

**HOTELLING WAS RIGHT ABOUT SNOB/CONGESTION GOODS
(ASYMPTOTICALLY)**

by

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Hotelling Was Right About Snob/Congestion Goods (Asymptotically)

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Abstract

We add congestion/snobbery to the Hotelling model of spatial competition. For any firm locations on opposite sides of the midpoint, a pure strategy price equilibrium exists and is unique if congestion costs are strong enough relative to transportation costs. The maximum distance between firms in any pure strategy symmetric location equilibrium declines toward zero as congestion costs increase relative to transportation costs. For any non-zero minimum distance between firms, high enough congestion costs relative to transportation costs guarantee that the unique pure strategy symmetric location equilibrium involves minimum differentiation. In this sense Hotelling was right about differentiation of snob/congestion goods.

1 Introduction

“Nobody goes there anymore; it’s too crowded.” — attributed to Yogi Berra

In the seminal Hotelling (1929) model of spatial competition,¹ two firms compete to sell homogeneous products. In stage one, they locate along a line segment; in stage two, they set their prices. Consumers located along the line choose the best deal based on the posted price and linear transportation costs.

Hotelling argued that, regardless of location, either firm would increase its profits by moving closer to its competitor. This force for conformity was termed the “principle of minimum differentiation” – the tendency of competing firms to make similar choices in geographical or product-characteristic dimensions. D’Aspremont et al. (1979), however,

¹See Gabszewicz and Thisse (1992) and Eiselt et al. (1993) for surveys of its extensive applications.

critiqued this principle’s validity within the Hotelling model. They showed that no pure strategy pricing equilibrium exists when the firms are sufficiently close together (but not at the same location) – under symmetry, within the inner quartiles.² Without knowing equilibrium profits, there would be no guarantee that firms located in the inner quartiles would want to move closer to each other.

Osborne and Pitchik (1987) essentially resolve the issue by proving existence of, and characterizing, a mixed strategy price equilibrium when firms are close to each other.³ In the pure strategy, symmetric location equilibrium they compute, both firms locate just inside the inner quartiles, each 0.23 of the distance from the midpoint. Equilibrium differentiation is substantial; evidently, the incentives to move closer together do not hold over much of the region where Hotelling’s price equilibrium fails to exist.

We analyze a class of goods for which Hotelling’s principle of minimum differentiation has greater validity:⁴ goods that exhibit congestion or snob appeal. In particular, consumers are supposed to face an additional cost proportional to the number of other consumers purchasing from the same retailer. One can think of these negative consumption externalities as arising from congestion for underlying technological reasons – for example, queuing costs at a restaurant or barber. One can also think of them as *brand* snobbery – disutility from others consuming the exact same good.⁵

We find that the greater are congestion costs relative to transportation costs, the more of Hotelling’s line contains a Hotelling-like pure strategy pricing equilibrium. The central interval where the equilibrium does not exist shrinks as congestion costs grow in importance, and in the limit vanishes. Thus, the Hotelling pricing equilibrium can be supported at any

²This non-existence is due to discontinuities in the profit function; the temptation exists for a store to cut its price just enough to steal the entire mass of customers located on the other side of its competitor.

³Much of the literature, beginning with d’Aspremont et al. (1979) and furthered by Economides (1986), replaces the linear transportation costs of Hotelling (1929) with convex transportation costs that restore the existence of a pure strategy price equilibrium at any set of firm locations. Substantial, and even maximum, differentiation typically characterizes the equilibrium outcome under these assumptions.

⁴The title borrows stylistically from Irmen and Thisse (1998), who examine an N -dimension version of the model and find conditions under which minimum differentiation holds in all but one of the dimensions.

⁵The differentiation results we obtain do not generally hold if disutility is also experienced when others consume *nearby* goods – we discuss this case in section 3.

locations where the firms locate on opposite sides of the midpoint, given sufficiently high congestion costs relative to transportation costs.

Congestion costs thus extend the portion of the line over which Hotelling logic applies. That is, wherever the Hotelling pricing equilibrium exists, firms can increase profits by moving closer to each other. Given that it exists everywhere but in a central interval whose width is shrinking toward zero as congestion costs grow in importance, the bound on differentiation in a pure strategy, symmetric location equilibrium declines with congestion costs and approaches zero.

Our last result adds to the model a positive minimum distance between firms, justifiable because firms take up non-negligible space or because of patent or copyright restrictions. Given this minimum distance, congestion costs that are sufficiently large relative to transportation costs guarantee that the unique pure strategy, symmetric location equilibrium involves minimum differentiation.

The following mechanism is behind these results. Congestion effects expand the location pairs over which a pure strategy price equilibrium exists. They do this by reducing the number of customers that can be attracted by a price cut, making deviation from the equilibrium less attractive. They thus extend the portion of the line over which the Hotelling logic holds, that is, where firms gain from moving closer to each other.

Though firms locate close to each other when congestion costs are high, profits can also be high, since equilibrium prices rise with the congestion cost. In a sense, (brand-)snob or congestion effects reduce the need for firms to differentiate their products in order to be guaranteed a substantial consumer base.

Externalities arising due to others' consumption of the same product have long been analyzed by economists. Leibenstein (1950) introduces and analyzes bandwagon/conformity effects and snob effects. There are, however, few attempts to integrate spatial competition and negative consumption externalities into a common framework. Kohlberg (1983) analyzes a setting in which firms compete on the basis of *location only*, and consumers

minimize the sum of their travel time and their waiting time (transportation and congestion costs). De Palma and Leruth (1989) add congestion effects to a *price only* setting of Bertrand price competition and find that firms can earn positive profits. We know of only a handful of studies that analyze price competition with congestion effects in a spatial model. Ahlin (1997) analyzes the same model that we do, but does not address firm differentiation, pricing equilibrium existence under asymmetric firm locations, or pricing equilibrium uniqueness. Grilo et al. (2001) introduce consumption externalities (positive or negative) into a Hotelling framework with quadratic transportation costs. They find that conformity can enhance price competition while snobbery dampens it. Our analysis differs from theirs in retaining Hotelling's assumption of linear transportation costs and focusing on equilibrium differentiation.⁶

This model suggests that similar restaurants, for example, may locate relatively close to each other since they are prone to congestion.⁷ It is consistent with phenomena like Greek Towns and Little Italys. Another potential application is fashion. Perhaps polo shirt consumers are content to differentiate themselves via a tiny alligator or a horse knit onto their shirt – the model predicts that the shirts themselves may be very similar. It would also predict that if snobbery in high fashion is brand-snobbery, designers may differentiate themselves only slightly. This would be consistent with fashion *trends*, in which all designers recycle similar fashions (e.g. 70's-era) simultaneously with relatively moderate twists, instead of a case in which all decades and all styles were always on offer from one top designer or another. Similarly, the model suggests an explanation for phenomena like a small number of featured colors per year in auxiliary wedding apparel.

