

DEFORMATIONS OF  $II_1$  FACTORS WITH APPLICATIONS  
TO THEIR STRUCTURAL THEORY

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*to my parents, Tom and Linda*

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## CHAPTER I

### INTRODUCTION

This work explores some aspects of the deformation/rigidity theory of  $\text{II}_1$  factors, initially developed by Sorin Popa in the early part of the previous decade [76]. The primary purpose of deformation/rigidity theory is to provide a theoretical framework for classifying certain subalgebras of the bounded operators on Hilbert space known as von Neumann algebras. The theory of von Neumann algebras was initiated by von Neumann and F. J. Murray in a series of papers written in the late 1930s and early 1940s [47, 48, 49, 50]. Those seminal investigations suggested a surprising diversity among an important subclass of von Neumann algebras known as  $\text{II}_1$  factors. Yet, the culminating achievement of the first four decades' work on the classification of  $\text{II}_1$  factors was the deep result of Connes [14] which demonstrated that an *a priori* large class of  $\text{II}_1$  factors—those which are “injective”—precisely constitutes the isomorphism class of the hyperfinite  $\text{II}_1$  factor  $R$  constructed by Murray and von Neumann.

Until the early 2000s, the classification of  $\text{II}_1$  factors focused on defining global properties of the factor—e.g., property Gamma of Murray and von Neumann [49], the Cowling–Haagerup constant [20], and Connes and Jones' property (T) [17]. The paradigm shift initiated by Popa was to attempt to classify the relative positions of certain subalgebras of a  $\text{II}_1$  factor in order to distinguish among factors which have different *local structure* rather than *global invariants*. The powerful techniques and strategies which allowed Popa to realize this change in perspective form the basis of the deformation/rigidity theory of  $\text{II}_1$  factors. While we will not dwell on any specific achievements, suffice it to say that Popa's theory is responsible for many advances in the theory of  $\text{II}_1$  factors, including several sweeping classification results for uncountable families of  $\text{II}_1$  factors, e.g., [39, 60, 61, 70]. Some of these results have dovetailed with a concurrent renaissance in the orbit equivalence theory of ergodic actions of countable discrete groups [29, 70, 71, 74, 77, 79], for instance, as detailed in the introduction of Chapter II.

## I.1 An outline of the dissertation

Besides the introduction, the dissertation is divided into three chapters, each chapter consisting of a self-contained research article. The article reproduced here as Chapter II was researched and written in collaboration with Jesse Peterson and, except for minor changes, will appear under the same title in a future volume of the journal “Ergodic Theory and Dynamical Systems” [65]. Chapter III reproduces an article that has appeared in the “Journal of Functional Analysis” [92] and is the sole work of the author. Chapter IV is taken from a recent collaboration with Ionut Chifan.

The common theme which unites these articles is a widening of the perspective and techniques on the deformation side of deformation/rigidity. The primary motivation for doing so is to accommodate natural deformations coming from geometry and Lie groups in order to study the structure of group and group-measure space von Neumann algebras. There is also some hope that the ideas and outlook developed here will be useful in building a unified perspective on various rigidity phenomena discovered in  $\text{II}_1$  factors, semi-simple Lie groups, and the theory of orbit equivalence of ergodic actions of countable discrete groups. We briefly describe the contents of each chapter.

The main result Chapter II shows that Popa’s Cocycle Superrigidity Theorem for Bernoulli actions [74, 77] holds for the class of  $L^2$ -rigid groups as defined by Peterson [64]. This provides new examples of superrigid groups as well as a common treatment of the property (T) and product group cases.  $L^2$ -rigidity may be viewed as a strong form of vanishing first  $L^2$ -Betti number, and, though we are not able to extend our techniques to cover this larger class, we are able to show that non-vanishing of the first  $L^2$ -Betti number is an obstruction to Popa superrigidity.

Chapter III extends a structural result, known as *strong solidity*, of Ozawa and Popa on group von Neumann algebras of discrete groups of motions of the hyperbolic plane [61] to groups of motions of  $n$ -dimensional (real or complex) hyperbolic space. The main innovation is the development of a complete theory of relative amenability for Hilbert bimodules which is used to extract more information about the von Neumann algebra from weaker deformations arising from higher-dimensional hyperbolic geometry. These techniques are also applicable in other situations, for instance, to obtain “spectral gap rigidity” for tensor products of non-amenable  $\text{II}_1$  factors or in establishing prime decomposition results.

Chapter IV reworks Ozawa’s theorem on the “solidity” of group von Neumann algebras of

Gromov hyperbolic groups [54] from the perspective of deformation/rigidity theory. Somewhat surprisingly, we are able to do so by working with weak “deformations” which do not preserve the von Neumann-algebraic structure, but remain controllable only on a dense  $C^*$ -subalgebra. As a new result, we are able to demonstrate strong solidity for all i.c.c. Gromov hyperbolic groups and all i.c.c. lattices in connected, simple Lie groups of rank one.

## I.2 Some preliminaries

Before proceeding further, we pause to acquaint the reader with some concepts and terminology for which some familiarity is implicitly assumed in the sequel. Let  $\mathcal{H}$  denote a Hilbert space and  $\mathbb{B}(\mathcal{H})$  the set of bounded operators on  $\mathcal{H}$ . A *von Neumann algebra*  $A \subset \mathbb{B}(\mathcal{H})$  is a self-adjoint subalgebra such that  $1_{\mathbb{B}(\mathcal{H})} \in A$  which is closed in the topology generated by the semi-norms  $\mathbb{B}(\mathcal{H}) \ni T \mapsto |\langle T\xi, \eta \rangle|$  for all  $\xi, \eta \in \mathcal{H}$ , i.e., the weak operator topology. In particular, a von Neumann algebra is closed in the (usual) operator norm topology on  $\mathbb{B}(\mathcal{H})$ . The bicommutant theorem of von Neumann alternately characterizes von Neumann algebras as those self-adjoint subalgebras  $A \subset \mathbb{B}(\mathcal{H})$  such that  $(A')' = A$ , where  $A' = \{T \in \mathbb{B}(\mathcal{H}) : Tx = xT, \forall x \in A\}$  is the commutant of  $A$  in  $\mathbb{B}(\mathcal{H})$ . A functional  $\phi \in A^*$  is said to be *normal* if it is weakly continuous on the unit ball  $(A)_1$  of  $A$ . By Sakai’s theorem [94], a von Neumann algebra  $A$  is a dual Banach space (in the norm topology). The weak\* topology on  $A$  is known as the *ultraweak topology*, and coincides with the weak topology on  $(A)_1$ .

A normal faithful state  $\tau \in A^*$  is said to be a trace if  $\tau(xy) = \tau(yx)$  for all  $x, y \in A$ : we say that a von Neumann algebra is *finite* if it admits a trace. A von Neumann algebra  $A$  is said to be a *factor* if the center of  $A$  consists of scalar multiples of the identity, i.e.,  $A' \cap A = \mathbb{C}1$ . A factor is said to be of type  $\text{II}_1$  if it is infinite-dimensional and possesses a (necessarily unique) trace. Examples of  $\text{II}_1$  factors are the group von Neumann algebra  $L\Gamma$  of any i.c.c.<sup>1</sup> countable discrete group  $\Gamma$  and the group-measure space construction  $L^\infty(X, \mu) \rtimes \Gamma$  of any free, ergodic, measure-preserving action of  $\Gamma$  on a probability space  $(X, \mu)$ : both of these constructions are detailed in section II.2. Let  $M$  be a  $\text{II}_1$  factor with trace  $\tau$ . The Hilbert space  $L^2(M)$  is defined to be the completion of  $\hat{M}$  (“ $\hat{\cdot}$ ” is the forgetful functor from algebras to vector spaces) under the norm induced by the inner product

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<sup>1</sup>A countable discrete group is said to be i.c.c. if every non-identity conjugacy class is infinite



$\langle \hat{x}, \hat{y} \rangle = \tau(y^*x)$  for all  $x, y \in M$ . We often use the convention  $\|x\|_2$  to denote  $\langle \hat{x}, \hat{x} \rangle^{1/2} = \tau(x^*x)^{1/2}$ . Every  $\text{II}_1$  factor  $M$  is canonically realized as a von Neumann subalgebra of  $\mathbb{B}(L^2(M))$  via the map  $x \mapsto L(x)$ , where  $L(x)\hat{y} = \widehat{xy}$ , for all  $x \in M, \hat{y} \in \hat{M}$ .

An *operator space* is a unital, self-adjoint, norm-closed linear subspace  $E \subset \mathbb{B}(\mathcal{H})$ . A  $*$ -linear map  $\varphi : E \rightarrow \mathbb{B}(\mathcal{H})$  is said to be *completely bounded* (c.b.) if  $\|\varphi\|_{\text{cb}} = \sup_{n \in \mathbb{N}} \|\varphi \otimes 1_n\| < \infty$ , where  $\varphi \otimes 1_n : E \otimes \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{B}(\mathcal{H}) \otimes \mathbb{M}_n(\mathbb{C})$ . Such a map is *contractive, completely positive* (c.c.p.) if  $\varphi \otimes 1_n$  is positive, for all  $n \in \mathbb{N}$ , and  $\|\varphi\|_{\text{cb}} = \|\varphi(1)\| \leq 1$ : *unital, completely positive* (u.c.p.) if additionally  $\varphi(1) = 1$ . Some of the most useful u.c.p. maps in the theory of  $\text{II}_1$  factors (or finite von Neumann algebras) are the conditional expectations onto their subalgebras. Let  $M$  be  $\text{II}_1$  factor and  $N \subset M$  be a von Neumann subalgebra. Then there is a unique u.c.p. map  $E_N : M \rightarrow N$  such that  $E_N(x) = x$ , for all  $x \in N$ , which preserves the trace on  $M$ . The map  $E_N$  is known as the conditional expectation from  $M$  to  $N$ .

## CHAPTER II

### ON COCYCLE SUPERRIGIDITY FOR GAUSSIAN ACTIONS

#### II.1 Introduction

A central motivating problem in the theory of measure-preserving actions of countable groups on probability spaces is to classify certain actions up to orbit equivalence, i.e., isomorphism of the underlying probability spaces such that the orbits of one group are carried onto the orbits of the other. When the groups are amenable this problem was completely settled in the early '80s (cf. [25, 26, 51, 16]): all free ergodic actions of countable, discrete, amenable groups are orbit equivalent. The nonamenable case, however, is much more complex and has recently seen a flurry of activity including a number of striking results. We direct the reader to the survey articles [76, 90] for a summary of these recent developments.

One direction which we highlight here is Popa's use of his deformation/rigidity techniques in von Neumann algebras to produce rigidity results for orbit equivalence (cf. [69, 71, 72, 73, 74, 77, 78]). One of the seminal results using these techniques is Popa's Cocycle Superrigidity Theorem [74, 77] (see also [27] and [97] for more on this) which obtains orbit equivalence superrigidity results by means of untwisting cocycles into a finite von Neumann algebra. In order to state this result we recall a few notions regarding groups.

A subgroup  $\Gamma_0 \subset \Gamma$  is *wq-normal* if there exists no intermediate subgroup  $\Gamma_0 \subset K \subsetneq \Gamma$  such that  $\gamma K \gamma^{-1} \cap K$  is finite for all  $\gamma \in \Gamma \setminus K$ . If  $\mathcal{U}$  is a class of Polish groups then a free, ergodic, measure-preserving action of a countable discrete group  $\Gamma$  on a standard probability space  $(X, \mu)$  is said to be  *$\mathcal{U}$ -cocycle superrigid* if any cocycle for the action  $\Gamma \curvearrowright (X, \mu)$  which is valued in a group contained in the class  $\mathcal{U}$  must be cohomologous to a homomorphism.  $\mathcal{U}_{\text{fin}}$  is used to denote the class of Polish groups which arise as closed subgroups of the unitary groups of  $\text{II}_1$  factors. In particular, the class of compact Polish groups and the class of countable discrete groups are both contained in  $\mathcal{U}_{\text{fin}}$ . The notions of wq-normality and the class  $\mathcal{U}_{\text{fin}}$  are due to Popa (cf. [71, 74]).

**Popa’s Cocycle Superrigidity Theorem** ([74], [77]) (for Bernoulli shift actions). Let  $\Gamma$  be a group which contains an infinite wq-normal subgroup  $\Gamma_0$  such that the pair  $(\Gamma, \Gamma_0)$  has relative property (T), or such that  $\Gamma_0$  is the direct product of an infinite group and a nonamenable group, and let  $(X_0, \mu_0)$  be a standard probability space. Then the Bernoulli shift action  $\Gamma \curvearrowright \prod_{g \in \Gamma} (X_0, \mu_0)$  is  $\mathcal{U}_{\text{fin}}$ -cocycle superrigid.

The proof of this theorem uses a combination of deformation/rigidity and intertwining techniques that were initiated in [70]. Roughly, if we are given a cocycle into a unitary group of a  $\text{II}_1$  factor, we may consider the “twisted” group algebra sitting inside of the group-measure space construction. The existence of rigidity can then be contrasted against natural malleable deformations from the Bernoulli shift in order to locate the “twisted” algebra inside of the group-measure space construction. Locating the “twisted” algebra allows us to “untwist” it, and, in so doing, untwist the cocycle in the process.

The existence of such s-malleable deformations (introduced by Popa in [72, 73]) actually occurs in a broader setting than the (generalized) Bernoulli shifts with diffuse core, but it was Furman [27] who first noticed that the even larger class of Gaussian actions are also s-malleable. The class of Gaussian actions has a rich structure, owing to the fact that every Gaussian action of a group  $\Gamma$  arises functorially from an orthogonal representation of  $\Gamma$ . The interplay between the representation theory and the ergodic theory of a group via the Gaussian action has been fruitfully exploited in the literature (cf. the seminal works of Connes and Weiss and of Schmidt, [18, 88, 89], *inter alios*).

In this chapter, we will explore  $\mathcal{U}_{\text{fin}}$ -cocycle superrigidity within the class of Gaussian actions. An advantage to our approach is that we develop a general framework for investigating cocycle superrigidity of such actions by using derivations on von Neumann algebras. The first theme we take up is the relation between the cohomology of group representations and the cohomology of their respective Gaussian actions. Under general assumptions, we show that cohomological information coming from the representation can be faithfully transferred to the cohomology group of the action with coefficients in the circle group  $\mathbb{T}$ . As a consequence, we obtain our first result, that the cohomology of the representation provides an obstruction to the  $\mathcal{U}_{\text{fin}}$ -cocycle superrigidity of the associated Gaussian action.

**Theorem II.1.1.** *Let  $\Gamma$  be a countable discrete group and  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K})$  a weakly mixing orthogonal representation. A necessary condition for the corresponding Gaussian action to be  $\{\mathbb{T}\}$ -cocycle superrigid is that  $H^1(\Gamma, \pi) = \{0\}$ .*

The Bernoulli shift action of a group is precisely the Gaussian action corresponding to the left-regular representation, and the circle group  $\mathbb{T}$  is contained in the class  $\mathcal{U}_{\text{fin}}$ . When combined with Corollary 2.4 in [67] which states that for a nonamenable group vanishing of the first  $\ell^2$ -Betti number is equivalent to  $H^1(\Gamma, \lambda) = \{0\}$  we obtain the following corollary.

**Corollary II.1.2.** *Let  $\Gamma$  be a countable discrete group. If  $\beta_1^{(2)}(\Gamma) \neq 0$  then the Bernoulli shift action is not  $\mathcal{U}_{\text{fin}}$ -cocycle superrigid.*

The second theme explored is the deformation/derivation duality developed by the first author in [64]. The flexibility inherent at the infinitesimal level allows us to offer a unified treatment of Popa's theorem in the case of generalized Bernoulli actions and expand the class of groups whose Bernoulli actions are known to be  $\mathcal{U}_{\text{fin}}$ -cocycle superrigid. As a partial converse to the above results, we have that an *a priori* stronger property than having  $\beta_1^{(2)}(\Gamma) = 0$ ,  $L^2$ -rigidity (see Definition II.2.13), is sufficient to guarantee  $\mathcal{U}_{\text{fin}}$ -cocycle superrigidity of the Bernoulli shift. For this result, and throughout this chapter we denote by  $L\Gamma$  the group von Neumann algebra of  $\Gamma$ , i.e.,  $L\Gamma$  is smallest von Neumann algebra in  $\mathcal{B}(\ell^2\Gamma)$  which contains the image of the left-regular representation  $\lambda : \Gamma \rightarrow \mathcal{U}(\ell^2\Gamma)$ .

**Theorem II.1.3.** *Let  $\Gamma$  be a countable discrete group. If  $L\Gamma$  is  $L^2$ -rigid then the Bernoulli shift action of  $\Gamma$  is  $\mathcal{U}_{\text{fin}}$ -cocycle superrigid.*

Examples of groups for which this holds are groups which contain an infinite normal subgroup which has relative property (T) or is the direct product of an infinite group and a nonamenable group, recovering Popa's Cocycle Superrigidity Theorem for Bernoulli actions of these groups.

We also obtain new groups for which Popa's theorem holds. For example, we show that the theorem holds for any generalized wreath product  $A_0 \wr_X \Gamma_0$ , where  $A_0$  is a non-trivial abelian group and  $\Gamma_0$  does not have the Haagerup property. Also, if  $L\Lambda$  is nonamenable and has property Gamma of Murray and von Neumann [49] then the theorem also holds for  $\Lambda$ .

We remark that it is an open question whether vanishing of the first  $\ell^2$ -Betti number characterizes groups whose Bernoulli actions are  $\mathcal{U}_{\text{fin}}$ -cocycle superrigid. For instance, it is unknown for

the group  $\mathbb{Z} \wr \mathbb{F}_2$ , which contains an infinite normal abelian subgroup and hence has vanishing first  $\ell^2$ -Betti number by [10].

## II.2 Preliminaries

We begin by reviewing the basic notions of Gaussian actions, cohomology of representations and actions, and closable derivations. Though our treatment of the last two topics is standard, our approach to Gaussian actions is somewhat non-standard, where we take a more operator-algebraic approach by viewing the algebra of bounded functions on the probability space as a von Neumann algebra acting on a symmetric Fock space. In the noncommutative setting of free probability this is the same as Voiculescu's approach in [99]. But first, let us recall a few basic definitions and concepts which constitute the basic language in which this chapter is written. Throughout, all Hilbert spaces are assumed to be separable.

**Definition II.2.1.** Let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  be a unitary representation, and denote by  $\pi^{op}$  the associated contragredient representation on the contragredient Hilbert space  $\mathcal{H}^{op}$  of  $\mathcal{H}$ . We say that  $\pi$ :

1. is *ergodic* if  $\pi$  has no non-zero invariant vectors;
2. is *weakly mixing* if  $\pi \otimes \pi^{op}$  is ergodic (equivalently,  $\pi \otimes \rho^{op}$  is ergodic for any unitary  $\Gamma$ -representation  $\rho$ );
3. is *mixing* if  $\langle \pi_\gamma(\xi), \eta \rangle \rightarrow 0$  as  $\gamma \rightarrow \infty$ , for all  $\xi, \eta \in \mathcal{H}$ ;
4. has *spectral gap* if there exists  $K \subset G$ , finite, and  $C > 0$  such that  $\|\xi - P(\xi)\| \leq C \sum_{k \in K} \|\pi_k(\xi) - \xi\|$ , for all  $\xi \in \mathcal{H}$ , where  $P$  is the projection onto the invariant vectors;
5. has *stable spectral gap* if  $\pi \otimes \pi^{op}$  has spectral gap (equivalently,  $\pi \otimes \rho^{op}$  has spectral gap for any unitary  $\Gamma$ -representation  $\rho$ );
6. is *amenable* if  $\pi$  is either not weakly mixing or does not have stable spectral gap.

Note that for an orthogonal representation  $\pi$  of  $\Gamma$  into a real Hilbert space  $\mathcal{K}$ , the associated unitary representation into  $\mathcal{K} \otimes \mathbb{C}$  is canonically isomorphic to its contragredient. Hence, in this

situation we may replace in the above definition “ $\pi \otimes \pi^{op}$ ” and “ $\pi \otimes \rho^{op}$ ” with “ $\pi \otimes \pi$ ” and “ $\pi \otimes \rho$ ”, respectively.

Let  $\Gamma \curvearrowright^\sigma (X, \mu)$  be an action of the countable discrete group  $\Gamma$  by  $\mu$ -preserving automorphisms of a standard probability space  $(X, \mu)$ . This yields a unitary representation  $\pi^\sigma : \Gamma \rightarrow \mathcal{U}(L_0^2(X, \mu))$  called the *Koopman representation* associated to  $\sigma$ . (Here  $L_0^2(X, \mu)$  denotes the orthogonal complement in  $L^2(X, \mu)$  to the subspace of the constant functions on  $X$ .) Note that the Koopman representation is the unitary representation associated to the orthogonal representation of  $\Gamma$  acting on the real-valued  $L^2$ -functions. We say that the action  $\sigma$  is ergodic (or weakly mixing, mixing, etc.) if the Koopman representation  $\pi^\sigma$  is in the sense of the above definition. An action  $\Gamma \curvearrowright^\sigma (X, \mu)$  is (*essentially*) *free* if, for all  $\gamma \in \Gamma$ ,  $\gamma \neq e$ ,  $\mu\{x \in X : \sigma_\gamma(x) = x\} = 0$ .

Given unitary representations  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  and  $\rho : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$ , we say that  $\pi$  is *contained* in  $\rho$  if there is a linear isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that  $\pi_\gamma = V^* \rho_\gamma V$ , for all  $\gamma \in \Gamma$ . We say that  $\pi$  is *weakly contained*  $\rho$  if for any  $\xi \in \mathcal{H}$ ,  $F \subset \Gamma$  finite, and  $\varepsilon > 0$ , there are  $\xi'_1, \dots, \xi'_n \in \mathcal{K}$  such that  $|\langle \pi_\gamma(\xi), \xi \rangle - \sum_{k=1}^n \langle \rho_\gamma(\xi'_k), \xi'_k \rangle| < \varepsilon$ , for all  $\gamma \in F$ . Note that amenability of a representation  $\pi$  is equivalent to  $\pi \otimes \pi^{op}$  weakly containing the trivial representation, which is equivalent with Bekka’s definition by Theorem 5.1 in [2].

Throughout this chapter, we will assume that a finite von Neumann algebra comes with a fixed trace, and by an inclusion of finite von Neumann algebras  $(M, \tau) \subset (\tilde{M}, \tilde{\tau})$  we mean an inclusion  $M \subset \tilde{M}$  such that  $\tilde{\tau}$  is a trace on  $\tilde{M}$  which agrees with  $\tau$  when restricted to  $M$ .

Associated to a measure preserving action  $\Gamma \curvearrowright^\sigma (X, \mu)$  of a countable discrete group  $\Gamma$  on a probability space  $(X, \mu)$  is a finite von Neumann algebra known as the group-measure space construction [47]. Note that  $\Gamma$  acts on  $L^\infty(X, \mu)$  (we will also denote this action by  $\sigma$ ) by the formula  $\sigma_\gamma(f) = f \circ \sigma_{\gamma^{-1}}$ , and since the action of  $\Gamma$  on  $X$  preserves the measure, this action on  $L^\infty(X, \mu)$  preserves the integral.

Consider the Hilbert space  $\mathcal{H} = \ell^2(\Gamma, L^2(X, \mu)) = \{\sum_{\gamma \in \Gamma} a_\gamma u_\gamma \mid a \in L^2(X, \mu), \sum_{\gamma \in \Gamma} \|a_\gamma\|_2^2 < \infty\}$ . On this Hilbert space we define a convolution operation by

$$(\sum_{\gamma \in \Gamma} a_\gamma u_\gamma) \cdot (\sum_{\lambda \in \Gamma} b_\lambda u_\lambda) = \sum_{\gamma, \lambda \in \Gamma} a_\gamma \sigma_\gamma(b_\lambda) u_{\gamma\lambda} \in \ell^1(\Gamma, L^1(X, \mu)).$$

If  $x \in \mathcal{H}$  such that  $x \cdot \eta \in \mathcal{H}$  for all  $\eta \in \mathcal{H}$  then by the Closed Graph Theorem, convolution by  $x$  describes a bounded operator on  $\mathcal{H}$ . We may then consider  $L^\infty(X, \mu) \rtimes \Gamma = \{x \in \mathcal{H} \mid x \cdot \eta \in \mathcal{H} \text{ for all } \eta \in \mathcal{H}\} \subset \mathcal{B}(\mathcal{H})$ .  $L^\infty(X, \mu) \rtimes \Gamma$  is a finite von Neumann algebra which contains  $L^\infty(X, \mu)$  as a von Neumann subalgebra and has a faithful normal tracial state given by

$$\tau(\sum_{\gamma \in \Gamma} a_\gamma u_\gamma) = \int a_e d\mu.$$

If  $(X, \mu)$  is a one-point probability space then the above construction gives rise to the group von Neumann algebra, which we will denote by  $L\Gamma$ . Note that in general, we always have  $L\Gamma \subset L^\infty(X, \mu) \rtimes \Gamma$  by considering the sums above for which  $a_\gamma$  is constant, for all  $\gamma \in \Gamma$ .

The connection between the group-measure space construction and orbit equivalence is due to Singer who showed in [93] that two free measure preserving actions  $\Gamma \curvearrowright (X, \mu)$  and  $\Lambda \curvearrowright (Y, \nu)$  are orbit equivalent if and only if there is an  $*$ -isomorphism  $\theta : L^\infty(X, \mu) \rtimes \Gamma \rightarrow L^\infty(Y, \nu) \rtimes \Lambda$  such that  $\theta(L^\infty(X, \mu)) = L^\infty(Y, \nu)$ .

The “representation theory” of a finite von Neumann algebra is captured in the structure of its bimodules, also called correspondences (cf. [5, 68]). The theory of correspondences of von Neumann algebras was first developed by Connes [15].

**Definition II.2.2.** Let  $(M, \tau)$  be a finite von Neumann algebra. An  $M$ - $M$  Hilbert bimodule is a Hilbert space  $\mathcal{H}$  equipped with a representation  $\pi : M \otimes_{\text{alg}} M^{op} \rightarrow B(\mathcal{H})$  which is normal when restricted to  $M$  and  $M^{op}$ . We write  $\pi(x \otimes y^{op})\xi$  as  $x\xi y$ .

An  $M$ - $M$  Hilbert bimodule  $\mathcal{H}$  is *contained* in an  $M$ - $M$  Hilbert bimodule  $\mathcal{K}$  if there is a linear isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that  $V(x\xi y) = xV(\xi)y$ , for all  $\xi \in \mathcal{H}$ ,  $x, y \in M$ ;  $\mathcal{H}$  is *weakly contained* in  $\mathcal{K}$  if for any  $\xi \in \mathcal{H}$ ,  $F \subset M$  finite, and  $\varepsilon > 0$ , there exist  $\xi'_1, \dots, \xi'_n \in \mathcal{K}$  such that  $|\langle x\xi y, \xi \rangle - \sum_{k=1}^n \langle x\xi'_k y, \xi'_k \rangle| < \varepsilon$ , for all  $x, y \in F$ . The *trivial bimodule* is the space  $L^2(M, \tau)$  with the bimodule structure induced by left and right multiplication; the *coarse bimodule* is the space  $L^2(M, \tau) \otimes L^2(M, \tau)$  with bimodule structure induced by left multiplication on the first factor and right multiplication on the second. The trivial and coarse bimodules play analogous roles in the theory of  $M$ - $M$  Hilbert bimodules to the roles played, respectively, by the trivial and left-regular representations in the theory of unitary representations of locally compact groups. Note that an

$M$ - $M$  correspondence  $\mathcal{H}$  contains the trivial correspondence if and only if  $\mathcal{H}$  has non-zero  $M$ -central vectors (a vector  $\xi$  is  $M$ -central if  $x\xi = \xi x$ , for all  $x \in M$ ).

Given  $\xi, \eta \in \mathcal{H}$ , note the maps  $M \ni x \mapsto \langle x\xi, \eta \rangle, \langle \xi x, \eta \rangle$  are normal linear functionals on  $M$ . A vector  $\xi \in \mathcal{H}$  is called *left (respectively, right) bounded* if there exists  $C > 0$  such that for every  $x \in M$ ,  $\|x\xi\| \leq C\|x\|_2$ , (resp.,  $\|\xi x\| \leq C\|x\|_2$ ). The set of vectors which are both left and right-bounded forms a dense subspace of  $\mathcal{H}$ . By [68], to  $\xi$ , a left-bounded vector, we can associate a completely-positive map  $\phi_\xi : M \rightarrow M$  such that for all  $x, y \in M$ ,  $\|x\xi y\| = \tau(x^*x\phi_\xi(yy^*))^{1/2}$ . If  $\xi$  is also right-bounded then this map is seen to naturally extend to a bounded operator  $\hat{\phi}_\xi : L^2(M, \tau) \rightarrow L^2(M, \tau)$ .

Given two  $M$ - $M$  Hilbert bimodules  $\mathcal{H}$  and  $\mathcal{K}$ , there is a well-defined tensor product  $\mathcal{H} \otimes_M \mathcal{K}$  in the category of  $M$ - $M$  Hilbert bimodules: see [68] for details.

**Definition II.2.3** (Compare with Definition II.2.1.). An  $M$ - $M$  Hilbert bimodule is said to:

1. be *weakly mixing* if  $\mathcal{H} \otimes_M \mathcal{H}^{op}$  does not contain the trivial  $M$ - $M$  Hilbert bimodule;
2. be *mixing* if for every sequence  $u_i \in \mathcal{U}(M)$  such that  $u_i \rightarrow 0$ , weakly, we have that

$$\lim_{i \rightarrow \infty} \sup_{\|x\| \leq 1} \langle u_i \xi x, \eta \rangle = \lim_{i \rightarrow \infty} \sup_{\|x\| \leq 1} \langle x \xi u_i, \eta \rangle = 0,$$

for all  $\xi, \eta \in \mathcal{H}$ ;

3. have *spectral gap* if there exist  $x_1, \dots, x_n \in M$  such that  $\|\xi - P(\xi)\| \leq \sum_{i=1}^n \|x_i \xi - \xi x_i\|$ , for all  $\xi \in \mathcal{H}$ , where  $P$  is the projection onto the central vectors;
4. have *stable spectral gap* if  $\mathcal{H} \otimes_M \mathcal{H}^{op}$  has spectral gap;
5. be *amenable* if it is either not weakly mixing or does not have stable spectral gap.

If  $\mathcal{H}$  is a mixing  $M$ -correspondence and  $\mathcal{K}$  an arbitrary  $M$ -correspondence, then  $\mathcal{H} \otimes_M \mathcal{K}$  (and also  $\mathcal{K} \otimes_M \mathcal{H}$ ) is mixing, since  $\hat{\phi}_{\xi \otimes_M \eta} = \hat{\phi}_\eta \circ \hat{\phi}_\xi$  if  $\xi$  and  $\eta$  are both left and right-bounded.

