Axionlike Dark Energy and Particle Decay in the Future of the Accelerating Universe

Written by

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Abstract

The 1998 discovery that the universe was accelerating in its expansion has yet to be explained theoretically, meriting the continual theoretical and observational study of this phenomena. In this thesis, we undergo a phenomenological study of the cosmological implications of this "dark energy" in two different ways.

In the first part of this thesis, we examine the cosmological evolution of ultralight axionlike (ULA) scalar fields with potentials of the form $V(\phi) = m^2 f^2 [1 - \cos(\phi/f)]^2$, with particular emphasis on the deviation in their behavior from the corresponding small-ϕ power-law approximations to these potentials: $V(\phi) \propto \phi^{2n}$. We show that in the slow-roll regime, when $\dot{\phi}^2/2 \ll V(\phi)$, the full ULA potentials yield a more interesting range of possibilities for quintessence than do the corresponding power law approximations. For rapidly oscillating scalar fields, we derive the equation of state parameter and oscillation frequency for the ULA potentials and show how they deviate from the corresponding power-law values. We derive an analytic expression for the equation of state parameter that better approximates the ULA value than does the pure power-law approximation.

In the second part, we study particle decay in the future of the accelerating universe. We generalize the result that in a cosmological constant dominated universe, the decay of matter into relativistic particles can never cause radiation to once again dominate over matter. We study both models of dark energy comprised of quintessence and cosmologies ending in a "big rip" in this context.

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Contents

Chapter 1

Introduction & Background

Dark energy is among the greatest mysteries of our time; although the accelerating expansion of the universe was confirmed through observation, the theoretical cause of this acceleration, dubbed 'dark energy,' eludes us. With observations suggesting nearly 70% of the energy density of the universe is in the form of this exotic negative pressure component, understanding its nature is an imperative step to understanding the nature of our universe. Additional motivation for the study of dark energy comes from the Hubble tension, the discrepancy between direct local measurements of the Hubble parameter and the value inferred within the ΛCDM model from measurements of the cosmic microwave background (CMB). It has recently been suggested that extra energy at the time of recombination, a so-called 'early dark energy' could resolve this discrepancy.

In the present chapter, we give background information on the theoretical cosmology necessary to carry out the calculations we wish to in the following chapters. In the second chapter, we undertake a study of axionlike dark energy. In the third chapter, we study particle decay in the future of the accelerating universe. The appendix contains derivatios.

1.1 Friedman-Robertson-Walker Cosmology

The most widely accepted standard cosmology is based on the *cosmological principle*, which tell us that the universe should look the same at all points and in all directions. The solution to Einstein's equation that corresponds to this principle is the Friedman-Robertson-Walker (FRW) metric, which assumes an isotropic and homogeneous universe. The FRW metric is given by [1]

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.1}$$

where $a(t)$ is the scale factor and r, θ and ϕ are comoving coordinates (coordinates in which an object moving freely with the expansion of the universe is at rest), and K defines the spatial curvature of the universe. We define $a(t)$ by

$$a(t) = \frac{d(t)}{d_0}, \tag{1.2}$$

where $d(t)$ is the proper distance at time t, and d_0 is the proper distance at a reference time t_0. t_0 is commonly set to be the present age of the universe, meaning that $a(t) = 1$ in present day.

We can also write the FRW metric as

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + f_K^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2] \tag{1.3}$$

$$f_K(\chi) = \begin{cases} \sin\chi, & K = +1 \\ \chi, & K = 0 \\ \sinh\chi & K = -1. \end{cases} \tag{1.4}$$

Then, we can introduce the Hubble constant, which characterizes the expansion rate of the universe, the observable size of the universe, and the expansion age of the universe,

$$H \equiv \frac{\dot{a}(t)}{a(t)}. \tag{1.5}$$

The scale factor can also be related to the redshift and frequency by the relation

$$a(t) = \frac{1}{1+z} = \frac{\lambda}{\lambda_0}. \tag{1.6}$$

We often model the matter and energy in the universe by a perfect fluid, and so we must obtain their evolution equations using Einstein's equation. Considering a perfect fluid to be the source term for Einstein's equations, we arrive at the following two equations

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{K}{a^2} \tag{1.7}$$

$$\dot{H} = -4\pi G(p+\rho) + \frac{K}{a^2}. \tag{1.8}$$

From these two equations, we can derive the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{1.9}$$

as well as

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3p), \tag{1.10}$$

where p is pressure and ρ is density.

We model the universe to be comprised of perfect fluids, with equations of state

$$w = p/\rho. \tag{1.11}$$

The fluids that comprise our universe are matter, radiation, and the mysterious "dark energy." We will assume (based off of observation) that the universe is nearly flat, that $K = 0$. Thus, the last term in equations (1.7) and (1.8) can be dropped.

We use Einstein's equations (1.7) and (1.8)to derive the evolution of the fluids in our universe. Taking $K = 0$ we get the following equation for the Hubble parameter

$$H = \frac{2/3}{(1+w)(t-t_0)} \tag{1.12}$$

Then, using $\dot{a}/a$, we can solve the differential equation for $a(t)$.

$$\frac{1}{a}da = \frac{2/3}{(1+w)(t-t_0)}dt \tag{1.13}$$

Integrating, we get

$$\ln(a) = \frac{2/3}{1+w}\ln(t-t_0) + C \tag{1.14}$$

If we wish to drop the constant, we can simply say,

$$a(t) \propto (t-t_0)^{\frac{2}{3(1+w)}} \tag{1.15}$$

Then, we can calculate the density by plugging in our expression for $a(t)$ and $\dot{a}(t)$ into equation (1.7). The time derivative of $a(t)$ is given by

$$\dot{a} = \frac{(2/3)a}{(1-w)(t-t_0)} \tag{1.16}$$

Writing t in terms of a,

$$(t-t_0) \propto a^{3(1+w)/2} \tag{1.17}$$

Then,

$$\dot{a} \propto \frac{a}{a^{3(1+w)/2}} \tag{1.18}$$

Using 1.7, we get

$$\rho \propto \left(\frac{1}{a^{3(1+w)/2}}\right) \tag{1.19}$$

Which means that

$$\rho \propto a^{-3(1+w)} \tag{1.20}$$

This solution is only valid when $w \neq -1$.

We can also express the flatness of the universe in terms of the sum of the density of each of the fluids, matter, radiation, and dark energy (which is also called vacuum energy). First, it is useful to define the critical density, ρ_c,

$$\rho_c = \frac{3H^2}{8\pi G}. \tag{1.21}$$

The critical density is the density required for a spatially flat universe. Then we can introduce the Ω_i parameters,

$$\Omega_i = \frac{\rho_i}{\rho_c}, \tag{1.22}$$

where i can represent the matter, radiation or vacuum energy (dark energy). In a flat universe,

$$\Omega_M + \Omega_R + \Omega_\Lambda = 1, \tag{1.23}$$

where Ω_M is the energy density of matter, Ω_R is the energy density of radiation, and Ω_Λ is the energy density of the vacuum energy.

1.2 Observational Evidence

In 1998, observations of Type Ia Supernovae revolutionized cosmology, showing that the universe was not, as Einstein had predicted, slowing down in its expansion, but instead speeding up [2, 3].

Understanding distance measures it of utmost importance to understanding observations in an expanding universe. There are many measures of distance, but among the most useful is the *luminosity distance*, which we denote d_L. The discussion from equations (1.24) - (1.28) loosely follow that in Ref. [4]. Suppose we wish to know the total energy flux, $\mathcal{F}$, of a source at a distance d away from the measurement point. The flux will be related to the absolute luminosity of the source by

$$\mathcal{F} = \frac{L_s}{4\pi d^2}. \tag{1.24}$$

We can generalize this to an expanding universe, defining the luminosity distance to be

$$d_L^2 \equiv \frac{L_s}{4\pi \mathcal{F}}. \tag{1.25}$$

Luminosity distance is able to provide a link between phenomena that can be observed and the energy densities of different components of the universe. In a flat geometry, the luminosity is related to energy density by [4]

$$d_L = \frac{(1+z)}{H_0} \int_0^z \frac{dz'}{\sqrt{\sum_i \Omega_i^{(0)} (1+z')^{3(1+\omega_i)}}}. \tag{1.26}$$

When $z \ll 1$, a much simpler approximation to this relation is given by

$$d_L \approx z/H_0. \tag{1.27}$$

The luminosity distance is also related to the apparent and absolute magnitudes of a source, m and M (respectively), by [5, 6]

$$m - M = 5 \log_{10} \left(\frac{d_L}{\text{Mpc}} \right) + 25. \tag{1.28}$$

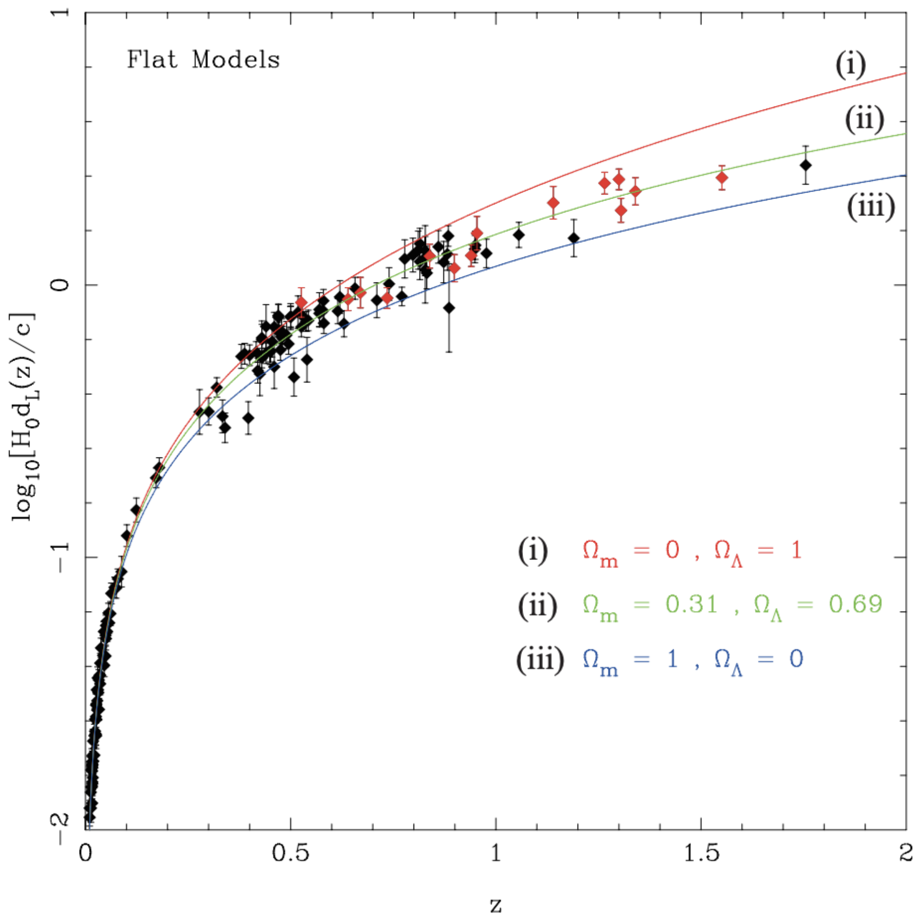

Figure 1.1: The luminosity distance $H_0 d_L$ versus the redshift in a flat cosmology. The Black points are data from [7], while the red are data from the Hubble Space Telescope. From Ref. [8]

