

DOMINANCE CRITERIA FOR CRITICAL-LEVEL GENERALIZED UTILITARIANISM

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Generalized Utilitarianism***

by

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Abstract

“Dominance Criteria for Critical-Level Generalized Utilitarianism”

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Social welfare dominance criteria based on critical-level generalized utilitarian social welfare functions are investigated. An analogue of a generalized Lorenz curve called a generalized concentration curve is introduced. For a fixed critical utility level c , a partial order of utility distributions based on these curves is defined and shown to coincide with the partial order obtained by declaring one utility distribution to be weakly preferred to a second if and only if the former is weakly preferred to the latter for all inequality averse critical-level c generalized utilitarian social welfare functions. An extension of this result that allows for a range of critical levels is also established.

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1. Introduction

While Dalton (1920) was the first to ground the measurement of income inequality on social welfare considerations, it was not until the pioneering articles of Kolm (1969), Atkinson (1970), Dasgupta, Sen, and Starrett (1973), and Rothschild and Stiglitz (1973) in the late 1960s and early 1970s that a systematic attempt was made to provide normative foundations for the measurement of inequality.¹ Much of the subsequent literature employs the framework and concepts introduced by Atkinson and Kolm. However, their theoretical analyses are restricted to comparisons of distributions for the same number of individuals. As Dasgupta, Sen, and Starrett (1973, p. 184) have observed, in order to make inequality comparisons across countries or time, it is necessary to consider populations of different size. A natural way to extend fixed population results to the variable population case is provided by the Dalton (1920) Principle of Population. This principle regards an income distribution and any replication of it as exhibiting the same degree of inequality. The dominance criteria based on Lorenz and generalized Lorenz curves satisfy this population replication principle. Following Dasgupta, Sen, and Starrett (1973), it is now standard practice to make comparisons involving different sized populations on the basis of Dalton's principle. However, by employing this principle, one is implicitly assuming that inequality and social welfare should be thought of in per capita terms and, hence, that population size is not a concern.

About the same time that the foundations of the modern theory of inequality measurement were being laid, there was a resurgence of interest in population ethics. See, for example, Dasgupta (1969). Particularly influential contributions to this literature were provided by Blackorby and Donaldson (1984) and Parfit (1984).² One of their central concerns is the question of determining under what circumstances should the addition of a person to a given population be regarded as being welfare improving. To answer this question, Blackorby and Donaldson (1984) proposed using a generalization of classical utilitarianism called critical-level generalized utilitarianism as a social objective.

To date, the ethical debates about the best way to evaluate distribu-

¹See Sen (1973) for an illuminating discussion of these articles.

²See Broome (2004) and Blackorby, Bossert, and Donaldson (2005) for recent monographs on population ethics.

tions of utilities for different population sizes has not had any impact on normatively-based inequality measurement. In this article, we make an initial attempt at integrating the literatures on population ethics and inequality measurement by investigating the implications for social welfare dominance criteria of making comparisons of distributions of utilities (or any other scalar attribute of well-being, such as income or wealth) using critical-level generalized utilitarian social welfare functions.³ Dominance criteria based on critical-level generalized utilitarianism provide an alternative to the generalized Lorenz dominance criterion of Tomić (1949), Kolm (1969), Rothschild and Stiglitz (1973), Shorrocks (1983), and Kakwani (1984). Our new dominance criteria are of interest in so far as one would like a measure of social welfare for the population as a whole, rather than some measure of welfare per capita, as is implicitly the case with generalized Lorenz dominance.

The generalized Lorenz dominance criterion provides a partial ordering of alternative income distributions for homogeneous populations.⁴ According to this criterion, one income distribution weakly dominates a second if the generalized Lorenz curve for the former lies nowhere below the generalized Lorenz curve for the latter. With a population of size n , for each fraction k/n of the population, $k = 0, \dots, n$, a generalized Lorenz curve plots one n th of the total income of the poorest k people against k/n , with linear interpolation used so that the curve is defined for all points $p \in [0, 1]$. This curve is simply the Lorenz curve scaled up by the mean income. This dominance criterion can be applied both when the size of the population is the same in both distributions and when it is not. Replicating a population and its distribution of incomes has no effect on the shape of a generalized Lorenz curve and, hence, as we have already noted, the generalized Lorenz criterion satisfies Dalton's Principle of Population.

In homogenous populations, everyone receives the same utility from a given amount of income. When this is the case, Kakwani (1984) and Shorrocks (1983) have shown that the average utility for one income distribution is no less than the average utility for a second income distribution for all continuous, increasing, concave utility functions if and only the former distribution generalized Lorenz dominates the latter. We shall henceforth refer to this result as the Kakwani-Shorrocks Theorem. More generally, assuming that

³A social welfare function is a real-valued function defined on distributions of utilities.

⁴For a good introduction to generalized Lorenz dominance, see Lambert (2001, Chapter 3).

the social welfare function is invariant to a replication of the distribution of utilities (and, hence, invariant to a replication of the income distribution), a straightforward extension of an argument developed for Lorenz domination by Dasgupta, Sen, and Starrett (1973) shows that it is sufficient for the equivalence between welfare dominance and generalized Lorenz dominance to hold that the social welfare function is increasing, symmetric, and quasiconcave for each population size. It is not necessary for the social welfare function to aggregate utilities by taking their average. Note that the replication invariance property of the social welfare function implies that overall social welfare is being measured in per capita terms.

The generalized Lorenz dominance criterion can also be applied to distributions of utility. In this case, the Kakwani–Shorrocks Theorem shows that one distribution of utilities is weakly preferred to a second distribution by all inequality averse average generalized utilitarian social welfare functions if and only if the former utility distribution generalized Lorenz dominates the latter. In its inequality averse formulation, average generalized utilitarianism applies a common continuous, increasing, concave transform to each person’s utility before averaging across individuals to form the social objective function (see Blackorby, Bossert, and Donaldson, 2005, p. 171). Average utilitarianism is simply the special case in which this function is defined using the identity transform.