⁶In independent work, Di Cintio (2006) adds the Grilo et al. (2001) consumption externality specification to the original Hotelling framework of linear transportation costs. He finds similar results to ours under significantly weaker conditions, which we believe are not sufficient.

⁷This applies to Yogi Berra restaurants as opposed to Gary Becker (1991) restaurants, where congestion effects dominate bandwagon effects: most likely all but a relatively small top tier.



Figure 1: Firm a located at distance x_a from left endpoint, firm b located at distance x_b from right endpoint.

2 Model and Results

The original Hotelling game has two stages. In the first stage, firms a and b simultaneously choose locations along a unit line. In the second stage, the firms simultaneously choose prices at which they will sell the homogeneous good, whose production cost is zero. A unit measure of consumers with inelastic unit demand, distributed uniformly on the line, then choose which firm to purchase from. They face a linear transportation cost and thus consider the firms' respective prices and locations. We will call an equilibrium in the second stage, i.e. an equilibrium in prices at fixed locations, a pricing equilibrium. We will call a subgame perfect equilibrium to the entire game a location equilibrium.

2.1 Pricing Equilibrium

Consider the second stage, and denote the distance of firms a and b from the left and right endpoints, respectively, as x_a and x_b ; see Figure 1. The cost to a consumer who buys from firm $j \in \{a, b\}$ includes the posted price, p_j , and a linear transportation cost proportional to $\tau > 0$ and the distance to firm j . To Hotelling's model we add a third cost due to congestion or snob effects. Specifically, if q_j is the quantity sold by store j , consumers experience an additional cost equal to κq_j , $\kappa \geq 0$, when purchasing from store j .

The disutility to a consumer located at point x on $[0, 1]$, if he buys from firm a , is

$$-U(x) = p_a + \tau|x - x_a| + \kappa q_a; \tag{1}$$

and if he buys from firm B , is

$$-U(x) = p_b + \tau|1 - x_b - x| + \kappa q_b. \quad (2)$$

Define $\delta \equiv 1 - x_a - x_b$ as the distance between the two firms. Equations 1 and 2 can be used to derive firm a demand as a function of prices p_a and p_b :

$$q_a = \begin{cases} 1 & \text{if } p_a \leq p_I \equiv p_b - \tau\delta - \kappa & | \text{region I} \\ \frac{p_b - p_a - \tau\delta + \kappa}{2\kappa} & \text{if } p_I \leq p_a \leq p_{II} \equiv p_b - \tau\delta - \kappa(1 - 2x_b) & | \text{region II} \\ \frac{p_b - p_a + \tau(1 + x_a - x_b) + \kappa}{2(\kappa + \tau)} & \text{if } p_{II} \leq p_a \leq p_{III} \equiv p_b + \tau\delta + \kappa(1 - 2x_a) & | \text{region III} \\ \frac{p_b - p_a + \tau\delta + \kappa}{2\kappa} & \text{if } p_{III} \leq p_a \leq p_{IV} \equiv p_b + \tau\delta + \kappa & | \text{region IV} \\ 0 & \text{if } p_{IV} \leq p_a & | \text{region V} \end{cases} \quad (3)$$

In regions I and V, price differences between the two firms are enough to overcome the transportation cost of traveling from one firm to the other ($\tau\delta$) and the disutility from consuming the same product as all other consumers (κ). Firm a or b captures the whole market. In region III, there is a point between the two firms such that all customers to the left (right) of this point prefer, and patronize, firm a (b). The novel cases are regions II and IV. In region II, for example, all consumers to the left of firm b strictly prefer firm a . Consumers to the right of firm b are indifferent between firms a and b , *conditional* on a specific number of them patronizing each firm; any other number would raise congestion effects at one site and make the other strictly preferable. Within this region, a lower p_a raises the amount of congestion that keeps these customers indifferent and leads to more customers patronizing firm a .

With strictly positive congestion costs, the demand curve is continuous, unlike in the original Hotelling model. Congestion limits demand swings that can be caused by arbitrarily small price changes. This can be seen in Figure 2, which graphs firm a 's demand and profit functions against its price for several values of κ/τ , with $(\kappa + \tau)$ fixed at one, $p_b = 1$, and