Type Ia Supernovae are thought to form in the same way irrelevant from where they form in the universe, and thus have an absolute magnitude that is not dependent on z. This allows them to be thought of as a standard candle, and means that measurements of their apparent magnitudes allow us to make conclusions about Ω_Λ. Results from an analysis based off of this theoretical principle suggesting a nonzero vacuum energy is shown in Fig. (1.1). This data analysis shows that a flat universe without dark energy is not compatible with experiment.

1.2.1 The Hubble tension

Apart from the current accelerated expansion of the universe, another phenomena potentially requiring a physical explanation is the 'Hubble tension,' the discrepancy between direct local measurements of the Hubble parameter and the value inferred from measurements of the cosmic microwave background (CMB) [9, 10]. This tension is either an indication of new physics, or unrecognized uncertainties. The possibility that a scalar field might contribute transiently to the energy density has been proposed as a possible solution to the Hubble tension. In these scalar field solutions to the Hubble tension, the universe is never dominated by the energy density of the scalar field; instead, the scalar field density reaches roughly 10% of the total density in the universe and then decays away.

1.3 Different Models of Dark Energy

1.3.1 The Cosmological Constant

The standard model for cosmology treats the dark energy component of the energy density as a cosmological constant. That is, as a component with a constant equation of state. While the cosmological constant is consistent with observations measuring the expansion rate of the universe, and may be the most simple model of dark energy, it is not without issues. The cosmological constant needs to take on an extremely finely tuned value in order for our universe to have been able to be conducive to life, and this necessary value is much larger than the vacuum energy motivated by particle physics.

Moreover, while data has been consistent with the cosmological constant, it has not been able to rule out the possibility of dark energy of many other forms, that still have the nearly same energy density that we see in observations.

1.3.2 Quintessence

One extension to the standard cosmological model is to consider dark energy as being a scalar field, minimally coupled to gravity, instead of a cosmological constant. This scalar field changes value with respect to time, but is the same at all points in space. The class of scalar fields with a potential leading to a late time inflation are known as quintessence. Other classes of scalar fields have been proposed as models of dark energy, but we will not go into those for the purpose of this thesis). The action of a scalar field as such is given by

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\boldsymbol{\nabla}\phi)^2 - V(\phi)\right], \tag{1.29}$$

where ϕ is the scalar field, and $V(\phi)$ is the ϕ-dependent potential. One can derive from this the equation of motion for such a scalar field,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi}. \tag{1.30}$$

The pressure and density of the scalar field is given by

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.31}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{1.32}$$

And the equation of state is then given by

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{1.33}$$

Chapter 2

Axionlike Dark Energy

Motivated by both the need for a theoretical understanding of dark energy, as well as for the Hubble tension, we study scalar fields with potentials of the form,

$$V(\phi) = m^2 f^2 [1 - \cos(\phi/f)]^n. \tag{2.1}$$

This class of scalar fields provides a particularly good fit to the Hubble data. The case where $n = 1$ is the well-studied axion potential, and following Ref. [11], we will refer to scalar fields with potentials given by Eq. (2.1) as ultralight axionlike (ULA) fields. Such potentials (with $n = 1$) were among the first proposed for quintessence [12] and have been extensively studied in that context. Larger values of n might arise from higher-order instanton corrections [13]. Other cosmological consequences of scalar fields with potentials given by Eq. (2.1) are examined in Refs. [14, 15, 16].

Most of these studies simply approximate the potential in Eq. (2.1) by the corresponding power law appropriate for $\phi \ll f$, i.e.,

$$V(\phi) = \frac{1}{2^n} m^2 f^2 (\phi/f)^{2n}. \tag{2.2}$$

This is an excellent approximation in the limit of small ϕ, but as shown in Ref. [17], the use of this approximation can cause the evolution of the scalar field to deviate significantly from the evolution for the full potential of Eq. (2.1). For this reason, we undertake here a general study of scalar field evolution with ULA potentials, with the goal of understanding the ways in which this evolution deviates from the corresponding power-law evolution. Because these fields have been applied as both models for quintessence and as solutions of the Hubble tension, we will keep our discussion as general as possible, considering both the slow-roll and oscillatory phases of the scalar field evolution.

In the next section, we examine the evolution of ULA scalar fields in detail. We first investigate the initial slow-roll thawing evolution of these models and then their behavior when they oscillate rapidly. We highlight the differences between the evolution of the ULA models and the corresponding power-law approximations, and we discuss how these differences impact both quintessence and Hubble

2.1 Evolution of ULA scalar fields

Now consider a scalar field ϕ evolving in a potential of the form Eq. (2.1) or (2.2). We will assume that any nonzero value of $\dot{\phi}$ is damped by Hubble friction, so that the scalar field is initially at rest ($\dot{\phi} \approx 0$, $w \approx -1$). As the field begins to thaw, it will start to roll downhill. At first, the field energy will be dominated by the potential, so that $\dot{\phi}^2/2 \ll V(\phi)$, but as $\dot{\phi}$ increases and $V(\phi)$ decreases, the field will eventually reach a state for which $V(\phi) \sim \dot{\phi}^2/2$. Finally, after the field reaches the bottom of the potential, it will undergo rapid oscillations, with frequency $\nu \gg H$.

While there is no general analytic solution for $\phi(t)$ in any of these regimes, there are well-known approximations for the two limiting cases: the initial slow-roll regime with $\dot{\phi}^2/2 \ll V(\phi)$, and the final oscillatory phase with $\nu \gg H$. These two cases are the subject of the next two subsections.

2.1.1 Slow rolling ULA fields

Here we will examine the initial "slow-rolling" phase of a field evolving in a ULA potential. As the field rolls downhill from its initial value in the potential $V(\phi)$, w slowly increases, but $\dot{\phi}^2/2 \ll V(\phi)$, so that w remains close to -1. The evolution of ϕ in this slow-rolling regime depends on the relative values of V, V' and V'', where

$$V' = \frac{dV}{d\phi}, \qquad V'' = \frac{d^2V}{d\phi^2} \tag{2.3}$$

Such fields can provide a natural mechanism to yield dark energy at late times with an equation of state $w \approx -1$. In the terminology of Ref. [18] these are "thawing" quintessence fields. If $\dot{\phi}^2/2$ never becomes large compared to $V(\phi)$, then the field evolves from $w = -1$ to a value of w only slightly greater than -1, consistent with current observations.

Assuming that w never diverges far from -1, Ref. [19] considered potentials satisfying the inflationary slow-roll conditions, namely

$$\left(\frac{V'}{V}\right)^2 \ll 1, \tag{2.4}$$

and

$$\frac{V''}{V} \ll 1, \tag{2.5}$$

while in Refs. [20, 21, 22], the condition on the potential given by Eq. (2.4) was retained, but condition (2.5) was relaxed.

When conditions (2.4) and (2.5) both are imposed on the potential, along with the thawing initial condition ($\dot{\phi} \approx 0$ at early times), it is possible to derive an approximate analytic solution for $w(a)$ that is independent of $V(\phi)$. This solution is [19].

$$1 + w(a) = (1 + w_0)\frac{\left[F(a) - (F(a)^2 - 1)\coth^{-1} F(a)\right]^2}{\left[F(1) - (F(1)^2 - 1)\coth^{-1} F(1)\right]^2}, \tag{2.6}$$

where w_0 is the value of w at the present. The function $F(a)$ is

$$F(a) = \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}}, \tag{2.7}$$

where $\Omega_{\phi 0}$ is the fraction of the total density at present contributed by the scalar field. With these definitions, $F(a) = 1/\sqrt{\Omega_\phi(a)}$ and $F(1) = 1/\sqrt{\Omega_{\phi 0}}$. Equation 2.6 is derived in appendix (A.1).

The evolution of $w(a)$ given by Eq. (2.6) is well-approximated by a roughly linear dependence of $w(a)$ on a: the Chevallier-Polarski-Linder [23, 24] parametrization. Specifically, we have

$$w(a) = w_0 + w_a(1 - a), \tag{2.8}$$

where the value of w_a is well-fit by [19].

$$w_a \approx -1.5(1 + w_0) \tag{2.9}$$

In Refs. [20, 21, 22] the condition on the potential given by Eq. (2.4) was retained, but condition (2.5) was relaxed, resulting in a wider range of possible behaviors. In this case, the evolution of w with scale factor is derived in appendix A.2 and given by [20, 21, 22]

$$1 + w(a) = (1 + w_0)a^{3(K-1)}\frac{[(F(a) + 1)^K(K - F(a)) + (F(a) - 1)^K(K + F(a))]^2}{[(F(1) + 1)^K(K - F(1)) + (F(1) - 1)^K(K + F(1))]^2}, \tag{2.10}$$

where the constant K is a function of V''/V evaluated at ϕ_i (the initial value of ϕ), namely,

$$K = \sqrt{1 - (4/3)V''(\phi_i)/V(\phi_i)}. \tag{2.11}$$

Now instead of a single functional form for $w(a)$ for a given value of w_0, Eq. (2.10) provides a family of solutions that depend on K. As K becomes large, these solutions thaw more slowly, i.e., w remains close to -1 until later in the evolution [22]. In the opposite limit, as $K \to 1$, the solution in Eq. (2.10) approaches the evolution given in Eq. (2.6). A graphical representation of the trajectories for $w(a)$ for various values of K is shown in Fig. (2.1) [22].

