

**SUBSTITUTION AND RISK AVERSION: IS RISK AVERSION IMPORTANT  
FOR UNDERSTANDING ASSET PRICES?**

by

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SUBSTITUTION AND RISK AVERSION: IS RISK AVERSION IMPORTANT FOR  
UNDERSTANDING ASSET PRICES?<sup>1</sup>

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This paper uses a recursive time-non-separable expected utility function to separate between the intertemporal elasticity of substitution (IES) and a measure of relative risk aversion to bets in terms of money (RAM). Risk premium does not require risk aversion. Changes in IES have large effects on asset prices but changes in risk aversion have only a small effect on asset prices. Assuming IES = 1 and allowing a wide range for the RAM coefficient (say between 0 and 10) is consistent with the cross-countries observation made by Lucas (2003) and the net of taxes and net of frictions rates of return estimated by McGrattan and Prescott (2003).

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## 1. INTRODUCTION

Aversion to risk and aversion to fluctuations are two distinct attributes of taste. Yet, in the standard time separable expected utility specification there is a single parameter that determines both the relative risk aversion (RA) and the intertemporal elasticity of substitution (IES). To separate between the two attributes researchers have stepped outside of the expected utility framework. See the well-known work of Selden (1978), Kreps and Porteus (1978), Epstein and Zin (1989) and Weil (1990). Here I attempt the separation between RA and IES within the expected utility framework.<sup>3</sup>

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<sup>3</sup> One of the advantages of using an expected utility function is that only an expected utility maximizer is not subject to the so-called Dutch book outcomes. To illustrate I consider the case in which a consumer prefers lottery  $a$  to lottery  $b$  ( $L^a \succ L^b$ ) and lottery  $a$  to lottery  $c$  ( $L^a \succ L^c$ ) but in contradiction to the independence axiom he also prefers the compound lottery in which the prizes are lottery  $b$  and lottery  $c$  to lottery  $a$ :  $L^d = \{L^b \text{ or } L^c\} \succ L^a$ .

Assume now that the consumer's initial endowment is lottery  $a$ . We offer to exchange lottery  $a$  for the compound lottery  $d$  for a small fee (1 dollar). Since  $L^d \succ L^a$  he accepts it. We then execute lottery  $d$ . If the outcome is lottery  $b$  we offer to exchange lottery  $b$  for lottery  $a$  for a fee of 1 dollar. Since  $L^a \succ L^b$  he accepts. If the outcome is lottery  $c$  we offer to exchange lottery  $c$  for lottery  $a$  for a fee of 1 dollar. Since  $L^a \succ L^c$  he accepts. Thus regardless of the outcome of lottery  $d$  we can get an additional dollar. The consumer has now his initial endowment minus 2 dollars. For more on this see, Yaari (1985) and Gollier (2001). For a useful survey on non-expected utility

I distinguish between measures of risk aversion to bets in terms of dated consumption and measures of risk aversion to bets in terms of money. Bets in terms of money (wealth) are resolved immediately before any irreversible consumption choice is made. Introspections about money bets require an assumption about borrowing and lending opportunities.

Bets in terms of dated consumption require a different thought experiments. We start from a non-random consumption path and then consider a bet that makes date  $t$  consumption a random variable holding consumption at all dates other than  $t$  constant. The attitude towards this type of bets does not require any assumption about the asset market. But introspection seems more difficult.

Here I separate the attitude towards money bets from IES using the recursive time non-separable (TNS) utility function<sup>4</sup>:

$$(1) \quad U(C_1, \dots, C_T; \alpha, \rho) = (1/\alpha) \left( \sum_{t=0}^T \beta^t (C_t)^\rho \right)^{\alpha/\rho} \quad \rho \neq 0 \text{ and } \alpha \neq 0 \text{ (ICES)}$$

$$U(C_1, \dots, C_T; \alpha, \rho) = (1/\alpha) \prod_{t=0}^T (C_t)^{\alpha \beta^t}, \quad \rho = 0 \text{ and } \alpha \neq 0 \text{ (ICD)}$$

$$U(C_1, \dots, C_T; \alpha, \rho) = \sum_{t=0}^T \beta^t \ln(C_t), \quad \rho = 0 \text{ and } \alpha = 0 \text{ (IL)}$$

where  $T+1$  is the horizon,  $0 < \beta < 1$  is the discount factor,

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functions and other "exotic" utility functions, see Backus, Routledge and Zin [2004].

<sup>4</sup> The utility function (1) satisfies Koopmans (1960) axioms and is hence recursive. See Becker and Boyd (1997) for a comprehensive discussion.

$IES = 1/(1-\rho)$  is the intertemporal elasticity of substitution and  $\alpha$  is a parameter that determines the relative risk aversion to money bets (RAM). Although all of the functions in (1) exhibit constant elasticity of substitution, I refer to the first as the intertemporal CES (ICES). I also use ICD for intertemporal Cobb-Douglas and IL for intertemporal log.

The intertemporal log utility function (IL) with  $IES = RAM = 1$  is widely used because it is consistent with cross-countries observations about the average rate of return of assets: It predicts an average interest rate equal to the subjective interest rate plus the expected rate of growth in consumption (see Lucas [2003]). Here I show that the ICD function preserves this prediction regardless of the RAM coefficient. This allow us to revisit Lucas' welfare calculations and examine whether the welfare cost of business cycles is sensitive to changes in the RAM coefficient (keeping  $IES = 1$ ).

It is also shown that the ICES utility function with  $RAM = 1$  is observationally equivalent to the standard power utility function: Both yield the same predictions about asset prices. The TNS utility function may therefore be used to generalize the intertemporal log (IL) utility function in various ways. We can keep  $RAM = 1$  and change only the  $IES$  by working with the standard power utility function. We can keep  $IES = 1$  and change  $RAM$  by working with the ICD utility function. We can change both by working with the ICES utility function.

Most of the analysis assumes a two periods horizon. A generalization to any finite horizon is available in the Appendix for the ICD-IL single asset case. This paper should therefore be read as a first step that focuses on conceptual issues.

I also worked out examples that use NIPA data. I find that:

(a) Risk premium does not require risk aversion; (b) Changes in the intertemporal elasticity of substitution (IES) parameter seem to have a large effect on asset prices and (c) Changes in the risk aversion parameter seem to have only a tiny effect on asset prices.

I start with a discussion of the qualitative results (a).

## 2. FLUCTUATIONS AVERSION AND RISK AVERSION

Introspection may help in making the distinction between fluctuations aversion and risk aversion. Would you prefer a smooth consumption path to a path that fluctuates around the same mean? In terms of Figure 1 the smooth consumption path *a* promises 3 units of consumption in every period. The fluctuating consumption path *d* starts from 3.5 units and then fluctuates between 3.5 and 2.5. If you prefer the path *a* then a time separable utility function predicts that you will also prefer a smooth consumption path of 2 (*e* in Figure 1) to a bet between a smooth consumption path of 3 and a smooth consumption path of 1 (*a* and *b* in Figure 1). In the time separable utility function aversion to fluctuations implies aversion to risk.

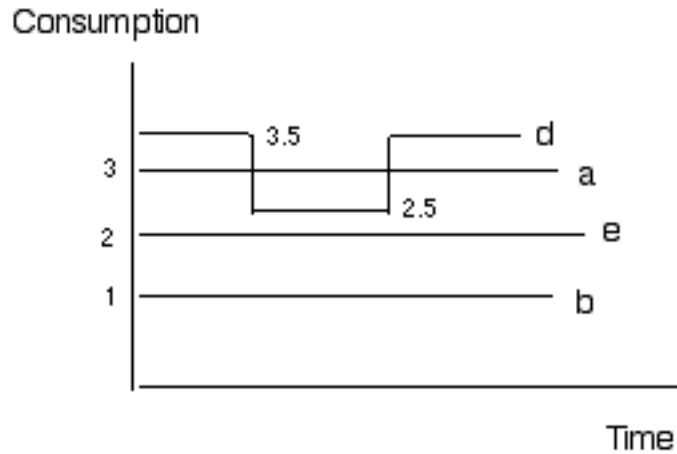


Figure 1

But aversion to fluctuations may have nothing to do with aversion to risk. It is possible that a consumer does not like fluctuations because they require changes in durables. To implement the path *d* one needs to change his house every period or to suffer from a mismatch between his house size and other components of consumption. In addition there are some irreversible choices (like the number of children) that have to be made early on (in most cases). For example, when facing a smooth consumption path one may choose to have 1 child if his permanent consumption is 1, 2 children if his permanent consumption is 2 and 3 children if his permanent consumption is 3. When facing the fluctuating consumption path *d* he may choose to have 3 children but may not enjoy them as much because they will complain whenever his consumption level drops to 2.5 and he has to cut on say the number of movies that they go to.

On the other hand if after a lottery between *a* and *b* he gets to know his permanent consumption early on he will make the optimal choice

of the number of children: He will choose one child if his permanent consumption turns out to be 1 and 3 children if it turns out to be 3.

