REPEATED GAMES WITH ASYNCHRONOUS MOVES

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Abstract

This paper studies a class of dynamic games, called repeated games with asynchronous moves, where not all players may revise their actions in every period. With state-dependent backwards induction, we introduce the concept of effective minimax in repeated games with asynchronous moves. A player's effective minimax value crucially depends on the asynchronous move structure in the repeated game, but not on the player's minimax or effective minimax value in the stage game. Any player's equilibrium payoffs are bounded below by his effective minimax value. We establish a folk theorem: when players are sufficiently patient, any feasible payoff vector where every player receives more than his effective minimax value can be approximated by a perfect equilibrium in the repeated game with asynchronous moves. This folk theorem integrates Fudenberg and Maskin's (1986) folk theorem for standard repeated games, Lagunoff and Matsui's (1997) anti-folk theorem for repeated pure coordination game with asynchronous moves, and Wen's (2002) folk theorem for repeated sequential games.

JEL Classification: C72, C73.

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1 Introduction

Players behave quite differently in a stage game and in its repeated game due to their different short-run and long-run objectives. When all players are sufficiently patient, repeated games admit almost all "reasonable" outcomes in equilibrium. This type of result is known as the folk theorem.[1] One seminal paper in this area is Fudenberg and Maskin (1986). Under certain conditions, Fudenberg and Maskin's folk theorem asserts that any feasible and strictly individually rational payoff of a stage game can be supported by a perfect equilibrium in the corresponding infinitely repeated game. Traditionally, Stage games are often modeled in normal-form and it is often assumed that players revise their actions simultaneously and synchronously in every period. Fudenberg and Tirole (1991) first raise the issues on simultaneous moves and asynchronous moves, which have attracted a lot of attention in the recent years. To the best of my knowledge, there are mainly two approaches to deal with these two separate but related issues.

To analyze the issue on simultaneous moves, repeated extensive-form stage games are studied by Rubinstein and Wolinsky (1995), and Sorin (1995). Sorin (1995) argues, if an extensive-form stage game satisfies the full dimensionality condition, Fudenberg and Maskin's folk theorem is valid on the repeated game of the normal-form representation of the stage game. Examples by Rubinstein and Wolinsky (1995), however, suggest a very different result when stage games do not satisfy the full-dimensionality condition. Wen (2002) studies a class of repeated extensive-form stage games, known as repeated sequential games. By extending the concept of effective minimax of Wen (1994) to sequential games, a folk theorem is established for repeated sequential games. Takahashi (2002) further extends this concept to all extensive games with almost perfect information, and shows that equilibrium payoffs are bounded below by these effective minimax values.

Regarding the issue on synchronous moves, repeated games with asynchronous move are

[1] For surveys on repeated games, see Aumann (1981), Fudenberg and Tirole (1991), Benoȋt and Krishna (1998).

investigated, leaded by Lagunoff and Matsui (1997) with the repeated pure coordination game with alternating moves. This paper studies general repeated games with asynchronous move where not all players may revise their actions in every period. With a state-dependent backwards induction technique, we first introduce the concept of effective minimax in repeated games with asynchronous moves. A player's effective minimax value in a repeated game crucially depends on the asynchronous move structure, but not on the players' standard or effective minimax value in the stage game. We then establish a folk theorem: when players are sufficiently patient, any feasible payoff vector where every player receives more than his effective minimax value in the repeated game can be approximated by a perfect equilibrium in the repeated game. Our folk theorem integrates Fudenberg and Maskin's (1986) folk theorem for standard repeated games, Lagunoff and Matsui's (1997) anti-folk theorem for repeated pure coordination game with asynchronous moves, and also Wen's (2002) folk theorem for repeated sequential games.

The literature on repeated games with asynchronous moves is growing rapidly in the last a few years. Lagunoff and Matsui (1997) first show that in the infinitely repeated pure coordination game, if two players alternate in revising their actions then the Pareto optimal outcome will be the only perfect equilibrium outcome in the repeated game. Lagunoff and Matsui (2002) further demonstrate that this type of anti-folk theorem phenomenon is fairly generic. The violation of the full dimensionality by the pure coordination game along cannot explain their findings. The asynchronous move structure plays a very important role. Haller and Lagunoff (2000b) investigate Markov perfect equilibrium in this class of repeated games. Yoon (2001) also studies asynchronously repeated games under the assumption that the players who do not move in any given period have to play the same actions (including mixed actions) from the past, and obtains the traditional folk theorem under certain conditions. In our model, however, the player who do not move in any given period have to play the realizations of their mixed strategies from the past. Repeated games with asynchronous moves belong to stochastic games studied by Dutta (1995), Haller and Lagunoff (2000a) and many others. Under the full dimensionality condition, Dutta (1995) proves a folk theorem based on players' minimax values that are formulated from equilibrium strategies from the entire stochastic game, which can be difficult to find.

This research is initiated from some of the questions raised in studying repeated sequential games. Some repeated games with asynchronous moves can be treated as repeated sequential games. For example, consider a repeated game of the following battle of the sexes:

$1 \setminus 2$	L	R
U	2, 1	0, 0
D	0, 0	1, 2

This game has two pure Nash equilibria, (U, L) and (D, R), and one mixed Nash equilibrium where every player receives his minimax value of 2/3. Now suppose that player 1 moves in every period but player 2 moves only once in every other period. Let δ be players' common discount factor per period. The game played over the two periods where player 1 moves in both periods and player 2 moves only in the first period is a sequential game by Wen (2002). Then this repeated game with asynchronous moves is a repeated sequential game with discount factor δ^2, where the sequential stage game depends on δ. In this two-period sequential stage game, player 2 payoff in the second period can be forced to zero. Therefore, player 2's minimax value in this two-period sequential game cannot be higher than 2/3, so 1/3 on average per period. On the other hand, player 1's payoff in the second period cannot not be lower than 1. So player 1's minimax payoff must be strictly greater than 2/3 on average per period. If player 2 moves only at 0, T, $2T$ and so on, player 2's payoffs can be forced as low as $2/(3T)$ on average, and player 1's payoffs cannot lower than his pure-strategy minimax value of one.[2]

Wen (2002) demonstrates how effective minimax value in sequential games can be used to explain Lagunoff and Matsui's (1997) anti-folk theorem. But the results from Wen (2002) are inadequate to deal with repeated games with asynchronous moves in general. Nevertheless, using effective minimax value from sequential games to analyze some repeated games with

[2]By feasibility in this game, player 2's equilibrium cannot be lower than 1/2 as 1's payoffs are not less than 1.

asynchronous moves motivates the state-dependent backwards induction technique developed here to derive effective minimax values in general repeated games with asynchronous moves. The contribution of this research is to understand this class of dynamic games and integrate a number of folk theorems for different repeated game models.

