Finite Time and Density Effects on Interacting Quantum Fields in Cosmological Spacetimes

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DEDICATION

To my beloved wife, Michele, your support is infinite in extent

and

To my cherished sons, Parviz and Oskar, the source of my motivation

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Chapter 1

Introduction

"... it is the effects due to the interaction of quantum free matter fields with a classical gravitational field."

A.A. Starobinsky

A covariant description of quantum fields in the dynamical spacetime of the early universe is essential to models for the origins of the observed matter content at late times, e.g. baryogenesis and dark matter production, and to models of the field(s) posited to drive inflation. However, in a non-stationary Friedmann–Robertson–Walker (FRW) spacetime the lack of Poincaré invariance, among other concerns, makes the applicability of the Minkowski space formalism of quantum fields suspect as there is no notion of a global vacuum state serving as the basis of a Fock space. Additional issues arise when considering the nature of the interactions believed to be responsible for the observed inhomogeneity and matter content of the universe as the current paradigm supposes that during some earlier period all quantum fields participated in both near-to and far-from-equilibrium interactions with respect to a primordial plasma. Here, the Minkowski space formulation of non-equilibrium quantum field theory can gives rise to appreciable corrections to the aforementioned interaction rates [1–5], and its extension to a non-stationary spacetime is nontrivial [6]. The calculation of a cosmological observable involving these early universe interactions is therefore usually carried out via a semiclassical approximation, i.e. classical Boltzmann equations augmented with thermally averaged interaction rates derived from the S-matrix associated with the irreducible representations of the standard model of particle physics in an effort to quantify particle production in a covariant generalization of Minkowski spacetime to an FRW spacetime background.

An alternate treatment may be carried out within the algebraic formulation of locally covariant quantum field theory as presented, for example, in Ref. [7]. This mathematically rigorous formalism is in general useful for clarifying conceptual issues related to and/or providing a foundation for the calculation of observables with traditionally heuristic justifications. In this work, we propose a non-traditional application of the formalism inspired by numerical calculations such as those found in Refs. [8, 9] where algebraic quantum field theory is employed in order to characterize the energy density of a free scalar field propagating in a non-stationary FRW spacetime. In other words, we seek to employ the established algebraic formalism in a concrete numerical calculation of a cosmological observable and not in the traditional pursuit of a rigorous proof of theorem. Though this numerical calculation may be computationally expensive, as compared to the standard formalism, meeting the requirement that cosmological observables be compatible with the semiclassical Einstein equation; i.e. the stress-energy tensor is the expectation value of a quantum state back-reacting on the metric of general relativity, would seem to justify the cost [10–13].

In the algebraic framework the following considerations make finite time intervals essential to formulating the physical states of interest in an FRW spacetime.

1. The lack of time-translation invariance does not allow for a unitary, one parameter group of time shift automorphisms on the algebra of observables, hence a two parameter family of automorphisms is required [14, 15]

2. Gravitationally induced excitation of the quantum matter field is a general feature of a non-stationary spacetime background [16, 17] where the resulting quantum energy density is only bound from below when smeared along a timelike curve [18, 19] such that the ground state is defined as a state of minimal smeared energy along a finite worldline of an isotropic observer [20, 21]

3. Interacting quantum fields are generally defined by an algebra generated by a time averaged perturbation in an arbitrarily small, yet finite, time slice [22]

Furthermore, the dense environment of quantum fields comprising the primordial plasma require careful consideration as the usual notions of thermal equilibrium and non-equilibrium dynamics become somewhat ambiguous in FRW spacetimes. For example, the work in Refs. [23–25] suggests observables computed in a manner consistent with the standard formulation of thermal quantum field theory in Minkowski spacetime may serve only as a reference for the properties of the observed state in FRW spacetimes.

Hence, we take the first step towards probing for a correction to an observable computed in the semiclassical approximation using the standard approach to particle physics. We derive, via algebraic quantum field theory in curved spacetime, an expression for the renormalized energy density of a free scalar field subjected during a finite time interval to the influence of a perturbative interaction while propagating in a classical, yet non-stationary, FRW spacetime. We claim that the resulting expression is, at least in principle, amenable to numerical calculation. In order to derive this expression we must begin with the general time evolution of the algebraic state. As there is no time-translation invariance in our cosmological model, we make use of a two-parameter family of propagators, including a time averaged perturbative interaction, resulting in a method analogous to the

Schwinger–Keldysh closed-time-path [26, 27], however extended to non-stationary spacetimes. The evolved state will then encode both the effects of the finite time intervals via the explicit construction of ground states and the influence of a dense environment via perturbative interactions; as well as renormalization constraints and ambiguities associated with curved spacetimes via techniques developed in the literature cited above.

To this end we allocate Ch. 2. to a review of the origins of Boltzmann kinetic theory in both the classical and quantum regime, including the non-equilibrium dynamics of the Schwinger–Keldysh formalism. In Ch. 3 we introduce kinetic theory in the standard cosmological context of cold dark matter and dark energy in the form of a cosmological constant. Here, we develop a very general model of neutral scalars propagating in a spatially flat FRW spacetime whose energy density is dominated by radiation, i.e. nearly massless relativistic degrees of freedom. Though simplistic, these scalars serve as proxy for models in which a hidden quantum sector containing the field associated with dark matter and an observed quantum sector containing fields comprising the primordial plasma, which simultaneously contribute to the classical gravitational curvature of the FRW spacetime, are connected during early times via a feebly coupled unstable quantum field known as a portal. We then review the semiclassical approximation of quantum Boltzmann kinetics in an expanding universe in the context of this toy model.

As an intermediate step to our main result we present in Sec. 3.2.2. the numerical results of Eq. (3.49) as corrections to the semiclassical approximation of the energy density of a cosmic relic found by employing the full quantum treatment in the standard formalism of non-equilibrium quantum field theory. In Sec. 3.4 we derive Eq. (3.132) as the main result of this work; i.e. the general form of the expectation value of the renormalized quantum energy density given the influence, during a finite interval of cosmological time, of a dense environment of perturbative quantum interactions and a non-stationary spacetime background; as derived for the first time by this author in Ref. [28]. We conclude with a discussion of this result and future works in Sec. 4.

Chapter 2

Kinetic Theory

"There is nothing more practical than a good theory."

K. Lewin

2.1 Classical Particles

In this section we review the classical theory of Lagrangian and Hamiltonian mechanics along with the extension to Boltzmann kinetics (see, e.g. Ref. [29–31] for a pedagogical introduction to the classical mechanics of a particle and its generalization to statistical ensembles of N particles).

2.1.1 Single Particle Dynamics

In the Lagrangian formulation of classical mechanics, the dynamics of a single particle with mass m are described by its Lagrangian function $L(\mathbf{q}, \dot{\mathbf{q}}; t)$ defined as

$$L(\mathbf{q}, \dot{\mathbf{q}}; t) := T(\dot{\mathbf{q}}) - V(\mathbf{q}) \tag{2.1}$$

given the kinetic energy $T(\dot{\mathbf{q}}) = m\dot{\mathbf{q}}^2/2$ and a potential $V(\mathbf{q})$ at some time t. Here, the 3-vectors $\mathbf{q} := \langle q_1, q_2, q_3 \rangle$ with a general coordinate q_i, where $\dot{q}_i := dq_i/dt$, are those of the standard Euclidean space $(\mathcal{M}_3, d)$, i.e. a 3-manifold $\mathcal{M}_3 := \mathbb{R}^3$ with metric d such that

$$ds^2 = \sum_{i=1}^{3} dx_i^2. \tag{2.2}$$

The action functional $S[L(\mathbf{q}, \dot{\mathbf{q}})]$ defined as

$$S[L(\mathbf{q}, \dot{\mathbf{q}})] := \int_{t_i}^{t_f} dt\ L(\mathbf{q}, \dot{\mathbf{q}}; t), \tag{2.3}$$

interpreted here as a path integral, is useful in expressing Hamilton's principal of stationary action

$$\delta S[L(\mathbf{q}, \dot{\mathbf{q}})] = \int_{t}^{t+dt} dt' \sum_{i}^{3} \left(\frac{\partial L(q_i, \dot{q}_i; t')}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial L(q_i, \dot{q}_i; t')}{\partial q_i} \delta q_i \right) = 0, \tag{2.4}$$

to first order in the infinitesimal variation δ where $\delta(t) = \delta(t + dt) = 0$, which gives rise to the Euler–Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L(q_i, \dot{q}_i; t)}{\partial \dot{q}_i}\right) = \frac{\partial L(q_i, \dot{q}_i; t)}{\partial q_i}. \tag{2.5}$$

The Hamiltonian function $H(\mathbf{q}, \mathbf{p}; t)$ is defined via the inverse Legendre transform

$$H(\mathbf{q}, \mathbf{p}; t) := \sum_{i=1}^{3} \dot{q}_i \left(\frac{\partial L(q_i, \dot{q}_i; t)}{\partial \dot{q}_i}\right) - L(q_i, \dot{q}_i; t), \tag{2.6}$$

given the 3-momentum $\mathbf{p} := \langle p_1, p_2, p_3 \rangle$ with $p_i = m\dot{q}_i$ as the canonical conjugate, such that

$$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{p}) + V(\mathbf{q}) \tag{2.7}$$

represents the total energy $E = T + V$ of the particle where, for example, a classical particle in a central gravitational field has an ascribed energy

$$H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{3} \frac{p_i^2}{2m} - \kappa G \left(\sum_{i=1}^{3} q_i^2\right)^{-1/2} \tag{2.8}$$

with κG as the gravitational coupling given G as Newton's constant and $[\kappa] = [m^2]$.

Hamilton's equations of motion are then

$$\dot{q}_i = \frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial p_i} \tag{2.9}$$

and

$$\dot{p}_i = -\frac{\partial H(\mathbf{q}, \mathbf{p})}{\partial q_i} \tag{2.10}$$

where the Hamiltonian uniquely defines the time evolution of states in the canonical phase space $\mathcal{M}_\Omega$, i.e. the cotangent bundle $T^*\mathcal{M}_3$ defined here as the set

$$\mathcal{M}_\Omega := \{\, \mathbf{q}(t), \mathbf{q}'(t + dt), \ldots, \mathbf{p}(t), \mathbf{p}'(t + dt), \ldots \,\}. \tag{2.11}$$

The Poisson bracket $[\cdot, \cdot]_P$, defined as

$$[v, w]_P := \sum_{i=1}^{3} \left(\frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}\right), \tag{2.12}$$

for any variables $v(\mathbf{q}, \mathbf{p}, ; t)$ and $w(\mathbf{q}, \mathbf{p}, ; t)$ leaves Hamilton's equations of motion invariant under the infinitesimal canonical transformations

$$q_i \rightarrow q_i + dq_i \tag{2.13}$$

$$p_i \rightarrow p_i + dp_i. \tag{2.14}$$

The canonical condition on the phase space $\mathcal{M}_\Omega$ requires

$$[q_i, q_j]_P = [p_i, p_j]_P = 0 \text{ and } [q_i, p_j]_P = -[p_i, q_j]_P = \delta_{ij} \tag{2.15}$$

as the fundamental Poisson brackets where δ_{ij} is the Kronecker delta. The canonical invariance of Eqs. (2.9) and (2.10) in our Hamiltonian system now allows us to consider the time evolution of some arbitrary constant of motion u where decomposition of the total differential is

$$\frac{du}{dt} = \sum_{i=1}^{3} \left(\frac{\partial u}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial u}{\partial p_i} \frac{dp_i}{dt} \right) + \frac{\partial u}{\partial t} \frac{dt}{dt} = 0 \tag{2.16}$$

such that

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{3} \left(\frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial p_i} \right) = [u, H]_P. \tag{2.17}$$

The Taylor expansion for $u(t)$ is then

$$\begin{aligned} u(t) &= u(0) + t \frac{du}{dt}\bigg|_{t=0} + \frac{t^2}{2!} \frac{d^2 u}{dt^2}\bigg|_{t=0} + \frac{t^3}{3!} \frac{d^3 u}{dt^3}\bigg|_{t=0} + \ \dots \\ &= u_0 + t[u, H]_P + \frac{t^2}{2!}[[u, H], H]_P + \frac{t^3}{3!}[[[u, H], H], H]_P + \ \dots \end{aligned} \tag{2.18}$$

where the Hamiltonian H is identified as the time translation generator for continuous infinitesimal canonical transformations and

$$u(t) = u_0 \exp(t[\cdot, H]_P). \tag{2.19}$$

2.1.2 Dynamics in a Statistical Ensemble

We now extend the dynamical formalism to a system of particles. Given an ensemble of N identical, yet distinguishable, particles the phase space $\mathcal{M}_\Omega$ is expanded to

$$\begin{aligned}\mathcal{M}_\Omega^N \quad &= \quad \{\mathbf{q}_1(t), \ldots, \mathbf{q}_N(t),\ \mathbf{p}_1(t), \ldots, \mathbf{p}_N(t); \\ &\qquad \mathbf{q}_1'(t+dt), \ldots, \mathbf{q}_N'(t+dt),\ \mathbf{p}_1'(t+dt), \ldots, \mathbf{p}_N'(t+dt)\ ;\ \ldots\}. \end{aligned} \tag{2.20}$$

where we define a phase space density ρ_N as

$$\rho_N := \frac{dN}{dV_\Omega} \tag{2.21}$$

such that at time t the number of Hamiltonian systems dN_S in a state S, corresponding to the infinitesimal phase space volume element

$$dV_S = d\mathbf{q}_1(t), \ldots, d\mathbf{q}_N(t) d\mathbf{p}_1(t), \ldots, d\mathbf{p}_N(t) \tag{2.22}$$

is

$$dN_S = \rho_{N_S}\, dV_S. \tag{2.23}$$

In accordance with the statistical, i.e. probabilistic, theory the expectation value of a random variable $\langle X \rangle$ on the probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ is the Lebesgue integral

$$\langle X \rangle = \int_\Omega d\mathcal{P}\ X \tag{2.24}$$

where the sample space Ω is the set of all possible outcomes, the σ-algebra $\mathcal{F} = \{\emptyset, \ldots, \Omega\}$ is the set of events containing no to all outcomes, and $\mathcal{P} : \mathcal{F} \to [0, 1]$ the map of events to probabilities with the normalization $\mathcal{P}(\Omega) = 1$; given $X : \Omega \to \mathcal{S}$ as a measurable function from the sample space to the outcome space $\mathcal{S} \subset \mathcal{F}$. In the context of our ensemble of systems, the probability element $dP_S := \mathcal{P}(S)dV_S$ corresponding to finding a state $S \in \mathcal{S}$, now thought of as the state space, is simply

$$dP_S = \frac{dN_S}{N} \tag{2.25}$$

such that the probability of finding a system in our ensemble with a state S between $\mathbf{q}_1, \ldots, \mathbf{q}_N$ and $\mathbf{q}_1', \ldots, \mathbf{q}_N'$ where $\mathbf{q}' = \mathbf{q} + d\mathbf{q}$ with momentum between $\mathbf{p}_1, \ldots, \mathbf{p}_N$ and