Under the TNS utility function the consumer may show aversion to fluctuations but not aversion to risk. This leads to the result that risk premium does not require risk aversion. A consumer who does not like fluctuations does not like uncertainty about his future income and the return on assets. But nevertheless he may be willing to accept bets that are resolved before any irreversible consumption choices are made. This argument is similar to the argument I used earlier, in Eden (1977, 1979), to account for the behavior of a gambler who buys insurance (the Friedman-Savage paradox).

### 3. BETS IN TERMS OF MONEY AND BETS IN TERMS OF CONSUMPTION

To compare the TNS utility function with other utility functions that have been used, it is useful to distinguish between bets in terms of money and bets in terms of dated consumption. As was said in the introduction, a bet in terms of money is a casino type bets that is resolved immediately. A bet in terms of date  $t$  consumption assumes that consumption at all dates other than  $t$  is given and we may evaluate it without any assumption about borrowing and lending opportunities.

The distinction between the two types of bets can be illustrated with the help of Figure 2 that assumes a two-period horizon ( $t = 0, 1$ ) and a zero interest rate. The maximum utility that the consumer can get when having the wealth 9, 10 or 11 is  $a$ ,  $e$  and  $b$  respectively, where I use these letters to denote numbers (the level of cardinal satisfaction). From observing the indifference map we know that:



$a < e < b$ . But we do not know by how much. The consumer will prefer a wealth of 10 with certainty to a random wealth {9 or 11 with equal probabilities} if  $e > (\frac{1}{2})a + (\frac{1}{2})b$ . This will occur for example, if  $a = 2$ ,  $e = 9$  and  $b = 10$ . Otherwise, he will prefer the bet (if for example,  $a = 8.5$ ,  $e = 9$  and  $b = 10$ ).

A bet in terms of second period consumption assumes that the level of first period consumption is fixed. For example, in Figure 2 a bet in terms of future consumption (that is of the same size as the money bet just described) has the outcomes {4 or 6}.

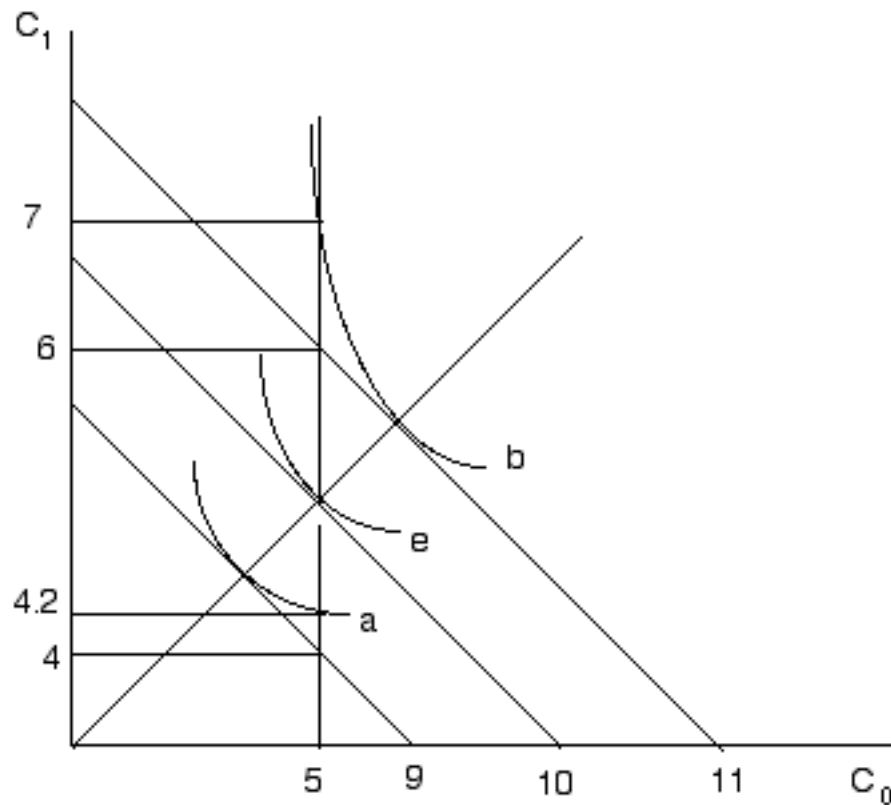


Figure 2

It is clear that the consumer will prefer the money bet {9, 11} to the future consumption bet {4, 6}. But the two bets are of different

relative size. The money bet is on 10% of wealth. The consumption bet is on 20% of consumption. The question is whether the consumer will prefer a money bet on  $x\%$  of wealth to a consumption bet on  $x\%$  of consumption. To answer this question I compare the relative risk aversion measures to the two kinds of bets. I start by showing that in the time separable case the coefficient of relative risk aversion is the same for the two kinds of bets.

I assume a  $T+1$  periods horizon. The consumer single period strictly concave utility function is  $U(C)$  and the discount factor is  $0 < \beta < 1$ . The consumer can lend and borrow at the gross interest rate  $R = 1/\beta$ . The consumer's problem when starting with the wealth  $w$  is:

$$(2) \quad V(w) = \max_{C_t} \sum_{t=0}^T \beta^t U(C_t) \quad \text{s.t.} \quad \sum_{t=0}^T R^t C_t = w.$$

The attitude towards bets in terms of money is determined by the property of the value function  $V(w)$ . Since,  $R = 1/\beta$  the solution to (2) is:  $C_t = w/(T+1)$  for all  $t$  and

$$(3) \quad V(w) = \sum_{t=0}^T \beta^t U[w/(T+1)] = (T+1)U[w/(T+1)]$$

Taking derivatives leads to:

$$(4) \quad V''(w)w/V'(w) = U''[w/(T+1)][w/(T+1)]/U'[w/(T+1)] = U''(c)c/U'(c)$$

Thus under the time separable utility function, the relative risk aversion for bets in terms of money is the same as the relative risk

aversion to bets in terms of consumption (at any date). An immediate implication is that relative risk aversion to money bets does not depend on age: When the individual advances with age, the horizon,  $T+1$ , gets shorter but consumption per period,  $w/(T+1)$ , does not change and therefore relative risk aversion does not change with age.

I now turn to show that the above result is special to the time-separable case.

#### 4. THE ATTITUDE TOWARDS RISK UNDER THE TIME-NON-SEPARABLE FUNCTION

To study the attitude towards risk of the TNS utility function (1) I define the value function:

$$(5) \quad V(w) = \max U(C_1, \dots, C_T; \alpha, \rho) \text{ s.t. } \sum_{t=0}^T R^t C_t = w.$$

As before I assume  $R = 1/\beta$  and therefore the solution to the maximization problem in (5) is  $C_t = w/(T+1)$  and the value function is:

$$(6) \quad V(w) = (1/\alpha)[w/(T+1)]^\alpha \left( \sum_{t=0}^T \beta^t \right)^{\alpha/\rho}, \quad (\text{ICES})$$

$$V(w) = (1/\alpha) [w/(T+1)]^\alpha \sum_{t=0}^T \beta^t, \quad (\text{ICD})$$

$$V(w) = \ln[w/(T+1)] \sum_{t=0}^T \beta^t, \quad (\text{IL})$$

The coefficient of relative risk aversion to bets in terms of money (RAM) is:

$$(7) \quad -V''(w)w/V'(w) = 1 - \alpha, \quad (\text{ICES})$$

$$-V''(w)w/V'(w) = 1 - \alpha \sum_{t=0}^T \beta^t, \quad (\text{ICD})$$

$$-V''(w)w/V'(w) = 1, \quad (\text{IL})$$

The coefficient of relative risk aversion to bets in terms of consumption (RAC) is:

$$(8) \quad -U_{tt}C_t/U_t = \frac{\alpha\rho/(\rho-1) \left( \sum_{t=0}^T \beta^t (C_t)^\rho \right)^{\alpha/\rho-2} (C_t)^\rho + 1 - 1/\rho}{\left( \sum_{t=0}^T \beta^t (C_t)^\rho \right)^{\alpha/\rho-1}}, \quad (\text{ICES})$$

$$-U_{tt}C_t/U_t = 1 - \alpha\beta^t, \quad (\text{ICD})$$

$$-U_{tt}C_t/U_t = 1, \quad (\text{IL}).$$

Comparing (7) to (8) we see that when the utility is not time separable the measure of risk aversion to proportional bets in terms of money is different from the measure of risk aversion to proportional bets in terms of consumption.

Note that the TNS utility function (1) achieves a separation between the attitude towards money bets and the ordinal properties of the utility function: The RAM coefficient does not depend on the

elasticity of substitution parameter  $\rho$ . But it does not separate between the attitude towards consumption bets and the ordinal properties: The RAC coefficient does depend on  $\rho$ .

Can we achieve a complete separation between the attitude towards both types of bets and the ordinal properties? The answer is in the negative. I now show this claim with the help of Figure 2. Assume for example, that  $RAM = 0$  and the consumer is indifferent between  $w = 10$  and the bet  $\{9, 11\}$ . Assume further that we know the indifference map. Then we can infer that the consumer is indifferent between the certain future consumption of 5 and a bet that promises 4.2 or 7 units of future consumption with equal probabilities. Thus once we know the attitude towards money bets and the indifference map we can infer the attitude towards consumption bets.