With these results, we can examine the evolution of w for a ULA scalar field that serves as dark energy. From Eq. (2.1) we derive

$$\left(\frac{V'}{V}\right)^2 = \left(\frac{n}{f}\right)^2 \frac{1 + \cos(\phi/f)}{1 - \cos(\phi/f)}, \tag{2.12}$$

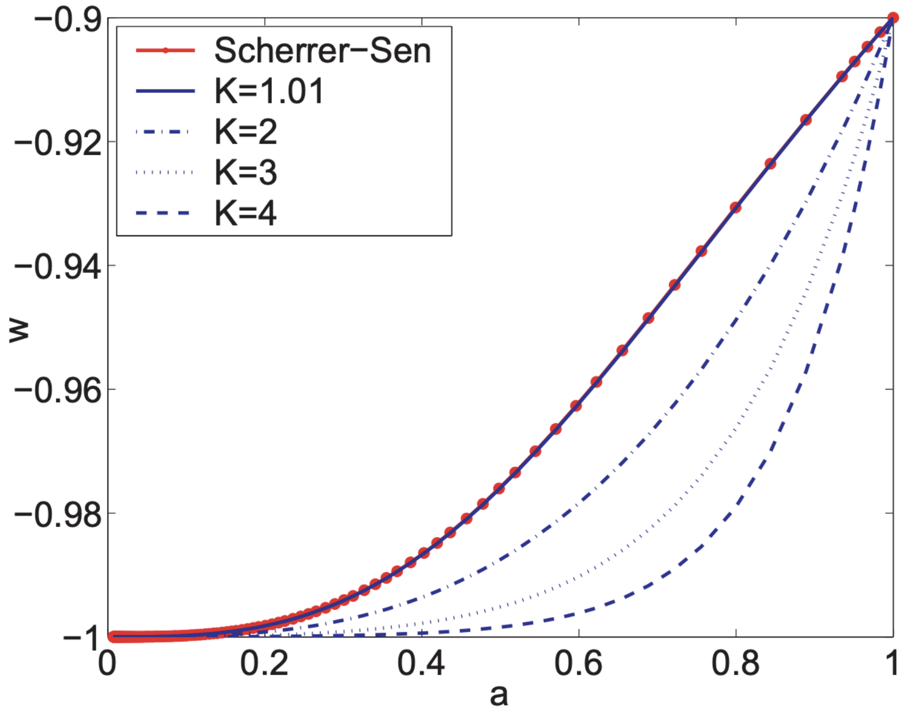

Figure 2.1: The trajectories for $w(a)$ for $K = 1.01, 2, 3, 4$, where K is given in Eq. (2.11), and $w(a)$ is given in Eq. (2.10). For these trajectories, $w = 0.9$, and $\Omega_{\phi 0} = 0.7$. The red curve (with filled circles) is from [19]. Figure from [22].

and

$$\frac{V''}{V} = \left(\frac{n}{f}\right)^2 \frac{1+\cos(\phi/f) - 1/n}{1-\cos(\phi/f)}, \tag{2.13}$$

We see that $(V'/V)^2 \ll 1$ is satisfied for two cases: either $f \gg n$, or $\phi_i/f \approx \pi$. Now consider the value of V''/V for these two cases. If $f \gg n$, then we have $V''/V \ll 1$. This is the model examined in Ref. [19], and it produces a single form for the evolution of $w(a)$, which is given in Eq. (2.6). On the other hand, if $\phi_i/f \approx \pi$, then Eq. (2.13) gives $V''/V = -(n/2f^2)$. In this case V''/V is necessarily negative, but it can be arbitrarily large or small, depending on the values of n and f. The corresponding value of K is

$$K = \sqrt{1 + \frac{2}{3}\frac{n}{f^2}}. \tag{2.14}$$

In this case, we can obtain the full range of possible trajectories for $w(a)$ given in Eq. (2.10), with $w(a)$ dependent on the value of n/f^2.

Contrast this to the behavior of the corresponding power-law potentials in Eq. (2.2). For these potentials, we have

$$\left(\frac{V'}{V}\right)^2 = \frac{4n^2}{\phi^2} \tag{2.15}$$

and

$$\frac{V''}{V} = \frac{2n(2n-1)}{\phi^2} \tag{2.16}$$

For these potentials, $(V'/V)^2 \ll 1$ when $\phi_i \gg n$; when this is the case, $V''/V \ll 1$ as well. Thus, the set of possible slow-roll quintessence evolutions is much more restricted than is the case for the ULA potentials. For slow-roll quintessence from power-law potentials, w always evolves as in Eq. (2.6), and never as in Eq. (2.10). Recall that these power-law potentials are the limiting case of the ULA potentials when $\phi \ll f$. This limiting case is consistent with slow-roll behavior when $n \ll \phi_i \ll f$.

For the case in which the ULA field acts as early dark energy to resolve the Hubble tension, the results in [19, 22, 20, 21] must be generalized to the case of a background (matter or radiation) dominated expansion. While this is a straightforward calculation, the results are of little utility in describing the resulting evolution of the density. The reason is that in the early dark energy models [25, 17, 26, 27, 15, 11] the regime of interest occurs after the scalar field has exited the slow-rolling regime and reached its maximum density relative to background density. If this occurs in the radiation-dominated era, then this maximum density relative to the background is achieved when $w = 1/3$, which is well beyond the validity of the slow-roll approach. However, the subsequent oscillation and decline of the scalar field energy density can be usefully described analytically, as shown in the next subsection.

2.1.2 Oscillating ULA fields

Rapidly oscillating scalar fields, in which the period of the scalar field oscillation is much less than the Hubble time, were first systematically explored by Turner [28], and subsequently by many others [29, 30, 31, 32, 33, 34, 35]. Consider a rapidly-oscillating scalar field with potential $V(\phi)$. We will consider only potentials symmetric about $\phi = 0$, so that the field oscillates between $\phi = -\phi_m$ and $\phi = +\phi_m$. Then the energy density of this field is equal to $\rho_\phi = V(\phi_m)$. Although ϕ_m and ρ_ϕ change with time, we will assume that this evolution is much slower than the oscillation frequency [28]

$$1 + w = 2\frac{\int_0^{\phi_m}[1 - V(\phi)/V(\phi_m)]^{1/2}d\phi}{\int_0^{\phi_m}[1 - V(\phi)/V(\phi_m)]^{-1/2}d\phi}. \tag{2.17}$$

For power-law potentials of the form given in Eq. (2.2), we obtain the standard result [28]

$$w = \frac{n-1}{n+1}. \tag{2.18}$$

Now consider the ULA potential (Eq. 2.1). To simplify our expressions, we make the change of variables $\Theta = \phi/f$ and obtain

$$1 + w = 2\frac{\int_0^{\Theta_m}[1 - (1-\cos\Theta)^n/(1-\cos\Theta_m)^n]^{1/2}d\Theta}{\int_0^{\Theta_m}[1 - (1-\cos\Theta)^n/(1-\cos\Theta_m)^n]^{-1/2}d\Theta}. \tag{2.19}$$

For $n = 1$, this result for $1 + w$ can be expressed in terms of elliptic integrals, but that provides little insight into the behavior of the equation of state parameter. Instead, we have numerically integrated Eq. (2.19) for $n =$ 1-3; the results for $1+w$ are shown in Figs. 2.2-2.4, respectively.

As expected, the equation of state parameters for the power-law and ULA potentials are identical at small Θ_m and diverge at large Θ_m. We can derive an analytic approximation for $1 + w$ for the ULA potentials at small Θ_m using the expression in Ref. [28] for potentials that diverge slightly from power-law behavior. Specifically, for potentials of the form

$$V(\phi) = a\phi^k(1 + \epsilon\phi^l), \tag{2.20}$$

with $\epsilon \ll 1$, the equation of state parameter is given by [28]

$$1 + w = \frac{2k}{k+2} + \epsilon(\phi_m)^l \frac{4l(l+1)}{(2l+k+2)(k+2)} \frac{\Gamma\left(\frac{k+2}{2k}\right)\Gamma\left(\frac{l+1}{k}\right)}{\Gamma\left(\frac{1}{k}\right)\Gamma\left(\frac{2l+k+2}{2k}\right)}, \tag{2.21}$$

plus higher-order terms, where ϕ_m is again the maximum value of ϕ attained in its oscillation.