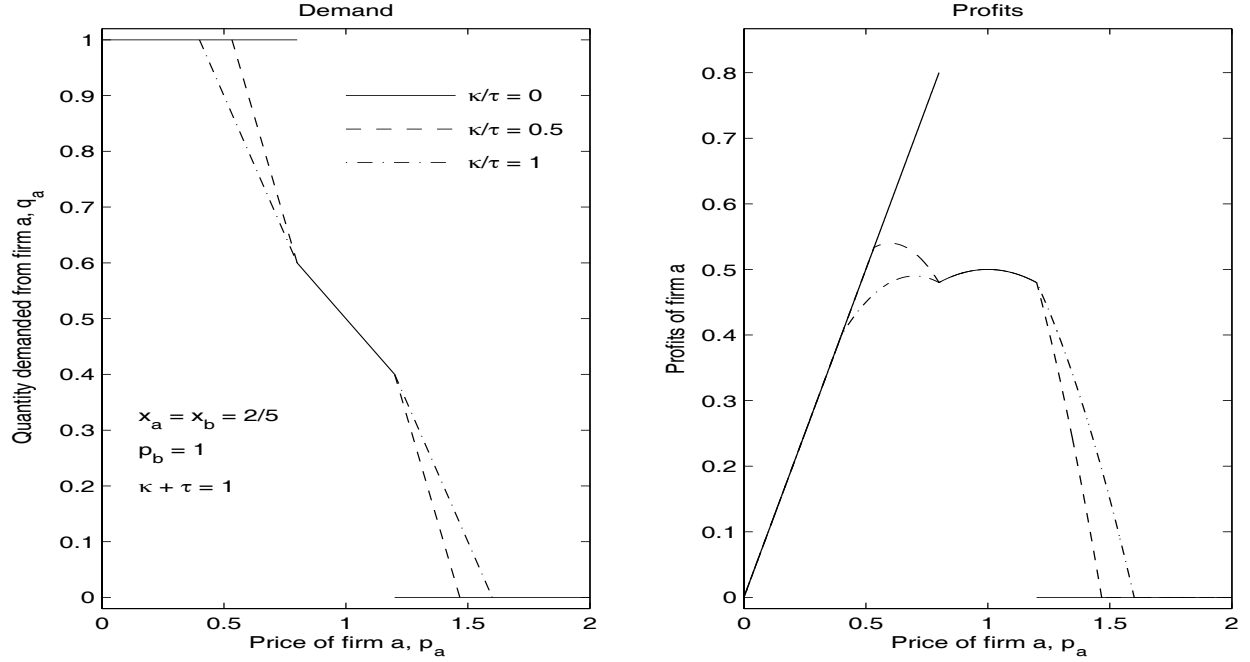


Figure 2: Firm a 's demand and profits as a function of its price p_a , for various κ/τ ratios, given $\kappa + \tau = 1$, $p_b = 1$, and $x_a = x_b = 2/5$. Parameter values and curve markings are the same in both panels.

$x_a = x_b = 2/5$. The original model, $\kappa/\tau = 0$, corresponds to the solid lines. Demand and profits are discontinuous, as is well-known. For example, firm a can lower its price below its competitor's just enough to cover the cost of transportation between the two firms and win the entire mass of consumers to the right of firm b . With congestion effects, however, demand responses to price cuts are continuous and dampened. This can be seen in the dashed and dash-dotted demand curves and profit functions of Figure 2, which correspond to $\kappa/\tau = 1/2$ and $\kappa/\tau = 1$, respectively.

Analysis of this pricing game reveals that, unlike in the original Hotelling model, a Nash equilibrium in pure strategies exists for any values of $x_a, x_b \in [0, 1/2)$. The key condition is that congestion effects are large enough relative to transportation costs.

Proposition 1. *Let $x_a, x_b \in [0, 1/2)$. There exist functions $B, C : [0, 1/2)^2 \rightarrow \Re$ such that the following hold. If and only if*

$$\kappa/\tau \geq B(x_a, x_b), \tag{A1}$$

the following Nash equilibrium exists:

$$p_a^* = \kappa + \tau + \tau \frac{(x_a - x_b)}{3} \quad p_b^* = \kappa + \tau + \tau \frac{(x_b - x_a)}{3}. \quad (4)$$

No other pure strategy Nash equilibrium exists. This equilibrium exists and is unique if

$$\kappa/\tau \geq C(x_a, x_b). \quad (A2)$$

Proof. See Appendix.

The pure strategy equilibrium, when it exists, involves a point of indifference between the two firms and strict preference for firm a (b) to the left (right) of this point. The equilibrium prices are identical to Hotelling's, except for the addition of κ . Higher congestion costs thus dampen competition and allow firms to charge more in equilibrium, as others have found (Grilo et al., 2001, for example).

In contrast to Hotelling's model, a pure strategy equilibrium exists regardless of how close to the midpoint the firms locate (x_a and x_b close to $1/2$), as long as congestion costs are sufficiently large relative to transportation costs. In the original Hotelling model, when x_a and x_b are close to $1/2$, the equilibrium does not exist because a small price reduction can capture all the competitor's customers. Congestion costs restore equilibrium by limiting the willingness of a large number of consumers to switch firms and thus muting the demand response to a lower price. This is apparent in the profit function graph of Figure 2; the candidate equilibrium is at $p_a = 1$, but only when κ/τ is high enough is it robust to a downward deviation.

The exact bound functions B and C are in the appendix (equations 6 and 7, respectively). These bounds increase the closer either firm gets to the other, and they approach infinity as either firm approaches the midpoint. Thus, greater congestion costs support the pure strategy pricing equilibrium over a larger set of locations. The locations for which this

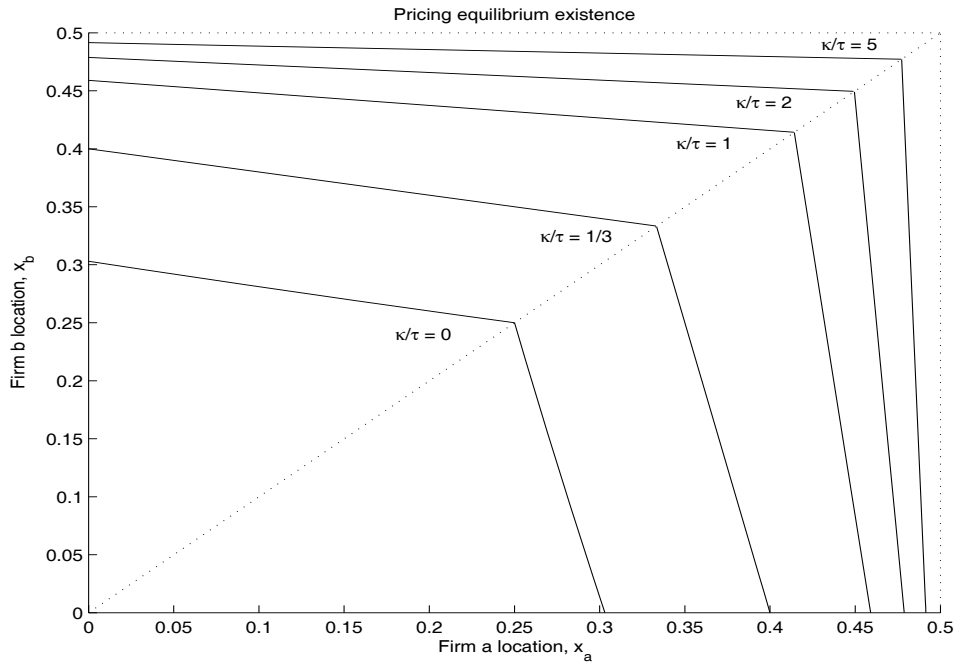


Figure 3: The pure strategy pricing equilibrium of Proposition 1 exists when (x_a, x_b) falls below the lines drawn, for various values of κ/τ .