We therefore must make a choice: either separate between IES and RAC or between IES and RAM. The literature has chosen the first. Here I choose the latter because most introspections and experiments are done in terms of money bets.

Restrictions on the parameters: I assume risk aversion to bets in terms of consumption. To get  $RAC > 0$ , I assume  $\rho < 0$  and  $\alpha > 0$  for the ICES case, and I assume  $\alpha < 1$  for the ICD case.

The assumption  $\rho \leq 0$  implies  $0 < IES \leq 1$ . This is consistent with the findings in Hall (1988), Campbell and Mankiw (1989) and Beaudry and Wincoop (1996) who estimated IES between zero and one.

The restrictions on the parameters imply restrictions on the combination of IES and risk aversion measures that we can entertain. When  $IES = 1$  we can have  $RAM > 1 - \sum_{t=0}^T \beta^t$  and  $RAC > 1 - \beta^t$ . When

IES < 1, we can have RAM < 1 and RAC > 0. Note that we can have RAM ≤ 0 and RAC > 0. This implies that a consumer may have preference to bets in terms of money but aversion to bets in terms of consumption.

In what follows I focus on bets in terms of money. Note that we can change the RAM coefficient by varying  $\alpha$ . But varying  $\alpha$  will not change the ordinal properties of the utility function. Therefore aversion to fluctuations is separated from aversion to risk.

RAM and age: Some people have priors about the way the RAM coefficient changes with age. In our TNS function only the ICD case implies that RAM changes with age. At age  $\tau$ ,  $RAM = 1 - \alpha \sum_{t=\tau}^T \beta^t$ . When  $\alpha > 0$ , RAM increases with age reaching a maximum of  $1 - \alpha\beta^T$  in the last period of one's life. When  $\alpha < 0$ , RAM decreases with age reaching a minimum of  $1 - \alpha\beta^T$  in the last period of one's life. When  $\alpha$  approaches zero RAM approaches 1 (the log utility case). Figure 3 illustrates the changes in risk aversion to proportional money bets over the life-cycle.

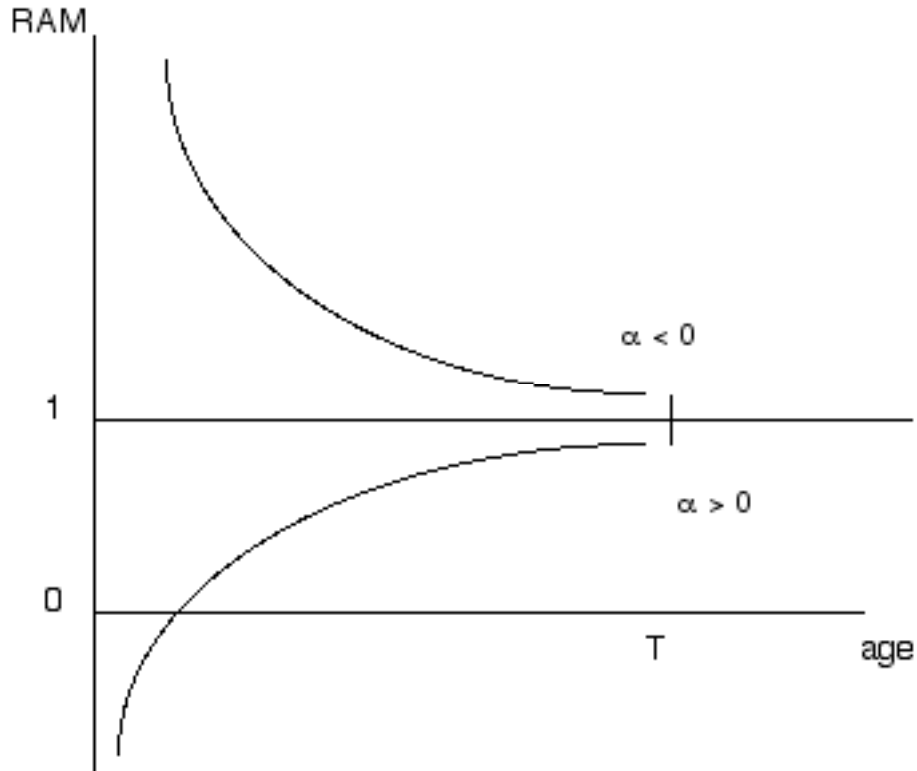


Figure 3: RAM and age for the ICD case

Thus, a prior about the way the RAM coefficient changes with age may help us in choosing the parameters of the TNS function.

##### 5. A TWO PERIODS SINGLE TREE ECONOMY

I now turn to assess the importance of the RAM coefficient for understanding asset prices - the question in the title. I start with a simple version of Lucas (1978) tree economy. There is a representative consumer who lives for two periods. He is born with an endowment of a tree that yields  $y$  units of consumption in the first period of his life and  $d_s$  units in the second period state  $s$ . After the first period dividends are distributed there is a market for trees. The price of a

tree is  $p$  and the representative consumer chooses (in the first period of his life) present consumption ( $C_0$ ) and the amount of trees ( $A$ ) subject to the budget constraint:

$$(9) \quad C_0 + pA = y + p$$

Consumption in the second period in state  $s$  is given by:

$$(10) \quad C_{1s} = Ad_s$$

Substituting (9) into (10) leads to:  $C_{1s} = d_s(y + p - C_0)/p$ . The consumer chooses  $C_0$  to solve:

$$(11) \quad \max_{C_0} \sum_{s=1}^S \Pi_s U[C_0, d_s(y + p - C_0)/p],$$

where  $\Pi_s$  is the probability of state  $s$ . The first order condition to (11) is:

$$(12) \quad \sum_{s=1}^S \Pi_s (U_{0s} - U_{1s}d_s/p) = 0$$

where  $U_{0s} = \partial U(C_0, C_{1s})/\partial C_0$  and  $U_{1s} = \partial U(C_0, C_{1s})/\partial C_{1s}$ .

The ICD-IL case:

We now assume the Cobb-Douglas case:  $U(C_0, C_1) = (1/\alpha)(C_0)^\alpha(C_1)^\delta$ , where  $\delta = \alpha\beta$ . In this case:



$$(13) \quad U_{0s} - d_s U_{1s}/p = (1/\alpha) \left( \frac{\alpha}{C_0} - \frac{\delta}{y+p-C_0} \right) (C_0)^\alpha \left( \frac{(y+p-C_0)d_s}{p} \right)^\delta$$

Therefore the first order condition (12) requires

$$\left( \frac{\alpha}{C_0} - \frac{\delta}{y+p-C_0} \right) = 0 \text{ and } C_0 = \alpha(y+p)/(\alpha+\delta).$$

To solve for  $p$  we substitute the market clearing condition  $C_0 = y$  in  $C_0 = \alpha(y+p)/(\alpha+\delta)$ . This leads to:

$$(14) \quad p = (\delta/\alpha)y = \beta y.$$

The asset pricing formula (14) can also be obtained for the IL case. The rate of return on the asset is:

$$(15) \quad D/p = D/\beta y = G/\beta,$$

where  $D = \sum_{s=1}^S \Pi_s d_s$  is expected dividends and  $G = 1 + g = D/y$  is the expected gross rate of growth of consumption. Since (7) implies  $RAM = 1 - \alpha(1 + \beta)$ , varying  $\alpha$  will change it without affecting the expected returns on the asset. We have thus shown,

Claim 1: When the representative agent's utility function is ICD-IL, the expected rate of return on the asset does not depend on the RAM measure of relative risk aversion and does not depend on the variance of the return. It depends only on the expected rate of growth in consumption ( $G$ ) and the time preference parameter  $\beta$ .

Claim 1 is generalized in the Appendix to the finite horizon case and to any monotonic transformation of the ICD utility function. Since a monotonic transformation does not change IES we conclude that IES = 1 leads to (14).

The ICES case:

I now consider the case in which the elasticity of substitution is less than unity ( $\rho < 0$ ) and  $U(C_0, C_1) = (1/\alpha)[(C_0)^\rho + \beta(C_1)^\rho]^{\alpha/\rho}$ . In this case the first order condition (12) implies:

$$(16) \quad p = \beta y^{1-\rho} \frac{\sum_{s=1}^S \Pi_s [y^\rho + \beta(d_s)^\rho]^{\alpha/\rho-1} (d_s)^\rho}{\sum_{s=1}^S \Pi_s [y^\rho + \beta(d_s)^\rho]^{\alpha/\rho-1}}$$

Thus when the elasticity of substitution is different from unity the price does depend on the RAM parameter  $\alpha$ .

Note that when  $\rho$  is small (16) is close to (14). Thus,

Claim 2: The ICES predicted asset price (16) is approximately equal to the ICD-IL predicted price (14) when  $\rho$  is close to zero (and IES is close to 1).

This says that under the TNS utility function (1) the asset price does not jump when we move from IES = 1 (ICD-IL function) to IES close to one (ICES function). This can be used to show that under (1) the asset price is a continuous function of  $\rho$ .

The standard power utility function:

I now assume the standard power (SP) utility function:

$$(17) \quad U(C_0, C_1) = (1/\rho)[(C_0)^\rho + \beta(C_1)^\rho] , \rho < 0.$$

Also here IES = 1/(1 - ρ). I restrict ρ < 0 to facilitate the comparison with the ICES function.