Expanding the ULA potential (Eq. 2.1) to the lowest order beyond pure power-law behavior gives

$$V(\Theta) = m^2 f^2 \frac{\Theta^{2n}}{2^n}\left[1 - \frac{n}{12}\Theta^2\right], \tag{2.22}$$

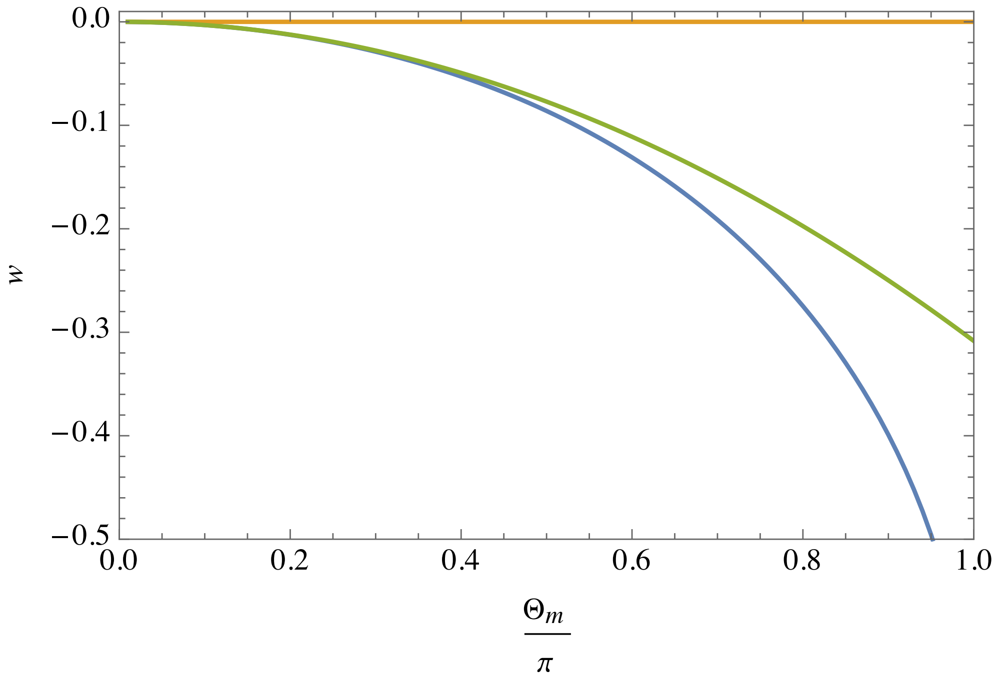

Figure 2.2: Equation of state parameter w as a function of Θ_m, the maximum value of the displacement angle, for the ULA potential (Eq. 2.1) with $n = 1$ (blue curve). The gold curve gives the equation of state parameter for the power-law potential that corresponds to the ULA potential in the limit of small Θ_m. The green curve is the approximate ULA equation of state parameter to quadratic order in Θ_m. Here and in Figs. 2.3 and 2.4, the blue curve approaches $w = -1$ as Θ_m/π goes to 1.

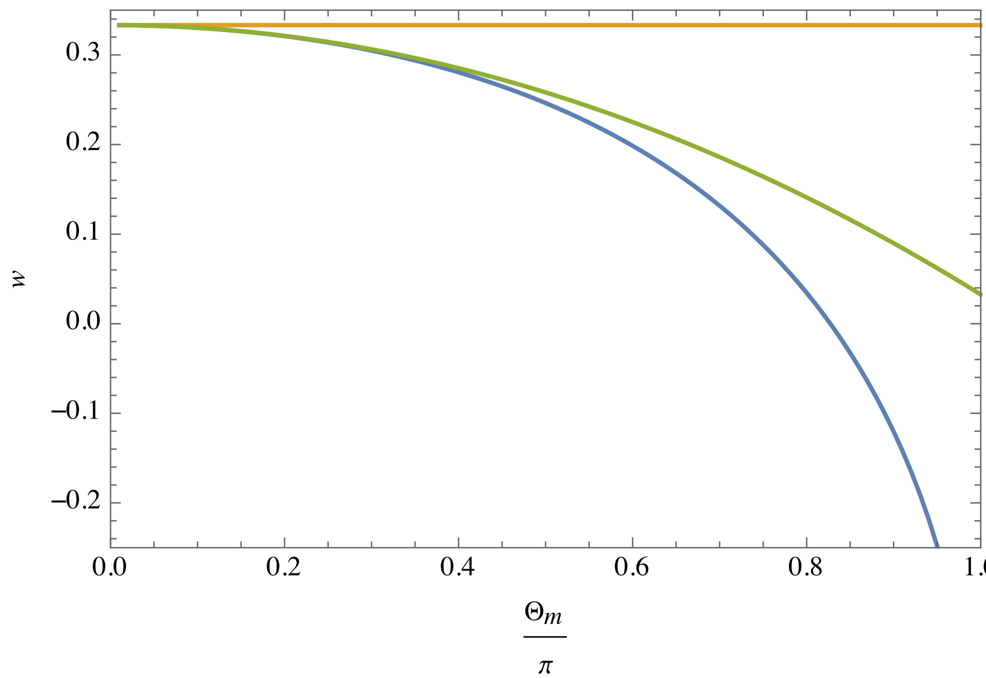

Figure 2.3: As Fig. 2.2, for $n = 2$.

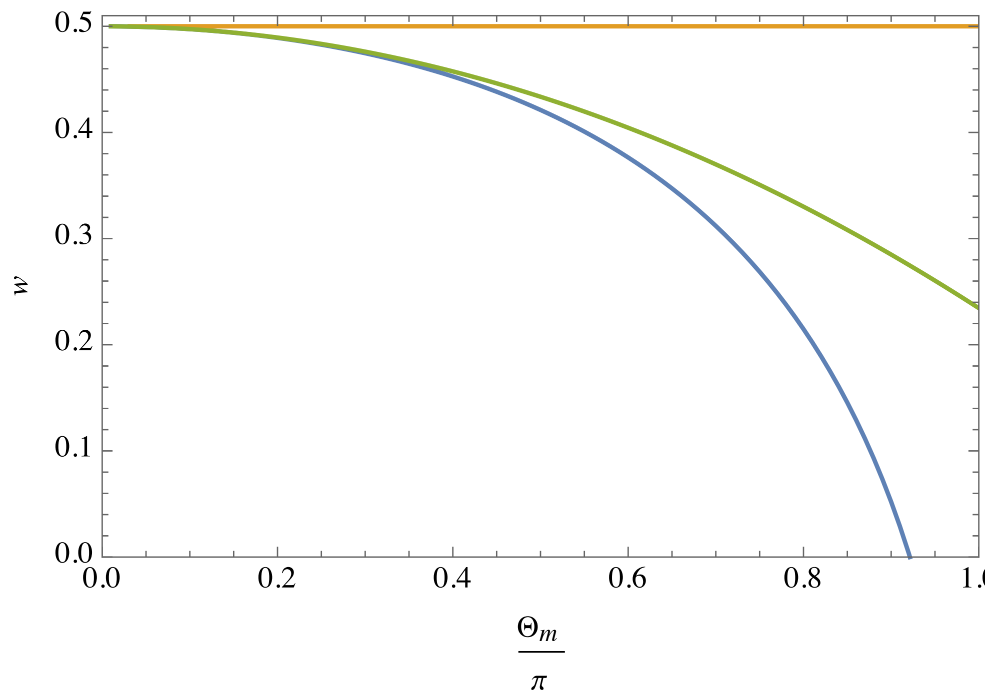

Figure 2.4: As Fig. 2.2, for $n = 3$.

from which we can derive the corresponding approximations to the equation of state parameter for small Θ_m, namely,

$$w \approx 0 - \frac{1}{32}\Theta_m^2, \qquad n = 1, \tag{2.23}$$

$$w \approx \frac{1}{3} - 0.0305\ \Theta_m^2, \qquad n = 2, \tag{2.24}$$

$$w \approx \frac{1}{2} - 0.0269\ \Theta_m^2, \qquad n = 3. \tag{2.25}$$

These expressions are displayed in Figs. 2.2-2.4. Note that the power-law value for w is a poor approximation in all of these cases once Θ/π increases beyond ~ 0.1. The leading-order perturbation does much better, giving very good agreement out to roughly $\Theta = \pi/2$.

Now consider the behavior of the oscillation frequency ν. For a potential $V(\phi)$, this frequency is given by [33, 34].

$$\nu = \left[\int_0^{\phi_m} 2\sqrt{\frac{2}{V(\phi_m) - V(\phi)}}d\phi\right]^{-1} \tag{2.26}$$

For the power-law potential in Eq. (2.2), this expression can be integrated exactly to give [33]

$$\nu = \frac{n\Gamma(\frac{1}{2} + \frac{1}{2n})}{\sqrt{\pi}\Gamma(\frac{1}{2n})2^{(n+1)/2}} m\Theta_m^{n-1}. \tag{2.27}$$

(See also the corresponding expression in Ref. [17]). The oscillation frequency for the ULA potential can be derived numerically by substituting the potential from Eq. (2.1) into the integral for the frequency given by Eq. (2.26). Figs. 4-6 provide a comparison of the oscillation frequency for the ULA potential with the oscillation frequency for the corresponding small-Θ_m power-law potential. As expected, the power law gives a good approximation for the oscillation frequency only for small values of Θ, roughly $\Theta_m/\pi = 0.2$.

Now consider the cosmological consequences of these results. For the ULA potentials with a rapidly-oscillating scalar field, the time-averaged value of w approaches -1 as Θ_m goes to π. In theory, such models could serve as dark energy (see, e.g., Ref. [33]). However, Johnson and Kamionkowski [35] have argued that such models are unstable to growth of inhomogeneities. Thus, quintessence models based on ULA potentials with high-frequency oscillations are not as plausible as the slow-roll quintessence models discussed in the previous section.

On the other hand, these rapidly oscillating ULA scalar fields form an important component of the early dark energy models that might resolve the Hubble tension [25, 17, 26, 27, 15, 11]. Furthermore, Smith et al. [17] argue that the ULA potentials with $n =$ 2-3 (with $n = 3$ providing the best fit) and large values of Θ_m (specifically $\Theta_m \gtrsim 2.5$) provide a better fit to the Hubble data than the power-law potentials corresponding to $\Theta_m \ll 1$.