As Blackorby, Bossert, and Donaldson (2005, p. 143) have noted, average utilitarianism “makes some stark trade-offs: an alternative with a population of any size in which each person is equally well off is ranked as worse than an alternative in which a single person experiences a trivially higher utility level.” The same observation can also be made about any social welfare function that is defined in per capita terms, such as average generalized utilitarianism. As another example of these questionable trade-offs, consider a poor country that experiences a marginal decrease in utility per capita holding the distribution of utilities unchanged as measured by the Lorenz criterion. According to the generalized Lorenz criterion, there has been a loss in social welfare, and this is true even if the population has increased substantially.

Classical utilitarianism does not fare much better, as it suffers from what Parfit (1984) has called the repugnant conclusion. A social welfare ranking of utility distributions is subject to the repugnant conclusion if any distribution in which everyone’s utility is positive, no matter how large, is socially worse than some other distribution for a larger population in which everyone’s util-

ity is arbitrarily close to zero.⁵ Average utilitarianism avoids the repugnant conclusion because the addition of an individual to the population is welfare improving only if his or her utility exceeds the initial average utility level.

Critical-level generalized utilitarianism was introduced by Blackorby and Donaldson (1984) in order to overcome these problems with average and classical utilitarianism. What distinguishes a critical-level population principle is the existence of a utility level c such that adding a person with this utility to any utility distribution is a matter of social indifference. The objective function for a critical-level generalized utilitarian social welfare function is obtained from the objective function for a generalized utilitarian social welfare function by subjecting the critical level to the same transform that is applied to utilities and subtracting this amount from each person's transformed utility before summing across individuals. By choosing the critical level to be positive, the repugnant conclusion is avoided. In our example of a poor country with declining per capita utility, a critical-level generalized utilitarian may regard this change as being a social improvement if the proportion of the added population with utilities above the critical level is sufficiently large. For a detailed discussion of critical-level generalized utilitarianism and related population principles, see Blackorby, Bossert, and Donaldson (2005).

As noted above, we are interested in developing social welfare dominance criteria for comparing distributions of utilities using inequality averse critical-level generalized utilitarian social welfare functions. These welfare dominance criteria can be given different interpretations. In one interpretation, there is a social planner whose preferences (as expressed by the social welfare function) agrees with the ethical norms underlying critical-level generalized utilitarianism when this principle exhibits inequality aversion. As in traditional welfare dominance analysis, this planner wants to propose a social welfare ranking that has widespread support, and so he identifies the partial order that is obtained by taking the intersection of all of the inequality-averse critical-level generalized utilitarian orderings of the utility distributions for a given critical level. Alternatively, we can suppose that every individual agrees that utility distributions should be ranked by a critical-level generalized utilitarian social welfare function, but they disagree about which transform should be applied to the individual utilities before aggregating. By taking the intersection of

⁵In most formal models of population ethics (e.g., Blackorby, Bossert, and Donaldson, 2005) zero utility represents a neutral life, with negative utilities corresponding to lives that are considered to be not worth living.

all such rankings for a given critical level, we obtain the same dominance relation as in the social planner interpretation of the problem.

Our main objective is to establish an analogue of the Kakwani–Shorrocks Theorem for critical-level generalized utilitarianism. To do this, we introduce a new graphical representation of a distribution called a generalized concentration curve. For distributions of utilities, this curve plots the sum of the utilities of the t individuals with the smallest utilities against t . For a given critical level c , we define a dominance criterion based on these generalized concentration curves and show that this dominance criterion identifies the same partial order of utility distributions as does the critical-level generalized utilitarian dominance criterion described above. We also extend our results to allow for the possibility that the critical level lies within some range, rather than is known for certain. This extension is based on a generalization of critical-level generalized utilitarianism introduced by Blackorby, Bossert, and Donaldson (1996) called critical-band generalized utilitarianism.

In Section 2, we consider welfare dominance based on average generalized utilitarianism and formally state the Kakwani–Shorrocks Theorem. In Section 3, we present our welfare dominance results for critical-level generalized utilitarianism when the critical level is known. We extend these results in Section 4 to the case in which the critical level is only known to lie in some interval. In Section 5, we offer some concluding remarks.

2. Welfare Dominance for Average Generalized Utilitarianism

A utility distribution for a population of size $n \in \mathbb{N}$ is a vector $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n$, where u_i is the i th person’s utility, $i = 1, \dots, n$, and \mathbb{N} is the set of positive integers.⁶ The set of possible utility distributions is $\mathcal{U} = \cup_{n \in \mathbb{N}} \mathbb{R}^n$. While we are interpreting the variable whose distribution is of interest to be utility, it can also be interpreted as being any other scalar attribute of well-being, such as income or wealth, provided that the critical level is defined in terms of this attribute, not utility.⁷ For all $\mathbf{u} \in \mathcal{U}$, $n(\mathbf{u})$ denotes the size of

⁶We shall also have occasion to consider the set of nonnegative integers, \mathbb{N}^* .

⁷In some of these alternative interpretations of the model, it may be natural to require all distributions to be nonnegative. The formal results presented here also hold with this restriction. Such a restriction is appropriate when the attribute being considered is wage income, but not when it is income from self employment or wealth. For example, in their study of Israeli kibbutzim, Amiel, Cowell, and Polovin (1996) found that the incomes and wealth of some kibbutzim were negative for some years in their sample.

the population in \mathbf{u} and \mathbf{u}_\uparrow denotes the vector in which the components of \mathbf{u} have been rearranged in a nondecreasing order. A *social welfare function* is a mapping $W: \mathcal{U} \rightarrow \mathbb{R}$.