The first order condition (12) implies in this case:

$$(18) \quad p = \sum_{s=1}^S \Pi_s U_{1s} d_s / U_0 = \beta y^{1-\rho} \sum_{s=1}^S \Pi_s (d_s)^\rho$$

Comparing (18) to (16) leads to the following Claim.

Claim 3: The ICES utility function with α close to zero (RAM close to unity) yields approximately the same predicted asset price as the standard power utility function with the same ρ (IES) parameter.

This Claim says that if we accept RAM = 1, we may work with standard power utility function to study the effect of variations in IES on the asset's price. I now turn to an example.

Example: In the ICES case the asset price (16) will in general depend on the amount of aggregate risk in the economy. To illustrate I now consider two hypothetical economies. In both economies the expected gross rate of change in income (consumption) is 1.02. In one economy

$G = 1.02$  with probability 1. In the other economy  $G$  is a random variable that can take two possible realizations: 1 and 1.04 with equal probabilities. Table 1 calculates the gross rate of return ( $D/p$ ) for alternative values of the elasticity of substitution parameter and the risk aversion parameter. The elasticity of substitution varies from 1 (the ICD case) to 0.333 and the coefficient of relative risk aversion varies from 1 to 0. In this example, changes in the elasticity of substitution have a large effect on gross returns while changes in risk aversion have a relatively small effect. We also note that when the elasticity of substitution is less than one, the expected rate of return on the asset in the risky economy is lower than the rate of return on the asset in the risk free economy.<sup>5</sup> The predictions of the standard power utility function are in the columns with  $RAM = 1$ .

Table 1\*: The TNS utility function with  $\beta = 1$ : ( $D/p$ ) as a function of IES and RAM

IES\RAM	G = 1.02		G = {1 or 1.04}	
	RAM=1 (SP)	RAM=0	RAM=1 (SP)	RAM=0
IES=1	1.02	1.02	1.02	1.02
IES=0.5	1.0404	1.0404	1.0400	1.0401
IES=0.333	1.0612	1.0612	1.0600	1.0603

\* The first column is the IES. The second column is  $D/p$  for the case  $G = 1.02$ . The third column assumes  $G = \{1 \text{ or } 1.04\}$ . Each column is divided into two: One for  $RAM = 1$  and one for  $RAM = 0$ . The predictions of the standard power (SP) utility function are the same as in the columns with  $RAM = 1$ .

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<sup>5</sup> Levhari and Srinivasan (1969) have shown that uncertainty may increase savings and may therefore lead to lower equilibrium interest rates.

I now turn to the case for IES = 1.

Lucas' case for IES = 1:

In his presidential address Lucas (2003) uses the standard power utility function and argues for a relative risk aversion coefficient of unity. Lucas' uses a well-known formula for an economy average return on capital under the power utility function preferences. Using our notation this formula (Equation [6] in Lucas [2003]) is:

$$(19) \quad r = (1/\beta - 1) + \gamma g,$$

where  $r$  is the interest rate,  $g = G - 1$ , is the growth rate of consumption,  $1/\beta - 1$  is the subjective interest rate ( $\rho$  in Lucas' notations) and  $\gamma = 1 - \rho$  is the power coefficient. Lucas argues that "...this formula makes it clear why fairly low  $\gamma$  values must be used. Per capita consumption growth in the United States is about 0.02 and the after-tax return on capital is around 0.05, so the fact that the subjective interest rate must be positive requires that  $\gamma$  be at most 2.5. Moreover, a value as high as 2.5 would imply much larger interest rate differential than those we see between fast-growing economies like Taiwan and mature economies like the United States. This is the kind of evidence that leads to the use of  $\gamma$  values at or near 1 in applications."<sup>6</sup>

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<sup>6</sup> Pages 6 and 7 in Lucas [2003] with some modifications due to difference in notation.

Under the TNS utility function (1) Claims 1 and 3 allow us to interpret Lucas' argument as an argument for IES = 1 and for the ICD-IL utility function. In particular, when  $\gamma = 1$  the formula (19) is an approximation of (15). In what follows I will therefore devote special attention to the ICD-IL utility function.

## 6. A TWO PERIODS MANY ASSETS ECONOMY

I now turn to the many assets economy. I endow the representative agent with  $n$  trees. These  $n$  trees yield a total of  $y$  units of consumption (fruits) in the first period. Tree  $i$  yields  $d_{is}$  units in the second period in state  $s$ . The budget constraint of the representative agent is now:

$$(20) \quad C_0 + \sum_{i=1}^n p_i A_i = y + \sum_{i=1}^n p_i$$

$$(21) \quad C_{1s} = \sum_{i=1}^n d_{is} A_i$$

The agent problem is:

$$(22) \quad \max_{A_i} E\{U(C_0, C_1)\} \text{ s.t. (20) and (21).}$$

Substituting the constraints in the objective function we can write (22) as:

$$(23) \quad \max_{A_i} \sum_{s=1}^S \Pi_s U(y + \sum_{i=1}^n p_i - \sum_{i=1}^n p_i A_i, \sum_{i=1}^n d_{is} A_i)$$

The first order condition for this problem is:

$$(24) \quad \sum_{s=1}^S \Pi_s (-U_{0s} p_i + U_{1s} d_{is}) = 0$$

I use  $D_s = \sum_{i=1}^n d_{is}$  for the aggregate dividends. I also assume that we can write the dividends of asset  $i$  in state  $s$  as a linear function of  $D_s$ :

$$(25) \quad d_{is} = a_i + b_i D_s + e_{is},$$

where  $\sum_{i=1}^n e_{is} = 0$  for all  $s$ ;  $\sum_{i=1}^n b_i = 1$  and  $\sum_{i=1}^n a_i = 0$ . We assume that the error terms  $e_{is}$  is determined by a zero sum purely distributive lottery, has zero mean and is independent of  $D_s$ . A riskless asset is an asset with non-random dividends. The market portfolio is an asset for which  $d_{is} = D_s$ . The assumption about the error terms insures that the expected return on an asset with  $b_i = 0$  is the same as the return on a riskless asset and the expected return on an asset with  $a_i = 0$  is the same as the expected returns on the market portfolio. I now show this for the ICD case.

The ICD case:

We now turn to the ICD case:  $U(C_0, C_1) = (1/\alpha)(C_0)^\alpha(C_1)^\delta$ . Using the first order condition (24) and the market clearing conditions  $C_0 = y$  and  $C_{1s} = D_s$ , we arrive at the equilibrium condition:

$$(26) \quad \sum_{s=1}^S \Pi_s \{-p_i (\sum_{i=1}^n d_{is})^\delta \alpha y^{\alpha-1} + d_{is} \delta (\sum_{i=1}^n d_{is})^{\delta-1} y^\alpha\} = 0$$

Substituting (25) into (26), rearranging and using the assumption that  $e_i$  does not depend on  $D$ , leads to:

$$(27) \quad p_i = \beta y \frac{\sum_{s=1}^S \Pi_s d_{is} (D_s)^{\alpha\beta-1}}{\sum_{s=1}^S \Pi_s (D_s)^{\alpha\beta}} = \beta y \frac{\sum_{s=1}^S \Pi_s (a_i + b_i D_s) (D_s)^{\alpha\beta-1}}{\sum_{s=1}^S \Pi_s (D_s)^{\alpha\beta}}$$

When  $a_i = 0$ ,  $p_i = \beta b_i y$ . For this asset,  $d_{is}/p_i = (b_i D_s + e_{is})/\beta b_i y$ . Taking expectations leads to the following Claim.

Claim 4: The rate of return on an asset that its dividends are proportional to the aggregate dividends ( $a_i = 0$ ) is  $G/\beta$ .

I now turn to show that risk premium does not require risk aversion.

Claim 5: When  $\delta < 1$ , the rates of return on all assets with  $b_i = 0$  is the same and is less than  $G/\beta$ .

Note that when  $\delta = \alpha\beta < 1$  the coefficient of risk aversion  $RAM = 1 - \alpha(1 + \beta)$  may be positive or negative. For example if  $\beta = 1$  and  $\alpha = 0.5$  then  $RAM = 0$ .

Proof: The rate of return on asset  $i$  is:

$$(28) \quad (a_i + b_i D_s + e_{is})/p_i = (1/\beta y)(a_i + b_i D_s + e_{is}) G(a_i, b_i),$$

where  $G(a_i, b_i) = \frac{1}{b_i + a_i \sum_{s=1}^S \Pi_s (D_s)^{\delta-1} / \sum_{s=1}^S \Pi_s (D_s)^{\delta}}$  is a non linear term. Since

we assume  $\delta < 1$ , the covariance between  $D$  and  $D^{\delta-1}$  is negative and



$$(29) \quad G(1, 0) = \frac{\sum_{s=1}^S \Pi_s(D_s)^\delta}{\sum_{s=1}^S \Pi_s(D_s)^{\delta-1}} = \frac{\sum_{s=1}^S \Pi_s(D_s)^{\delta-1} D_s}{\sum_{s=1}^S \Pi_s(D_s)^{\delta-1}} = \frac{Cov(D^{\delta-1}, D)}{\sum_{s=1}^S \Pi_s(D_s)^{\delta-1}} + \sum_{s=1}^S \Pi_s D_s$$

$$< \sum_{s=1}^S \Pi_s D_s.$$

Substituting this in (28) and taking expectations leads to the conclusion that the expected rate of return on any asset with  $b_i = 0$  is less than  $G/\beta$ .  $\square$

The intuition is in the observation that when  $\delta < 1$ ,  $RAC = 1 - \delta > 0$  and the representative consumer is averse to uncertainty about future consumption. He will therefore hold the market portfolio rather than the risk free asset only if there is a risk premium.