equilibrium exists are graphed for various values of κ/τ in Figure 3.⁸

Since the conditions involve κ/τ , what is important is that congestion costs are high *relative* to transportation costs, not necessarily high in absolute terms. Indeed, κ/τ can achieve any positive value while the equilibrium price ($\kappa + \tau$, under symmetric locations) is fixed. This alleviates concerns that the assumption of inelastic consumer demand is restrictive for high values of κ/τ .

The proof involves checking best responses for all possible pure strategy equilibria; only one candidate emerges at which both firms are locally best-responding. Next, it involves finding conditions for no profitable deviation to exist from this candidate equilibrium. As in the Hotelling model, the only potentially profitable deviations are downward; however, unlike in the original Hotelling model, a downward deviation that does not capture the

⁸Regardless of the relative importance of congestion costs, there is no pure strategy pricing equilibrium when both firms locate on the same half of the line (including the midpoint). In these cases, there is no stable set of prices with the point of indifference between firms, or if there is, higher congestion costs raise the locationally disadvantaged firm's deviation payoff at least as much as its equilibrium payoff.

entire market may be the optimal deviation (see the dashed profit function in Figure 2). The bound function B comes from directly comparing firms' equilibrium profits with their deviation profits to ensure the latter are not greater.

Condition A1 guaranteeing existence of the pure strategy equilibrium is as weak as possible. Condition A2 is stronger than condition A1 and guarantees existence *and uniqueness* of the pure strategy equilibrium. That is, it rules out additional mixed strategy equilibria. The proof (inspired by Osborne and Pitchik, 1987) uses dominance arguments to limit a firm's pricing support in any equilibrium to a single point. These arguments require quasi-concavity of firm i 's profit function when $p_j < p_j^*$; necessary and sufficient for this is condition A2. Quasi-concavity is not necessary to guarantee existence; for example, the dash-dotted profit function of Figure 2 has no profitable deviation from $p_a = p_a^* = 1$ though not quasi-concave. Similarly, quasi-concavity may not be necessary for uniqueness. Nonetheless, the qualitative result is the same under the stronger condition: any location pair $x_a, x_b \in [0, 1/2)$ has as its unique pricing equilibrium the Hotelling one as long as κ/τ is large enough.

2.2 Location Equilibrium

The pricing equilibrium results have implications for equilibrium firm locations. Specifically, congestion costs extend the portion of the line over which the Hotelling equilibrium exists and minimum differentiation logic applies.

Proposition 2. *The distance between firms in a pure strategy, symmetric location equilibrium is less than*

$$\frac{1}{1 + 2\frac{\kappa}{\tau}}.$$

Assume in addition that a pure strategy equilibrium is played in the pricing game if one exists. Then the distance between firms in a pure strategy, symmetric location equilibrium is

less than

$$\begin{aligned} \frac{1-\frac{\kappa}{\tau}}{2} & \quad \text{if } \kappa/\tau \leq 1/3 \\ 1 - 2\frac{\kappa}{\tau}(\sqrt{1+\frac{\tau}{\kappa}} - 1) & \quad \text{if } \kappa/\tau \geq 1/3 \end{aligned}$$

Proof. The first expression is the inverse of condition A2 with symmetry imposed and written in terms of distance. Thus, for any symmetric firm locations $x_a = x_b = x$ with distance apart strictly greater than the first expression, condition A2 holds with strict inequality. By Proposition 1, there is a unique pricing equilibrium for any firm locations in a neighborhood of (x, x) , with firm a profits given by expression 5 (in the Appendix). Note from this expression that firm a would raise second-stage profits by locating closer to firm b . Thus this cannot be a location equilibrium.

If the distance between symmetrically located firms is strictly greater than the second expression, condition A1 holds with strict inequality. Thus there exists a pure strategy pricing equilibrium that is unique among pure strategy equilibria. The assumption that pure strategy equilibria are selected first along with the above logic rules out this outcome as a location equilibrium. ■

Thus, the greater are congestion/snob effects relative to transportation costs, the closer firms must be in any pure strategy, symmetric location equilibrium. Asymptotically, the distance between them goes to zero.⁹ Proposition 2's upper bounds on firm differentiation are graphed against κ/τ in Figure 4. The case of $\kappa/\tau = 0$ is already known: symmetric locations more than 0.5 apart are ruled out (d'Aspremont et al., 1979) and the pure strategy symmetric location equilibrium appears to involve the firms locating about 0.46 apart (Osborne and Pitchik, 1987). Proposition 2 makes clear that as κ/τ increases, a smaller and smaller central interval of the unit line remains in which any symmetric location equilibrium must exist; thus maximum firm differentiation declines toward zero. The assumption that pure strategy

⁹Note that $\lim_{x \rightarrow \infty} x(\sqrt{1+1/x} - 1) = 1/2$.