$\mathbf{p}'_1, \ldots, \mathbf{p}'_N$ for $\mathbf{p}' = \mathbf{p} + d\mathbf{p}$ at time t is then

$$dP_S = \frac{1}{N}\,\rho(\mathbf{q}_1, \ldots, \mathbf{q}_N;\ \mathbf{p}_1, \ldots, \mathbf{p}_N;\ t)d\mathbf{q}_1, \ldots, d\mathbf{q}_N d\mathbf{p}_1, \ldots, d\mathbf{p}_N \tag{2.26}$$

with the obligatory normalization

$$\int_\Omega dP = \int d\mathbf{q}_1, \ldots, d\mathbf{q}_N\ d\mathbf{p}_1, \ldots, d\mathbf{p}_N\ \mathcal{P}(\mathbf{q}_1, \ldots, \mathbf{q}_N;\ \mathbf{p}_1, \ldots, \mathbf{p}_N;\ t) = 1. \tag{2.27}$$

Hence, the expected value of the energy $\langle E_S \rangle$ corresponding to the system in state S is specified as

$$\langle\ E_S\ \rangle = \frac{1}{N}\int d\mathbf{q}_1, \ldots, d\mathbf{q}_N d\mathbf{p}_1, \ldots, d\mathbf{p}_N \Big\{ \rho(\mathbf{q}_1, \ldots, \mathbf{q}_N;\ \mathbf{p}_1, \ldots, \mathbf{p}_N;\ t)\ H_N(\mathbf{q}_1, ..., \mathbf{q}_N, \mathbf{p}_1, ..., \mathbf{p}_N) \Big\} \tag{2.28}$$

given $H_N(\mathbf{q}_1, ..., \mathbf{q}_N, \mathbf{p}_1, ..., \mathbf{p}_N) := \sum_{n=1}^{N} H(\mathbf{q}_n, \mathbf{p}_n)$ as the N-particle Hamiltonian of the system.

In what follows our ensemble is taken to be a dilute gas of N particles with mass m such that the Hamiltonian for the n^{th} particle is

$$H(\mathbf{q}_n, \mathbf{p}_n) = \sum_i^3 \frac{p_{n,i}^2}{2m}. \tag{2.29}$$

Liouville's theorem of a conserved phase space volume under infinitesimal canonical transformations, or

$$d\mathbf{q}_1, \ldots, d\mathbf{q}_N d\mathbf{p}_1, \ldots, d\mathbf{p}_N = d\mathbf{q}'_1, \ldots, d\mathbf{q}'_N d\mathbf{p}'_1, \ldots, d\mathbf{p}'_N, \tag{2.30}$$

in a Hamiltonian system with a velocity independent potential, implies

$$\rho(\mathbf{q}_1, \ldots, \mathbf{q}_N;\ \mathbf{p}_1, \ldots, d\mathbf{p}_N;\ t) = \rho(\mathbf{q}'_1, \ldots, \mathbf{q}'_N;\ \mathbf{p}'_1, \ldots, \mathbf{p}'_N;\ t') \tag{2.31}$$

such that $\rho(\mathbf{q}_1, \ldots, \mathbf{q}_N;\ \mathbf{p}_1, \ldots, d\mathbf{p}_N;\ t)$ is constant in the absence of interactions between systems and the continuity equation may be re-expressed as the Liouville equation

$$\frac{\partial \rho_N}{\partial t} = [\rho_N, H_N], \tag{2.32}$$

i.e. the equation of motion for the phase space density with time translations generated by the system's Hamiltonian where

$$\rho_N(t) \quad = \quad \rho_N(0)\exp(t[\cdot, H_N]) \tag{2.33}$$

via Eq. (2.19).

2.1.3 Boltzmann Kinetic Theory

Boltzmann's theory of a dilute gas then breaks Liouville's symmetry with the introduction of a kinetic term that accounts for interactions between the systems. Adding this term to Liouville's equation, the so called Boltzmann collision kernel $\mathcal{C}[\rho(t)]$, represents the classical elastic scattering of particles in our gas such that

$$\frac{d\rho}{dt} = \mathcal{C}[\rho(t)]. \tag{2.34}$$

Modeling the scattering processes via complicated, correlated multi-particle states is cumbersome at best and intractable as $N \to \infty$. Hence, Boltzmann simplifies the formulation by invoking the *Stosszahlansatz* of uncorrelated initial momenta with a factorizable multi-particle phase space density where

$$\rho(\mathbf{q};\ \mathbf{p}_1, \ldots, \mathbf{p}_N;\ t) = f_1(\mathbf{q}, \mathbf{p}_1;\ t) f_2(\mathbf{q}, \mathbf{p}_2;\ t) \ldots f_N(\mathbf{q}, \mathbf{p}_N;\ t) \tag{2.35}$$

for $f_n(\mathbf{q}, \mathbf{p}_n;\ t)$ as the distribution function for particles with momentum between $\mathbf{p}_n$ and $\mathbf{p}_n + d\mathbf{p}_n$.

Time evolution is now tractable and observables may be related to $f_1(\mathbf{q}, \mathbf{p}_1;\ t)d\mathbf{q}d\mathbf{p}_1$ as the mean number of particles dN_1 with momentum between $\mathbf{p}_1$ and $\mathbf{p}_1 + d\mathbf{p}_1$ where, for example, the number density n_1 is defined as

$$n_1(\mathbf{q}, t) := \int d\mathbf{p}_1\ f_1(\mathbf{q}, \mathbf{p}_1;\ t) \tag{2.36}$$

or the mean number of particles per unit volume.

For simplicity, we consider the nonzero probability for elastic scattering given the multi-particle distribution

$$\rho(\mathbf{q};\ \mathbf{p}_\Phi, \mathbf{p}_\phi;\ t) = f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t) f_\phi(\mathbf{q}, \mathbf{p}_\phi;\ t) \tag{2.37}$$

of a dilute gas of point particles where Φ corresponds to the incident particle with momentum $\mathbf{p}_\Phi$ and ϕ to the target particle with momentum $\mathbf{p}_\phi$. In addition to the *Stosszahlansatz*, a dilute gas refers to the following simplifying assumptions.

1. Multiple particle collisions are extremely rare as compared to two-body collisions, hence two-body scattering is the dominant mechanism for evolving $f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)$ in time

2. The effect of any external force $\mathbf{F} := \dot{\mathbf{p}}$ on the two-body scattering cross section is negligible

3. The time between collisions is much longer than the interaction time of a collision

4. $f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)$ doesn't vary over time or distance intervals relevant to particle interactions

We may now write the Boltzmann equation as

$$\left(\frac{\partial f_\Phi}{\partial t} + \frac{\mathbf{p}_\Phi}{m} \cdot \frac{\partial f_\Phi}{\partial \mathbf{q}} + \mathbf{F} \cdot \frac{\partial f_\Phi}{\partial \mathbf{p}_\Phi}\right) = \mathcal{C}[f_\Phi(t)]. \tag{2.38}$$

In general, we may express the Boltzmann equation for the distribution function f_Φ as

$$\widehat{\mathbf{L}}\ f_\Phi = \widehat{\mathbf{C}}\ f_\Phi \tag{2.39}$$

where the $\widehat{\mathbf{L}}$ is the Liouville operator as defined by the left-hand-side (LHS) of Eq. (2.38) and $\widehat{\mathbf{C}}$ the collision operator corresponding to a specified interaction.

In order to determine the kernel $\mathcal{C}[f_\Phi(t)]$ we focus on a fixed unit of volume between $\mathbf{q}$ and $\mathbf{q} + d\mathbf{q}$. Changes to the momentum distribution $f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)$ in our fixed volume arise from both scattering into and out of the momentum range $\mathbf{p}_\Phi$ and $\mathbf{p}_\Phi + d\mathbf{p}_\Phi$ in a time $t' - t$. We begin with the probability of loss $P^>$, or scattering from $f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi$ to $f_\Phi(\mathbf{q}, \mathbf{p}'_\Phi;\ t')d\mathbf{q}d\mathbf{p}'_\Phi$ where $\mathbf{p}' \neq \mathbf{p} + d\mathbf{p}$. Defined as

$$dP^> := \frac{dN'}{I_\Phi}, \tag{2.40}$$

the differential probability is simply the mean number of scattered Φ' and ϕ' particles per the flux I_Φ of incident Φ particles. For our dilute gas of point particles, the incident flux per unit volume is

$$\begin{aligned} I_\Phi &= |\mathbf{p}_\Phi - \mathbf{p}_\phi| dN_\Phi \\ &= |\mathbf{p}_\Phi - \mathbf{p}_\phi|\ f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi \end{aligned} \tag{2.41}$$

given $|\mathbf{p}_\Phi - \mathbf{p}_\phi|$ as the relative momentum. The differential probability for scattering per unit time, or the differential scattering cross section

$$d\sigma := \frac{dP}{(t'-t)}, \tag{2.42}$$

is then

$$\begin{aligned} d\sigma(\mathbf{p}_\Phi \mathbf{p}_\phi \to \mathbf{p}'_\Phi \mathbf{p}'_\phi) &= \frac{dN'}{I_\Phi(t'-t)} \\ &= \frac{f_\Phi(\mathbf{q}, \mathbf{p}'_\Phi;\ t')d\mathbf{q}d\mathbf{p}'_\Phi\ f_\phi(\mathbf{q}, \mathbf{p}'_\phi;\ t')d\mathbf{q}d\mathbf{p}'_\phi}{|\mathbf{p}_\Phi - \mathbf{p}_\phi|\ f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi\ (t'-t)} \end{aligned} \tag{2.43}$$

such that the total cross section is now

$$\sigma(\mathbf{p}_\Phi \mathbf{p}_\phi \to \mathbf{p}'_\Phi \mathbf{p}'_\phi) = \int d\mathbf{p}'_\Phi \int d\mathbf{p}'_\phi \frac{f_\Phi(\mathbf{q}, \mathbf{p}'_\Phi;\ t')d\mathbf{q}\ f_\phi(\mathbf{q}, \mathbf{p}'_\phi;\ t')d\mathbf{q}}{|\mathbf{p}_\Phi - \mathbf{p}_\phi|\ f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi\ (t'-t)}. \tag{2.44}$$

The interaction rate $\Gamma^>$ is then expressed as the product of the scattering cross section and the relative incident flux of Φ particles in a volume between $\mathbf{q}$ and $\mathbf{q} + d\mathbf{q}$ with momentum between $\mathbf{p}_\Phi$ and $\mathbf{p}_\Phi + d\mathbf{p}_\Phi$ such that

$$\Gamma^> = I_\Phi\ \sigma(\mathbf{p}_\Phi \mathbf{p}_\phi \to \mathbf{p}'_\Phi \mathbf{p}'_\phi) \tag{2.45}$$

and the time rate of change in $f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi$ due to scatterings out of the momentum range between $\mathbf{p}_\Phi$ and $\mathbf{p}_\Phi + d\mathbf{p}_\Phi$ is thus the product of the interaction rate and the number of target ϕ particles available per unit volume integrated over all possible momenta of the target and scattered particles, or

$$\frac{\partial}{\partial t}\left[f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)d\mathbf{q}_\Phi d\mathbf{p}_\Phi\right]^> := \int d\mathbf{p}_\phi \int d\mathbf{p}'_\Phi \int d\mathbf{p}'_\phi\ \Gamma^>\ f_\phi(\mathbf{q}_\phi, \mathbf{p}_\phi;\ t)d\mathbf{q}d\mathbf{p}_\phi. \tag{2.46}$$

Similarly, for $dP^< := dN/I'_\Phi$ we find

$$\frac{\partial}{\partial t}\left[f_\Phi(\mathbf{q}, \mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi\right]^< := \int d\mathbf{p}'_\phi \int d\mathbf{p}_\Phi \int d\mathbf{p}_\phi\ \Gamma^<\ f_\phi(\mathbf{q}, \mathbf{p}'_\phi;\ t)d\mathbf{q}d\mathbf{p}'_\phi. \tag{2.47}$$

Hence, the total time evolution associated with the kernel $\mathcal{C}[f_\Phi(t)]$ is the difference of time dependent gains $(<)$ and losses $(>)$ that represent Φ particles scattering into and out of the distribution such that

$$\mathcal{C}\left[f_\Phi(\mathbf{q},\mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi\right] = \frac{\partial}{\partial t}\left[f_\Phi(\mathbf{q},\mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi\right]^{<} - \frac{\partial}{\partial t}\left[f_\Phi(\mathbf{q},\mathbf{p}_\Phi;\ t)d\mathbf{q}d\mathbf{p}_\Phi\right]^{>}. \tag{2.48}$$

By simple substitution we find

$$\left.\frac{\partial f_\Phi}{\partial t}\right|^{<} = \int d\mathbf{p}'_\phi \int d\mathbf{p}_\Phi \int d\mathbf{p}_\phi \,|\mathbf{p}'_\Phi - \mathbf{p}'_\phi|\, \sigma(\mathbf{p}_\Phi\mathbf{p}_\phi \leftarrow \mathbf{p}'_\Phi\mathbf{p}'_\phi)\ f'_\Phi f'_\phi \tag{2.49}$$

$$\left.\frac{\partial f_\Phi}{\partial t}\right|^{>} = \int d\mathbf{p}_\phi \int d\mathbf{p}'_\Phi \int d\mathbf{p}'_\phi \,|\mathbf{p}_\Phi - \mathbf{p}_\phi|\, \sigma(\mathbf{p}_\Phi\mathbf{p}_\phi \rightarrow \mathbf{p}'_\Phi\mathbf{p}'_\phi)\ f_\Phi f_\phi \tag{2.50}$$

where $f := f(\mathbf{q},\mathbf{p};\ t)$ and $f' := f(\mathbf{q}',\mathbf{p}';\ t')$ as shorthand notation.

