$. Backward induction then shows that for any $k=2,3,...T$, the equilibrium price function will have the form:

$$\widetilde{p}_{T-k}=\frac{\alpha_{T-k}[1-\lambda_{T-k}]-\gamma\lambda_{T-(k+1)}}{\beta+\alpha_{T-k}(1-\lambda_{T-k})}\widetilde{p}_{T-(k+1)}-\frac{u_{T-k}}{\beta+\alpha_{T-k}(1-\lambda_{T-k})} \quad \text{(P.1.6)}$$

where

$$\lambda_{T-(k+1)} = \frac{\alpha_{T-(k+1)}[1-\lambda_{T-(k+1)}] - \gamma\lambda_{T-(k+1)}}{\beta + \alpha_{T-k}[1-\lambda_{T-k}]} \tag{P.1.7}$$

$$\pi_{T-k} = \frac{-1}{\beta + \alpha_{T-k}[1-\lambda_{T-k}]} \tag{P.1.8}$$

To establish the third recursive relationship (2.16), note that the one-period-ahead forecast variance at $T-k$ is

$$E_{T-k}[(\widetilde{p}_{T-(k-1)} - E_{T-k}\widetilde{p}_{T-(k-1)})^2]. \tag{P.1.9}$$

Substituting (2.13) into (P.1.9), we have

$$(\widetilde{p}_{T-(k-1)} - E_{T-k}\widetilde{p}_{T-(k-1)})^2 = [\pi_{T-(k-1)}u_{T-(k-1)}]^2. \tag{P.1.10}$$

Hence,

$$\sigma^2_{T-(k-1),1} = \sigma^2_u[\pi_{T-(k-1)}]^2; \tag{P.1.11}$$

Substituting this into (2.8) yields

$$\alpha_{T-k} = \frac{\widehat{K}}{\sigma^2_u[\pi_{T-(k-1)}]^2} \tag{P.1.12}$$

Substitution of (2.15) for $\pi_{T-(k-1)}$ into this yields (2.16).

QED ■

We can now state the following turnpike proposition:

Proposition 2 *Assume parameter values are such that two real roots exist as solutions to equation (2.8). For any* $t > 0$ *and any arbitrarily small* $\varepsilon > 0$*, there exists a horizon-length* $T > t$ *sufficiently large such that for a given sequence* $\{u_t\}$, $\mid \widetilde{p}_t - p_{1,t} \mid < \varepsilon$.

Proof. What we need to establish is that $\lim_{(T-k)\to\infty} \lambda_{T-k} = \lambda_1$. To this end, note that (2.14) and (2.16) form a coupled difference-equation system in $\{\lambda_{T-k}, \alpha_{T-k}\}$. Let $T-k=k$, $T-(k+1)=k+1$, and so on, for notational ease. Furthermore, define $x_k = \alpha_k(1-\lambda_k)$. Using this definition, rearrange (2.14) as:

$$\lambda_{k+1} = \frac{x_{k+1}}{\beta + \gamma + x_k} \tag{P.2.1}$$

Hence,

$$1 - \lambda_{k+1} = \frac{\beta + \gamma + x_k - x_{k+1}}{\beta + \gamma + x_k} \tag{P.2.2}$$

(2.16) can be written as

$$\alpha_{k+1} = K[\beta + x_k]^2 \tag{P.2.3}$$

By a rearrangement of the the definition of x, we have:

$$\alpha_{k+1} = \frac{x_{k+1}}{1-\lambda_{k+1}} \tag{P.2.4}$$

So,

$$\frac{x_{k+1}}{1-\lambda_{k+1}} = K[\beta + x_k]^2 \tag{P.2.5}$$

Combine (P.2.2) and (P.2.5) to yield:

$$x_{k+1} = \frac{K[\beta + x_k]^2(\beta+\gamma+x_k)}{K[\beta+x_k]^2+\beta+\gamma+x_k} \equiv \phi(x_k) \tag{P.2.6}$$

The properties of ϕ are:

$$\phi(0) = \frac{K\beta^2(\beta+\gamma)}{K\beta^2+\beta+\gamma} > 0; \tag{P.2.7}$$

$$\phi' = \frac{K^2(\beta+x)^4+2(\beta+x)(\beta+\gamma+x)^2}{\{K(\beta+x)^2+\beta+\gamma+x\}^2} > 0 \tag{P.2.8}$$

$$\lim_{x\to\infty}\phi' = +\infty \tag{P.2.9}$$

where the subscript on x is suppressed. Given the definition of x, if two real roots λ_1 and λ_2 exist, then there must be two real critical values that solve (P.2.6). That is, $\phi(x_k)$ intersects the locus $x_{k+1} = x_k$ at two points, say x_1 and x_2, $0 < x_1 < x_2$. Because $\phi(0) > 0$ and ϕ is monotonically increasing in x, $\phi(x_k) > x_k$ for $x_k < x_1$, and $\phi' < 1$ at $x_k = x_1$. Now, the value of x_{T-1} is computed from the boundary conditions on λ_{T-1} and α_{T-1}, and is

$$x_{T-1} = \frac{K\beta^2(\beta+\gamma)}{K\beta^2+\beta+\gamma} \tag{P.2.10}$$

which is equal to $\phi(0)$. Now, $\phi(x)$ is upward-sloping, so it must be that $\phi(0) < x_1$, because $\phi(x) = x$ at x_1. Hence, $x_{t-1} < x_1$. Because $\phi(x)$ is monotonically increasing, and because $x_{T-2} = \phi(x_{T-1})$, it must be that $x_{T-1} < x_{T-2} < x_1$. Now $x_{T-3} = \phi(x_{T-2}) < x_1$. Again, by monotonicity of ϕ, $x_{T-3} < x_{T-4} < x_1$. Because ϕ is continuous and monotonic, it must be that $x_{T-k} < x_{T-(k+1)} < x_1, k = 1, 2, 3, ..., T$, and $\lim_{k\to\infty} \mid x_1 - x_{T-k} \mid = 0$. Hence, $\lim_{k\to\infty} x_k = x_1$. Because the steady-state value of x and steady-state value of λ are monotonically related by the following relationship:

$$\bar{\lambda} = \frac{\bar{x}}{\beta+\gamma+\bar{x}} \tag{P.2.11}$$

it must be that $\lim_{(T-k)\to\infty} \lambda_{T-k} = \lambda_1 . Q.E.D.$ ■

In the above proof, the $\phi(x_k)$ and $x_{k+1} = x_k$ relationships form a phase diagram. In the following figure, we construct such a phase diagram, with

x_{k+1} on vertical axis and x_k on horizontal. The parameter values used are: $\beta = .1, \gamma = .4, K = 2.5$. The intersections of the two curves occur at $\{x = .0052, .947\}$. Note that if the starting value for $\phi(x)$ were not tied down by the terminal condition, one might be fooled into thinking that the larger root of x could be reached if only the starting value $\phi(x)$ were greater than the value of the lower intersection of $\phi(x) = x$.