Let  $\mathcal{H}$  and  $\mathcal{K}$  be  $M$ - $M$  correspondences, and denote by  $\mathcal{H}_0$  and  $\mathcal{K}_0$  the set of right-bounded vectors in  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. For  $\xi, \eta \in \mathcal{H}_0$ , denote by  $(\xi|\eta)$  the element of  $M$  such that  $\langle \xi x, \eta y \rangle = \tau(y^*(\xi|\eta)x)$ , for all  $x, y \in M$  (by normality of the map  $z \mapsto \langle \xi z, \eta \rangle$ , there exists such a



$(\xi|\eta) \in L^1(M, \tau)$ ; right-boundedness of  $\xi$  and  $\eta$  implies  $(\xi|\eta) \in M$ . It is clear that  $(\cdot|\cdot)$  is a bilinear map  $\mathcal{H}_0 \times \mathcal{H}_0 \rightarrow M$  such that  $(\xi|\xi) \geq 0$  and  $(\xi|\xi) = 0$  if and only if  $\xi = 0$ , for all  $\xi \in \mathcal{H}_0$ . For  $\xi \in \mathcal{H}_0$  and  $\eta \in \mathcal{K}_0$ , define the linear map  $T_{\xi, \eta} : \mathcal{H}_0 \rightarrow \mathcal{K}_0^{op}$  by  $T_{\xi, \eta}(\cdot) = (\cdot|\xi)\eta^{op}$ . It is not hard to check that  $T_{\xi, \eta}$  is a bounded with  $\|T_{\xi, \eta}\| \leq \|(\xi|\xi)\| \|(\eta|\eta)\|$ ; hence,  $T_{\xi, \eta}$  extends to a bounded operator  $\mathcal{H} \rightarrow \mathcal{K}^{op}$ . Let  $\mathcal{L}_M^2(\mathcal{H}, \mathcal{K})$  be the subspace of  $B(\mathcal{H}, \mathcal{K}^{op})$  which is the closed span of all operators of the form  $T_{\xi, \eta}$  under the Hilbert norm  $\|T_{\xi, \eta}\|_{\mathcal{L}_M^2} = \tau((\xi|\xi)(\eta|\eta))^{1/2}$ . Moreover,  $\mathcal{L}_M^2(\mathcal{H}, \mathcal{K})$  is equipped with a natural  $M$ - $M$  Hilbert bimodule structure given by  $(x \otimes y^{op})(T_{\xi, \eta}) = T_{x\xi, y\eta}$  identifying it with  $\mathcal{H} \otimes_M \mathcal{K}^{op}$ . Note that if  $T \in \mathcal{L}_M^2(\mathcal{H}, \mathcal{K})$ , then  $(T^*T)^{1/2} \in \mathcal{L}_M^2(\mathcal{H}, \mathcal{H})$ .

**Proposition II.2.4.** An  $M$ - $M$  correspondence  $\mathcal{H}$  is weakly mixing if and only if for any  $M$ - $M$  correspondence  $\mathcal{K}$ ,  $\mathcal{H} \otimes_M \mathcal{K}^{op}$  does not contain the trivial correspondence.

*Proof.* The reverse implication is trivial. Conversely, suppose there exists  $\mathcal{K}$  such that  $\mathcal{H} \otimes \mathcal{K}^{op}$  contains an  $M$ -central vector. Identifying  $\mathcal{H} \otimes_M \mathcal{K}^{op}$  with  $\mathcal{L}_M^2(\mathcal{H}, \mathcal{K})$ , let  $T \in \mathcal{L}_M^2(\mathcal{H}, \mathcal{K})$  be an  $M$ -central vector. Then  $(T^*T)^{1/2} \in \mathcal{L}_M^2(\mathcal{H}, \mathcal{H})$  is an  $M$ -central vector; hence,  $\mathcal{H}$  is not weakly mixing.  $\square$

## II.2.1 Gaussian actions

Let  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$  be an orthogonal representation of a countable discrete group  $\Gamma$ . The aim of this section is to describe the construction of a measure-preserving action of  $\Gamma$  on a non-atomic standard probability space  $(X, \mu)$  such that  $\mathcal{H}$  is realized as a subspace of  $L_{\mathbb{R}}^2(X, \mu)$  and  $\pi$  is contained in the Koopman representation  $\Gamma \curvearrowright L_0^2(X, \mu)$ . The action  $\Gamma \curvearrowright (X, \mu)$  is referred to as the *Gaussian action* associated to  $\pi$ . We give an operator-algebraic alternative construction of the Gaussian action similar to Voiculescu's construction of free semi-circular random variables.

Given a real Hilbert space  $\mathcal{H}$ , the  $n$ -symmetric tensor  $\mathcal{H}^{\odot n}$  is the subspace of  $\mathcal{H}^{\otimes n}$  fixed by the action of the symmetric group  $S_n$  by permuting the indices. For  $\xi_1, \dots, \xi_n \in \mathcal{H}$ , we define their symmetric tensor product  $\xi_1 \odot \dots \odot \xi_n \in \mathcal{H}^{\odot n}$  to be  $\frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \dots \otimes \xi_{\sigma(n)}$ . Denote

$$\mathfrak{S}(\mathcal{H}) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} (\mathcal{H} \otimes \mathbb{C})^{\odot n},$$

with  $\Omega$  the vacuum vector and having renormalized inner product such that  $\|\xi\|_{\mathfrak{S}(\mathcal{H})}^2 = n!\|\xi\|^2$ , for  $\xi \in \mathcal{H}^{\odot n}$ .

For  $\xi \in \mathcal{H}$  let  $x_\xi$  be the *symmetric creation operator*,

$$x_\xi(\Omega) = \xi, \quad x_\xi(\eta_1 \odot \cdots \odot \eta_k) = \xi \odot \eta_1 \odot \cdots \odot \eta_k,$$

and its adjoint,  $\frac{\partial}{\partial \xi} = (x_\xi)^*$

$$\frac{\partial}{\partial \xi}(\Omega) = 0, \quad \frac{\partial}{\partial \xi}(\eta_1 \odot \cdots \odot \eta_k) = \sum_{i=1}^k \langle \xi, \eta_i \rangle \eta_1 \odot \cdots \odot \widehat{\eta}_i \odot \cdots \odot \eta_k.$$

Let

$$s(\xi) = \frac{1}{2}(x_\xi + \frac{\partial}{\partial \xi}),$$

and note that it is an unbounded, self-adjoint operator on  $\mathfrak{S}(\mathcal{H})$ .

The *moment generating function*  $M(t)$  for the Gaussian distribution is defined to be

$$M(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(tx) \exp(-x^2/2) dx = \exp(t^2/2).$$

It is easy to check that if  $\|\xi\| = 1$  then

$$\langle s(\xi)^n \Omega, \Omega \rangle = M^{(n)}(0) = \frac{(2k)!}{2^k k!},$$

if  $n = 2k$  and 0 if  $n$  is odd. Hence,  $s(\xi)$  may be regarded as a Gaussian random variable.

Note that if  $\xi, \eta \in \mathcal{H}$  then  $s(\xi)$  and  $s(\eta)$  commute, moreover, if  $\xi \perp \eta$ , then  $\langle s(\xi)^m s(\eta)^n \Omega, \Omega \rangle = \langle s(\xi)^m \Omega, \Omega \rangle \langle s(\eta)^n \Omega, \Omega \rangle$ , for all  $m, n \in \mathbb{N}$ ; thus,  $s(\xi)$  and  $s(\eta)$  are independent random variables.

From now on we will use the convention  $\xi_1 \xi_2 \dots \xi_k$  to denote the symmetric tensor  $\xi_1 \odot \xi_2 \odot \cdots \odot \xi_k$ .

Let  $\Xi$  be a basis for  $\mathcal{H}$  and

$$\mathcal{S}(\Xi) = \{\Omega\} \cup \{s(\xi_1)s(\xi_2) \dots s(\xi_k)\Omega : \xi_1, \xi_2, \dots, \xi_k \in \Xi\}.$$

**Lemma II.2.5.** *The set  $\mathcal{S}(\Xi)$  is a (non-orthonormal) basis of  $\mathfrak{S}(\mathcal{H})$ .*

*Proof.* We will show that  $\xi_1 \dots \xi_k \in \text{span}(\mathcal{S}(\Xi))$ , for all  $\xi_1, \dots, \xi_k \in \mathcal{H}$ . We have  $\Omega \in \text{span}(\mathcal{S}(\Xi))$ . Also, since  $s(\xi)\Omega = \xi$ ,  $\mathcal{H} \subset \text{span}(\mathcal{S}(\Xi))$ . Now as  $s(\xi_1) \dots s(\xi_k)\Omega = P(\xi_1, \dots, \xi_k)$  is a polynomial in  $\xi_1, \dots, \xi_k$  of degree  $k$  with top term  $\xi_1 \dots \xi_k$ , the result follows by induction on  $k$ .  $\square$

Let  $u(\xi_1, \dots, \xi_k) = \exp(\pi i s(\xi_1) \dots s(\xi_k))$  and  $u(\xi_1, \dots, \xi_k)^t = \exp(\pi i t s(\xi_1) \dots s(\xi_k))$ . Denote by  $A$  the von Neumann algebra generated by all such  $u(\xi_1, \dots, \xi_k)$ , which is the same as the von Neumann algebra generated by the spectral projections of the unbounded operators  $s(\xi_1) \dots s(\xi_k)$ .

**Theorem II.2.6.** *We have that  $L^2(A, \tau) \cong \mathfrak{S}(\mathcal{H})$ , and  $A$  is a maximal abelian  $*$ -subalgebra of  $B(\mathfrak{S}(\mathcal{H}))$  with faithful trace  $\tau = \langle \cdot, \Omega \rangle$ . In particular,  $A$  is a diffuse abelian von Neumann algebra.*

*Proof.* By Lemma II.2.5,  $A \mapsto A\Omega$  is an embedding of  $A$  into  $\mathfrak{S}(\mathcal{H})$ . By Stone's Theorem

$$\lim_{t \rightarrow 0} \frac{u(\xi_1, \dots, \xi_k)^t - 1}{\pi i t} \Omega = s(\xi_1) \dots s(\xi_k) \Omega;$$

hence,  $A\Omega$  is dense in  $\mathfrak{S}(\mathcal{H})$ . This implies that  $A$  is maximal abelian in  $B(\mathfrak{S}(\mathcal{H}))$ .  $\square$

There is a natural strongly continuous embedding  $\mathcal{O}(\mathcal{H}) \hookrightarrow \mathcal{U}(\mathfrak{S}(\mathcal{H}))$  given by

$$T \mapsto T^{\mathfrak{S}} = 1 \oplus \bigoplus_{n=1}^{\infty} T^{\otimes n}.$$

It follows that there is an embedding  $\mathcal{O}(\mathcal{H}) \rightarrow \text{Aut}(A, \tau)$ ,  $T \mapsto \sigma_T$ , which can be identified on the unitaries  $u(\xi_1, \dots, \xi_k)$  by

$$\sigma_T(u(\xi_1, \dots, \xi_k)) = \text{Ad}(T^{\mathfrak{S}})(u(\xi_1, \dots, \xi_k)) = u(T(\xi_1), \dots, T(\xi_k)).$$

Thus for an orthogonal representation  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$ , there is a natural action  $\sigma^\pi : \Gamma \rightarrow \text{Aut}(A, \tau)$  given by  $\sigma_\gamma^\pi(u(\xi_1, \dots, \xi_k)) = u(\pi_\gamma(\xi_1), \dots, \pi_\gamma(\xi_k)) = \text{Ad}(\pi_\gamma^{\mathfrak{S}})(u(\xi_1, \dots, \xi_k))$ . The action  $\sigma^\pi$  is the *Gaussian action* associated to  $\pi$ .

We have that ergodic properties which remain stable with respect to tensor products transfer from  $\pi$  to  $\sigma^\pi$ .

**Proposition II.2.7.** In particular, for a subgroup  $H \leq \Gamma$ ,  $\sigma^\pi|_H$  possesses any of the following properties if and only if  $\pi|_H$  does:

1. weak mixing;
2. mixing;
3. stable spectral gap;
4. being contained in a direct sum of copies of the left-regular representation;
5. being weakly contained in the left-regular representation.

For Gaussian actions, stable properties are equivalent to their “non-stable” counterparts. The following proposition serves as a prototype of such a result, showing that ergodicity implies stable ergodicity, i.e., weak mixing.

**Theorem II.2.8.**  $\Gamma \curvearrowright^{\sigma^\pi} (A, \tau)$  is ergodic if and only if  $\pi$  is weakly mixing.

*Proof.* The reverse implication follows from Proposition II.2.7. Conversely, suppose there exists  $\xi \in \mathcal{H}^{\otimes 2}$  such that for all  $\gamma \in \Gamma$ ,  $\pi_\gamma^2(\xi) = \xi$ . Viewing  $\xi$  as a Hilbert-Schmidt operator on  $\mathcal{H}$ , let  $|\xi| = (\xi\xi^*)^{1/2}$ . Since the map  $\xi \otimes \eta \mapsto \eta \otimes \xi$  is the same as taking the adjoint of the corresponding Hilbert-Schmidt operator, we have that  $|\xi| \in \mathcal{H}^{\odot 2}$  and  $\pi_\gamma(|\xi|) = |\xi|$ . By functional calculus, there exists  $\lambda > 0$ , such that  $\eta = E_\lambda(|\xi|) \neq 0$  is a finite rank operator. Hence,  $\eta = \eta_{11} \odot \eta_{12} + \cdots + \eta_{m1} \odot \eta_{m2} \in \mathcal{H}^{\odot 2}$  with  $\eta_{i1} \odot \eta_{i2} \perp \eta_{j1} \odot \eta_{j2}$  for  $i \neq j$ . But then  $u = \prod_{i=1}^n u(\eta_{i1}, \eta_{i2}) \in A$ , a non-trivial unitary and  $\sigma_\gamma^\pi(u) = u$ . Hence,  $\sigma^\pi$  is not ergodic.  $\square$

## II.2.2 Cocycles from representations and from actions

Let  $\mathcal{K}$  be a real Hilbert space and  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K})$  an orthogonal representation of a countable discrete group  $\Gamma$ .

**Definition II.2.9.** A 1-cocycle is a map  $b : \Gamma \rightarrow \mathcal{K}$  satisfying the cocycle identity  $b(\gamma_1\gamma_2) = \pi_{\gamma_1}b(\gamma_2) + b(\gamma_1)$ , for all  $\gamma_1, \gamma_2 \in \Gamma$ . A 1-cocycle is a coboundary if there exists  $\eta \in \mathcal{K}$  such that  $b(\gamma) = \pi_\gamma\eta - \eta$ , for all  $\gamma \in \Gamma$ .

It is a well-known fact (cf. [3]) that a 1-cocycle  $b$  is a coboundary if and only if  $\sup_{\gamma \in \Gamma} \|b(\gamma)\| < \infty$ . Let  $Z^1(\Gamma, \pi)$  and  $B^1(\Gamma, \pi)$  denote, respectively, the vector space of all 1-cocycles and the subspace of coboundaries. The *first cohomology space*  $H^1(\Gamma, \pi)$  of the representation  $\pi$  is then defined to be  $Z^1(\Gamma, \pi)/B^1(\Gamma, \pi)$ .

Let  $\Gamma \curvearrowright^\sigma (X, \mu)$  be an ergodic, measure-preserving action on a standard probability space  $(X, \mu)$ , and let  $\mathbb{A}$  be a Polish topological group.

**Definition II.2.10.** A *1-cocycle* is a measurable map  $c : \Gamma \times X \rightarrow \mathbb{A}$  satisfying the cocycle identity  $c(\gamma_1\gamma_2, x) = c(\gamma_1, \sigma_{\gamma_2}(x))c(\gamma_2, x)$ , for all  $\gamma_1, \gamma_2 \in \Gamma$ , a.e.  $x \in X$ . A pair of 1-cocycles  $c_1, c_2$  are *cohomologous* (written  $c_1 \sim c_2$ ) if there exists a measurable map  $\xi : X \rightarrow \mathbb{A}$  such that  $\xi(\sigma_\gamma(x))c_1(\gamma, x)\xi(x)^{-1} = c_2(\gamma, x)$  for all  $\gamma \in \Gamma$ , a.e.  $x \in X$ . A 1-cocycle is a *coboundary* if it is cohomologous to the cocycle which is identically 1.

Let  $Z^1(\Gamma, \sigma, \mathbb{A})$  and  $B^1(\Gamma, \sigma, \mathbb{A})$  denote, respectively, the space of all 1-cocycles and the subspace of coboundaries. The *first cohomology space*  $H^1(\Gamma, \sigma, \mathbb{A})$  of the action  $\sigma$  is defined to be  $Z^1(\Gamma, \sigma, \mathbb{A})/\sim$ . Note that if  $\mathbb{A}$  is abelian,  $Z^1(\Gamma, \sigma, \mathbb{A})$  is endowed with a natural abelian group structure and  $H^1(\Gamma, \sigma, \mathbb{A}) = Z^1(\Gamma, \sigma, \mathbb{A})/B^1(\Gamma, \sigma, \mathbb{A})$ . To any homomorphism  $\rho : \Gamma \rightarrow \mathbb{A}$  we can associate a cocycle  $\tilde{\rho}$  by  $\tilde{\rho}(\gamma, x) = \rho(\gamma)$ . Using terminology developed by Popa (cf. [74]), a 1-cocycle  $c$  is said to *untwist* if there exists a homomorphism  $\rho : \Gamma \rightarrow \mathbb{A}$  such that  $c$  is cohomologous to  $\tilde{\rho}$ . To any 1-cocycle  $c \in Z^1(\Gamma, \sigma, \mathbb{A})$ , we can associated two 1-cocycles  $c_\ell, c_r \in Z^1(\Gamma, \sigma \times \sigma, \mathbb{A})$  given by  $c_\ell(\gamma, (x, y)) = c(\gamma, x)$  and  $c_r(\gamma, (x, y)) = c(\gamma, y)$ . It is easy to check that  $c$  untwists only if  $c_\ell$  is cohomologous to  $c_r$ ; if  $\sigma$  is weakly mixing, Theorem 3.1 in [74] establishes the converse.

N.B. For brevity, we will drop the “1” when discussing 1-cocycles of representations or actions.

The pertinence of the 1-cohomology of group actions to ergodic theory is that it provides a natural – and rather powerful – technical framework for the orbit equivalence theory of free ergodic actions of countable discrete groups. We give a brief account of this connection: details may found in, for instance, [105].

Consider two free, ergodic, measure-preserving actions  $\Gamma \curvearrowright^\sigma (X, \mu)$  and  $\Lambda \curvearrowright^\rho (Y, \nu)$  of count-

able discrete groups  $\Gamma$  and  $\Lambda$  on respective standard probability spaces  $(X, \mu)$  and  $(Y, \nu)$ .

**Definition II.2.11.** The actions  $\Gamma \curvearrowright^\sigma (X, \mu)$  and  $\Lambda \curvearrowright^\rho (Y, \nu)$  are *orbit equivalent* if there exists a measurable isomorphism  $\Phi : X \rightarrow Y$  such that  $\Phi(\Gamma x) = \Lambda \Phi(x)$  for a.e.  $x \in X$ . The actions are *conjugate* if there exists an isomorphism of groups  $\phi : \Gamma \rightarrow \Lambda$  and a measurable isomorphism  $\Phi : X \rightarrow Y$  such that  $\Phi(\gamma x) = \phi(\gamma)\Phi(x)$  for all  $\gamma \in \Gamma$ , a.e.  $x \in X$ .

It is clear that orbit equivalence is weaker than conjugacy; in fact, strictly weaker by the classification of amenable group actions mentioned in the introduction. Given an orbit equivalence  $\Phi$  from  $\Gamma \curvearrowright (X, \mu)$  to  $\Lambda \curvearrowright (Y, \nu)$ , we would like to describe how far  $\Phi$  departs from implementing a conjugacy. Since the actions are free, for almost every  $x \in X$ , for every  $\gamma \in \Gamma$  there exists a unique  $\lambda \in \Lambda$  such that  $\Phi(\gamma x) = \lambda\Phi(x)$ . One can easily verify that the map  $c : \Gamma \times X \rightarrow \Lambda$  which selects the  $\lambda$  for the corresponding pair  $(\gamma, x)$  is almost everywhere well-defined and measurable. From the fact the  $\Phi$  preserves orbits, it follows that  $c$  is a cocycle, the *Zimmer cocycle*, associated to  $\Phi$ . It is a straightforward exercise to check that the Zimmer cocycle  $c$  will untwist if and only if  $\Phi$  is implemented by a conjugacy; i.e., one can find a isomorphism of groups  $\psi : \Gamma \rightarrow \Lambda$  and a measurable isomorphism  $\Psi : X \rightarrow Y$  such that  $\Phi(\gamma x) = \psi(\gamma)\Psi(x)$  for all  $\gamma \in G$ , a.e.  $x \in X$ , cf. [105].

This strategy of conceptualizing orbit equivalence theory in the broader context of cohomology is particularly useful when one wants to show some flavor of orbit equivalence *rigidity* holds for an action  $\Gamma \curvearrowright^\sigma (X, \mu)$ ; that is, given some nice class of group actions  $\mathcal{L}$ , of which, say,  $\Lambda \curvearrowright (Y, \nu)$  is a representative, any orbit equivalence between  $\Gamma \curvearrowright (X, \mu)$  and  $\Lambda \curvearrowright (Y, \nu)$  is implemented by a conjugacy. To do so, it is sufficient (and often much simpler) to demonstrate that the action is *superrigid* – that, it is merely the target *group*  $\Lambda$ , and not the *action*  $\Lambda \curvearrowright (Y, \nu)$  which is relevant – which, in practice, amounts to showing that every cocycle  $c \in Z^1(\Gamma, \sigma, \Lambda)$  untwists.

### II.2.3 Closable derivations

We review here briefly some general properties of closable derivations on a finite von Neumann algebra and set up some notation to be used in the sequel. For a more detailed discussion see [24], [63], [64], or [61].

**Definition II.2.12.** Let  $(N, \tau)$  be a finite von Neumann algebra and  $\mathcal{H}$  be an  $N$ - $N$  correspondence.

A *derivation*  $\delta$  is an unbounded operator  $\delta : L^2(N, \tau) \rightarrow \mathcal{H}$  such that the domain of  $\delta$ ,  $D(\delta)$ , is a  $\|\cdot\|_2$ -dense  $*$ -subalgebra of  $N$ , and  $\delta(xy) = x\delta(y) + \delta(x)y$ , for each  $x, y \in D(\delta)$ . A derivation is *closable* if it is closable as an operator and *real* if  $\mathcal{H}$  has an antilinear involution  $\mathcal{J}$  such that  $\mathcal{J}(x\xi y) = y^*\mathcal{J}(\xi)x^*$ , and  $\mathcal{J}(\delta(z)) = \delta(z^*)$ , for each  $x, y \in N$ ,  $\xi \in \mathcal{H}$ ,  $z \in D(\delta)$ .

If  $\delta$  is a closable derivation then by [24]  $D(\bar{\delta}) \cap N$  is again a  $*$ -subalgebra and  $\delta|_{D(\bar{\delta}) \cap N}$  is again a derivation. We will thus use the slight abuse of notation by saying that  $\bar{\delta}$  is a closed derivation.

To every closed real derivation  $\delta : N \rightarrow \mathcal{H}$ , we can associate a semigroup deformation  $\Phi^t = \exp(-t\delta^*\bar{\delta})$ ,  $t > 0$ , and a resolvent deformation  $\zeta_\alpha = (\alpha/(\alpha + \delta^*\bar{\delta}))^{1/2}$ ,  $\alpha > 0$ . Both of these deformations are of unital, symmetric, completely-positive maps; moreover, the derivation  $\delta$  can be recovered from these deformations [86, 87].

We also have that the deformation  $\Phi^t$  converges uniformly on  $(N)_1$  as  $t \rightarrow 0$  if and only if the deformation  $\zeta_\alpha$  converges uniformly on  $(N)_1$  as  $\alpha \rightarrow \infty$ .

**Definition II.2.13** (Definition 4.1 in [64]). Let  $(N, \tau)$  be a finite von Neumann algebra.  $N$  is  *$L^2$ -rigid* if given any inclusion  $(N, \tau) \subset (M, \tilde{\tau})$ , and any closable real derivation  $\delta : M \rightarrow \mathcal{H}$  such that  $\mathcal{H}$  when viewed as an  $N$ - $N$  correspondence embeds in  $(L^2N \bar{\otimes} L^2N)^{\oplus \infty}$ , we then have that the associated deformation  $\zeta_\alpha$  converges uniformly to the identity in  $\|\cdot\|_2$  on the unit ball of  $N$ .

We point out here that our definition above is formally stronger than the one given in [64]. Specifically, there it was assumed that  $\mathcal{H}$  embedded into the coarse bimodule as an  $M$ - $M$  bimodule rather than an  $N$ - $N$  bimodule. However, this extra condition was not used in [64], and since the above definition has better stability properties (see Theorem II.6.3) we have chosen to use the same terminology.

Examples of nonamenable groups which do not give rise to  $L^2$ -rigid group von Neumann algebras are groups such that the first  $\ell^2$ -Betti number is positive. These are, in fact, the only known examples, and  $L^2$ -rigidity should be viewed as a von Neumann analog of vanishing first  $\ell^2$ -Betti number.

Showing that a group von Neumann algebra is  $L^2$ -rigid can be quite difficult in general since one has to consider derivations which may not be defined on the group algebra. Nonetheless, there are certain situations where this can be verified.

**Theorem II.2.14** (Corollary 4.6 in [64]). *Let  $\Gamma$  be a nonamenable countable discrete group. If  $L\Gamma$  is weakly rigid, non-prime, or has property Gamma of Murray and von Neumann, then  $L\Gamma$  is  $L^2$ -rigid.*

We give another class of examples below (see also [60], [61], or [66]). The gap between group von Neumann algebras which are known to be  $L^2$ -rigid and groups with vanishing first  $\ell^2$ -Betti number is, however, quite large. For example, as we mentioned in the introduction, the wreath product  $\mathbb{Z} \wr \mathbb{F}_2$  is a group which has vanishing first  $\ell^2$ -Betti number but for which it is not known whether the group von Neumann algebra is  $L^2$ -rigid.

### II.3 Deformations

In this section and Section II.5 we will discuss the interplay between one-parameter groups of automorphisms or, more generally, semigroups of completely positive maps of finite factors (deformations) and their infinitesimal generators (derivations). The motivation for studying deformations at the infinitesimal level is that it allows for the creation of other related deformations of the algebra. And while Popa's deformation/rigidity machinery requires uniform convergence of the original deformation on some target subalgebra, it is often more feasible to demonstrate uniform convergence of a related deformation, then transfer those estimates back to the original.

We begin by recalling Popa's notion of an  $s$ -malleable deformation, and give some examples of such deformations that have appeared in the literature.

**Definition II.3.1** (Definition 4.3 in [74]). Let  $(M, \tau)$  be a finite von Neumann algebra such that  $(M, \tau) \subset (\tilde{M}, \tilde{\tau})$ , where  $(\tilde{M}, \tilde{\tau})$  is another finite von Neumann algebra. A pair  $(\alpha, \beta)$ , consisting of a point-wise strongly continuous one-parameter family  $\alpha : \mathbb{R} \rightarrow \text{Aut}(\tilde{M}, \tilde{\tau})$  and an involution  $\beta \in \text{Aut}(\tilde{M}, \tilde{\tau})$  is called an  *$s$ -malleable deformation* of  $M$  if:

1.  $M \subset \tilde{M}^\beta$ ;
2.  $\alpha_t \circ \beta = \beta \circ \alpha_{-t}$ ; and
3.  $\alpha_1(M) \perp M$ .



### II.3.1 Popa's deformation

The following deformation was used by Popa in [74] to obtain cocycle superrigidity for generalized Bernoulli actions of property (T) groups.

Let  $(A, \tau)$  be a finite diffuse abelian von Neumann algebra and  $u, v \in A \otimes A$  be generating Haar unitaries for  $A \otimes 1, 1 \otimes A \subset A \otimes A$ , respectively. Set  $w = u^*v$ . Choose  $h \in A \otimes A$  self-adjoint such that  $\exp(\pi ih) = w$ , and let  $w^t = \exp(\pi it h)$ . Since  $\{w\}'' \perp A \otimes 1, 1 \otimes A$ , we have that for any  $t$ ,  $w^t u$  and  $w^t v$  are again Haar unitaries. Moreover, since  $w \in \{w^t u, w^t v\}''$ ,  $\{w^t u, w^t v\}$  is a pair of generating Haar unitaries in  $A \otimes A$ . Hence there is a well-defined one-parameter family  $\alpha : \mathbb{R} \rightarrow \text{Aut}(A \otimes A, \tau \otimes \tau)$  given by

$$\alpha_t(u) = w^t u, \quad \alpha_t(v) = w^t v.$$

The family  $\alpha$ , together with the automorphism  $\beta$  given by

$$\beta(u) = u, \quad \beta(v) = u^2 v^*,$$

is seen to be an s-malleable deformation of  $A \otimes 1 \subset A \otimes A$ .

**Definition II.3.2.** Let  $(P, \tau)$  be a finite von Neumann algebra and  $\sigma : \Gamma \rightarrow \text{Aut}(P, \tau)$  a  $\Gamma$ -action.  $\Gamma \curvearrowright^\sigma (P, \tau)$  is an *s-malleable action* if there exists an s-malleable deformation  $(\alpha, \beta)$  of  $(P, \tau)$  such that  $\beta$  and  $\alpha_t$  commute with  $\sigma_\gamma \otimes \sigma_\gamma$  for all  $t \in \mathbb{R}$ ,  $\gamma \in \Gamma$ .

For any countable discrete group there is a canonical example of an s-malleable action, the *Bernoulli shift*. Let  $(A, \tau) = (L^\infty(\mathbb{T}, \lambda), \int \cdot d\lambda)$ ,  $(X, \mu) = \prod_{g \in \Gamma} (\mathbb{T}, \lambda)$ , and  $(B, \tau') = \bigotimes_{\gamma \in \Gamma} (A, \tau)$ . The Bernoulli shift is the natural action  $\Gamma \curvearrowright^\sigma (X, \mu)$  defined by shifting indices:  $\sigma_{\gamma_0}((x_\gamma)_\gamma) = (x_\gamma)_{\gamma_0 \gamma} = (x_{\gamma_0^{-1} \gamma})_\gamma$ . Defining

$$\tilde{\alpha}_t((\tilde{x}_\gamma)_\gamma) = (\alpha_t(\tilde{x}_\gamma))_\gamma$$

and

$$\tilde{\beta}((\tilde{x}_\gamma)_\gamma) = (\beta(\tilde{x}_\gamma))_\gamma,$$

for  $(\tilde{x}_\gamma)_\gamma \in \tilde{B} = \bigotimes_{\gamma \in \Gamma} (A \otimes A) \cong B \otimes B$ , we see that  $(\tilde{\alpha}, \tilde{\beta})$  is an s-malleable deformation of  $B$

which commutes with the Bernoulli  $\Gamma$ -action.

### II.3.2 Ioana's deformation

The deformation described below was first used by Ioana [36] in the case when the base space is nonamenable, and later used by Chifan and Ioana [11] in part to obtain solidity of  $L^\infty(X, \mu) \rtimes_\sigma \Gamma$ , whenever  $L\Gamma$  is solid and  $\Gamma \curvearrowright^\sigma (X, \mu)$  is the Bernoulli shift. (A finite von Neumann algebra  $M$  is solid [54] if for every von Neumann subalgebra  $B \subset M$ ,  $B' \cap M$  is amenable whenever  $B$  does not have minimal projections.) Their deformation is inspired by the free product deformation used in [38]. A similar deformation has also been previously used by Voiculescu in [103].