The classical time reversal symmetry $(t'-t) \rightarrow (t-t')$ of the equations of motion gives the same probability to scatter from an initial momenta $\mathbf{p}'_\Phi$ into the momenta range between $\mathbf{p}_\Phi$ and $\mathbf{p}_\Phi + d\mathbf{p}_\Phi$, i.e.

$$\sigma(\mathbf{p}_\Phi\mathbf{p}_\phi \leftarrow \mathbf{p}'_\Phi\mathbf{p}'_\phi)d\mathbf{p}_\Phi d\mathbf{p}_\phi = \sigma(\mathbf{p}_\Phi\mathbf{p}_\phi \rightarrow \mathbf{p}'_\Phi\mathbf{p}'_\phi)d\mathbf{p}'_\Phi d\mathbf{p}'_\phi, \tag{2.51}$$

and

$$|\mathbf{p}_\Phi - \mathbf{p}_\phi| = |\mathbf{p}'_\Phi - \mathbf{p}'_\phi|. \tag{2.52}$$

given the conserved momentum of elastic collisions. The simplified collision term is thus

$$\mathcal{C}[f_\Phi(t)] = \int d\mathbf{p}_\phi \int d\mathbf{p}'_\Phi \int d\mathbf{p}'_\phi \,(f'_\phi f'_\Phi - f_\phi f_\Phi)\, |\mathbf{p}_\Phi - \mathbf{p}_\phi|\, \sigma(\mathbf{p}_\Phi\mathbf{p}_\phi \rightarrow \mathbf{p}'_\Phi\mathbf{p}'_\phi) \tag{2.53}$$

such that the complete Boltzmann equation is written as the integrodifferential equation

$$\begin{aligned}&\left(\frac{\partial f_\Phi}{\partial t} + \frac{\mathbf{p}_\Phi}{m_\Phi}\cdot\frac{\partial f_\Phi}{\partial \mathbf{q}} + \mathbf{F}_\Phi \cdot \frac{\partial f_\Phi}{\partial \mathbf{p}_\Phi}\right) = \\ &\int d\mathbf{p}_\phi \int d\mathbf{p}'_\Phi \int d\mathbf{p}'_\phi \,(f'_\phi f'_\Phi - f_\phi f_\Phi)\, |\mathbf{p}_\Phi - \mathbf{p}_\phi|\, \sigma(\mathbf{p}_\Phi\mathbf{p}_\phi \rightarrow \mathbf{p}'_\Phi\mathbf{p}'_\phi).\end{aligned} \tag{2.54}$$

Observables of interest are related to f_Φ, hence we require solutions to Eq. (2.54). We first consider the case of kinetic equilibrium in our simplified model, i.e. $\mathcal{C}[f_\Phi(t)] = 0$.

This condition is evaluated via the Boltzmann $\mathcal{H}$-theorem

$$\frac{d\mathcal{H}_B}{dt} \leq 0 \tag{2.55}$$

given

$$\mathcal{H}_B := \int d\mathbf{p}_\Phi \; f(\mathbf{q}, \mathbf{p}_\Phi;\; t) \log\Big(f(\mathbf{q}, \mathbf{p}_\Phi;\; t) \Big). \tag{2.56}$$

Equality in Eq. (2.55) describes kinetic equilibrium as a constant number of particles $f_\Phi^{EQ} d\mathbf{p}_\Phi$ in the available state f_Φ^{EQ} for time scales longer than $t' - t$, i.e. $\mathcal{H}_B$ is a form of entropy. Here,

$$\begin{aligned} \left.\frac{d\mathcal{H}_B}{dt}\right|_{EQ} &= -\frac{1}{2} \int d\mathbf{p}_\Phi \; d\mathbf{p}_\phi \Big\{ |\mathbf{p}_\Phi - \mathbf{p}_\phi| \; \sigma(\mathbf{p}_\Phi \mathbf{p}_\phi \to \mathbf{p}'_\Phi \mathbf{p}'_\phi) \\ &\quad \times \Big[f_\Phi^{EQ} f_\phi^{EQ} - f_\Phi'^{EQ} f_\phi'^{EQ} \Big] \Big[\log\Big(f_\Phi^{EQ} f_\phi^{EQ} \Big) - \log\Big(f_\Phi'^{EQ} f_\phi'^{EQ} \Big) \Big] \Big\} \\ &= 0 \iff f_\Phi^{EQ} f_\phi^{EQ} = f_\Phi'^{EQ} f_\phi'^{EQ} \end{aligned} \tag{2.57}$$

or equivalently $\log f_\phi^{EQ} + \log f_\Phi^{EQ} = \log f_\phi'^{EQ} + \log f_\Phi'^{EQ}$. As an additional constraint, we include the conserved kinetic energy of elastic collisions where $E_\Phi + E_\phi = E'_\Phi + E'_\phi$ such that

$$f_\Phi^{EQ} = A \exp(-\beta E_\Phi) \tag{2.58}$$

as the Maxwell–Boltzmann distribution for the dimensionless normalization constant A and parameter $\beta \geq 0$, given $[\beta] = [E]^{-1}$. We now encounter the *Umkehreinwand*, or the Loschmidt, paradox of deducing the approach to equilibrium as an irreversible process from an initial assumption of time reversal symmetry. This is a strong indication that the initial *Stosszahlansatz* of uncorrelated momenta does not persist once collisions begin, otherwise it would be equally probable by time reversal symmetry to begin in equilibrium and evolve to a state that violates the $\mathcal{H}$-theorem. Notice that no reference to temperature and/or thermodynamics is required to arrive at the preceding kinetic results; however, for the purpose of analogy with the quantum theory in the next section we introduce the partition function $Z(\beta)$ defined as

$$Z(\beta) := \frac{1}{N! h^{3N}} \int d\mathbf{q}_1 d\mathbf{q}_2 ... d\mathbf{q}_N \; d\mathbf{p}_1 d\mathbf{p}_2 ... d\mathbf{p}_N \; \exp(-\beta H_N) \tag{2.59}$$

given Planck's parameter $h > 0$ with $[h] = [Et]$.

For departures from kinetic equilibrium we may invoke the relaxation time assumption. Here, we may simply assume that the effect of the kernel $\mathcal{C}[f_\Phi(t)]$ is to restore $f_\Phi(t)$ to local equilibrium at a rate proportional to the collision frequency $\nu := (t' - t)^{-1}$, i.e.

$$\mathcal{C}[f_\Phi(t)] = \nu\,[f_\Phi(t) - f_\Phi^{EQ}] \tag{2.60}$$

such that

$$f_\Phi(t) = f_\Phi^{EQ} + [f_\Phi(0) - f_\Phi^{EQ}]\exp(-\Gamma t) \tag{2.61}$$

given $\Gamma := \nu^{-1}$ as the relation rate.

2.2 Quantum Fields

In this section, the classical kinetic theory is extended to the standard operator formalism of relativistic quantum theory associated with the field theoretic interactions of particle physics; to include open quantum systems in the context of non-equilibrium quantum field theory (see, e.g. Refs. [32–34] for a pedagogical introduction to quantum field theory and the standard model of particle physics, Refs. [35, 36] for an introduction to thermal quantum field theory, and Refs. [37, 38] for an overview of open quantum systems both near-to and far-from-equilibrium).

2.2.1 Relativistic Field Theory

We begin with the classical Lagrangian density for a manifestly Lorentz covariant scalar field $\Phi(x_\mu)$ as a relativistic system with a continuous set of degrees of freedom

$$\mathscr{L}[\Phi(x_\mu)] := \frac{1}{2}\partial_\mu\Phi(x_\mu)\partial^\mu\Phi(x^\mu) - \frac{1}{2}M_\Phi^2\Phi^2(x_\mu) - \frac{1}{2}\mathscr{V}[\Phi(x_\mu)]. \tag{2.62}$$

Here, the 4-vector coordinate $x^\mu := \langle x_0, x_1, x_2, x_3\rangle$ with 4-momentum $p^\mu := \langle p_0, p_1, p_2, p_3\rangle$, where $x_0 := t$ and $p_0 := \omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + M_\Phi^2}$, are those of a Minkowski spacetime $(\mathcal{M}_0, \eta)$, i.e. a globally hyperbolic Lorentzian manifold $\mathcal{M}_0 := \mathbb{R}\times\mathbb{R}^3$ with flat metric η given in the standard form as

$$ds^2 = dt^2 - \sum_{i=1}^{3} dx_i^2 \tag{2.63}$$

where the relativistic constant $c = 1$ and $\partial_\mu := \partial/\partial x^\mu$. The term $\mathscr{V}[\Phi(x_\mu)]$ is a potential density corresponding to field theoretic interactions and will be taken to be of the general form

$$\mathscr{V}[\Phi(x_\mu)] = J(x_\mu)\Phi(x_\mu) \tag{2.64}$$

with $J(x_\mu)$ an external current sourcing the field $\Phi(x_\mu)$.

Invoking Hamilton's principle of stationary action

$$\delta S[\mathscr{L}(\Phi)] = \int d^4x \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \delta(\partial_\mu \Phi) + \frac{\partial \mathscr{L}}{\partial \Phi} \delta\Phi \right) = 0, \tag{2.65}$$

where $\delta\Phi$ vanishes at the boundary, again leads to the Euler–Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \right) = \frac{\partial \mathscr{L}}{\partial \Phi}. \tag{2.66}$$

In the case of a free field, where $\mathscr{V}[\Phi(x_\mu)] = 0$ with vanishing source, these equations of motion reduce to the Klein–Gordon equation

$$(\Box_\eta + M_\Phi^2)\Phi = 0 \tag{2.67}$$

with the d'Alembertian operator defined as $\Box_\eta := \partial_\mu^2$. Here, spacetime translations of the field Φ on $(\mathcal{M}_0, \eta)$, i.e.

$$\Phi(x_\mu) \to \Phi(x_\mu + \delta x_\mu) = \Phi(x_\mu) + \delta x_\nu \partial_\nu \Phi(x_\mu) \tag{2.68}$$

to first order in δx_μ, corresponds to a global symmetry of the action

$$\delta S[\mathscr{L}(x_\mu + \delta x_\mu)] = \delta x_\nu \int d^4x \, \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi \right) = 0 \tag{2.69}$$

such that

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \eta_{\mu\nu} \mathscr{L} \right) = 0 \tag{2.70}$$

or $\partial_\mu T_{\mu\nu} = 0$ in analogy with Noether's theorem given

$$T_{\mu\nu} := \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \eta_{\mu\nu} \mathscr{L} \tag{2.71}$$

as the conserved, canonical stress-energy tensor. The functional derivative $\Pi(x_\mu)$, as the canonical conjugate to the free field $\Phi(x_\mu)$, is defined

$$\Pi(x_\mu) := \frac{\partial \mathscr{L}[\Phi(x_\mu)]}{\partial[\partial_t \Phi(x_\mu)]} \tag{2.72}$$

such that the Hamiltonian density may be found via the transform

$$\mathscr{H}[\Phi(x_\mu), \Pi(x_\mu)] = \Pi(x_\mu) \partial_t \Phi(x_\mu) - \mathscr{L}[\Phi(x_\mu), \partial_t \Phi(x_\mu)]. \tag{2.73}$$

Here, the Hamiltonian may be identified with the component T_{00} as the energy density $\mathscr{E}_\Phi$ given

$$\mathscr{E}_\Phi := \Pi \frac{\partial \mathscr{L}}{\partial \Pi} - \mathscr{L}. \tag{2.74}$$

Solutions to the Klein–Gordon equation, Eq. (2.67), are classical plane waves with the general form

$$\Phi(x_\mu) = \int \frac{d^3p}{(2\pi)^3} \Big(a(\vec{p}) \exp(-ip^\mu x_\mu) + a^*(\vec{p}) \exp(ip^\mu x_\mu) \Big) \tag{2.75}$$

with $a \in \mathbb{C}$ and $p^\mu x_\mu = \omega_{\vec{p}}\, x_0 - \vec{p} \cdot \vec{x}$.

2.2.2 Canonical Quantization of a Relativistic Field

Canonical quantization is achieved by promoting the conjugate variables to operators, i.e. $\Phi \to \hat{\Phi}$ and $\Phi \to \hat{\Pi}$, satisfying the commutation relation

$$[\hat{\Phi}(t, \vec{x}), \hat{\Pi}(t, \vec{y})] = i\delta^3(\vec{x} - \vec{y}) \tag{2.76}$$

where $\delta^3(\vec{x})$ is the 3-dimensional Dirac distribution and Planck's constant $\hbar = 1$. The free quantum field $\hat{\Phi}(x_\mu)$ is defined

$$\hat{\Phi}(t, \vec{x}) := \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \Big(\hat{a}_{\vec{p}} \exp(-i\omega_{\vec{p}}\, t) + \hat{a}^\dagger_{\vec{p}} \exp(i\omega_{\vec{p}}\, t) \Big) \exp(i\vec{p} \cdot \vec{x}) \tag{2.77}$$

such that

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{p}}] = [\hat{a}^\dagger_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] = 0 \;\text{ and }\; [\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{p}}] = (2\pi)^3 \delta^3(\vec{p} - \vec{k}). \tag{2.78}$$

Here, $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{p}}$ are the creation and annihilation operators acting on the Fock space $\mathcal{F}(\mathcal{H}_0) := \bigoplus_{n=0}^{\infty} \mathcal{H}_0^{\otimes n}$ given $\mathcal{H}_0$ as the one-particle Hilbert space with unique vector $|\Phi_{\vec{p}}\rangle$ defined via the free field vacuum, i.e. the unique vector $|0\rangle$ defined

$$\hat{a}_{\vec{p}}\,|0\rangle := 0 \tag{2.79}$$

and subject to the normalization condition

$$\langle 0|0\rangle = 1, \tag{2.80}$$

such that

$$|\Phi_{\vec{p}}\rangle := \sqrt{2\omega_{\vec{p}}}\, \hat{a}^\dagger_{\vec{p}}\,|0\rangle \tag{2.81}$$

with

$$\langle \Phi_{\vec{k}} | \Phi_{\vec{p}} \rangle = 2\sqrt{\omega_{\vec{k}}\omega_{\vec{p}}}\ \langle 0|\hat{a}_{\vec{k}}\ \hat{a}^{\dagger}_{\vec{p}}|0\rangle = 2\omega_{\vec{k}}\ (2\pi)^3\delta^3(\vec{k}-\vec{p}). \tag{2.82}$$