2.1. Generalized Lorenz Dominance

The *generalized Lorenz curve* for a utility distribution $\mathbf{u} \in \mathcal{U}$ is a function $GL_{\mathbf{u}}: [0, 1] \rightarrow \mathbb{R}$ defined as follows. For each $p \in [0, 1]$, let k_p be the smallest integer $k_p \in \{0, \dots, n(\mathbf{u}) - 1\}$ such that $\frac{k_p}{n(\mathbf{u})} \leq p \leq \frac{(k_p+1)}{n(\mathbf{u})}$ and let $\lambda_p \in [0, 1]$ be the unique number for which $p = (1 - \lambda_p)\frac{k_p}{n(\mathbf{u})} + \lambda_p\frac{(k_p+1)}{n(\mathbf{u})}$. Then, for all $p \in [0, 1]$,

$$GL_{\mathbf{u}}(p) = \frac{1}{n(\mathbf{u})} \left[\sum_{i=0}^{k_p} u_{\uparrow i} + \lambda_p [u_{\uparrow k_p+1} - u_{\uparrow k_p}] \right], \quad (2.1)$$

where $u_{\uparrow 0} = 0$.

The generalized Lorenz curve is well-defined and convex for all $\mathbf{u} \in \mathcal{U}$. In contrast, the Lorenz curve for \mathbf{u} is not defined if the mean utility is zero and it is concave if the mean is negative. If some utilities are negative, then initially a generalized Lorenz curve has a negative slope. For a perfectly equal distribution, the slope is constant and equal to the mean.⁸ Note that when $p = \frac{k}{n(\mathbf{u})}$ for some $k \in \{1, \dots, n(\mathbf{u})\}$ (in which case $k_p = k - 1$), the formula in (2.1) simplifies to

$$GL_{\mathbf{u}}(p) = \frac{1}{n(\mathbf{u})} \sum_{i=1}^k u_{\uparrow i}. \quad (2.2)$$

The *generalized Lorenz dominance* criterion is the partial order \succsim^{GL} on \mathcal{U} for which one utility distribution weakly dominates a second utility distribution if the generalized Lorenz curve for the first distribution lies nowhere below that of the second.⁹

Generalized Lorenz Dominance. For all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$,

$$\mathbf{u} \succsim^{\text{GL}} \mathbf{u}' \Leftrightarrow GL_{\mathbf{u}}(p) \geq GL_{\mathbf{u}'}(p) \text{ for all } p \in [0, 1]. \quad (2.3)$$

⁸See Amiel, Cowell, and Polovin (1996) and Jenkins and Jäntii (2005) for further discussion of the properties of a Lorenz curve when \mathbf{u} has one or more negative components.

⁹For any binary relation \succsim on \mathcal{U} , the corresponding asymmetric and symmetric factors are denoted by \succ and \sim , respectively.

If $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^n$, then (2.3) simplifies to:

$$\mathbf{u} \succsim^{\text{GL}} \mathbf{u}' \Leftrightarrow \sum_{i=1}^k u_{\uparrow i} \geq \sum_{i=1}^k u'_{\uparrow i} \text{ for all } k \in \{1, \dots, n\}. \quad (2.4)$$

2.2. Average Generalized Utilitarian Dominance

An average generalized utilitarian social welfare function is characterized by a *utility transform* $g: \mathbb{R} \rightarrow \mathbb{R}$. The transform g that is applied to the individual utilities permits the social value of utility to diverge from its individual value. We assume that $g \in \mathcal{C}$, the set of increasing concave functions for which $g(0) = 0$.¹⁰ The social welfare function for *average generalized utilitarianism* with utility transform g is given by

$$W_g^{\text{A}}(\mathbf{u}) = \frac{1}{n(\mathbf{u})} \sum_{i=1}^{n(\mathbf{u})} g(u_i), \quad \forall \mathbf{u} \in \mathcal{U}. \quad (2.5)$$

By requiring g to be concave, the social welfare function is weakly inequality averse. If g is the identity mapping, then W_g^{A} is the social welfare function for *average utilitarianism*.

Consider any pair of distributions $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$. The change in social welfare ΔW_g^{A} that results from a change in the distribution from \mathbf{u}' to \mathbf{u} is

$$\Delta W_g^{\text{A}}(\mathbf{u}, \mathbf{u}') = W_g^{\text{A}}(\mathbf{u}) - W_g^{\text{A}}(\mathbf{u}'). \quad (2.6)$$

The *average generalized utilitarian dominance* partial order \succsim^{A} on \mathcal{U} is defined by taking the intersection of the orderings of the utility distributions in \mathcal{U} for all average generalized utilitarian social welfare functions. That is, it identifies the ordered pairs of utility distributions for which the change in social welfare ΔW_g^{A} is nonnegative no matter how inequality averse the social welfare function is as measured by the utility transform g .

Average Generalized Utilitarian Dominance. For all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$,

$$\mathbf{u} \succsim^{\text{A}} \mathbf{u}' \Leftrightarrow \Delta W_g^{\text{A}}(\mathbf{u}, \mathbf{u}') \geq 0 \text{ for all } g \in \mathcal{C}. \quad (2.7)$$

¹⁰As we shall see, the assumption that $g(0) = 0$ is a harmless normalization. If utilities are restricted to be nonnegative and, thus, g is only defined on \mathbb{R}_+ , in order to ensure that g is continuous, it is also necessary to assume that g is continuous at the origin.