Given a finite von Neumann algebra  $(B, \tau)$ , let  $\tilde{B} = B * LZ$ , the free product of the von Neumann algebras  $B$  and  $L\mathbb{Z}$ . If  $u \in \mathcal{U}(L\mathbb{Z})$  is a generating Haar unitary, choose an  $h \in LZ$  such that  $\exp(\pi ih) = u$ , and let  $u^t = \exp(\pi it h)$ . Define the deformation  $\alpha : \mathbb{R} \rightarrow \text{Aut}(\tilde{B}, \tilde{\tau})$  by

$$\alpha_t = \text{Ad}(u^t).$$

Let  $\beta \in \text{Aut}(\tilde{B}, \tilde{\tau})$  be defined by

$$\beta|_B = \text{id} \quad \text{and} \quad \beta(u) = u^*.$$

It is easy to check that  $(\alpha, \beta)$  is an s-malleable deformation of  $B$ .

If a countable discrete group  $\Gamma$  acts on a countable set  $S$  then we may consider the *generalized Bernoulli shift action* of  $\Gamma$  on  $\otimes_{s \in S} B$  given by  $\sigma_\gamma(\otimes_{s \in S} b_s) = \otimes_{s \in S} b_{\gamma^{-1}s}$ . We then have that  $\otimes_{s \in S} B \subset \otimes_{s \in S} \tilde{B}$  and  $(\otimes_{s \in S} \alpha, \otimes_{s \in S} \beta)$  gives a s-malleable deformation of  $\otimes_{s \in S} B$ .

### II.3.3 Malleable deformations of Gaussian actions

We will now construct the canonical s-malleable deformation of a Gaussian action which is given in Section 4.3 of [27], and give an explicit description of its associated derivation. To begin, let  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$  be an orthogonal representation,  $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ , and  $\tilde{\pi} = \pi \oplus \pi$ . If  $\sigma^\pi : \Gamma \rightarrow \text{Aut}(A, \tau)$  is the Gaussian action associated with  $\pi$ , then the Gaussian action associated to  $\tilde{\pi}$  is naturally identified with the action  $\sigma^\pi \otimes \sigma^\pi$  on  $A \otimes A$ . Let  $\tilde{\sigma}^\pi = \sigma^\pi \otimes \sigma^\pi$ .

Let  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , the operator which gives  $\tilde{\mathcal{H}}$  the structure of a complex Hilbert space, and consider the one-parameter unitary group  $\theta_t = \exp(\frac{\pi t}{2}J)$ . Since  $\theta_t$  commutes with  $\tilde{\pi}$ , there is a well-defined one-parameter group  $\alpha : \mathbb{R} \rightarrow \text{Aut}(A \otimes A, \tau \otimes \tau)$  which commutes with  $\tilde{\sigma}^\pi$  namely,

$$\alpha_t = \sigma_{\theta_t} = \text{Ad}(\exp(\frac{\pi t}{2}J)^\mathfrak{S}).$$

Let  $\rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ , and observe  $\rho \circ \theta_{-t} = \theta_t \circ \rho$ . Hence,

$$\beta = \sigma_\rho = \text{Ad}(\rho^\mathfrak{S})$$

conjugates  $\alpha_t$  and  $\alpha_{-t}$ . Finally notice that  $\theta_1(\mathcal{H} \oplus 0) = 0 \oplus \mathcal{H}$ , which gives  $\alpha_1(A \otimes 1) = 1 \otimes A$ . The pair  $(\alpha, \beta)$  is an s-malleable deformation of the action  $\sigma^\pi$ .

Let  $T \in B(\tilde{\mathcal{H}})$  be skew-adjoint. Associate to  $T$  the unbounded skew-adjoint operator  $\partial(T)$  on  $\mathfrak{S}(\mathcal{H})$  defined by

$$\partial(T)(\Omega) = 0, \quad \partial(T)(\xi_1 \dots \xi_n) = \sum_{i=1}^n \xi_1 \dots T(\xi_i) \dots \xi_n.$$

We have that if  $U(t) = \exp(tT) \in \mathcal{O}(\mathcal{H})$ , then

$$\lim_{t \rightarrow 0} \frac{U(t)^\mathfrak{S} - I}{t} = \partial(T).$$

Let  $\delta : A \otimes A \rightarrow L^2(A \otimes A)$  be the derivation defined by

$$\delta(x) = [x, \partial(T)] = \lim_{t \rightarrow 0} \frac{\sigma_{U(t)}(x) - x}{t}.$$

Taking  $T$  to be the operator  $J$  defined above, gives us the derivation which is the infinitesimal generator of the s-malleable deformation of the Gaussian action described in this section. From the relation  $\delta(\cdot) = [\cdot, \partial(J)]$ , we see that the  $*$ -algebra generated by the operators  $s(\xi)$  forms a core for  $\delta$ .

Letting  $\delta_0 = \delta|_{A \otimes 1}$ , we have that

$$\Phi^t = \exp(-t\delta_0^* \bar{\delta}_0) = \exp(-tE_{A \otimes 1} \circ \delta^* \bar{\delta}) = \exp(tE_{A \otimes 1} \circ \delta^2).$$

We compute

$$E_{A \otimes 1} \circ \delta^2(s(\xi_1) \dots s(\xi_k)) = -k s(\xi_1) \dots s(\xi_k).$$

Hence,

$$\Phi^t(s(\xi_1) \dots s(\xi_k)) = (1 - e^{-kt})s(\Omega) + e^{-kt}s(\xi_1) \dots s(\xi_k).$$

## II.4 Cohomology of Gaussian actions

In this section, we obtain Theorem II.1.1 and its corollary. We do so by using a construction (cf. [30], [62], [89]) which, given an orthogonal representation and a cocycle, produces a  $\mathbb{T}$ -valued cocycles for the associated Gaussian action. We then show that these cocycles do not untwist by applying the above deformation.

Let  $b : \Gamma \rightarrow \mathcal{H}$  be a cocycle for an orthogonal representation  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$  and  $\Gamma \curvearrowright^\sigma (A, \tau) = (L^\infty(X, \mu), \int \cdot d\mu)$  be the Gaussian action associated to  $\pi$ , as described in section III.2.2. Viewing  $\mathcal{H}$  as a subset of  $L^2_{\mathbb{R}}(X, \mu)$ , Parthasarathy and Schmidt in [62] constructed the cocycle  $c : \Gamma \times X \rightarrow \mathbb{T}$  by the rule

$$c(\gamma, x) = \exp(ib(\gamma^{-1}))(x).$$

We write  $\omega_\gamma$  for the element of  $\mathcal{U}(L^\infty(X, \mu))$  given by  $\omega_\gamma(x) = c(\gamma, \gamma^{-1}x)$ . The cocycle identity for  $c$  then transforms to the formula  $\omega_{\gamma_1\gamma_2} = \omega_{\gamma_1}\sigma_{\gamma_1}(\omega_{\gamma_2})$ , for all  $\gamma_1, \gamma_2 \in \Gamma$ . Moreover,  $c$  is cohomologous to a homomorphism if and only if there is a unitary element  $u \in \mathcal{U}(L^\infty(X, \mu))$  such that  $\gamma \mapsto u\omega_\gamma\sigma_\gamma(u^*)$  is a homomorphism, i.e., each  $u\omega_\gamma\sigma_\gamma(u^*)$  is fixed by the action of the group.

A routine calculation shows that  $\tau(\omega_\gamma) = \int c(\gamma, x)d\mu(x) = \exp(-\|b(\gamma)\|^2/2)$ . In particular, this shows that the representation associated to the positive-definite function  $\varphi(\gamma) = \exp(-\|b(\gamma)\|^2/2)$  is naturally isomorphic to the twisted Gaussian action  $\omega_\gamma\sigma_\gamma$ .

**Theorem II.4.1.** *Using the notation above, if  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$  is weak mixing, (so that  $\sigma$  is ergodic) and if  $b$  is an unbounded cocycle, then  $c$  does not untwist.*

*Proof.* Since  $\sigma$  is ergodic, if  $c$  were to untwist then there would exist some  $u \in \mathcal{U}(A)$  such that  $u\omega_\gamma\sigma_\gamma(u) \in \mathbb{T}$ , for all  $\gamma \in \Gamma$ . It would then follow that any deformation of  $A$  which commutes with the action of  $\Gamma$  must converge uniformly on the set  $\{\omega_\gamma \mid \gamma \in \Gamma\}$ . Indeed, this is just a consequence of the fact that completely positive deformations become asymptotically  $A$ -bimodular.

However, if we apply the deformation  $\alpha_t$  from Section II.3.3 then we can compute

$$\begin{aligned} & \langle \alpha_{2t/\pi}(\omega_\gamma \otimes 1), \omega_\gamma \otimes 1 \rangle \\ &= \langle \exp(i(\cos t)b(\gamma^{-1})) \otimes \exp(-i(\sin t)b(\gamma^{-1})), \exp(ib(\gamma^{-1})) \otimes 1 \rangle \\ &= \exp((1 - \cos t)^2 \|b(\gamma)\|^2/2 + (\sin^2 t) \|b(\gamma)\|^2/2) \\ &= \exp(-(1 - \cos t) \|b(\gamma)\|^2) \end{aligned}$$

This will converge uniformly for  $\gamma \in \Gamma$  if and only if the cocycle  $b$  is bounded and hence the result follows.  $\square$

**Corollary II.4.2.** *The exponentiation map described above induces an injective homomorphism  $H^1(\Gamma, \pi) \rightarrow H^1(\Gamma, \sigma, \mathbb{T})/\chi(\Gamma)$ , where  $\chi(\Gamma)$  is the character group of  $\Gamma$ .*

*Proof.* It is easy to see that if two cocycles in  $Z^1(\Gamma, \pi)$  are cohomologous then the resulting cocycles for the Gaussian action will also be cohomologous. This shows that the map described above is well defined.

The above theorem, together with the fact that this map is a homomorphism, shows that this map is injective.  $\square$

Since a nonamenable group has vanishing first  $\ell^2$ -Betti number if and only if it has vanishing first cohomology into its left regular representation [4], [67], we derive the following corollary.

**Corollary II.4.3.** *Let  $\Gamma$  be a nonamenable countable discrete group, and let  $\Gamma \curvearrowright^\sigma (X, \mu)$  be the Bernoulli shift action. If  $\beta_1^{(2)}(\Gamma) \neq 0$  then  $H^1(\Gamma, \sigma, \mathbb{T}) \neq \chi(\Gamma)$ , where  $\chi(\Gamma)$  is the group of characters. In particular,  $\Gamma \curvearrowright^\sigma (X, \mu)$  is not  $\mathcal{U}_{\text{fin}}$ -cocycle superrigid.*

## II.5 Derivations

In this section we continue our investigation of deformations, but this time on the infinitesimal level.

### II.5.1 Derivations from s-malleable deformations

Let  $(M, \tau)$  be a finite von Neumann algebra, and let  $\alpha : \mathbb{R} \rightarrow \text{Aut}(M, \tau)$  be a pointwise strongly continuous one-parameter group of automorphisms. Let  $\delta$  be the infinitesimal generator of  $\alpha$ , *i.e.*,  $\exp(t\delta) = \alpha_t$ . For  $f \in L^1(\mathbb{R})$  define the bounded operator  $\alpha_f : M \rightarrow M$  by

$$\alpha_f(x) = \int_{-\infty}^{\infty} f(s)\alpha_s(x)ds.$$

It can be checked that if  $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$  and  $f' \in L^1(\mathbb{R})$ , then

$$\delta \circ \alpha_f(x) = -\alpha_{f'}(x).$$

Also if  $x \in M \cap D(\delta)$ , then we have that

$$\alpha_t(x) - x = \int_0^t \delta \circ \alpha_s(x)ds = \int_0^t \alpha_s(\delta(x))ds.$$

**Theorem II.5.1.** *Suppose that for every  $\varepsilon > 0$ , there exists  $f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$  such that  $f' \in L^1(\mathbb{R})$  and  $\sup_{x \in (M)_1} \|\alpha_f(x) - x\|_2 \leq \varepsilon/4$ . Then  $\alpha_t$  converges  $\|\cdot\|_2$ -uniformly to the identity on  $(M)_1$  as  $t \rightarrow 0$ .*

*Proof.* We need only show for every  $\varepsilon > 0$  that there exists some  $\eta > 0$  such that for all  $t < \eta$ ,  $\sup_{x \in (M)_1} \|\alpha_t(x) - x\|_2 \leq \varepsilon$ . Let  $\tilde{x} = \alpha_f(x)$ . We have that  $\|\alpha_t(x) - x\|_2 \leq \|\alpha_t(\tilde{x}) - \tilde{x}\|_2 + \varepsilon/2$ . Since  $\delta \circ \alpha_f$  is defined everywhere,  $\delta \circ \alpha_f : M \rightarrow L^2(M, \tau)$  is bounded. In fact,  $\|\delta \circ \alpha_f\| \leq \|f'\|_{L^1}$ . Now, since  $\tilde{x} \in D(\delta)$ , we have  $\alpha_t(\tilde{x}) - \tilde{x} = \int_0^t \alpha_s(\delta(\tilde{x}))ds$ . Hence  $\|\alpha_t(\tilde{x}) - \tilde{x}\|_2 \leq t\|f'\|_{L^1}$ . Choosing  $\eta = \varepsilon(2\|f'\|_{L^1})^{-1}$  does the job. □

**Corollary II.5.2.** *If  $\varphi_t = \exp(-t\delta^*\delta)$  converges uniformly to the identity as  $t \rightarrow 0$ , then so does  $\alpha_t$ .*

*Proof.* Let  $f_t(s) = \frac{1}{\sqrt{4\pi t}} \exp(-s^2/4t)$ ; then,  $\varphi_t(x) = \int_{-\infty}^{\infty} f_t(s)\alpha_s(x)ds$  follows by completing the square.  $\square$

## II.5.2 Tensor products of derivations

We describe here the notion of a tensor product of derivations; see also Section 6 of [64].

Consider  $N_i, i \in I$  a family of finite von Neumann algebras with normal faithful traces  $\tau_i$ . If  $\delta_i : N_i \rightarrow \mathcal{H}_i$  is a family of closable real derivations into Hilbert bimodules  $\mathcal{H}_i$  with domains  $D(\delta_i)$  then we may consider the dense  $*$ -subalgebra  $D(\delta) = \otimes_{i \in I}^{\text{alg}} D(\delta_i) \subset N = \overline{\otimes}_{i \in I} N_i$ .

We denote by  $\hat{N}_j$  the tensor product of the  $N_i$ 's obtained by omitting the  $j$ th index so, fixing an arbitrary order in which the successive tensor powers are resolved in  $N$ , we have a natural identification  $N = \hat{N}_j \overline{\otimes} N_j$  for each  $j \in I$ . Let  $\mathcal{H} = \bigoplus_{j \in I} \mathcal{H}_j \overline{\otimes} L^2(\hat{N}_j)$  which is naturally a Hilbert bimodule because of the identification  $N = \hat{N}_j \overline{\otimes} N_j$ .

The tensor product of the derivations  $\delta_i, i \in I$  is defined to be the derivation  $\delta = \bigotimes_{i \in I} \delta_i : D(\delta) \rightarrow \mathcal{H}$  which satisfies

$$\delta(\otimes_{i \in I} x_i) = \bigoplus_{j \in I} (\delta_j(x_j) \otimes_{i \in I, i \neq j} x_i).$$

This is well defined as only finitely many of the  $x_i$ 's are not equal to 1 and hence the right hand side is a finite sum.

If  $\Phi_i^t = \exp(-t\delta_i^* \overline{\delta_i})$  is the semigroup deformation associated to  $\delta_i$  then one easily checks that the semigroup deformation associated to  $\delta$  is  $\Phi^t = \bigotimes_{i \in I} \Phi_i^t : N \rightarrow N$ . A similar formula holds for the resolvent deformation. Note that by viewing the Hilbert bimodule associated with  $\Phi^t$  and using the usual ‘‘averaging trick’’ (e.g. Theorem 4.2 in [68]) it follows that  $\Phi^t$  will converge uniformly in  $\|\cdot\|_2$  to the identity on  $(N)_1$  if and only if each  $\Phi_i^t$  converges uniformly in  $\|\cdot\|_2$  to the identity on  $(N_i)_1$  and moreover this convergence is uniform in  $i \in I$ .

## II.5.3 Derivations from generalized Bernoulli shifts

We use here the notation in Section III.2.2 above. Given a real Hilbert space  $\mathcal{H}$ , we consider the new Hilbert space  $\mathcal{H}' = \mathbb{R}\Omega_0 \oplus \mathcal{H}$ . If  $\xi \in \mathcal{H}$  is a non-zero element we denote by  $P_\xi$  the rank one projection onto  $\xi$ . We denote by  $\tilde{\mathcal{H}}$  the tensor product (complex) Hilbert space  $\mathcal{H} \otimes \mathfrak{S}(\mathcal{H}')$

Let  $N \in \mathbb{N} \cup \{\infty\}$  be the dimension of  $\mathcal{H}$  and consider an orthonormal basis  $\beta = \{\xi_n\}_{n=1}^N$  for  $\mathcal{H}$ .

We then define a left action of  $A$ , the von Neumann algebra generated by the spectral projections of  $s(\xi)$ ,  $\xi \in \mathcal{H}$ , on  $\tilde{\mathcal{H}}$  such that for each  $\xi \in \mathcal{H}$ ,  $s(\xi)$  acts on the left (as an unbounded operator) by

$$\ell_\beta(s(\xi)) = \text{id} \otimes s(\xi).$$

We also define a right action of  $A$  on  $\tilde{\mathcal{H}}$  such that for each  $\xi \in \mathcal{H}$ ,  $s(\xi)$  acts on the right by extending linearly the formula

$$r_\beta(s(\xi))(\xi_n \otimes \eta) = P_{\xi_n}(\xi) \otimes S(\Omega_0)\eta + \xi_n \otimes s(\xi - P_{\xi_n}(\xi))\eta, \quad (\text{II.5.1})$$

for each  $1 \leq n \leq N, \eta \in \mathfrak{S}(\mathcal{H}')$ .

These formulas define unbounded self-adjoint operators on  $\tilde{\mathcal{H}}$  in general; however, by functional calculus they extend to give commuting normal actions of  $A$  on  $\tilde{\mathcal{H}}$ .

Moreover, if  $T \in \mathcal{O}(\mathcal{H}) \subset \mathcal{O}(\mathcal{H}')$ , then we have that for any  $\xi \in \mathcal{H}$

$$\ell_{T\beta}(s(T\xi)) = \ell_{T\beta}(\sigma_T(s(\xi))) = \text{Ad}(T \otimes T^\mathfrak{S})\ell_\beta(s(\xi)).$$

Also,

$$r_{T\beta}(s(T\xi)) = r_{T\beta}(\sigma_T(s(\xi))) = \text{Ad}(T \otimes T^\mathfrak{S})(r_\beta(s(\xi))).$$

From here on we will denote the left action of  $A$  on  $\tilde{\mathcal{H}}$  by  $\ell_\beta(a)x = a \cdot_\beta x$  and the right action by  $r_\beta(a)x = x \cdot_\beta a$ . By extending the formulas above to  $A$  we have the following lemma.

**Lemma II.5.3.** *Using the notation above, consider the inclusion  $\mathcal{O}(\mathcal{H}) \subset \mathcal{U}(\tilde{\mathcal{H}})$  given by  $T \mapsto \tilde{T} = T \otimes T^\mathfrak{S}$ . Then for each  $T \in \mathcal{O}(\mathcal{H})$ ,  $x, y \in A$ , and  $\tilde{\xi} \in \tilde{\mathcal{H}}$ , we have  $\tilde{T}(x \cdot_\beta \tilde{\xi} \cdot_\beta y) = \sigma_T(x) \cdot_{T\beta}(\tilde{T}\tilde{\xi}) \cdot_{T\beta} \sigma_T(y)$ .*

**Remark II.5.4.** While we will not use this in the sequel, an alternate way to view the  $A$ - $A$  Hilbert bimodule structure on  $\tilde{\mathcal{H}}$  is as follows. Given our basis  $\beta = \{\xi_n\}_{n=1}^N \subset \mathcal{H}$ , consider the probability space  $(X, \mu) = \Pi_n(\mathbb{R}, g)$  where  $g$  is the Gaussian measure on  $\mathbb{R}$ . We can identify  $A = L^\infty(X, \mu)$ , and we denote by  $\pi_n \in L^2(X, \mu)$  the projection onto the  $n$ th copy of  $(\mathbb{R}, g)$  so that the  $\pi_n$ 's are I.I.D. Gaussian random variables.

We embed  $\mathcal{H}$  into  $L^2(X, \mu)$  linearly by the map  $\eta$  such that  $\eta(\xi_n) = \pi_n$  given an orthogonal



transformation  $T \in \mathcal{O}(\mathcal{H})$ , we associate to  $T$  the unique measure-preserving automorphism  $\sigma_T \in \text{Aut}(A)$  such that  $\sigma_T(\eta(\xi)) = \eta(T\xi)$ , for all  $\xi \in \mathcal{H}$ .

For each  $k$  we denote by

$$A_k = \left( \bigotimes_{n < k} L^\infty(\mathbb{R}, g) \right) \otimes (L^\infty(\mathbb{R}, g) \otimes L^\infty(\mathbb{R}, g)) \otimes \left( \bigotimes_{n > k} L^\infty(\mathbb{R}, g) \right),$$

and we view  $L^2(A_k)$  as an  $A$ - $A$  bimodule so that

$$(\otimes_n a_n) \cdot x = \left( \bigotimes_{n < k} a_n \otimes (a_k \otimes 1) \otimes \bigotimes_{n > k} a_n \right) x$$

and

$$x \cdot (\otimes_n a_n) = x \left( \bigotimes_{n < k} a_n \otimes (1 \otimes a_k) \otimes \bigotimes_{n > k} a_n \right),$$

for  $x \in L^2(A_k)$ .

Consider the  $A$ - $A$  Hilbert bimodule  $\bigoplus_k L^2(A_k)$ , and note that it is canonically identified with the Hilbert space  $\mathcal{H} \otimes L^2(A_1) \cong \mathcal{H} \otimes L^2(\mathbb{R}, g) \otimes L^2(A) \cong \tilde{\mathcal{H}}$  in a way which preserves the  $A$ - $A$  bimodule structure. Under this identification the inclusion  $\mathcal{O}(\mathcal{H}) \subset \mathcal{U}(\mathcal{H} \otimes L^2(\mathbb{R}, g) \otimes L^2(A))$  becomes  $T \mapsto T \otimes \text{id} \otimes \sigma_T$ .

We now consider the algebra  $A_0 \subset L^2(A)$  of square summable operators generated by  $s(\xi)$ ,  $\xi \in \mathcal{H}$ , and define a derivation  $\delta_\beta$  on  $A_0$  by setting

$$\delta_\beta(s(\xi)) = \xi \otimes \Omega \in \tilde{\mathcal{H}},$$

for each  $\xi \in \mathcal{H}$ . Note that the formula for  $\delta_\beta(s(\xi))$  does not depend on the basis  $\beta$ , but the bimodule structure that we are imposing on  $\mathcal{H}$  does depend on  $\beta$ . If  $\xi_0, \xi_1, \dots, \xi_k \in \beta$  such that  $\xi_0$  is orthogonal to the vectors  $\xi_1, \dots, \xi_k$  then it follows that  $\delta_\beta(s(\xi_1) \cdots s(\xi_k))$  is a  $s(\xi_0)$ -central vector and hence by induction on  $k$  it follows that  $\delta_\beta$  is well defined. Also, since  $\delta_\beta$  extends to a bounded operator on  $\overline{\text{sp}}\{s(\xi_1) \cdots s(\xi_k) \mid \xi_1, \dots, \xi_k \in \mathcal{H}\}$  for each  $k$  it follows that  $\delta_\beta$  is a closable operator and if we still denote by  $\delta_\beta$  the closure of this operator we have that  $x \mapsto \|\delta_\beta(x)\|^2$  is a quantum Dirichlet form on  $L^2(A)$  (see [24, 86, 87]).

In particular, it follows from [24] that  $D(\delta_\beta) \cap A$  is a weakly dense  $*$ -subalgebra and  $\delta_\beta|_{D(\delta_\beta) \cap A}$

is a derivation.

Note that if we identify  $\tilde{\mathcal{H}}$  with  $\bigoplus_k L^2(A_k)$  as above then  $\delta_\beta$  can also be viewed as the tensor product derivation  $\delta_\beta = \bigotimes_k \delta_k$  where  $\delta_k : L^2(\mathbb{R}, g) \rightarrow L^2(\mathbb{R}, g) \overline{\otimes} L^2(\mathbb{R}, g)$  is the difference quotient derivation for each  $k$ , i.e.,  $\delta_k(f)(x, y) = \frac{f(x) - f(y)}{x - y}$ .

**Lemma II.5.5.** *Using the above notation,  $\delta_\beta$  is a densely defined closed real derivation,  $s(\mathcal{H}) \subset D(\delta_\beta)$ ,  $\delta_\beta \circ s : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$  is an isometry, and for all  $T \in \mathcal{O}(\mathcal{H})$ ,  $\sigma_T(D(\delta_\beta)) = D(\delta_{T\beta})$ , and  $\delta_{T\beta}(\sigma_T(a)) = \tilde{T}(\delta_\beta(a))$ , for all  $a \in D(\delta_\beta)$ .*

*Proof.* The fact that  $s(\mathcal{H}) \subset D(\delta_\beta)$ , and that  $\delta_\beta \circ s$  is an isometry follows from the formula  $\delta_\beta(s(\xi)) = \xi \otimes \Omega$  above.

Moreover for  $\xi \in \mathcal{H}$  we have

$$\begin{aligned} \delta_{T\beta}(\sigma_T(s(\xi))) &= T\xi \otimes \Omega \\ &= (T \otimes T^\mathfrak{S})(\xi \otimes \Omega) = \tilde{T}\delta_\beta(s(\xi)). \end{aligned}$$

By Lemma II.5.3 this formula then extends to  $A_0$ , and since  $\tilde{T}$  acts on  $\tilde{\mathcal{H}}$  unitarily and  $A_0$  is a core for  $\delta_\beta$  we have that  $\sigma_T(D(\delta_\beta)) = D(\delta_{T\beta})$  and this formula remains valid for  $a \in D(\delta_\beta)$ .  $\square$

Given an action of a countable discrete group  $\Gamma$  on a countable set  $S$  we may consider the generalized Bernoulli shift action of  $\Gamma$  on  $(X, \mu) = \prod_{s \in S} (\mathbb{R}, g)$  given by  $\gamma(r_s)_{s \in S} = (r_{\gamma^{-1}s})_{s \in S}$ . If we set  $\mathcal{H} = \ell^2 S$  and consider the corresponding representation  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$  then the generalized Bernoulli shift can be viewed as the Gaussian action corresponding to  $\pi$ . Moreover we have that the canonical basis  $\beta = \{\delta_s\}_{s \in S}$  is invariant to the representation, i.e.,  $\pi_\gamma \beta = \beta$ , for all  $\gamma \in \Gamma$ .

In this case by Lemma II.5.5 we have that  $D(\delta_\beta)$  is  $\sigma_\gamma$  invariant for all  $\gamma \in \Gamma$  and  $\delta_\beta(\sigma_\gamma(a)) = \tilde{\pi}_\gamma(\delta_\beta(a))$ , for all  $\gamma \in \Gamma$ ,  $a \in D(\delta_\beta)$ , where  $\tilde{\pi} : \Gamma \rightarrow \mathcal{U}(\tilde{\mathcal{H}})$  is the unitary representation given by  $\tilde{\pi} = \pi \otimes \pi^\mathfrak{S}$ . If we denote by  $N = A \rtimes \Gamma$  the corresponding group-measure space construction then using Lemma II.5.3 we may define an  $N$ - $N$  Hilbert bimodule structure on  $\mathcal{K} = \tilde{\mathcal{H}} \otimes \ell^2 \Gamma$  which satisfies

$$(au_{\gamma_1})(\xi \otimes \delta_{\gamma_0})(bu_{\gamma_2}) = (a \cdot_\beta (\tilde{\pi}_{\gamma_1} \xi) \cdot_\beta \sigma_{\gamma_1 \gamma_0}(b)) \otimes \delta_{\gamma_1 \gamma_0 \gamma_2},$$

for all  $a, b \in A$ ,  $\gamma_0, \gamma_1, \gamma_2 \in \Gamma$ , and  $\xi \in \tilde{\mathcal{H}}$ . We may then extend  $\delta_\beta$  to a closable derivation  $\delta : *\text{-Alg}(D(\delta_\beta) \cap A, \Gamma) \rightarrow \mathcal{K}$  such that  $\delta(au_\gamma) = \delta_\beta(a) \otimes u_\gamma$ , for all  $a \in D(\delta_\beta)$ ,  $\gamma \in \Gamma$ .

As above we denote by  $\zeta_\alpha : N \rightarrow N$  the unital, symmetric, c.p. resolvent maps given by  $\zeta_\alpha = (\alpha/(\alpha + \delta^*\bar{\delta}))^{1/2}$ , for  $\alpha > 0$ .

Note that if  $M$  is a finite von Neumann algebra then we let  $\Gamma$  act on  $M$  trivially and we may extend the derivation  $\delta$  to  $(A\bar{\otimes}M) \rtimes \Gamma \cong (A \rtimes \Gamma)\bar{\otimes}M$  by considering the tensor product derivation of  $\delta$  with the trivial derivation (identically 0) on  $M$ . In this case the corresponding deformation of resolvent maps is just  $\zeta_\alpha \otimes \text{id}$ .

**Lemma II.5.6.** *Consider Ioana's deformation  $\alpha_t$  on  $A$  corresponding to generalized Bernoulli shift as described above in Section II.3.2. If  $M$  is a finite von Neumann algebra and  $B \subset (A\bar{\otimes}M) \rtimes \Gamma$  is a subalgebra such that  $\zeta_\alpha$  converges uniformly to the identity on  $(B)_1$  as  $\alpha \rightarrow 0$  then  $\alpha_t$  converges uniformly to the identity on  $(B)_1$  as  $t \rightarrow 0$ .*

*Proof.* The infinitesimal generator of Ioana's deformation cannot be identified with  $\delta$  as the  $\alpha_t$ 's will converge uniformly on the algebra generated by  $s(\xi)$  for each  $\xi \in \beta$ , and  $\zeta_\alpha$  will not have this property. However, it is not hard to check using the fact that both derivations arise as tensor product derivations that if  $\zeta_\alpha^0$  are the resolvent maps corresponding to the infinitesimal generator of  $\alpha_t$  then we have the inequality  $\tau(\zeta_\alpha(a)a^*) \leq 2\tau(\zeta_\alpha^0(a)a^*)$ , for all  $a \in A$ . Hence the lemma follows from Lemma 2.1 in [64] and Corollary II.5.2 above.  $\square$

**Remark II.5.7.** It can be shown in fact that the deformation coming from the derivation above, Ioana's deformation, and the s-malleable deformation from the Gaussian action, are successively weaker deformations. That is to say that one deformation converging uniformly on a subset of the unit ball implies that the next deformation must also converge uniformly.