$\frac{(2.5)(.1+x)^2(.5+x)}{2.5(.1+x)^2+.5+x}, x$

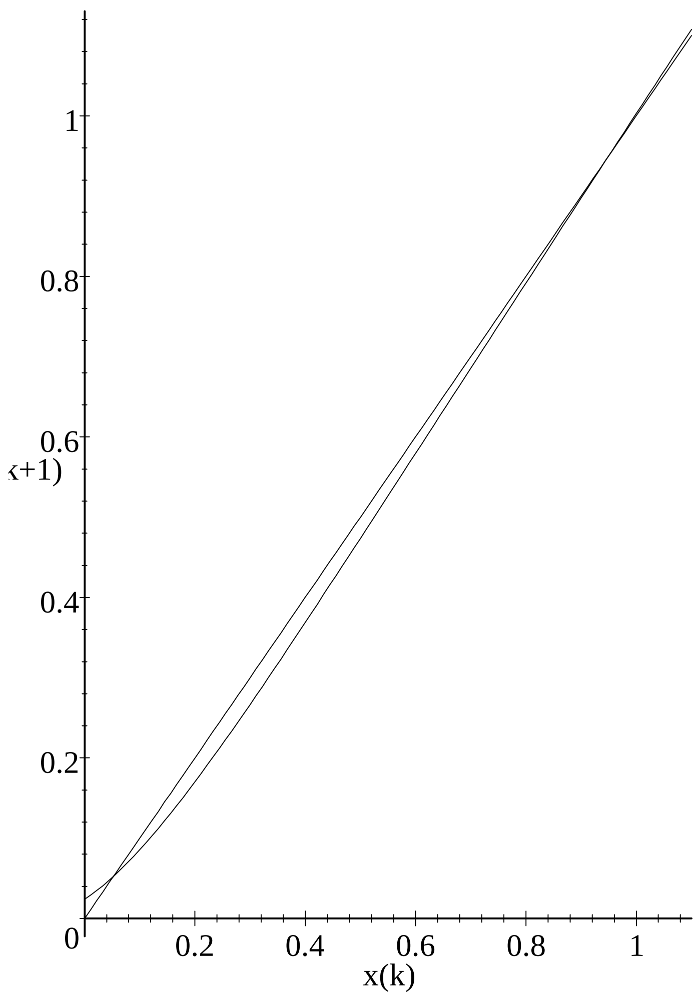

Figure 1

Note that this selection criteria eliminates the larger root whether or not it is stable, i.e., whether or not $\lambda_2 \gtrless 1$. Both the speculative bubble solutions and the more problematic case of multiple stable solutions are handled by this approach.

2.3 The Learning Approach

In the only other explicit analysis of the Muth model, Evans (1986, 1989) classified the two solutions to the infinite-horizon Muth model according to whether or not they were what he called "e-stable", the "e" a mnemonic for "expectational." His concept of e-stability asked whether or not agents who start with incorrect expectations (of a particular sort) but update expectations (in meta-time) according to a learning-like rule eventually converge to the correct rational expectations solution. This approach, like the finite-horizon model here, gives rise to recursive relationships between the parameters of the price function, albeit in meta-time. Also as with the selection criterion developed here, the equilibrium selected on the basis of e-stability was the equilibrium associated with the smaller of the two roots to the fundamental infinite-horizon characteristic equation.[12]

While it is interesting that both of these selection criteria pick the same solution, some differences between the approaches should also be noted. The criteria proposed here could perhaps be called an equilibrium approach: at every moment, all agents are assumed rational and knowledgeable. The e-stability criteria is based on adjustment from a diseqilibrium starting point, with agents engaging in "reasonable" mental updating rules.

3 Dynamic Duopoly with Time-Dependent Demand

In this section, we analyze a model related to Fershtman and Kamien (1987) and Driskill and McCafferty (2001). Fershtman and Kamien studied duopolistic competition in a homogeneous good under the assumption that the good's "current desirability is an exponentially weighted function of accumulated past consumption."[13] Such utility functions, first introduced by Ryder and Heal (1973), posit that instantaneous utility is a function of both current flow consumption, denoted here by u, and the exponentially-weighted sum of past consumption of the good, denoted here by z. Driskill and McCafferty (2001) studied monopoly and oligopoly supply of such an 'experience" good using the specific functional form of the utility function introduced by Becker and Murphy (1988). Our purpose here is to illustrate that a particular variant of such an infinite-horizon dynamic game can yield multiple stable equilibria. That is, the dynamic path

[12] Evans (1989) also showed that real-time least-squares learning corresponds to e-stability convergence.

[13] Fershtman and Kamien (1987), p. 1151.

of z approaches a finite steady-state value along either of the permissible equilibrium paths. We then show that only one of these equilibria is the limit of the associated finite-horizon game as the horizon goes to infinity.

3.1 The Infinite-Horizon Model

3.1.1 Consumer Behavior

Assume a continuum of identical agents distributed over the unit interval. For notational convenience, consumer subscripts are omitted. For each individual, "consumption capital" is given as

$$z(t) = \int_{\tau=-\infty}^{\tau=t} u(\tau)e^{-s(t-\tau)}d\tau. \tag{3.1}$$

or, in differential form, as

$$\dot{z} = u - sz \tag{3.2}$$

where s is a positive constant. Following Becker and Murphy (1988) and Driskill and McCafferty(2001), we assume a quadratic instantaneous utility function that is also quasilinear with respect to a non-experience good.

$$v(t) = \alpha_0 u - \frac{\alpha}{2}u^2 + \delta z + x \tag{3.3}$$

where x is the non-experience good[14]. For our purposes, we assume that $\alpha_0 > 0$, $\alpha > 0$, and $\delta \lesseqqgtr 0$. [15]. An individual's budget constraint is given by

$$R = x + pu \tag{3.4}$$

where the price.of x is normalized to one and p is the price of the experience good. In the infinite-horizon model, consumers choose u to maximize the present discounted value of instantaneous utility:

$$\max_u V = \int_0^\infty v(t)e^{-rt}dt \tag{3.5}$$

subject to (3.2) and (3.4). First-order conditions for this problem are

$$\begin{aligned}
\frac{\partial L}{\partial u} &= \alpha_0 - \alpha u + \delta z - p + \lambda = 0 && \text{(3.6.i)}\\
\frac{-\partial L}{\partial z} &= \lambda - r\lambda = -\delta u^* + \lambda s && \text{(3.6.ii)}\\
\lim_{T\to\infty} \lambda(T)e^{-rT} &= 0. && \text{(3.6.iii)}
\end{aligned}$$

[14] Becker and Murphy (1988) and Driskill and McCafferty (2001) used a slightly more general utility function. Because our purpose is to illustrate a particular possibiblity, the simpler version is suitable.