The rest of this paper is organized as follows. Section 2 introduces the model and reviews the concept of effective minimax in normal-form games. Section 3 presents two examples, with and without the NEU condition respectively, to illustrate how to derive a player's effective minimax value in repeated games with asynchronous moves. Based on the insight from these two examples, Section 4 introduces the effective minimax value for every player, and studies some of its properties. Section 5 presents a folk theorem for this class of repeated games and Section 6 concludes the paper.

2 The Model

Consider a finite n-player game in normal-form, $G = \{A_i, u_i(\cdot);\ i \in I\}$, where I is the set of n players, A_i is the set of player i's *pure actions*, and $u_i(\cdot) : \times_{j\in I} A_j \to \mathbf{R}$ is player i's *payoff function* for all $i \in I$. Denote $A = \times_{j\in I} A_j$ and $u(\cdot) \equiv (u_1(\cdot), \ldots, u_n(\cdot))$. A player i's mixed action is denoted as $\sigma_i \in \Sigma_i$, where Σ_i is the set of player i's mixed actions. Let $u_i(\cdot)$ also denote player i's expected payoff from a mixed action profile. $\Sigma = \times_{j\in I} \Sigma_j$ is the set of mixed action profiles. The normal-form game G is refereed as the stage game.

The *set of feasible payoffs* in G is the convex hull of $u(A)$, denoted by $F \subset \mathbf{R}^n$. $\forall\, i \in I$, a generic $\sigma \in \Sigma$ is decomposed as $\sigma = (\sigma_i, \sigma_{-i})$.[3] Player i's standard mixed *minimax value* in G is defined as[4]

$$m_i^s = \min_{\sigma_{-i}} \max_{a_i \in A_i} \quad u_i(a_i, \sigma_{-i}). \tag{1}$$

$m^s = (m_1^s, \ldots, m_n^s)$ is refereed as the standard mixed minimax point in game G. Any

[3] A pure action profile can be decomposed in the same way.

[4] It does not lose any generality to formulate player i's minimax value as in (1) where player i maximizes his payoff with respect to his pure actions, rather than his mixed actions. Player i's pure minimax value is formulated in a same way but the minimization is taken over the other players' pure actions.

payoff vector that strictly (weakly) dominates the standard mixed minimax point is strictly (weakly) *individually rational*. The set of feasible and strictly individually rational payoffs in game G is then $F^* = F \cap \{v \in \mathbf{R}^n | \; v >> m\}$.[5] In the repeated game of G with discounting, Fudenberg and Maskin's folk theorem asserts that if $n = 2$ or F^* has a dimension of n, every payoff vector in F^* can be supported by a perfect equilibrium of the repeated game when players are sufficiently patient.

In the repeated game of G with asynchronous moves, assume that not all players may revise their actions in every period. In period $t = 0$, all players choose their actions. In period $t > 0$, only a subset of players may revise their actions. Let $I_t \subset I$ denote the set of players who move in period t. Accordingly, players who are not in I_t cannot move in period t. $\{I_t\}_{t=0}^{\infty}$ is refereed as the asynchronous move structure. Repeated games with asynchronous moves includes Fudenberg and Maskin's (1986) model with $I_t = I$ for all $t \geq 0$, Lagunoff and Matsui's (1997) model with $I_t = \{1\}$ for all odd t and $I_t = \{2\}$ for all even t, and Wen's (2002) model with $I_t = I_{T+t}$ for all $t \geq 0$ and a finite T.

Players may play mixed actions whenever they move. It is assumed that players observe all past actions, including past mixed actions. Then a history at t consists of all the past actions, denoted by $h_t = (\sigma^0, \sigma^1, \ldots, \sigma^{t-1})$ such that $\sigma^0 \in \Sigma$ and

$$\sigma_i^s \text{ is a pure action in the support of } \sigma_i^{s-1} \text{ if } i \notin I_s \text{ for } 0 < s < t. \tag{2}$$

Condition (2) is the consistency requirement under the asynchronous moves. It means that the player who does not move in period s must play the same action as before. Without loss of generality, a history in period t can be simplified as

$$h_t = (\sigma_I, \sigma_{I_1}, \sigma_{I_2}, \ldots, \sigma_{I_{t-1}}), \tag{3}$$

where σ_{I_s} represents the actions of those in I_s at s for $0 \leq s \leq t-1$. Histories are important to identify the deviator and to coordinate punishment actions in the future. The set of

[5] $v >> m$ means $v_i > m_i$ for all $i \in I$.

histories in period t is denoted as H_t. H_0 contains only the null history at the beginning of the game.

A strategy profile specifies actions for those in I_t after any h_t. Player i's strategies is a function $f_i : \bigcup_{i \in I_t} H_t \to \Sigma_i$. Any strategy profile $f = (f_1, f_2, \ldots, f_n)$ induces a unique distribution on the set of outcome paths. A outcome path $\pi = (a^0, a^1, \cdots, a^t, \cdots)$ has to satisfies the consistency requirement as well imposed by asynchronous move structure, which is $a_i^t = a_i^{t-1}$ for all t and $i \notin I_t$. Players' average payoffs from outcome path π are

$$u(\pi) = (1 - \delta) \sum_{t=0}^{\infty} \delta^t u(a^t), \tag{4}$$

where $\delta \in (0, 1)$ is players' common discount factor per period. Players evaluate their strategies based on their expected payoffs from induced outcome paths. The equilibrium concept adopted in this paper is (subgame) perfect equilibrium, that constitutes a Nash equilibrium in every subgame of the repeated game after any finite history.

In this rest of this section, we review the concept of effective minimax value in normal-form games introduced by Wen (1994). According to Abreu, Dutta and Smith (1994), players i and j have equivalent utilities if $\exists\ \alpha_{ij} > 0$ and some β_{ij} such that

$$u_j(\sigma) \;=\; \alpha_{ji} \cdot u_i(\sigma) + \beta_{ji}, \quad \forall\ \sigma \in \Sigma. \tag{5}$$

$\forall\ i \in I$, let E_i be the set of players who have equivalent utilities to player i. From the definition, $E_i = E_j$ if and only if i and j have equivalent utilities. Game G satisfies the Non-Equivalent Utility (NEU) condition if each $E_i = \{i\}$ is singleton. Without the NEU condition, players in E_i do not want to minimize player i's payoff in minimaxing player i. Accordingly, player i's *effective minimax value* from G is defined as

$$m_i^e = \min_{\sigma} \left[\max_{j \in E_i} \max_{a_j} \quad u_i(\sigma) \right]. \tag{6}$$

m_i^e is the minimum of player i's payoff under the best unilateral deviation by those in E_i. Under the NEU condition, m_i^e from (6) simply reduces to m_i^s of (1) since $E_i = \{i\}$. From any solution to (6), player i's payoff is less than m_i^e in general. However, under the best unilateral deviation by those in E_i, player i's payoff is m_i^e.