When we restrict the bimodule structure on  $\mathcal{K}$  to the subalgebra  $L\Gamma$  we see that this is exactly the bimodule structure coming from the representation  $\tilde{\pi} = \pi \otimes \pi^\mathfrak{S}$ , this give rise to the following lemma:

**Lemma II.5.8.** *Using the notation above, given  $H < \Gamma$  we have the following:*

1.  ${}_{LH}\mathcal{K}_{LH}$  embeds into a direct sum of coarse bimodules if and only if  $\pi|_H$  embeds into a direct sum of left regular representations.
2.  ${}_{LH}\mathcal{K}_{LH}$  weakly embeds into a direct sum of coarse bimodules if and only if  $\pi|_H$  weakly embeds into a direct sum of left regular representations.

3.  ${}_{LH}\mathcal{K}_{LH}$  has stable spectral gap if and only if  $\pi|_H$  has stable spectral gap.
4.  ${}_{LH}\mathcal{K}_{LH}$  is a mixing correspondence if and only if  $\pi|_H$  is a mixing representation.
5.  ${}_{LH}\mathcal{K}_{LH}$  is weakly mixing if and only if  $\pi|_H$  is weakly mixing.

## II.6 $L^2$ -rigidity and $\mathcal{U}_{\text{fin}}$ -cocycle superrigidity

In this section we use the tools developed above to prove  $\mathcal{U}_{\text{fin}}$ -cocycle superrigidity for the Bernoulli shift action which we view as the Gaussian action corresponding to the left-regular representation.

To prove that a cocycle untwists we use the same general setup as Popa in [74]. In particular, we use the fact that for a weakly mixing action, in order to show that a cocycle untwists it is enough to show that the corresponding s-malleable deformation converges uniformly on the “twisted” subalgebra of the crossed product algebra. The main difference in our approach is that to show that the s-malleable deformation converges uniformly it is enough by Lemma II.5.6 to show that the deformation coming from the Bernoulli shift derivation converges uniformly. This allows us to use the techniques developed in [63], [64], [61], and [66] to analyze the cocycle on the level of the base space itself rather than the exponential of the space where the properties can be somewhat hidden.

**Theorem II.6.1.** *Let  $\Gamma$  be a countable discrete group. If  $L\Gamma$  is  $L^2$ -rigid then the Bernoulli shift action with diffuse core of  $\Gamma$  is  $\mathcal{U}_{\text{fin}}$ -cocycle superrigid.*

*Proof.* Let  $\mathcal{G} \in \mathcal{U}_{\text{fin}}$ , then  $\mathcal{G} \subset \mathcal{U}(M)$  as a closed subgroup where  $M$  is a finite separable von Neumann algebra. Let  $c : \Gamma \times X \rightarrow \mathcal{G}$  be a cocycle where  $X$  is the probability space of the Gaussian action. Consider  $A = L^\infty(X)$ , and  $\omega : \Gamma \rightarrow \mathcal{U}(A \overline{\otimes} M)$  given by  $\omega_\gamma(x) = c(\gamma, \gamma^{-1}x)$  the corresponding unitary cocycle for the action  $\tilde{\sigma}_\gamma = \sigma_\gamma \otimes \text{id}$ . Note that  $\omega_{\gamma_1\gamma_2} = \omega_{\gamma_1} \tilde{\sigma}_{\gamma_1}(\omega_{\gamma_2})$ , for all  $\gamma_1, \gamma_2 \in \Gamma$ . Here we view a unitary element in  $A \overline{\otimes} M$  as map from  $X$  to  $\mathcal{U}(M)$ . More explicitly, let  $L^\infty(X; M)$  be the space of  $\mu$ -measure classes of norm bounded functions from  $X$  to  $M$  which are measurable with respect to the strong topology on  $M$ . This is a  $*$ -algebra under pointwise multiplication and is equipped with a trace  $\tilde{\tau}(f) = \int \tau(f(x)) d\mu(x)$ . It is easy to see that any unitary in  $L^\infty(X; M)$  must take values in  $\mathcal{U}(M)$  almost everywhere. Also, for any  $B \subset X$ , measurable, and  $x \in M$ , it can be shown that the map which sends  $1_B \otimes x$  to the simple function supported on  $B$  which takes the value  $x$  extends to a strongly continuous  $*$ -isomorphism of the

algebras  $A\overline{\otimes}M$  and  $L^\infty(X; M)$  (see [74] for a detailed explanation).

As noted above, the Bernoulli shift action with diffuse core is precisely the Gaussian action corresponding to the left-regular representation; hence, by Lemma II.5.8 we have that as an  $L\Gamma$ - $L\Gamma$  Hilbert bimodule  $\mathcal{K}$  embeds into a direct sum of coarse correspondences. If we denote by  $\widetilde{L\Gamma}$  the von Neumann algebra generated by  $\{\tilde{u}_\gamma\} = \{\omega_\gamma u_\gamma\}$  then the bimodule structure of  $\widetilde{L\Gamma}$  ( $\cong L\Gamma$ ) on  $\mathcal{K}$  is the same as the bimodule structure of  $L\Gamma$  on the correspondence coming from the representation  $\gamma \mapsto \text{Ad}(\omega_\gamma) \circ \tilde{\pi}_\gamma$  on  $\tilde{\mathcal{H}} \otimes L^2M$ . The  $A\overline{\otimes}M$  bimodule structure on  $\tilde{\mathcal{H}} \otimes L^2M = \mathcal{H} \otimes \mathfrak{S}(\mathcal{H}') \otimes L^2M$  decomposes as a direct sum of bimodules  $\mathcal{H} \otimes \mathfrak{S}(\mathcal{H}') \otimes L^2M = \bigoplus_{\xi \in \beta} \mathfrak{S}(\mathcal{H}') \otimes L^2M$  where the bimodule structure on each copy of  $\mathfrak{S}(\mathcal{H}') \otimes L^2M$  is given by Equation (II.5.1), and under this decomposition we have  $\text{Ad}(\omega_\gamma) \circ \tilde{\pi}_\gamma = \pi_\gamma \otimes (\text{Ad}(\omega_\gamma) \circ \pi_\gamma^\mathfrak{S})$ . Therefore by Fell's absorption principle this representation is an infinite direct sum of left-regular representations; hence, we have that  $\mathcal{K}$  also embeds into a direct sum of coarse correspondences when  $\mathcal{K}$  is viewed as an  $\widetilde{L\Gamma}$ - $\widetilde{L\Gamma}$  Hilbert bimodule.

Since  $L\Gamma$  is  $L^2$ -rigid we have that the corresponding deformation  $\zeta_\alpha$  converges uniformly to the identity map on  $(\widetilde{L\Gamma})_1$ , by Lemma II.5.6 we have that a corresponding s-malleable deformation also converges uniformly to the identity on  $(\widetilde{L\Gamma})_1$ . Thus, by Theorem 3.2 in [74] the cocycle  $\omega$  is cohomologous to a homomorphism.  $\square$

We end this chapter with some examples of groups for which the hypothesis of the Theorem II.6.1 is satisfied.

It follows from [64] that if  $N$  is a nonamenable  $\text{II}_1$  factor which is non-prime, has property Gamma, or is  $w$ -rigid, then  $N$  is  $L^2$ -rigid. We include here another class of  $L^2$ -rigid finite von Neumann algebras, this class includes the group von Neumann algebras of all generalized wreath product groups  $A_0 \wr_X \Gamma_0$  where  $A_0$  is an infinite abelian group and  $\Gamma_0$  does not have the Haagerup property, or  $\Gamma_0$  is a non-amenable direct products of infinite groups. This is a special case of a more general result which can be found in [66].

**Theorem II.6.2.** *Let  $\Gamma$  be a countable discrete group which contains an infinite normal abelian subgroup and either does not have the Haagerup property or contains an infinite subgroup  $\Gamma_0$  such that  $L\Gamma_0$  is  $L^2$ -rigid, then  $L\Gamma$  is  $L^2$ -rigid.*

*Proof.* We will use the same notation as in [64]. Suppose  $(M, \tau)$  is a finite von Neumann algebra

with  $L\Gamma \subset M$ , and  $\delta : M \rightarrow L^2M \overline{\otimes} L^2M$  is a densely defined closable real derivation.

Since the maps  $\eta_\alpha$  converge pointwise to the identity we may take an appropriate sequence  $\alpha_n$  such that the map  $\phi : \Gamma \rightarrow \mathbb{R}$  given by  $\phi(\gamma) = \sum_n 1 - \tau(\eta_{\alpha_n}(u_\gamma)u_\gamma^*)$  is well defined. If the deformation  $\eta_\alpha$  does not converge uniformly on any infinite subset of  $\Gamma$  then the map  $\phi$  is not bounded on any infinite subset and hence defines a proper, conditionally negative definite function on  $\Gamma$  showing that  $\Gamma$  has the Haagerup property.

Therefore if  $\Gamma$  does not have the Haagerup property then there must exist an infinite set  $X \subset \Gamma$  on which the deformation  $\eta_\alpha$  converges uniformly. Similarly, if  $\Gamma_0 \subset \Gamma$  is an infinite subgroup such that  $L\Gamma_0$  is  $L^2$ -rigid then we have that the deformation  $\eta_\alpha$  converges uniformly on the infinite set  $X = \Gamma_0$ .

Let  $A \subset \Gamma$  be an infinite normal abelian subgroup. If there exists an  $a \in A$  such that  $a^X = \{xax^{-1} | x \in X\}$  is infinite, then we have that the deformation  $\eta_\alpha$  converges uniformly on this set, and by applying the results in [64] it follows that  $\eta_\alpha$  converges uniformly on  $A \subset LA$ . Since  $A$  is a subgroup in  $\mathcal{U}(LA)$  which generates  $LA$  it then follows that  $\eta_\alpha$  converges uniformly on  $(LA)_1$  and hence also on  $(L\Gamma)_1$  since  $A$  is normal in  $\Gamma$ .

If  $a \in A$  and  $a^X$  is finite then there exists an infinite sequence  $\gamma_n \in X^{-1}X$  such that  $[\gamma_n, a] = e$ , for each  $n$ . Thus if  $a^X$  is finite for each  $a \in A$  then by taking a diagonal subsequence we construct a new sequence  $\gamma_n \in X^{-1}X$  such that  $\lim_{n \rightarrow \infty} [\gamma_n, a] = e$ . Since  $\eta_\alpha$  also converges uniformly on  $X^{-1}X$  we may again apply the results in [64] to conclude that  $\eta_\alpha$  converges uniformly on  $A$  and hence on  $(L\Gamma)_1$  as above.  $\square$

It has been pointed out to us by Adrian Ioana that in light of Corollary 1.3 in [12] the above argument is sufficient to show that for a lattice  $\Gamma$  in a connected Lie group which does not have the Haagerup property, we must have that  $L\Gamma$  is  $L^2$ -rigid.

We also show that  $L^2$ -rigidity is stable under orbit equivalence. The proof of this uses the diagonal embedding argument of Popa and Vaes [80].

**Theorem II.6.3.** *Let  $\Gamma_i \curvearrowright (X_i, \mu_i)$  be free ergodic measure preserving actions for  $i = 1, 2$ . If the two actions are orbit equivalent and  $L\Gamma_1$  is  $L^2$ -rigid then  $L\Gamma_2$  is also  $L^2$ -rigid.*

*Proof.* Suppose  $L\Gamma_2 \subset M$  and  $\delta : M \rightarrow \mathcal{H}$  is a closable real derivation such that  $\mathcal{H}$  as an  $L\Gamma_2$  bimodule embeds into a direct sum of coarse bimodules. Let  $N = L^\infty(X_1, \mu_1) \rtimes L\Gamma_1 = L^\infty(X_2, \mu_2) \rtimes$

$L\Gamma_2$  and consider the  $N\overline{\otimes}M$  bimodule  $\tilde{\mathcal{H}} = L^2N\overline{\otimes}\mathcal{H}$ . If we embed  $N$  into  $N\overline{\otimes}M$  by the linear map  $\alpha$  which satisfies  $\alpha(au_\gamma) = au_\gamma \otimes u_\gamma$  for all  $a \in L^\infty(X_2, \mu_2)$ , and  $\gamma \in \Gamma_2$ , then when we consider the  $\alpha(N)$ - $\alpha(N)$  bimodule  $\tilde{\mathcal{H}}$  we see that this bimodule is contained in a direct sum of the bimodule  $L^2\langle\alpha(N), \alpha(L^\infty(X_1, \mu_1))\rangle$  coming from the basic construction of  $(\alpha(L^\infty(X, \mu)) \subset \alpha(N))$ . Indeed, this follows because the completely positive maps corresponding to left and right bounded vectors of the form  $1 \otimes \xi \in L^2N\overline{\otimes}\mathcal{H}$  are easily seen to live in  $L^2\langle\alpha(N), \alpha(L^\infty(X_1, \mu_1))\rangle$ .

The  $\alpha(N)$ - $\alpha(N)$  bimodule  $L^2\langle\alpha(N), \alpha(L^\infty(X_1, \mu_1))\rangle$  is an orbit equivalence invariant and is canonically isomorphic to the bimodule coming from the left regular representation of  $\Gamma_1$  (see for example Section 1.1.4 in [70]). It therefore follows that  $\tilde{\mathcal{H}}$  when viewed as an  $\alpha(L\Gamma_1)$  bimodule embeds into a direct sum of coarse bimodules.

We consider the closable derivation  $0 \otimes \delta : N\overline{\otimes}M \rightarrow \tilde{\mathcal{H}}$  as defined in Section II.5.2 and use the fact that  $L\Gamma_1$  is  $L^2$ -rigid to conclude that the corresponding deformation  $\text{id} \otimes \eta_\alpha$  converges uniformly on the unit ball of  $\alpha(N)$ , (note that  $\text{id} \otimes \eta_\alpha$  is the identity on  $\alpha(L^\infty(X_1, \mu_1)) = \alpha(L^\infty(X_2, \mu_2))$ ). In particular,  $\text{id} \otimes \eta_\alpha$  converges uniformly on  $\{\alpha(u_\gamma) \mid \gamma \in \Gamma_2\}$  which shows that  $\eta_\alpha$  converges uniformly on  $\{u_\gamma \mid \gamma \in \Gamma_2\}$ . As this is a group which generates  $L\Gamma_2$  we may then use a standard averaging argument to conclude that  $\eta_\alpha$  converges uniformly on the unit ball of  $L\Gamma_2$ , (see for example Theorem 4.1.7 in [68]).  $\square$

**Remark II.6.4.** The above argument will also work to show that the “ $L^2$ -Haagerup property” (see [64]) is preserved by orbit equivalence. In particular, this gives a new way to show that the von Neumann algebra of a group  $\Gamma$  which is orbit equivalent to free groups is solid in the sense of Ozawa [54], i.e.  $B' \cap L\Gamma$  is amenable whenever  $B \subset L\Gamma$  does not have minimal projections. Solidity of group von Neumann algebras for groups which are orbit equivalent to free groups was first shown by Sako in [84].

We also note that by [22] any group which is orbit equivalent to a free group will have the complete metric approximation property. It will no doubt follow by using the techniques in [61] that the von Neumann algebra of a group  $\Gamma$  which is orbit equivalent to a free group will be strongly solid (see Definition III.1.1).

Examples of groups which are orbit equivalent to a free group can be found in [28], and [6].

## CHAPTER III

### STRONG SOLIDITY FOR GROUP FACTORS FROM LATTICES IN $\mathrm{SO}(N, 1)$ AND $\mathrm{SU}(N, 1)$

#### III.1 Introduction

In their paper [60], Ozawa and Popa brought new techniques to bear on the study of free group factors which allowed them to show that these factors possess a powerful structural property, what they called “strong solidity.”

**Definition III.1.1** (Ozawa–Popa [60]). A  $\mathrm{II}_1$  factor  $M$  is *strongly solid* if for any diffuse amenable subalgebra  $P \subset M$  we have that  $\mathcal{N}_M(P)''$  is amenable.

As usual,  $\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : uPu^* = P\}$  denotes the *normalizer* of  $P$  in  $M$ . It can be seen that every nonamenable  $\mathrm{II}_1$  subfactor of a strongly solid  $\mathrm{II}_1$  factor is non-Gamma, prime and has no Cartan subalgebras. (A maximal abelian  $*$ -subalgebra  $A \subset M$  is a Cartan subalgebra if  $\mathcal{N}_M(A)'' = M$ .) Thus, Ozawa and Popa’s result broadened and offered a unified approach to the two main results on the structure of free group factors hitherto known: Voiculescu’s [101] pioneering result, which showed that the free group factors  $L\mathbb{F}_n$ ,  $2 \leq n \leq \infty$ , have no Cartan subalgebras, and Ozawa’s [54] seminal work on “solid” von Neumann algebras, which showed that every nonamenable  $\mathrm{II}_1$  subfactor of a free group factor is non-Gamma and prime. Moreover, they exhibited the first, and so far only, examples of  $\mathrm{II}_1$  factors with a unique Cartan up to unitary conjugacy; namely, the group-measure space constructions of free ergodic profinite actions of groups with property  $(\mathrm{HH})^+$  [61]; e.g., nonamenable free groups. This improved on the ground-breaking work of Popa [70], which gave examples of  $\mathrm{II}_1$  factors with a unique “HT-Cartan” subalgebra up to unitary equivalence; e.g.,  $L(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}))$ .

By incorporating ideas and techniques of Peterson [64], Ozawa and Popa [61] were later able to extend the class of strongly solid factors to, in particular, all group factors of i.c.c. lattices in  $\mathrm{PSL}(2, \mathbb{R})$  or  $\mathrm{PSL}(2, \mathbb{C})$ . Other examples of strongly solid factors were subsequently constructed by Houdayer [34] and by Houdayer and Shlyakhtenko [35].

By a *lattice* we mean a discrete subgroup  $\Gamma < G$  of some Lie group with finitely many connected components such that  $G/\Gamma$  admits a regular Borel probability measure invariant under left



translation by  $G$ . The main goal of this chapter will be to demonstrate the following result:

**Theorem III.1.2.** *If  $\Gamma$  is an i.c.c. lattice in  $\mathrm{SO}(n, 1)$  or  $\mathrm{SU}(n, 1)$ , then  $L\Gamma$  is strongly solid.*

These factors are already known by the work of Ozawa and Popa [61] to have no Cartan subalgebras. Since  $\mathrm{SO}(n, 1)$  and  $\mathrm{SU}(n, 1)$  are simple Lie groups with finite center, Borel’s density theorem via Theorem 6.5 in [20] shows that every  $\gamma \in \Gamma$  which is not in the center of  $G$  has infinite  $\Gamma$ -conjugacy class, so examples of i.c.c. lattices abound. In the  $\mathrm{SO}(n, 1)$  case, the restriction of the lattice subgroup  $\Gamma$  to the connected component of the identity  $\mathrm{SO}(n, 1)_0$  is always i.c.c.  $\mathrm{SO}(n, 1)_0$  having trivial center, and all results in this chapter will hold for these groups as well. In particular, we have that  $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{SO}(2, 1)_0 \cong \mathrm{SU}(1, 1)$  and  $\mathrm{PSL}(2, \mathbb{C}) \cong \mathrm{SO}(3, 1)_0$ , so Theorem III.1.2 recovers the main result in Ozawa–Popa [61]. Finally, notice that if  $G$  is a Lie group with finite center and finitely many connected components which is locally isomorphic to  $\mathrm{SO}(n, 1)$ , then it is a finite-to-one covering of—hence, a finite extension of— $\mathrm{SO}(n, 1)_0$ . Cohomological induction, combined with the techniques below, will then be sufficient to show that the group von Neumann algebra of any i.c.c. lattice in such a Lie group is also strongly solid.

The proof follows the same strategy as Ozawa and Popa’s in [60, 61]. Though, instead of working with closable derivations, we use a natural one-parameter family of deformations first constructed by Parthasarathy and Schmidt [62]. The derivations Ozawa and Popa consider appear as the infinitesimal generators of these deformations (so, the approaches are largely equivalent), but by using the deformations we avoid some of the technical issues which arise when working with derivations.

The main difficulty in obtaining Theorem III.1.2 for lattice factors in  $\mathrm{SO}(n, 1)$  or  $\mathrm{SU}(m, 1)$  when  $n \geq 4$  or  $m \geq 2$  is that the bimodules which admit good deformations/derivations are themselves too weak to allow one to deduce the amenability of the normalizer algebra e.g., strong solidity. However, sufficiently large tensor powers of these bimodules can be used to deduce strong solidity. Unfortunately, derivation techniques perturb the original bimodules slightly, and the behavior of tensor powers of the perturbed bimodules becomes unclear. To circumvent this problem, we first notice that Ozawa and Popa’s techniques actually allow one to deduce a kind of relative amenability of the normalizer subalgebra with respect to the bimodule, given in terms of an “invariant mean”. We then use a result of Sauvageot [85] to obtain from the invariant mean an almost invariant

sequence of vectors in the bimodule. Since the property of having an almost invariant sequence of vectors is stable under taking tensor powers, we are able to transfer relative amenability to a large tensor power of the bimodule in order to deduce amenability of the normalizer algebra.

We remark that as a corollary to the techniques used in the proof of Theorem III.1.2, we are able to strengthen a result of Houdayer [34] on free product group factors admitting no Cartan subalgebras.

**Theorem III.1.3.** *Let  $\Gamma$  be a nonamenable, countable, discrete group which has the complete metric approximation property (Definition III.2.6). If  $\Gamma \cong \Gamma_1 * \Gamma_2$  decomposes as a non-trivial free product, then  $L\Gamma$  has no Cartan subalgebras. Moreover, if  $N \subset L\Gamma$  is a nonamenable subfactor which has a Cartan subalgebra, then there exists projections  $p_1, p_2$  in the center of  $N' \cap L\Gamma$  such that  $p_1 + p_2 = 1$  and unitaries  $u_1, u_2 \in \mathcal{U}(M)$  such that  $u_i N p_i u_i^* \subset L\Gamma_i \subset L\Gamma$ ,  $i \in \{1, 2\}$ .*

## III.2 Preliminaries

We collect in this section the necessary definitions, concepts and results needed for the proofs of Theorems III.1.2 and III.1.3.

### III.2.1 Representations, correspondences, and weak containment

Let  $\Gamma$  be a countable discrete group and  $\pi, \rho$  be unitary representations of  $\Gamma$  into separable Hilbert spaces  $\mathcal{H}_\pi$  and  $\mathcal{H}_\rho$ , respectively.

**Definition III.2.1.** We say that  $\rho$  is *weakly contained* in  $\pi$  if for any  $\varepsilon > 0$ ,  $\xi \in \mathcal{H}_\rho$  and any finite subset  $F \subset \Gamma$ , there exist vectors  $\xi'_1, \dots, \xi'_n \in \mathcal{H}_\pi$  such that  $|\langle \rho(\gamma)\xi, \xi \rangle - \sum_{i=1}^n \langle \pi(\gamma)\xi'_i, \xi'_i \rangle| < \varepsilon$  for all  $\gamma \in F$ .

A representation  $\pi$  is said to be *tempered* if it is weakly contained in the left-regular representation, and *strongly  $\ell^p$*  [91] if for any  $\varepsilon > 0$ , there exists a dense subspace  $\mathcal{H}_0 \subset \mathcal{H}$  such that for all  $\xi, \eta \in \mathcal{H}_0$  the matrix coefficient  $\langle \pi(\gamma)\xi, \eta \rangle$  belongs to  $\ell^{p+\varepsilon}(\Gamma)$ . By a theorem of Cowling, Haagerup and Howe [21], a representation which is strongly  $\ell^2$  is tempered. As was pointed out in [91], applying standard Hölder estimates to the matrix coefficients, we obtain that if  $\pi$  is strongly  $\ell^p$  for some  $p \geq 2$ , then for all  $n > p/2$ ,  $\pi^{\otimes n}$  is strongly  $\ell^2$ , hence tempered.

In the theory of von Neumann algebras, correspondences play an analogous role to unitary representations in the theory of countable discrete groups. For von Neumann algebras  $N$  and  $M$ , recall that an  $N$ - $M$  correspondence is a  $*$ -representation  $\pi$  of the algebraic tensor  $N \odot M^o$  into the bounded operators on a Hilbert space  $\mathcal{H}$  which is normal when restricted to both  $N$  and  $M^o$ . We will denote the restrictions of  $\pi$  to  $N$  and  $M^o$  by  $\pi_N$  and  $\pi_{M^o}$ , respectively. When the  $N$ - $M$  correspondence  $\pi$  is implicit for the Hilbert space  $\mathcal{H}$ , we will use the notation  $x\xi y$  to denote  $\pi(x \otimes y^o)\xi$ , for  $x \in N$ ,  $y \in M$  and  $\xi \in \mathcal{H}$ .

**Definition III.2.2.** Let  $\pi : N \odot M^o \rightarrow \mathbb{B}(\mathcal{H}_\pi)$ ,  $\rho : N \odot M^o \rightarrow \mathbb{B}(\mathcal{H}_\rho)$  be correspondences. We say that  $\rho$  is *weakly contained* in  $\pi$  if for any  $\varepsilon > 0$ ,  $\xi \in \mathcal{H}_\rho$  and any finite subsets  $F_1 \subset N$ ,  $F_2 \subset M$ , there exist vectors  $\xi_1, \dots, \xi_n \in \mathcal{H}_\pi$  such that  $|\langle x\xi y, \xi \rangle - \sum_{i=1}^n \langle x\xi_i^o y, \xi_i^o \rangle| < \varepsilon$  for all  $x \in F_1$ ,  $y \in F_2$ .

There is a well-known functor from the category whose objects are (separable) unitary representations of  $\Gamma$  and morphisms weak containment to the one of  $L\Gamma$ - $L\Gamma$  correspondences and weak containment, cf. [80], which translates the representation theory of  $\Gamma$  into the theory of  $L\Gamma$ - $L\Gamma$  correspondences. The construction is as follows. Given  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  a unitary representation, let  $\mathcal{H}^\pi$  be the Hilbert space  $\mathcal{H}_\pi \otimes \ell^2\Gamma$ . Then, the maps  $u_\gamma(\xi \otimes \eta) = \pi(\gamma)\xi \otimes u_\gamma\eta$ ,  $(\xi \otimes \eta)u_\gamma = \xi \otimes (\eta u_\gamma)$  extend to commuting normal representations of  $L\Gamma$  and  $(L\Gamma)^o$  on  $\mathcal{H}^\pi$ : the former by Fell's absorption principle, the latter trivially. This functor is well-behaved with respect to tensor products; i.e.,  $\mathcal{H}^{\pi \otimes \rho} \cong \mathcal{H}^\pi \otimes_{L\Gamma} \mathcal{H}^\rho$  as  $L\Gamma$ - $L\Gamma$  correspondences for any unitary  $\Gamma$ -representations  $\pi$  and  $\rho$ . We refer the reader to [1, 5, 68] for the theory of tensor products of correspondences and the basic theory of correspondences in general.

For a  $\text{II}_1$  factor  $M$  there are two canonical correspondences: the *trivial correspondence*,  $L^2(M)$  with  $M$  acting by left left and right multiplication, and the *coarse correspondence*,  $L^2(M) \otimes L^2(\bar{M})$  with  $M$  acting by left multiplication of the left copy of  $L^2(M)$  and right multiplication on the right copy. When  $M = L\Gamma$  for some countable discrete group, the trivial and coarse correspondences are the correspondences induced respectively by the trivial and left regular representations of  $\Gamma$ .

### III.2.2 Cocycles and the Gaussian construction

In this section,  $\mathcal{H}$  will denote a *real* Hilbert space which we will fix along with a orthogonal representation  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$  of some countable discrete group  $\Gamma$ .

**Definition III.2.3.** A cocycle is a map  $b : \Gamma \rightarrow \mathcal{H}$  satisfying the cocycle relation

$$b(\gamma\gamma') = b(\gamma) + \pi(\gamma)b(\gamma'), \text{ for all } \gamma, \gamma' \in \Gamma.$$

Given  $\pi$  and  $\mathcal{H}$ , there is a canonical standard probability space  $(X, \mu)$  and a canonical measure-preserving action  $\Gamma \curvearrowright^\sigma (X, \mu)$  such that there is a Hilbert space embedding of  $\mathcal{H}$  into  $L^2_{\mathbb{R}}(X, \mu)$  intertwining  $\pi$  and the natural representation induced on  $L^2_{\mathbb{R}}(X, \mu)$  by  $\sigma$ . This is known as the *Gaussian construction*, cf. [65] or [89]. It is well-known that the natural  $\Gamma$  representation  $\sigma_0$  on  $L^2_0(X, \mu) = L^2(X, \mu) \ominus \mathbb{C}1_X$  inherits all “stable” properties from  $\pi$ , cf. [65]. In particular,  $\sigma_0^{\otimes n}$  is tempered if and only if  $\pi^{\otimes n}$  is tempered for any  $n \geq 1$ .

It was discovered by Parthasarathy and Schmidt [62] that cocycles also fit well into the framework of the Gaussian construction, inducing one-parameter families of deformations (i.e., cocycles) of the action  $\sigma$ . To be precise:

**Theorem III.2.4** (Parthasarathy–Schmidt [62]). *Let  $b : \Gamma \rightarrow \mathcal{H}$  be a cocycle, then there exists a one-parameter family  $\omega_t : \Gamma \rightarrow \mathcal{U}(L^\infty(X, \mu))$ ,  $t \in \mathbb{R}$  such that:*

$$\omega_t(\gamma\gamma') = \omega_t(\gamma)\sigma_\gamma(\omega_t(\gamma')), \text{ for all } \gamma, \gamma' \in \Gamma; \text{ and} \tag{III.2.1}$$

$$\int \omega_t(\gamma) d\mu = \exp(-(t\|b(\gamma)\|)^2), \text{ for all } \gamma \in \Gamma. \tag{III.2.2}$$

### III.2.3 Weak compactness and the CMAP

**Definition III.2.5** (Ozawa–Popa [60]). Let  $(P, \tau)$  be a finite von Neumann algebra equipped with a trace  $\tau$ , and  $G \curvearrowright^\sigma P$  be an action of a group  $G$  on  $P$  by  $\tau$ -preserving  $*$ -automorphisms. We say that the action  $\sigma$  is *weakly compact* if there exists a net of unit vectors  $(\eta_k) \in L^2(P \bar{\otimes} \bar{P}, \tau \otimes \bar{\tau})_+$  such that:

1.  $\|\eta_k - (v \otimes \bar{v})\eta_k\| \rightarrow 0$ , for all  $v \in \mathcal{U}(P)$ ;
2.  $\|\eta_k - (\sigma_g \otimes \bar{\sigma}_g)\eta_k\| \rightarrow 0$ , for all  $g \in G$ ; and
3.  $\langle (x \otimes 1)\eta_k, \eta_k \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_k, \eta_k \rangle$ , for all  $x \in P$ .

**Definition III.2.6.** A  $\text{II}_1$  factor  $M$  is said to have the *complete metric approximation property* (CMAP) if there exists a net  $(\varphi_i)$  of finite rank, normal, completely bounded maps  $\varphi_i : M \rightarrow M$  such that  $\limsup \|\varphi_i\|_{\text{cb}} \leq 1$  and such that  $\|\varphi_i(x) - x\|_2 \rightarrow 0$ , for all  $x \in M$ .