[15] $\delta > 0$ corresponds to what Becker and Murphy (1988) denote as the addictive case.

where L is the current-value Hamiltonian, λ is the current-value costate variable and the asterisk on u denotes an optimal value. Consumers have perfect foresight and know that the equilibrium values of u are a linear function of the state, z :

$$u(t) = \gamma_0 + \gamma z(t) \tag{3.7}$$

where γ_0 and γ are as-yet-to-be-determined parameters. Integrating (3.7) from t to ∞ and substituting the resulting expression into (3.6.ii) yields the consumers' decision rule. This rule is the instantaneous demand curve that constrains firms at any moment in time:

$$p(t) = \chi_0 - \alpha u + \chi z \tag{3.8}$$

where

$$\chi_0 = \frac{\chi\gamma_0 + (r+s)\alpha_0 - \beta_0}{r+s} \tag{3.9.i}$$

$$\chi = \frac{\widehat{A}}{r+2s-\gamma} \equiv \psi(\gamma) \tag{3.9.ii}$$

and $\widehat{A} \equiv \delta(r+2s)$.

Equation (3.8) describes a demand curve that is downward-sloping at any moment but that shifts as z changes. Whether it shifts in or out for an increase in z depends on the sign of χ, which is in equilibrium a function of all the structural parameters of the model.

Equations (3.8) are a pair of relationships between (χ, γ) and (χ_0, γ_0). Firm behavior will provide another pair of relationships between these pairs of variables. These four relationships will then determine the equilibrium values of (χ, γ) and (χ_0, γ_0) as functions of structural parameters.

3.1.2 Firm Behavior

Firms are assumed to have the linear cost function

$$C(u_i) = cu_i; i = 1, 2;\ c \geq 0. \tag{3.10}$$

where $u_i \geq 0$ is the ith firm's rate of production. Each firm attempts to maximize

$$J^i = \int_0^\infty e^{-rt}[p(t)u_i(t) - cu_i(t)]dt;\ i = 1, 2. \tag{3.11}$$

subject to (3.2) and (3.8). Firms are assumed to pick strategies from a Markov strategy space:

Definition: The Markov strategy space for firm i is the set

$$S_i = \{u_i(z,t) \mid u_i(z,t) \text{ is continuous and differentiable for all } (z,t) \text{ and } u_i(z,t) \geq 0.\}$$

These strategies describe decision rules that are a function of the state, z, and time, t. The first-order conditions for the i^{th} firm is thus

$$\begin{aligned} p + u_i \frac{\partial p}{\partial u_i} - c + \lambda &= p - \alpha u_i - c + \lambda_i = 0 && (3.12.\text{i}) \\ \dot{\lambda}_i &= \lambda_i [r + s - \sum_{j \neq i} \frac{\partial u_j(z,t)}{\partial z}] && (3.12.\text{ii}) \\ & \quad -\chi u_i + \alpha \sum_{j \neq i} \frac{\partial u_j(z,t)}{\partial z}; \\ \lim_{T \to \infty} \lambda_i(T) e^{-rT} &= 0 && (3.12.\text{iii}) \end{aligned}$$

Assume the firms' strategies are symmetric and a linear function of z :

$$u_i(z,t) = m_0 + mz(t); m_0 \in R; m \in R. \quad (3.13)$$

With this assumption, time-differentiating (3.12.i), substituting this result into (3.12.ii) and substituting (3.12.i) into this expression for λ_i does in fact yield a linear relationship between u_i and z. This is the firm's strategy. Aggregating this strategy over both firms yields a linear relationship between u and z. Equating coefficients between this relationship and (3.7) yields the promised second pair of relationships between (χ, γ) and (χ_0, γ_0) :

$$\begin{aligned} \chi &= \frac{\frac{3}{2}\alpha\gamma(r+2s) - 2\alpha\gamma^2}{r + 2s - \gamma} \equiv \phi(\gamma); && (3.14.\text{i}) \\ -\{r + s - \frac{1}{2}\gamma\}(c - \chi_0) &= \gamma_0 \{\frac{3}{2}\alpha\}\{r + s - \frac{1}{2}\gamma\} + \frac{\chi}{2} + \alpha\gamma && (3.14.\text{ii}) \end{aligned}$$

Once the value of γ that simultaneously solves (3.9.ii) and (3.14.i) is found, it can be used to solve recursively all other endogenous variables in the model, in particular γ_0 and χ. Knowledge of the values of these endogenous parameters allows one to describe the equilibrium strategies of the two firms and the equilibrium decision rule of the consumers. Note that, because $\dot{z} = u - sz$, asymptotic stability of the model would require that $\gamma < s$.

Equating (3.9.ii) to (3.13.i) yields the following quadratic equation in γ:

$$\gamma^2 - \frac{3}{4}(r + 2s)\gamma + A = 0, \; A \equiv \frac{\widehat{A}}{2\alpha}. \quad (3.15)$$

For the roots to this equation to be real, we need the following restriction on the parameters of the model:

$$A < \frac{9}{64}(r + 2s)^2 \quad (3.16)$$

We assume throughout that (3.16) is satisfied. Denote the smaller root of (3.15) as γ_1 and the larger as γ_2. If $A \leq 0$, $\gamma_1 \leq 0$ and $\gamma_2 > s$. If $A > 0$, two possibilities emerge. First, if $0 < A < \frac{3}{4}rs + \frac{1}{2}rs^2$, then $0 < \gamma_1 < s < \gamma_2$.[16] On the other hand, if $\frac{3}{4}rs + \frac{1}{2}rs^2 < A < \frac{9}{64}(r+2s)^2$ and $2s > 3r$, then $0 < \gamma_1 < \gamma_2 < s$. For example, if $s = 2,\ r = 1$, and $A = \frac{224.75}{64}$, then $\gamma_1 = \frac{29}{16}$ and $\gamma_2 = \frac{31}{16}$.

Because most of the literature has viewed as less problematic the case of multiple equilibria where only one of the equilibria is stable, our primary interest is in situations in which multiple stable equilibria exist, that is, in situations in which parameter values are such that $0 < \gamma_1 < \gamma_2 < s$. Note, though, that our results apply to the case where one stable and one unstable equilibria exist, that is, for the case where parameters are such that $\gamma_1 < s < \gamma_2$

We are now prepared to describe and characterize the equilibrium of the above game.