$m^e = (m_1^e, \cdots, m_n^e)$ is the effective minimax point in G. Generally, we have $m^e \geq m^s$. Any payoff vector that strictly (weakly) dominates m^e is strictly (weakly) *Equivalent Utility Class* (EUC) rational. Wen's (1994) folk theorem asserts that any feasible and strictly EUC rational payoff vector can be supported by a perfect equilibrium of the repeated game of G when δ is large enough, without any condition on the stage game G.

3 Two Examples

We now consider two examples to motivate the concept of effective minimax values in repeated games with asynchronous moves. The first example is the repeated battle of the sexes game with alternating moves. The battle of the sexes game satisfies the NEU condition. The second example is the repeated pure coordination game with alternating moves studied by Lagunoff and Matsui (1997). The pure coordination game does not satisfy the NEU condition. These two examples illustrate how to calculate a player's lowest possible equilibrium payoff, effective minimax value, in repeated games with asynchronous moves when stage games satisfies and does not satisfy the NEU condition, respectively.

3.1 The Battle of the Sexes

Reconsider the battle of the sexes (BOS) game given in the introduction. There are two pure Nash equilibria, (U, L) and (D, R), and one mixed Nash equilibrium in which player 1 plays U with probability 2/3 and player 2 plays L with probability 1/3. Each player has the same minimax value of 2/3 in the mixed Nash equilibrium. The BOS game satisfies the full dimensionality condition, hence satisfies the NEU condition. Now suppose both players move simultaneously at $t = 0$, player 1 moves at all odd t, and player 2 moves at all even t. Accordingly, $I_0 = I$, $I_t = \{1\}$ for all odd t and $I_t = \{2\}$ for all even t. This repeated game satisfies the FPI condition.

As in Rubinstein's (1982) alternating-offer bargaining game, this repeated BOS game is structurally cyclic for every two periods, except at $t = 0$. With a state-dependent backwards induction technique as in Shaked and Sutton (1984), we derive a player's lowest possible perfect equilibrium payoff in this repeated BOS game with asynchronous moves.

Player 1's lowest possible equilibrium payoff at any odd t depends the initial state at t, either L or R, which player 2's action at $t-1$. So denote player 1's lowest possible equilibrium payoffs as $m_1(L)$ and $m_1(R)$ respectively. Due to the two-period cyclic structure, player 1's lowest possible equilibrium payoff in period $t+2$ will be also $m_1(a_2)$, depending on the realization of player 2' action $a_2 \in \{L,\ R\}$ at $t+1$. Player 1's equilibrium payoffs from $a_1 \in \{U, D\}$ at t then cannot be less than

$$m_1(a_1|L) = (1-\delta)u_1(a_1, L) + \delta \min_{\sigma_2 \in \Sigma_2} E_{\sigma_2}\left[(1-\delta)u_1(a_1, a_2(\sigma_2)) + \delta m_1(a_2(\sigma_2))\right], \tag{7}$$

where E_{σ_2} represents the expected value calculated from σ_2, and $a_2(\sigma_2)$ is the realization of σ_2 which becomes the initial state at $t+2$. The first term of (7) is player 1's payoff during period t, and the second term of (7) is player 1's lowest possible equilibrium payoff at $t+1$. Player 1's payoffs in period with initial state L cannot be less than

$$m_1(L) = \max_{\sigma_1 \in \Sigma_1} E_{\sigma_1} m_1(a_1(\sigma_1)|L), \tag{8}$$

where E_{σ_1} is the expected value calculated based on σ_1 and $a_1(\sigma_2)$ is the realization of σ_1. Given players choose probability distributions over two alternatives in the optimization problems of (7) and (8), these optimization problems simply become the optimal value from the two alternatives. Substituting player 1's payoffs into (7) and (8), we have

$$\begin{aligned} m_1(L) &= \max\left\{ m_1(U|L),\ m_1(D|L)\right\}, \text{ where} && (9)\\ m_1(U|L) &= (1-\delta)2 + \delta\min\left\{ (1-\delta)2 + \delta m_1(L),\ \delta m_1(R)\right\} && (10)\\ m_1(D|L) &= \delta\min\left\{ \delta m_1(L),\ (1-\delta) + \delta m_1(R)\right\}, && (11) \end{aligned}$$

By the same argument, if the initial state at any odd t is R, we have

$$m_1(R) = \max\left\{ m_1(U|R),\ m_1(D|R)\right\}, \text{ where} \tag{12}$$

$$m_1(U|R) \;=\; \delta \min\Big\{ (1-\delta)2 + \delta m_1(L),\ \delta m_1(R) \Big\}, \tag{13}$$

$$m_1(D|R) \;=\; (1-\delta) + \delta \min\Big\{ \delta m_1(L),\ (1-\delta) + \delta m_1(R) \Big\}. \tag{14}$$

Solving equations (9)—(14), we have the following proposition:

Proposition 1 *During the infinitely repeated BOS game with alternating moves, player's perfect equilibrium payoffs at any odd t with initial state $a_2 \in \{L, R\}$ are not less than $m_1(a_2)$, where*

$$m_1(L) = \frac{(1-\delta)(2+\delta^2)}{1-\delta^4}, \qquad m_1(R) = \frac{(1-\delta)(1+2\delta^2)}{1-\delta^4}. \tag{15}$$

Proof: To solve equations (9)—(14), we need to consider the following three cases:

Case 1: Assume that $\delta m_1(L) \geq (1-\delta) + \delta m_1(R)$.

Then $2(1-\delta) + \delta m_1(L) \geq \delta m_1(R)$. (10) (11), (13) and (14) become

$$m_1(U|L) = 2(1-\delta) + \delta^2 m_1(R), \qquad m_1(U|R) = \delta^2 m_1(R),$$
$$m_1(D|L) = (1-\delta)\delta + \delta^2 m_1(R), \qquad m_1(D|R) = (1-\delta^2) + \delta^2 m_1(R).$$

Note that $m_1(U|R) < m_1(D|R)$ and $m_1(D|L) < m_1(U|L)$ for all $\delta \in (0,1)$. (9) and (12) yield $m_1(R) = 1$ and $m_1(L) = 1 + (1-\delta)^2$, which contradict the inequality for Case 1. Therefore, Case 1 is impossible.