If  $\Gamma$  is an i.c.c. countable discrete group, then  $L\Gamma$  has the CMAP if and only if the Cowling-Haagerup constant of  $\Gamma$ ,  $\Lambda_{\text{cb}}(\Gamma)$ , equals 1, and if  $\Gamma$  is a lattice in  $G$ , then  $\Lambda_{\text{cb}}(\Gamma) = \Lambda_{\text{cb}}(G)$ , cf. §12.3 of [7] and [32].

**Theorem III.2.7** (Ozawa–Popa, Theorem 3.5 of [60]). *Let  $M$  be a  $\text{II}_1$  factor which has the CMAP. Then for any diffuse amenable  $*$ -subalgebra  $A \subset M$ ,  $\mathcal{N}_M(A)$  acts weakly compactly on  $A$  by conjugation.*

### III.3 Amenable correspondences

**Definition III.3.1** (Anantharaman-Delaroche [1]). An  $N$ - $M$  correspondence  $\mathcal{H}$  is called (*left*) *amenable* if  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  weakly contains the trivial  $N$ - $N$  correspondence.

The concept of amenability for correspondences is the von Neumann algebraic analog of the concept of amenability of a unitary representation of a locally compact group due to Bekka [2]. As was observed by Bekka, amenability of the representation  $\pi$  is equivalent to the existence of a state  $\Phi$  on  $\mathbb{B}(\mathcal{H})$  satisfying  $\Phi(\pi(g)T) = \Phi(T\pi(g))$  for all  $g \in G$ ,  $T \in \mathbb{B}(\mathcal{H})$ . One can ask if a similar criterion holds for amenable correspondences. When  $M$  is a  $\text{II}_1$  factor, we will show that this indeed is the case if we replace  $\mathbb{B}(\mathcal{H})$  with the von Neumann algebra  $\mathcal{N} = \mathbb{B}(\mathcal{H}) \cap \pi_{M \circ}(M^{\circ})'$ . That is, we obtain the following characterization of amenable correspondences:

**Theorem III.3.2** (Compare with Theorem 2.1 in [60]). *Let  $\mathcal{H}$  be an  $N$ - $M$  correspondence with  $N$  finite with normal faithful trace  $\tau$  and  $M$  a  $\text{II}_1$  factor. Let  $P \subset N$  be a von Neumann subalgebra. Then the following are equivalent:*

1. *there exists a net  $(\xi_n)$  in  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  such that  $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$  for all  $x \in N$  and  $\|[u, \xi_n]\| \rightarrow 0$  for all  $u \in \mathcal{U}(P)$ ;*
2. *there exists a  $P$ -central state  $\Phi$  on  $\mathcal{N}$  such that  $\Phi$  is normal when restricted to  $N$  and faithful when restricted to  $\mathcal{Z}(P' \cap N)$ ;*

3. there exists a  $P$ -central state  $\Phi$  on  $\mathcal{N}$  which restricts to  $\tau$  on  $N$ .

We do so by constructing a normal, faithful, semi-finite (tracial) weight  $\bar{\tau}$  on  $\mathcal{N}$  which canonically realizes  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  as  $\mathcal{L}^2(\mathcal{N}, \bar{\tau})$ . An identical construction to the one we propose has already appeared in the work of Sauvageot [85] for an arbitrary factor  $M$ . However, we present an elementary approach in the  $\text{II}_1$  case.

Before presenting the details, we pause here to illustrate how Theorem III.3.2 generalizes (relative) amenability for  $\text{II}_1$  factors. Let  $M$  be a  $\text{II}_1$  factor and  $\mathcal{H} = L^2 M \otimes L^2 \bar{M}$  be the coarse  $M$ - $M$  correspondence. We then have explicitly that  $\mathcal{M} = \mathbb{B}(\mathcal{H}) \cap (M^\circ)' = \mathbb{B}(L^2 M) \bar{\otimes} M$ ; hence,  $\mathcal{M}$  has an  $M$ -central state which restricts to the trace on  $M$  if and only if the trace on  $M$  extends to a hypertrace on  $\mathbb{B}(L^2 M)$ , i.e.,  $M$  is amenable. Similarly, if  $P, Q \subset M$  are von Neumann subalgebras and  $\mathcal{H} = L^2 \langle M, e_Q \rangle$ , where  $\langle M, e_Q \rangle$  denotes the basic construction of Jones [40], then  $\langle M, e_Q \rangle \subset \mathbb{B}(\mathcal{H}) \cap (M^\circ)'$ . So, if  $P \subset M$  satisfies one of the conditions in the above theorem, then  $P$  is relatively amenable to  $Q$  inside  $M$  ( $P \triangleleft_M Q$ ) in the sense of Theorem 2.1 in [60]. Conversely, if  $P \triangleleft_M Q$ , then using condition (4) in Theorem 2.1 of [60], one can construct such a functional  $\Phi$  on  $\mathbb{B}(\mathcal{H}) \cap (M^\circ)'$  as in condition (3) in the above theorem for  $P \subset M$ .

We recall from Chapter II that the right-bounded vectors form a dense subspace of  $\mathcal{H}$  which we will denote by  $\mathcal{H}_b$ . We also recall, regarding  $\mathcal{H}$  as a right Hilbert  $M$ -module, that we can define a natural  $M$ -valued inner product on  $\mathcal{H}_b$ , which we will denote  $(\xi|\eta) \in M$  for  $\xi, \eta \in \mathcal{H}_b$ , by setting  $(\xi|\eta)$  to be the Radon-Nikodym derivative of the normal functional  $x \mapsto \langle \xi x, \eta \rangle$ .

Let  $\mathcal{N} = \mathbb{B}(\mathcal{H}) \cap \pi_{M^\circ}(M^\circ)'$ . (For instance, if  $\tilde{M} \supset M$  is a tracial inclusion of  $\text{II}_1$  factors and  $\mathcal{H} = L^2(\tilde{M})$ , considered as a Hilbert  $M$ - $M$  bimodule in the natural way, then we have that  $(\hat{x}|\hat{y}) = E_M(y^*x)$  for all  $x, y \in \tilde{M}$  and  $\mathbb{B}(\mathcal{H}) \cap (M^\circ)' = \langle \tilde{M}, e_M \rangle$ , where  $E_M$  is the trace preserving conditional expectation from  $\tilde{M}$  onto  $M$ .) For  $\xi, \eta \in \mathcal{H}_b$ , let  $T_{\xi, \eta} : \mathcal{H}_b \rightarrow \mathcal{H}_b$  be the ‘‘rank one operator’’ given by  $T_{\xi, \eta}(\cdot) = \xi(\cdot|\eta)$ . Then  $T_{\xi, \eta}$  extends to a bounded operator with  $\|T_{\xi, \eta}\|_\infty^2 \leq \|(\xi|\xi)\|_\infty \|(\eta|\eta)\|_\infty$  [81]. Notice that  $T_{\xi, \xi} \geq 0$  and that  $T_{\xi, \xi}$  is a projection if  $(\xi|\xi) \in \mathcal{P}(M)$ . Since  $T_{\xi, \eta} \pi_{M^\circ}(x) = \pi_{M^\circ}(x) T_{\xi, \eta}$  for all  $x \in M^\circ$ , we have that  $T_{\xi, \eta} \in \mathcal{N}$ . It is easy to see that the span of all such operators  $T_{\xi, \eta}$  is a  $*$ -subalgebra of  $\mathbb{B}(\mathcal{H})$  which we will denote by  $\mathcal{N}_f$ . Noticing that for any  $S \in \mathcal{N}$ ,  $S(\mathcal{H}_b) \subset \mathcal{H}_b$ , we have that  $S T_{\xi, \eta} = T_{S\xi, \eta}$  and  $T_{\xi, \eta} S = T_{\xi, S^*\eta}$ . It follows that  $\mathcal{N}_f$  is an ideal of  $\mathcal{N}$  which can be considered as the analog of the finite rank operators in  $\mathbb{B}(\mathcal{H})$ . The following

lemma further cements this analogy.

**Lemma III.3.3.** *If  $M$  is a  $\text{II}_1$  factor, we have that  $\mathcal{N}'_f \cap \mathbb{B}(\mathcal{H}) = \pi_{M^\circ}(M^\circ)$ ; hence,  $\mathcal{N}_f + \mathbb{C}1_{\mathbb{B}(\mathcal{H})}$  is weakly dense in  $\mathcal{N}$ .*

*Proof.* The inclusion  $\pi_{M^\circ}(M^\circ) \subset \mathcal{N}'_f \cap \mathbb{B}(\mathcal{H})$  is trivial. Conversely, let  $T \in \mathcal{N}'_f \cap \mathbb{B}(\mathcal{H})$  and choose a non-zero  $\zeta \in \mathcal{H}_b$  such that  $(\zeta|\zeta) = p \in \mathcal{P}(M)$ . (One can always find such a  $\zeta$  as  $\mathcal{H}$  has an orthonormal basis of right bounded vectors as a right Hilbert  $M$ -module.) Choose a sequence  $\eta_i \in \mathcal{H}_b$  such that  $\|\eta_i - T\zeta\| \rightarrow 0$  and let  $y_i = (\eta_i|\zeta)$ . Then for every  $\xi \in \mathcal{H}_b$  we have that  $\|T(\xi p) - \xi y_i\| \leq \|T_{\xi, \zeta}\|_\infty \|\eta_i - T\zeta\| \leq \|(\xi|\xi)\|_\infty^{1/2} \|\eta_i - T\zeta\|$  so, the sequence  $(\pi_{M^\circ}(y_i^\circ))$  converges in the strong topology to  $T \circ \pi_{M^\circ}(p^\circ)$ . Hence,  $T \circ \pi_{M^\circ}(p^\circ) \in \pi_{M^\circ}(M^\circ)'' = \pi_{M^\circ}(M^\circ)$ . Since  $M$  is a  $\text{II}_1$  factor, by repeating the argument with  $\zeta' = \zeta u$  for  $u \in \mathcal{U}(M)$  and using standard averaging techniques, we conclude that there exists  $y_T \in M$  such that  $T\xi = \xi y_T$  for all  $\xi \in \mathcal{H}$ . Thus  $T = \pi_{M^\circ}(y_T^\circ)$ .  $\square$

Now, consider an element  $\varphi \in M_*$ , and define a functional  $\bar{\varphi} \in (\mathcal{N}_f)^*$  by  $\bar{\varphi}(T_{\xi, \eta}) = \varphi((\xi|\eta))$ . It is easy to see that  $\bar{\varphi}$  is normal on  $\mathcal{N}_f$  and so, by the preceding lemma, may be extended to a normal semi-finite weight on  $\mathcal{N}$ . Hence, we may construct for each such  $\varphi$  a noncommutative  $L^p$ -space over  $\mathcal{N}$ ,  $\mathcal{L}^p(\mathcal{N}, \bar{\varphi}) = \{T \in \mathcal{N} : \|T\|_p = \bar{\varphi}(|T|^p)^{1/p} < \infty\}$ . If  $M$  is a  $\text{II}_1$  factor with trace  $\tau$ , then  $\bar{\tau}$  is a normal, faithful, semi-finite trace on  $\mathcal{N}$  and we denote  $\mathcal{L}^p(\mathcal{N}, \bar{\tau})$  simply by  $\mathcal{L}^p(\mathcal{N})$ . In the case of  $\mathcal{L}^2(\mathcal{N})$ , we compute that  $\|T_{\xi, \eta}\|_2^2 = \tau((\xi|\xi)(\eta|\eta)) = \langle \xi(\eta|\eta), \xi \rangle$ . This shows that the map which sends  $T_{\xi, \eta}$  to the elementary  $M$ -tensor  $\xi \otimes_M \bar{\eta} \in \mathcal{H} \otimes_M \bar{\mathcal{H}}$  extends to an  $N$ - $N$  bimodular Hilbert space isometry from  $\mathcal{L}^2(\mathcal{N})$  to  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$ . We are now ready to prove the motivating result in this section.

*Proof of Theorem III.3.2.* The proof of (1)  $\Leftrightarrow$  (3) follows the usual strategy. For (1)  $\Rightarrow$  (3), we have that there exists a net  $(\xi_n)$  of vectors in  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  such that  $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(x)$  for all  $x \in N$  and  $\|[u, \xi_n]\| \rightarrow 0$  for all  $u \in \mathcal{U}(P)$ . Viewing  $\xi_n$  as an element of  $\mathcal{L}^2(\mathcal{N})$ , let  $\Phi_n \in \mathcal{N}_*$  be given by  $\Phi_n(T) = \bar{\tau}(\xi_n \xi_n^* T)$  for any  $T \in \mathcal{N}$ . Then, by the generalized Powers-Størmer inequality (Theorem IX.1.2 in [95]), we have that  $|\Phi_n(x) - \tau(x)| \rightarrow 0$  for all  $x \in N$  and  $\|\text{Ad}(u)\Phi_n - \Phi_n\|_1 \rightarrow 0$  for all  $u \in \mathcal{U}(P)$ . Taking a weak cluster point of  $(\Phi_n)$  in  $\mathcal{N}^*$  gives the required  $N$ -tracial  $P$ -central state on  $\mathcal{N}$ . Conversely, given such a state  $\Phi$ , we can find a net  $(\eta_n)$  in  $\mathcal{L}^1(\mathcal{N})_+$  such that  $\Phi_n(T) = \bar{\tau}(\eta_n T)$

weakly converges to  $\Phi$ . In fact, by passing to convex combinations we may assume  $\|[u, \eta_n]\|_1 \rightarrow 0$  for all  $u \in \mathcal{U}(P)$ . By another application of the generalized Powers-Størmer inequality, it is easy to check that  $\xi_n = \eta_n^{1/2} \in \mathcal{L}^2(\mathcal{N}) \cong \mathcal{H} \otimes_M \bar{\mathcal{H}}$  satisfies the requirements of (1).

We now need only show (2)  $\Rightarrow$  (3) as (3)  $\Rightarrow$  (2) is trivial. But this is exactly the averaging trick found in the proof of (2)  $\Rightarrow$  (1) in Theorem 2.1 of [60]. We repeat the argument here for the sake of completeness. Since  $\Phi$  is normal on  $N$ , we have that for some  $\eta \in \mathcal{L}^1(N)_+$ ,  $\Phi(x) = \tau(\eta x)$  for all  $x \in N$ . In fact,  $\eta \in \mathcal{L}^1(P' \cap N)_+$  since  $\Phi$  is  $P$ -central. Denoting by  $\mathcal{F}$  the net of finite subsets of  $\mathcal{U}(P' \cap N)$  under inclusion, for any  $\varepsilon > 0$ , we set  $\eta_F = \frac{1}{|F|} \sum_{u \in F} u \eta u^*$  and  $\xi_{F,\varepsilon} = (\chi_{[\varepsilon, \infty)}(\eta_F)) \eta_F^{-1/2}$ . We now let  $\Psi_{F,\varepsilon}(T) = \frac{1}{|F|} \sum_{u \in F} \Phi(u \xi_{F,\varepsilon} T \xi_{F,\varepsilon} u^*)$ . Note that  $\Psi_{F,\varepsilon}$  is still  $P$ -central. Now it is easy to see that  $\lim_{\mathcal{F}, \varepsilon} \chi_{[\varepsilon, \infty)}(\eta_F) = z$ , where  $z$  is the central support of  $\eta$ . But by the faithfulness of  $\Phi$  on  $\mathcal{Z}(P' \cap N)$ , we see that  $z = 1$ . Hence, any weak cluster point of  $(\Psi_{F,\varepsilon})_{\mathcal{F}, \varepsilon}$  in  $\mathcal{N}^*$  is a  $P$ -central state which when restricted to  $N$  is  $\tau$ .  $\square$

**Corollary III.3.4** (generalized Haagerup's criterion for amenability). *Let  $N, M$  be  $\text{II}_1$  factors and  $\mathcal{H}$  an  $N$ - $M$  correspondence. If  $P \subset N$  is a von Neumann subalgebra, then  $\mathcal{H}$  is left amenable over  $P$  (in the sense of Theorem III.3.2) if and only if for every non-zero projection  $p \in \mathcal{Z}(P' \cap N)$  and finite subset  $F \subset \mathcal{U}(P)$ , we have*

$$\left\| \sum_{u \in F} u p \otimes \overline{u p} \right\|_{\mathcal{H} \otimes_M \bar{\mathcal{H}}, \infty} = |F|,$$

where  $\|\cdot\|_{\mathcal{H} \otimes_M \bar{\mathcal{H}}, \infty}$  denotes the operator norm on  $\mathbb{B}(\mathcal{H} \otimes_M \bar{\mathcal{H}})$ .

*Proof.* Since we have obtained a ‘‘hypertrace’’ characterization of amenability for correspondences in Theorem III.3.2, the result follows by the same arguments as in Lemma 2.2 in [31].  $\square$

**Definition III.3.5** (cf. Definition 1.3 in [65]). Let  $M$  be a  $\text{II}_1$  factor,  $\mathcal{H}$  an  $M$ - $M$  correspondence and  $P^c$  the orthogonal projection onto the subspace  $\mathcal{H}^c = \{\xi \in \mathcal{H} : x \xi = \xi x, \forall x \in M\}$ . The correspondence  $\mathcal{H}$  has *spectral gap* if for every  $\varepsilon > 0$  there exist  $\delta > 0$  and  $x_1, \dots, x_n \in M$  such that if  $\|x_i \xi - \xi x_i\| < \delta, i = 1, \dots, n$ , then  $\|\xi - P^c \xi\| < \varepsilon$ . The correspondence  $\mathcal{H}$  has *stable spectral gap* if  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  has spectral gap.

Note that if  $\mathcal{H}$  has stable spectral gap, then  $\mathcal{H}$  is amenable if and only if  $(\mathcal{H} \otimes_M \bar{\mathcal{H}})^c \neq \{0\}$ . Hence, we say an  $M$ - $M$  correspondence  $\mathcal{H}$  is *nonamenable* if it has stable spectral gap and



$(\mathcal{H} \otimes_M \bar{\mathcal{H}})^c = \{0\}$ . The following theorem is the analog of Lemma 3.2 in [77] for the category of correspondences. N.B. Stable spectral gap as defined in [77] corresponds to our definition of nonamenability.

**Theorem III.3.6.** *Let  $M$  be a  $\text{II}_1$  factor and  $\mathcal{H}$  an  $M$ - $M$  correspondence. Then  $\mathcal{H}$  is nonamenable if and only if  $\mathcal{H} \otimes_M \bar{\mathcal{K}}$  has spectral gap and for any  $M$ - $M$  correspondence  $\mathcal{K}$ .*

*Proof.* Let  $\mathcal{H}_b, \mathcal{K}_b$  denote subspaces of right-bounded vectors in  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. Given  $\xi \in \mathcal{H}_b$  and  $\eta \in \mathcal{K}_b$ , by the same arguments as above we can define a bounded operator  $T_{\xi, \eta} : \mathcal{K} \rightarrow \mathcal{H}$  by  $T_{\xi, \eta}(\cdot) = \xi(\cdot|\eta)$ . As above, one may check that  $\|(T^*T)^{1/2}\|_2 = \|\xi \otimes_M \bar{\eta}\| = \|(TT^*)^{1/2}\|_2$  so that  $\mathcal{H} \otimes_M \mathcal{K}$  is isometric to a Hilbert-normed subspace of the bounded right  $M$ -linear operators from  $\mathcal{H}$  to  $\mathcal{K}$ , which we denote  $\mathcal{L}^2(\mathcal{H}, \mathcal{K})$ . Moreover, this identification is natural with respect to the  $M$ - $M$  bimodular structure on  $\mathcal{L}^2(\mathcal{H}, \mathcal{K})$  given by  $xT_{\xi, \eta}y = T_{x\xi, y^*\eta}$ .

We need now only prove the forward implication, as the converse is trivial. Let us fix some arbitrary  $M$ - $M$  correspondence  $\mathcal{K}$ . From Proposition 1.4 in [65], we have that if  $(\mathcal{H} \otimes_M \bar{\mathcal{H}})^c = \{0\}$ , then  $(\mathcal{H} \otimes_M \bar{\mathcal{K}})^c = \{0\}$ . So, by way of contradiction, we may assume that for every  $\varepsilon > 0$  and  $x_1, \dots, x_n \in M$ , there exists a unit vector  $\xi \in \mathcal{H} \otimes_M \bar{\mathcal{K}}$  such that  $\|x_i\xi - \xi x_i\|_2 \leq \varepsilon$ ,  $i = 1, \dots, n$ . Without loss of generality, we may assume  $x_1, \dots, x_n$  are unitaries. Viewing  $\xi$  as an element of  $\mathcal{L}^2(\mathcal{H}, \mathcal{K})$ , let  $\eta = (\xi^*\xi)^{1/2} \in \mathcal{L}^2(\mathcal{H})$ . By the generalized Powers-Størmer inequality, we have  $\|x_i\eta x_i^* - \eta\|_2^2 \leq 2\|x_i\xi x_i^* - \xi\|_2 \leq 2\varepsilon$ ,  $i = 1, \dots, n$ . Hence,  $\eta \in \mathcal{L}^2(\mathcal{H}) \cong \mathcal{H} \otimes_M \bar{\mathcal{H}}$  is a unit vector such that  $\|x_i\eta - \eta x_i\|_2 \leq \sqrt{2\varepsilon}$ ,  $i = 1, \dots, n$ . Thus,  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  does not have spectral gap, a contradiction.  $\square$

### III.4 Proofs of main theorems

In this section we prove our main result, from which will follow Theorems III.1.2 and III.1.3. To begin, let  $\Gamma$  be an i.c.c. countable discrete group which admits an unbounded cocycle  $b : \Gamma \rightarrow \mathcal{K}$  for some orthogonal representation  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{K})$ . Let  $\Gamma \curvearrowright^\sigma (X, \mu)$  be the Gaussian construction associated to  $\pi$  as described in section III.2.2 and  $\{\omega_t : t \in \mathbb{R}\}$  be the one-parameter family of cocycles associated to  $b$  as given by Theorem III.2.4. Let  $\alpha_t$  be the  $*$ -automorphism of  $\tilde{M} = L^\infty(X, \mu) \rtimes \Gamma$  defined by  $\alpha_t(au_\gamma) = a\omega_t(\gamma)u_\gamma$  for all  $a \in L^\infty(X, \mu)$ ,  $\gamma \in \Gamma$ . Finally, we set  $M = L\Gamma$ , and we denote by  $\mathcal{H}$  the  $M$ - $M$  bimodule  $L_0^2(X, \mu) \otimes \ell^2\Gamma$  with the usual bimodule structure; i.e.,

the one defined by  $u_\gamma(\xi \otimes \eta) = \sigma(\gamma)\xi \otimes u_\gamma\eta$ ,  $(\xi \otimes \eta)u_\gamma = \xi \otimes (\eta u_\gamma)$  for all  $\xi \in L_0^2(X, \mu)$ ,  $\eta \in \ell^2\Gamma$  and  $\gamma \in \Gamma$ .

**Theorem III.4.1.** *With the assumptions and notations as above, suppose  $P \subset M$  is a diffuse von Neumann subalgebra such that  $\mathcal{N}_M(P)$  acts weakly compactly on  $P$  via conjugation. Let  $Q = \mathcal{N}_M(P)''$ . If either: (1)  $b$  is a proper cocycle; or (2)  $\pi$  is a mixing representation and  $\alpha_t$  does not converge  $\|\cdot\|_2$ -uniformly to the identity on  $(Qp)_1$  for any projection  $p \in \mathcal{Z}(Q' \cap M)$  as  $t \rightarrow 0$ , then the  $M$ - $M$  correspondence  $\mathcal{H}$  is left amenable over  $Q$  in the sense of satisfying Theorem III.3.2 for  $Q \subset M$ .*

*Proof.* In the case of (1), since  $b$  is proper, it is easy to see by formula III.2.2 that  $E_{L\Gamma} \circ \alpha_t$  restricted to  $L\Gamma$  is compact for all  $t > 0$ . Hence, by the proof of Theorem 4.9 in [60], for any  $K \geq 8$ , any non-zero projection  $p \in \mathcal{Z}(Q' \cap M)$ , and any finite subset  $F \subset \mathcal{N}_M(P)$ , we can find a vector  $\xi_{p,F} \in \mathcal{H} \otimes L^2(\bar{M})$  such that  $\|x\xi_{p,F}\| \leq \|x\|_2$  for all  $x \in M$ ,  $\|p\xi_{p,F}\| \geq \|p\|_2/K$  and  $\|[u \otimes \bar{u}, \xi_{p,F}]\| < 1/|F|$  for all  $u \in F$ .

In the case of (2), we need only demonstrate that our assumptions imply the existence of such a net  $(\xi_{p,F})$  as in case (1) for some  $K \geq 8$  then argue commonly for both sets of assumptions. By contradiction if such a net  $(\xi_{p,F})$  did not exist for any  $K \geq 8$ , the proof of Theorem 4.9 in [60] shows that for every  $0 < \delta = K^{-1} \leq 1/8$  and for any  $t > 0$  sufficiently small we have

$$\|E_M \circ \alpha_t(up)\|_2 \geq (1 - 6\delta)\|p\|_2 \quad (\text{III.4.1})$$

for all  $u \in \mathcal{U}(P)$ . Now, the operators  $(E_M \circ \alpha_t)_{t \geq 0}$  can be seen to form a one-parameter semigroup of unital, tracial completely-positive maps (cf. Example 2.2 in [64]) such that  $0 \leq E_M \circ \alpha_t \leq \text{id}$  for all  $0 \leq t < \infty$ . This implies that

$$\|x\|_2^2 - \|E_M \circ \alpha_t(x)\|_2^2 \geq \|x - E_M \circ \alpha_t(x)\|_2^2. \quad (\text{III.4.2})$$

Hence, if  $\alpha_t$  does not converge uniformly on  $(Pp)_1$ , we have that  $\alpha_t$  cannot converge uniformly on  $\mathcal{U}(P)p$ , so there exists  $c > 0$  such that for every  $t > 0$  sufficiently small, there exists  $u_t \in \mathcal{U}(P)$  such that

$$\|E_M \circ \alpha_t(u_t p)\|_2 \leq \sqrt{1 - c^2}\|p\|_2.$$

However, this contradicts the inequality III.4.1 for  $\delta$  sufficiently small.

To conclude the discussion of case (2), we have that  $\pi$  is mixing and, by the previous paragraph,  $\alpha_t$  converges  $\|\cdot\|_2$ -uniformly on  $(Pp)_1$ . We will show that  $\alpha_t$  converges uniformly on  $(Qp)_1$ , which contradicts our assumptions on  $\alpha_t$ . Since there is a natural trace-preserving automorphism  $\beta \in \text{Aut}(\tilde{M})$  which pointwise fixes  $M$  and such that  $\beta \circ \alpha_t = \alpha_{-t} \circ \beta$  for all  $t \in \mathbb{R}$  (cf. [65]), by Popa's transversality lemma [77], it is enough to show that  $\|\alpha_t(x) - E_M \circ \alpha_t(x)\|_2 \rightarrow 0$  uniformly on  $(Qp)_1$ . Notice that  $\delta_t(x) = \alpha_t(x) - E_M \circ \alpha_t(x) = (1 - E_M)(\alpha_t(x) - x)$  is a (bounded) derivation  $\delta_t : M \rightarrow \mathcal{H}$ . Since  $\pi$  is mixing, by [66] we have that  $\mathcal{H}$  is a compact correspondence; hence, by Theorem 4.5 in [64]  $\delta_t \rightarrow 0$  uniformly in  $\|\cdot\|_2$ -norm on  $(Qp)_1$  as  $t \rightarrow 0$ .

Now, fixing a suitable  $K \geq 8$  and proceeding commonly for both cases, let  $\mathcal{F}$  be the net of finite subsets of  $\mathcal{N}_M(P)$  under inclusion. We define the state  $\Phi_p$  on  $\mathcal{N} = \mathbb{B}(\mathcal{H}) \cap \pi_{M^\circ}(M^\circ)'$  by

$$\Phi_p(T) = \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)p\xi_{p,F}, p\xi_{p,F} \rangle,$$

where  $\text{Lim}_{\mathcal{F}}$  is an arbitrary Banach limit. It is easy to see by the properties of  $\xi_{p,F}$  that  $\Phi_p$  is normal on  $M$ . Proceeding as in Lemma 5.3 in [61], we then have that for all  $u \in \mathcal{N}_M(P)$ ,

$$\begin{aligned} \Phi_p(u^*Tu) &= \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)up\xi_{p,F}, up\xi_{p,F} \rangle \\ &= \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)p(u \otimes \bar{u})\xi_{p,F}, p(u \otimes \bar{u})\xi_{p,F} \rangle \\ &= \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle (T \otimes 1)p(u \otimes \bar{u})\xi_{p,F}(u \otimes \bar{u})^*, p(u \otimes \bar{u})\xi_{p,F}(u \otimes \bar{u})^* \rangle \\ &= \Phi_p(T). \end{aligned} \tag{III.4.3}$$

Hence, we have that  $\Phi_p([x, T]) = 0$  for all  $x$  in the span of  $\mathcal{N}_M(P)$  and  $T \in \mathcal{N}$ . But we have that

$$\begin{aligned} |\Phi_p(Tx)| &\leq \|T\|_\infty \left| \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \langle xp\xi_{p,F}, p\xi_{p,F} \rangle \right| \\ &\leq \|T\|_\infty \text{Lim}_{\mathcal{F}} \frac{1}{\|p\xi_{p,F}\|_2^2} \|xp\xi_{p,F}\|_2 \leq \frac{K}{\|p\|_2} \|T\|_\infty \|x\|_2 \end{aligned} \tag{III.4.4}$$

and similarly for  $|\Phi_p(xT)|$ . Thus, by Kaplansky's density theorem we have that  $\Phi_p$  is a  $Q$ -central state.

To summarize, for every non-zero projection  $p \in \mathcal{Z}(Q' \cap M)$ , we have obtained a state  $\Phi_p$  on

$\mathcal{N}$  such that  $\Phi_p(p) = 1$ ,  $\Phi_p$  is normal on  $M$  and  $\Phi_p$  is  $Q$ -central. A simple maximality argument then shows that there exists a state  $\Phi$  on  $\mathcal{N}$  which is normal on  $M$ ,  $Q$ -central, and faithful on  $\mathcal{Z}(Q' \cap M)$ . Thus,  $\Phi$  satisfies condition (2) of Theorem III.3.2, and we are done.  $\square$

Keeping with the same notations, assume now that the orthogonal representation  $b : \Gamma \rightarrow \mathcal{O}(\mathcal{K})$  is such that there exists an  $K > 0$  such that  $\pi^{\otimes K}$  is weakly contained in the left regular representation. As was pointed out in section III.2.2, the representation induced on  $L_0^2(X, \mu)$  by  $\Gamma \curvearrowright^\sigma (X, \mu)$  also has this property. Let  $\mathcal{H}_\sigma = L_0^2(X, \mu)$  so that  $\mathcal{H} = \mathcal{H}_\sigma \otimes \ell^2\Gamma$  is the  $M$ - $M$  correspondence induced by the representation  $\sigma$ . Denote by  $\tilde{\mathcal{H}}^n$  the  $M$ - $M$  correspondence  $((\mathcal{H}_\sigma \otimes \bar{\mathcal{H}}_\sigma)^{\otimes n}) \otimes \ell^2\Gamma$  with the natural bimodule structure. It is straightforward to check that  $\mathcal{H} \otimes_M \bar{\mathcal{H}} \cong \mathcal{H}_\sigma \otimes \ell^2\Gamma \otimes \bar{\mathcal{H}}_\sigma$  and that  $(\mathcal{H} \otimes_M \bar{\mathcal{H}}) \otimes_M \cdots \otimes_M (\mathcal{H} \otimes_M \bar{\mathcal{H}})$  for  $n + 1$  copies is isomorphic to  $\mathcal{H} \otimes_M (\tilde{\mathcal{H}}^n) \otimes_M \bar{\mathcal{H}}$  as  $M$ - $M$  bimodules. Hence, the  $M$ -tensor product of  $K$  copies of  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  is weakly contained in the coarse  $M$ - $M$  correspondence.