Observables of the theory include correlation functions of the field operators, i.e. the expectation value $\langle 0|\,\hat{\Phi}(x_1)\hat{\Phi}(x_2)...\hat{\Phi}(x_n)\,|0\rangle$. Here, we focus on the two-point function

$$\begin{aligned} \langle 0|\,\hat{\Phi}(x_\mu)\hat{\Phi}(y_\mu)\,|0\rangle &= \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3p}{(2\pi)^3}\frac{\langle 0|\,\hat{a}_{\vec{k}}\ \hat{a}^{\dagger}_{\vec{p}}\,|0\rangle}{2\sqrt{\omega_{\vec{k}}\ \omega_{\vec{p}}}}\,\exp(i[k^\mu x_\mu - p^\mu y_\mu]) \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\,\exp(ik^\mu[x_\mu - y_\mu]) \end{aligned} \tag{2.83}$$

via the normalization in Eq. (2.82). In addition, the expectation value of the field's energy in the vacuum state $\langle E_\Phi\rangle_0$ is found via the time-independent free field Hamiltonian operator $\widehat{H}_\Phi$, expressed as

$$\widehat{H}_\Phi = \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\left(\hat{a}^{\dagger}_{\vec{k}}\,\hat{a}_{\vec{k}} + \frac{1}{2}\delta^3_{\vec{k}}(\vec{0})\right), \tag{2.84}$$

such that

$$\langle E_\Phi\rangle_0 := \langle 0|\widehat{H}_\Phi|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\ \delta^3_{\vec{k}}(\vec{0}) \tag{2.85}$$

which is infinite. Here, removal of the infinite vacuum energy is carried out via normal ordering where $:\widehat{O}(\hat{a}_{\vec{k}},\hat{a}^{\dagger}_{\vec{k}}):$ for some operator $\widehat{O}$ means $\hat{a}_{\vec{k}}\ \hat{a}^{\dagger}_{\vec{k}} \rightarrow \hat{a}^{\dagger}_{\vec{k}}\ \hat{a}_{\vec{k}}$ regardless of the commutation relations in Eq. (2.78). For example,

$$:\hat{a}^{\dagger}_{\vec{k}}\,\hat{a}_{\vec{k}} + \hat{a}_{\vec{k}}\,\hat{a}^{\dagger}_{\vec{k}} := 2\hat{a}^{\dagger}_{\vec{k}}\,\hat{a}_{\vec{k}} \tag{2.86}$$

such that the normal ordered Hamiltonian is

$$:\widehat{H}_\Phi := \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\ \hat{a}^{\dagger}_{\vec{k}}\,\hat{a}_{\vec{k}} \tag{2.87}$$

and Eq. (2.85) becomes

$$\langle 0|:\widehat{H}_\Phi:|0\rangle = 0 \tag{2.88}$$

which is equivalent to

$$\langle 0|:\widehat{H}_\Phi:|0\rangle = \langle 0|\widehat{H}_\Phi|0\rangle - \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}}\ \delta^3_{\vec{k}}(\vec{0})\,\langle 0|0\rangle\,, \tag{2.89}$$

i.e. subtraction of a reference state.

2.2.3 Time-Dependent Perturbation Theory

The free field quantum analog of classical time evolution in the Hamiltonian formulation is

$$\partial_t \hat{\Phi}(t, \vec{x}) = -i[\hat{\Phi}, \widehat{H}_\Phi] \tag{2.90}$$

with solution

$$\hat{\Phi}(t, \vec{x}) \;\; = \;\; \exp\biggl(i \int_{t_0}^{t} dt' \; \widehat{H}_\Phi \biggr) \hat{\Phi}(t_0, \vec{x}) \exp\biggl(- i \int_{t_0}^{t} dt' \; \widehat{H}_\Phi \biggr) \tag{2.91}$$

in analogy with Eq. (2.19). Perturbative dynamics are introduced via a time-dependent interaction term of the general form

$$\widetilde{V}(t) = \widetilde{J}(t, \vec{x}) \hat{\Phi}(t, \vec{x}), \tag{2.92}$$

corresponding to Eq. (2.64), with $\widetilde{J}(t, \vec{x}) := \kappa J(t, \vec{x})$ given the perturbative coupling parameter $\mathbb{R} \ni \kappa \ll 1$ and $[J] = [M_\Phi^3]$. The time-dependent Hamiltonian $\widehat{H}(t)$ is now the sum

$$\widehat{H}(t) = \widehat{H}_\Phi + \widetilde{V}[\hat{\Phi}(t)]. \tag{2.93}$$

The field operator $\hat{\Phi}^J(t, \vec{x})$, with dynamics perturbed by the non vanishing source $J(t, \vec{x})$, now satisfies the Heisenberg equation of motion

$$\partial_t \hat{\Phi}^J(t) \;\; = \;\; -i[\hat{\Phi}^J(t_0), \widehat{H}(t)] \tag{2.94}$$

with solution

$$\hat{\Phi}^J(t) \;\; = \;\; \widehat{S}^{\,\dagger}(t, t_0) \hat{\Phi}(t_0)^J \widehat{S}(t, t_0) \tag{2.95}$$

such that

$$\partial_t \widehat{S}(t, t_0) \;\; = \;\; \widehat{H}(t) \widehat{S}(t, t_0) \tag{2.96}$$

given t_0 as an arbitrary reference time, at which the Heisenberg field above corresponds to the time independent Schrödinger field $\hat{\Phi}(\vec{x})$. Here, the interacting field is related to the free field via

$$\hat{\Phi}^J(t) \;\; = \;\; \widehat{U}^{\,\dagger}(t, t_0) \, \hat{\Phi}(t) \, \widehat{U}(t, t_0). \tag{2.97}$$

where

$$\widehat{U}(t,t_0) \;:=\; \exp\Big(i\int_{t_0}^{t} dt'\,\widehat{H}_\Phi\Big)\widehat{S}(t,t_0) \tag{2.98}$$

and

$$\partial_t\widehat{U}(t,t_0) \;=\; \exp\Big(i\int_{t_0}^{t} dt'\,\widehat{H}_\Phi\Big)\widetilde{V}[\hat{\Phi}(t)]\,\exp\Big(-i\int_{t_0}^{t} dt'\,\widehat{H}_\Phi\Big)\widehat{U}(t,t_0). \tag{2.99}$$

The solution to the equation above is found via the Dyson series

$$\widehat{U}(t,t_0) \;=\; 1+\sum_{n+1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2...\int_{t_0}^{t}dt_n\,\mathrm{T}\Big[V[\hat{\Phi}(t_1)]V[\hat{\Phi}(t_2)]...V[\hat{\Phi}(t_n)]\Big] \tag{2.100}$$

where, for convenience, we define $V[\hat{\Phi}(t)]$ as

$$V[\hat{\Phi}(t)] \;:=\; \exp\Big(i\int_{t_0}^{t} dt'\,\widehat{H}_\Phi\Big)\widetilde{V}[\hat{\Phi}(t)]\,\exp\Big(-i\int_{t_0}^{t} dt'\,\widehat{H}_\Phi\Big) \tag{2.101}$$

such that

$$\widehat{U}(t,t_0) \;=\; \mathrm{T}\Big[\exp\Big(-i\int_{t_0}^{t} dt'\;V[\hat{\Phi}(t')]\Big)\Big]. \tag{2.102}$$

The time ordering operation T[...] places operators at earlier time prior to those at late times, e.g.

$$\mathrm{T}[\hat{\Phi}(t)] \;=\; \hat{\Phi}(t) \tag{2.103}$$

$$\mathrm{T}[\hat{\Phi}(t)\hat{\Phi}(t')] \;=\; \hat{\Phi}(t)\hat{\Phi}(t')\theta(t-t')+\hat{\Phi}(t')\hat{\Phi}(t)\theta(t'-t) \tag{2.104}$$

given $\theta(t)$ as the Heaviside step function. The two-point function for the time ordered product of free fields is then written as

$$\begin{aligned}\langle 0|\,\mathrm{T}[\hat{\Phi}(x_\mu)\hat{\Phi}(y_\mu)]\,|0\rangle \;&=\; \langle 0|\,\hat{\Phi}(t_x)\hat{\Phi}(t_y)\,|0\rangle\,\theta(t_x-t_y)+\langle 0|\,\hat{\Phi}(t_y)\hat{\Phi}(t_x)\,|0\rangle\,\theta(t_y-t_x)\\ &=\; \int\frac{d^3k}{(2\pi)^3}\Big\{\frac{1}{2\omega_{\vec{k}}}\exp(-i\vec{k}\cdot[\vec{x}-\vec{y}])\\ &\quad\times\Big[\exp(i\omega_{\vec{k}}(t_x-t_y))\theta(t_y-t_x)+\exp(-i\omega_{\vec{k}}(t_x-t_y))\theta(t_x-t_y)\Big]\Big\}\end{aligned} \tag{2.105}$$

where

$$\left[\exp(i\omega_{\vec{k}}(t_x - t_y))\theta(t_y - t_x) \quad + \quad \exp(-i\omega_{\vec{k}}(t_x - t_y))\theta(t_x - t_y)\right] = \lim_{\epsilon\to 0} i\frac{\omega_{\vec{k}}}{\pi}\int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2 - \omega_{\vec{k}}^2 + i\epsilon}\exp(i\omega[t_x - t_y]) \tag{2.106}$$

such that the free time ordered two-point function, known as the Feynman propagator $D_F(x_\mu, y_\mu)$, is now defined

$$\begin{aligned} D_F(x_\mu, y_\mu) &:= \langle 0|\, \mathrm{T}[\hat{\Phi}(x_\mu)\hat{\Phi}(y_\mu)]\, |0\rangle \\ &= i\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2 - M_\Phi^2 + i\epsilon}\exp(ik^\mu[x_\mu - y_\mu]). \end{aligned} \tag{2.107}$$

where $k_0 \neq \omega_{\vec{k}}$. Here, the contour integral in Eq. (2.106) closes in the upper half plane for the pole at $-\omega_{\vec{k}} + i\epsilon$ and closes in the lower half plane for the pole at $\omega_{\vec{k}} - i\epsilon$.

The interacting vacuum $|\Omega\rangle$ is then defined via the invariant vector $|0\rangle$ in the asymptotic limit

$$|\Omega\rangle := n^- \widehat{U}(0, t_0 \to -\infty)\, |0\rangle \tag{2.108}$$

$$\langle\Omega| := n^+ \langle 0|\, \widehat{U}(t \to \infty, 0) \tag{2.109}$$

with the normalization constants $n^-, n^+ \in \mathbb{R}$ such that the normalization condition

$$\langle\Omega|\Omega\rangle = 1 \tag{2.110}$$

implies

$$n^+ n^- = \langle 0|\widehat{U}(\infty, -\infty)|0\rangle\,. \tag{2.111}$$

Hence, the general expression for interacting time ordered products is

$$\begin{aligned} \langle\Omega|\ \mathrm{T}[\hat{\Phi}^J(x_1)\hat{\Phi}^J(x_2)\ \ldots\ \hat{\Phi}^J(x_n)]\, |\Omega\rangle \ &= \\ \langle 0|\ \mathrm{T}\Big[\hat{\Phi}(x_1)\hat{\Phi}(x_2)\ \ \ldots\ \ \Phi(x_n)\exp\Big(i\int_{-\infty}^{\infty} d^4x\, V[\Phi(x)]\Big)\Big]\, |0\rangle& \\ \times\ \Big\{\langle 0|\ \mathrm{T}\Big[\exp\Big(i\int_{-\infty}^{\infty} d^4x\, V[\Phi(x)]\Big)\Big]\, |0\rangle\Big\}^{-1}&. \end{aligned} \tag{2.112}$$

2.2.4 Asymptotic States and the S-matrix

In the quantum formulation the two-body scattering of the classical theory is interpreted as evolving incoming asymptotic two-particle states to outgoing asymptotic two-particle states, e.g. $|\Phi_{\vec{p}}\,,\phi_{\vec{k}}\rangle_{-\infty} \to |\Phi'_{\vec{p}}\,,\phi'_{\vec{k}}\rangle_{\infty}$, where $|\Phi_{\vec{p}}\,,\phi_{\vec{k}}\rangle_{-\infty} := |\Phi_{\vec{p}}\rangle_{-\infty} \otimes |\phi_{\vec{k}}\rangle_{-\infty}$. This is the so called "in-out" formalism. The scattering matrix, or S-matrix, is then defined via the Lehmann–Symanzik–Zimmermann reduction formula

$$\begin{aligned}\langle\Phi'_{\vec{p}}\,,\phi'_{\vec{k}}|\,\mathbf{S}\,|\Phi_{\vec{p}}\,,\phi_{\vec{k}}\rangle_{\pm\infty} \;:=\; & \int d^4x_1 \int d^4x_2 \int d^4x_3 \int d^4x_4 \,\Big\{ \\ & \times \exp(-ip_{\Phi'}x_1)(\Box_1 + m^2_{\Phi'})\exp(-ik_{\phi'}x_2)(\Box_2 + m^2_{\phi'}) \\ & \times \exp(ip_{\Phi}x_3)(\Box_3 + m^2_{\Phi})\exp(ik_{\phi}x_4)(\Box_4 + m^2_{\phi}) \\ & \times \;\langle\Omega|\;\mathrm{T}[\hat{\Phi}'(x_1)\hat{\phi}'(x_2)\hat{\Phi}(x_3)\hat{\phi}(x_4)]\,|\Omega\rangle \Big\} \end{aligned} \tag{2.113}$$

where, for example, the Schwinger–Dyson equation gives

$$\begin{aligned}(\Box_1 + m^2_{\Phi'})\,\langle\Omega|\;\mathrm{T}[\hat{\Phi}'(x_1)\hat{\phi}'(x_2)\hat{\Phi}(x_3)\hat{\phi}(x_4)]\,|\Omega\rangle \;=\; & \\ \langle\Omega|\;\mathrm{T}[\,\partial_{\Phi'}V[\Phi'(x_1)]\;\hat{\phi}'(x_2)\hat{\Phi}(x_3)\hat{\phi}(x_4)]\,|\Omega\rangle \;-\; & i\delta^4(x_1-x_2)\,\langle\Omega|\;\mathrm{T}[\hat{\Phi}(x_3)\hat{\phi}(x_4)]\,|\Omega\rangle \\ i\delta^4(x_1-x_3)\,\langle\Omega|\;\mathrm{T}[\hat{\phi}'(x_2)\hat{\phi}(x_4)]\,|\Omega\rangle \;-\; & i\delta^4(x_1-x_4)\,\langle\Omega|\;\mathrm{T}[\hat{\phi}'(x_2)\hat{\Phi}(x_3)]\,|\Omega\rangle\,. \end{aligned} \tag{2.114}$$

Here, we have dropped the superscript J for notational convenience.