Case 2: Assume that $2(1-\delta) + \delta m_1(L) \leq \delta m_1(R)$.

Then $\delta m_1(L) \leq (1-\delta) + \delta m_1(R)$. (10), (11), (13) and (14) become

$$m_1(U|L) = 2(1-\delta^2) + \delta^2 m_1(L), \qquad m_1(U|R) = 2(1-\delta)\delta + \delta^2 m_1(L),$$
$$m_1(D|L) = \delta^2 m_1(L), \qquad m_1(D|R) = (1-\delta) + \delta^2 m_1(L).$$

Note that $m_1(U|L) > m_1(D|L)$ for all $\delta \in (0,1)$. (9) and (12) yield $m_1(L) = m_1(R) = 2$, which contradict the inequality for Case 2. Therefore, Case 2 is also impossible.

Case 3: Assume that $\delta m_1(L) \leq (1-\delta) + \delta m_1(R)$ and $2(1-\delta) + \delta m_1(L) \geq \delta m_1(R)$.

Then (10), (11), (13) and (14) become

$$m_1(U|L) = 2(1-\delta) + \delta^2 m_1(R), \qquad m_1(U|R) = \delta^2 m_1(L),$$
$$m_1(D|L) = \delta^2 m_1(R), \qquad m_1(D|R) = (1-\delta) + \delta^2 m_1(L).$$

With those four possible values, (9) and (12) yield

$$\begin{aligned} m_1(L) &= \max\left\{ \delta^2 m_1(L),\ (1-\delta)2 + \delta^2 m_1(R) \right\}, \\ m_1(R) &= \max\left\{ (1-\delta) + \delta^2 m_1(L),\ \delta^2 m_1(R) \right\}. \end{aligned}$$

Then there are three sub cases to consider.

Case 3.1: Assume that $\delta^2 m_1(L) \geq (1-\delta)2 + \delta^2 m_1(L)$.

Then $(1-\delta) + \delta^2 m_1(L) \geq \delta^2 m_1(R)$. (9) yields $m_1(L) = 0$ or 1. If $m_1(L) = 0$ then $m_1(R) < 0$, which is impossible. If $m_1(L) = 1$ then $m_1(R) = 1 - \delta + \delta^2$. However, the set values of $m_1(L)$ and $m_1(R)$ contradict the inequality for Case 3.1. Therefore, Case 3.1 is impossible.

Case 3.2: Assume that $(1-\delta) + \delta^2 m_1(L) \leq \delta^2 m_1(R)$.

Then $\delta^2 m_1(L) \leq (1-\delta)2 + \delta^2 m_1(L)$. (13) yields $m_1(R) = 0$ or 1. If $m_1(R) = 0$ then $m_1(L) = 2(1-\delta)$, which contradicts the inequality that defines case 3.2. If $m_1(R) = 1$ then $m_1(L) = 2(1-\delta) + \delta^2$, which also contradict the inequality for Case 3.2. Therefore, Case 3.2 is impossible.

Case 3.3: Assume that $\delta^2 m_1(L) \leq (1-\delta)2 + \delta^2 m_1(L)$ and $(1-\delta) + \delta^2 m_1(L) \geq \delta^2 m_1(R)$.

Then (9) and (12) become

$$m_1(L) = 2(1-\delta) + \delta^2 m_1(R) \quad \text{and} \quad m_1(R) = (1-\delta) + \delta^2 m_1(L),$$

which give the values of $m_1(L)$ and $m_1(R)$ from (15) in Proposition 1. **Q.E.D.**

It is straightforward to see that $m_1(L) > m_1(R) > 3/4 > 2/3$ for all $\delta \in (0,1)$. Both $m_1(L)$ and $m_1(R)$ are decreasing with respect to δ and have the same limit of 3/4 as δ goes to 1. Player 1's equilibrium payoffs at any odd t are not less than 3/4, which is strictly greater than 2/3, player 1's mixed minimax value in the BOS game.

At the beginning of this repeated BOS game, player 1's equilibrium payoffs cannot be lower than

$$m_1^0 = \min_\beta \max_\alpha \left[\beta[(1-\delta)2\alpha + \delta m_1(L)] + (1-\beta)[(1-\delta)(1-\alpha) + \delta m_1(R)] \right], \tag{16}$$

where α is the probability that player 1 plays U and β is the probability that player 2 plays L at $t = 0$. Substituting (15) into (16), we have

Proposition 2 *At the beginning of the repeated BOS game with alternating moves, player 1's perfect equilibrium payoffs cannot be less than*

$$m_1^0 = \frac{(1-\delta)(2 + 4\delta + \delta^3 + 2\delta^4)}{3(1-\delta^4)}. \tag{17}$$

Player 1 plays U with probability 2/3 and player 2 plays L with probability 1/3 at $t = 0$.[6]

m_1^0 from (17) is continuous and strictly greater than 2/3 for all $\delta \in (0, 1)$. As δ goes to 1, m_1^0 converges to 3/4 as well. This implies that player 1's equilibrium payoffs cannot be less than 3/4 at $t = 0$ when two player are sufficiently patient. In order to force player 1's payoff to be sufficiently close to 3/4, they play the mixed Nash equilibrium at $t = 0$, then follow a 4-period cycle. For example, from the initial state L, player 1 plays U, then player 2 plays R, then player 1 plays D, and then player 2 plays L which leads the initial state L again. Over this four-period cycle, player 1 receives 2 during the first period, 1 during the third period, and 0 during the second and last periods. As δ goes to 1, player 1 receives 3/4 per period on average.

Next, we derive player 2's lowest possible equilibrium payoff at $t = 0$. By symmetry, player 2's payoffs at $t = 2$ cannot be less than $m_2(U) = m_1(R)$ or $m_2(D) = m_1(L)$, depends on player 1's action at $t = 1$. At $t = 1$, player 1 chooses either U or D to minimize player 2's continuation payoff. By backwards induction, player 2's payoffs at $t = 1$ cannot be less

[6] As matter of a fact, any of player 1's mixed action at $t = 0$ will yield the same value of m_1^0 for given player 2's mixed action. However, player 1's mixed action will affect player 2's optimal action at $t = 0$.

than $m_2'(L)$ or $m_2'(R)$ if player 2 plays L or R at $t = 0$ respectively, where

$$m_2'(L) = \min\Big\{(1-\delta)2 + \delta m_2(U),\ \delta m_2(D)\Big\} = \delta m_2(D) = \frac{\delta(1-\delta)(2+\delta^2)}{1-\delta^4}, \tag{18}$$

$$m_2'(R) = \min\Big\{\delta m_2(U),\ (1-\delta) + \delta m_2(D)\Big\} = \delta m_2(U) = \frac{\delta(1-\delta)(1+2\delta^2)}{1-\delta^4}. \tag{19}$$