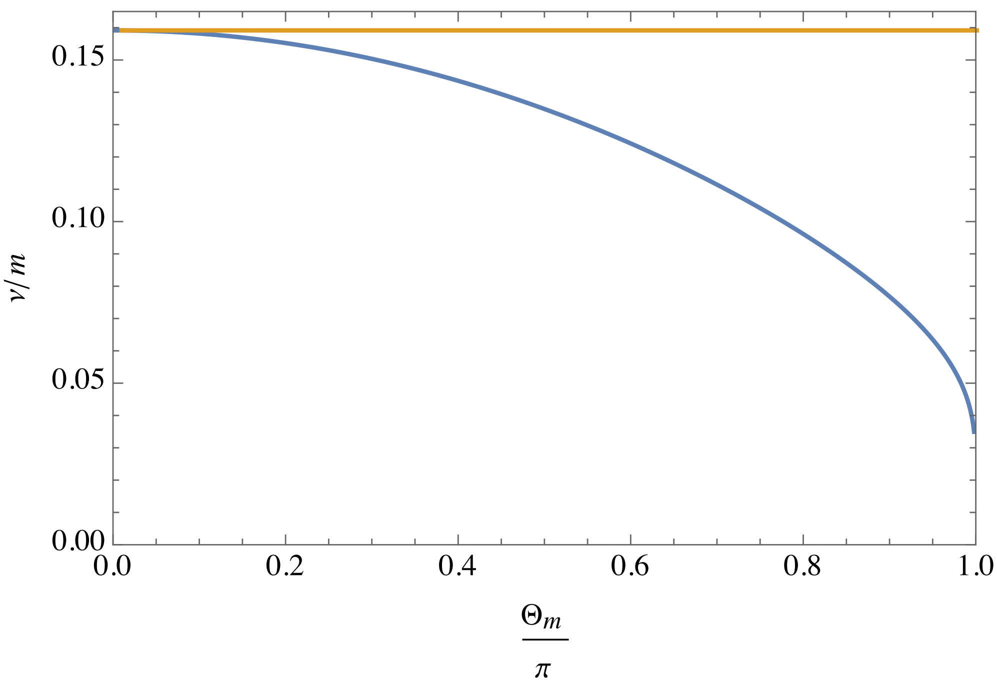

Figure 2.5: Oscillation frequency ν as a function of Θ_m, the maximum value of the displacement angle, for the ULA potential (Eq. 2.1) with $n = 1$ (blue curve). The gold curve gives the oscillation frequency for the power-law potential that corresponds to the ULA potential in the limit of small Θ_m.

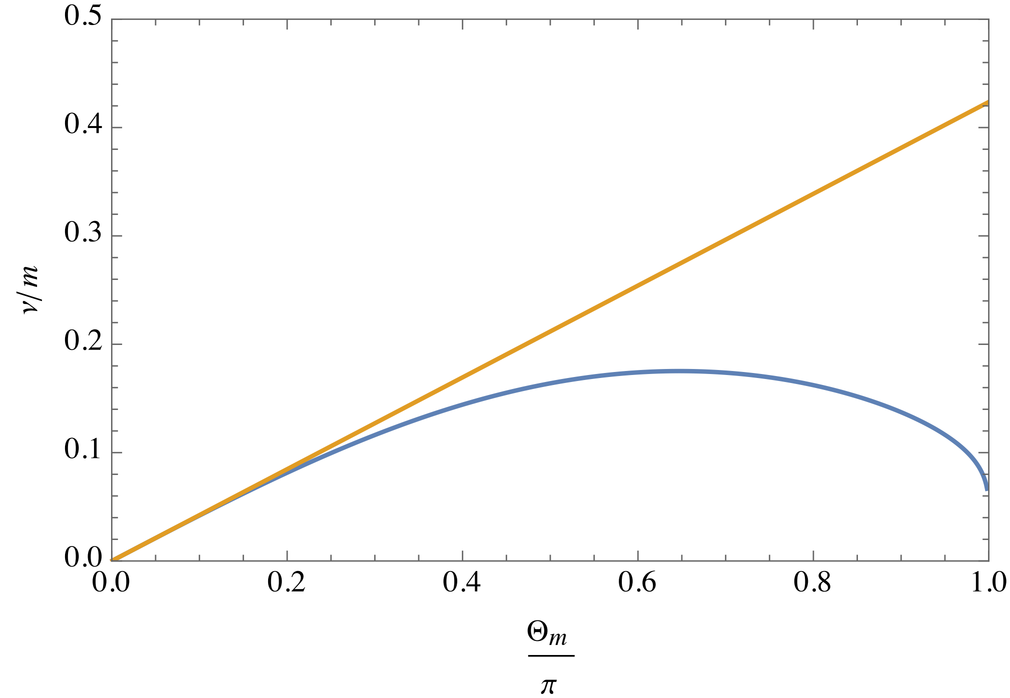

Figure 2.6: As Fig. 2.5, for $n = 2$.

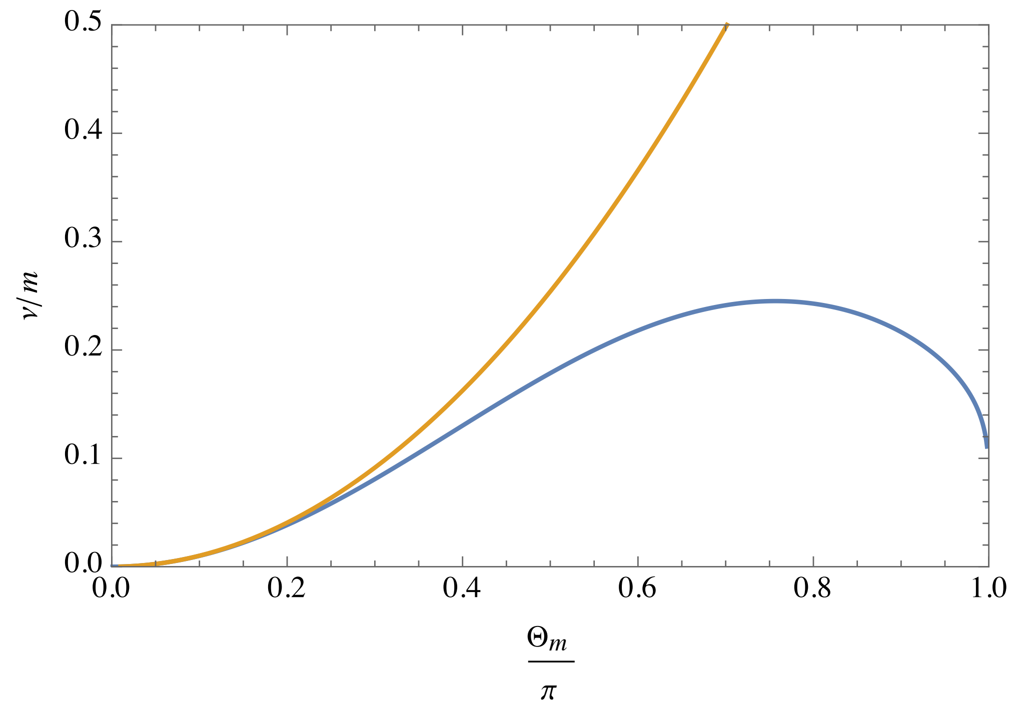

Figure 2.7: As Fig. 2.5, for $n = 3$.

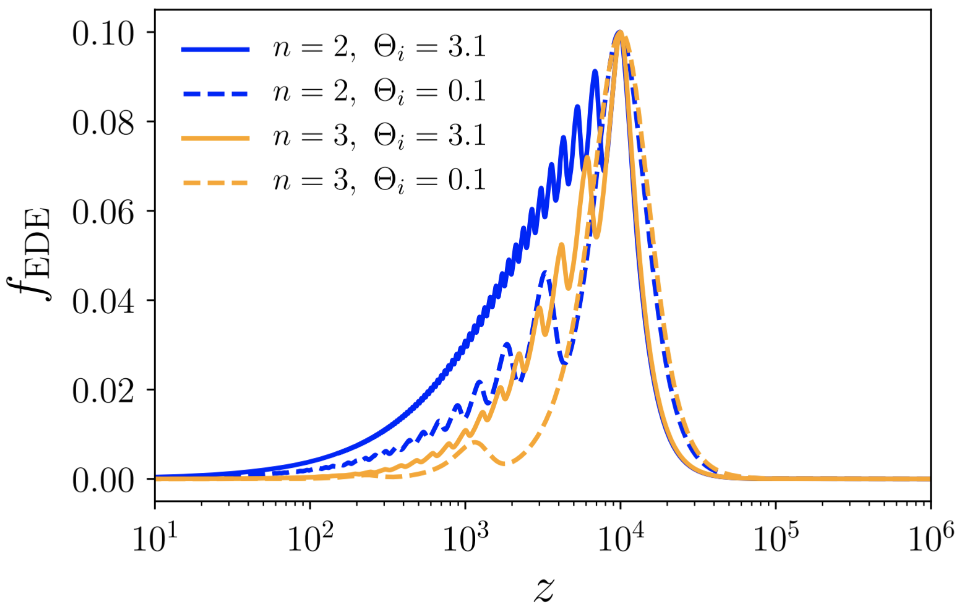

Figure 2.8: The evolution of the total energy density in the early dark energy (EDE) as a function of redshift. Fig (2) of Ref. [17]

.

Our results provide some insight into the evolutionary behavior of the models examined in Ref. [17]. In particular, the fact that w decreases with increasing Θ_m for the ULA potentials results in a slower decrease of the oscillation-averaged density at low redshift for large values of Θ_m. This is clearly the case in the numerical simulations of Ref. [17], shown in Fig. 2.8

These simulations also suggest that the oscillation frequency is much larger for the $n = 2$ and $n = 3$ ULA potentials with initial values of Θ_m near π than for very small initial values of Θ_m. This behavior is also apparent in our results (Figs. 2.6 and 2.7). Further, for $n =$ 2-3, large initial values of Θ_m yield an oscillation frequency that increases as the scalar field energy density (and therefore Θ_m) decreases, which is the opposite of the evolution of ν for very small initial values of Θ_m.

2.2 Discussion

Our analysis of the scalar field evolution for the full ULA potentials (Eq. 2.1) shows many interesting differences from the scalar field evolution for the corresponding small-ϕ power-law approximations (Eq. 2.2). While both sets of models can serve as slow-roll thawing quintessence, only the full ULA potentials can yield hilltop-style evolution of w, corresponding to a much richer set of evolutionary behaviors.

For rapidly-oscillating scalar fields, the oscillation-averaged value for the equation of state parameter w corresponding to the power-law approximation diverges from the value of w in the ULA potential for $\phi/(f\pi) > 0.1$, with the ULA potential giving a value for w much smaller than the corresponding power-law potential, and $w \to -1$ as $\phi/f \to \pi$. Similarly, the power-law approximation for the oscillation frequency ν diverges from the ULA frequency for $\phi/(f\phi) > 0.2$, with the ULA potential yielding a smaller oscillation frequency. Further, the dependence of the oscillation frequency on the oscillation amplitude ϕ_m is more complex for the ULA potentials; for $n = 2$ and $n = 3$, the oscillation frequency increases with ϕ_m, reaches a maximum, and then decreases as $\phi_m/f \to \pi$.