The terms proportional to $\delta(x - x_n)$ are the quantum corrections to the classical correlation functions. However, explicit calculations involving those corrections encounter divergences, starting at order κ^2, that must be removed. Hence, we employ the framework of renormalized perturbation theory in which we add formally infinite counterterms $\delta[\mathcal{O}(\kappa^2)]$ to the Lagrangian, e.g. $\hat{\Phi} \to \sqrt{Z_\Phi}\hat{\Phi}$ given $Z_\Phi := 1 + \delta_\Phi$ as field strength renormalization, such that the new Lagrangian is

$$\mathscr{L}[\hat{\Phi}(x_\mu)] = Z_\Phi \frac{1}{2}\partial_\mu\hat{\Phi}(x_\mu)\partial^\mu\hat{\Phi}(x_\mu) - Z_M Z_\Phi \frac{1}{2}M^2_\Phi\hat{\Phi}^2(x_\mu) - Z_\kappa\frac{\kappa}{2}V[\hat{\Phi}(x_\mu)]. \tag{2.115}$$

Here, the propagator associated with the renormalized, interacting time ordered products is

$$\langle\Omega|\;\mathrm{T}[Z_\Phi\hat{\Phi}(x_\mu)\hat{\Phi}(y_\mu)]\,|\Omega\rangle \;=\; i\int \frac{d^4p}{(2\pi)^4}\int dq^2\,\frac{\tilde{\sigma}(q^2)}{p^2 - q^2 + i\epsilon}\exp(ip^\mu[x_\mu - y_\mu]) \tag{2.116}$$

in the Källén–Lehmann representation. The spectral function $\tilde{\sigma}(q^2)$, given as

$$\tilde{\sigma}(q^2) = -\frac{1}{\pi}\,\mathrm{Im}[\Pi(q^2)] = Z_\Phi\,\delta(q^2 - m_p^2) + \sigma(q^2) \tag{2.117}$$

where $\Pi(q^2)$ is the dressed propagator and m_p is the mass at the pole corresponding to the single-particle state of the free theory and $\sigma(q^2)$, contains a discontinuity associated with the dynamical interactions of the theory.

The LHS of Eq. (2.113) corresponds to the matrix element $S_{\Phi\phi} := \langle \Phi'_{\vec{p}}\,, \phi'_{\vec{k}}|\,\mathbf{S}\,|\Phi_{\vec{p}}\,, \phi_{\vec{k}}\rangle$ such that, after performing the integrals on the right-hand-side (RHS), we may rewrite the expression as

$$S_{\Phi\phi} = (2\pi)^4\delta^4(p'_\Phi + k'_\phi - p_\Phi - k_\phi)i\mathscr{M}_{2\to 2} \tag{2.118}$$

where $\mathscr{M}_{2\to 2}$ is the so called probability amplitude and encodes the probabilities associated with the dynamics of the interacting time ordered products. In analogy with the classical theory, the quantum differential scattering cross section $d\sigma$ is defined

$$d\sigma(\Phi\phi \to \Phi'\phi') := \frac{dP}{\Delta t|\mathbf{v}_\Phi - \mathbf{v}_\phi|V^{-1}} \tag{2.119}$$

where dP is the differential probability given by

$$dP = (2\pi)^4\delta^4(p'_\Phi + k'_\phi - p_\Phi - k_\phi)|\mathscr{M}_{2\to 2}|^2\frac{\Delta t}{V}\frac{d^3p'_\Phi}{(2\pi)^3}\frac{d^3p'_\phi}{(2\pi)^3}\frac{1}{16\,\omega_{\vec{p}}\,\omega_{\vec{k}}\,\omega'_{\vec{p}}\,\omega'_{\vec{k}}} \tag{2.120}$$

where $\Delta t := (t - t_0) \to \infty$ in the limit of asymptotic states and $V \to \infty$ is the volume associated with the normalization factors in Eq. (2.82). Hence, the scattering cross section is expressed as

$$\sigma(\Phi\phi \to \Phi'\phi') = \int\frac{d^3p'_\Phi}{(2\pi)^3}\int\frac{d^3k'_\phi}{(2\pi)^3}\,|\mathscr{M}_{2\to 2}|^2\frac{(2\pi)^4\delta^4(p'_\Phi + k'_\phi - p_\Phi - k_\phi)}{16|\mathbf{v}_\Phi - \mathbf{v}_\phi|\omega_{\vec{p}}\,\omega_{\vec{k}}\,\omega'_{\vec{p}}\,\omega'_{\vec{k}}}. \tag{2.121}$$

Additionally, the theory allows for the decay of quantum fields interpreted as the one-body to two-body scattering process $|\Phi_{\vec{k}}\rangle_{-\infty} \to |\phi_{\vec{p}}\,, \phi_{\vec{q}}\rangle_\infty$ where the differential decay rate $d\Gamma(\Phi \to \phi\phi)$ is defined

$$d\Gamma(\Phi \to \phi\phi) := \frac{dP}{\Delta t} \tag{2.122}$$

such that

$$\Gamma(\Phi \to \phi\phi) = \int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\,|\mathscr{M}_{1\to 2}|^2\frac{(2\pi)^4\delta^4(k - p - q)}{8\,\omega_{\vec{k}}\,\omega_{\vec{p}}\,\omega_{\vec{q}}}. \tag{2.123}$$

The unitarity of the S-matrix, i.e. $\mathbf{S}^\dagger\mathbf{S} = \mathbb{1}$, has implications important for quantum kinetics. Consider the generic process $|A\rangle_{-\infty} \to |B\rangle_{\infty}$. Now let $\mathbf{S} = \mathbb{1} + i\mathscr{T}$ such that

$$\langle A|\, \mathscr{T}\, |B\rangle = (2\pi)^4\delta^4(p_A - k_B) i\mathscr{M}_{A\to B} \tag{2.124}$$

in accordance with Eq. (2.118) above. Here, $i(\mathscr{T}^\dagger - \mathscr{T}) = \mathscr{T}^\dagger\mathscr{T}$ gives the generalized optical theorem

$$\mathscr{M}_{A\to B} - \mathscr{M}^*_{B\to A} = i(2\pi)^4 \sum_X \int \frac{d^3k_X}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}\, \mathscr{M}_{A\to X}\mathscr{M}^*_{B\to X}\delta^4(p_A - k_X). \tag{2.125}$$

If we take $|A\rangle = |B\rangle = |\Phi_{\vec{p}}\,,\phi_{\vec{k}}\rangle$ and $|X\rangle = |\Phi'_{\vec{p}}\,,\phi'_{\vec{k}}\rangle$, as in the two-body scattering case, then in the center-of-mass (CM) reference frame

$$\text{Im}[\mathscr{M}(\Phi_{\vec{p}}\,,\phi_{\vec{k}})] = 2E_{CM}|\vec{p}|\sigma(\Phi\phi \to \Phi'\phi'). \tag{2.126}$$

Similarly, for $|A\rangle = |B\rangle = |\Phi_{\vec{k}}\rangle$ and $|X\rangle = |\phi_{\vec{p}}\,,\phi_{\vec{q}}\rangle$ we find

$$\text{Im}[\mathscr{M}(\Phi_{\vec{k}})] = M_\Phi\Gamma(\Phi \to \phi\phi) \tag{2.127}$$

in the rest frame of Φ. In this case, the optical theorem may be expressed as

$$\text{Im}[\Pi(m_p)] = m_p\,\Gamma(\Phi \to \phi\phi). \tag{2.128}$$

S-matrix elements may also be calculated via path integral methods. Here, the states are expressed in the field basis as a complete set of eigenstates, à la Schrödinger, such that

$$\hat{\Phi}(\vec{x})\,|\Phi;t\rangle = \Phi(\vec{x})\,|\Phi;t\rangle \tag{2.129}$$
$$\hat{\Pi}(\vec{x})\,|\Pi;t\rangle = \Pi(\vec{x})\,|\Pi;t\rangle \tag{2.130}$$

with $\Phi(\vec{x})$ and $\Pi(\vec{x})$ the eigenfunctions associated with the field operator and its canonical conjugate. The vacuum matrix element $\langle 0;\infty|0;-\infty\rangle$ is calculated by first summing over a complete set of intermediate states

$$\langle 0;t|0;t_0\rangle = \int \mathcal{D}\Phi_1(t,\vec{x})\mathcal{D}\Phi_n(t,\vec{x})\ \langle 0|\,\text{exp}(-i\delta t\hat{H}(t_n)\,|\Phi_n\rangle\,\langle\Phi_n|\,...\,|\Phi_1\rangle\,\langle\Phi_1|\,\text{exp}(-i\delta t\hat{H}(t_0)\,|0\rangle\,\Big\} \tag{2.131}$$

for n infinitesimal time intervals δt, with $t_m := t_0 + m\delta t$ and $t_n := t$. Here,

$$
\begin{aligned}
\langle \Phi_{m+1}| \exp(-i\delta t \hat{H}(t_m) |\Phi_m\rangle &= \int \mathcal{D}\Pi_m \langle \Phi_{m+1}|\Pi_m\rangle \langle \Pi_m| \exp\Big(- i\delta t \int d^3x \widehat{\mathscr{H}}(t_m, \vec{x})\Big) |\Phi_m\rangle \\
&= C \exp\Big(- i\delta t \int d^3x\, \mathscr{L}[\hat{\Phi}(t_m, \vec{x})]\Big), \qquad (2.132)
\end{aligned}
$$

with C a normalization constant, such that

$$
\langle 0; \infty | 0; -\infty\rangle = C \int \mathcal{D}\Phi(t, \vec{x}) \exp(-iS[\mathcal{L}(\Phi)]) \qquad (2.133)
$$

with the boundary conditions $\Phi(t, \vec{x}) \to \Phi(\vec{x})$ in the asymptotic limit $t \to \pm\infty$. By now inserting fields into the path integral, i.e

$$
C \int \mathcal{D}\Phi(t, \vec{x}) \exp(-iS[\mathcal{L}(\Phi)]) \;\; \to \;\; C \int \mathcal{D}\Phi(t, \vec{x}) \exp(-iS[\mathcal{L}(\Phi)])\Phi(t, \vec{x}),
$$

and imposing the normalization condition of the asymptotic, interacting vacuum $|\Omega\rangle_{\pm\infty}$ we may express interacting time ordered products as

$$
\begin{aligned}
\langle \Omega|\ \mathrm{T}[\hat{\Phi}(x_1)\hat{\Phi}(x_2)...\hat{\Phi}(x_n)] |\Omega\rangle &= C \int \mathcal{D}\Phi(t, \vec{x})\ \Phi(x_1)\Phi(x_2)...\Phi(x_n) \exp(-iS[\mathcal{L}(\Phi)]) \\
&\quad \times \Big[C \int \mathcal{D}\Phi(t, \vec{x}) \exp(-iS[\mathcal{L}(\Phi)])\Big]^{-1}. \qquad (2.134)
\end{aligned}
$$

The generating functional $Z[\widetilde{J}(t, \vec{x})]$, i.e. the vacuum amplitude in the presence of our source $\widetilde{J}(t, \vec{x})$, is defined as

$$
Z[\widetilde{J}(t, \vec{x})] = \int \mathcal{D}\Phi(t, \vec{x}) \exp\Big(- iS[\mathcal{L}_0(\Phi)] + i \int d^4x\ \widetilde{J}(t, \vec{x})\Phi(t, \vec{x})\Big) \qquad (2.135)
$$

where $\mathcal{L}_0(\Phi)$ is the free field Lagrangian. Now, the n^{th} order functional derivative gives

$$
\begin{aligned}
\frac{\partial^n Z[\widetilde{J}(x)]}{\partial\widetilde{J}(x_1)\partial\widetilde{J}(x_2)...\partial\widetilde{J}(x_n)} &= i^n \int \mathcal{D}\Phi(x) \Big\{ \exp\Big(- iS[\mathcal{L}_0(\Phi)] + i \int d^4x\ \widetilde{J}(x)\Phi(x)\Big) \\
&\quad \times \Big(\Phi(x_1)\Phi(x_2)...\Phi(x_n)\Big)\Big\} \qquad (2.136)
\end{aligned}
$$

such that

$$\frac{(-i)^n}{Z[0]} \frac{\partial^n Z[\widetilde{J}(x)]}{\partial \widetilde{J}(x_1)\partial \widetilde{J}(x_2)...\partial \widetilde{J}(x_n)}\bigg|_{\widetilde{J}=0} = \int \mathcal{D}\Phi(x)\, \exp\bigg(- iS[\mathcal{L}_0(\Phi)]\bigg)\Phi(x_1)\Phi(x_2)...\Phi(x_n) \times \bigg[\int \mathcal{D}\Phi(x)\, \exp\bigg(- iS[\mathcal{L}_0(\Phi)]\bigg)\bigg]^{-1} = \langle\Omega|\ \mathrm{T}[\hat{\Phi}(x_1)\hat{\Phi}(x_2)...\hat{\Phi}(x_n)]\,|\Omega\rangle \tag{2.137}$$

via Eq. (2.134).

Of course, matrix elements are readily calculated via the diagrammatic methods of Feynman, as detailed in Ch. 6. of Ref [32]. As the ultimate goal of this work is to compute observables without regard to the particle physics interpretation of quantum field theory we do not cover this method here.