2.3. The Kakwani–Shorrocks Theorem

The Kakwani (1984)–Shorrocks (1983) Theorem shows that one distribution of utilities generalized Lorenz dominates a second if and only if the former average generalized utilitarian dominates the latter.¹¹

Proposition 1. *For all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$, $\mathbf{u} \succ^{\text{GL}} \mathbf{u}' \Leftrightarrow \mathbf{u} \succ^{\text{A}} \mathbf{u}'$.*

Proposition 1 is a variable population extension of the fixed population version of this theorem due to Tomić (1949).¹² It is instructive to see how it is possible to use Tomić’s Theorem to establish Proposition 1. Consider any $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{u}' \in \mathbb{R}^{n'}$. Let $\hat{\mathbf{u}}$ be the utility distribution obtained by replicating \mathbf{u} n' times. Similarly, $\hat{\mathbf{u}}'$ is obtained by replicating \mathbf{u}' n times. By construction, $\mathbf{u} \sim^{\text{GL}} \hat{\mathbf{u}}$ and $\mathbf{u}' \sim^{\text{GL}} \hat{\mathbf{u}}'$. Therefore, $\mathbf{u} \succ^{\text{GL}} \mathbf{u}' \Leftrightarrow \hat{\mathbf{u}} \succ^{\text{GL}} \hat{\mathbf{u}}'$. For any $g \in \mathcal{C}$, $W_g^{\text{A}}(\mathbf{u}) = W_g^{\text{A}}(\hat{\mathbf{u}})$ and $W_g^{\text{A}}(\mathbf{u}') = W_g^{\text{A}}(\hat{\mathbf{u}}')$. Hence, $\mathbf{u} \succ^{\text{A}} \mathbf{u}' \Leftrightarrow \hat{\mathbf{u}} \succ^{\text{A}} \hat{\mathbf{u}}'$. Because $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}'$ are both in $\mathbb{R}^{nn'}$, Tomić’s Theorem implies that $\hat{\mathbf{u}} \succ^{\text{GL}} \hat{\mathbf{u}}' \Leftrightarrow \hat{\mathbf{u}} \succ^{\text{A}} \hat{\mathbf{u}}'$. Proposition 1 then follows from these equivalences. Thus, as in Dasgupta, Sen, and Starrett (1973) and Shorrocks (1983), replications of two utility distributions are used in order to reduce any variable population comparison to one in which the population size is fixed.

3. Welfare Dominance for Critical-Level Generalized Utilitarianism

3.1. Critical-Level Generalized Utilitarian Dominance

A critical-level generalized utilitarian social welfare function is characterized by a *critical level* $c \in \mathbb{R}$ and a utility transform $g \in \mathcal{C}$. For any utility distribution $\mathbf{u} \in \mathcal{U}$, adding an individual to the population with utility level

¹¹See Kakwani (1984, Theorem 1). This theorem is not stated explicitly in Shorrocks (1983), but it follows from the Corollary to his Theorem 2 and his remarks in footnote 7 and on page 8. Neither Kakwani nor Shorrocks assume that $g(0) = 0$. However, if g does not satisfy this property, it can be made to do so by replacing $g(t)$ with $\bar{g}(t) = g(t) - g(0)$ for all $t \in \mathbb{R}$ without affecting the sum in (2.5).

¹²See Marshall and Olkin (1979, Theorem 4.B.2). Versions of Tomić’s Theorem have also been established by Kolm (1969) and Rothschild and Stiglitz (1973). Kakwani and Shorrocks appear to be the first to have applied the generalized Lorenz dominance criterion to populations of different size.

c is a matter of social indifference. The social welfare function for *critical-level generalized utilitarianism* with critical level c and utility transform g is given by

$$W_{c,g}(\mathbf{u}) = \sum_{i=1}^{n(\mathbf{u})} [g(u_i) - g(c)], \quad \forall \mathbf{u} \in \mathcal{U}.^{13} \quad (3.1)$$

If g is the identity mapping, then $W_{c,g}$ is the social welfare function for *critical-level utilitarianism* with critical level c . If, furthermore, $c = 0$, we then have *classical (total) utilitarianism*. Analogous to (2.6), for all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$, let

$$\Delta W_{c,g}(\mathbf{u}, \mathbf{u}') = W_{c,g}(\mathbf{u}) - W_{c,g}(\mathbf{u}'). \quad (3.2)$$

The *critical-level c generalized utilitarian dominance* partial order \succsim_c^{CL} on \mathcal{U} is defined by taking the intersection of the orderings of the utility distributions in \mathcal{U} for all critical-level generalized utilitarian social welfare functions when the critical level is fixed at c . Analogous to the construction of the average generalized utilitarian dominance relation, it identifies the ordered pairs of utility distributions for which the change in social welfare $\Delta W_{c,g}$ is nonnegative regardless of the degree of inequality aversion exhibited by the social welfare function, i.e., regardless of the degree of concavity of the transform g .

Critical-Level c Generalized Utilitarian Dominance. For all $c \in \mathbb{R}$, for all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$,

$$\mathbf{u} \succsim_c^{\text{CL}} \mathbf{u}' \Leftrightarrow \Delta W_{c,g}(\mathbf{u}, \mathbf{u}') \geq 0 \text{ for all } g \in \mathcal{C}. \quad (3.3)$$

For comparisons of two utility distributions with the same sized population, average generalized utilitarian and critical-level c generalized utilitarian dominance are equivalent criteria because the terms involving the critical level cancel in the welfare difference in (3.2) and the sign of the welfare difference in (2.6) is unaffected if average utility is replaced by the sum of utilities. However, this equivalence no longer holds if the two distributions are for populations of different size.

¹³For an axiomatization of this population principle, see Blackorby, Bossert, and Donaldson (1998, Theorem 2).

3.2. Critical-Level Generalized Concentration Curve Dominance

Generalized Lorenz dominance employs replications of distributions in order to reduce a variable population comparison to a fixed population equivalent. However, replicating a utility distribution is not a matter of social indifference for any critical-level generalized utilitarian rule. Nevertheless, adding individuals with utility equal to the critical level is. This observation provides a basis for identifying a new dominance criterion that can be used to establish an analogue to Proposition 1 for critical-level c generalized utilitarian dominance. Furthermore, this criterion coincides with the generalized Lorenz dominance partial ordering for fixed population comparisons.