I now turn to a numerical example. As in the example of Table 1 the rate of growth in aggregate dividends (consumption) is 1 or 1.04 with equal probabilities and  $\beta = 1$ . I consider three assets and use the following notation:

$R^b$  = the return on an asset with  $a_i = 1$  and  $b_i = 0$  (the risk free return);

$R^1$  = the return on an asset with  $a_i = 0$  and  $b_i = 1$ ;

$R^2$  = the return on an asset with  $a_i = -2$  and  $b_i = 3$ .

As we can see from Table 2 the rate of return on the market portfolio  $R^1$  does not depend on the RAM coefficient and is equal to  $G/\beta = 1.02$  in our example. The rate of return on the risk free asset is lower and the difference (the risk premium) increases with RAM. The risk premium under risk neutrality is less than one tenth of a percent. When RAM is close to 1 (the log utility case) the risk premium is a little

over one tenth of a percent. When RAM is 3, the risk premium is about two tenth of a percent.

Table 2: Rates of Returns under the TNS utility function with IES = 1  
(ICD-IL;  $\beta = 1$ )

RAM = $1-2\alpha$	$R^1$ $a_i = 0; b_i = 1$ $d_i = \{1, 1.04\}$	$R^2$ $a_i = -2; b_i = 3$ $d_i = \{1, 1.12\}$	$R^b$ $a_i = 1; b_i = 0$ $d_i = \{1, 1\}$	$R^1 - R^b$	$R^2 - R^b$
0	1.02	1.0204	1.0198	0.0002	0.0006
1	1.02	1.0207	1.0196	0.0004	0.0011
2	1.02	1.0211	1.0194	0.0006	0.0017
3	1.02	1.0215	1.0192	0.0008	0.0023
10	1.02	1.0241	1.0179	0.0021	0.0062

The IL case:

For the log case the asset pricing formula is given by:

$$(30) \quad p_i = \beta y \sum_{s=1}^S \Pi_s d_{is} / D_s$$

and can be obtained as the limit of (27).

The ICES case:

I now turn to the case in which the elasticity of substitution is less than unity ( $\rho < 0$ ) and the utility function is:

$U(C_0, C_1) = (1/\alpha)[(C_0)^\rho + \beta(C_1)^\rho]^{\alpha/\rho}$ . In this case, the first order condition (24) is:

$$(31) \quad p_i = \beta y^{1-\rho} \frac{\sum_{s=1}^S \Pi_s [(y)^\rho + \beta(D_s)^\rho]^{\alpha/\rho-1} (D_s)^{\rho-1} d_{is}}{\sum_{s=1}^S \Pi_s [(y)^\rho + \beta(D_s)^\rho]^{\alpha/\rho-1}}$$

In Table 3 I use (31) to calculate the rates of returns under the assumption that  $\alpha$  is close to zero and  $RAM = 1 - \alpha$  is close to unity. We see that changes in the elasticity of substitution make a big difference both to the levels of the rates of return and to the risk premia.

Table 3: Rates of Returns under the TNS utility function with  $RAM = 1$  (SP;  $\beta = 1$ )

IES =	$R^1$	$R^2$	$R^b$	$R^1 - R^b$	$R^2 - R^b$
$1/(1-\rho)$	$a_i=0; b_i=1$ $d_i=\{1, 1.04\}$	$a_i=-2; b_i=3$			
1	1.02	1.0207	1.0196	0.0004	0.0011
0.5	1.04	1.0415	1.0392	0.0008	0.0023
0.333	1.0600	1.0623	1.0588	0.0012	0.0035
0.1	1.1982	1.2069	1.1936	0.0045	0.0132
0.01	1.9988	2.0740	1.9612	0.0377	0.1128

Since Claim 3 holds for this many assets case, the rates of return in Table 3 are the same as the rates of return under the standard power (SP) utility function that impose  $RAM = 1/IES$ . Comparing Tables 2 and 3 reveal the importance of the RAM coefficient under the ICD-IL function

relative to the SP function. The results for  $RAM = IES = 1$  are the same in both Tables because both share the IL utility function. The relevant comparison for  $RAM = 2$  in Table 2 is with  $IES = 0.5$  in Table 3. Both the rates of returns and the risk premiums are larger under the SP utility function. The difference in the response to the change in the RAM coefficient may be explained by the fact that in the SP utility function we are changing both RAM and IES and therefore the effect is stronger.

I now turn to modify Claim 2 for the many assets case and to promote the view that the TNS utility function is an extension of the IL function.

Claim 6: When  $\rho$  and  $\alpha$  are close to zero ( $IES$  and  $RAM$  are close to unity), the ICES predicted asset prices (31), the ICD predicted prices (27) and the IL predicted prices (30) are approximately the same.

Claim 6 says that small variation from the log utility function will lead to small variations in asset prices. As was said in the introduction the TNS utility function allows for the extension of the IL function (with  $IES = RAM = 1$ ) in various directions. We can keep  $RAM = 1$  and change only the IES by working with the standard power utility function. We can keep  $IES = 1$  and change  $RAM$  by working with the ICD utility function. We can change both by working with the ICES utility function.

"Ordinal certainty equivalent preferences":

As was said in the introduction, Selden (1978) has proposed a non-expected utility function that separates between the elasticity of substitution and risk aversion. Kreps and Porteus (1978) and Epstein and Zin (1989) have extended Selden's analysis to the multi-period case in a time-consistent manner. I now show that when  $IES = 1$ , Selden's procedure may be observationally equivalent to the ICD-IL utility function.

Selden evaluates consumption paths in two stages. He first uses a "certainty equivalence function" to substitute a certainty equivalent for the random future consumption and then an "aggregator function" to evaluate current consumption and the certainty equivalence of future consumption.

To illustrate, let  $C$  denotes current consumption and  $x$  denotes a random future consumption. The consumer first uses the certainty equivalence function  $\mu$  to convert  $x$  to a scalar:  $Z = \mu(x)$ . He then uses the aggregator function  $G(C, Z)$  to evaluate the consumption path. In this formulation  $IES$  is determined by the properties of the aggregator function  $G$  while  $RAC$  is determined by the properties of the certainty equivalence function  $\mu$ .

I now turn to the special case:

$$(32) \quad G(C, Z) = \log(C) + \log(Z); \quad Z = (Ex^\sigma)^{1/\sigma} \text{ where } 0 \neq \sigma < 1.$$

In (32) the aggregator function is logarithmic and as in Epstein and Zin (1991), the certainty equivalence function is of the CES type. For the single asset case, the consumer's problem is:

$$(33) \quad \max_A \log(y + p - pA) + \beta \log\left\{\left[\sum_{s=1}^S \Pi_s (D_s A)^\sigma\right]^{1/\sigma}\right\}$$

The first order condition for this problem is (14). Thus as in the log expected utility case the price of the asset depends only on current dividends ( $p = \beta y$ ) and not on the certainty equivalent of future consumption. Therefore risk aversion and aggregate risk do not affect the price of the asset and the expected return.

For the many asset case, the consumer problem under Selden's utility function is:

$$(34) \quad \max_{A_i} \log\left[y + \sum_{i=1}^n p_i - \sum_{i=1}^n p_i A_i\right] + \beta \log\left\{\left[\sum_{s=1}^S \Pi_s \left(\sum_{i=1}^n d_{is} A_i\right)^\sigma\right]^{1/\sigma}\right\}$$

The equilibrium prices (which we obtain after substituting  $A_i = 1$  and in the first order conditions) are:

$$(35) \quad p_i = \beta y \frac{\sum_{s=1}^S \Pi_s d_{is} (D_s)^{\sigma-1}}{\sum_{s=1}^S \Pi_s (D_s)^\sigma}$$

This is exactly the formula (27). We have thus shown the following Claim.

**Claim 7:** The Selden-Epstein-Zin utility function (32) and the ICD utility function are observationally equivalent (in the sense that they both yield the same asset prices) when  $\sigma = \alpha\beta$ .

Under the ICD utility function the relative risk aversion measure for bets in terms of second period consumption is:  $RAC = 1 - \alpha\beta$ . Thus we may interpret the coefficient  $\sigma$  in (32) as a measure of RAC. We may also use Table 2 to get the predictions of (32) about asset returns.

Note that  $RAC = 1 - \beta\alpha = (RAM + 1/\beta)/(1/\beta + 1)$ . Therefore a unit change in RAC is equivalent to roughly 2 units change in RAM and this will make the RAC measure of risk aversion look more important than our RAM measure. For example, in Table 2 with  $\beta = 1$ , RAM varies from 0 to 10 while RAC varies from 0.5 to 5.5.