We emphasize that the standard axion potential ($n = 1$) has been investigated previously, as has the behavior of the $n > 1$ potentials in the limit where they are well-approximated by a power-law potential. What is new here is the treatment of the latter cases with the full ULA potential. Furthermore, our quintessence modeling in Sec. II.A. is only approximate; it would be interesting to investigate the behavior of these models in the slow-roll regime with a full numerical integration of the equations of motion to more precisely determine their suitability as models for dark energy.

Chapter 3

Decaying Dark Matter

The second portion of this thesis will deal with the different, but related, idea of decaying matter in the universe. In Ref. [36], Scherrer and Krauss showed the surprising result that radiation can never dominate over matter in a cosmological constant dominated universe. This is surprising, as it is not true when the universe is not dominated by a cosmological constant. In this portion of the thesis, we seek to generalize this result to a class of models more general than a cosmological constant dominated universe.

3.1 Decay Evolution

We consider a non-relativistic component of the energy density, ρ_M (matter), which decays into a relativistic component, ρ_R (radiation). If the decay products do not have significant interactions with anything else, the equations for the evolution of matter and radiation are

$$\frac{d\rho_M}{dt} = -3H\rho_M - \rho_M/\tau \tag{3.1}$$

$$\frac{d\rho_R}{dt} = -4H\rho_R + \rho_M/\tau, \tag{3.2}$$

where τ is the lifetime of the decay, and H is again the Hubble parameter. Take r to be the ratio of the radiation density to the matter density,

$$r \equiv \frac{\rho_R}{\rho_M}. \tag{3.3}$$

We study the evolution of r as a function of time, to investigate whether or not it increases above 1. We can combine Eq. (3.1) and (3.2) to obtain

$$\frac{dr}{dt} = \frac{1}{\tau} + \left(\frac{1}{\tau} - H\right) r. \tag{3.4}$$

3.2 Particle Decay in Different Cosmologies

3.2.1 Cosmological Constant

Scherrer and Krauss investigated the decay of matter in a cosmological constant dominated universe [36]. In the matter dominated era, $r \to \infty$, as long as $H < 1/\tau$. In a vacuum energy dominated state, the Hubble parameter approaches the constant value

$$H_\Lambda = \left(\frac{8\pi G\rho_\Lambda}{3}\right)^{1/2}. \tag{3.5}$$

Following [36], we define t_Λ, the 'time of no return,' to be

$$t_\Lambda \equiv 1/H_\Lambda. \tag{3.6}$$

Plugging this into Eq. (3.4), we can obtain the following analytic equation for $r(t)$ [36]

$$\left(1 - \frac{\tau}{t_\Lambda}\right) r = \exp\left\{\left[\frac{1}{\tau} - \frac{1}{t_\Lambda}\right] t\right\} - 1. \tag{3.7}$$

Notably, if $\tau > t_\Lambda$, r asymptotically approaches a constant, given by [36]

$$r(t \to \infty) = \frac{t_\Lambda}{\tau - t_\Lambda}, \tag{3.8}$$

whereas if $\tau < t_\Lambda$, then $r \to \infty$ as $t \to \infty$.

In our present universe, which can be approximated as a mix of matter and a cosmological constant, the decay equation (3.4) is not analytically solvable. Instead, we numerically integrate the equation. The long term behavior of $r(t)$ depends on the quantity, t_Λ/τ. We show the behavior for different values of t_Λ/τ in Fig. (3.1).

3.2.2 Quintessence

The Hubble parameter for quintessence is given by

$$H = \frac{(2/3)}{(w+1)t}. \tag{3.9}$$

We solve eq. (3.4) numerically in this case, arriving at a solution of $r(t)$ that depends on τ. In this case, $r(t)$ goes to infinity for all values of τ, meaning that radiation can dominate over matter in this universe. We show this behavior in Fig (3.2).

3.2.3 The Big Rip

In the "big rip" cosmology, the equation of state of dark energy approaches -1 from the negative side; that is, $w_\phi < -1$. Matter with an energy density of less than -1 is called "phantom matter."

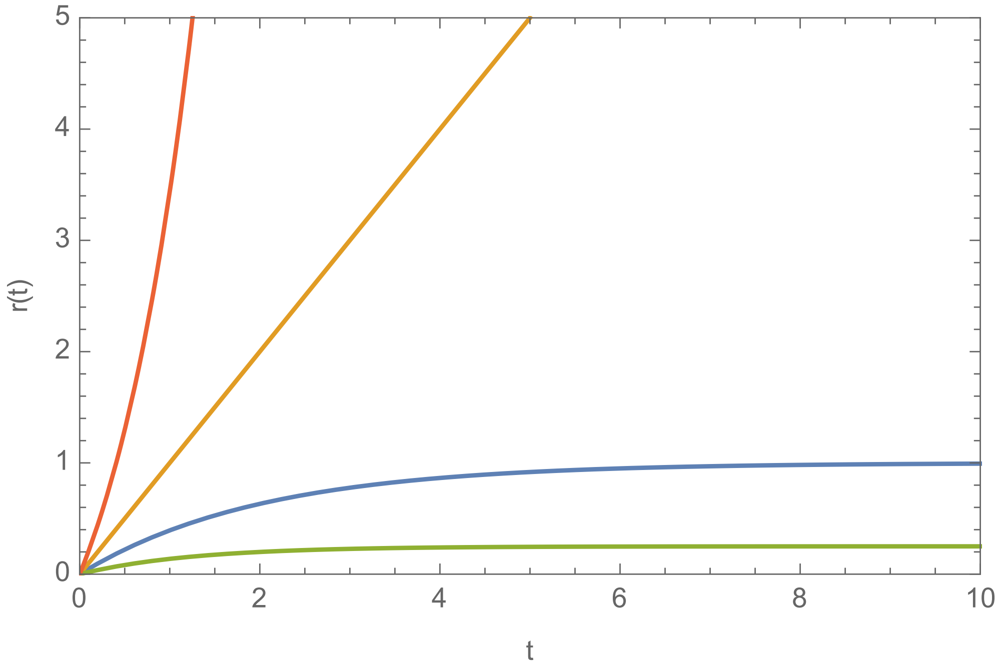

Figure 3.1: $r(t)$ in a cosmological constant dominated universe. From bottom to top, the curves correspond to values of $t_\Lambda/\tau = 0.2$, 0.5, 1, 2.

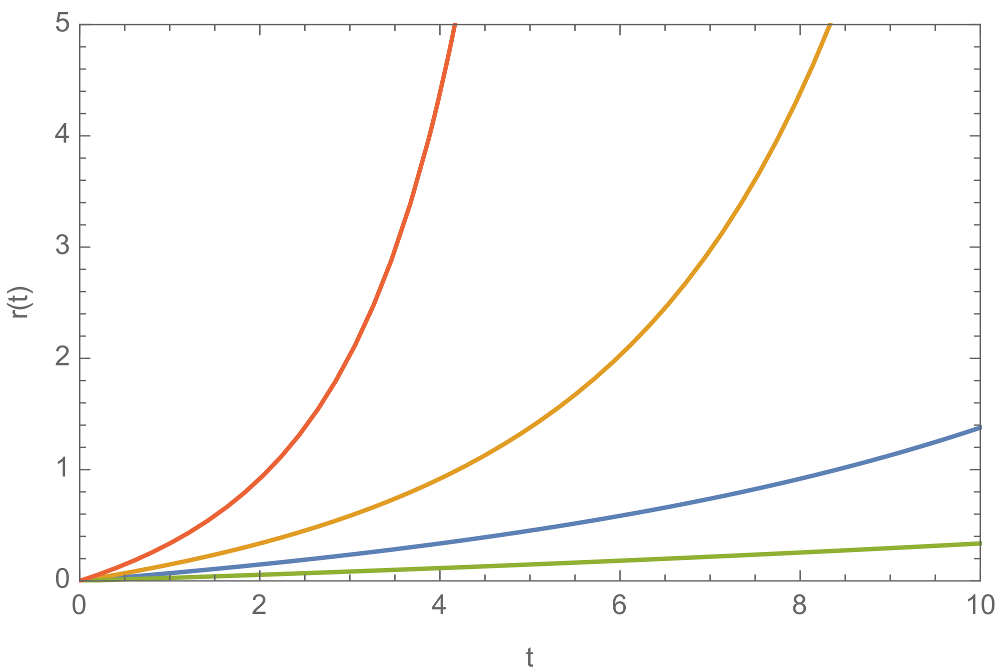

Figure 3.2: $r(t)$ in a quintessence dominated universe. From bottom to top, the curves correspond to values of $1/\tau = 0.2$, 0.5, 1, 2.

In this cosmology, H takes the form

$$H = \frac{(2/3)(1/t_m)}{-w + (1+w)(t/t_m)} \tag{3.10}$$

Thus, the differential equation for r is

$$\frac{dr}{dt} = \frac{1}{\tau} + \left(\frac{1}{\tau} - \frac{(2/3)(1/t_m)}{-w + (1+w)(t/t_m)}\right) r. \tag{3.11}$$

We can rearrange this equation to combine the parameters τ and t_m. Multiplying each term by t_m, we get

$$\frac{dr}{dt} = \frac{t_m}{\tau} + \left(\frac{t_m}{\tau} - \frac{2/3}{-w + (1+w)(t/t_m)}\right) r. \tag{3.12}$$

Then, we can define $\tilde{t}$ to be t/t_m, which removes the final instance of t_m not appearing as multiplied by $1/\tau$. Now, we can clearly see that the behavior of $r(t)$ does not depend on both parameters separately, but a combination of them. This simplifies greatly our systematic understanding of how $r(\tilde{t})$ behaves for different parameter values.

The end of the universe corresponds to the time when H goes to infinity. For this calculation, we will take $w = -1.1$, which means that $H = \infty$ when $\tilde{t} = 11$, so we restrict our attention to $\tilde{t} < 11$. We show the behavior of this in Figs. (3.3) and (3.4), for different values of t_m/τ. Fig. (3.3) shows the behavior for $t_m/\tau = 0.2, 0.5, 1$ and (3.4) shows the behavior for $t_m/\tau = 2$. The behavior is shown in two different figures because of the disparity in $r(t)$ vales once $t_m/\tau > 1$.