For any utility distribution, a generalized concentration curve plots the sum of the utilities of the t individuals with the smallest utilities against t , with linear interpolation used so that the curve is defined for non-integer values of t . Formally, the *generalized concentration curve* for $\mathbf{u} \in \mathcal{U}$ is the function $GC_{\mathbf{u}}: [0, n(\mathbf{u})] \rightarrow \mathbb{R}$ defined as follows. For each $t \in [0, n(\mathbf{u})]$, let k_t be the smallest integer $k_t \in \{0, \dots, n(\mathbf{u}) - 1\}$ such that $k_t \leq t \leq k_t + 1$ and let $\lambda_t \in [0, 1]$ be the unique number for which $t = (1 - \lambda_t)k_t + \lambda_t(k_t + 1)$. Then, for all $t \in [0, n(\mathbf{u})]$,

$$GC_{\mathbf{u}}(t) = \sum_{i=0}^{k_t} u_{\uparrow i} + \lambda_t [u_{\uparrow k_t+1} - u_{\uparrow k_t}]. \quad (3.4)$$

The generalized concentration curve is well-defined and convex for all $\mathbf{u} \in \mathcal{U}$. If some utilities are negative, then initially this curve has a negative slope. For a perfectly equal distribution, the slope is constant and equal to the common utility value. When $t \in \{1, \dots, n(\mathbf{u})\}$, the formula in (3.4) simplifies to

$$GC_{\mathbf{u}}(t) = \sum_{i=1}^t u_{\uparrow i}. \quad (3.5)$$

The construction of a generalized concentration curve is illustrated in Figure 1. The solid lines in this diagram show the generalized concentration curves for the distributions $\mathbf{u}^1 = (5, 5, 10, 15, 25)$ and $\mathbf{u}^2 = (-20, 0, 20, 40)$.

Had we plotted the fraction of the total utility for the population as a whole that is obtained by the t individuals with the smallest utilities against t , the resulting curve would be formally equivalent to what is known in the literature on the measurement of industrial concentration as a concentration

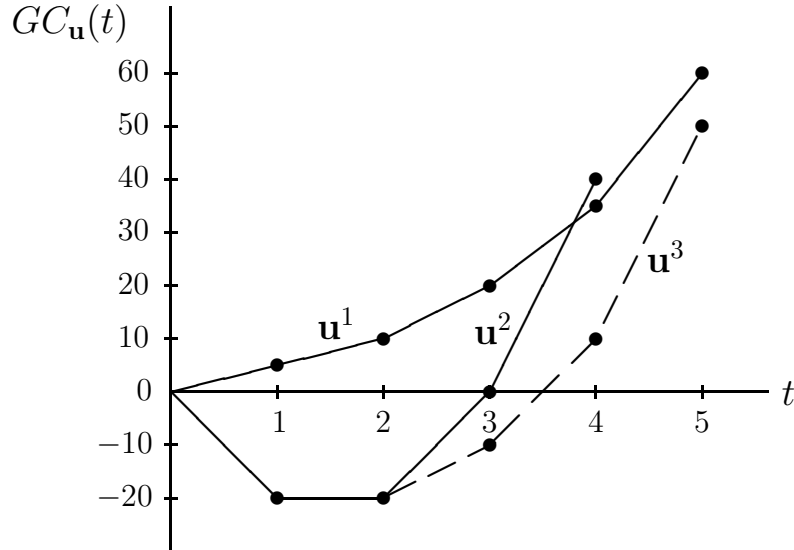


Figure 1: Generalized Concentration Curves

curve. See, for example, Blackorby, Donaldson, and Weymark (1982).¹⁴ In this application, firms correspond to individuals and output or sales correspond to incomes. Thus, a generalized concentration curve is obtained from a concentration curve by multiplying by the mean of the variable being considered, just as a generalized Lorenz curve is obtained from a Lorenz curve by multiplying by the mean, at least when the mean is not zero.

For utility distributions $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^n$, the generalized concentration curve for \mathbf{u} lies nowhere below that of \mathbf{u}' if and only if the generalized Lorenz curve for \mathbf{u} lies nowhere below that of \mathbf{u}' . However, when the number of individuals differ in two utility distributions, whether their generalized concentration curves cross or not is of no significance because they have different domains of definition. We are interested in defining a dominance criterion for generalized concentration curves that can be applied to both fixed and variable population comparisons when the critical level of utility is a given fixed value c . This is accomplished by regarding generalized concentration curves for two utility distributions with different population sizes as being

¹⁴Strictly speaking, the concentration curves used to measure industrial concentration plot the sum of the t largest utilities against t and, hence, are concave functions.

in the same equivalence class of the dominance relation if the only difference in the two distributions is that one of them has additional individuals with the critical utility level.¹⁵ In order to compare two generalized concentration curves for different populations, we augment the utility distribution for the smaller population with a sufficient number of individuals with the critical level of utility until the population sizes in the two distributions are the same.

For any $c \in \mathbb{R}$, the *critical-level c generalized concentration curve dominance* criterion is the partial order \succ_c^{GC} on \mathcal{U} defined as follows. One utility distribution weakly dominates a second utility distribution according to this criterion if after augmenting the distribution for the smaller population (if necessary) as described above, the generalized concentration curve for the first (possibly augmented) distribution lies nowhere below that of the (possibly augmented) second distribution. To define this dominance criterion formally, we need to introduce some further definitions.

For all $\mathbf{u} \in \mathcal{U}$, $c \in \mathbb{R}$, and $\bar{n} \in \mathbb{N}^*$, the *augmented utility distribution* $\mathbf{u}_{c,\bar{n}}$ is defined by setting $\mathbf{u}_{c,\bar{n}} = (\mathbf{u}, c\mathbf{1}_{\bar{n}})$, where $\mathbf{1}_{\bar{n}}$ is the vector of ones in $\mathbb{R}^{\bar{n}}$. Note that $\mathbf{u} = \mathbf{u}_{c,\bar{n}}$ when $\bar{n} = 0$. For all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$, let $\bar{n}(\mathbf{u}, \mathbf{u}') = 0$ if $n(\mathbf{u}) \geq n(\mathbf{u}')$ and $\bar{n}(\mathbf{u}, \mathbf{u}') = n(\mathbf{u}') - n(\mathbf{u})$ otherwise.