At the beginning of this repeated game, player 2's equilibrium payoffs cannot be less than

$$m_2^0 = \min_\alpha \max_\beta \Big[\beta[(1-\delta)\alpha + \delta m_2'(L)] + (1-\beta)[(1-\delta)(1-\alpha)2 + \delta m_2'(R)]\Big], \tag{20}$$

where α is the probability that player 1 chooses U and where β is the probability that player 2 chooses L at $t = 0$. Substituting (18) and (19) into (20), we have

Proposition 3 *At the beginning of the repeated BOS game with alternating moves, player 2's average perfect equilibrium payoffs cannot be less than*

$$m_2^0 = \frac{(1-\delta)(2+\delta^2)(1+2\delta^2)}{3(1-\delta^4)}. \tag{21}$$

For player 2 to receive a payoff sufficiently close to m_2^0, player 1 plays U with a probability of $(2+\delta^2)/(3+3\delta^2)$ at $t = 0$, then two player will follow another four-period cycle. Note that $m_2^0 \to 3/4$ as $\delta \to 1$. Propositions 2 and 3 imply that any perfect equilibrium payoff vector in the repeated game must weakly dominate $(3/4, 3/4)$ as $\delta \to 1$. Feasible and strictly individually rational payoff vectors that do not dominate $(3/4, 3/4)$ cannot be supported by perfect equilibrium in the repeated game.

3.2 The Pure Coordination

Consider the following pure coordination (PC) game:

$1 \setminus 2$	L	R
U	$a,\ a$	$b,\ b$
D	$c,\ c$	$d,\ d$

where $a > \max\{b, c, d\}$. So (U, L) is one pure Nash equilibrium. Two players have equivalent utilities and $\max\{b, c, d\}$ is either player's effective minimax value. Lagunoff and Matsui (1997) study the infinitely repeated PC game with alternating moves where $I_0 = I$, $I_t = \{1\}$

for all odd t, and $I_t = \{2\}$ for all even t, and show that the only perfect equilibrium outcome in the repeated game is (a, a) in every period for all $\delta \in (0, 1)$.

We consider this repeated game again to demonstrate how to derive a player's lowest possible equilibrium payoff when the NEU condition is violated. We find that a player's lowest possible equilibrium payoff is a when the initial state is either U or L (Theorem 1 of Lagunoff and Matsui (1997)), or sufficiently close to a as $\delta \to 1$ when the initial state is either D or R (Theorem 2 of Lagunoff and Matsui (1997)).

Since two players have equivalent utilities, player 2 now would maximize player 1's payoff. This changes the optimization problems in (7). Corresponding to (9)—(14), for this repeated PC game, we have

$$m_1(L) = \max\left\{ m_1(U|L),\ m_1(D|L) \right\}, \quad \text{and} \quad m_1(R) = \max\left\{ m_1(U|R),\ m_1(D|R) \right\},$$

where

$$\begin{aligned}
m_1(U|L) &= (1-\delta)a + \delta \max\left\{ (1-\delta)a + \delta m_1(L), (1-\delta)b + \delta m_1(R) \right\}, \\
m_1(D|L) &= (1-\delta)c + \delta \max\left\{ (1-\delta)c + \delta m_1(L), (1-\delta)d + \delta m_1(R) \right\}, \\
m_1(U|R) &= (1-\delta)b + \delta \max\left\{ (1-\delta)a + \delta m_1(L), (1-\delta)b + \delta m_1(R) \right\}, \\
m_1(D|R) &= (1-\delta)d + \delta \max\left\{ (1-\delta)c + \delta m_1(L), (1-\delta)d + \delta m_1(R) \right\}.
\end{aligned}$$

It is immediate that $m_1(L) = a$. Given $m_1(L) = a$, $m_1(R) \geq m_1(U|R) \geq (1-\delta)b + \delta a$. As $\delta \to 1$, $m_1(R) \to a$. At the beginning of the repeated game, player 1's equilibrium payoffs will not be less than

$$m_1^0 \;=\; \max_{a_2 \in \{L,R\}} \max_{a_1 \in \{U,D\}} [(1-\delta)u_1(a_1, a_2) + \delta m_1(a_2)] \;=\; a.$$

Since two players has equivalent utilities, player 2's perfect equilibrium payoffs will not be less than a either at $t = 0$. We now state these results as Proposition 4.

Proposition 4 *In the repeated PC game with alternating moves, a player's equilibrium payoff is a when the initial state is either U or L. As two players become sufficiently patient* *enough, a player's equilibrium payoffs will be sufficiently close to a for any initial state. At the beginning of the game, either player's equilibrium payoff is equal to a.*

4 Effective Minimax Values

The two examples from the previous section demonstrate that a player's lowest possible equilibrium payoff in a repeated game crucially depends on the asynchronous move structure. We now derive a player's lowest possible equilibrium payoff in a repeated game with asynchronous moves.

$\forall\ I' \subset I$, let $\sigma_{I'}$ and $a_{I'}$ be generic mixed and pure action profiles by those in I' respectively, and let $\sigma_{-I'}$ and $a_{-I'}$ be generic mixed and pure action profiles by those who are not in I'. Recall that the players in I_t move simultaneously at t. The actions by the other players will be fixed at $a^t_{-I_t}$ during period t. $a^t_{-I_t}$ is the initial state of the continuation game at t. An initial state at t is a pure action profile by those who are not in I_t since all past mixed actions have realized at t. If $I_t = I$, the initial state at t is denoted as $\emptyset$. For any $t \geq 0$ and $i \in I$, denote player i's lowest possible equilibrium payoff as $m^t_i(a_{-I_t})$ at t where the initial state is $a^t_{-I_t}$.