**Theorem III.4.2.** *With the assumptions and notations as above, including those assumed for Theorem III.4.1, suppose  $P \subset M$  is a diffuse von Neumann subalgebra such that  $\mathcal{N}_M(P)$  acts weakly compactly on  $P$  via conjugation. Then  $Q = \mathcal{N}_M(P)''$  is amenable.*

*Proof.* Let  $p$  be a non-zero projection in  $\mathcal{Z}(Q' \cap M)$ . By the proofs of Theorems III.4.1 and III.3.2, it follows that we can find a net  $(\xi_n)$  in  $\mathcal{H} \otimes_M \bar{\mathcal{H}}$  such that  $\langle x\xi_n, \xi_n \rangle \rightarrow \tau(pxp)/\tau(p)$  for all  $x \in M$  and  $\|[u, \xi_n]\| \rightarrow 0$  for all  $u \in \mathcal{U}(Q)$ . In fact, without loss of generality we may assume that  $\langle x\xi_n, \xi_n \rangle = \tau(pxp)/\tau(p) = \langle \xi_n x, \xi_n \rangle$  for all  $x \in M$  (see the proof of Proposition 3.2 in [60]). In particular,  $(\xi_n)$  is uniformly left and right bounded. Let  $\tilde{\xi}_n$  be the  $M$ -tensor product of  $K$  copies of  $\xi_n$ . Then the net  $(\tilde{\xi}_n)$  may be seen to satisfy the same properties. Since  $(\tilde{\xi}_n)$  are vectors in a correspondence weakly contained in the coarse  $M$ - $M$  correspondence, we have that for any finite subset  $F \subset \mathcal{U}(Q)$  that

$$\left\| \sum_{u \in F} up \otimes \bar{u}p \right\|_{M \bar{\otimes} M} \geq \lim_n \left\| \sum_{u \in F} u \tilde{\xi}_n u^* \right\| = |F|.$$

Hence, by Haagerup's criterion [31],  $Q$  is amenable.  $\square$

**Remark III.4.3.** Let  $M$  be a  $\text{II}_1$  factor and  $\delta$  a closable real derivation from  $M$  into an  $M$ - $M$  correspondence  $\mathcal{H}$  (cf. [64]). Suppose  $P \subset M$  is a von Neumann subalgebra and  $\mathcal{N}_M(P)$  acts weakly compactly on  $P$  by conjugation. Let  $Q = \mathcal{N}_M(P)''$ . One can show that if  $\delta^* \bar{\delta}$  has compact resolvents,

then  $\mathcal{H}$  is left amenable over  $Q$ . The proof is identical to the proof of Theorem B in [61], using the generalized Haagerup's criterion (Corollary III.3.4). In particular, this sharpens Theorem A in [61]: any  $\text{II}_1$  factor  $M$  with the CMAP admitting such a derivation into a nonamenable correspondence has no Cartan subalgebras.

We are now ready to prove Theorems III.1.2 and III.1.3.

*Proof of Theorem III.1.2.* We need only check the cases  $\text{SO}(n, 1)$ ,  $n \geq 4$ , and  $\text{SU}(m, 1)$ ,  $m \geq 2$ , as  $\text{SO}(1, 1)$  is amenable and the remaining cases were dealt with by Ozawa–Popa [61]. If  $\Gamma$  is an i.c.c. lattice in  $\text{SO}(n, 1)$  for  $n \geq 3$  or  $\text{SU}(m, 1)$  for  $m \geq 2$ , then Theorems 1.9 in [91] shows that  $\Gamma$  possesses an unbounded cocycle into some strongly  $\ell^p$  representation for  $p \geq 2$ . By Theorem 3.4 in the same, any unbounded cocycle for such a lattice is proper. Since  $L\Gamma$  has the CMAP by [9] and [32], by Theorems III.2.7 and III.4.2 the result obtains.  $\square$

*Proof of Theorem III.1.3.* For the first assertion, since  $\Gamma \cong \Gamma_1 * \Gamma_2$ ,  $\Gamma$  admits a canonical unbounded cocycle  $b : \Gamma \rightarrow \bigoplus^\infty \ell^2 \Gamma$  into a direct sum of left regular representations. The left regular representation is mixing and  $L\Gamma$ - $L\Gamma$  correspondence associated to the left regular representation (the coarse correspondence) is amenable if and only if  $\Gamma$  is amenable. So, if  $L\Gamma$  did admit a Cartan subalgebra, then by Theorems III.2.7 and III.4.1 the deformation  $\alpha_t$  of  $L\Gamma$  obtained from the cocycle  $b$  would have to converge uniformly on  $(L\Gamma)_1$  as  $t \rightarrow 0$ . But this contradicts that  $b$  is unbounded.

For the second assertion, if a non-amenable  $\text{II}_1$  subfactor  $N \subset L\Gamma$  admits a Cartan subalgebra, then we have that  $\alpha_t$  converges  $\|\cdot\|_2$ -uniformly on the unit ball of  $N$ , since  $N$  has the CMAP and the coarse  $L\Gamma$ - $L\Gamma$  correspondence viewed as an  $N$ - $N$  correspondence embeds into a direct sum of coarse  $N$ - $N$  correspondences. Let  $\tilde{\Gamma} = \Gamma * \mathbb{F}_2$ , where  $\mathbb{F}_2$  is the free group on two generators. Let  $u_1, u_2 \in L\mathbb{F}_2$  be the canonical generating unitaries and  $h_1, h_2 \in L\mathbb{F}_2$  self-adjoint elements such that  $u_j = \exp(\pi i h_j)$ ,  $j = 1, 2$ . Define  $u_j^t = \exp(\pi i t h_j)$ ,  $j = 1, 2$ , and let  $\theta_t$  be the  $*$ -automorphism of  $L\tilde{\Gamma}$  given by  $\theta_t = \text{Ad}(u_1^t) * \text{Ad}(u_2^t)$ . It follows from Lemma 5.1 in [64] and Corollary 4.2 in [65] that  $\theta_t$  converges uniformly in  $\|\cdot\|_2$ -norm on the unit ball of  $N \subset L\Gamma \subset L\tilde{\Gamma}$  as  $t \rightarrow 0$ . An examination of the proof of Theorem 4.3 in [38] shows that this is the only condition necessary for the theorem to obtain. Our result then follows directly from Theorem 5.1 in [38].  $\square$

## CHAPTER IV

### ON THE STRUCTURE OF $\text{II}_1$ FACTORS OF NEGATIVELY CURVED GROUPS

#### IV.1 Introduction

In a conceptual *tour de force* Ozawa established a broad property for group factors of Gromov hyperbolic groups—what he termed solidity—which essentially allowed him to reflect the “small cancelation” property such a group enjoys in terms of its associated von Neumann algebra.

**Ozawa’s Solidity Theorem** ([54]). If  $\Gamma$  is an i.c.c. Gromov hyperbolic group, then  $L\Gamma$  is *solid* i.e.,  $A' \cap L\Gamma$  is amenable for every diffuse von Neumann subalgebra  $A \subset L\Gamma$ .

Remarkable in its generality, Ozawa’s argument relies on a subtle interplay between  $C^*$ -algebraic and von Neumann algebraic techniques [7].

In a significant step forward, using his deformation/rigidity theory [76], Popa was able to offer an alternate, elementary proof of solidity for free group factors: more generally, for factors admitting a “free malleable deformation” [75]. Popa’s approach exemplifies the usage of spectral gap rigidity arguments which subsequently opened up many new directions in deformation/rigidity theory, cf. [76, 75, 77]. Of particular importance to what follows here, his techniques brought the necessary perspective to attack the Cartan subalgebra problem for free group factors in later work with Ozawa [60, 61].

A new von Neumann-algebraic approach to solidity was developed by Peterson in his important paper on  $L^2$ -rigidity [64]. Essentially, Peterson was able to exploit the “negative curvature” of the free group on two generators  $\mathbb{F}_2$ , in terms of a proper 1-cocycle into the left regular representation, to rule out the existence of large relative commutants of diffuse subalgebras of  $L\mathbb{F}_2$ .

**Peterson’s Solidity Theorem** ([64]). If  $\Gamma$  is an i.c.c. countable discrete which admits a proper 1-cocycle  $b : \Gamma \rightarrow \mathcal{H}_\pi$  for some unitary representation  $\pi$  which is weakly  $\ell^2$  (i.e., weakly contained in the left regular representation), then  $L\Gamma$  is solid.

It was later realized in [92] that many of the explicit unbounded derivations (i.e., the ones constructed from 1-cocycles) that Peterson works with have natural dilations which are malleable

deformations of their corresponding (group) von Neumann algebras.

However, the non-vanishing of 1-cohomology of  $\Gamma$  with coefficients in the left regular representation does not reflect the full spectrum of negative curvature phenomena in geometric group theory as evidenced by the existence of non-elementary hyperbolic groups with Kazhdan’s property (T), cf. [3]. In their fundamental works on the rigidity of group actions [45, 46], Monod and Shalom proposed a more inclusive cohomological definition of negative curvature in group theory which is given in terms of non-vanishing of the second-degree bounded cohomology for  $\Gamma$  with coefficients in the left-regular representation. Relying on Monod’s work in bounded cohomology [44], we will make use of a related condition, which is the existence of a proper *quasi*-1-cocycle on  $\Gamma$  into the left-regular representation (more generally, into a representation weakly contained in the left-regular representation), cf. [44, 96]. By a result of Mineyev, Monod, and Shalom [43], this condition is satisfied for any hyperbolic group—the case of vanishing first  $\ell^2$ -Betti number is due to Mineyev [42].

## Statement of results

We now state the main results of this chapter, in order to place them within the context of previous results in the structural theory of group von Neumann algebras. We begin with the motivating result of the chapter, which unifies the solidity theorems of Ozawa and Peterson.

**Theorem A.** Let  $\Gamma$  be an i.c.c. countable discrete group which is exact and admits a proper quasi-1-cocycle  $q : \Gamma \rightarrow \mathcal{H}_\pi$  for some weakly- $\ell^2$  unitary representation  $\pi$  (more generally,  $\Gamma$  is exact and belongs to the class  $\mathcal{QH}_{\text{reg}}$  of Definition IV.2.4). Then  $L\Gamma$  is solid.

In particular, all Gromov hyperbolic groups are exact, cf. [83], and admit a proper quasi-1-cocycle for the left-regular representation [43]. Moreover, all bi-exact groups are in the class  $\mathcal{QH}_{\text{reg}}$  (section IV.5), so the previous result fully recovers Ozawa’s Solidity Theorem.

Following Ozawa’s and Peterson’s work on solidity, there was some hope that similar techniques could be used to approach to the Cartan subalgebra problem for group factors of hyperbolic groups, generalizing Voiculescu’s celebrated theorem on the absence of Cartan subalgebras for free group factors [101]. However, the Cartan problem for general hyperbolic groups would remain intractable until the breakthrough approach of Ozawa and Popa through Popa’s deformation/rigidity theory

resolved it in the positive for the group factor of any discrete group of isometries of the hyperbolic plane [61]. In fact, they were able to show that any such  $\text{II}_1$  factor  $M$  is *strongly solid*, i.e., for every diffuse, amenable von Neumann subalgebra  $A \subset M$ ,  $\mathcal{N}_M(A)'' \subset M$  is an amenable von Neumann algebra, where  $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$ . Using the techniques developed by Ozawa and Popa [60, 61] and a recent result of Ozawa [58], we obtain the following strengthening of Theorem A.

**Theorem B.** Let  $\Gamma$  be an i.c.c. countable discrete group which is exact, weakly amenable, and admits a proper quasi-1-cocycle into a weakly  $\ell^2$  representation. Then  $L\Gamma$  is strongly solid. Also, if  $\Gamma$  is i.c.c., exact, weakly amenable, and admits a proper quasi-1-cocycle into a non-amenable representation (see Definition II.2.1), then  $L\Gamma$  has no Cartan subalgebras.

Appealing to Ozawa’s proof of the weak amenability of hyperbolic groups [56], Theorem B allows us to fully resolve in the positive the strong solidity problem—hence the Cartan problem—for i.c.c. hyperbolic groups and for lattices in connected rank one simple Lie groups.

**Corollary C.** Let  $\Gamma$  be an i.c.c. countable discrete group which is either Gromov hyperbolic or a lattice in  $\text{Sp}(n, 1)$  or the exceptional group  $F_{4(-20)}$ , then  $L\Gamma$  is strongly solid.

The strong solidity problem for the other rank one simple Lie groups—those locally isomorphic to  $\text{SO}(n, 1)$  or  $\text{SU}(n, 1)$ —was resolved for  $\text{SO}(2, 1)$ ,  $\text{SO}(3, 1)$ , and  $\text{SU}(1, 1)$  by the work of Ozawa and Popa [61] and, in the general case, by the work of the second author [92]. The corollary follows directly from Theorem B in the co-compact (i.e., uniform) case: in the non-uniform case, we must appeal to a result of Shalom (Theorem 3.7 in [91]) on the integrability of lattices in connected simple rank one Lie groups. A natural question to ask is whether our techniques can be extended to demonstrate strong solidity of the group factor of any i.c.c. countable discrete group which is relatively hyperbolic [52] to a family of amenable subgroups, e.g., Sela’s limit groups [23].

Beyond solidity results, we highlight that the techniques developed in this chapter also enable strong decomposition results for products of groups. The following result is a direct generalization of a theorem of Peterson (Corollary 6.2 in [64]) and harmonizes well with prime decomposition results achieved by Ozawa and Popa [59].

**Theorem D.** Let  $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$  be a non-trivial product of exact, i.c.c. countable discrete groups such that  $\Gamma_i$  admits a proper quasi-1-cocycle into a weakly  $\ell^2$  representation (or,  $\Gamma_i \in \mathcal{QH}_{\text{reg}}$ ),



$1 \leq i \leq n$ . If  $N = N_1 \overline{\otimes} \cdots \overline{\otimes} N_m$ , and  $L\Gamma \cong N$ , then  $n = m$  and there exist  $t_1, \dots, t_n > 0$  with  $t_1 \cdots t_n = 1$  so that, up to a permutation,  $(L\Gamma_i)^{t_i} \cong N_i$ ,  $1 \leq i \leq n$ .

## On the method of proof

We attempt to chart a “middle path” between the results of Ozawa and Peterson by recasting Ozawa’s approach to solidity effectively as a deformation/rigidity argument. We do so by finding a “cohomological” analog of Ozawa’s class  $\mathcal{S}$  [55], which we refer to as the class  $\mathcal{QH}_{\text{reg}}$ . The class  $\mathcal{QH}_{\text{reg}}$  has many affinities with (strict) cohomological characterizations of negative curvature proposed by Monod and Shalom [46] and Thom [96].

The main advantage to working from the cohomological perspective is that it allows one to construct “deformations” of  $L\Gamma$ . Though these “deformations” no longer map  $L\Gamma$  into itself, we are still able to control the convergence of these deformations on a weakly dense  $C^*$ -subalgebra of  $L\Gamma$  namely, the reduced group  $C^*$ -algebra  $C_\lambda^*(\Gamma)$ . Our method then borrows Ozawa’s insight of using local reflexivity to pass from  $C_\lambda^*(\Gamma)$  to the entire von Neumann algebra.

### IV.2 Cohomological-type properties and negative curvature

Let  $\Gamma$  be a countable discrete group. Recall that a *length function*  $|\cdot| : \Gamma \rightarrow \mathbb{R}_{\geq 0}$  is a map satisfying:

(1)  $|\gamma| = 0$  if and only if  $\gamma = e$  is the identity; (2)  $|\gamma^{-1}| = |\gamma|$ , for all  $\gamma \in \Gamma$ ; and (3)  $|\gamma\delta| \leq |\gamma| + |\delta|$ , for all  $\gamma, \delta \in \Gamma$ .

**Definition IV.2.1.** Let  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H}_\pi)$  be an orthogonal representation of a countable discrete group  $\Gamma$ . A map  $q : \Gamma \rightarrow \mathcal{H}_\pi$  is called an *array* if: (1)  $\gamma \mapsto \|q(\gamma)\|$  is proper; (2)  $\pi_\gamma(q(\gamma^{-1})) = -q(\gamma)$  (i.e.,  $q$  is *anti-symmetric*); and (3) for every finite subset  $F \subset \Gamma$  there exists  $K \geq 0$  such that

$$\|\pi_\gamma(q(\delta)) - q(\gamma\delta)\| \leq K, \tag{IV.2.1}$$

for all  $\gamma \in F$ ,  $\delta \in \Gamma$  (i.e.,  $q$  is *boundedly equivariant*). It is an easy exercise to show that for any array  $q$  there exists a length function on  $\Gamma$  which bounds  $\|q(\gamma)\|$  from above. An array  $q : \Gamma \rightarrow \mathcal{H}_\pi$  is said to be *uniform* if there exists a proper length function  $|\cdot|$  on  $\Gamma$  and an increasing function

$\rho : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that  $\rho(t) \rightarrow \infty$  as  $t \rightarrow \infty$  and such that

$$\rho(|\gamma^{-1}\delta|) \leq \|q(\gamma) - q(\delta)\|,$$

for all  $\gamma, \delta \in \Gamma$ .

*Remark IV.2.2.* In the preceding definition we could as well have relaxed the condition of strict anti-symmetry to merely the condition that  $\|\pi_\gamma(q(\gamma^{-1})) + q(\gamma)\|$  is bounded. However, it is easy to check that for any such function  $q$ , there exists an array  $\tilde{q}$  which is a bounded distance from  $q$ ; namely,  $\tilde{q}(\gamma) = \frac{1}{2}(q(\gamma) - \pi_\gamma(q(\gamma^{-1})))$ . This observation is essentially due to Andreas Thom [96].

Our primary examples of (uniform) arrays will be quasi-1-cocycles.

**Definition IV.2.3.** Let  $\Gamma$  be a countable discrete group and  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$  be an orthogonal representation of  $\Gamma$  on a real Hilbert space  $\mathcal{H}$ . A map  $q : \Gamma \rightarrow \mathcal{H}$  is called a *quasi-1-cocycle* for the representation  $\pi$  if one can find a constant  $K \geq 0$  such that for all  $\gamma, \lambda \in \Gamma$  we have

$$\|q(\gamma\lambda) - q(\gamma) - \pi_\gamma(q(\lambda))\| \leq K. \tag{IV.2.2}$$

We denote by  $D(q)$  the *defect* of the quasi-1-cocycle  $q$ , which is the infimum of all  $K$  satisfying equation IV.2.2. Notice that when the defect is zero the quasi-1-cocycle  $q$  is actually a 1-cocycle for  $\pi$  [3]. In the sequel, we will drop the “1” and refer to (quasi-)1-cocycles as (quasi-)cocycles. Again, without (much) loss of generality we will require a quasi-cocycle  $q$  to be anti-symmetric, since every quasi-cocycle  $q$  is a bounded distance from some anti-symmetric quasi-cocycle  $\tilde{q}$ , cf. [96], which will suffice for our purposes.

We now proceed to describe some “cohomological” properties of countable discrete groups which capture many aspects of negative curvature from the perspective of representation theory.

**Definition IV.2.4.** We say that a countable discrete group  $\Gamma$  is in the class  $\mathcal{QH}$  if it admits an array  $q : \Gamma \rightarrow \mathcal{H}_\pi$  for some non-amenable unitary representation  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$ . If the representation  $\pi$  can be chosen to be weakly  $\ell^2$ , then we say that  $\Gamma$  belongs to the class  $\mathcal{QH}_{\text{reg}}$ .

Note that the class  $\mathcal{QH}_{\text{reg}}$  generalizes the class  $\mathcal{D}_{\text{reg}}$  of Thom [96] and that the class  $\mathcal{QH}$  contains all groups having Ozawa and Popa’s property (HH) [61].

**Proposition IV.2.1.** The following statements are true.

1. If  $\Gamma_1, \Gamma_2 \in \mathcal{QH}$ , then so are  $\Gamma_1 \times \Gamma_2$  and  $\Gamma_1 * \Gamma_2$ .
2. If  $\Gamma_1, \Gamma_2 \in \mathcal{QH}_{\text{reg}}$ , then  $\Gamma_1 * \Gamma_2 \in \mathcal{QH}_{\text{reg}}$ .
3. If  $\Gamma_1, \Gamma_2 \in \mathcal{QH}_{\text{reg}}$  are non-amenable, then  $\Gamma_1 \times \Gamma_2 \notin \mathcal{QH}_{\text{reg}}$ .
4. If  $\Gamma$  is a lattice in a simple connected Lie group with real rank one, then  $\Gamma \in \mathcal{QH}_{\text{reg}}$ .
5. If  $\Gamma \in \mathcal{QH}$ , then  $\Gamma$  is not inner amenable. If in addition  $\Gamma$  is weakly amenable, then  $\Gamma$  has no infinite normal amenable subgroups.

Statement (5) is essentially Proposition 2.1 in [61] combined with Theorem A in [58].

*Proof.* Statements (1) and (2) follow exactly as they do for groups which admit a proper cocycle into some non-amenable (respectively, weakly  $\ell^2$ ) unitary representation, cf. [96].

We prove statement (3) under the weaker assumption that  $\Gamma \cong \Lambda \times \Sigma$ , where  $\Lambda$  is non-amenable and  $\Sigma$  is an arbitrary infinite group. By contradiction, assume  $\Gamma$  admits a proper, anti-symmetric, boundedly equivariant map  $q : \Gamma \rightarrow \ell^2(\Gamma)$  (by inspection, the same argument will hold for  $q : \Gamma \rightarrow \mathcal{H}_\pi$  for any weakly  $\ell^2$  unitary representation  $\pi$ ). Since the action of  $\Lambda$  on  $\ell^2(\Gamma)$  has spectral gap and admits no non-zero invariant vectors, there exists a finite, symmetric subset  $S \subset \Lambda$  and  $K' > 0$  such that

$$\|\xi\| \leq K' \sum_{s \in S} \|\lambda_s(\xi) - \xi\|$$

for all  $\xi \in \ell^2(\Gamma)$ . Let  $K'' \geq 0$  be a constant so that inequality IV.2.1 is satisfied for  $S \subset \Gamma$ , and set

$K = \max\{K', K''\}$ . We then have for any  $g \in \Sigma$  that:

$$\begin{aligned}
\|q(g)\| &\leq K \sum_{s \in S} \|\lambda_s(q(g)) - q(g)\| \\
&\leq K \sum_{s \in S} \|q(sg) - q(g)\| + K^2|S| \\
&= K \sum_{s \in S} \|q(gs) - q(g)\| + K^2|S| \tag{IV.2.3} \\
&= K \sum_{s \in S} \|-\lambda_{gs}(q(s^{-1}g^{-1})) + \lambda_g(q(g^{-1}))\| + K^2|S| \\
&= K \sum_{s \in S} \|\lambda_{s^{-1}}(q(g^{-1})) - q(s^{-1}g^{-1})\| + K^2|S| \leq 2K^2|S|
\end{aligned}$$

Hence,  $\|q(g)\|$  is bounded on  $\Sigma$ , which contradicts that  $q$  is proper.

For statement (4), it is well known that any co-compact lattice in a simple Lie group with real rank one is Gromov hyperbolic; hence, by [43] admits a proper quasi-cocycle into the left-regular representation. A result of Shalom, Theorem 3.7 in [91], shows that any lattice in such a Lie group is integrable, and therefore  $\ell^1$ -measure equivalent to any other lattice in the same Lie group. It is easy to check that having a proper quasi-cocycle into the left-regular representation is invariant under  $\ell^1$ -measure equivalence, cf. Theorem 5.10 in [96].

In order to prove statement (5), we assume by contradiction that  $\Gamma$  is inner amenable, i.e., there exists a state  $\varphi$  on  $\ell^\infty(\Gamma)$  such that  $\varphi \perp \ell^1(\Gamma)$  and  $\varphi \circ \text{Ad}(\gamma) = \varphi$  for all  $\gamma \in \Gamma$ . Let  $q : \Gamma \rightarrow \mathcal{H}_\pi$  be an array into a non-amenable representation  $\pi$ . Define a u.c.p. map  $T : \mathfrak{B}(\mathcal{H}_\pi) \rightarrow \ell^\infty(\Gamma)$  by  $T(x)(\gamma) = \frac{1}{\|q(\gamma)\|^2} \langle xq(\gamma), q(\gamma) \rangle$ . Similarly to the proof of statement (3), by anti-symmetry and bounded equivariance, for every  $\gamma \in \Gamma$ , there exists  $K \geq 0$  such that

$$\|q(\gamma^{-1}\delta\gamma) - \pi_{\gamma^{-1}}(q(\delta))\| \leq K, \tag{IV.2.4}$$

for all  $\delta \in \Gamma$ . Since  $q$  is proper, this implies that the state  $\Phi = \varphi \circ T$  on  $\mathfrak{B}(\mathcal{H}_\pi)$  is  $\text{Ad}(\pi)$ -invariant. However, this contradicts the fact that  $\pi$  is a non-amenable representation. The remaining assertion follows by Theorem A in [58].

□

We conclude this section with some final remarks and questions about the classes  $\mathcal{QH}$  and

$\mathcal{QH}_{\text{reg}}$ .

*Question IV.2.5.* Are the classes  $\mathcal{QH}$  and  $\mathcal{QH}_{\text{reg}}$  closed under measure equivalence?

**Proposition IV.2.2.** The group  $\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z})$  is in the class  $\mathcal{QH}_{\text{reg}}$ .

Since arguments of Burger and Monod [8] demonstrate that  $\mathbb{Z}^2 \rtimes \text{SL}(2, \mathbb{Z})$  is not in the class  $\mathcal{D}_{\text{reg}}$  (in fact, admits no proper quasi-cocycle into *any* representation), as a corollary to Proposition IV.2.2, we have that the class  $\mathcal{QH}_{\text{reg}}$  is strictly larger than the class  $\mathcal{D}_{\text{reg}}$ . For a proof of the proposition, see section IV.6.

In light of a result of Cornuier, Stalder, and Valette, which shows that  $\mathbb{Z} \wr \mathbb{F}_2$  admits a proper cocycle for some weakly- $\ell^2$  representation [19], it would be interesting to settle the following:

*Question IV.2.6.* If  $\Gamma \in \mathcal{QH}_{\text{reg}}$ , is  $\mathbb{Z} \wr \Gamma \in \mathcal{QH}_{\text{reg}}$  as well?

### IV.3 Deformations of the uniform Roe algebra

#### IV.3.1 Schur multipliers and the uniform Roe algebra.

Using exponentiation, we now describe a canonical way to associate to an array  $q : \Gamma \rightarrow \mathcal{H}_\pi$  a family of multipliers  $\mathbf{m}_t$  on  $\mathfrak{B}(\ell^2(\Gamma))$ . First notice that the kernel  $(\gamma, \delta) \mapsto \|q(\gamma) - q(\delta)\|^2$  is conditionally negative definite (cf. Section 11.2 in [83] or Appendix D in [7]) and therefore by Schoenberg's theorem [83], for every  $t \in \mathbb{R}$ , the kernel  $\kappa_t(\gamma, \delta) = \exp(-t^2 \|q(\gamma) - q(\delta)\|^2)$  is positive definite. Hence for every  $t$  there is a unique unital, completely positive (u.c.p.) map  $\mathbf{m}_t : \mathfrak{B}(\ell^2(\Gamma)) \rightarrow \mathfrak{B}(\ell^2(\Gamma))$  called a *Schur multiplier*, such that

$$\mathbf{m}_t([x_{\gamma, \delta}]) = [\kappa_t(\gamma, \delta)x_{\gamma, \delta}], \tag{IV.3.1}$$

for all  $x \in \mathfrak{B}(\ell^2(\Gamma))$ .

If  $\Gamma$  is a group then the *uniform Roe algebra*  $C_u^*(\Gamma)$  is defined as the  $C^*$ -subalgebra of  $\mathfrak{B}(\ell^2(\Gamma))$  generated by  $C_\lambda^*(\Gamma)$  and  $\ell^\infty(\Gamma)$ . Notice that if one considers the action  $\Gamma \curvearrowright^\lambda \ell^\infty(\Gamma)$  by left translation, then the uniform algebra  $C_u^*(\Gamma)$  can be canonically identified with the reduced crossed product  $C^*$ -algebra  $\ell^\infty(\Gamma) \rtimes_{\lambda, r} \Gamma$ . Let  $\mathcal{F}_0$  denote the net of unital, symmetric, finite subsets of  $\Gamma$ . Given a finite subset  $F \in \mathcal{F}_0$ , we define the operator space of *F-width operators*  $X(F)$  to be the space of bounded operators  $x \in \mathfrak{B}(\ell^2 \Gamma)$  such that  $x_{\gamma, \delta} = 0$  whenever  $\gamma^{-1}\delta \in \Gamma \setminus F$ . Since it is easy

to check that  $X(F) = \ell^\infty(\Gamma)\mathbb{C}[F]\ell^\infty(\Gamma)$ , we have  $C_u^*(\Gamma) = \overline{\text{span}\{X(F) : F \in \mathcal{F}_0\}}^{\|\cdot\|}$ . Our interest in the uniform Roe algebra stems from the fact that it is, in practical terms, the smallest  $C^*$ -algebra which contains  $C_\lambda^*(\Gamma)$  and which is invariant under the class of Schur multipliers associated to  $\Gamma$ .

**Proposition IV.3.1.**  $\mathfrak{m}_t(C_r^*(\Gamma)) \subset C_u^*(\Gamma)$ .

*Proof.* Let  $x \in C_r^*(\Gamma)$ , then there exists a sequence  $(x_n)$  of elements of  $\mathbb{C}[\Gamma]$  such that  $\|x_n - x\| \rightarrow 0$ . It is easy to see that if  $x_n$  is supported on the set  $F_n \in \mathcal{F}_0$ , then  $\mathfrak{m}_t(x_n) \in X(F_n)$ . Since  $\|\mathfrak{m}_t(x_n) - \mathfrak{m}_t(x)\| \rightarrow 0$ , we have that  $\mathfrak{m}_t(x) \in C_u^*(\Gamma)$ .  $\square$

We will also heavily use the following observation of Roe on the convergence of  $\mathfrak{m}_t$  on  $C_u^*(\Gamma)$ , cf. Lemma 4.27 in [83].

**Proposition IV.3.2.** If  $q$  is an array, then for all  $x \in C_u^*(\Gamma)$ , we have that  $\|\mathfrak{m}_t(x) - x\|_\infty \rightarrow 0$  as  $t \rightarrow 0$ .