2.2.5 Ensembles of Quantum States

We now extend the quantum theory to a statistical ensemble of states. The quantum analog of the classical distribution function $\rho(\mathbf{q}, \mathbf{p}; t)$ is the density matrix operator $\hat{\rho}(t)$, defined in the field basis via the spectral decomposition

$$\hat{\rho}_X(t) := \sum_{n=0}^{\infty} w_n\, \widehat{U}^{\dagger}(t, t_0)\, |X_n(t_0)\rangle \langle X_n(t_0)|\, \widehat{U}(t, t_0) \tag{2.138}$$

with positive weighting functions w_n. Normalization is then required such that

$$\mathrm{Tr}\, [\hat{\rho}_X(t)] = 1. \tag{2.139}$$

Here, Tr $[\hat{\rho}_X^{\,2}(t)] \leq 1$, where the identity holds in the case of a pure state, i.e.

$$\hat{\rho}_X(t) = \ |X(t)\rangle \langle X(t)|\, , \tag{2.140}$$

and inequality describes the generic mixed state characterized by the convex linear combination

$$\hat{\rho}_X(t) = \lambda \hat{\rho}_Y(t) + (1 - \lambda)\hat{\rho}_Z(t) \tag{2.141}$$

for $\lambda \in [0, 1]$. Its time evolution is given by the now familiar expression

$$\partial_t \hat{\rho}_X(t) = i[\hat{\rho}_X(t), \hat{H}(t)] \tag{2.142}$$

as the Liouville–von Neumann equation of motion with solution

$$\hat{\rho}_X(t) = \hat{\rho}_X(t_0) - i \int_{t_0}^{t} dt' \, [\widehat{H}(t'), \hat{\rho}_X(t')]. \tag{2.143}$$

Observables in the interaction picture, e.g. correlation functions of the interacting field operators, are given by

$$\langle \widehat{X}(t, \vec{x}_1) \widehat{X}(t, \vec{x}_2) ... \widehat{X}(t, \vec{x}_n) \rangle_\rho = \text{Tr} \, [\widehat{X}(t, \vec{x}_1) \widehat{X}(t, \vec{x}_2) ... \widehat{X}(t, \vec{x}_n) \; \hat{\rho}_X(t)] \tag{2.144}$$

where permutations are permitted via the cyclic property of the trace. Kinetic equilibrium of the quantum ensemble is defined via the Kubo–Martin–Schwinger (KMS) condition

$$\langle \widehat{X}^\dagger(t_0) \widehat{X}(t) \rangle_\rho = \langle \widehat{X}(t_0) \widehat{X}^\dagger(t + i\beta) \rangle_\rho \tag{2.145}$$

such that

$$\hat{\rho}_X^{EQ} := \frac{\exp(-\beta \widehat{H}_X)}{\text{Tr} \exp(-\beta \widehat{H}_X)} \tag{2.146}$$

is time-independent. Here, Tr $\exp(-\beta \widehat{H}_X)$ is the normalization factor, in analogy with the classical partition function $Z(\beta)$ in Eq. (2.59), such that the equilibrium quantum number density

$$\mathcal{N}_{\vec{k}}^{EQ} := \langle \hat{a}_{\vec{k}}^\dagger \, \hat{a}_{\vec{k}} \rangle_{\rho_X^{EQ}} = \text{Tr} \left[\frac{\hat{a}_{\vec{k}}^\dagger \, \hat{a}_{\vec{k}} \exp(-\beta \widehat{H}_X)}{\text{Tr} \exp(-\beta \widehat{H}_X)} \right] = \frac{1}{\exp(\beta \omega_{\vec{k}}) - 1} \tag{2.147}$$

is given by the Bose–Einstein distribution. Similarly, the von Neumann entropy $S[\hat{\rho}_X(t)]$ is defined

$$S[\hat{\rho}_X(t)] := -\text{Tr} \, [\hat{\rho}_X(t) \log \hat{\rho}_X(t)] \tag{2.148}$$

in analogy with Eq. (2.56) such that $S[\hat{\rho}_X(t)] \geq 0$, where once again equality corresponds to the pure state, and

$$S[\widehat{U}^\dagger(t, t_0) \hat{\rho}_X(t_0) \widehat{U}(t, t_0)] = S[\hat{\rho}_X(t_0)]. \tag{2.149}$$

Here, we take $0 \log 0 := 0$ and $\hat{\rho}_X$ to be an unbounded operator on a densely defined subspace $\mathcal{D}(\mathcal{H}_x) \subset \mathcal{H}_X$ if $\dim \mathcal{H}_X = \infty$.

Dynamical far-from-equilibrium interactions are generally framed via the formulation of an open quantum system, i.e. the Hilbert space of the ensemble is initially factorized into that of a **system** of interest, and a coupled **environment**, such that $\mathcal{H}_{SE} \rightarrow \mathcal{H}_S \otimes \mathcal{H}_E$. The density matrix of the total ensemble $\hat{\rho}_{SE}(t_0)$ shares this structure at the initial time t_0 such that

$$\hat{\rho}_{SE}(t_0) := \hat{\rho}_S(t_0) \otimes \hat{\rho}_E(t_0) \tag{2.150}$$

and evolves unitarily via

$$\hat{\rho}_{SE}(t) := \widehat{U}^{\dagger}_{SE}(t, t_0)\ \hat{\rho}_S(t_0) \otimes \hat{\rho}_E(t_0)\ \widehat{U}_{SE}(t, t_0) \tag{2.151}$$

given

$$\widehat{U}_{SE} = \mathrm{T}\Big[\exp\Big(- i \int_{t_0}^{t} dt'\ \widehat{H}_{SE}(t')\Big)\Big] \tag{2.152}$$

where

$$\widehat{H}_{SE}(t) = \widehat{H}_S \otimes \mathbf{I}_E + \widehat{H}_E \otimes \mathbf{I}_S + V[\widehat{X}_S(t), \widehat{X}_E(t)] \tag{2.153}$$

with $\mathbf{I}_X$ as the identity on $\mathcal{H}_X$. The reduced system dynamics, i.e. the dynamics of the system via interactions with the environment over the time interval $(t - t_0)$ is obtained by the partial trace as the map $\mathrm{Tr}_E : \mathcal{T}(\mathcal{H}_S \otimes \mathcal{H}_E) \rightarrow \mathcal{T}(\mathcal{H}_S)$ where $\mathcal{T}(\mathcal{H}_X)$ is the Banach space of possibly unbounded operators over $\mathcal{H}_X$, such that

$$\hat{\rho}_S(t) = \mathrm{Tr}_E[\widehat{U}^{\dagger}_{SE}(t, t_0)\ \hat{\rho}_S(t_0) \otimes \hat{\rho}_E(t_0)\ \widehat{U}_{SE}(t, t_0)]. \tag{2.154}$$

2.2.6 Non-equilibrium Quantum Field Theory

In an attempt at brevity we focus here on the one-body to two-body decay process $|\Phi_{\vec{k}}\rangle \rightarrow |\phi_{\vec{p}}\,, \phi_{\vec{q}}\rangle$. We take $\hat{\Phi}$ as the system of interest and the environment to be the composite field operator $\hat{\psi} := \hat{\phi}\hat{\phi}$. The factorized Schrödinger picture density operator in the field basis, including all three fields, is given at the initial time t_0 as

$$\hat{\rho}_{\Phi,\psi}(t_0) = \hat{\rho}_{\Phi}(t_0) \otimes \hat{\rho}_{\psi}, \tag{2.155}$$

where $\hat{\rho}_{\Phi}(t)$ is the density operator for the system of Φ-field states $|\Phi_{\vec{k}}\rangle$ in causal contact with the environment of ψ-field states $|\psi_{\vec{p},\vec{q}}\rangle := |\phi_{\vec{p}}\rangle \otimes |\phi_{\vec{q}}\rangle$; from here forward referred to as the Φ-system and its ψ-environment respectively. While the tensor product in Eq.

(2.155) represents the pure state of the composite density operator, we take the initial state of the ψ-environment to be of the KMS form. Hence, the density operator in the Born approximation will maintain the a general form

$$\hat{\rho}_{\psi} := \frac{1}{Z(\beta)}\exp[-\beta \widehat{H}_{\psi}] \tag{2.156}$$

where the quantum partition function is defined

$$Z(\beta) := \mathrm{Tr}_{\psi}\, \exp[-\beta \widehat{H}_{\psi}]. \tag{2.157}$$

Passing to the interaction picture, the unitary time evolution of the total ensemble from the common reference time $t_0 = 0$ to a final time t_f is then

$$\hat{\rho}_{\Phi,\psi}(t_f) = \widehat{U}^{\dagger}(t_f, 0)\; \hat{\rho}_{\Phi,\psi}(0)\; \widehat{U}(t_f, 0) \tag{2.158}$$

where

$$\widehat{U}(t_f, 0) = \mathrm{T}\Bigg\{\, \exp\Bigg[-i(\widehat{H}_{\Phi} + \widehat{H}_{\psi})t_f - i\int_0^{t_f} dt_x \int d^3x\; \widetilde{J}(\vec{x}, t_x)\hat{\psi}(\vec{x}, t_x)\Bigg]\Bigg\} \tag{2.159}$$

with $\widetilde{J}(\vec{x}, t_x) := \kappa M_{\Phi}\hat{\Phi}(\vec{x}, t_x)$ given $[\hat{\psi}] = [M_{\Phi}^2]$. We now let $M_{\Phi} = 1$ GeV for notational convenience. In accordance with the partition function the generating functional is given by the partial trace

$$Z_{\psi}[\hat{\Phi}^{+}, \hat{\Phi}^{-}; t_f] := \mathrm{Tr}_{\psi}\Bigg[\,\widehat{U}(\hat{\Phi}^{+}; t_f, 0)\; \hat{\rho}_{\Phi,\psi}(0)\; \widehat{U}(\hat{\Phi}^{-}; 0, t_f)\,\Bigg]. \tag{2.160}$$

Here, we double the degrees of freedom

$$\hat{\Phi} \rightarrow \begin{bmatrix} \hat{\Phi}^{+} \\ \hat{\Phi}^{-} \end{bmatrix} \text{ and } \hat{\psi} \rightarrow \begin{bmatrix} \hat{\psi}^{+} \\ \hat{\psi}^{-} \end{bmatrix} \tag{2.161}$$

where $\widehat{U}(\hat{\Phi}^{+}; t_f, 0)$ and $\widehat{U}(\hat{\Phi}^{-}; 0, t_f)$ are now interpreted as forward and backward time evolution operators on the closed-time-path contour $\mathcal{C}$ of Fig 2.1. This is the so called Schwinger–Keldysh "in-in" formalism where the time evolution operators take as an in-state the ensemble at some initial time t_0, evolve out to some macroscopic final time t_f on the positive branch of the contour, and then evolve the ensemble backwards over the negative branch to the initial time t_0. It is important to note the time dependence of the source operators, i.e. if $\hat{\Phi}^{+}(t_y) = \hat{\Phi}^{+}(t_x) = \hat{\Phi}^{-}(t_x) = \hat{\Phi}^{-}(t_y)$ we recover equilibrium

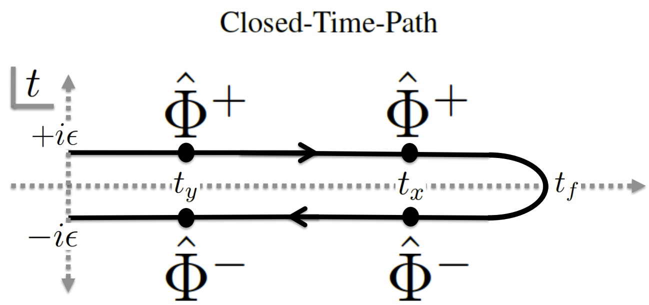

Figure 2.1: Closed-time-path contour $\mathcal{C}$ of the "in-in" formalism via the forward and backward time evolution operators $\widehat{U}(\hat{\Phi}^+; t_f, 0)$ and $\widehat{U}(\hat{\Phi}^-; 0, t_f)$ given the sources $\hat{\Phi}^+$ and $\hat{\Phi}^-$ respectively.

while $\hat{\Phi}^+(t_y) \neq \hat{\Phi}^+(t_x) \neq \hat{\Phi}^-(t_x) \neq \hat{\Phi}^-(t_y)$ allows for the analysis of field theoretic non-equilibrium quantum dynamics.

Invoking the path integral representation, we may put the generating functional into the standard form

$$\begin{aligned} Z_{\mathcal{C}}[\hat{\Phi}] &= \int \mathcal{D}[\hat{\psi}(\vec{x}, t_x)] \exp\Big(i \int_{\mathcal{C}} dt_x \int d^3x \left\{ \mathscr{L}_0[\hat{\psi}(\vec{x}, t_x)] + \widetilde{J}(\vec{x}, t_x)\hat{\psi}(\vec{x}, t_x) \right\} \Big) \\ &= \exp\Big(- i\frac{\kappa^2}{2} \int_{\mathcal{C}} dt_x \int_{\mathcal{C}} dt_y \int d^3x \int d^3y \; \hat{\Phi}_{\mathcal{C}}(\vec{x}, t_x) D_{\mathcal{C}}(\vec{x}, t_x; \vec{y}, t_y) \hat{\Phi}_{\mathcal{C}}(\vec{y}, t_y) \Big). \end{aligned} \tag{2.162}$$

where

$$D_{\mathcal{C}}(\vec{x}, t_x; \vec{y}, t_y) := \begin{bmatrix} D_{++}(\vec{x}, t_x; \vec{y}, t_y) & D_{+-}(\vec{x}, t_x; \vec{y}, t_y) \\ D_{-+}(\vec{x}, t_x; \vec{y}, t_y) & D_{--}(\vec{x}, t_x; \vec{y}, t_y) \end{bmatrix} \tag{2.163}$$

$$\hat{\Phi}_{\mathcal{C}}(\vec{x}, t_x) := \begin{bmatrix} \hat{\Phi}^+(\vec{x}, t_x), & \hat{\Phi}^-(\vec{x}, t_x) \end{bmatrix} \tag{2.164}$$

$$\hat{\phi}_{\mathcal{C}}(\vec{y}, t_y) := \begin{bmatrix} \hat{\Phi}^+(\vec{y}, t_y) \\ \hat{\Phi}^-(\vec{y}, t_y) \end{bmatrix} \tag{2.165}$$

given the closed-time-path contour $\mathcal{C}$. We may now find $D_{\pm\pm}(\vec{x}, t_x; \vec{y}, t_y)$ via the functional derivatives

$$iD_{\pm\pm}(\vec{x}, t_x; \vec{y}, t_y) = \frac{(-i)^2}{\kappa^2 Z(0)} \frac{\partial^2 Z[\hat{\Phi}^+, \hat{\Phi}^-]}{\partial\hat{\Phi}^{\pm}(\vec{x}, t_x)\partial\hat{\Phi}^{\pm}(\vec{y}, t_y)}\Big|_{\Phi^{\pm} = 0} \tag{2.166}$$

such that

$$D_{++}(\vec{x}-\vec{y};t_x-t_y)+D_{--}(\vec{x}-\vec{y};t_x-t_y)=D_{+-}(\vec{x}-\vec{y};t_x-t_y)+D_{-+}(\vec{x}-\vec{y};t_x-t_y) \tag{2.167}$$

with

$$\begin{aligned} iD_{++}(\vec{x}-\vec{y};t_x-t_y) &= \langle \mathrm{T}[\hat{\psi}(\vec{x},t_x)\hat{\psi}(\vec{y},t_y)]\rangle - \langle\hat{\psi}(\vec{x},t_x)\rangle\langle\hat{\psi}(\vec{y},t_y)\rangle\,; \\ &\quad \text{for } t_x,t_y\in[+i\epsilon,t_f] \end{aligned} \tag{2.168}$$

$$\begin{aligned} iD_{--}(\vec{x}-\vec{y};t_x-t_y) &= \langle \overline{\mathrm{T}}[\hat{\psi}(\vec{x},t_x)\hat{\psi}(\vec{y},t_y)]\rangle - \langle\hat{\psi}(\vec{y},t_y)\rangle\langle\hat{\psi}(\vec{x},t_x)\rangle\,; \\ &\quad \text{for } t_x,t_y\in[t_f,-i\epsilon] \end{aligned} \tag{2.169}$$

$$\begin{aligned} iD_{+-}(\vec{x}-\vec{y};t_x-t_y) &= \langle\hat{\psi}(\vec{y},t_y)\hat{\psi}(\vec{x},t_x)\rangle - \langle\hat{\psi}(\vec{y},t_y)\rangle\langle\hat{\psi}(\vec{x},t_x)\rangle\,; \\ &\quad \text{for } t_x\in[+i\epsilon,t_f],\, t_y\in[t_f,-i\epsilon] \end{aligned} \tag{2.170}$$

$$\begin{aligned} iD_{-+}(\vec{x}-\vec{y};t_x-t_y) &= \langle\hat{\psi}(\vec{x},t_x)\hat{\psi}(\vec{y},t_y)\rangle\langle\hat{\psi}-(\vec{x},t_x)\rangle\langle\hat{\psi}(\vec{y},t_y)\rangle\,; \\ &\quad \text{for } t_x\in[t_f,-i\epsilon],\, t_y\in[+i\epsilon,t_f]. \end{aligned} \tag{2.171}$$