Critical-Level c Generalized Concentration Curve Dominance. For all $c \in \mathbb{R}$, for all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$,

$$\mathbf{u} \succ_c^{\text{GC}} \mathbf{u}' \Leftrightarrow GC_{\mathbf{u}_{c,\bar{n}(\mathbf{u},\mathbf{u}')}}(t) \geq GC_{\mathbf{u}'_{c,\bar{n}(\mathbf{u}',\mathbf{u})}}(t) \text{ for all } t \in [0, \max\{n(\mathbf{u}), n(\mathbf{u}')\}]. \quad (3.6)$$

This definition can be illustrated using the utility distributions in Figure 1. To compare \mathbf{u}^1 and \mathbf{u}^2 , it is necessary to augment \mathbf{u}^2 by adding a single individual with the critical utility level c . For concreteness, let $c = 10$. The resulting distribution is $\mathbf{u}^3 = \mathbf{u}_{10,1}^2 = (-20, 0, 20, 40, 10)$. The corresponding concentration curve coincides with that of \mathbf{u}^2 for $t \leq 2$ and then shifts to the right for higher values of t , as indicated by the dashed line in the diagram. We thus have $\mathbf{u}^1 \succ_{10}^{\text{GC}} \mathbf{u}^2$ even though the generalized concentration curves for \mathbf{u}^1 and \mathbf{u}^2 intersect. Note, however, that if the critical level exceeds 20, then \mathbf{u}^1 and \mathbf{u}^2 are not comparable using this dominance criterion.

¹⁵In their analysis of the measurement of industrial concentration, Blackorby, Donaldson, and Weymark (1982) introduced the corresponding property for concentration curves for the special case in which $c = 0$. They called their property “zero output independence.” They did not consider other values of c .

3.3. An Equivalence Theorem

We now show that for any value of the critical level c , the partial order of the utility distributions in \mathcal{U} obtained using critical-level c generalized concentration curve dominance is equivalent to that obtained using the critical-level generalized utilitarian dominance criterion for the same value of the critical level.

Proposition 2. *For any $c \in \mathbb{R}$, for all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$, $\mathbf{u} \succ_c^{\text{GC}} \mathbf{u}' \Leftrightarrow \mathbf{u} \succ_c^{\text{CL}} \mathbf{u}'$.*

Proof. Consider any $c \in \mathbb{R}$ and $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$. From (3.6), we have that $\mathbf{u} \succ_c^{\text{GC}} \mathbf{u}' \Leftrightarrow \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{GC}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})}$. For a fixed population, it follows from their definitions that generalized Lorenz dominance coincides with critical-level c generalized concentration curve dominance for any value of $c \in \mathbb{R}$. Thus, $\mathbf{u} \succ_c^{\text{GC}} \mathbf{u}' \Leftrightarrow \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{GL}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})}$. To complete the proof, we show that the latter relation holds if and only if $\mathbf{u} \succ_c^{\text{GC}} \mathbf{u}'$.

For any critical-level generalized utilitarian rule, adding an individual with utility equal to the critical level is a matter of social indifference. Hence, $\mathbf{u} \sim_c^{\text{CL}} \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \mathbf{u}'$ and $\mathbf{u}' \sim_c^{\text{CL}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})}$. The transitivity of \succ_c^{CL} then implies that $\mathbf{u} \succ_c^{\text{CL}} \mathbf{u}' \Leftrightarrow \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{CL}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})}$. But for fixed population comparisons, for any $g \in \mathcal{C}$, an average generalized utilitarian rule with utility transform g ranks utility distributions in exactly the same way as a critical-level generalized utilitarian rule for the same utility transform regardless of the value of the critical level. Therefore, $\mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{CL}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})} \Leftrightarrow \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{A}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})}$. By Proposition 1, $\mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{A}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})} \Leftrightarrow \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{GL}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})}$. It then follows from these equivalences that $\mathbf{u} \succ_c^{\text{CL}} \mathbf{u}' \Leftrightarrow \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \succ_c^{\text{GL}} \mathbf{u}'_{c, \bar{n}(\mathbf{u}', \mathbf{u})}$. \square

As we have observed, for any value of the critical level c , $\mathbf{u} \succ_c^{\text{CL}} \mathbf{u}' \Leftrightarrow \mathbf{u} \succ_c^{\text{GL}} \mathbf{u}'$ whenever $n(\mathbf{u}) = n(\mathbf{u}')$. However, when $n(\mathbf{u}) \neq n(\mathbf{u}')$, $\mathbf{u} \succ_c^{\text{CL}} \mathbf{u}'$ is neither a necessary or sufficient condition for $\mathbf{u} \succ_c^{\text{GL}} \mathbf{u}'$. For example, suppose that $n(\mathbf{u}') > n(\mathbf{u})$ and that $\mathbf{u} \sim_c^{\text{CL}} \mathbf{u}'$. By Proposition 2, we therefore have $\mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')} \sim_c^{\text{GL}} \mathbf{u}'$. However, if $u_i > c$ for all $i \in \{1, \dots, n(\mathbf{u})\}$, then by Proposition 1, we have $\mathbf{u} \succ_c^{\text{GL}} \mathbf{u}_{c, \bar{n}(\mathbf{u}, \mathbf{u}')}$, from which it follows that $\mathbf{u} \succ_c^{\text{GL}} \mathbf{u}'$. A similar argument shows that $\mathbf{u}' \succ_c^{\text{GL}} \mathbf{u}$ if $u_i < c$ for all $i \in \{1, \dots, n(\mathbf{u})\}$.