### IV.3.2 Construction of the extended Roe algebra $C_u^*(\Gamma \curvearrowright^\sigma X)$ .

Let  $\Gamma \curvearrowright^\sigma (X, \mu)$  be a measure preserving action on a probability space  $X$ . By abuse of notation, we still denote by  $\sigma$  the Koopman representation of  $\Gamma$  on  $L^2(X, \mu)$  induced by the action  $\sigma$ . Then consider the Hilbert space  $L^2(X, \mu) \otimes \ell^2(\Gamma)$  and for every  $\gamma \in \Gamma$  define a unitary  $u_\gamma \in \mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$  by the formula

$$u_\gamma(\xi \otimes \delta_h) = \sigma_\gamma(\xi) \otimes \delta_{\gamma h},$$

where  $\xi \in L^2(X)$  and  $h \in \Gamma$ . Consider the algebra  $L^\infty(X \times \Gamma, \mu \times c) \subset \mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$ , where  $c$  is the counting measure on  $\Gamma$ . Then the *extended Roe algebra*  $C_u^*(\Gamma \curvearrowright^\sigma X)$  is defined as the  $C^*$ -algebra generated by  $L^\infty(X \times \Gamma)$  and the unitaries  $u_\gamma$  inside  $\mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$ . Notice that when  $X$  consists of a point, our definition recovers the regular uniform Roe algebra i.e.,  $C_u^*(\Gamma \curvearrowright^\sigma X) = C_u^*(\Gamma)$ .

As in the case of the uniform Roe algebra, we will see that  $C_u^*(\Gamma \curvearrowright^\sigma X)$  can be realized as a reduced crossed product algebra. Specifically, we consider the action  $\Gamma \curvearrowright L^\infty(X \times \Gamma)$  given by  $\lambda_\gamma^\sigma(f)(x, h) = f(\gamma^{-1}x, \gamma^{-1}h)$  where  $f \in L^\infty(X \times \Gamma)$ ,  $x \in X$  and  $\gamma, h \in \Gamma$ . Then we show that  $C_u^*(\Gamma \curvearrowright^\sigma X)$  is naturally identified with the reduced crossed product algebra corresponding to this action and the faithful representation  $L^\infty(X \times \Gamma) \subset \mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$ .

**Proposition IV.3.3.**  $C_u^*(\Gamma \curvearrowright^\sigma X) \cong L^\infty(X \times \Gamma) \rtimes_{\lambda_{\sigma, r}} \Gamma$ .

*Proof.* Consider the operator  $U : L^2(X) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma) \rightarrow L^2(X) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma)$  defined by  $U(\xi \otimes \delta_k \otimes \delta_h) = \sigma_h(\xi) \otimes \delta_k \otimes \delta_{hk}$ , where  $\xi \in L^2(X)$  and  $\gamma, h \in \Gamma$ . One can check this is a unitary, and we will show that it implements a spatial isomorphism between the two algebras. For this purpose we will be seeing  $C_u^*(\Gamma \curvearrowright X)$  as the  $C^*$ -algebra generated by  $L^\infty(X \times \Gamma)$  and  $u_\gamma$  inside  $\mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma))$ , where

$$\begin{aligned} f(\xi \otimes \delta_k \otimes \delta_h) &= (f(\cdot, h)\xi) \otimes \delta_k \otimes \delta_h \\ u_\gamma(\xi \otimes \delta_k \otimes \delta_h) &= \sigma_\gamma(\xi) \otimes \delta_k \otimes \delta_{\gamma h}, \end{aligned} \tag{IV.3.2}$$

for all  $f \in L^\infty(X \times \Gamma)$ ,  $\gamma, h, k \in \Gamma$  and  $\xi \in L^2(X)$ .

Using the formula for  $U$  in combination with equations (IV.3.2), we have

$$\begin{aligned} U(1 \otimes \lambda_\gamma)(\xi \otimes \delta_k \otimes \delta_h) &= U(\xi \otimes \delta_k \otimes \delta_{\gamma h}) \\ &= \sigma_{\gamma h}(\xi) \otimes \delta_k \otimes \delta_{\gamma h k} \\ &= u_\gamma(\sigma_h(\xi) \otimes \delta_k \otimes \delta_{hk}) \\ &= u_\gamma U(\xi \otimes \delta_k \otimes \delta_h); \end{aligned}$$

hence,  $U(1 \otimes \lambda_\gamma)U^* = u_\gamma$  for all  $\gamma \in \Gamma$ .

Also, we consider the representation of  $L^\infty(X \times \Gamma)$  on  $\mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma) \otimes \ell^2(\Gamma))$  given by  $\pi(f)(\xi \otimes \delta_k \otimes \delta_h) = \lambda_{h^{-1}}^\sigma(f)(\xi \otimes \delta_k) \otimes \delta_h$  for every  $f \in L^\infty(X \times \Gamma)$ .

Combining this with equations IV.3.2 we see that

$$\begin{aligned} U\pi(f)(\xi \otimes \delta_k \otimes \delta_h) &= U((\lambda_{h^{-1}}^\sigma(f)(\xi \otimes \delta_k) \otimes \delta_h)) \\ &= U(\sigma_{h^{-1}}(f(\cdot, hk))\xi \otimes \delta_k \otimes \delta_h) \\ &= f(\cdot, hk)\sigma_h(\xi) \otimes \delta_k \otimes \delta_{h\gamma} \\ &= fU(\xi \otimes \delta_k \otimes \delta_h). \end{aligned}$$

Therefore, for all  $f \in L^\infty(X \times \Gamma)$  we have  $U\pi(f)U^* = f$ , and from the discussion above we conclude that  $U(L^\infty(X \times \Gamma) \rtimes_{\lambda^{\sigma, r}} \Gamma)U^* = C_u^*(\Gamma \curvearrowright X)$ .  $\square$

For further reference we keep in mind the following diagram of canonical inclusions:

$$\begin{array}{ccccc}
L^\infty(X) \rtimes_{\sigma,r} \Gamma & \subset & L^\infty(X \times \Gamma) \rtimes_{\lambda^\sigma,r} \Gamma & = & C_u^*(\Gamma \curvearrowright^\sigma X) \\
\cup & & \cup & & \\
C_r^*(\Gamma) & \subset & \ell^\infty(\Gamma) \rtimes_{\lambda,r} \Gamma & = & C_u^*(\Gamma)
\end{array} \tag{IV.3.3}$$

Note there exists a conditional expectation  $E : L^\infty(X \times \Gamma) \rightarrow \ell^\infty(\Gamma)$  defined by  $E(f)(\gamma) = \int_X f(x, \gamma) d\mu(x)$ . This map is clearly  $\Gamma$ -equivariant and thus it extends to a conditional expectation  $\tilde{E} : C_u^*(\Gamma \curvearrowright X) \rightarrow C_u^*(\Gamma)$  by letting  $\tilde{E}(\sum_\gamma x_\gamma u_\gamma) = \sum_\gamma E(x_\gamma) u_\gamma$  for any  $\sum_\gamma x_\gamma u_\gamma \in C_u^*(\Gamma \curvearrowright X)$  with  $x_\gamma \in L^\infty(X \times \Gamma)$ .

### IV.3.3 A path of automorphisms of the extended Roe algebra associated with the Gaussian action

Following [65], any group representation  $\pi : \Gamma \rightarrow \mathcal{H}_\pi$ , gives rise to a measure-preserving action  $\Gamma \curvearrowright^\sigma (X, \mu)$  called the Gaussian action. We consider the associated extended Roe algebra  $C_u^*(\Gamma \curvearrowright^\sigma X)$  and below we indicate a procedure to construct a one-parameter family  $(\alpha_t)_{t \in \mathbb{R}}$  of  $*$ -automorphisms of  $C_u^*(\Gamma \curvearrowright^\sigma X)$ . Specifically,  $\alpha_t$  is obtained by exponentiating an array  $q : \Gamma \rightarrow \mathcal{H}_\pi$  in a similar way to the construction of the malleable deformation of  $L\Gamma$  from a cocycle  $b$  as carried out in §3 of [92]. Crucially, this family will be continuous with respect to the uniform norm as  $t \rightarrow 0$  (Lemma IV.3.6).

Following the construction presented in §1.2 of [92], given an array  $q : \Gamma \rightarrow \mathcal{H}_\pi$ , there exists a one-parameter family of maps  $v_t : \Gamma \rightarrow \mathcal{U}(L^\infty(X, \mu))$  defined by  $v_t(\gamma)(x) = \exp(itq(\gamma)(x))$  where  $\gamma \in \Gamma$ ,  $x \in X$ . Using similar computations as in [65, 92] one can verify that we have the following properties:

#### Proposition IV.3.4.

$$\text{If } \pi \text{ is weakly-}\ell^2 \text{ then the Koopman representation } \pi_{\sigma|L_0^2(X,\mu)} \tag{IV.3.4}$$

is also weakly- $\ell^2$ ;

$$\int v_t(\gamma)(x) v_t(\delta)^*(x) d\mu(x) = \kappa_t(\gamma, \delta) \text{ for all } \gamma, \delta \in \Gamma. \tag{IV.3.5}$$



These maps give rise naturally to a path of operators  $V_t \in \mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$  by letting  $V_t(\xi \otimes \delta_\gamma) = v_t(\gamma)\xi \otimes \delta_\gamma$  for every  $\xi \in L^2(X)$  and  $\gamma \in \Gamma$ . For further reference we summarize below some basic properties of  $V_t$ .

**Proposition IV.3.5.** For every  $t, s \in \mathbb{R}$  we have the following properties:

1.  $V_t V_s = V_{t+s}$ ,  $V_t V_t^* = V_t^* V_t = 1$
2.  $JV_t J = V_t$  where  $J : L^2(L^\infty(X) \rtimes \Gamma) \rightarrow L^2(L^\infty(X) \rtimes \Gamma)$  is Tomita's conjugation.

*Proof.* The first part follows directly from the definitions, so we leave the details to the reader. To get the second part it suffices to verify that the two operators coincide on vectors of the form  $\xi \otimes \delta_\gamma \in L^2(X) \otimes \ell^2(\Gamma)$ . Employing the formulas for  $J$ ,  $V_t$ ,  $v_t$  and then using the fact that  $q$  is anti-symmetric we see that

$$\begin{aligned}
JV_t J(\xi \otimes \delta_\gamma) &= JV_t((\sigma_{\gamma^{-1}}(\xi^*)) \otimes \delta_{\gamma^{-1}}) \\
&= J(v_t(\gamma^{-1})\sigma_{\gamma^{-1}}(\xi^*) \otimes \delta_{\gamma^{-1}}) \\
&= (\sigma_\gamma(v_{-t}(\gamma^{-1}))\xi) \otimes \delta_\gamma \\
&= (\exp(-it\pi_\gamma(q(\gamma^{-1})))\xi) \otimes \delta_\gamma \\
&= (\exp(itq(\gamma))\xi) \otimes \delta_\gamma \\
&= V_t(\xi \otimes \delta_\gamma),
\end{aligned}$$

which finishes the proof. □

Since  $V_t$  is a unitary on  $L^2(X) \otimes \ell^2(\Gamma)$ , we may consider an inner automorphism  $\alpha_t$  of  $\mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$  by letting  $\alpha_t(x) = V_t x V_t^*$  for all  $x \in \mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$ . Notice that this forms a family of inner automorphism of the extended Roe algebra. Moreover, when restricting to the uniform Roe algebra one can recover from  $\alpha_t$  the multipliers introduced above:  $E \circ \alpha_t(x) = \mathfrak{m}_t(x)$  for all  $x \in C_u^*(\Gamma)$ .

However, one can see right away that these automorphisms do not move the group von Neumann algebra  $L\Gamma$  into itself. Hence, applying the deformation/rigidity arguments at the level of von Neumann algebra  $L\Gamma$  is rather inadequate. As we will see in the next section, this difficulty is overcome by working with the reduced  $C^*$ -algebra  $C_\lambda^*(\Gamma)$  rather than  $L\Gamma$ . The following result

underlines that the path  $\alpha_t$  is a deformation at the  $C^*$ -algebraic level i.e., with respect to the operatorial norm.

**Lemma IV.3.6.** *Assuming the notations above, for every  $x \in C_\lambda^*(\Gamma)$  we have*

$$\|(\alpha_t(x) - x) \cdot e\|_\infty \rightarrow 0 \text{ as } t \rightarrow 0; \quad (\text{IV.3.6})$$

$$\|(\alpha_t(JxJ) - JxJ) \cdot e\|_\infty \rightarrow 0 \text{ as } t \rightarrow 0, \quad (\text{IV.3.7})$$

where  $\|\cdot\|_\infty$  denotes the operatorial norm in  $\mathfrak{B}(L^2(X) \otimes \ell^2(\Gamma))$ . Here  $e$  denotes the orthogonal projection on  $\ell^2(\Gamma)$ .

*Proof.* Since elements in  $C_\lambda^*(\Gamma)$  can be approximated in the uniform norm by finitely supported elements, using the triangle inequality it suffices to show (IV.3.6) only for  $x = \sum_{g \in F} x_g u_g$ , a finite sum. Fix an arbitrary  $\xi = \sum_\gamma \xi_\gamma \otimes \delta_\gamma \in \mathbb{C} \otimes \ell^2(\Gamma)$ . Using the formula for  $\alpha_t$  in combination with Cauchy-Schwartz inequality, we have

$$\begin{aligned} \|(\alpha_t(x) - x)\xi\|^2 &= \left\| \sum_{g \in F} \sum_{\gamma \in \Gamma} x_g (V_t u_g V_{-t} - u_g) (\xi_\gamma \otimes \delta_\gamma) \right\|^2 \\ &\leq \left( \sum_{g \in F} |x_g|^2 \right) \left( \sum_{g \in F} \left\| \sum_{\gamma \in \Gamma} (V_t u_g V_{-t} - u_g) (\xi_\gamma \otimes \delta_\gamma) \right\|^2 \right) \\ &= \|x\|_2^2 \sum_{g \in F} \left\| \sum_{\gamma \in \Gamma} (V_t u_g V_{-t} - u_g) (\xi_\gamma \otimes \delta_\gamma) \right\|^2 \end{aligned} \quad (\text{IV.3.8})$$

Applying the definitions and the formula for  $V_t$  we see that  $u_g(\xi_\gamma \otimes \delta_\gamma) = \xi_\gamma \otimes \delta_{g\gamma}$  and  $V_t u_g V_{-t}(\xi_\gamma \otimes \delta_\gamma) = v_t(g\gamma) \sigma_g(v_{-t}(\gamma)) \xi_\gamma \otimes \delta_{g\gamma}$ . Therefore, continuing in (IV.3.8) we obtain

$$\begin{aligned} &= \|x\|_2^2 \sum_{g \in F} \left\| \sum_{\gamma \in \Gamma} \xi_\gamma (v_t(g\gamma) \sigma_g(v_{-t}(\gamma)) - 1) \otimes \delta_{g\gamma} \right\|^2 \\ &= \|x\|_2^2 \sum_{g \in F} \sum_{\gamma \in \Gamma} |\xi_\gamma|^2 \|v_t(g\gamma) \sigma_g(v_{-t}(\gamma)) - 1\|^2 \\ &= 2\|x\|_2^2 \sum_{g \in F} \sum_{\gamma \in \Gamma} |\xi_\gamma|^2 (1 - \tau(v_t(g\gamma) \sigma_g(v_{-t}(\gamma)))) . \end{aligned} \quad (\text{IV.3.9})$$

On the other hand, the same computations as in the proof of (IV.3.5) together with inequality

(IV.2.1) imply that, there exist  $K \geq 0$  such that for every  $g \in F$  and  $\gamma \in \Gamma$  we have

$$\begin{aligned} \tau(v_t(g\gamma)\sigma_g(v_{-t}(\gamma))) &= \int_X \exp(it(q(g\gamma) - \pi_g(q(\gamma)))(x)) d\mu(x) \\ &= \exp(-t^2\|q(g\gamma) - \pi_g(q(\gamma))\|^2) \\ &\geq \exp(-t^2K). \end{aligned} \tag{IV.3.10}$$

Thus, combining (IV.3.8), (IV.3.9) and (IV.3.10) we conclude that, for all  $\xi \in \mathbb{C} \otimes \ell^2(\Gamma)$ , we have

$$\|(\alpha_t(x) - x)\xi\|^2 \leq 2\|x\|_2^2\|\xi\|^2|F|(1 - \exp(-t^2K)),$$

which further implies

$$\|(\alpha_t(x) - x) \cdot e\|_\infty \leq 2\|x\|_2^2|F|(1 - \exp(-t^2K)). \tag{IV.3.11}$$

Since  $F$  is finite, then  $\exp(-t^2K) \rightarrow 1$  as  $t \rightarrow 0$ , and IV.3.6 follows from (IV.3.11).

It remains to show (IV.3.7). Since  $[e, J] = 0$ , by Proposition IV.3.5 we see that  $(\alpha_t(JxJ) - JxJ) \cdot e = J((\alpha_t(x) - x) \cdot e)J$ . Therefore, (IV.3.7) follows from (IV.3.6) because  $J$  is an anti-linear isometry.  $\square$

Next we show that the path of unitaries  $V_t$  satisfies a “transversality” property very similar to Lemma 2.1 in [77]. Our proof follows closely the proof of Lemma 3.1 in [98]: we include it here only for the sake of completeness.

**Lemma IV.3.7.** *If  $V_t$  is the unitary defined above, then for all  $\xi \in \mathbb{C} \otimes \ell^2(\Gamma)$  and all  $t \in \mathbb{R}$  we have*

$$2\|V_t(\xi) - e \cdot V_t(\xi)\|^2 \geq \|\xi - V_t(\xi)\|^2. \tag{IV.3.12}$$

*Proof.* Fix  $\xi \in \mathbb{C} \otimes \ell^2(\Gamma)$  and assume that it can be written as  $\xi = \sum_\gamma \xi_\gamma \otimes \delta_\gamma$  with  $\xi_\gamma$  scalars. Straightforward computations show that  $V_t(x) = \sum_\gamma \xi_\gamma v_t(\gamma) \otimes \delta_h$  and  $e \cdot V_t(\xi) = \sum_\gamma \xi_\gamma \tau(v_t(\gamma)) 1 \otimes \delta_\gamma$ ;

thus, the left side of (IV.3.12) is equal to

$$\begin{aligned}
2\|V_t(\xi) - e \cdot V_t(\xi)\|^2 &= 2(\|V_t(\xi)\|^2 - \|e \cdot V_t(\xi)\|^2) \\
&= 2 \sum_{\gamma} |\xi_{\gamma}|^2 \|v_t(\gamma)\|^2 - |\xi_h|^2 |\tau(v_t(\gamma))|^2 \\
&= 2 \sum_{\gamma} |\xi_{\gamma}|^2 (1 - |\tau(v_t(\gamma))|^2).
\end{aligned} \tag{IV.3.13}$$

Applying the same formulas as above, we see that the right side of (IV.3.12) is equal to

$$\begin{aligned}
\|\xi - V_t(\xi)\|^2 &= \|\xi\|^2 + \|V_t(\xi)\|^2 - 2\operatorname{Re}\langle V_t(\xi), \xi \rangle \\
&= 2(\|\xi\|^2 - \operatorname{Re}\langle V_t(\xi), \xi \rangle) \\
&= 2 \sum_{\gamma} |\xi_{\gamma}|^2 (1 - \tau(v_t(\gamma))).
\end{aligned} \tag{IV.3.14}$$

Since we have  $\tau(v_t(\gamma)) = \exp(-t^2\|q(\gamma)\|^2) \geq \exp(-2t^2\|q(\gamma)\|^2) = |\tau(v_t(\gamma))|^2$ , the conclusion follows from (IV.3.13) and (IV.3.14).  $\square$

The multipliers  $\mathfrak{m}_t$  arising from a proper quasi-cocycle behave in some sense as compact operators on  $L\Gamma$  i.e.,  $\mathfrak{m}_t$  is continuous from the weak operator topology to the strong operator topology on bounded sets. This result will be used crucially in the next section.

**Proposition IV.3.8.** Let  $\mathfrak{m}_t$  be the Schur multiplier associated to some proper quasi-cocycle on  $\Gamma$ . If  $v_k \in L\Gamma$  is a bounded net of elements such that  $v_k$  converges to  $v$  weakly, then for every  $t > 0$  and every  $\xi \in L^2(\Gamma)$  we have that

$$\|\mathfrak{m}_t(v_k)\xi - \mathfrak{m}_t(v)\xi\| \rightarrow 0. \tag{IV.3.15}$$

*Proof.* For simplicity we may assume that  $\xi = \delta_{\gamma}$  with  $\gamma \in \Gamma$  and  $v = 0$ . Let  $v_k = \sum_h \mu_h^k u_h$  with  $\mu_h^k$  scalars. Then applying the formula for  $\mathfrak{m}_t$  we have that

$$\begin{aligned}
\|\mathbf{m}_t(v_k)\xi\|^2 &= \|eV_t v_k V_{-t} \delta_\gamma\|^2 = \left\| \sum_h \mu_h^k eV_t u_h V_{-t} \delta_\gamma \right\|^2 \\
&= \left\| \sum_h \mu_h^k \tau(v_t(h\gamma)) \sigma_h(v_{-t}(\gamma)) 1 \otimes \delta_{h\gamma} \right\|^2 \\
&= \sum_h |\mu_h^k|^2 |\tau(v_t(h\gamma)) \sigma_h(v_{-t}(\gamma))|^2 \\
&= \sum_h |\mu_h^k|^2 \exp(-t^2 \|q(h\gamma) - \pi_h(q(\gamma))\|^2) \\
&\leq \sum_h |\mu_h^k|^2 \exp\left(-\frac{t^2}{2} \|q(h)\|^2 + t^2 D(q)\right)
\end{aligned} \tag{IV.3.16}$$

Fix  $\varepsilon > 0$ . Since  $q$  is proper there exists a finite set  $F_\varepsilon \subset \Gamma$  such that  $\frac{2}{t^2} \ln\left(\frac{2 \exp(t^2 D(q))}{\varepsilon}\right) \leq \|q(h)\|^2$  for all  $h \in \Gamma \setminus F_\varepsilon$ . This obviously implies that

$$\exp\left(-\frac{t^2}{2} \|q(h)\|^2 + t^2 D(q)\right) \leq \frac{\varepsilon}{2} \tag{IV.3.17}$$

for all  $h \in \Gamma \setminus F_\varepsilon$ . Also, since  $v_k$  is weakly convergent to 0 and  $F_\varepsilon$  is finite, there exists  $k_\varepsilon$  such that for all  $k \geq k_\varepsilon$  and all  $h \in F_\varepsilon$  we have

$$|\mu_h^k| \leq \left( \frac{\varepsilon}{2|F_\varepsilon| \max_{h \in F_\varepsilon} \exp\left(-\frac{t^2}{2} \|q(h)\|^2 + t^2 \Delta(q)\right)} \right)^{\frac{1}{2}} \tag{IV.3.18}$$

Using (IV.3.16), (IV.3.17), and (IV.3.18) together with  $\sum_h |\mu_h^k|^2 = 1$ , for all  $k \geq k_\varepsilon$  we have

$$\begin{aligned}
\|\mathbf{m}_t(v_k)\xi\|^2 &\leq \sum_h |\mu_h^k|^2 \exp\left(-\frac{t^2}{2} \|q(h)\|^2 + t^2 D(q)\right) \\
&\leq \left( \sum_{h \in F_\varepsilon} |\mu_h^k|^2 \right) \max_{h \in F_\varepsilon} \exp\left(-\frac{t^2}{2} \|q(h)\|^2 + t^2 D(q)\right) + \frac{\varepsilon}{2} \left( \sum_{h \in \Gamma \setminus F_\varepsilon} |\mu_h^k|^2 \right) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

giving the desired conclusion.  $\square$

#### IV.4 Proofs of the Main Results

We begin by proving Theorem A on the solidity of  $L\Gamma$  for groups admitting proper quasi-cocycles. The strategy of proof will essentially follow Peterson’s approach (Theorem 4.3 in [64]), formally replacing the family of almost derivations with the one-parameter group  $\alpha_t$  constructed in §IV.3.3. We note that unlike the proofs of solidity by Popa [75] and Vaes [98]—which also make use of one-parameter automorphism groups—we cannot directly appeal to spectral gap arguments and must, like Peterson, make fundamental use of Haagerup’s criterion for amenability. We state Haagerup’s criterion here for the convenience of the reader.

**Proposition IV.4.1** (Haagerup, Lemma 2.2 in [31]). Let  $M$  be a  $\text{II}_1$  factor. A von Neumann subalgebra  $N \subset M$  is amenable if and only if for every non-zero projection  $p \in \mathcal{Z}(N)$  and every finite set of unitaries  $u_1, u_2, \dots, u_n \in \mathcal{U}(Np)$  we have

$$\left\| \sum_{i=1}^n u_i \otimes \bar{u}_i \right\|_\infty = n. \tag{IV.4.1}$$

*Proof of Theorem A.* Let  $N \subset L\Gamma$  be a nonamenable von Neumann subalgebra, and set  $A = N' \cap L\Gamma$ . Let  $p \in \mathcal{Z}(N)$  be a non-zero projection and  $F \subset \mathcal{U}(Np)$  be a finite set of unitaries. Suppose  $A$  is diffuse, in which case  $V_t$  cannot converge uniformly on  $(Ap)_1$ . Indeed, for every  $\sum_\gamma \mu_\gamma \delta_\gamma \in \ell^2(\Gamma)$  it is easy to calculate that

$$\|e \cdot V_t(\sum_\gamma \mu_\gamma \delta_\gamma)\|^2 = \sum_\gamma |\mu_\gamma|^2 \exp(-2t^2 \|q(\gamma)\|^2),$$

so  $e \cdot V_t$  is a compact operator on  $\ell^2(\Gamma)$  for all  $t > 0$ . The fact then follows, since we have that  $\|V_t(x) - x\| \geq \|e \cdot V_t(x) - x\|$  for all  $x \in \ell^2(\Gamma)$ . Therefore, there exists  $c > 0$  such that for every  $t > 0$  there exists  $x_t \in (Ap)_1$  such that  $\|V_t(\hat{x}_t) - \hat{x}_t\| \geq c$ . Let us denote  $V_t(\hat{x}_t)$  by  $\zeta_t$  and define  $\xi_t$  to be  $\zeta_t - e(\zeta_t)$ . By Lemma IV.3.7, we have that  $\|\xi_t\| \geq \frac{c}{2}$ .

Let  $E \subset L\Gamma$  be the operator system spanned by  $\{p\} \cup F \cup F^*$ : by local reflexivity, there exists a net  $(\varphi_i)_{i \in I}$  of contractive completely positive maps  $\varphi_i : E \rightarrow C_\lambda^*(\Gamma)$  such that  $\varphi_i \rightarrow \text{id}_E$  pointwise-ultraweakly. In fact, by passing to convex combinations of the  $\varphi_i$ ’s, we may assume that  $\varphi_i(u) \rightarrow u$  in the strong\* topology for all  $u \in F$ . Fixing  $i \in I$ , we have that for all  $u \in F$

$$\begin{aligned}
& \lim_{t \rightarrow 0} \|\varphi_i(u)\xi_t\varphi_i(u^*) - \xi_t\| \\
&= \lim_{t \rightarrow 0} \|(1 - e)(\varphi_i(u)\zeta_t\varphi_i(u^*) - \zeta_t)\| \\
&\leq \lim_{t \rightarrow 0} \|\varphi_i(u)J\varphi_i(u)J(\zeta_t) - \zeta_t\| \\
&\leq \lim_{t \rightarrow 0} \|\alpha_t(\varphi_i(u))\alpha_t(J\varphi_i(u)J)(\zeta_t) - \zeta_t\| + 2 \lim_{t \rightarrow 0} \|(\alpha_{-t}(\varphi_i(u)) - \varphi_i(u))\hat{x}_t\| \\
&\leq \lim_{t \rightarrow 0} \|V_t(\varphi_i(u)x_t\varphi_i(u^*) - x_t)\| + 2 \lim_{t \rightarrow 0} \|(\alpha_t(\varphi_i(u)) - \varphi_i(u)) \circ e\|_\infty \\
&\leq \lim_{t \rightarrow 0} \|\varphi_i(u)x_t\varphi_i(u^*) - x_t\|_2 \\
&\leq \lim_{t \rightarrow 0} \|ux_tu^* - x_t\|_2 + 2 \lim_{t \rightarrow 0} \|x_t\|_\infty \|\varphi_i(u) - u\|_2 \\
&\leq 2\|\varphi_i(u) - u\|_2
\end{aligned} \tag{IV.4.2}$$

Given  $\varepsilon > 0$ , let us choose  $i \in I$  such that  $\sum_{u \in F} \|\varphi_i(u) - u\|_2 \leq \frac{\varepsilon}{4}$ . It then follows from the calculation above that there exists  $t > 0$  such that

$$\begin{aligned}
\left\| \sum_{u \in F} u \otimes \bar{u} \right\|_\infty &\geq \frac{\|\sum_{u \in F} u\xi_t u^*\|}{\|\xi_t\|} \\
&\geq \frac{\|\sum_{u \in F} \varphi_i(u)\xi_t\varphi_i(u^*)\|}{\|\xi_t\|} \geq |F| - \varepsilon
\end{aligned}$$

Hence, by Haagerup's criterion we have that  $N$  is amenable, a contradiction. □

*Proof of Theorem B.* The proof follows the proof of Theorem B in [61]. Let  $P \subset L\Gamma = M$  be a diffuse amenable von Neumann subalgebra,  $\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : uPu^* = P\}$ ,  $N = \mathcal{N}_M(P)''$ , and fix  $p \in N' \cap M$  a projection. Since  $\Gamma$  is weakly amenable, by Theorem B in [58], we have that  $P$  is weakly compact in  $M$ . That is, there exists a net of unit vectors  $(\eta_n)_{n \in \mathbb{N}}$  in  $L^2(M) \otimes L^2(\bar{M})$  such that:

1.  $\|\eta_n - (v \otimes \bar{v})\eta_n\| \rightarrow 0$ , for all  $v \in \mathcal{U}(P)$ ;
2.  $\|[u \otimes \bar{u}, \eta_n]\| \rightarrow 0$ , for all  $u \in \mathcal{N}_M(P)$ ; and
3.  $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n \rangle$ , for all  $x \in M$ .

Fix  $t > 0$  and denote by  $\tilde{\eta}_{n,t} = (V_t \otimes 1)(p \otimes 1)\eta_n \in \mathcal{H} \otimes L^2(M)$ ,  $\zeta_{n,t} = (e \otimes 1)\tilde{\eta}_{n,t} = (e \cdot V_t \otimes 1)(p \otimes 1)\eta_n$ , and  $\xi_{n,t} = \tilde{\eta}_{n,t} - \zeta_{n,t}$ . We will begin by showing that

$$\lim_n \|\xi_{n,t}\| \geq \frac{5}{12} \|p\|_2. \quad (\text{IV.4.3})$$

Using the triangle inequality, we have that

$$\begin{aligned} \|\tilde{\eta}_{n,t} - (e \circ \alpha_t(v) \otimes \bar{v})\zeta_{n,t}\| &\leq \|\tilde{\eta}_{n,t} - (e \circ \alpha_t(v) \otimes \bar{v})\tilde{\eta}_{n,t}\| + \|\xi_{n,t}\| \\ &\leq \|\zeta_{n,t} - (e \circ \alpha_t(v) \otimes \bar{v})\tilde{\eta}_{n,t}\| + 2\|\xi_{n,t}\| \\ &\leq \|\tilde{\eta}_{n,t} - (\alpha_t(v) \otimes \bar{v})\tilde{\eta}_{n,t}\| + 2\|\xi_{n,t}\| \\ &\leq \|\eta_n - (v \otimes \bar{v})\eta_n\| + 2\|\xi_{n,t}\| \end{aligned}$$

for all  $v \in \mathcal{U}(P)$  and all  $n$ .