Given $m^{t+1}_i(\cdot)$ for all possible initial state at $t+1$, consider players' strategic behavior during period t. Players in I_t move simultaneously as in a normal-form game. In any perfect equilibrium, player i's payoff at t can be less than the sum of his payoff during period t and his lowest possible equilibrium payoff at $t+1$. Based on the concept of effective minimax in normal-form games, player i's equilibrium payoffs at t cannot be less than

$$\begin{aligned} m^t_i(a_{-I_t}) = \\ \min_{\sigma_{I_t}} \left\{ \max_{j \in E_i \cap I_t} \max_{a_j \in A_j} \left[(1-\delta) u_i(\sigma_{I_t}, a_{-I_t}) + \delta E_{\sigma_{I_t}} m^{t+1}_i(a'_{(-I_{t+1}) \cap I_t}, a_{-(I_{t+1} \cup I_t)}) \right] \right\}, \quad (22) \end{aligned}$$

where $E_{\sigma_{I_t}}$ represents the expected value calculated based on the probability distributions induced by σ_{I_t} on the initial states at $t+1$, and $a'_{(-I_{t+1}) \cap I_t}$ represents the realization of σ_{I_t} by those who do not move at $t+1$. Different from effective minimax in a normal-form given by (6), the objective function in (22) is player i's continuation payoff at t. The players who move at t minimizes player i's continuation payoff, under the best unilateral deviation by any player who has equivalent utilities with player i and moves at t. The initial state at $t+1$ evolve as the players in I_t revise their actions. The actions by those who do not moves at t and $t+1$ remain unchanged in the initial state at $t+1$. (22) also applies at the beginning of repeated game where the initial state is $\emptyset$. From the properties of effective minimax value in normal-form games and (22), we have the following theorem:

Theorem 1 *For any discount factor $\delta \in (0,1)$, when the initial state at t is a_{-I_t}, player i's average equilibrium payoffs at t are not less than $m_i^t(a_{-I_t})$, as defined by (22).*

$m_i^t(a_{-I_t}^t)$ is called player i's *effective minimax value*, which depends on t, the initial state at t and the discount factor δ. In standard repeated games with discounting where $I_t = I$ for all $t \geq 0$, there is only one initial state $\emptyset$. Player i's effective minimax value of (22) simply reduces to his effective minimax value in G of (6).

If the repeated game has cyclic, i.e., $I_t = I_{t+T}$ for all $t \geq 0$ and a finite T, then the repeated game can be treated as a repeated sequential game of Wen (2002). The game from the first T periods is treated as a stage sequential game, which depends on the discount factor. The initial states at 0, T, $2T$ etc. are $\emptyset$. In this case, $m_i^0(\emptyset) = m_i^{kT}(\emptyset)$ for any integer k. $m_i^0(\emptyset)$ of (22) is then player i's effective minimax value in the corresponding sequential game, under proper adjustment of measurement.

If G satisfies the NEU condition, $m_i^t(\cdot)$ from (22) cannot be higher than player i's pure minimax value in G, regardless the asynchronous move structure in the repeated game. The reason for this observation is, minimaxing player i with pure actions is always feasible. However, $m_i^t(\cdot)$ could be either higher or lower than m_i^e, depending on the asynchronous move structure. Recall the repeated BOS game where player 1 moves in every period and player 2 moves only once for every T periods. Player 2's effective minimax value cannot be more than $1/T$ which is much less than his mixed minimax value 2/3 in the BOS game. On the other hand, player 1's effective minimax value cannot be less than $(T-1)/T$ which is more than his mixed minimax value 2/3. If G does not satisfies the NEU condition, $m_i^t(\cdot)$ could be even higher than player i's pure minimax value in G, such as in the repeated PC game with alternating moves.

It is generally difficult to solve $m_i^t(\cdot)$ directly from (22). When the repeated game is cyclic, even without $t = 0$, i.e., $I_t = I_{t+T}$ for all $t \geq 1$ and a finite T, solving (22) analytically is possible as demonstrated by our two examples. In this case, $m_i^t(\cdot) = m_i^{t+T}(\cdot)$. Recursively substitution of (22) will solve $m_i^t(\cdot)$. Although $m_i^t(\cdot)$ depends the t and initial state at t, under the FPI condition, the limit of $m_i^t(\cdot)$ as $\delta \to 1$, however, does not depend on t and the initial state at t. The intuition is, under the FPI condition, it is always possible to witch to the state which is the worst for player i in a finite number periods. As $\delta \to 1$, player i's payoffs during those finite transition periods becomes negligible. What matters is the limit of player i's effective minimax value from this worst state in the future.

Lemma 1 *Under the FPI condition, as $\delta \to 1$, the limit of $m_i^t(\cdot)$ does not depend on t and the initial state.*

Proof: Under the FPI condition, $\exists$ a finite T such that every player revises his action at least once between t and $t+T$ for any t. Therefore, any state at $t+T$ is reachable from any state at t. Without loss of generality, let $\hat{a}_{-I_{t+T}}$ denote the "minimax" state at $t+T$ for players E_i in the sense that[7]

$$\lim_{\delta \to 1} m_i^{t+T}(\hat{a}_{-I_{t+T}}) = \min_{a_{-(I_{t+T} \cup E_i)}} \max_{a_{-I_{t+T} \cap E_i}} \lim_{\delta \to 1} m_i^{t+T}(a_{-I_{t+T}}) \equiv m_i. \tag{23}$$

First we show that the limit of $m_i^t(\cdot)$ is not less than m_i from any state at t. Note that (22) eventually leads to some states at $t+T$. As δ goes to 1, player i's payoff would not be less than m_i given that players in E_i choose their actions optimally. For $\delta < 1$, dynamic programming of (22) may lead to some states at $t+T$ which could be worse than $\hat{a}_{-I_{t+T}}$ for player i. In the limit as $\delta \to 1$, the players in E_i would choose their actions optimally

[7] We did not explicitly write $t+T$ in the limit. In fact, the limit does not depend on $t+T$.

between t and $t+T$. Recursively applying (22) from t to $t+T$ yields

$$\begin{aligned} m_i^t(a_{-I_t}) \;\; &= \;\; \max_{j\in E_i\cap I_t}\max_{a_j\in A_j}\Big[(1-\delta)u_i(\hat{\sigma}^t_{I_t}(a_{-I_t}),a_{-I_t})+ \\ &\max_{j\in E_i\cap I_{t+1}}\max_{a_j\in A_j}\Big[(1-\delta)\delta u_i(\hat{\sigma}^{t+1}_{I_{t+1}}(a_{-I_{t+1}}(\hat{\sigma}^t_{I_t},a_{-I_t})),(\hat{\sigma}^t_{I_t},a_{-I_t})) \\ &+\cdots+\delta^{t+T^*}E_{\hat{\sigma}^t_{-I_t}}\cdots E_{\hat{\sigma}^{t+T}_{-I_{t+T}}}m_i^{t+T}(\cdot)\Big]\Big], \end{aligned} \tag{24}$$

where $a_{I_{t+1}}(\hat{\sigma}^t_{I_t},a_{-I_t})$ represents the states at $t+1$ resulted from state a_{-I_t} at t and $\hat{\sigma}^s_{I_s}$ for $t\le s\le t+T$. Note that the first T terms of (24) have a common factor $(1-\delta)$, which go to 0 as $\delta\to 1$. By (23) and (24), $\lim_{\delta\to 1} m_i^t(\cdot)\ge m_i$.