Hence,

$$\begin{aligned} Z_\psi[\hat{\Phi}^+,\hat{\Phi}^-] &= \exp\Big(-\frac{\kappa^2}{2}\int_{+i\epsilon}^{t_f}dt_x\int_{+i\epsilon}^{t_x}dt_y\int d^3x\int d^3y\,\Big\{ \\ &\quad \hat{\Phi}^+(\vec{x},t_x)\hat{\Phi}^+(\vec{y},t_y)D^>(\vec{x}-\vec{y};t_x-t_y) \\ &+ \hat{\Phi}^-(\vec{x},t_x)\hat{\Phi}^-(\vec{y},t_y)D^<(\vec{x}-\vec{y};t_x-t_y) \\ &- \hat{\Phi}^-(\vec{x},t_x)\hat{\Phi}^+(\vec{y},t_y)D^>(\vec{x}-\vec{y};t_x-t_y) \\ &- \hat{\Phi}^+(\vec{x},t_x)\hat{\Phi}^-(\vec{y},t_y)D^<(\vec{x}-\vec{y};t_x-t_y)\Big\}\Big) \end{aligned} \tag{2.172}$$

upon the relabeling of the connected correlation functions

$$D^{>}(\vec{x}-\vec{y};t_x-t_y) \;:=\; iD_{\pm\pm}(\vec{x}-\vec{y};t_x-t_y) \tag{2.173}$$

$$D^{<}(\vec{x}-\vec{y};t_x-t_y) \;:=\; iD_{\pm\pm}(\vec{y}-\vec{x};t_y-t_x). \tag{2.174}$$

To all orders, the dynamics of the ψ-environment are contained in the spectral function

$$\sigma(\vec{k},k_0;\beta)=\frac{\pi}{2}\int\frac{d^3p}{(2\pi)^3\omega_{\vec{p}}\,\omega_{\vec{q}}}\Bigg\{\;[1+n_{\vec{p}}+n_{\vec{q}}]\,[\delta(k_0-\omega_{\vec{p}}-\omega_{\vec{q}})-\delta(k_0+\omega_{\vec{p}}+\omega_{\vec{q}})] + [n_{\vec{q}}-n_{\vec{p}}]\,[\delta(k_0-\omega_{\vec{p}}+\omega_{\vec{q}})-\delta(k_0+\omega_{\vec{p}}-\omega_{\vec{q}})]\Bigg\} \tag{2.175}$$

where $\vec{q}=|\vec{k}-\vec{p}|$ and

$$n_{\vec{p}}=\frac{1}{\exp[\beta\omega_{\vec{p}}]-1}\;;\;n_{\vec{q}}=\frac{1}{\exp[\beta\omega_{\vec{q}}]-1}. \tag{2.176}$$

Here, the four delta functions correspond to all four processes, and their inverse, available in the plasma:

$$\begin{aligned}
|\Phi_{\vec{k}}\rangle \to |\phi_{\vec{p}}\,,\phi_{\vec{q}}\rangle \quad &\text{and} \quad |\phi_{\vec{p}}\,,\phi_{\vec{q}}\rangle \to |\Phi_{\vec{k}}\rangle\\
|\phi_{\vec{p}}\rangle \to |\Phi_{\vec{k}}\,,\phi_{\vec{q}}\rangle \quad &\text{and} \quad |\Phi_{\vec{k}}\,,\phi_{\vec{q}}\rangle \to |\phi_{\vec{p}}\rangle\\
|\phi_{\vec{q}}\rangle \to |\Phi_{\vec{k}}\,,\phi_{\vec{p}}\rangle \quad &\text{and} \quad |\Phi_{\vec{k}}\,,\phi_{\vec{p}}\rangle \to |\phi_{\vec{q}}\rangle\\
|\Omega\rangle \to |\Phi_{\vec{k}}\,,\phi_{\vec{p}}\,,\phi_{\vec{q}}\rangle \quad &\text{and} \quad |\Phi_{\vec{k}}\,,\phi_{\vec{p}}\,,\phi_{\vec{q}}\rangle \to |\Omega\rangle\,.
\end{aligned}$$

Given the relation

$$\langle[\hat{\psi}(\vec{x},t_x),\;\hat{\psi}(\vec{y},t_y)]\rangle=\int\frac{d^3k}{(2\pi)^3}\int\frac{dk_0}{2\pi}\,\sigma(\vec{k},k_0;\beta)\,\exp[-ik_0(t_x-t_y)+i\vec{k}\cdot(\vec{x}-\vec{y})] \tag{2.177}$$

where

$$\langle[\hat{\psi}(\vec{x},t_x),\;\hat{\psi}(\vec{y},t_y)]\rangle=D^{>}(\vec{x}-\vec{y};t_x-t_y)-D^{<}(\vec{x}-\vec{y};t_x-t_y) \tag{2.178}$$

such that

$$\sigma(\vec{k},k_0;\beta)=D^{>}(k_0,\vec{k};\beta)-D^{<}(k_0,\vec{k};\beta) \tag{2.179}$$

and

$$D^{>}(\vec{k},k_0;\beta)=D^{<}(\vec{k},k_0;\beta)\,\exp[\beta k_0] \tag{2.180}$$

satisfies the KMS relation, then

$$D^{>}(\vec{k}, k_0; \beta) \;=\; \sigma(\vec{k}, k_0; \beta)\,[1 + n(k_0)] \tag{2.181}$$

$$D^{<}(\vec{k}, k_0; \beta) \;=\; \sigma(\vec{k}, k_0; \beta)\, n(k_0) \tag{2.182}$$

for

$$n(k_0) = \frac{1}{\exp[\beta k_0] - 1}. \tag{2.183}$$

Following the formulation of Boyanovsky in Ref. [39] we may now ascribe an effective action, to second order in κ, in the form of

$$S_{Eff}[\hat{\Phi}^{+}, \hat{\Phi}^{-}; \beta] := \int_{+i\epsilon}^{t_f} dt_x \int d^3x \left\{ \mathscr{L}_0[\hat{\Phi}^{+}(\vec{x}, t_x)] - \mathscr{L}_0[\hat{\Phi}^{-}(\vec{x}, t_x)] \right\} + F[\hat{\Phi}^{+}, \hat{\Phi}^{-}; \beta] \tag{2.184}$$

with $F[\hat{\Phi}^{+}, \hat{\Phi}^{-}; \beta]$ as the so called Feynman–Vernon influence phase of Ref. [40] such that

$$Z_{\psi}[\hat{\Phi}^{+}, \hat{\Phi}^{-}; \beta] = \exp\bigg(iF[\hat{\Phi}^{+}, \hat{\Phi}^{-}; \beta] \bigg). \tag{2.185}$$

Using the Fourier transforms

$$\int d^3x\; \hat{\Phi}^{\pm}(\vec{x}, t_x)\, \exp(i\vec{k}\cdot\vec{x}) \;=\; \hat{\Phi}^{\pm}(\vec{k}, t_x) \tag{2.186}$$

and

$$\int d^3y\; \hat{\Phi}^{\pm}(\vec{y}, t_y)\, \exp(-i\vec{k}\cdot\vec{y}) \;=\; \int \frac{d\omega_{\vec{k}}}{2\pi}\, \hat{\Phi}^{\pm}(-\vec{k}, \omega_{\vec{k}})\, \exp[-i\omega_{\vec{k}} t_y] \tag{2.187}$$

produces the expression

$$\begin{aligned} F[\hat{\Phi}^{+}, \hat{\Phi}^{-}; \beta] \;=\; & \lim_{\epsilon \downarrow 0}\, i\kappa^2 \int_{+i\epsilon}^{t_f} dt_x \int_{+i\epsilon}^{t_x} dt_y \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega_{\vec{k}}}{2\pi} \Bigg\{ \\ & \exp[-i\omega_{\vec{k}} t_x]\, \exp[i(\omega_{\vec{k}} - k_0)(t_x - t_y)] \\ & \times \Bigg[\hat{\Phi}^{+}(\vec{k}, t_x)\hat{\Phi}^{+}(-\vec{k}, \omega_{\vec{k}}) D^{>}(\vec{k}, k_0; \beta) \\ & -\; \hat{\Phi}^{-}(\vec{k}, t_x)\hat{\Phi}^{+}(-\vec{k}, \omega_{\vec{k}}) D^{>}(\vec{k}, k_0; \beta) \\ & +\; \hat{\Phi}^{-}(\vec{k}, t_x)\hat{\Phi}^{-}(-\vec{k}, \omega_{\vec{k}}) D^{<}(\vec{k}, k_0; \beta) \\ & -\; \hat{\Phi}^{+}(\vec{k}, t_x)\hat{\Phi}^{-}(-\vec{k}\omega_{\vec{k}}) D^{<}(\vec{k}, k_0; \beta) \Bigg] \Bigg\}. \end{aligned} \tag{2.188}$$

Here,

$$\lim_{\epsilon\downarrow 0}\int_{+i\epsilon}^{t_x} dt_y \,\exp[i(\omega_{\vec{k}}-k_0)(t_x-t_y)] \;=\; \frac{i}{(\omega_{\vec{k}}-k_0)}\Big(1-\exp[i(\omega_{\vec{k}}-k_0)t_x]\Big)$$

such that we may decompose the influence phase

$$F[\Phi^+,\Phi^-;\beta] = F_1[\Phi^+,\Phi^-;\beta] + F_2[\Phi^+,\Phi^-;\beta] \tag{2.189}$$

into a unitary fluctuation term

$$\begin{aligned} F_1[\hat{\Phi}^+,\hat{\Phi}^-;\beta] \;=\;& i\kappa^2\int_0^{t_f} dt_x \int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega_{\vec{k}}}{2\pi}\exp(-i\omega_{\vec{k}}t_x)\Big\{ \\ & \hat{\Phi}^+(\vec{k},t_x)\hat{\Phi}^+(-\vec{k},\omega_{\vec{k}})\Delta_1^>(\vec{k},\omega_{\vec{k}};\beta) \\ -\;& \hat{\Phi}^-(\vec{k},t_x)\hat{\Phi}^+(-\vec{k},\omega_{\vec{k}})\Delta_1^>(\vec{k},\omega_{\vec{k}};\beta) \\ +\;& \hat{\Phi}^-(\vec{k},t_x)\hat{\Phi}^-(-\vec{k},\omega_{\vec{k}})\Delta_1^<(\vec{k},\omega_{\vec{k}};\beta) \\ -\;& \hat{\Phi}^+(\vec{k},t_x)\hat{\Phi}^-(-\vec{k},\omega_{\vec{k}})\Delta_1^<(\vec{k},\omega_{\vec{k}};\beta)\Big\} \end{aligned} \tag{2.190}$$

and a nonunitary dissipation term

$$\begin{aligned} F_2[\hat{\phi}^+,\hat{\phi}^-;\beta] \;=\;& i\kappa^2\int_0^{t_f} dt_x \int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega_{\vec{k}}}{2\pi}\exp(-i\omega_{\vec{k}}t_x)\Big\{ \\ & \hat{\Phi}^+(\vec{k},t_x)\hat{\Phi}^+(-\vec{k},\omega_{\vec{k}})\Delta_2^>(\vec{k},\omega_{\vec{k}};\beta) \\ -\;& \hat{\Phi}^-(\vec{k},t_x)\hat{\Phi}^+(-\vec{k},\omega_{\vec{k}})\Delta_2^>(\vec{k},\omega_{\vec{k}};\beta) \\ +\;& \hat{\Phi}^-(\vec{k},t_x)\hat{\Phi}^-(-\vec{k},\omega_{\vec{k}})\Delta_2^<(\vec{k},\omega_{\vec{k}};\beta) \\ -\;& \hat{\Phi}^+(\vec{k},t_x)\hat{\Phi}^-(-\vec{k},\omega_{\vec{k}})\Delta_2^<(\vec{k},\omega_{\vec{k}};\beta)\Big\} \end{aligned} \tag{2.191}$$

given

$$\Delta_1^{>(<)}(\vec{k},\omega_{\vec{k}};\beta) \;:=\; i\int\frac{dk_0}{2\pi}\,D^{>(<)}(\vec{k},k_0;\beta)\left(\frac{1-\cos[(\omega_{\vec{k}}-k_0)t_x]}{(\omega_{\vec{k}}-k_0)}\right) \tag{2.192}$$

and

$$\Delta_2^{>(<)}(\vec{k},\omega_{\vec{k}};\beta) \;:=\; \int\frac{dk_0}{2\pi}\,D^{>(<)}(\vec{k},k_0;\beta)\left(\frac{\sin[(\omega_{\vec{k}}-k_0)t_x]}{(\omega_{\vec{k}}-k_0)}\right). \tag{2.193}$$

Having framed our effective field theory as an open quantum system we should then expect a Markovian master equation (MME) as a description of the entropically irreversible reduced dynamics of the Φ-system. Here, we introduce a quasifree quantum dynamical semigroup as the family of maps $\mu_t : \mathcal{T}(\mathcal{H}_\Phi) \to \mathcal{T}(\mathcal{H}_\Phi)$ such that

$$\partial_t \, \hat{\rho}_\Phi(t) = \widehat{\mathfrak{L}} \, \hat{\rho}_\Phi(t); \tag{2.194}$$

where the unbounded operator $\widehat{\mathfrak{L}}$, with dense domain $\mathcal{D}(\widehat{\mathfrak{L}}) \subset \mathcal{H}_\Phi$, is the generator of a one-parameter, completely positive Markov semigroup

$$\mu_t = \exp(\, \widehat{\mathfrak{L}} \, t \,) \tag{2.195}$$

and $\mu_t \circ \mu_s = \mu_{s+t}$. In this context $\widehat{\mathfrak{L}}$ is of course the Linblad superoperator where the MME is of the well known Linblad form [41]

$$\widehat{\mathfrak{L}} \, \hat{\rho}_\Phi(t) = -i[\widehat{H}_\Phi, \hat{\rho}_\Phi(t)] + \sum_n \, \widehat{L}_n \, \hat{\rho}_\Phi(t) \widehat{L}_n^\dagger - \frac{1}{2} \sum_n \, \widehat{L}_n^\dagger \widehat{L}_n \, \hat{\rho}_\Phi(t). \tag{2.196}$$