Consequently, since by (3) we have  $\|\tilde{\eta}_{n,t}\| = \|p\|_2$ , then using the triangle inequality again we get

$$\|(e \circ \alpha_t(v) \otimes \bar{v})\zeta_{n,t}\| \geq \|p\|_2 - 2\|\xi_{n,t}\| - \|\eta_n - (v \otimes \bar{v})\eta_n\|. \quad (\text{IV.4.4})$$

Since the operator  $e \cdot V_t$  is compact on  $L^2(M)$ , one can find a finite set  $F \subset \Gamma$  such that  $\|(P_F \otimes 1)(e \cdot V_t \otimes 1) - e \cdot V_t \otimes 1\|_\infty \leq \frac{\|p\|_2}{6}$ ; here  $P_F$  denotes the orthogonal projection on the span of  $F$ . Hence using the formula  $e \circ \alpha_t = \mathbf{m}_t$  together (IV.4.4) and triangle inequality we obtain

$$\|(\mathbf{m}_t(v) \otimes 1)(P_F \otimes 1)\zeta_{n,t}\| \geq \frac{5}{6} \|p\|_2 - 2\|\xi_{n,t}\| - \|\eta_n - (v \otimes \bar{v})\eta_n\|, \quad (\text{IV.4.5})$$

for all  $v \in \mathcal{U}(P)$  and all  $n \in \mathbf{N}$ .

On the other hand, consider the decomposition  $(p \otimes 1)\eta_n = \sum_{g,h} \mu_{g,h}^n \delta_g \otimes \delta_h$  with  $\mu_{g,h}^n$  scalars. Applying the formula for the multiplier  $\mathbf{m}_t$  together with Cauchy-Schwartz inequality, we obtain



$$\begin{aligned}
& \|(\mathbf{m}_t(v) \otimes 1)(P_F \otimes 1)\zeta_{n,t}\|^2 \\
&= \sum_h \left\| \sum_{g \in F} e^{-t^2 \|q(g)\|^2} \mu_{g,h}^n \mathbf{m}_t(v) \delta_g \right\|^2 \\
&\leq \sum_h \left( \sum_{g \in F} e^{-2t^2 \|q(g)\|^2} |\mu_{g,h}^n|^2 \right) \left( \sum_{g \in F} \|\mathbf{m}_t(v) \delta_g\|^2 \right) \\
&\leq \|(p \otimes 1)\eta_n\|^2 \left( \sum_{g \in F} \|\mathbf{m}_t(v) \delta_g\|^2 \right) = \|p\|_2^2 \sum_{g \in F} \|\mathbf{m}_t(v) \delta_g\|^2,
\end{aligned} \tag{IV.4.6}$$

for all  $v \in \mathcal{U}(P)$  and all  $n \in \mathbf{N}$ . Thus, combining (IV.4.5) with (IV.4.6) and taking an arbitrary Banach limit  $\text{Lim}_n$ , by relation (1), we obtain that

$$\left( \|p\|_2^2 \sum_{g \in F} \|\mathbf{m}_t(v) \delta_g\|^2 \right)^{\frac{1}{2}} \geq \frac{5}{6} \|p\|_2 - 2 \text{Lim}_n \|\xi_{n,t}\|, \tag{IV.4.7}$$

for all  $v \in \mathcal{U}(P)$ . This shows that the limit  $\text{Lim}_n \|\xi_{n,t}\| \geq \frac{5}{12} \|p\|_2$ . Indeed, since  $P$  is diffuse there exists a sequence of unitaries  $v_k \in \mathcal{U}(P)$  which converges weakly to 0. Then by Proposition IV.3.8 the infimum achieved by the left side of (IV.4.7) is 0, and we are done.

Let  $\mathcal{H} = L_0^2(X) \otimes \ell^2(\Gamma)$  which as we remarked before is weakly contained as an  $M$ -bimodule in the coarse bimodule. Following the same argument as in Theorem B of [61] we define a state  $\psi_t$  on  $\mathcal{N} = \mathfrak{B}(\mathcal{H}) \cap \rho(M^{op})'$ . Explicitly,  $\psi_t(x) = \text{Lim}_n \frac{1}{\|\xi_{n,t}\|^2} \langle (x \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle$  for every  $x \in \mathcal{N}$ . Next we prove the following technical result

**Lemma IV.4.2.** *For every  $\varepsilon > 0$  and every finite set  $K \subset C_\lambda^*(\Gamma)$  with  $\text{dist}_{\|\cdot\|_2}(y, (N)_1) \leq \varepsilon$  for all  $y \in K$  one can find  $t_\varepsilon > 0$  and a finite set  $L_{K,\varepsilon} \subset \mathcal{N}_M(P)$  such that*

$$| \langle ((yx - xy) \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle | \leq 4\varepsilon + 2 \sum_{v \in L_{K,\varepsilon}} \| [v \otimes \bar{v}, \eta_n] \|, \tag{IV.4.8}$$

for all  $y \in K$ ,  $\|x\|_\infty \leq 1$ ,  $t_\varepsilon > t > 0$ , and  $n$ .

*Proof.* Fix  $\varepsilon > 0$  and  $y \in K$ . Since  $N = \mathcal{N}_M(P)''$  by the Kaplansky density theorem there exists a

finite set  $F_y = \{v_i\} \subset \mathcal{N}_M(P)$  and scalars  $\mu_i$  such that  $\|\sum_i \mu_i v_i\|_\infty \leq 1$  and

$$\|y - \sum_i \mu_i v_i\|_2 \leq \varepsilon. \quad (\text{IV.4.9})$$

Also using Proposition IV.3.6 one can find a positive number  $t_\varepsilon > 0$  such that for all  $t_\varepsilon > t > 0$

$$\|y - \alpha_{-t}(y)\|_\infty \leq \varepsilon. \quad (\text{IV.4.10})$$

Next we will proceed in several steps to show inequality (IV.4.8). First we fix  $t_\varepsilon > t > 0$ . Then, using the triangle inequality in combination with

$\|x\|_\infty \leq 1$ , (IV.4.10), and the  $M$ -bimodularity of  $1 - e = e^\perp$ , we see that

$$\begin{aligned} & |\langle (x \otimes 1)\xi_{n,t}, (y^* \otimes 1)\xi_{n,t} \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle| \\ & \leq \|\alpha_{-t}(y^*) - y^*\|_\infty + |\langle (x \otimes 1)\xi_{n,t_\varepsilon}, (e^\perp V_{t_\varepsilon} y^* p \otimes 1)\eta_n \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle| \\ & \leq \varepsilon + |\langle (x \otimes 1)\xi_{n,t}, (e^\perp V_t y^* p \otimes 1)\eta_n \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle| \end{aligned}$$

Furthermore, Cauchy-Schwartz inequality together with (3) and (IV.4.9) allow us to see that the last quantity above is smaller than

$$\begin{aligned} & \leq \varepsilon + \|((y^* p - \sum_i \bar{\mu}_i v_i^*) p \otimes 1)\eta_n\| + |\sum_i \mu_i \langle (xy \otimes 1)\xi_{n,t}, (e^\perp V_t v_i^* p \otimes 1)\eta_n \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle| \\ & \leq 2\varepsilon + |\sum_i \mu_i \langle (x \otimes \bar{v}_i^*)\xi_{n,t}, (e^\perp V_t p v_i^* \otimes \bar{v}_i^*)\eta_n \rangle - \langle (xy \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle| \end{aligned}$$

Applying successively the triangle inequality and using (IV.4.9),  $v_i$  being a unitary, and (IV.4.10) we have that the previous quantity is smaller than

$$\begin{aligned}
&\leq 2\varepsilon + \sum_i \|[v_i^* \otimes \bar{v}_i^*, \eta_n]\| + \left| \sum_i \mu_i \langle (x \otimes \bar{v}_i^*) \xi_{n,t}, \xi_{n,t} (v_i^* \otimes \bar{v}_i^*) \rangle - \langle (xy \otimes 1) \xi_{n,t}, \xi_{n,t} \rangle \right| \\
&\leq 2\varepsilon + \sum_i \|[v_i^* \otimes \bar{v}_i^*, \eta_n]\| + \left| \sum_i \mu_i \langle (x \otimes \bar{v}_i^*) \xi_{n,t} (v_i \otimes \bar{v}_i), \xi_{n,t} \rangle - \langle (xy \otimes 1) \xi_{n,t}, \xi_{n,t} \rangle \right| \\
&\leq 2\varepsilon + 2 \sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \left| \langle (xe^\perp V_t (\sum_i \mu_i v_i) p \otimes 1) \eta_n, \xi_{n,t} \rangle - \langle (xy \otimes 1) \xi_{n,t}, \xi_{n,t} \rangle \right| \\
&\leq 2\varepsilon + \left\| \left( (up - \sum_i \mu_i v_i) p \otimes 1 \right) \eta_n \right\| + 2 \sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \\
&\quad + \left| \langle (xe^\perp V_t y p \otimes 1) \eta_n, \xi_{n,t} \rangle - \langle (xy \otimes 1) \xi_{n,t}, \xi_{n,t} \rangle \right| \\
&\leq 3\varepsilon + 2 \sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\| + \|y - a_{-t}(y)\|_\infty \leq 4\varepsilon + 2 \sum_i \|[v_i \otimes \bar{v}_i, \eta_n]\|
\end{aligned}$$

In conclusion, (IV.4.8) follows from the previous inequalities by taking  $L_{K,\varepsilon} = \cup_{y \in K} Fy$ .  $\square$

**Lemma IV.4.3.** *For every  $\varepsilon > 0$  and  $F_0 \subset \mathcal{U}(N)$  finite set there exist  $F_0 \subset F \subset M$  finite set, a c.c.p. map  $\varphi_{F,\varepsilon} : \text{span}(F) \rightarrow C_\lambda^*(\Gamma)$ , and  $t_\varepsilon > 0$  such that*

$$|\psi_{t_\varepsilon}(\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up)) - \psi_{t_\varepsilon}(x)| \leq 47\varepsilon, \quad (\text{IV.4.11})$$

for all  $u \in F_0$  and  $\|x\|_\infty \leq 1$ .

*Proof.* Fix  $\varepsilon > 0$ . Denote by  $F = \{up, u^*p\} \cup F_0 \cup F_0^*$  and  $E = \text{span}(F)$  and by local reflexivity, we may choose a c.c.p. map  $\varphi_{F,\varepsilon} : E \rightarrow C_\lambda^*(\Gamma)$  such that for all  $u \in F$

$$\|\varphi_{F,\varepsilon}(up) - up\|_2 \leq \varepsilon. \quad (\text{IV.4.12})$$

This shows in particular that  $\text{dist}_{\|\cdot\|_2}(\varphi_{F,\varepsilon}(up), (N)_1) \leq \varepsilon$  for all  $u \in F$ . Therefore applying the previous lemma for  $K = \{\varphi_{F,\varepsilon}(up) \mid u \in F\} \subset C_\lambda^*(\Gamma)$  there exists a  $t_\varepsilon > 0$  and a finite set  $K' \subset \mathcal{N}_M(P)$  such that for all  $u \in F$ , all  $\|x\|_\infty \leq 1$ , and all  $n$  we have

$$\left| \langle ((\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up) - x \varphi_{F,\varepsilon}(up) \varphi_{F,\varepsilon}(up)^*) \otimes 1) \xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon} \rangle \right| \leq 4\varepsilon + 2 \sum_{v \in K'} \|[v \otimes \bar{v}, \eta_n]\| \quad (\text{IV.4.13})$$

Also using Proposition IV.3.6, after shrinking  $t_\varepsilon$  if necessary, we can assume in addition that for all  $u \in F$  we have

$$\|\varphi_{F,\varepsilon}(up) - \alpha_{-t_\varepsilon}(\varphi_{F,\varepsilon}(up))\|_\infty \leq \varepsilon. \quad (\text{IV.4.14})$$

Hence, using triangle inequality together with (IV.4.13) and Cauchy-Schwartz inequality, we have that

$$\begin{aligned} & | \langle (\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up) \otimes 1) \xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon} \rangle - \langle (x \otimes 1) \xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon} \rangle | \\ & \leq 4\varepsilon + 2 \sum_i \|[v \otimes \bar{v}, \eta_n]\| + | \langle (x(\varphi_{F,\varepsilon}(up) \varphi_{F,t_\varepsilon}(up)^* - 1) \otimes 1) \xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon} \rangle | \\ & \leq 4\varepsilon + 2 \sum_v \|[v \otimes \bar{v}, \eta_n]\| + \|x\|_\infty \|(\varphi_{F,\varepsilon}(up) \varphi_{F,t_\varepsilon}(up)^* - 1) \otimes 1\| \|\xi_{n,t_\varepsilon}\| \\ & \leq 4\varepsilon + 2 \sum_v \|[v \otimes \bar{v}, \eta_n]\| + 2\|\alpha_{-t_\varepsilon}(\varphi_{F,\varepsilon}(up)) - \varphi_{F,\varepsilon}(up)\|_\infty \\ & \quad + \|(V_{t_\varepsilon}(\varphi_{F,\varepsilon}(up) \varphi_{F,\varepsilon}(up)^* - 1)p \otimes 1)\eta_n\| \end{aligned}$$

Furthermore, using (IV.4.12) together with Cauchy-Schwartz inequality, (3) and (IV.4.14) we see that the last quantity above is smaller than

$$\begin{aligned} & \leq 6\varepsilon + 2 \sum_v \|[v \otimes \bar{v}, \eta_n]\| + \|(\varphi_{F,\varepsilon}(up) \varphi_{F,\varepsilon}(up)^* - 1)p \otimes 1\| \|\eta_n\| \\ & \leq 6\varepsilon + 2 \sum_v \|[v \otimes \bar{v}, \eta_n]\| + \|\varphi_{F,\varepsilon}(up) \varphi_{F,\varepsilon}(up)^* - p\|_2 \\ & \leq 6\varepsilon + 2 \sum_v \|[v \otimes \bar{v}, \eta_n]\| + 2\|\varphi_{F,\varepsilon}(up) - up\|_2 \\ & \leq 8\varepsilon + 2 \sum_v \|[v \otimes \bar{v}, \eta_n]\|. \end{aligned}$$

Altogether the above sequence of inequalities shows that

$$| \langle (\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up) \otimes 1) \xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon} \rangle - \langle (x \otimes 1) \xi_{n,t_\varepsilon}, \xi_{n,t_\varepsilon} \rangle | \leq 8\varepsilon + 2 \sum_{v \in K'} \|[v \otimes \bar{v}, \eta_n]\|,$$

and combining this with (2) and (IV.4.3) we obtain

$$\begin{aligned} |\psi_{t_\varepsilon}(\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up)) - \psi_{t_\varepsilon}(x)| &\leq \text{Lim}_n \left( \frac{8\varepsilon + 2 \sum_v \|[v \otimes \bar{v}, \eta_n]\|}{\|\xi_{n,t_\varepsilon}\|^2} \right) \\ &\leq \frac{8\varepsilon}{\left(\frac{5}{12}\right)^2} < 47\varepsilon, \end{aligned}$$

which finishes the proof.  $\square$

For the remaining part of the proof we mention that one can use Haagerup criterion to show that  $N$  is amenable. In fact the reasoning in Theorem B in [61] applies verbatim in our case and we leave the details to the reader.  $\square$

*Proof of Corollary C.* In the case that  $\Gamma$  is hyperbolic, a result of Ozawa shows that  $\Gamma$  is weakly amenable [56]. In the case that  $\Gamma$  is a lattice in  $\text{Sp}(n, 1)$ , choose a co-compact lattice  $\Lambda < \text{Sp}(n, 1)$ . We have that  $\Lambda$  is Gromov hyperbolic; hence, by [43]  $\Lambda$  belongs to the class  $\mathcal{QH}_{\text{reg}}$ . A result of Shalom (Theorem 3.7 in [91]) shows that  $\Gamma < \text{Sp}(n, 1)$  is integrable, thus  $\ell^1$ -measure equivalent to  $\Lambda$ . This implies that  $\Gamma \in \mathcal{QH}_{\text{reg}}$ . The work of Cowling and Haagerup [20] shows that  $\text{Sp}(n, 1)$  is weakly amenable, which implies, by an unpublished result of Haagerup, that any lattice in  $\text{Sp}(n, 1)$  is also weakly amenable. Therefore, the hypotheses of Theorem B are satisfied.  $\square$

*Proof of Theorem D.* Let  $M = L\Gamma = L\Gamma_1 \bar{\otimes} \cdots \bar{\otimes} L\Gamma_n$ ,  $M_i = L\Gamma_i$ , and for each  $\Gamma_i$ ,  $1 \leq i \leq n$ , let  $q_i : \Gamma_i \rightarrow \mathcal{H}_i$  be a proper array for some weakly- $\ell^2$  unitary representation  $(\mathcal{H}_i, \pi_i)$ . Let  $V_t^i \in \mathcal{U}(\mathcal{H}^i)$ , where  $\mathcal{H}^i = L^2(X_i) \otimes \ell^2(\Gamma_i)$ , be the one-parameter family of unitaries as defined in §IV.3.3 with the attendant family of automorphisms  $\alpha_t^i$  of  $\mathfrak{B}(\mathcal{H}^i)$ . Let us denote by  $\tilde{\mathcal{H}}^i$  the Hilbert space  $\mathcal{H}^i \otimes L^2(\hat{M}_i)$  with the natural  $M$ - $M$  bimodular structure. We extend  $V_t^i$  to the unitary  $\tilde{V}_t^i = V_t^i \otimes 1 \in \mathcal{U}(\tilde{\mathcal{H}}^i)$ .

Define  $\mathcal{F}$  to be the net of finite sets of unitaries in  $\mathcal{U}(N)$ . Note that by local reflexivity, for each  $\varepsilon > 0$  and  $F \in \mathcal{F}$ , there exists a c.c.p. map  $\varphi_{F,\varepsilon} : E \rightarrow C_\lambda^*(\Gamma)$ , where  $E$  is the spanned by  $\{1\} \cup F \cup F^*$ , such that  $\sum_{u \in F} \|\varphi_{F,\varepsilon}(u) - u\|_2 \leq \varepsilon$ .

To begin, suppose that  $B \subset L\Gamma$  is a  $\text{II}_1$  subfactor whose relative commutant  $N = B' \cap L\Gamma$  is a non-amenable factor. Let's assume that by way of contradiction, for all  $1 \leq i \leq n$ ,  $\|\tilde{V}_t^i(\hat{x}) - \hat{x}\|_2$  does not converge uniformly as  $t \rightarrow 0$ , where  $x$  ranges in  $(B)_1$ . By the proof of Theorem A, there would exist  $c > 0$  such that, for every  $\varepsilon > 0$ , every finite subset  $F \subset \mathcal{U}(N)$ , and every  $1 \leq i \leq n$ , there

would be a vector  $\xi_{F,\varepsilon}^i \in \tilde{\mathcal{H}}^i$  satisfying: (1)  $\|\xi_{F,\varepsilon}^i\|_2 \geq c$  and (2)  $\sum_{u \in F} \|\varphi_{F,\varepsilon}(u)\xi_{F,\varepsilon}^i - \xi_{F,\varepsilon}^i\|_2 \leq \varepsilon$ . Setting  $\mathcal{M}_i = \mathfrak{B}(\tilde{\mathcal{H}}^i) \cap (M^o)'$ , we define a state  $\psi_{F,\varepsilon}^i$  on  $\mathcal{M}_i$  by

$$\psi_{F,\varepsilon}^i(x) = \frac{1}{\|\xi_{F,\varepsilon}^i\|_2^2} \langle x\xi_{F,\varepsilon}^i, \xi_{F,\varepsilon}^i \rangle \quad (\text{IV.4.15})$$

An easy computation shows that  $|\psi_{F,\varepsilon}^i(x) - \psi_{F,\varepsilon}^i(\varphi_{F,\varepsilon}(u)x\varphi_{F,\varepsilon}(u^*))| \leq c^{-1}\varepsilon\|x\|_\infty$ , for all  $x \in \mathcal{M}_i$ ,  $u \in F$ . Since  $\psi_{F,\varepsilon}^i \in (\mathcal{M}_i)^*$ , it belongs to the closed convex hull of states of the form  $\sigma_\eta(x) = \langle x\eta, \eta \rangle$ , where  $\eta \in L^2(\mathcal{M}_i)$ . Hence, by the generalized Powers–Størmer inequality (cf. §2.1 of [60]) we conclude that

$$\lim_{\varepsilon \rightarrow 0} \left\| \sum_{u \in F} \varphi_{F,\varepsilon}(u) \otimes \overline{\varphi_{F,\varepsilon}(u)} \right\|_{L^2(\mathcal{M}_i), \infty} = |F| \quad (\text{IV.4.16})$$

Canonically identifying  $L^2(\mathcal{M}_i)$  with  $\tilde{\mathcal{H}}^i \otimes_M \overline{\tilde{\mathcal{H}}^i}$  as in §2 of [92], we have that

$$\left\| \sum_{u \in F} u \otimes \bar{u} \right\|_{\tilde{\mathcal{H}}^i \otimes_M \overline{\tilde{\mathcal{H}}^i}, \infty} \geq \lim_{\varepsilon \rightarrow 0} \left\| \sum_{u \in F} \varphi_{F,\varepsilon}(u) \otimes \overline{\varphi_{F,\varepsilon}(u)} \right\|_{L^2(\mathcal{M}_i), \infty} = |F|.$$

Hence, by the generalized Haagerup’s criterion for amenability ([92], Corollary 2.4), we have that  $\tilde{\mathcal{H}}^i$  is a left  $N$ -amenable Hilbert  $M$ – $M$  bimodule. That is, there exists a net of unital vectors  $\xi_n^i \in \tilde{\mathcal{H}}^i \otimes_M \overline{\tilde{\mathcal{H}}^i} \cong (\mathcal{H}^i \otimes_M \overline{\mathcal{H}^i}) \otimes L^2(\hat{M}_i)$  such that: (1)  $\langle x\xi_n^i, \xi_n^i \rangle = \tau(x) = \langle \xi_n^i x, \xi_n^i \rangle$ , for all  $x \in M$ ; and (2)  $\|[y, \xi_n^i]\| \rightarrow 0$ , for all  $y \in N$ . Now, since  $\tilde{\mathcal{H}}^1 \otimes_M \overline{\tilde{\mathcal{H}}^1} \otimes_M \cdots \otimes_M \tilde{\mathcal{H}}^n \otimes_M \overline{\tilde{\mathcal{H}}^n}$  may be seen to be weakly contained in  $L^2(M) \otimes L^2(\bar{M})$ , by the proof of Theorem 3.2 in [92] it follows that  $N$  is an amenable  $\text{II}_1$  factor, a contradiction.

Therefore, we must have that  $\|\tilde{V}_t^i(\hat{x}) - \hat{x}\|_2 \rightarrow 0$  uniformly on  $(B)_1$  for some  $1 \leq i \leq n$ . Since  $e_M \circ \tilde{V}_t^i$  is  $\hat{M}_i$ – $\hat{M}_i$  bimodular and compact when restricted  $L^2(M_i)$ , an examination of the proof on Theorem 6.2 in [70] shows that  $B \preceq_M \hat{M}_i$ . The result then follows by appealing to Proposition 12 of [59].

□

#### IV.5 Amenable actions, (bi-)exactness, and local reflexivity

**Definition IV.5.1** ([33]). Let  $\Gamma$  be a countable discrete group and  $\Gamma \curvearrowright X$  be an action of  $\Gamma$  by homeomorphisms on a compact topological space  $X$ . The action  $\Gamma \curvearrowright X$  is said to be (*topologically*)

amenable if there exists a sequence  $(\xi_n)$  of continuous maps  $\xi_n : X \rightarrow \ell^2(\Gamma)$  such that  $\xi_n \geq 0$ ,  $\|\xi_n(x)\|_2 = 1$ , for all  $x \in X$ ,  $n \in \mathbb{N}$ , and

$$\sup_{x \in X} \|\lambda_\gamma(\xi_n(x)) - \xi_n(\gamma x)\|_2 \rightarrow 0, \quad (\text{IV.5.1})$$

for all  $\gamma \in \Gamma$ .

**Proposition IV.5.1** (Higson and Roe [33]). A countable discrete group  $\Gamma$  has Guoliang Yu's *property A* [104] if and only if  $\Gamma$  acts amenably on its Stone-Ćech boundary  $\beta'\Gamma = \beta\Gamma \setminus \Gamma$ .

Property A is equivalent, cf. [83], to the nuclearity of  $C_u^*(\Gamma)$  which is, in turn, equivalent to the exactness of  $C_\lambda^*(\Gamma)$  by a result of Ozawa [53].

For the purposes of this chapter, the crucial property implied by exactness is that  $C_\lambda^*(\Gamma)$  is a locally reflexive  $C^*$ -algebra, cf. [7], Chapter 9.

**Definition IV.5.2.** A  $C^*$ -algebra  $A$  is said to be *locally reflexive* if for every finite-dimensional operator system  $E \subset A^{**}$ , there exists a net  $(\varphi_i)_{i \in I}$  of contractive completely positive (c.c.p.) maps  $\varphi_i : E \rightarrow A$  which converge to the identity in the pointwise-ultraweak topology.

We conclude this section with some remarks concerning amenable actions and the class  $\mathcal{QH}_{\text{reg}}$ .

**Definition IV.5.3** (Ozawa [7, 55]). A countable discrete group  $G$  is said to be *bi-exact* if it admits a sequence  $\xi_n : \beta'\Gamma \rightarrow \ell^2(\Gamma)$  of continuous maps such that  $\xi_n \geq 0$ ,  $\|\xi_n(x)\|_2 = 1$ , for all  $x \in \beta'\Gamma$ ,  $n \in \mathbb{N}$ , which satisfy

$$\sup_{x \in \beta'\Gamma} \|\lambda_\gamma(\xi_n(x)) - \xi_n(\gamma x \delta)\|_2 \rightarrow 0, \quad (\text{IV.5.2})$$

for all  $\gamma, \delta \in \Gamma$ .

It is easy to see that if  $\Gamma$  is bi-exact in the sense of Definition 15.1.2 of [7] if and only if  $\Gamma$  is bi-exact in the sense of the above definition. By the same proof that “property A  $\Rightarrow$  coarse embeddability into Hilbert space” (cf. [7, 83]), we have the following.

**Proposition IV.5.2.** If  $\Gamma$  is bi-exact, then it admits a uniform array into  $\ell^2(\Gamma)^{\oplus \infty}$ . In particular,  $\Gamma$  is exact and belongs to the class  $\mathcal{QH}_{\text{reg}}$ .

IV.6 A proof of Proposition IV.2.2

The aim of this section is to establish that  $\Gamma = \mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$  belongs to the class  $\mathcal{QH}_{\mathrm{reg}}$ . Appealing to Theorem A then furnishes an alternate proof of the solidity of  $L\Gamma$ , the main result of [57]. As in [57], our proof will make use of the amenability of the natural action of  $\mathrm{SL}(2, \mathbb{Z})$  on  $\mathrm{SL}(2, \mathbb{R})/T \cong \mathbb{RP}^1$ , where  $T$  is the group of upper-triangular  $2 \times 2$  real matrices.

To begin, note that  $\Gamma_0 = \mathrm{SL}(2, \mathbb{Z})$  admits a proper cocycle  $b : \Gamma_0 \rightarrow \ell^2(\Gamma_0)$  with respect to the left regular representation. Let  $\pi$  be the representation of  $\Gamma$  on  $\ell^2(\Gamma_0)$  obtained by pulling the left regular representation of  $\Gamma_0$  back along the quotient  $\Gamma \rightarrow \Gamma/\mathbb{Z}^2 \cong \Gamma_0$ , so that  $\pi$  is weakly contained in the left regular representation of  $\Gamma$ . Let  $p : \mathbb{Z}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{RP}^1$  be the projection defined by  $p((x, y)) = x/y$ , and note that  $p$  is equivariant with respect to the natural actions of  $\mathrm{SL}(2, \mathbb{Z})$  on  $\mathbb{Z}^2$  and  $\mathbb{RP}^1$ .

Given a sequence of continuous maps  $\xi_n : \mathbb{RP}^1 \rightarrow \ell^2(\Gamma_0)$  satisfying Definition IV.5.1, define the maps  $\xi'_n : \mathbb{Z}^2 \rightarrow \ell^2(\Gamma_0)$  by

$$\xi'_n(z) = \sigma(z)\xi_n(p(z)),$$

for  $z = (z_1, z_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ , and  $\xi'_n(z) = 0$ , otherwise. Here  $\sigma(z) = 1$ , if  $-\pi/2 < \arg(z) \leq \pi/2$ , and  $\sigma(z) = -1$ , otherwise. Note that for any  $a \in \mathbb{Z}^2$  we have

$$\limsup_{z \rightarrow \infty} \|\xi'_n(z) - \xi'_n(z + a)\|_2 = 0, \tag{IV.6.1}$$

for all  $n \in \mathbb{N}$ .

Now, consider finite, symmetric generating subsets  $S' \subset \Gamma_0$  and  $S'' \subset \mathbb{Z}^2$ . Define  $S_1 = S' \cup S''$  and  $S_{k+1} = S_k \cup (S_1)^{k+1}$  for all  $k \in \mathbb{N}$ . By equations IV.5.1 and IV.6.1, there exists an increasing sequence of finite, symmetric subsets  $F_1 \subset F_2 \subset \dots \subset F_k \subset \dots \subset \mathbb{Z}^2$  such that  $\bigcup_{k=1}^{\infty} F_k = \mathbb{Z}^2$  and a subsequence  $(n_k)$  such that

$$\sup_{s \in S_k} \sup_{g \in \mathbb{Z}^2 \setminus F_k} \|\pi_s(\xi'_{n_k}(g)) - \xi'_{n_k}(s \cdot g)\|_2 \leq \frac{1}{2^k}, \tag{IV.6.2}$$

where  $s \cdot g$  is the natural  $\Gamma$ -action on  $\mathbb{Z}^2$ . Define a map  $\partial : \mathbb{Z}^2 \rightarrow \ell^2(\mathbb{N}; \ell^2(\Gamma_0)) = \mathcal{H}$  by  $\partial(z)(k) = \xi'_{n_k}(z)$ , if  $z \notin F_k$ , and 0, otherwise. It is then straightforward to check that  $\partial$  is proper, anti-



symmetric, and boundedly  $\Gamma$ -equivariant. For  $(z, \gamma) \in \mathbb{Z}^2 \times \mathrm{SL}(2, \mathbb{Z})$  we define the map  $q((z, \gamma)) = b(\gamma) \oplus \partial(z) \in \ell^2(\Gamma_0) \oplus \mathcal{H}$ . It is easy to see that  $q$  is an array into the weakly  $\ell^2$  representation  $\pi \oplus \pi^{\oplus \infty}$ . Thus,  $\mathbb{Z}^2 \times \mathrm{SL}(2, \mathbb{Z}) \in \mathcal{QH}_{\mathrm{reg}}$  and we are done.

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