3.1.3 The Markov Nash Equilibrium

Definition: A Markov Nash equilibrium for the above game is:

1. A decision rule $u^* = h(p, z, t)$ that satisfies the consumer's dynamic optimization problem $\forall z, t$;
2. A pair of Markov strategies $\{u_1^*, u_2^*\} \in S_1 \times S_2$ such that for every possible initial condition $\{z_0, t_0\}$:
$$J^i(u_i^*, u_j^*) \geq J^j(u_i, u_j^*)$$
for every $u_i \in S_i,\ i, j = 1, 2; i \neq j$;
3. A market-clearing condition that requires $\forall z, t$,
$$u^* = u_1^* + u_2^*$$

Note that the market-clearing condition implies that there exists an equilibrium price function $p = H(z, t)$, implicitly defined by equating $h(p, z, t)$ to $\sum_i u_i^*(z, t)$.

Proposition 3 *Assume (3.16) is satisfied. Let $u_{i,k}^*$ denote the i^{th} firm's equilibrium strategy in the k^{th} equilibrium, u_k^* denote the consumers' equilibrium decision rule in the k^{th} equilibrium, and p_k^* denote the equilibrium price in the k^{th} equilibrium. There exist two symmetric Markov equilibria for the preceding game, each described by the following equilibrium firm strategies and consumers' decision rules:*

$$\begin{aligned} u_{i,1}^* &= \frac{\gamma_{0,1}^*}{2} + \frac{\gamma_1^*}{2}z;\ i = 1, 2; \\ u_{i,2}^* &= \frac{\gamma_{0,2}^*}{2} + \frac{\gamma_2^*}{2}z;\ i = 1, 2; \\ p_1^* &= \chi_{0,1} - \alpha u_1^* + \chi_1 z; \\ p_2^* &= \chi_{0,2} - \alpha u_2^* + \chi_2 z \end{aligned}$$

[16] Note that $\frac{3}{4}rs + \frac{1}{2}rs^2 < \frac{9}{64}(r+2s)^2$.

where

$$\gamma_1^* = \frac{\frac{3}{4}(r+2s) - \left(\frac{9}{16}(r+2s)^2 - 4A\right)^{\frac{1}{2}}}{2};$$
$$\gamma_2^* = \frac{\frac{3}{4}(r+2s) + \left(\frac{9}{16}(r+2s)^2 - 4A\right)^{\frac{1}{2}}}{2};$$
$$\chi_k^* = \frac{\frac{3}{2}\alpha\gamma_k^*(r+2s) - 2\alpha\left(\gamma_k^*\right)^2}{r+2s-\gamma_k^*}; k = 1,2;$$

and $\gamma_{0,k}^*$ *and* $\chi_{0,k}^*$ *solve*

$$-\{r+s-\frac{1}{2}\gamma_k^*\}(c-\chi_{0,k}^*) = \gamma_{0,k}^*\{\frac{3}{2}\alpha\}\{r+s-\frac{1}{2}\gamma_k^*\} + \frac{\chi_k^*}{2} + \alpha\gamma_k^*;$$
$$\chi_{0,k}^* = \frac{\chi_k^*\gamma_{0,k}^* + (r+s)\alpha_0 - \beta_0}{r+s};$$
$$k = 1,2.$$

Proof. Substitution of the consumers' decision rule and the firm strategies into the consumers and firms first-order conditions, respectively, give rise to equations (3.9) and (3.14), respectively. ∎

We now move on to the finite-horizon version of this model and show that only the infinite-horizon equilibrium associated with the smaller value of γ^*,i.e., γ_1^*, is the limit of the associated finite-horizon equilibrium.

3.2 The Finite-Horizon game.

Assume a horizon of length T. The first-order conditions for the consumer differ from those in the infinite-horizon case only in that, in the finite-horizon case, $\lambda(T) = 0$. This means that at T consumers act myopically and choose consumption so as to equate price to instantaneous marginal utility:

$$p(T) = \alpha_0 - \alpha u(T) + \delta z(T) \tag{3.17}$$

The other difference between the finite and infinite-horizon game is that the equilibrium output function in the finite-horizon game has time-varying parameters:

$$u^*(t) = \gamma_0(t) + \gamma(t)z(t). \tag{3.18}$$

Following the same steps as in the infinite-horizon case, the consumer's first-order conditions can be manipulated to yield the following instantaneous demand curve:

$$p(t) = \chi_0(t) - \alpha u(t) + \chi(t)z(t) \tag{3.19}$$

where the constraints on $\chi_0(t)$, $\chi(t)$, $\gamma_0(t)$, and $\gamma(t)$ obey the following differential equations:

$$\dot{\chi} = \chi(r+2s-\gamma) - \widehat{A} \tag{3.20.i}$$
$$\dot{\chi}_0 = \chi_0(r+s) - \beta_0 - (r+s)\alpha_0 - \gamma_0\chi \tag{3.20.ii}$$

Note that (317) and (3.19) together imply the following terminal conditions on $\{\chi_0(T), \chi(T)\}$:

$$\chi_0(T) = \alpha_0 \tag{3.21.i}$$
$$\chi(T) = \delta \tag{3.21.ii}$$

Turning to firm behavior, the only difference between the first-order conditions for firms in the finite-horizon and infinite-horizon case is that, in the finite-horizon case, $\lambda_i(T) = 0$. This means that at T, firms act as one-shot profit maximizers and choose output such that instantaneous marginal revenue equals instantaneous marginal cost. The other key difference is that firm strategies have time-varying parameters:

$$u_i = m_0(t) + m(t)z(t), \; i = 1, 2. \tag{3.22}$$

Repeated time-differentiation and substitution of the first-order conditions along with use of the equilibrium price and output functions yields the following differential equations that must be obeyed by the parameters $\{\chi_0(t), \chi(t), \gamma_0(t), \gamma(t)\}$:

$$\dot{\gamma} = \frac{-\gamma^2 + \gamma(r+2s)\left(\frac{3}{4}\right) - A}{\left(\frac{3}{2}\right)\alpha} \tag{3.23.i}$$

$$\dot{\gamma}_0 = \frac{\chi_0\left(\frac{1}{2}\right)\gamma}{\left(\frac{3}{2}\right)\alpha} + \frac{\gamma_0[\left(\frac{3}{2}\alpha\right)\left(r+2s-\left(\frac{3}{2}\gamma\right)\right) + \left(\frac{1}{4}\alpha\gamma\right) - \left(\frac{1}{2}c\gamma\right) - (r+s)(\alpha_0 - c)]}{\left(\frac{3}{2}\right)\alpha} \tag{3.23.ii}$$