## 7. INCOMPLETE MARKETS

The examples in Table 1-3 suggest that asset prices are not sensitive to changes in the RAM coefficient but are sensitive to changes in IES. I now turn to examine this conclusion for the case in which there is idiosyncratic risk that cannot be insured.

I assume  $N$  households indexed  $h$ . There are  $n+N$  types of trees:  $n$  types (of physical capital) are traded and  $N$  types (of human capital) are not traded. Each household starts with a portfolio of  $n+1$  trees one tree from each of the traded-physical-capital type and human capital. The aggregate per capita amount of fruit (income) in state  $s$  is  $D_s$ .

The amount of dividends from trees of type  $1, \dots, n$  is given by (25) and is repeated here for convenience.

$$(36) \quad d_{is} = a_i + b_i D_s + e_{is}$$

where  $\sum_{i=1}^n e_{is} = 0$  and  $e_{is}$  are independent of  $D_s$ . The amount of dividends from human capital  $H_s^h$  is given by:

$$(37) \quad H_s^h = a_h + b_h D_s + u_s^h$$

where  $\sum_{h=1}^N u_s^h = 0$  and the  $u_s^h$  are independent of  $D_s$ . Per capita income is given by:

$$(38) \quad D_s = \sum_{i=1}^n d_{is} + (1/N) \sum_{h=1}^N H_s^h$$

We may think in terms of three independent lotteries that occur at the beginning of period 1. The first lottery determines the aggregate per capita magnitude  $D$ . The second is a zero sum lottery that determines  $e$  and the third is a zero sum lottery that determines  $u$ . A state of nature  $s$  is a description of the outcome of all three lotteries.<sup>7</sup> I assume  $a_h = 0$  and  $b_h = 0.7$  for all  $h$ . It is also assumed that  $\sum_{i=1}^n b_i = 0.3$  and  $\sum_{i=1}^n a_i = 0$ .

Household  $h$  consumption is:

$$(39) \quad C_0^h + \sum_{i=1}^n p_i A_i^h = y + \sum_{i=1}^n p_i$$

$$(40) \quad C_{1s}^h = \sum_{i=1}^n d_{is} A_i^h + H_s^h$$

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<sup>7</sup> Since the lotteries are independent the number of states of nature is:  $S = L_1 \times L_2 \times L_3$  where  $L_i$  is the number of possible realizations of lottery  $i$ .



where  $A_i^h$  is household h choice of the quantity of asset i ( $i=1, \dots, n$ ).

Since labor share is 0.7, the typical agent problem can now be written as:

$$(41) \quad \max_{A_i} \sum_{s=1}^S \Pi_s U(y + \sum_{i=1}^n p_i(1 - A_i), \sum_{i=1}^n d_{is}A_i + 0.7D_s + u_s),$$

where the superscript h is suppressed. The first order condition for this problem is still given by (24).

Using symmetry all consumers will make the same first period consumption choice and therefore the clearing of the first period consumption market requires:  $C_0 = y$ . Symmetry also implies that consumption of household h in the second period is given by  $C_s^h = D_s + u_s^h$ . The first order condition (24) should hold for all h and therefore I suppress the superscript h and write  $C_{1s} = D_s + u_s$  for the representative consumer. Substituting this in the first order condition (24) leads to the following pricing formula:

$$(42) \quad p_i = \beta y \frac{\sum_{s=1}^S \Pi_s (a_i + b_i D_s) (D_s + u_s)^{\alpha\beta - 1}}{\sum_{s=1}^S \Pi_s (D_s + u_s)^{\alpha\beta}}$$

I now turn to a numerical example in which aggregate consumption may take the realizations 1 and 1.04. For each realization of the aggregate consumption we add a bet in which the typical household can win or lose 0.08 units. This is consistent with the standard deviations

of aggregate consumption in the data (0.02) and the Deaton-Paxon estimate of the standard deviation of individual consumption (0.08).<sup>8</sup>

Table 4 presents the results of the numerical example. The risk premia in Table 4 are almost identical to the risk premia in Table 2. The rates of return themselves are not. The rate of return on the market portfolio is now declining in the RAM coefficient.

It thus seems that allowing for incomplete markets will affect our estimate of  $\beta$  but will have little or no effect on our estimate of the risk premia.

Table 4: Predicted Rates of Returns when markets are incomplete  
(IES = 1;  $\beta = 1$  and  $C_1 = \{1 \pm 0.08, 1.04 \pm 0.08\}$ )

RAM = $1-2\alpha$	$R^1$ $a_i = 0; b_i = 1$ $d_i = \{1, 1.04\}$	$R^2$ $a_i = -2; b_i = 3$ $d_i = \{1, 1.12\}$	$R^b$ $a_i = 1; b_i = 0$ $d_i = \{1, 1\}$	$R^1 - R^b$	$R^2 - R^b$
0	1.017	1.017	1.017	0.0002	0.0006
1	1.014	1.014	1.013	0.0004	0.0011
2	1.011	1.012	1.010	0.0006	0.0017
3	1.008	1.009	1.007	0.0008	0.0023
10	0.987	0.991	0.985	0.0022	0.0063

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<sup>8</sup> Deaton and Paxson (1994) finds that the variance of log consumption within each age cohort increases by 0.07 every decade in the US (page 446). Their random walk assumption in equations (1) - (3) imply a variance in consumption of 0.007 per year which is roughly equal to a standard deviation of 0.08.

I now turn to the effect of changes in RAM on welfare calculations.

## 8. WELFARE CALCULATIONS

In his presidential address Lucas (2003) attempt to assess the gains from stabilization policy. His argument for choosing a relative risk aversion of unity used equation (19) and was interpret here under the TNS utility function as an argument for IES = 1 and for the ICD-IL utility function. This interpretation allow us to vary the RAM measure without violating the implication about the average rate of return in the economy which will remain  $G/\beta$  regardless of the choice of RAM.

To examine the effect of RAM on welfare (holding constant IES = 1) I start with a consumption path of  $C_0 = 1$  and  $C_1 = \{1 \text{ or } 1.04\}$ . Following Lucas I calculate the required compensation ( $\lambda$ ) for the consumption risk, where  $\lambda$  solves:

$$(43) \quad \left(\frac{1}{2}\right)U(1 + \lambda, 1 + \lambda) + \left(\frac{1}{2}\right)U[1 + \lambda, 1.04(1 + \lambda)] = U(1, 1.02)$$

Thus the consumer is fully compensated for the risk if his consumption in all periods and states of nature is increased by a fraction of  $\lambda$ .

Assuming the ICD utility function with  $\beta = 1$  leads to:

$$(44) \quad \lambda = [2(1.02)^\alpha / (1.04^\alpha + 1.00^\alpha)]^{1/2\alpha} - 1.$$

The second column of Table 5 reports the required compensation in percentage terms ( $100\lambda$ ) for various levels of RAM. Not surprising, risk aversion matters. For example, going from RAM = 0 to RAM = 1 doubles the required compensation. But as in Lucas (2003) all the magnitudes are a small fraction of a percent.

Table 5: Required compensations in percentage terms ( $100\lambda$ );

$C_1 = \{1 \text{ or } 1.04\}$

Ram	2 periods, 1 shock	2 periods, 2 shocks	3 periods, 3 shocks	2 periods, 1 shock SD = 0.08
0	0.005	0.005	0.009	0.077
1	0.010	0.029	0.038	0.154
2	0.014	0.053	0.068	0.232
3	0.019	0.077	0.098	0.309
10	0.053	0.245	0.307	0.839

To check for robustness I also considered cases in which consumption follows a random walk. The third column in Table 5 reports the required compensation when both current and future consumption are random:  $C_0 = \{1 \text{ or } 1.04\}$  and  $C_1 = \{C_0 \text{ or } 1.04C_0\}$ . In this case the required compensation are substantially higher relative to the single shock case but are still a fraction of a percent. The three periods random walk case, reported in the fourth column assumes:  $C_0 = \{1 \text{ or } 1.04\}$ ,  $C_1 = \{C_0 \text{ or } 1.04C_0\}$  and  $C_2 = \{C_1 \text{ or } 1.04C_2\}$ . In this case the welfare cost is larger than in the previous case. This suggests

that adding shocks whose effect are being eliminated by "good policy" increases the welfare gain. Note also that the required compensation is almost proportional to the RAM coefficient.