Blackorby, Bossert, and Donaldson (2003, p. 375) have introduced a generalization of critical-level generalized utilitarianism called *number-sensitive critical-level generalized utilitarianism*. (See also Blackorby, Bossert, and

Donaldson, 2005, p. 168.) With this principle, the critical level is permitted to depend on the population size, but not on the individual utilities. It is straightforward to extend Proposition 2 so that it applies to number-sensitive critical-level generalized utilitarianism. Note that the augmentation procedure used above to convert two distributions into distributions for the same sized population only involves augmenting the distribution for the smaller population by adding an appropriate number of individuals with the fixed critical level. If, however, the critical level is permitted to depend on population size, then it is these number-sensitive critical levels that are used when adding individuals. For example, if the smaller population contains 10 people and the larger contains 12, then two people are added to the smaller population distribution, one person with the critical level for a population of size 10 and one person with the critical level for a population of size 11.

4. Critical-Band Generalized Utilitarian Dominance

Blackorby, Bossert, and Donaldson (1996, 2005) have considered a generalization of critical-level c generalized utilitarian dominance that allows for there to be a range of critical levels. Consider any $\underline{c}, \bar{c} \in \mathbb{R}$ with $\underline{c} < \bar{c}$. The interval $[\underline{c}, \bar{c}]$ is interpreted as being the smallest *band* in which it is known that the critical level lies. *Critical-band $[\underline{c}, \bar{c}]$ generalized utilitarian dominance* is the partial order $\succsim_{[\underline{c}, \bar{c}]}^{\text{CB}}$ obtained by taking the intersection of the critical-level generalized utilitarian partial orders for all $c \in [\underline{c}, \bar{c}]$. In other words, the critical-level generalized dominance criteria must agree for all values of the critical level in this interval in order for this criterion to rank utility distributions.

Critical-Band $[\underline{c}, \bar{c}]$ Generalized Utilitarian Dominance. For all $\underline{c}, \bar{c} \in \mathbb{R}$ with $\underline{c} < \bar{c}$, for all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$,

$$\mathbf{u} \succsim_{[\underline{c}, \bar{c}]}^{\text{CB}} \mathbf{u}' \Leftrightarrow \mathbf{u} \succsim_c^{\text{CL}} \mathbf{u}' \text{ for all } c \in [\underline{c}, \bar{c}].^{16} \quad (4.1)$$

In Proposition 3, we show that this partial order is equivalent to the partial order that is obtained by taking the intersection of the critical-level c

¹⁶In their definition of critical-band generalized utilitarian dominance, Blackorby, Bossert, and Donaldson (1996, 2005) replace the closed interval $[\underline{c}, \bar{c}]$ with an arbitrary bounded interval. An axiomatization of their version of this population principle may be found in Blackorby, Bossert, and Donaldson (2005, Theorem 7.12).

generalized utilitarian partial orders for the critical levels \underline{c} and \bar{c} that define the endpoints of the band. It then follows from Proposition 2 that this partial order is also the intersection of the partial orders defined by critical-level c generalized concentration curve dominance for these two values of c .

Proposition 3. *For all $\underline{c}, \bar{c} \in \mathbb{R}$ with $\underline{c} < \bar{c}$, for all $\mathbf{u}, \mathbf{u}' \in \mathcal{U}$, the following conditions are equivalent:*

- (i) $\mathbf{u} \succ_{[\underline{c}, \bar{c}]}^{\text{CB}} \mathbf{u}'$,
- (ii) $[\mathbf{u} \succ_{\underline{c}}^{\text{CL}} \mathbf{u}' \text{ and } \mathbf{u} \succ_{\bar{c}}^{\text{CL}} \mathbf{u}']$, and
- (iii) $[\mathbf{u} \succ_{\underline{c}}^{\text{GC}} \mathbf{u}' \text{ and } \mathbf{u} \succ_{\bar{c}}^{\text{GC}} \mathbf{u}']$.

Proof. It follows trivially from the definition of $\succ_{[\underline{c}, \bar{c}]}^{\text{CB}}$ that (i) implies (ii).

We now show that (ii) implies (i). Suppose that (ii) holds. Consider any $c \in [\underline{c}, \bar{c}]$, $g \in \mathcal{C}$, $n, m \in \mathbb{N}$, $\mathbf{u} \in \mathbb{R}^n$, and $\mathbf{u}' \in \mathbb{R}^m$. There are two cases to consider.

Case 1. Suppose that $n \geq m$. By assumption,

$$\sum_{i=1}^n [g(u_i) - g(\bar{c})] \geq \sum_{i=1}^m [g(u'_i) - g(\bar{c})]. \quad (4.2)$$

Equivalently,

$$\sum_{i=1}^n g(u_i) \geq \sum_{i=1}^m g(u'_i) + (n - m)g(\bar{c}). \quad (4.3)$$

Because $n - m \geq 0$, $c \leq \bar{c}$, and the function g is increasing, (4.3) implies that

$$\sum_{i=1}^n g(u_i) \geq \sum_{i=1}^m g(u'_i) + (n - m)g(c), \quad (4.4)$$

or, equivalently,

$$\sum_{i=1}^n [g(u_i) - g(c)] \geq \sum_{i=1}^m [g(u'_i) - g(c)]. \quad (4.5)$$

Case 2. Suppose that $m > n$. By assumption,

$$\sum_{i=1}^n [g(u_i) - g(\underline{c})] \geq \sum_{i=1}^m [g(u'_i) - g(\underline{c})]. \quad (4.6)$$

Equivalently,

$$\sum_{i=1}^n g(u_i) + (m - n)g(\underline{c}) \geq \sum_{i=1}^m g(u'_i). \quad (4.7)$$

Because $m - n \geq 0$, $c \geq \underline{c}$, and the function g is increasing, (4.7) implies that

$$\sum_{i=1}^n g(u_i) + (m - n)g(c) \geq \sum_{i=1}^m g(u'_i), \quad (4.8)$$

which is equivalent to (4.5).

Thus, (4.5) holds in both cases. Because c is an arbitrary element of $[\underline{c}, \bar{c}]$, we have therefore shown that $\mathbf{u} \succsim_c^{\text{CL}} \mathbf{u}'$ for all $c \in [\underline{c}, \bar{c}]$. That is, (i) holds.