Equations (3.20) and (3.23) constitute a system of differential equations in the four parameters $\chi_0(t)$, $\chi(t)$, $\gamma_0(t)$, and $\gamma(t)$. The system can be solved recursively starting with (3.23.i), which only involves γ, and then moving to (3.20.i), and thence to (3.20.ii) and (3.23.ii). By construction,the critical values of these parameters are the values of the parameters from the autonomous infinite-horizon problem. Terminal conditions are derived from the first-order conditions at T:

$$\gamma(T) = \frac{\delta}{\left(\frac{3}{2}\alpha\right)}; \tag{3.24.i}$$

$$\gamma_0(T) = \frac{(\alpha_0 - c)}{\left(\frac{3}{2}\alpha\right)} \tag{3.24.ii}$$

We can now state the following proposition:

Proposition 4 *Assume (3.16) is satisfied. Let* $\widetilde{z}, \widetilde{u}$, *and* $\widetilde{p}$ *denote the equilibrium stock of consumption capital, equilibrium industry output (and consumption), and equilibrium price, respectively, in the finite-horizon game. For any* z_0, $t > 0$ *and any arbitrarily small* $\varepsilon > 0$, *there exists a horizon length* $T > t$ *sufficiently large such that* $|\widetilde{z}(t) - z_1(t)| < \varepsilon, |\widetilde{u}(t) - u_1^*(t)| < \varepsilon$, *and* $|\widetilde{p}(t) - p_1^*(t)| < \varepsilon$.

Proof. We need to show that the parameters $\chi_0(t)$, $\chi(t)$, $\gamma_0(t)$, and $\gamma(t)$ spend more and more time "close" to the critical values $\chi^*_{0,1}$, χ^*_1, $\gamma^*_{0,1}$, and γ^*_1. Because of the recursive nature of the problem, it will suffice to show that $\gamma(t) \to \gamma^*_1$ as $T \to \infty$. Consider (3.23.i). Denote the r.h.s. as $\theta(\gamma)$. The relevant properties of θ are:

$$\begin{aligned}
\theta(0) &= -\frac{2A}{3\alpha}; \\
\theta' &= \frac{-2\gamma + \frac{3}{4}(r+2s)}{\frac{3\alpha}{2}}; \\
\theta'' &= \frac{-4}{3\alpha} < 0; \\
\theta(\gamma(T)) &= \frac{-\left(\frac{2\delta}{3\alpha}\right)^2}{\frac{3\alpha}{2}} < 0; \\
\theta'(\gamma(T)) &= \frac{-4\delta}{3\alpha} + \frac{3}{4}(r+2s) > 0.
\end{aligned}$$

That is, $\theta(\gamma)$ is a concave function with a value of zero at γ^*_1 and γ^*_2. Furthermore,because $\theta(\gamma(T)) < 0$ and $\theta'(\gamma(T))$, it must be that $\gamma(T) < \gamma^*_1$. Hence, $\forall t \in (0,T), \dot{\gamma} < 0$. Hence, as $T \to \infty, \gamma(t) \to \gamma^*_1$. ■

The above relationship between $\theta(\gamma(T))$, $\theta(\gamma)$, and γ is illustrated in Figure 2:

$-2x^2 + 2x - .5$

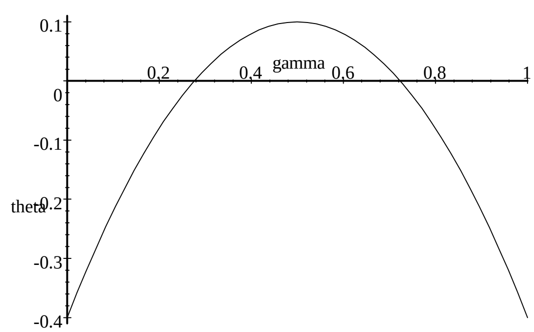

Figure 2

As in the Muth model example, here the limit of the finite horizon equilibrium is the infinite-horizon equilibrium associated with the smaller root of the fundamental characteristic equation. Also as in the Muth example, the selection criteria rules out both stable and unstable solutions Unlike the Muth example, though, the key interactions in the model involve strategic behavior. In additional contrast to the Muth model, the lack of any stochastic features means that criteria such as McCallum's minimum variance condition are simply inapplicable here.

4 Overlapping Generations Example

In this section we consider a model similar to those developed by Krugman (1991) and Matsuyama (1991). Both were interested in the problem of industrialization in an environment where an agglomeration externality existed in the manufacturing sector but not in the traditional agricultural sector. Both made explicit the forecasting and investment decision made by individuals when deciding in which sector to work. Krugman's model posited a positive linear relationship between the size of the manufacturing sector and the marginal product of a worker in that sector. Adjustment costs kept everyone from moving at once from one sector to another. Multiple equilibria existed in part because having everyone in either sector could be rational.

Matsuyama's model employed an overlapping-generations framework in which heterogeneous individuals must make an irrevocable choice at birth in which sector to work. In contrast to Krugman, he focused on a nonlinear agglomeration externality. Like Krugman, he found that multiple equilibria were possible.

The model we use here borrows heterogeneity and the overlapping-generations framework from Matsuyama and the linear agglomeration externality from Krugman. We first focus on parameter values of the model that allow us to focus on multiple equilibria that arise directly from the self-fulfilling nature of rational expectations. What we find is that only one of these equilibria are the limit of the associated finite-horizon model. We then look at parameter values that generate multiple equilibria more directly related to the types studied by Krugman and Matsuyama. What we show is that the finite-horizon selection criterion doesn't select one equilibrium precisely because there is not a unique finite-horizon limit.

4.1 The model.

Consider a small open economy that produces a traditional agrarian good and a manufactured good, both of which are sold at constant world prices normalized to one. At every moment a generation of size ρ is born. Each member faces a constant probability of death equal to ρ. At the moment of birth, individuals must make an irrevocable choice of whether to work in the traditional agrarian sector and receive a constant flow income R_a, or to work in the manufacturing sector and receive a flow income R_m that depends on the total number of workers in the manufacturing sector. Let u denote the size of the group of members of a generation that choose the manufacturing sector and z denote the size of the total group of people in the manufacturing sector. Members of each generation are heterogeneous in their productivity in the manufacturing sector. Perfect annuity markets exist, there are no bequest motives, and everyone can borrow and lend at the fixed world interest rate r, which is also each individual's rate of time preference.