Allowing for incomplete markets may increase the welfare gain. If by "good policy" we eliminate aggregate risk we may also greatly reduce the number of markets required for completeness. This may therefore improve the allocation of diversifiable risk in the economy. In the last column of Table 5 I assume that the "good policy" eliminates all risk in a two periods one shock economy assuming:  $C_1 = \{0.94 \text{ or } 1.1\}$  initially and then by "good policy" is converted to  $C_1 = 1.02$ . The assumed standard deviation of consumption is thus 0.08 and is consistent with the Deaton-Paxson estimate discussed above. Note that if  $RAM = 10$  the welfare gain is 0.8%. This starts to look like real money.<sup>9</sup>

#### 9. RATES OF RETURN FOR HYPOTHETICAL CLAIMS UNDER THE ICD-IL FUNCTION

Under the ICD-IL function the expected return on the market portfolio does not depend on the RAM coefficient. But the RAM coefficient does affect the expected rates of return on claims on parts of GDP that are not proportional to consumption. To get a sense of the

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<sup>9</sup> It has also been argued that a good policy may improve production efficiency. For example, it is possible that the consumer is not averse to fluctuations in consumption but is averse to fluctuations in labor supply. It is also possible that average capacity utilization will improve as a result of policy. For a recent survey of the literature see Barlevy (n.d.).

importance of the RAM coefficient, I consider now hypothetical claims on (a) GDP, (b) the wage bill, (c) non-wage income (profits) and (d) corporate profits, all in real per-capita terms.

Our first task is to express equation (25) in terms of rates of change. Equation (25) is conditional on all the information available at time  $t$  and we may therefore write:

$$(45) \quad d_{it+1} = a_{it+1} + b_{it+1}D_{t+1} + e_{it+1},$$

where the coefficients are time dependent. I normalize  $y = 1$  and assume:  $a_{it+1} = a_i d_{it}$  and  $b_{it+1} = b_i d_{it}$ . This means that the predicted share in the pie is proportional to the time  $t$  share. Dividing (45) by  $d_{it}$  yields:

$$(46) \quad G_{it+1} = a_i + b_i G_{t+1} + \varepsilon_{it+1},$$

where  $G_{it+1} = d_{it+1}/d_{it}$  is the gross rate of growth in asset  $i$  dividends,  $G_{t+1} = D_{t+1}$  is the gross rate of growth in consumption and  $\varepsilon_{it+1} = e_{it+1}/d_{it}$  is an error term. I also assume that  $\varepsilon_{it+1}$  has a zero mean and is not correlated with  $G_{t+1}$ . The time invariant coefficients  $a_i$  and  $b_i$  can therefore be estimated from running the regression (46).

Note that multiplying the coefficients  $a_i$  and  $b_i$  by the same constant does not change the expected rate of return (28). Therefore after estimating the regression coefficients in (46) we can plug the coefficients directly (without multiplying it by  $d_{t_i}$ ) in (28) to compute the predicted gross rate of return on asset  $i$ .

Equation (46) requires data on the gross rates of change of flows (fruits) and these data are easier to get than data on prices. For

example there is no market for slaves and therefore no data on the price of human capital defined as body plus the knowledge embodied in it. But we can predict the gross rate of return on human capital even without observing its price. Similarly and maybe more relevant, we do not observe the price of unincorporated equity. But nevertheless we can predict the rate of return on it if we observe the flow of profits it yields.

I use NIPA US post war data (from January 1948 to January 2004) taken from the Saint Louis Fed web page to compute the gross rate of growth in real per capita terms of the following variables: consumption (c), wage earnings (w), corporate profits (pr), GDP (y) and non-wage income (y-w). The detail of the calculations of these variables and the description of the data are in Appendix c.

Table 6 provides summary statistics for the annual data. All rates of change are close to 2%. The smallest rate is for the wage bill (1.6%) and the highest is for corporate profits (2.2%). The standard deviation is in the range 0.02 - 0.04 except for corporate profits where it is much higher (0.16).

Table 6: Summary Statistics about Annual Per Capita Gross Rates of Change

	Average	Standard deviation
Consumption (c)	1.019	0.02
GDP (y)	1.018	0.03
Wage earnings (w)	1.016	0.03
Profits (y-w)	1.020	0.04
Corporate Profits (pr)	1.022	0.16

Table 7 provides the regression results from running (46). Most intercepts are small and barely significant. The intercept on corporate profits is an exception.

Table 7\*: Regressions of the rate of change of asset i on the rate of change in consumption

Dependent var.	Intercept	Slope	Rsquare
y	-0.19 (0.10)	1.18 (0.10)	0.73
w	0.07 (0.14)	0.93 (0.14)	0.46
y-w	-0.45 (0.14)	1.44 (0.14)	0.67
pr	-2.06 (0.93)	3.03 (0.91)	0.17

\* Standard errors in parentheses.



Table 2 may therefore provide a good approximation for the rates of return on the four hypothetical claims when  $\beta = 1$ . The expected return on the market portfolio ( $R^1$ ) is a good approximation for the returns on claims on the wage bill, non-wage income and GDP. The expected return on the more risky portfolio ( $R^2$ ) is an estimate of the return on a claim on corporate profits.

The prediction of the model for various  $\beta$  can be approximated by multiplying Table 2 by  $\beta$ . This is done in Table 8 for  $\beta = 1.025$ .

Table 8: Predicted Rates of Returns (IES = 1;  $\beta = 1.025$ )

RAM = $1-2\alpha$	$R^1$ $a_i = 0; b_i = 1$ $d_i = \{1, 1.04\}$	$R^2$ $a_i = -2; b_i = 3$	$R^b$ $a_i = 1; b_i = 0$	$R^1 - R^b$	$R^2 - R^b$
0	1.0455	1.0459	1.0453	0.0002	0.0006
1	1.0455	1.0463	1.0451	0.0004	0.0012
2	1.0455	1.0466	1.0449	0.0006	0.0017
3	1.0455	1.0470	1.0447	0.0008	0.0023
10	1.0455	1.0497	1.0433	0.0022	0.0064

The expected rates of returns in Table 8 are consistent with the estimates in McGrattan and Prescott (2003) who took an explicit account of taxes and frictions and found average returns in the 4-5 percent range. The expected rate of return on the market portfolio is 1.0455. The expected rate of return on a claim on corporate profits is 1.0459 when RAM = 0 and 1.05 when RAM = 10. The corresponding risk premia on

the more risky portfolio (corporate profits) are 0.05% and 0.6% respectively.

#### 10. CONCLUDING REMARKS

I used a time-non-separable (TNS) recursive expected utility function to separate between the intertemporal elasticity of substitution (IES) and the measure of relative risk aversion to bets in terms of money (RAM). In the examples I worked out, variations in the RAM coefficient have a small effect on asset prices relative to variations in the IES coefficient. When IES changes from 1 to 0.5 the rate on the market portfolio doubles in our example (Table 1) almost regardless of whether the RAM coefficient is 1 or 0.

In some cases the TNS utility function leads to predictions about asset prices that are the same as the predictions of other functions that have been used. The standard power (SP) utility function is observationally equivalent to TNS with  $RAM = 1$ . Selden's certainty equivalent approach is observationally equivalent to our approach when  $IES = 1$  and the certainty equivalent function is the CES like function used by Epstein and Zin (1991). It may be useful to view the TNS utility function as an extension of the intertemporal log (IL). Starting from  $IES = RAM = 1$ , we can keep  $RAM = 1$  and change the IES by working with the standard power utility function. We can keep  $IES = 1$  and change  $RAM$  by working with the ICD utility function. We can change both by working with the ICES utility function. Small variations from the log utility function will lead to small variations in asset prices.

Lucas' observations about cross-countries average interest rates are consistent with the prediction of the ICD-IL utility function with  $IES = 1$ . Under the ICD-IL utility function, the expected rate of return on the market portfolio is  $G/\beta$  regardless of the RAM coefficient and of the amount of aggregate risk.

Risk premia under the ICD-IL utility function are less sensitive to changes in the RAM coefficient than risk premia under the SP function. Risk premia are larger under the ICD function when  $RAM = 0$  and are smaller when  $RAM > 1$  (Tables 2 and 3). This may be explained by the fact that when we change the parameter in the SP function we are changing both the IES and the RAM and their combined effect is about twice as much as the effect of just changing the RAM coefficient. This may be a reason why risk aversion looks more important when using the standard power utility function.

Allowing for incomplete markets does not change risk-premia in the ICD example we worked out. But it does affect the rates of returns on the assets and introduces a negative relationship between the rates of return and the RAM coefficient. This may be the result of a precautionary savings type behavior.

Not surprisingly changes in the RAM coefficient affect the calculation of the welfare gains from eliminating business cycle risks. This point is well recognized by Lucas (2003) and other authors on this subject. Lucas objects to a power parameter of the SP function that deviates substantially from unity on the ground that it violates the above mentioned cross-countries observation. The ICD utility function allows for variations in the RAM coefficient that do not violate the cross-countries observations.

Data on flows can be used to compute the rates of returns on various claims. I used post war US NIPA data and found that claims on the wage bill and on total profits are close to a claim on the market portfolio (aggregate consumption). But a claim on corporate profits is more risky than a claim on the market portfolio. The predictions of the ICD-IL utility function are consistent with the findings in McGrattan and Prescott (2003) but cannot account for the original Mehra and Prescott (1985) puzzle.

#### APPENDIX A: A FINITE HORIZON SINGLE ASSET ICD ECONOMY

I now consider an economy in which the representative agent lives for  $T$  periods. At  $t = 0$  he gets endowment of one tree that provides fruits for  $T$  periods and then dies (together with the agent).