The equivalence of (iii) with both (i) and (ii) now follows immediately from Proposition 2. \square

5. Concluding Remarks

Social welfare dominance criteria provide a way of partially ordering distributions based on widely shared value judgements. In practice, the dominance criterion that is most commonly employed is the generalized Lorenz partial order. Implicitly, this dominance criterion measures social welfare in per capita terms. More precisely, as the Kakwani–Shorrocks Theorem establishes, it coincides with the averaged generalized utilitarian dominance criterion. However, as we have noted, per capita measures of social welfare make some trade-offs that many would find unpalatable when the size of the population is subject to variation. Critical-level generalized utilitarianism was introduced as a way of overcoming these concerns. The critical-level generalized utilitarian welfare dominance criterion introduced here measures differences in social welfare in aggregate, not per capita, terms.

We have also introduced a new geometric construction, the generalized concentration curve for a distribution, that is a natural analogue for critical-level generalized utilitarianism of a generalized Lorenz curve. For a given value of the critical level c , we have used generalized concentration curves to define a new dominance relation, the critical-level c generalized utilitarian partial order, and shown that it coincides with the critical-level generalized utilitarian partial order for this value of the critical level, thereby providing an analogue of the Kakwani–Shorrocks Theorem for this population principle. Furthermore, we have used critical-band generalized utilitarianism to extend this result so as to allow for a range of critical levels.

We have framed our discussion in terms of distributions of utilities. However, in empirical applications of our dominance criteria, for practical reasons, income is most likely to be chosen as the measure of individual well-being. Implementation of our proposal will then require the specification of a critical income level or critical income band. In this interpretation of our model, the critical level is the level of income for which it is a matter of social indifference to add an additional individual with this amount of income. For most societies, this level will be below the observed average income of the population. It is also likely to be below what is regarded as an appropriate value for an absolute poverty line. Given the lack of an obvious choice for the critical income level, the use of a critical band is an attractive option, as then any distributional comparisons that can be made using our approach to constructing a dominance partial order will not be overly sensitive to the exact specification of the critical level. In any event, the choice of the critical level calls for further investigation, which is beyond the scope of this article.

Sen (1973, p. 76) has argued that “[t]reating inequality as a quasi-ordering [i.e. as a partial ordering] has much to be commended from the normative as well as the descriptive point of view.” The same can be said for social welfare comparisons. The critical-level and critical-band generalized utilitarian welfare dominance criteria introduced here provide alternatives to generalized Lorenz dominance. They are alternatives that we think have much “to be commended.”

References

- Amiel, Y., Cowell, F., Polovin, A., 1996. Inequality among the kibbutzim. *Economica* 63, S63–S85.
- Atkinson, A. B., 1970. On the measurement of inequality. *Journal of Economic Theory* 2, 244–266.
- Blackorby, C., Bossert, W., Donaldson, D., 1996. Quasi-orderings and population ethics. *Social Choice and Welfare* 13, 129–150.
- Blackorby, C., Bossert, W., Donaldson, D., 1998. Uncertainty and critical-level population principles. *Journal of Population Economics* 11, 1–20.
- Blackorby, C., Bossert, W., Donaldson, D., 2003. The axiomatic approach to population ethics. *Politics, Philosophy & Economics* 2, 342–381.
- Blackorby, C., Bossert, W., Donaldson, D., 2005. *Population Issues in Social Choice Theory, Welfare Economics, and Ethics*. Cambridge University Press, Cambridge.

- Blackorby, C., Donaldson, D., 1984. Social criteria for evaluating population change. *Journal of Public Economics* 25, 13–33.
- Blackorby, C., Donaldson, D., Weymark, J. A., 1982. A normative approach to industrial-performance evaluation and concentration indices. *European Economic Review* 19, 89–121.
- Broome, J., 2004. *Weighing Lives*. Oxford University Press, Oxford.
- Dalton, H., 1920. The measurement of the inequality of incomes. *Economic Journal* 30, 348–361.
- Dasgupta, P., 1969. On the concept of optimum population. *Review of Economic Studies* 36, 295–318.
- Dasgupta, P., Sen, A., Starrett, D., 1973. Notes on the measurement of inequality. *Journal of Economic Theory* 6, 180–187.
- Jenkins, S. P., Jäntii, M., 2005. Methods for summarizing and comparing wealth distributions. ISER Working Paper No. 2005-05, Institute for Social and Economic Research, University of Essex.
- Kakwani, N., 1984. Welfare rankings of income distributions. In: Basmann, R. L., Rhodes, Jr., G. F. (Eds.), *Advances in Econometrics, Volume 3. Economic Inequality: Measurement and Policy*. JAI Press, Greenwich, Conn., pp. 191–213.
- Kolm, S.-C., 1969. The optimal production of social justice. In: Margolis, J., Guitton, H. (Eds.), *Public Economics: An Analysis of Public Production and Consumption and Their Relations to the Private Sector*. Macmillan, London, pp. 145–200.
- Lambert, P. J., 2001. *The Distribution and Redistribution of Income*, 3rd Edition. Manchester University Press, Manchester.
- Marshall, A. W., Olkin, I., 1979. *Inequalities: Theories of Majorization and Its Applications*. Academic Press, New York.
- Parfit, D., 1984. *Reasons and Persons*. Oxford University Press, Oxford.
- Rothschild, M., Stiglitz, J. E., 1973. Some further results on the measurement of inequality. *Journal of Economic Theory* 6, 188–204.
- Sen, A., 1973. *On Economic Inequality*. Clarendon Press, Oxford.
- Shorrocks, A. F., 1983. Ranking income distributions. *Economica* 50, 3–17.
- Tomić, M., 1949. Gauss' theorem concerning the center of gravity and its application. *Vesnik Društva Mathematicara i Fizicara Narodne Republike Srbije* 1, 31–40, in Serbo-Croatian.