4.1.1 The Infinite-Horizon Model.

First consider the infinite-horizon version of this model. Our goal here is to demonstrate that for some parameter values and initial conditions, there exist at least two equilibria. One of these has an interior steady state, that is, a steady state in which both sectors of the economy operate. In the other equilibrium, the steady state has all members of the economy working in the manufacturing sector.

A member of generation x, i.e., someone born at time $t = x$, maximizes expected utility:

$$\max_c \int_{t=x}^{t=\infty} e^{-\rho(t-x)} U(c) e^{-r(t-x)} dt \tag{4.1}$$

subject to his or her budget constraint:

$$\dot{w} = (r+\rho)w + R_i - c_i, \; i = a, m \tag{4.2}$$

where r is the constant rate of interest at which people can borrow and lend, w is non-human wealth and c_i is consumption. The optimal program requires consumption to be constant. Hence, the present discounted value of this constant rate of consumption equals the present discounted value of lifetime resources. If one stays in the agrarian sector, the present discounted value of lifetime resources is

$$\int_{t=x}^{t=\infty} R_a e^{-(r+\rho)(t-x)} dt = \frac{R_a}{r+\rho} \tag{4.3}$$

where, as noted, R_a is a positive constant.

If one opts for the manufacturing sector, the present discounted value of lifetime resources is

$$\int_{t=x}^{t=\infty} R_m(z) e^{-(r+\rho)(t-x)} dt \tag{4.4}$$

Because consumption is constant, each member of generation x chooses in which sector to locate based on which choice gives him or her the highest present value of lifetime resources. Assume

$$R_m(z) = \hat{R}_m + \widehat{A} z(t), \; \widehat{A} \in R^+ \tag{4.5}$$

where $\widehat{A}$ is a positive constant and, for each individual, $\widehat{R}_m$ is a positive constant. Assume, though, that individuals in any generation are heterogeneous in terms of $\hat{R}_m$ and that the distribution of $\hat{R}_m$ over members of a generation is uniform with highest value $\bar{R}_m$ and lowest value $\underline{R}_m$. Without loss of generality, think of the members of any generation as being sequenced in order of decreasing values of $\hat{R}_m$. The relationship between $\hat{R}_m$ for the last member of a generation that locates in the manufacturing sector and the size of the group of members of a generation that locate in the manufacturing sector is thus

$$\hat{R}_m = \bar{R}_m - \widehat{a}u,\ \widehat{a} \equiv \frac{\overline{R}_m - \underline{R}_m}{\rho}. \tag{4.6}$$

For any generation x, an (interior) equilibrium distribution of its members across the agrarian and manufacturing sectors is determined by equality of lifetime resources in either the agrarian or manufacturing sector for the marginal member:

$$\int_{t=x}^{t=\infty} [\widehat{R}_m + \widehat{A}z(t)]e^{-(r+\rho)(t-x)}dt = \frac{R_a}{r+\rho} \tag{4.7}$$

Using (4.6) and rearranging yields

$$u(x) = R + A\int_{t=x}^{t=\infty} z(t)e^{-(r+\rho)(t-x)}dt; \tag{4.9}$$

where for notational convenience we have made the following definitions of new variables

$$R \equiv \frac{\overline{R}_m - R_a}{a};\ a \equiv \frac{\widehat{a}}{r+\rho};\ A \equiv \frac{\widehat{A}}{a} \tag{4.9}$$

The forecasting problem faced by members of generation x is thus determination of the occupational choices of all future generations. Putting (4.9) in differential form by taking the derivative with respect to x, we have:

$$\dot{u} = (r+\rho)u - Az - R(r+\rho) \tag{4.10}$$

The dynamics of manufacturing sector growth are governed by new additions to the sector and by deaths of members of the sector. Hence,

$$\dot{z} = u - \rho z \tag{4.11}$$

Equations (4.10) and (4.11) thus determine the evolution of u and z through time. The solution to this coupled system is given by:

$$u = \overline{u} + (\rho+\theta_1)B_{21}e^{\theta_1 t} + (\rho+\theta_2)B_{22}e^{\theta_2 t}; \tag{4.12.i}$$

$$z = \overline{z} + B_{21}e^{\theta_1 t} + B_{22}e^{\theta_2 t}. \tag{4.12.ii}$$

where B_{21} and B_{22} are constants and

$$\overline{u} = \frac{\rho(r+\rho)R}{\rho(r+\rho) - A}; \tag{4.13.i}$$

$$\overline{z} = \frac{(r+\rho)R}{\rho(r+\rho) - A}; \tag{4.13.ii}$$

$$\theta_1 = \frac{1}{2}\left\{r - \sqrt{r^2 - 4[A - \rho(r+\rho)]}\right\}; \tag{4.13.iii}$$

$$\theta_2 = \frac{1}{2}\left\{r + \sqrt{r^2 - 4[A - \rho(r+\rho)]}\right\}; \tag{4.iv}$$

Because our goal is to investigate models in which there may be two equilibria, we first assume:

$$A < \rho(r + \rho) \tag{4.14}$$

This insures that θ_1 and θ_2 are real and that $\overline{u}$ and $\overline{z}$ are both positive. Note that, with this assumption

$$\theta_1 < 0;\ \theta_2 > 0. \tag{4.15}$$

Two solutions. The determination of the B_{ij}'s requires boundary conditions. One boundary condition inherent in the structure of the model is an initial value of z, say z_0, the stock of workers in the manufacturing sector at some initial time t_0. A second "natural" boundary condition is not obvious, though. One possibility that might occur to some people would be a requirement that $\lim_{t\to\infty} z(t) = \overline{z}$.[17] In this case, $B_{22} = 0$. Denote the complete solution under this boundary condition by labeling u and z in this case as u_1 and z_1. The solution is

$$u_1 = \overline{u} + (\rho + \theta_1)(z_0 - \overline{z})e^{\theta_1 t}; \tag{4.16.i}$$
$$z_1 = \overline{z} + (z_0 - \overline{z})e^{\theta_1 t}. \tag{4.16.ii}$$

This is clearly a solution: it satisfies (4.9) for any arbitrary feasible value z_0. The question is whether there are other solutions.