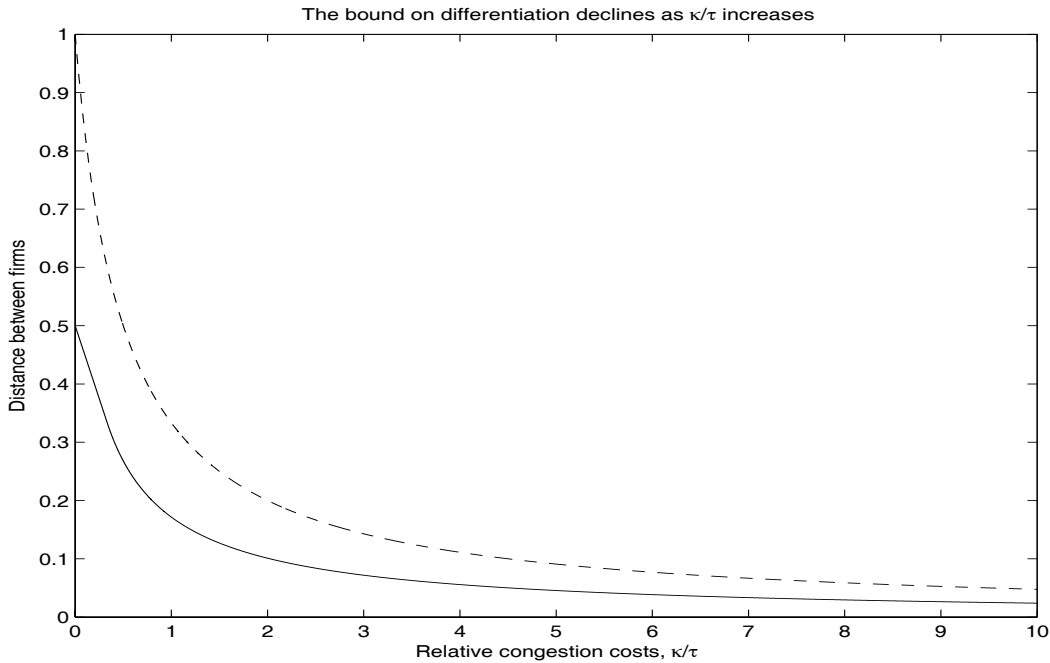


Figure 4: Upper bounds on differentiation in any symmetric location equilibrium are graphed against κ/τ . The tighter, solid-line bound uses the assumption that pure strategy pricing equilibria are selected if they exist; the dashed-line bound does not. Higher relative congestion costs κ/τ reduce possible differentiation, in the limit to zero.

equilibria are selected in the pricing stage allows us to bound equilibrium differentiation more tightly.¹⁰ However, even without this assumption, maximum firm differentiation declines toward zero as congestion costs become more important.

Proposition 2 essentially provides a negative result, ruling out location equilibria everywhere but in a central interval of the line. It would be ideal provide a positive result, i.e. to demonstrate the existence of a symmetric location equilibrium in the central interval not ruled out by Proposition 2. Basic results can be applied to show existence of a mixed strategy equilibrium in the second (pricing) stage within this interval.¹¹ However, gaining a sufficient handle on the pricing equilibrium payoffs to guide backward induction and guarantee a location equilibrium has not been accomplished analytically even in the original Hotelling model. Osborne and Pitchik (1987) make remarkable progress, but partially rely on computation.

¹⁰Most of the literature besides Osborne and Pitchik (1987) focuses on pure strategy equilibria.

¹¹Theorem 1.3 of Fudenberg and Tirole (1991) can be used since the payoff functions are continuous.

Their work gives optimism, however, that a location equilibrium within the central interval does exist.

We do not solve this difficult problem here. Instead, we make a small additional modification to the model. Assume there is some exogenous minimum distance between firms, $\delta_{min} > 0$. If Hotelling's line is interpreted as physical distance, this positive minimum differentiation can represent the impossibility of firms co-locating since firms take up space. If the line is taken to represent characteristic space, this minimum differentiation can represent copyright or patent restrictions. Given this assumption, we show the following.

Proposition 3. *The unique pure strategy, symmetric location equilibrium involves minimum differentiation, that is the distance between firms equal to δ_{min} , as long as*

$$\begin{aligned} \delta_{min} \in (1/3, 1] \quad \text{and} \quad \kappa/\tau \geq \frac{1-\delta_{min}}{2\delta_{min}}, \quad \text{or} \\ \delta_{min} \in (0, 1/3] \quad \text{and} \quad \kappa/\tau \geq \frac{(3-\delta_{min})^2}{12\delta_{min}}. \end{aligned}$$

Proof. See Appendix.

Thus, as long as there is some inability of firms to co-locate and provided congestion costs are strong enough relative to transportation costs, the unique pure strategy, symmetric location equilibrium involves minimum differentiation. In this sense, Hotelling was right – the principle of minimum differentiation holds, at least for goods exhibiting snob appeal or congestion, asymptotically.

The role of the assumed minimum differentiation is straightforward – it rules out locational deviations into the central interval where the pure strategy pricing equilibrium does not exist. The condition on κ/τ when $\delta_{min} > 1/3$ is simply condition A2 modified for symmetry and expressed in terms of distance between firms. The condition on κ/τ for $\delta_{min} \leq 1/3$ is stronger than condition A2. The reason is that when $\delta_{min} \leq 1/3$, one firm can deviate by leapfrogging the other and locating on the same side of the line. This would not seem especially profitable; however, when both firms are on the same side of the line there is no pure strategy pricing equilibrium so we cannot calculate deviation payoffs. Our strategy

is to bound them, using dominance arguments to bound prices charged at such locations and using the price bounds to bound deviation profits. Less restrictive bounds on κ/τ are available under assumptions that a) pure strategy pricing equilibria are chosen if they exist and/or that b) firms must locate on their own half of the line. However, the qualitative flavor of the bounds does not change: they increase in δ_{min} and approach infinity as δ_{min} approaches zero.

3 Discussion

It may seem paradoxical that the more pronounced is snobbery, the more similar the products firms will end up marketing. Note, however, that our model is of brand-snobbery, in that snobbery costs apply only to consumers frequenting the exact same firm. The minimum differentiation results apply in this context.