On the other hand, $\hat{a}_{-I_{t+T}}$ is reachable from any state at t since every player revises his action at least once from t to $t+T$, even when players in E_i revise their actions optimally in term of the state at $t+T$. For example, during these T periods, players always select their actions in $\hat{a}_{-I_{t+T}}$. However, choosing $\hat{a}_{T_{t+T}}$ by those who are not in E_i may do not minimize player i's payoff under the best unilateral deviation by those in E_i. Therefore, we have

$$\begin{aligned} m_i^t(a_{-I_t}) \le &\max_{j\in E_i\cap I_t}\max_{a_j\in A_j}\Big[(1-\delta)u_i(\hat{a}_{I_t},a_{-I_t})+ \\ &\max_{j\in E_i\cap I_{t+1}}\max_{a_j\in A_j}\Big[(1-\delta)\delta u_i(\hat{a}_{I_{t+1}\cup I_t},a_{-I_{t+1}\cup I_t)})+\cdots+\delta^{t+T}m_i^{t+T}(\hat{a}_{-I_{t+T}})\Big]\Big]. \end{aligned} \tag{25}$$

$\hat{a}_{I_{t+T}}$ denotes any action profile for simplicity. As $\delta\to 1$, the right hand side of (25) converges to m_i. Therefor $\lim_{\delta\to 1} m_i^t(a_{-I_t})\le m_i^t$, which concludes the proof. **Q.E.D.**

Lemma 1 simplifies our folk theorem for repeated games with asynchronous moves in a great deal. Players' effective minimax values roughly stay at the same in repeated game with asynchronous moves under the FPI condition. Since there are only finite possible asynchronous move structures under FPI condition, there must be one finite-period asynchronous move structure that repeats itself infinitely many times. Such a finite-period asynchronous move structure determines players' effective minimax values in the limit.

Since our folk theorem addresses the limiting set of perfect equilibrium payoffs as $\delta\to 1$, what really matters is then the limits of players' effective minimax values. Let $m=(m_1,\cdots,m_n)$ be the *effective minimax point* in the repeated game with asynchronous moves.

Theorem 1 implies that any payoff vector that is strictly dominated by m cannot be an equilibrium outcome when players are sufficiently patient. Therefore, m is a limiting lower bound of perfect equilibrium payoffs in the repeated game. By convention, a payoff vector is strictly (weakly) *Equivalent Utility Class* (EUC) rational if it strictly (weakly) dominates m. Theorem 1 asserts that any perfect equilibrium payoff vector must at least weakly EUC rational as $\delta \to 1$.

Another issue in the folk theorem is the feasibility. Not every payoff vector in F is attainable in a repeated game with asynchronous moves. For example, feasible payoff vector (1.5, 1.5) can never be achieved in the repeated BOS game with alternating moves, since switching form (2, 1) to (1, 2) must involve (0, 0) once as two players never revise their actions at the same time. Nevertheless, vector (1.5, 1.5) can be approximated arbitrarily closely when δ is sufficiently large. For example, players play (U, L) for T periods, then either (U, R) or (D, L) depends on who moves in period $T + 1$, then (D, R) forever. The average payoff vector from such an outcome path is then

$$(1 - \delta^T)(2, 1) + \delta^{T+1}(1, 2) \to (1.5, 1.5),$$

as δ goes to one, while T depends on T so that δ^T and δ^{T+1} converge to 1/2. Dutta (1995) shows that for a general stochastic game, any feasible payoff vector can be approximated arbitrarily closely when players are sufficiently patient. The same is true here since repeated games with asynchronous are stochastic games by definition.

5 The Folk Theorem

In this section, we establish the folk theorem for repeated games with asynchronous moves. The folk asserts that any feasible and strictly EUC rational payoff vector in a repeated game with asynchronous moves can be approximated by a perfect equilibrium when the players are sufficiently patient. The statement of this folk theorem reflects the fact that the effective minimax point depends on the asynchronous move structure and a payoff vector can only be approximated. Given Theorem 1, our folk theorem characterizes almost all perfect equilibrium payoffs when players are sufficiently patient. Payoff vectors that are covered by this folk theorem are those that are weakly but not strictly EUC rational.

Theorem 2 (The Folk Theorem) *Under the FPI condition,* $\forall\, v \in F$ *such that* $v >> m$*,* $\forall\, \varepsilon > 0$*,* $\exists\, \underline{\delta} \in (0,1)$ *such that for all* $\delta > \underline{\delta}$*, the repeated game has a perfect equilibrium where players' average payoff vector is within* ε *of* v*.*

Proof: The proof is constructive. For any feasible and strictly EUC rational payoff vector, first we provide a strategy profile where players' average payoff vector is arbitrarily close to this target payoff vector, then derive a set of sufficient conditions under which the strategy profile is subgame perfect, and lastly prove that when the discount factor is large enough all the sufficient conditions hold to ensure the subgame perfection of the strategy profile.

For any $v \in F$ such that $v >> m$, there is a set n personalized punishment payoff vectors $(v^1, v^2, \ldots, v^n)$, such that

$\forall\, i \in I$, v^i is feasible and strictly EUC rational: $v^i \in F$ and $v^i >> m$;

$\forall\, i \in I$, player i strictly prefers v to v^i: $v_i > v_i^i$;

for $j \notin E_i$, player i strictly prefers v^j to v^i: $v_i^j > v_i^i$.

Players with equivalent utilities share the same personalized punishment payoff vector. It is easy to construct n personalized punishment vectors if G satisfies the full dimensionality condition. No matter whether players moves synchronously or not, $(v, v^1, \cdot, v^n)$ are defined in terms of payoffs from G. Therefore, finding them does not pose any new challenge for repeated games with asynchronous moves. $\forall\, \varepsilon > 0$ and $(v, v^1, \cdots, v^n)$, define

$$\varepsilon^* = \min\left\{ \varepsilon,\ \min_{i\in I}\left(\frac{v_i^i - m_i}{3}\right),\ \min_{i\in I}\left(\frac{v_i - v_i^i}{3}\right),\ \min_{i\neq j}\left(\frac{v_i^j - v_i^i}{3}\right)\right\} > 0. \tag{26}$$

Suppose outcome paths $(\pi,\ \pi^1, \cdots, \pi^n)$ approximate $(v, v^1, \cdots, v^n)$ as $\delta \to 1$, respectively. For $\varepsilon^* > 0$ by (26), $\exists\, \underline{\delta}_1 \in (0,1)$ such that for all $\delta \geq \underline{\delta}_1$, players' continuation payoffs from π in any period are within ε^* of v, and players' continuation payoffs from π^i in any period game are within ε^* of v^i.