Consider the following possibility of another equilibrium. Could there be a value of z, say $\widehat{z}$, such that if at time τ the economy were to find itself with $\widehat{z}$, then every member of every generation from τ onward would find it optimal to join the manufacturing sector? If this were to be the case, then for $t \geq \tau$, $u(t) = \rho$ and $\dot{z} = \rho - \rho z$. Hence, under this scenario, we can solve for $z(t)$ by substituting $\rho = u$ into (4.11). The solution is:

$$z(t) = 1 - e^{\rho(\tau - t)} + z(\tau)e^{\rho(\tau - t)},\ \tau \leq t. \tag{4.17}$$

If, at τ, it is optimal for all members of generation τ to join the manufacturing sector, then (4.9) must be satisfied with equality for $u(\tau) = \rho$ and for $z(t)$ given by (4.17):

$$\rho = R + A\int_{t=\tau}^{t=\infty} [z(\tau)e^{-(r+2\rho)(t-\tau)} + e^{-(r+\rho)(t-\tau)} - e^{-(r+2\rho)(t-\tau)}]dt \tag{4.18}$$

Solving for $z(\tau)$ yields

$$\widehat{z} = \frac{(\rho - R)(r + 2\rho)}{A} - \frac{\rho}{r + \rho} \tag{4.19}$$

[17] McCallum (1999,p. 621) calls this the "popular 'saddle path' or 'stability'criterion." In this model, though, the stable arm of the saddle path is clearly not the only 'stable' solution, where stability is referring to non-explosiveness.

Now, for this to be a sensible value, we need $0 < \widehat{z} < 1$. Substituting (4.19) into these inequalities and rearranging gives the following parameter restrictions necessary for this to be the case:

$$\frac{(\rho - R)(r + 2\rho)}{(r+\rho)(r+2\rho)+\rho} < \frac{A}{r+\rho} < \frac{r(r+\rho)}{\rho} + 2R + 2\rho \tag{4.20}$$

Note that the right-hand-side of (4.20) is satisfied whenever $A < \rho(r+\rho)$, the assumption made earlier in (4.14).

Let us denote the equilibrium values of u and z in this equilibrium as u_2 and z_2. We can summarize the above findings in the following proposition:

Proposition 5 *Assume* $\frac{(\rho-R)(r+2\rho)}{(r+\rho)(r+2\rho)+\rho} < \frac{A}{r+\rho} < \rho$. *Let* $\widehat{z}$ *be given by (4.19). If the initial condition* $z(\tau) \geq \widehat{z}$ *is satisfied, an equilibrium for the above model for all* $t \geq \tau$ *is:*

$$\begin{aligned} u_2(t) &= \rho; \\ z_2(t) &= 1 + (z(\tau) - 1)e^{-\rho(t-\tau)}; \\ t &\geq \tau \end{aligned}$$

Note that, in this equilibrium, $\lim_{t\to\infty} z_2(t) = 1$. In the steady state, all of the population is in manufacturing.

Existence of this equilibrium requires that the economy start at time τ with a sufficiently large stock of people in the manufacturing sector. One could think of this as being created by Stalinesque forced resettlements of an agrarian population into the manufacturing sector, or by the offering of subsidies over a sufficient length of time to induce enough people to move to the manufacturing sector so that at some time τ, $z(\tau) = \widehat{z}$.

An example. Consider the following parameter values:

$$A = \frac{1}{2};\ \rho = \frac{3}{4};\ R = \frac{1}{8};\ r = \frac{1}{4}. \tag{4.21}$$

With these values, we have:

$$\overline{z} = \frac{1}{2};\ \overline{u} = \frac{3}{8};\ \widehat{z} = \frac{46}{56} \tag{4.22}$$

Hence, if $z_0 \geq \frac{46}{56}$, then both of the above equilibria exist. The diagram of the phase plane for this system is shown in Figure 3, in which the $\dot{u} = 0$ locus is displayed as a dotted line and the $\dot{z} = 0$ locus is displayed as a thicker solid line.

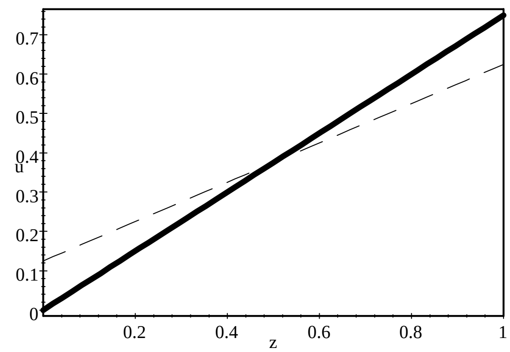

Figure 3

4.1.2 The finite-horizon model

Now consider the finite-horizon version of the above model. Assume a horizon of length T. The key difference introduced by this assumption of a finite horizon is that the occupational choice of members of generation T involves no forecast of future generations' behavior. This provides the second boundary condition to go with the condition that $z(0) = z_0$.

First consider the forecast of the occupational choice made by members of generation T. For a given $z(T)$, the size of the group of generation T that choose the manufacturing sector is found by equating the instantaneous earnings of the marginal worker in the manufacturing sector to the instantaneous earnings of the agrarian sector:

$$u(T) = R + (\frac{A}{r+\rho})z(T). \tag{4.23}$$

Now, from (4.12),

$$u(T) = \overline{u} + (\rho + \theta_1)B_{21}e^{\theta_1 T} + (\rho + \theta_2)B_{22}e^{\theta_2 T}; \tag{4.24.i}$$

$$z(T) = \overline{z} + B_{21}e^{\theta_1 T} + B_{22}e^{\theta_2 T} \tag{4.24.ii}$$

Substitution of (4.24) into (4.23) produces one equation that relates B_{21} and B_{22}. The other relationship between B_{21} and B_{22} comes from the initial condition that $z(0) = z_0$. Solving these two equations yields:

$$B_{21} = \frac{(\overline{z} - \overline{u} + R)e^{-\theta_2 T} + (z_0 - \overline{z})(\rho + \theta_2 - \frac{A}{r+\rho})}{(\rho + \theta_1 - \frac{A}{r+\rho})e^{(\theta_1 - \theta_2)T} - (\rho + \theta_2 - \frac{A}{r+\rho})}; \tag{4.25.i}$$

$$B_{22} = (z_0 - \overline{z}) - B_{21}. \tag{4.25.ii}$$

We can now state and prove the following turnpike theorem:

Proposition 6 *In the finite-horizon model, let $\widetilde{u}(t)$ denote the equilibrium values of the size of generation t that chooses manufacturing and let $\widetilde{z}(t)$ denote* *the size of the manufacturing sector at t . For any z_0, $t > 0$ and any arbitrarily small $\varepsilon > 0$, there exists a horizon length $T > t$ sufficiently large such that $|\widetilde{z}(t) - z_1(t)| < \varepsilon$, and $|\widetilde{u}(t) - u_1(t)| < \varepsilon$.*