In a different model of snobbery, however, consumers may experience disutility from others consuming nearby products also. For example, let the snob disutility be proportional to the number of customers of a product *and* to (one minus) the distance of that product from a consumer's chosen product. Then snobbery disutility for a customer of store i would be $\kappa q_i + \kappa(1 - \delta)q_j$, which equals $\kappa(1 - \delta) + \kappa\delta q_i$, since $q_a + q_b = 1$. That is, this model turns out to be isomorphic to the one we analyzed with a modified congestion cost $\kappa' = \kappa\delta$. Thus Proposition 1 goes through with the bounds for κ/τ divided by $\delta = 1 - x_a - x_b$. However, Proposition 2 does not go through. This is because equilibrium second-stage profits are, for firm a (see equation 5 in the appendix):

$$\frac{[\kappa' + \tau (1 + \frac{x_a - x_b}{3})]^2}{2(\kappa' + \tau)} = \frac{[\kappa(1 - x_a - x_b) + \tau (1 + \frac{x_a - x_b}{3})]^2}{2[\kappa(1 - x_a - x_b) + \tau]}.$$

This is *decreasing* in x_a for κ/τ large, meaning firm a would prefer to move away from firm b . Thus, not surprisingly a more continuous kind of snobbery would push toward maximum rather than minimum differentiation.

Nonetheless, congestion and brand snobbery as we model them ease competitive pressures and push firms toward one another. The more important are these costs relative to transportation costs, the closer minimum differentiation comes to holding; in the limit, it comes arbitrarily close. The simple addition of linear congestion costs gives Hotelling's original argument greater validity.

The model provides interesting testable implications. It suggests that differentiation between firms depends on the type of good being sold and its associated externalities. Variation across goods in the degree of congestion or brand-snobbery should be related to variation in product differentiation. Exploring this relationship empirically is left for future work.

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A Proofs

Proof of Proposition 1. We first show the existence of $B(x_a, x_b)$ such that $\kappa/\tau \geq B(x_a, x_b)$ is equivalent to (p_a^*, p_b^*) constituting a Nash equilibrium. Fix $p_b = p_b^*$. It can be verified that p_a^* is in the interior of region III (of equation 3) and is the best response to p_b in this region. It can also be shown that firm a profits are decreasing in price for $p_a \geq p_{III}$. Thus the only potential profitable deviations are downward.

Some algebra verifies that iff $\kappa/\tau \geq M_h \equiv (x_a + 2x_b)/[3(1 - 2x_b)]$, firm a 's profits are monotonically increasing up to p_{II} ; this guarantees no profitable downward deviation. If $\kappa/\tau \leq M_l \equiv (x_a + 2x_b)/3$, the best response below p_{II} is p_I . Deviation profits also equal p_I , since demand is one. Using the expressions for p_a^* and p_b^* , and demand from equation 3, equilibrium profits can be written

$$\frac{[\kappa + \tau (1 + \frac{x_a - x_b}{3})]^2}{2(\kappa + \tau)}. \quad (5)$$

Some algebra shows that deviation to p_I does not strictly raise profits iff

$$\kappa/\tau \geq K_l \equiv \frac{x_a + 2x_b}{3} - (1 - b) + 2\sqrt{b \left(\frac{x_a + 2x_b}{3} \right)}.$$

Iff $M_l \leq \kappa/\tau \leq M_h$, the best response below p_{II} is in the interior of region II: $(p_b - \tau\delta - \kappa)/2$, with associated deviation profits of

$$\frac{[\kappa + \tau \left(\frac{x_a + 2x_b}{3} \right)]^2}{2\kappa}.$$

This is not strictly higher than equilibrium profits iff

$$\kappa/\tau \geq K_h \equiv \frac{x_b \left(\frac{2x_a + x_b}{3} \right) - (1 - 2x_b) + (1 - x_b)\sqrt{\left(\frac{2x_a + x_b}{3} \right)^2 + 1 - 2x_b}}{2(1 - 2x_b)}.$$

Summarizing, firm a has no profitable deviation iff one of three sets of conditions holds: $K_l \leq \kappa/\tau \leq M_l$, $\max\{M_l, K_h\} \leq \kappa/\tau \leq M_h$, or $\kappa/\tau \geq M_h$. One can show that $M_l, K_h \leq M_h$; thus the latter two conditions are equivalent to $\max\{M_l, K_h\} \leq \kappa/\tau$. Also, if $x_a \leq (3 - 6x_b - 5x_b^2)/4x_b$, then both K_l and K_h are less than M_l , and the conditions are equivalent to $K_l \leq \kappa/\tau$. If $x_a \geq (3 - 6x_b - 5x_b^2)/4x_b$, then both K_l and K_h are greater than M_l , and the conditions are equivalent to $K_h \leq \kappa/\tau$. Symmetric reasoning gives rise to symmetric conditions for firm b to be best-responding. Condition A1 thus boils down to

$$\kappa/\tau \geq \begin{cases} \frac{x_i + 2x_j}{3} - (1 - x_j) + 2\sqrt{x_j \left(\frac{x_i + 2x_j}{3} \right)} & \text{if } x_i \leq \frac{3 - 6x_j - 5x_j^2}{4x_j} \\ \frac{x_j \left(\frac{2x_i + x_j}{3} \right) - (1 - 2x_j) + (1 - x_j)\sqrt{\left(\frac{2x_i + x_j}{3} \right)^2 + 1 - 2x_j}}{2(1 - 2x_j)} & \text{if } x_i \geq \frac{3 - 6x_j - 5x_j^2}{4x_j} \end{cases} \quad (6)$$

for $(i, j) = (a, b)$ and $(i, j) = (b, a)$. This bound is finite for $x_a, x_b \in [0, 1/2)$.

Conversely, it is clear that if condition 6 is not met, firm a or b has a strictly profitable deviation downward. Thus the existence of this equilibrium implies the stated conditions.

We next argue that there are no pure strategy equilibria beside the proposition's. It is clear that firm a pricing in region I cannot be an equilibrium; firm b could lower its price to earn strictly positive, instead of zero, profits. Writing out best responses, it is clear that firms a and b cannot be simultaneously best-responding when firm a is pricing in the interior of region II (and thus firm b is pricing in the interior of its analogous region IV). One can also show that if firm a is pricing on the boundary between regions II and III ($p_a = p_{II}$), firm a can gain by raising its price or firm b can gain by lowering its price. It is also straightforward to show that there are no mutual local best responses in the interior of region III besides (p_a^*, p_b^*) . Finally, symmetric arguments can be applied to rule out equilibria in which firm a prices in IV or V (i.e. firm b prices in analogous regions I or II).