Let $\rho^i(\cdot)$ be a path that solves $m_i^t(\cdot)$, depending on the initial state at t. As we argued, player i's payoff from $\rho^i(\cdot)$ is less than $m_i^t(\cdot)$ in general. Only under the best unilateral deviation by those in E_i in every period, player i's payoff will be equal to $m_i^t(\cdot)$. By Lemma 1, $\exists\, \underline{\delta}_2 \in (0,1)$ such that $m_i^t(\cdot)$ is within ε^* of m_i for all $\delta > \underline{\delta}_2$ at any t with any initial state.

Define $M = \max_{i,a\in A} |u_i(a)|$, which is finite by the assumptions on the stage game G. A player will never receive more than M nor less than $-M$ in any period.

Similar to a simple strategy profile of Abreu (1988), we consider a strategy profile defined by the target path π and punishment paths for n players in any period. Player i's punishment path at t contains three phases. The first phase is the effective minimax phase from $\rho^i(\cdot)$, the second phase is the transition phase and the third phase is the settling phase to path π^i. Player i's average payoff from such a punishment path is, at most,

$$(1-\delta^T)(m_i+\varepsilon^*) + (\delta^T - \delta^{T+T'})M + \delta^{T+T'}(v_i^i + \varepsilon^*), \tag{27}$$

where phase 1 has T periods and phase 2 has T' periods. T depends on i, t and initial state at t. Under the FPI condition, T' is finite. Since player i's payoff from the first phase is less than that form the later two phase, T is chosen large enough so that player i's payoff form his punishment path is bounded from above by $m_i + \varepsilon^*$.

The strategy profile begins with the target path π. If player i unilaterally deviates from any on-going path (either the target path π or the punishment path for any player), the strategy profile starts player i's punishment path for the corresponding period and initial state. This means that if a player deviates during his own punishment path, his will punishment again with his punishment path starting from the next period.

Now we provide a set of sufficient conditions for the strategy profile described above to be a perfect equilibrium of the repeated game. First of all, δ needs to be higher than $\max\{\underline{\delta}_1, \underline{\delta}_2\}$ so that what we did so far are valid.

During the target path π, if player $i \in I_t$ deviates at t then his payoff will be no more than M during period t. The strategy profile then switches player i's punishment path at $t+1$ from which player i's payoff is not more than will not be higher than (27). If player does not deviate then his continuation payoff will not be less than $v_i - \varepsilon^*$. A sufficient condition for player i not to deviate from the target path π is

$$(1-\delta)M + \delta\left[(1-\delta^T)(m_i+\varepsilon^*) + (\delta^T - \delta^{T^*})M + \delta^{T+T^*}(v_i^i+\varepsilon)\right] \leq v_i - \varepsilon^*. \tag{28}$$

During player j's punishment path for $j \notin E_i$, player i's unilateral deviation will trigger player i's punishment path as before. Note player i's payoff during the current period after deviation cannot be higher than M. So player i's average payoff from deviation cannot be higher than

$$(1-\delta)M + \delta\left[(1-\delta^T)(m_i+\varepsilon^*) + (\delta^T - \delta^{T+T'})M + \delta^{T+T'}(v_i^i+\varepsilon^*)\right] \tag{29}$$

If player i does not deviate in player j's punishment path, his payoff will be at least

$$(1-\delta^{T+T'})(-M) + \delta^{T+T'}(v_i^j - \varepsilon^*). \tag{30}$$

A sufficient condition for player i not to deviate in player j's punishment path is (29) $\leq$ (30).

During player j's punishment path where $j \in E_i$,[8] player i's unilateral deviation will trigger player i's punishment path. During the first phase, player i's payoff from his unilateral deviation cannot be higher than $m_i^t(\cdot)$ from definition. Player i's deviation however delays the second and third phases where player i has higher payoffs. So any unilateral deviation during the first phase is never beneficial to player i. If player i deviates at t in the second or the third phase, his payoff will not be more than M during period t and $(m_i+\varepsilon)$ at $t+1$. Player i's payoff from such a deviation is, at most, $(1-\delta)M + \delta(m_i+\varepsilon^*)$. During the last two phases, player i receives at least $-M$ for T' periods, followed by $v_i^i - \varepsilon^*$. Therefore, a sufficient for player i not to deviate in the second and third phases is

$$(1-\delta)M + \delta(m_i+\varepsilon^*) \leq (1-\delta^{T'})(-M) + \delta^{T'}(v_i^i - \varepsilon^*). \tag{31}$$

Recall the definition of ε^* in (26). As $\delta \to 1$, the left hand side of (28) converges to $v_i^i + \varepsilon^*$ which is strictly less than $v_i + \varepsilon^*$. For $j \notin E_i$, the limit of (29) is $v_i^i + \varepsilon^*$, which is strictly less

[8]For $j \in E_i$, player i and player j have the same punishment path at any t with any initial state.

than the limit of (30) $v_i^j - \varepsilon^*$. Similarly, (31) holds as $\delta \to 1$ from $m_i + \varepsilon^* < v_i^i - \varepsilon^*$. Therefore, $\exists\ \underline{\delta} > \max\{\underline{\delta}_1, \underline{\delta}_2\}$ such that for all $\delta \geq \underline{\delta}$, inequalities (28) and (31) hold, and (29) $\leq$ (30). In other words, all the sufficient conditions for the strategy profile to be a subgame perfect equilibrium are satisfied for $\delta \geq \underline{\delta}$. The equilibrium payoff vector is calculated from π, which is within $\varepsilon \leq \varepsilon^*$ of v for all $\delta \geq \underline{\delta}$. **Q.E.D.**

6 Concluding Remarks

This paper studies repeated games with asynchronous moves where players may not revise their actions in every period. Repeated games with asynchronous moves integrate a number of repeated game models in the literature. Based on the idea that players who have equivalent utilities should not minimize each other, the concept of effective minimax value is introduced based on the repeated game. A player's effective minimax value turns out to be the infimum of this player's equilibrium payoffs in the limit as the discount goes to one. It crucially depends on players' asynchronous move structure in the repeated game, but does not have obvious relationship with his effective or standard minimax value in the stage game. Generally speaking, if a players revises his actions less frequent, his effective minimax value will be lower. These results clarify some of the misunderstanding for this class of dynamic games. A folk theorem is established for repeated games with asynchronous moves in this paper. The effective minimax value formulated in this paper reduces its counterpart in repeated sequential games (Wen (2002)) and repeated game with synchronous moves (Wen (1994)). Theorem 1 in this paper also implies the anti-folk theorem of Lagunoff and Matsui (1997) in repeated pure coordination game with alternating moves.

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