Proof. Because $\theta_2 > 0$ and $\theta_1 < 0$, $\lim_{T\to\infty} B_{21} = (z_0 - \overline{z})$. This is turn implies that $\lim_{T\to\infty} B_{22} = 0$. Hence,

$$\begin{aligned} \lim_{T\to\infty} \widetilde{u}(t) &= \overline{u} + (\rho + \theta_1)(z_0 - \overline{z})e^{\theta_1 t}; \\ \lim_{T\to\infty} \widetilde{z}(t) &= \overline{z} + (z_0 - \overline{z})e^{\theta_1 t}. \end{aligned}$$

Q.E.D. ■

4.1.3 Discussion

The above turnpike result implies that, whether or not the economy ever achieves a size of the manufacturing sector equal to $\widehat{z}$, the only equilibrium that is the limit of the equilibrium of the finite-horizon model is $\{u_1(t), z_1(t)\}$, the equilibrium in which, for any feasible initial value of $z(0)$, u and z approach their critical values $\{\overline{u}, \overline{z}\}$ along the saddlepath. There is a sense of "perfectness" to this equilibrium that is not shared by the other: a disturbance to the stock of workers in the manufacturing sector, perhaps caused by some unanticipated exogenous shock or by non-optimal behavior by a measurable sequence of generations, does not change the solution.

In terms of the debate about history versus expectations, the above result tells us that if people do coordinate their belief structures around the finite-horizon model, then affecting history with policies designed to get the economy to a position with a critical size of the manufacturing sector, namely $\widehat{z}$, would not lead to the equilibrium in which all generations thereafter had all members choosing manufacturing. Expectations based on backward induction from the last generation's optimal choice always leads to the equilibrium described by $\{u_1(t), z_1(t)\}$.

4.1.4 A cyclical equilibrium.

In the above example, the limit of the finite-horizon model converged to a unique equilibrium. There are parameter values for this model, though, for which this is not the case. This example illustrates that for some model specifications, there may not be a unique limit to the associated finite-horizon model, and the coordination-of-beliefs problem remains.

Assume now that $R < 0$ and $A > (\rho - R)(r + \rho)$. This case corresponds closely to the model studied by Krugman (1991) in that, as in his model, one equilibrium is clearly: everyone in the manufacturing sector ($u = \rho, z = 1$), and another is everyone in the traditional sector ($u = 0, z = 0$). Furthermore, another equilibrium is $\{\overline{u}, \overline{z}\}$ where these values are given in (4.13.i) and (4.13.ii).

In this case, the roots to the characteristic equation are imaginary, and the solutions to the model are now given by:

$$\begin{aligned} z(t) &= \overline{z} + e^{rt}[k_1 \cos\omega t + k_2 \sin\omega t]; \\ u(t) &= \overline{u} + e^{rt}[(k_1\omega + \rho k_2)\sin\omega t + (\rho k_1 - k_2\omega)\cos\omega t]; \\ \omega &\equiv [A - \rho(r+\rho)]^{\frac{1}{2}} \end{aligned}$$

where k_1 and k_2 are constants to be determined from boundary conditions. Again, the initial condition that arises naturally from the economics of the problem is an exogenous specification of $z(0) = z_0$. This gives us $k_1 = z_0 - \overline{z}$. The second boundary condition, though, is simply that the last generation behave optimally, and choose $u(T) = R + \frac{A}{r+\rho}z(T)$. Unfortunately, in this case of cyclical equilibrium paths, the $\lim_{T\to\infty} k_2(T)$ is not necessarily unique. To illustrate this point most easily, consider the case of $r = 0$ and $z_0 = \overline{z}$. In this case, u and z oscillate with constant amplitude. While any particular choice of T does provide the necessary second boundary condition that provides a complete solution, this solution does not approach a unique limit as $T \to \infty$: because of the constant amplitude of the solutions, as T takes on different values over a segment of time equal to 2π radians, an infinite number of solutions are generated, and each of these solutions is then replicated at time intervals for T equal to 2π radians.

5 Summary and Conclusion.

Note some of the similarities between all three of the above examples. As noted in the introduction, all are characterized by a fundamental second-order dynamic equation. These equations all arise in part from an equilibrium condition at any moment in time for which the value of the equilibrating variable at that moment depends upon the future adjacent-moment's value of that equilibrating variable. In the Muth model, for example, this variable is price. In the overlapping-generations model, it is the size of generation that chooses the manufacturing sector. At any time t, we can think of price as equilibrating demand and supply at that moment. This equilibrium price, though, depends upon the expectation of next-period's price. The rational expectations assumption can be thought of as an assumption that people assume that next-period's price is determined as that price for which equilibrium in that period is satisfied. Of course, that next-period equilibrium price depends on the following-period price, and so on for as far into the future as the model permits. In an infinite-horizon model, this means that there is always one more price to be determined-the next-period price- than there are equilibrium conditions.

A similarity between the first two models is that whether or not the largest characteristic root of the fundamental second-order dynamic equation is stable or explosive depends on parameter values. The finite-horizon selection criterion, though, always picked the smaller root. This highlights that the existence of bubble solutions is a manifestation of the generic property of dynamic rational expectations models that, in an infinite-horizon model, there is always one more equation than there is unknowns.

Seen in this light, multiple equilibria in dynamic rational expectations models are a fundamental feature of infinite-horizon models. For such models, any selection criteria for choosing among the multiple equilibria must in some way provide an additional condition that makes the model determined.

It is useful to compare and contrast the finite-horizon selection criterion with learning approaches. Learning approaches provide the needed additional boundary condition by positing an initial disequilibrium value for some parameter in the model and then positing an updating rule. One similarity between the two approaches is that they both select the same equilibrium in the Muth model, the model for which an explicit comparison can be made. Another similarity is that they both have, in the language of game theory, an element of "perfectness" as providing the "refinement". The specifics, though, are different. In the learning approach, this "perfectness" is associated with "trembles" by atomistic agents about what are the parameters or the values of the rational expectations solution.[18] In the finite-horizon approach, the "perfectness" arises directly because the finite horizon leads to construction of a backward-induction solution.

The key difference between the two approaches is that the criterion espoused in this paper is built upon components more familiar to economists than ideas about how people learn. The learning approach, though, surely has great intuitive appeal to many economists as an accurate description of how the world works.

Finally, we should emphasize that the observation that all dynamic rational expectations are short one boundary condition does not imply that once such a boundary condition is supplied, all dynamic rational expectations models have a unique solution. For the examples studied here (except the one case that generated a cyclical constant-amplitude solution), this is the case. These examples were chosen, though, to highlight the possibility that *some* of these models have a unique solution when subject to this selection criterion. Other models, such as Howitt and McAfee (1992), in which imposition of a learning refinement did not eliminate multiple equilibria, may have multiple equilibria that arise for different reasons.

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[18] See Guesnerie (2002) for a discussion of this.

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