Finally, we argue that under a stronger condition than condition A1, namely condition A2:

$$\kappa/\tau \geq \frac{x_i + 2x_j}{3(1 - 2x_j)} \quad (7)$$

for $(i, j) = (a, b)$ and $(i, j) = (b, a)$, the equilibrium exists and is unique.¹² Existence is clear from above (since this bound equals M_h). We also have established that there are no other pure strategy equilibria. Consider a candidate mixed strategy equilibrium with pricing supports $[\underline{p}_a, \bar{p}_a]$ and $[\underline{p}_b, \bar{p}_b]$.

We first show that $\bar{p}_i \leq p_i^*$, $i = a, b$. Note that $\underline{p}_i \geq 0$, since firm i can guarantee zero profits. Define $p_i^*(p_j)$ as the best response of firm i to price p_j . Define $p_{III}^*(p_j)$ as the optimal price p_i given p_j using the region-III expression for demand: $p_{III}^*(p_j) = [p_j + \tau(1 + x_i - x_j) + \kappa]/2$. Define $p_{II}^*(p_j)$ and $p_{IV}^*(p_j)$ similarly: $p_{II}^*(p_j) = (p_j - \tau\delta + \kappa)/2$ and $p_{IV}^*(p_j) = (p_j + \tau\delta + \kappa)/2$. Simple algebra verifies that if $\bar{p}_a \leq p_{III}^*(\bar{p}_b)$ and $\bar{p}_b \leq p_{III}^*(\bar{p}_a)$, then $\bar{p}_i \leq p_i^*$, $i = a, b$.

Assume not, e.g. that $\bar{p}_a > p_{III}^*(\bar{p}_b)$. Now the best response function for firm a follows perhaps $p_{IV}^*(p_b)$, then perhaps $p_{III}^*(p_b)$, then $p_{III}^*(p_b)$; then it either jumps down to $p_{II}^*(p_b)$ and follows it until joining $p_I(p_b)$ forever, or jumps down to $p_I(p_b)$ and follows it forever. Thus it is continuous and increasing everywhere except at the downward jump. Examination of the best response function slopes and the profit functions makes clear that the best response to p_b is no greater than $p_{III}^*(p_b)$, and that profits are declining in p_a beyond $p_{III}^*(p_b)$, provided $p_I(p_b) \leq p_{III}^*(p_b)$, i.e. $p_b \leq Z_b \equiv 3\kappa + \tau(3 - x_a - 3x_b)$. Thus, since $p_{III}^*(p_j)$ is increasing, the assumption that $\bar{p}_b \leq Z_b$ guarantees that $p_{III}^*(\bar{p}_b)$ dominates all prices $p_i \in (p_{III}^*(\bar{p}_b), \bar{p}_a]$ for any firm- b price $p_b \in [\underline{p}_b, \bar{p}_b]$. Further, it strictly dominates prices in $[\bar{p}_b - \epsilon, \bar{p}_b]$ for some $\epsilon > 0$, since under condition 7 $p_{III}^*(\bar{p}_b) < p_{IV}^*(\bar{p}_b)$. This establishes that $\bar{p}_a > p_{III}^*(\bar{p}_b)$ is a contradiction. Similar reasoning gives that $\bar{p}_a \leq Z_a \equiv 3\kappa + \tau(3 - 3x_a - x_b)$ guarantees $\bar{p}_b \leq p_{III}^*(\bar{p}_a)$. Thus $\bar{p}_i \leq Z_i$, $i = a, b$, implies that $\bar{p}_i \leq p_i^*$, $i = a, b$.

Consider next $\bar{p}_i > Z_i$, $i = a, b$. This implies that $p_i^*(\bar{p}_j) = p_I(\bar{p}_j) > p_{III}^*(\bar{p}_j)$, $i = a, b$. This guarantees that $p_a^*(\bar{p}_b) < \bar{p}_a$ or $p_b^*(\bar{p}_a) < \bar{p}_b$. First, assume $p_a^*(\bar{p}_b) < \bar{p}_a$. For $p_b \leq Z_b$, the previous paragraph argues that the best response of a is no greater than $p_{III}^*(p_b)$ and that profits are declining in p_a beyond $p_{III}^*(p_b)$. For $p_b > Z_b$, the best response of a is $p_I(p_b)$ and one can show that profits are declining in p_a beyond $p_I(p_b)$. Since $p_{III}^*(p_b)$ and $p_I(p_b)$

¹²This part of the proof follows some strategies similar to or inspired by Osborne and Pitchik (1987).

are less than $p_I(\bar{p}_b)$, for $p_b \leq \bar{p}_b$, $p_I(\bar{p}_b) = p_a^*(\bar{p}_b)$ dominates all prices $p_a \in (p_I(\bar{p}_b), \bar{p}_a]$ for any firm- b price $p_b \in [\underline{p}_b, \bar{p}_b]$. Further, it strictly dominates prices in $[\bar{p}_b - \epsilon, \bar{p}_b]$ for some $\epsilon > 0$, since $p_I(\bar{p}_b) < p_{IV}(\bar{p}_b)$. This establishes that $\bar{p}_a > p_a^*(\bar{p}_b)$ cannot occur; symmetric reasoning gives that $\bar{p}_b > p_b^*(\bar{p}_a)$ cannot occur. Thus, $\bar{p}_i > Z_i$ for $i = a, b$ is impossible.

Consider next $\bar{p}_a \leq Z_a$ and $\bar{p}_b > Z_b$. By above reasoning, $\bar{p}_a \leq Z_a$ implies that $\bar{p}_b \leq p_{III}^*(\bar{p}_a)$. But, condition A2 and $\bar{p}_a \leq Z_a$ ensure that $p_{III}^*(\bar{p}_a) \leq Z_b$, so it cannot be that $\bar{p}_b \leq p_{III}^*(\bar{p}_a)$ and $\bar{p}_b > Z_b$. Similarly, $\bar{p}_b \leq Z_b$ and $\bar{p}_a > Z_a$ can be ruled out. Thus, $\bar{p}_i \leq p_i^*$, $i = a, b$.