I allow a general dividend (income) process. It is assumed that the representative agent at  $t = 0$  assigns positive probabilities,  $\pi_s$ , to all states  $s = 1, \dots, S$ . Over time he updates this probabilities when he learns that some states did not occur. The set of possible states at time  $t$  (the information available at time  $t$ ) is denoted by  $I_t$ . The updated probability of state  $s$  is denoted by  $(\pi_s | I_t)$ . Note that  $(\pi_s | I_t) = 0$  if  $s \notin I_t$ . The agent also knows the information that he will have at time  $j > t$  if state  $s$  occurred. This information is denoted by  $I_{j|s}$ . At time  $t$  the choices of  $(A_0, \dots, A_{t-1})$  was already made. Since there is one tree per agent we assume  $A_j = 1$  for  $j < t$ . The agent chooses  $A_t$  and makes a contingent plan that specifies the amount of trees he will own at future dates:  $(A_{t+1|s}, \dots, A_{T-1|s})$ . The agent has to choose  $A_{j|s} = A_{j|s}$  if at time  $j$  he cannot distinguish between the two states. Thus, he

faces the informational constraint:  $A_{js} = A_{j's'}$  if  $s, s' \in I_{js}$ . Assuming an ICD utility function we can state the time  $t$  problem as follows.

$$(A1) \quad V_t(k_{t-1}, I_t) = \max_{A_t, A_{t+1s}, \dots, A_{T-1s}}$$

$$k_{t-1}(d_t + p_t - A_t p_t)^{\alpha\beta^t} \sum_{s=1}^S (\pi_s | I_t) [A_t(d_{t+1s} + p_{t+1s}) - A_{t+1s} p_{t+1s}]^{\alpha\beta^t} k_{t+1s}$$

$$\text{s.t.}$$

$$k_{t-1} = \prod_{j=0}^{t-1} (d_j)^{\alpha\beta^j}$$

$$k_{t+1s} = \prod_{j=t+2}^T [A_{j-1s}(d_{js} + p_{js}) - A_{js} p_{js}]^{\alpha\beta^j}$$

$$A_{js} = A_{j's'} \text{ if } s, s' \in I_{js}$$

I now define equilibrium as follows.

Equilibrium at time  $t$  is a vector  $(A_t, A_{t+1s}, \dots, A_{T-1s}, \dots, A_{t+1s}, \dots, A_{T-1s};$

$p_t, p_{t+1s}, \dots, p_{T-1s}, \dots, p_{t+1s}, \dots, p_{T-1s})$  such that

(a) given prices  $(p_t, p_{t+1s}, \dots, p_{T-1s})$ , the quantity vector

$(A_t, A_{t+1s}, \dots, A_{T-1s})$  solves (A1) and

(b) market clearing:  $A_t = 1$  and  $A_{js} = 1$  for all  $j > t$  and all  $s$ .

I now generalize the asset pricing formula (10) to the finite horizon case.

Claim A: Equilibrium prices at time  $t$  are given by:

$$(A2) \quad p_t = (\beta + \beta^2 + \dots + \beta^{T-t})d_t \text{ and} \\ p_{j_s} = (\beta + \beta^2 + \dots + \beta^{T-j})d_{j_s} \text{ for all } t < j < T$$

Note that when  $T = \infty$  (A2) implies  $p_t = d_t/\rho$ , where the subjective interest rate  $1 + \rho = 1/\beta$ . This formula is in the logarithmic preference example in Ljungqvist and Sargent (2000, page 239).

Proof: When  $T = 1$ , there is trade in the asset only in period  $t = T - 1 = 0$  and (A2) coincides with (17). We now proceed by induction. We assume that equilibrium prices when the horizon is  $T-t-1$  (at time  $t+1$ ) satisfy (A2) and show that equilibrium prices when the horizon is  $T-t$  (at time  $t$ ) satisfy (A2).

Given our induction hypothesis we can write the problem (A1) as:

$$(A3) \quad V(k_{t-1}; I_t) =$$

$$\max_{A_t} k_{t-1} (d_t + p_t - A_t p_t)^{\alpha\beta^t} \sum_{s=1}^S (\pi_s | I_t) [A_t (d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1}} k_{t+1s}$$

Now  $k_{t+1s} = \prod_{j=t+2}^T (d_{j_s})^{\alpha\beta^j}$  is a constant and  $p_{t+1s} = (\beta + \beta^2 + \dots + \beta^{T-t-1})d_{t+1s}$ . Note

that the assumption  $A_{t+1s} = 1$  follows from the induction hypothesis.

The first order condition for the problem (A3) is:

$$(A4) \quad -\alpha\beta^t p_t (d_t + p_t - A_t p_t)^{\alpha\beta^t - 1} \sum_{s=1}^S (\pi_s | I_t) [A_t (d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1}} k_{t+1s} \\ + (d_t + p_t - A_t p_t)^{\alpha\beta^t} \sum_{s=1}^S (\pi_s | I_t) \alpha\beta^{t+1} (d_{t+1s} + p_{t+1s}) [A_t (d_{t+1s} + p_{t+1s}) - p_{t+1s}]^{\alpha\beta^{t+1} - 1} k_{t+1s} = 0$$

Substituting  $A_t = 1$  and  $p_{t+1s} = (\beta + \beta^2 + \dots + \beta^{T-t-1})d_{t+1s}$  in (A4) leads to:

$$(A5) \quad p_t = (\beta + \beta^2 + \dots + \beta^{T-t})d_t$$

This completes the proof.  $\square$

We can now use Claim A to compute the rate of return on the asset as follows.

$$(A6) \quad \begin{aligned} & (d_{t+1s} + p_{t+1s})/p_t = \\ & = (1 + \beta + \beta^2 + \dots + \beta^{T-t-1})d_{t+1s} / (\beta + \beta^2 + \dots + \beta^{T-t})d_t = d_{t+1s}/\beta d_t \end{aligned}$$

Using  $G_t = \sum_{s=1}^S (\pi_s | I_t) (d_{t+1s}/d_t)$  to denote the expected consumption growth we can write the expected rate of return at time  $t$  as:

$$(A7) \quad G_t/\beta = G_t(1 + \rho),$$

where  $\rho$  is the subjective rate of interest. This is exactly the formula (15) that we got in the two periods horizon.

#### APPENDIX B: MONOTONIC TRANSFORMATION OF THE COBB-DOUGLAS UTILITY FUNCTION

In Table 1 we have seen that the prediction of the log utility function about the average return in the economy is the same as the prediction of the Cobb-Douglas functions. We now show that this is also

the case for other monotonic transformation of the Cobb-Douglas function.

We assume a utility function  $F(U)$ , where  $F' > 0$ . The problem (14) is now:

$$(B1) \quad \max_{C_0} \sum_{s=1}^S \Pi_s F\{U[C_0, d_s(y + p - C_0)/p]\}$$

The first order condition for this problem is:

$$(B2) \quad \sum_{s=1}^S \Pi_s F'_s (U_{0s} - d_s U_{1s}/p) = 0$$

where  $F'_s = F'\{U[C_0, d_s(y + p - C_0)/p]\}$ . In general a monotonic transformation will change the price of a tree. In the Cobb-Douglas case  $C_0 = \alpha(y+p)/(\alpha+\delta)$  and

$$(B3) \quad U_{0s} - d_s U_{1s}/p = 0 \text{ for all } s.$$

It follows that a monotonic transformation that changes the derivatives  $F'_s$  will not change  $p$ .

We may now consider the family of utility functions that are monotonic transformation of the log utility function. This is a much larger family than the Cobb-Douglas utility function. It includes for example,  $[\ln(C_0) + \ln(C_1)]^\gamma$ . We can now generalize Claim 1 as follows.

**Claim B:** If the utility function of the representative agent is a monotonic transformation of the log utility function, then the expected rate of return in a single asset economy is  $G/\beta$ .



## APPENDIX C: DATA

I took the following series from the St. Louis Fed web site.

Population (POP): Civilian Labor Force (M, SA),

Wage bill (NW): Compensation of Employees: Wages and Salary Accruals (Q, SAAR),

Consumption (NC): Personal Consumption Expenditures (Q, SAAR)

Price level (P): Gross Domestic Product Chain-type Price Index

Corporate Profits (NPR): Corporate Profits After Tax with Inventory

Valuation Adjustment (IVA) and Capital Consumption Adjustment (CCADi)

Nominal GDP (NGDP): Gross Domestic Product, 1 Decimal

These data are available from January 1948 until January 2004. The data are available on a quarterly basis (except for population which is given on a monthly basis and was converted to a quarterly series). The data are in billions of current dollars and were divided by the price level and by population to obtain real per capita magnitudes:

$W = NW/P(POP)$  real per capita wage earnings

$C = NC/P(POP)$  real per capita consumption

$PR = NPR/P(POP)$  real per capita Corporate Profits

$Y = NGDP/P(POP)$  real per capita GDP

$Y-W = (NGDP-NW)/P(POP)$  real per capita non wage income

I computed the following gross rates of change:  $c_t = C_t/C_{t-1}$ ,

$w_t = W_t/W_{t-1}$ ,  $pr_t = PR_t/PR_{t-1}$ ,  $y_t = GDP_t/GDP_{t-1}$ ,  $(Y-w)_t = (Y-W)_t/(Y-W)_{t-1}$ .

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