3.3 Discussion

There is still much to be done in gaining an understanding of why these different cosmologies behave the way they do. In particular, the big rip cosmology results are exceptionally surprising. Gaining a better understanding of these results will be the objective of the author's next research project. These preliminary results are very interesting, as they indicate different behaviors for different possibilities of modeling dark energy.

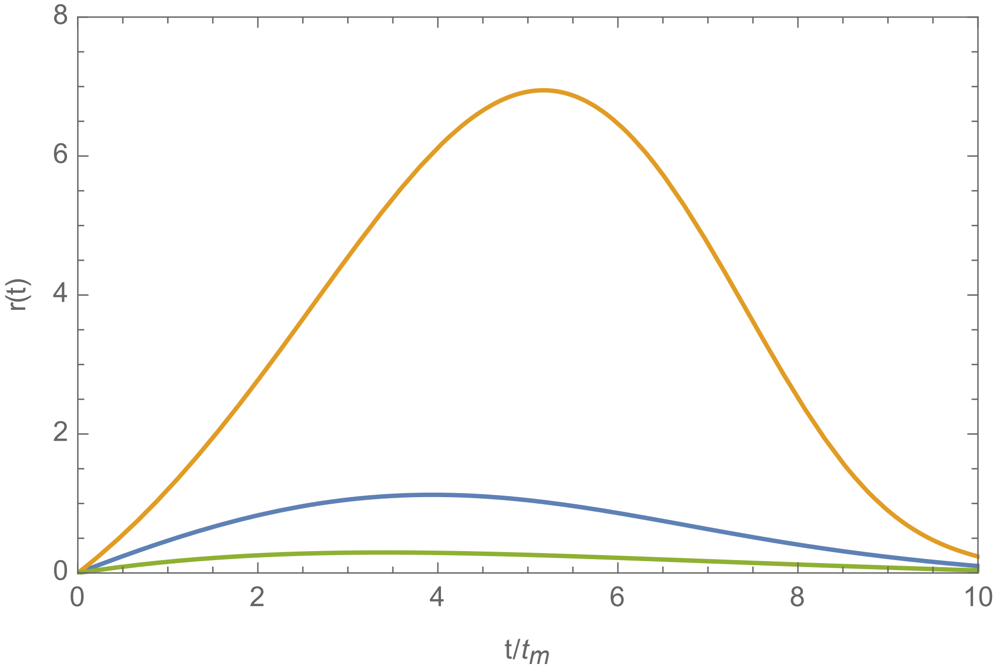

Figure 3.3: $r(t)$ for a big rip cosmology. From bottom to top, the curves correspond to values of $t_m/\tau = 0.2$, 0.5, 1.

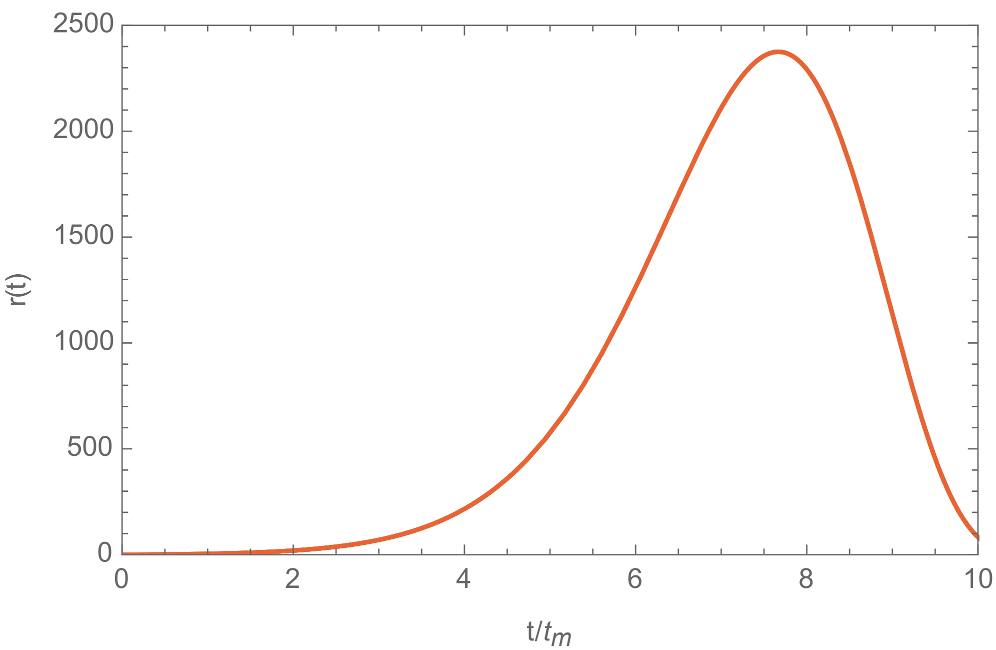

Figure 3.4: $r(t)$ for a big rip cosmology, with $t_m/\tau = 2$.

Appendix A

Derivations

A.1 Equation 2.6

This derivation follows [19, 37, 38, 39]. Assume that we have a scalar field that obeys the slow roll conditions described in Eq. (2.4) and (2.5), and also a flat universe with only matter and dark energy, such that

$$\Omega_M + \Omega_\Lambda = 1. \tag{A.1}$$

In this case, the equation of motion of the scalar field, 1.30 and the equation for the Hubble parameter 1.7 can be transformed to simpler forms by performing the following change of variables

$$x = \phi'/\sqrt{6} \tag{A.2}$$

$$y = \sqrt{V(\phi)/(3H^2)} \tag{A.3}$$

$$\lambda = -\frac{1}{V}\frac{dV}{d\phi} \tag{A.4}$$

With the prime denoting a derivative with respect to $\ln(a)$,

$$\phi' \equiv \frac{d\phi}{d\ln a} \tag{A.5}$$

$$= \frac{d\phi}{da}\frac{da}{d\ln a} \tag{A.6}$$

$$= a\frac{d\phi}{da}. \tag{A.7}$$

Then, we obtain [19]

$$x^2 + y^2 = \Omega_\phi. \tag{A.8}$$

And we have that the equation of state is

$$\lambda = 1 + w = \frac{2x}{x^2 + y^2}. \tag{A.9}$$

Then, the Hubble equation and the evolution equation REF become

$$x' = -3x + \lambda\sqrt{\frac{3}{2}}y^2 + \frac{3}{2}x[1 + x^2 - y^2] \tag{A.10}$$

$$y' = -\lambda\sqrt{\frac{3}{2}}xy + \frac{3}{2}y[1 + x^2 - y^2] \tag{A.11}$$

$$\lambda' = -\sqrt{6}\lambda^2(\Gamma - 1)x, \tag{A.12}$$

where

$$\Gamma \equiv V\frac{d^2V}{d\phi^2}/\left(\frac{dV}{d\phi}\right)^2. \tag{A.13}$$

Now, it is helpful to rewrite these equations to be completely in terms of observable quantities, Ω_ϕ, γ and λ. We can use Eqs. (A.8) and (A.9) to obtain these new equations

$$\gamma' = -3\gamma(2 - \gamma) + \lambda(2 - \gamma)\sqrt{3\gamma\Omega_\phi} \tag{A.14}$$

$$\Omega_\phi' = 3(1 - \gamma)\Omega_\phi(1 - \Omega_\phi) \tag{A.15}$$

$$\lambda' = -\sqrt{3}\lambda^2(\Gamma - 1)\sqrt{\gamma\Omega_\phi}. \tag{A.16}$$

If we transform our dependent variable from a to $\Omega_\phi(a)$, our equations simplify again

$$\frac{d\gamma}{d\Omega_\phi} = \frac{\gamma'}{\Omega_\phi'} = \frac{-3\gamma(2 - \gamma) + \lambda(2 - \gamma)\sqrt{3\gamma\Omega_\phi}}{3(2 - \gamma)\Omega_\phi(1 - \Omega_\phi}. \tag{A.17}$$

This change of variables is only valid if, for all points, $d\Omega_\phi/da \neq 0$, which we have in the present case.

Now, we make two assumptions in order to yield a relatively simple solution. First, we assume that $\gamma << 1$ (i.e. $w \approx 1$). Then we assume that γ is approximately constant

$$\gamma \approx \gamma_0. \tag{A.18}$$

This assumption follow from the slow-roll conditions.

Then, we can rewrite Eq. (A.17) by replacing γ with γ_0 and only keeping first order terms in λ (as others will be nearly zero),

$$\frac{d\gamma}{d\Omega_\phi} = -\frac{2\gamma}{\Omega_\phi(1 - \Omega_\phi)} + \frac{2}{3}\lambda_0\frac{\sqrt{3\gamma}}{(1 - \Omega_\phi)\sqrt{\Omega_\phi}}. \tag{A.19}$$

Then, we can transform this equation into a linear differential equation with the change of variables $s^2 = \gamma$. We can get an equation for $ds/d\Omega_\phi$,

$$\frac{ds}{d\Omega_\phi} = \frac{ds}{d\gamma}\frac{d\gamma}{d\Omega_\phi} \tag{A.20}$$

But we know

$$\frac{d\gamma}{ds} = 2s, \tag{A.21}$$

So then we can write our equation for $ds/d\Omega_\phi$,

$$\frac{ds}{d\Omega_\phi} = -\frac{s}{\Omega_\phi(1-\Omega_\phi)} + \frac{1}{3}\frac{\sqrt{3}}{(1-\Omega_\phi)\sqrt{\Omega_\phi}} \tag{A.22}$$

Then, the resulting solution, expressed in terms of w, is

$$1 + w = \frac{\lambda_0^2}{3}\left[\frac{1}{\sqrt{\Omega_\phi}} - \left(\frac{1}{\Omega_\phi} - 1\right)\tanh^{-1}\sqrt{\Omega_\phi}\right]^2 \tag{A.23}$$

$$= \frac{\lambda_0^2}{3}\left[\frac{1}{\sqrt{\Omega_\phi}} - \left(\frac{1}{\Omega_\phi} - 1\right)\ln\left(\frac{1+\sqrt{\Omega_\phi}}{1-\sqrt{\Omega_\phi}}\right)\right]^2. \tag{A.24}$$