Given condition 7, $\bar{p}_a \leq p_a^*$, and $\bar{p}_b \leq p_b^*$, and for any $p_j \in [\underline{p}_j, \bar{p}_j]$, one can show that firm- i profits are quasi-concave in p_i and have a unique maximum $p_i^*(p_j)$, which is continuously and strictly increasing in p_j . This allows us to establish that $\underline{p}_i \geq p_i^*(\underline{p}_j)$ and $\bar{p}_i \leq p_i^*(\bar{p}_j)$, $(i, j) = (a, b), (b, a)$. Assume not, e.g. that $p_i^*(\bar{p}_j) < \bar{p}_i$. Then $p_i^*(\bar{p}_j)$ dominates all p_i in $[p_i^*(\bar{p}_j), \bar{p}_i]$ for all $p_j \in [\underline{p}_j, \bar{p}_j]$, strictly so for p_j in a neighborhood of \bar{p}_j . This is a contradiction. Similarly, if $\underline{p}_i < p_i^*(\underline{p}_j)$, then $p_i^*(\underline{p}_j)$ dominates all p_i in $[\underline{p}_i, p_i^*(\underline{p}_j)]$ for all $p_j \in [\underline{p}_j, \bar{p}_j]$, strictly so for p_j in a neighborhood of \underline{p}_j .

Note that $p_i^*(p_j)$, for $p_j \in [0, p_j^*]$, is continuous and piecewise linear with slopes (where they exist) equal to 1/2 or 1; this is because for $p_j \in [0, p_j^*]$, $p_i^*(p_j)$ follows perhaps $p_{IV}^*(p_j)$, then perhaps $p_{III}^*(p_j)$, then $p_{II}^*(p_j)$. Thus, for some $\lambda_i, \lambda_j \in [1/2, 1]$, this fact gives rise to the equalities in the following set of claims:

$$p_i^*(\bar{p}_j) - p_i^*(\underline{p}_j) = \lambda_i(\bar{p}_j - \underline{p}_j) \leq \lambda_i[p_j^*(\bar{p}_i) - p_j^*(\underline{p}_i)] = \lambda_i\lambda_j(\bar{p}_i - \underline{p}_i) \leq \lambda_i\lambda_j[p_i^*(\bar{p}_j) - p_i^*(\underline{p}_j)].$$

The inequalities come from the previous paragraph's result that $\underline{p}_i \geq p_i^*(\underline{p}_j)$ and $\bar{p}_i \leq p_i^*(\bar{p}_j)$. There are two potential conclusions from this set of claims. One is that $\underline{p}_i = \bar{p}_i$, $i = a, b$, in which case the proof is complete. If this is not true, then $\lambda_i = 1$, $i = a, b$; also, $\underline{p}_a = p_a^*(\underline{p}_b)$ and $\underline{p}_b = p_b^*(\underline{p}_a)$ (also using the previous paragraph's result). This contradicts the fact that (p_a^*, p_b^*) is the unique pure strategy equilibrium. ■

Proof of Proposition 3. If $\delta_{min} \in (1/3, 1]$, the condition $\kappa/\tau \geq (1 - \delta_{min})/(2\delta_{min})$ (which merely rewrites condition 7 in terms of distance and using symmetry) guarantees that the Hotelling-like pricing equilibrium exists and is unique everywhere the firms may symmetrically locate and at all possible deviations from permissible symmetric locations. Thus, there is no pure strategy, symmetric location equilibrium where differentiation is not minimal, since firms can increase profits by moving closer to each other (as argued in the proof of Proposition 2); but being minimally differentiated is an equilibrium, since the only possible deviation is away from the other firm and thus lowers profits.

If $\delta_{min} \leq 1/3$, the condition on κ/τ is stronger than condition 7, so by the previous paragraph's arguments no pure strategy, symmetric location equilibrium exists that is not minimally differentiated; and no deviation away from the other firm in the symmetric minimum differentiation equilibrium can raise profits. However, a new kind of deviation from the minimum differentiation equilibrium is possible when $\delta_{min} \leq 1/3$: deviation to a location on the other side of the rival firm. In this case, both firms are located on the same side of the line's midpoint, and no pure strategy pricing equilibrium exists. We thus cannot calculate the deviation payoff. Our strategy here is to bound it.

Consider the candidate, minimum differentiation equilibrium where $x_a = x_b = (1 - \delta_{min})/2$. Each firm's profits equal $(\kappa + \tau)/2$. Consider a deviation by firm a to some location $x_a \in [1 - x_b + \delta_{min}, 1]$, i.e. $x_a \in [1/2 + 3\delta_{min}/2, 1]$. Competition at these locations is isomorphic to competition when $x_a \in [0, 1/2 - 3\delta_{min}/2]$ and $x_b = (1 + \delta_{min})/2$, so we consider the latter.

First, one can show that if firms are randomizing over pricing supports $[p_i, \bar{p}_i]$, $i = a, b$, then $\bar{p}_i \leq p_i^*$, $i = a, b$, where the p_i^* are calculated using the deviation locations. This uses the exact reasoning of the part of the proof of Proposition 1 that establishes the upper bound of the pricing support; the only difference is that whenever condition 7 is used there, here we substitute this proof's condition on κ/τ (for $\delta_{min} \leq 1/3$).

Next we let $p_b = p_b^*$. Since $\bar{p}_b \leq p_b^*$, and since firm a profits are increasing in firm b 's price p_b , for any price p_a , this allows for an upper bound on firm a profits. One can show that firm a profits are maximized the closer it is to firm b , regardless of p_a ; so, we set $x_a = 1/2 - 3\delta_{min}/2$. Finally, one can show that under the condition on κ/τ , firm a 's best response to p_b is in the interior of region II and leads to profits

$$\frac{(2\kappa + \tau - \tau\delta/3)^2}{8\kappa}.$$

This is no greater than equilibrium profits $(\kappa + \tau)/2$ given the proof's condition on κ/τ . ■