FIBERED UNIVERSAL ALGEBRA FOR FIRST-ORDER LOGICS

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MATHEMATICS

Fibered Universal Algebra for First-order Logics

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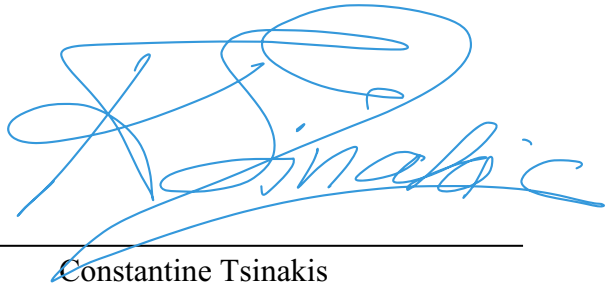
Dissertation under the direction of Professor Constantine Tsinakis

We extend Lawvere-Pitts prop-categories (aka. hyperdoctrines) to develop a general framework for providing "algebraic" semantics for nonclassical first-order logics. This framework includes a natural notion of substitution, which allows first-order logics to be considered as structural closure operators just as propositional logics are in abstract algebraic logic (AAL). In the spirit of AAL, which considers non-standard propositional connectives, we weaken the usual adjointness conditions on the quantifiers to model a variety of non-classical quantifiers. We show many results for the prop-categorical semantics extend to this setting and provide examples of non-standard quantifiers which fit the framework.

We then establish an extension of the homomorphism theorem from universal algebra for generalized prop-categories and characterize two natural closure operators on the prop-categorical semantics. The first closes a class of structures (which are interpreted as morphisms of prop-categories) under the satisfaction of their common first-order theory and the second closes a class of prop-categories under their associated first-order consequence. It turns out, these closure operators have characterizations that closely mirror Birkhoff's characterization of the closure of a class of algebras under the satisfaction of their common equational theory and Blok and Jonsson's characterization of closure under equational consequence, respectively. These "algebraic" characterizations of the first-order closure operators are unique to the prop-categorical semantics. They do not have analogs, for example, in the Tarskian semantics for classical first-order logic. The prop-categories we

consider are much more general than traditional intuitionistic prop-categories or triposes (I.e., topos representing indexed partially ordered sets). Nonetheless, to the best of our knowledge, these results are still new, even when restricted to these special classes of prop-categories.

Approved  Date June 11 2023

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