If we take the limit where $\gamma \ll 1$, then Eq. (A.15) can be solved to obtain

$$\Omega_\phi = \left[1 + \left(\Omega_{\phi 0}^{-1} - 1\right)a^{-3}\right]^{-1} \tag{A.25}$$

$$1 + w = (1 + w_0)\left[\sqrt{1 + (\Omega_{\phi 0}^{-}1 - 1)a^{-3}} - (\Omega_{\phi 0}^{-1})a^{-3}\tanh\frac{1}{\sqrt{1 + (\Omega_{\phi 0}^{-}1 - 1)a^{-3}}}\right]^2$$
$$\times\left[\frac{1}{\sqrt{\Omega_{\phi 0}}} - \left(\frac{1}{\Omega_{\phi 0}} - 1\right)\tanh^{-1}\sqrt{\Omega_\phi 0}\right]^{-2} \tag{A.26}$$

And then by defining $F(a)$ as

$$F(a) = \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}}, \tag{A.27}$$

we get the desired equation

$$1 + w(a) = (1 + w_0)\frac{\left[F(a) - (F(a)^2 - 1)\coth^{-1}F(a)\right]^2}{\left[F(1) - (F(1)^2 - 1)\coth^{-1}F(1)\right]^2}, \tag{A.28}$$

A.2 Equation 2.10

We derive the evolution of w as a function of the scale factor, following that given in [22]. We wish to solve the differential equation for ϕ given in Eq. (1.30). To eliminate the $\dot{\phi}$ term, we make the change of variables

$$\phi(t) = u(t)/a(t)^{3/2}. \tag{A.29}$$

Then

$$\dot{\phi}(t) = \frac{-3}{2}u(t)a(t)^{-5/2}\dot{a}(t) + \dot{u}(t)a(t)^{-3/2} \tag{A.30}$$

$$\ddot{\phi} = \ddot{u}(t)a(t)^{-3/2} - \dot{u}(t)(3a(t)^{-5/2}\dot{a}(t) + u(t)\frac{-3}{2}\left(a(t)^{-5/2}\ddot{a}(t) + \frac{-5}{2}a(t)^{-7/2}\dot{a}(t)\right) \tag{A.31}$$

Substituting these values into Eq. (1.30) and multiplying by $a(t)^{3/2}$, we get

$$\ddot{u}(t) - 3\dot{u}(t)\frac{\dot{a}(t)}{a(t)} - \frac{3}{2}u(t)\left(a(t)^{-1}\ddot{a}(t) + \frac{-5}{2}a(t)^{-2}\dot{a}(t)\right)$$
$$+ 3\left(\frac{\dot{a}(t)}{a(t)}\right)\left(\frac{-3}{2}u(t)a(t)^{-1}\dot{a}(t) + \dot{u}(t)\right) + a^{3/2}V'(u/a^{3/2}) = 0. \tag{A.32}$$

The $\dot{u}(t)$ terms cancel, and we then get,

$$\ddot{u} - \frac{3}{2}\left[\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2\right]u + a^{3/2}V'(u/a^{3/2}) = 0 \tag{A.33}$$

Plugging in

$$\frac{\dot{a}}{a} = \sqrt{\rho_T/3} \tag{A.34}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_T + 3p_T), \tag{A.35}$$

where ρ_T and p_T are the total density and total pressure, respectively, we arrive at the following equation

$$\ddot{u}(t) - \frac{3}{4}\rho_{\phi 0}u + a^{3/2}V'(u/a^{3/2}) = 0 \tag{A.36}$$

This equation has a simple form in a ΛCDM universe, in which p_T is constant. In our case, we are considering a universe with matter and dark energy. For the dark energy, w is always near -1, so the pressure is nearly constant. Since the matter is pressureless, the total pressure is approximately

$$p_t \approx -\rho_{\phi 0}, \tag{A.37}$$

where $\rho_{\phi 0}$ is the density of the dark energy. Then, equation $A.36$ takes the form

$$\ddot{u} - \frac{3}{4}\rho_{\phi 0}u + a^{3/2}V'(u/a^{3/2}) = 0. \tag{A.38}$$

This equation holds whenever the Hubble parameter is approximately that of ΛCDM. (i.e., when the potential term of the scalar field is much bigger than the kinetic term, so that $w \approx -1$). In our case, with a scalar field of the form of Eq. (2.1), this happens when the field is near the top of its potential hill. Then, we can approximate the potential by expanding it at its maximum, which we denote ϕ_*,

$$V(\phi) = V(\phi_*) + 1/2V''(\phi_*)\phi^2 + O(\phi^3)..., \tag{A.39}$$

where $O(\phi^3)$ denotes higher order terms that are negligible. Plugging this into Eq. (A.38), we get

$$\ddot{u} + \left[V''(\phi_*) - (3/4)V(\phi_*)\right] u = 0. \tag{A.40}$$

Then, defining

$$k \equiv \sqrt{(3/4)V(\phi_*) - V''(\phi_*)}, \tag{A.41}$$

we find the solution to Eq. (A.40) to be

$$u = A \sinh kt + B \cosh kt. \tag{A.42}$$

If we assume that $w \approx -1$, then the scale factor is well approximated by its value in the ΛCDM model. Starting with the Hubble equation, (1.7), and assuming that

$$\rho = \rho_{\Lambda 0} + \rho_{M0} a(t)^{-3}, \tag{A.43}$$

We get an equation for $a(t)$ that can be integrated in terms of hyperbolic functions. Once simplified, we can write the scale factor as

$$a(t) = \left[\frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}}\right]^{1/3} \sinh^{2/3} t/t_\Lambda, \tag{A.44}$$

where t_Λ is defined as

$$t_\Lambda = \frac{2}{\sqrt{3\rho_{\phi 0}}} = \frac{2}{\sqrt{3V(\phi_*)}}. \tag{A.45}$$

Then, we can find the solution for $\phi(t)$ using Eq. (A.29):

$$\phi(t) = \left[\frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}}\right]^{1/2} \frac{A \sinh kt + B \cosh kt}{\sinh t/t_\Lambda}. \tag{A.46}$$

For our initial condition, we require that

$$\phi(0) = \phi_i, \tag{A.47}$$

where ϕ_i is a fixed constant. However, plugging in zero for time leads to a divergence. What we really require is that

$$\lim_{t \to 0} \phi(t) = \phi_i, \tag{A.48}$$

which can clearly only happen if $B = 0$. Then, we need to pick A so that

$$\phi(t) = \frac{\phi_i}{kt_\Lambda} \frac{\sinh(kt)}{\sinh(t/t_\Lambda)}. \tag{A.49}$$

Now, we write out the equation of state for the dark energy, expressed as $1+w$. Any equation of state for quintessence can be written as

$$1+w = 1 + \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{A.50}$$

$$= \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{A.51}$$

$$= \frac{\dot{\phi}^2}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{A.52}$$

$$= \frac{\dot{\phi}^2}{\rho_\phi} \tag{A.53}$$

Then, we take

$$\rho_\phi \approx \rho_{\phi 0} \approx V(\phi_*), \tag{A.54}$$

where the first approximation comes from the assumption that the density is nearly constant (i.e. ≈ -1) and the second approximation comes from the assumption that the fact that $\dot{\phi}^2$ is small when the field is at the top of its potential hill. Now we use Eqs. (A.49) and (A.50) to get the full expression for $1+w$ for our specific case. The first time derivative of ϕ is

$$\dot{\phi} = \frac{\phi_i}{kt_\Lambda}\frac{k\sinh(t/t_\Lambda)\cosh(kt) - (1/t_\Lambda)\sinh(kt)\cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)}. \tag{A.55}$$

Then,

$$1+w = \frac{1}{V(\phi_*)}\left(\frac{\phi_i^2}{kt_\Lambda}\right)^2\left[\frac{k\sinh(t/t_\Lambda)\cosh(kt) - (1/t_\Lambda)\sinh(kt)\cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)}\right]^2. \tag{A.56}$$

Moreover, we can simplify a bit by writing

$$\frac{1}{V(\phi_*)}\frac{1}{t_\Lambda} = \frac{1}{V(\phi_*)}\frac{3V(\phi_*)}{4} = \frac{3}{4}. \tag{A.57}$$

And then we arrive at

$$1+w = \frac{3}{4}\frac{\phi_i^2}{k^2}\left[\frac{k\sinh(t/t_\Lambda)\cosh(kt) - (1/t_\Lambda)\sinh(kt)\cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)}\right]^2. \tag{A.58}$$

We can use Eq. A.44 to express this equation in terms of a, giving us

$$1+w(a) = (1+w_0)a^{-3}\frac{\left[\sqrt{\Omega_{\phi 0}}kt_\Lambda\cosh kt(a) - \sqrt{(1-\Omega_{\phi 0})a^{-3} + \Omega_{\phi 0}}\sinh\left[kt(a)\right]\right]^2}{\left[\sqrt{\Omega_{\phi 0}}kt_\Lambda\cosh\left(kt_0\right) - \sinh\left(kt_0\right)\right]^2}. \tag{A.59}$$

We derive $t(a)$ and t_0 from Eq. A.44

$$t(a) = t_\Lambda\sinh^{-1}\sqrt{\left(\frac{\Omega_{\phi 0}a^3}{1-\Omega_{\phi 0}}\right)}, \tag{A.60}$$

$$t_0 = t_\Lambda \tanh^{-1}\left(\sqrt{\Omega_{\phi 0}}\right). \tag{A.61}$$

Then, defining $K \equiv k t_\Lambda$ and

$$F(a) = \sqrt{1 + (\Omega_{\phi 0}^{-1} - 1)a^{-3}}, \tag{A.62}$$

we get the desired equation

$$1 + w(a) = (1 + w_0)a^{3(K-1)} \frac{[(F(a)+1)^K(K - F(a)) + (F(a)-1)^K(K + F(a))]^2}{[(F(1)+1)^K(K - F(1)) + (F(1)-1)^K(K + F(1))]^2}. \tag{A.63}$$

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