The complete positivity of μ_t ensures the form

$$\mu_t(\, \hat{\rho}_\Phi(0) \,) = \sum_n \, \widehat{M_n} \, \hat{\rho}_\Phi(0) \, \widehat{M_n^\dagger} \tag{2.197}$$

where the Kraus operators $\widehat{M_n}$ satisfy $\sum_n \, \widehat{M_n} \widehat{M_n^\dagger} \leq \mathbb{1}$, such that

$$\mu_t(\, \hat{\rho}_\Phi(0) \,) = \text{Tr}_\psi \Big[\hat{U}^{-1}(t,0) \, \hat{\rho}_\Phi(0) \otimes \hat{\rho}_\psi \, \hat{U}(t,0) \Big], \tag{2.198}$$

with the continued assumption of factorization throughout the unitarity evolution of the total ensemble $\hat{\rho}_{\Phi,\psi}(t)$. Given

$$\exp(\, iF[\hat{\Phi}^+, \hat{\Phi}^-; \beta] \,) = \text{Tr}_\psi \Big[\widehat{U}^{-1}(\hat{\Phi}^-; t_f, 0) \, \hat{\rho}_{\Phi,\psi}(0) \, \widehat{U}(\hat{\Phi}^+; t_f, 0) \Big] \tag{2.199}$$

established by Eqs. (2.160) and (2.185), we now review the main result of Ref. [39] via the relation

$$\hat{\rho}_\Phi(t_f) = \exp(iF[\hat{\Phi}^+, \hat{\Phi}^-; \beta]) \tag{2.200}$$

such that to order κ^2 Eq. (2.200) may be brought to the time-local form of a Bloch–Redfield master equation

$$
\begin{aligned}
\partial_t \, \hat{\rho}_\phi(t) \;\; &= \;\; \kappa^2 \int_0^t dt_y \int d^3y \int d^3x \, \Big\{ \\
& \qquad \hat{\Phi}^-(\vec{x},t)\hat{\Phi}^+(\vec{y},t_y) \; \hat{\rho}_\Phi(t) D^>(\vec{x}-\vec{y};t-t_y) \\
& \quad - \;\; \hat{\Phi}^+(\vec{x},t)\hat{\Phi}^+(\vec{y},t_y) \; \hat{\rho}_\phi(t) D^>(\vec{x}-\vec{y};t-t_y) \\
& \quad + \;\; \hat{\rho}_\Phi(t) \; \hat{\Phi}^+(\vec{x},t)\hat{\Phi}^-(\vec{y},t_y) D^<(\vec{x}-\vec{y};t-t_y) \\
& \quad - \;\; \hat{\rho}_\Phi(t) \; \hat{\Phi}^-(\vec{x},t)\hat{\phi}^-(\vec{y},t_y) D^<(\vec{x}-\vec{y};t-t_y) \Big\}. \qquad (2.201)
\end{aligned}
$$

To write Eq. (2.201) as an MME of Linblad form we first transform to the spectral representation via the relations of the previous subsection and time order the source terms $\hat{\Phi}^{\pm}$ with respect to $\hat{\rho}_\Phi$, e.g. $\hat{\Phi}^+\hat{\Phi}^-\hat{\rho}_\Phi \to \hat{\Phi}^+\hat{\rho}_\Phi\hat{\Phi}^-$. The Markov approximation is then made by first replacing the time coordinate t_y with the interval $\delta t := t - t_y$, i.e. the interval over which memory effects may be ignored, and then carrying out the integral over dt_y with a memoryless upper bound $t \to \infty$ such that

$$
\lim_{t\to\infty} \int_0^t dt_y \, \exp[-i(k_0 - \omega_{\vec{k}})(t-\delta t)] \;\; = \;\; i\text{PV}[...] + \pi\delta(\omega_{\vec{k}} - k_0). \qquad (2.202)
$$

Additionally, the secular approximation is invoked such that the contributions of rapidly dephasing terms proportional to $\exp[\pm i\omega_{\vec{k}}(t+t_y)]$ are ignored. We may now write the Linblad master equation

$$
\begin{aligned}
\partial_t \, \hat{\rho}_\Phi(t) \;\; &= \;\; \int \frac{d^3k}{(2\pi)^3)} \Big\{ - i\delta\omega_{\vec{k}} [\, \hat{a}^\dagger_{\vec{k}} \, \hat{a}_{\vec{k}}, \;\; \hat{\rho}_\Phi(t) \,] \\
& \quad - \frac{\Gamma^>_{\vec{k}}}{2} \Big[\hat{a}^\dagger_{\vec{k}} \, \hat{a}_{\vec{k}} \, \hat{\rho}_\phi(t) + \hat{\rho}_\Phi(t) \, \hat{a}^\dagger_{\vec{k}} \, \hat{a}_{\vec{k}} - 2 \, \hat{a}_{\vec{k}} \, \hat{\rho}_\Phi(t) \, \hat{a}^\dagger_{\vec{k}} \Big] \\
& \quad - \frac{\Gamma^<_{\vec{k}}}{2} \Big[\hat{a}_{\vec{k}} \, \hat{a}^\dagger_{\vec{k}} \, \hat{\rho}_\phi(t) + \hat{\rho}_\Phi(t) \, \hat{a}_{\vec{k}} \, \hat{a}^\dagger_{\vec{k}} - 2 \, \hat{a}^\dagger_{\vec{k}} \, \hat{\rho}_\Phi(t) \, \hat{a}_{\vec{k}} \Big] \Big\}. \qquad (2.203)
\end{aligned}
$$

Here,

$$
\delta\omega_{\vec{k}} \;\; = \;\; \text{PV}\Big[\frac{\kappa^2}{2\omega_{\vec{k}}} \int \frac{dk_0}{2\pi} \, \frac{\sigma(k_0,\vec{k};\beta)}{(\omega_{\vec{k}} - k_0)} \Big] = \frac{\text{Re } \Pi(\omega_{\vec{k}};\beta)}{2\omega_{\vec{k}}} \qquad (2.204)
$$

and

$$\frac{\Gamma_{\vec{k}}}{2}^{>(<)} = \frac{\kappa^2}{\omega_{\vec{k}}} \int \frac{dk_0}{2\pi}\, \sigma(k_0, \vec{k}; \beta)[n(k_0) + \Xi^{>(<)}]\pi\delta(\omega_{\vec{k}} - k_0)$$
$$= -\frac{\text{Im}\,\Pi(\omega_{\vec{k}}; \beta)}{2\omega_{\vec{k}}}[n(\omega_{\vec{k}}) + \Xi^{>(<)}] \tag{2.205}$$

where $\Pi(\omega_{\vec{k}}; \beta)$ contains the discontinuity associated with dynamics contained in the spectral function and $\Xi^{>(<)} := 1(0)$. In accordance with the optical theorem, Eq. (2.128), we may now write

$$\text{Im}\,\Pi(\omega_{\vec{k}}; \beta) = \omega_{\vec{k}}\, \Gamma_{\vec{k}}. \tag{2.206}$$

given $\Gamma_{\vec{k}} := \Gamma^{<}_{\vec{k}} - \Gamma^{>}_{\vec{k}}$. The fluctuating Hamiltonian like term $\delta\omega_{\vec{k}}$ and dissipative non-Hamiltonian terms $\Gamma^{>(<)}_{\vec{k}}$ correspond to the decomposed influence action terms $F_1[\hat{\Phi}^+, \hat{\Phi}^-; \beta]$ and $F_2[\hat{\Phi}^+, \hat{\Phi}^-; \beta]$ respectively. One sees immediately that though we have imposed the Born approximation along with the nontrivial approximation of secular, Markovian evolution there remain in this formulation dissipative nonlocal terms valid for the descriptions of the non-unitary entropically irreversible evolution of the Φ-system, i.e. $\partial_t S[\hat{\rho}_\Phi(t)] > 0$.

Taking the derivative with respect to time of the expectation value of the number operator $\widehat{N}_{\vec{k}} := \hat{a}^\dagger_{\vec{k}}\, \hat{a}_{\vec{k}}$ via the trace

$$\partial_t \langle \widehat{N}_{\vec{k}}(t) \rangle = \text{Tr}\,[\hat{a}^\dagger_{\vec{k}}\, \hat{a}_{\vec{k}}\, \partial_t \hat{\rho}_\phi(t)] \tag{2.207}$$

we find

$$\partial_t \mathscr{N}_{\vec{k}}(t) = [\mathscr{N}_{\vec{k}}(t) + 1]\, \Gamma^{<}_{\vec{k}} - \mathscr{N}_{\vec{k}}(t)\, \Gamma^{>}_{\vec{k}} \tag{2.208}$$

with solution

$$\mathscr{N}_{\vec{k}}(t) = n(\omega_{\vec{k}}) + [\mathscr{N}_{\vec{k}}(0) - n(\omega_{\vec{k}})]\, \exp(-\Gamma_{\vec{k}}\, t) \tag{2.209}$$

as Boltzmann's equation in agreement with the classical theory; where $\mathscr{N}_{\vec{k}}(t) := \langle \hat{N}_{\vec{k}}(t) \rangle$ is the statistical number density of the asymptotic state $|\Phi_{\vec{k}}\rangle_{\pm\infty}$ found via transition rates $\Gamma^{>(<)}_{\vec{k}}$ derived to include quantum corrections of order κ^2 using the "in-in" formalism of non-equilibrium quantum field theory.

There is of course an alternate formulation of quantum kinetic theory that is compatible with the graphs of Feynman, the so called "imaginary-time" formalism of Matsubara [42] (see Ref. [43] for a derivation of the quantum Boltzmann equation in the Matsubara framework).

Chapter 3

Quantum Kinetic Theory in Cosmological Spacetime

"... there is no natural definition of particles in a general curved spacetime."

R.M. Wald

3.1 Cosmology

In this section we review the standard cosmological model and kinetic mechanisms of the early universe posited as the origin of quantum matter relics (see, e.g. Refs. [44–47] for an introductory overview of the standard cosmology as well as the general kinetic mechanisms and their associated observational constraints).

3.1.1 Cosmological Basics

We take our cosmology to be that of the Λ-CDM model. This is a universe in which the total energy density is comprised of that of both the relativistic and non-relativistic matter of the standard model of particle physics, cold dark matter (CDM) as non-relativistic matter whose non-gravitational interactions with the standard model are taken to be feeble, and a cosmological constant Λ as the so called dark energy contribution; as well as an early period of cosmological inflation per the observational results found in Refs. [48, 49]. The globally hyperbolic spacetime $(\mathcal{M}_\Sigma, \mathbf{g})$ is taken to be the Lorentzian manifold $\mathcal{M}_\Sigma$ with Cauchy surface Σ_0 and metric $\mathbf{g}$. Here $\mathcal{M}_\Sigma := \mathbb{R} \times \Sigma_0$ is a spatially flat FRW spacetime with metric written in the familiar form

$$ds^2 = dt^2 - a_t^2 d\Sigma_0^2, \tag{3.1}$$

such that the D'alembertian becomes

$$\Box_{\mathrm{g}} = \partial_t^2 + 3H_t\partial_t + \frac{\nabla_\Sigma^2}{a_t^2}. \tag{3.2}$$

The Ricci scalar is then

$$R = 6\left(\frac{\ddot{a}_t}{a_t} + \frac{\dot{a}_t^2}{a_t^2}\right) \tag{3.3}$$

for $a_t := a(t) > 0$ given $a : \mathbb{R} \to \mathbb{R}$ as the scale factor and

$$H_t := \frac{\dot{a}_t}{a_t} \tag{3.4}$$

is the Hubble parameter. We may pass to conformal time η via the relation $dt = a_t d\eta$ where the metric becomes

$$ds^2 = a_t^2[d\eta^2 - d\Sigma_0^2]. \tag{3.5}$$

This transformation allows for the expansion of the domain of a_t into $(-\infty, \eta_0)$ with an asymptotically de Sitter (dS) spacetime $(\widetilde{\mathcal{M}_\Sigma}, \widetilde{\mathbf{g}})$ where $\widetilde{\mathbf{g}} = (\mathbf{\Omega}/a_t)^2\mathbf{g}$, given the map $\mathbf{\Omega} : \mathcal{M}_\Sigma \to \mathbb{R}^+$, such that $a_t = \exp\left(H_\Lambda t\right)$ for $\eta \in [-\infty, \eta_0]$ corresponds to an early period of inflation with H_Λ a constant. $\widetilde{\mathcal{M}_\Sigma}$ then contains a cosmological past horizon $\mathfrak{J}^-$ as a boundary, i.e. a smooth geodescially complete hypersurface diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$ at $\eta \to -\infty$, with coordinates $(v = t + r, \theta, \phi)$ and metric of Bondi form

$$\widetilde{ds}^2|_{\mathfrak{J}^-} = 2d\mathbf{\Omega}dv + d\mathbb{S}^2. \tag{3.6}$$

The abundance of a cosmological relic $\Omega_{X,\infty}$ is defined as the ratio of the late time energy density of the relic X to the so called critical energy density $\mathscr{E}_C$, or

$$\Omega_{X,\infty} := \frac{\mathscr{E}_{X,\infty}}{\mathscr{E}_C}, \tag{3.7}$$

where

$$\mathscr{E}_C := \frac{3H_t^2}{8\pi G}. \tag{3.8}$$

Here, $\mathscr{E}$ is the T_0^0 component of the FRW stress-energy tensor T_ν^μ where

$$T_\nu^\mu = \text{diag}\left[\mathscr{E}, -p, -p, -p\right] \tag{3.9}$$

such that $d(a_t^3\mathscr{E}) = -pd(a_t^3)$ gives the Clausius relation. The pressure p contributes to cosmological expansion via the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}R\,\mathrm{g}_{\mu\nu} = 8\pi G\,T_{\mu\nu} + \Lambda\,\mathrm{g}_{\mu\nu}, \tag{3.10}$$

where $R_{\mu\nu}$ is the Ricci tensor, which then reduces to Eq. (3.8) as the Friedmann equation given the conditions of homogeneity and isotropy. The equation of state $p = w\mathscr{E}$ then describes the evolution of the free energy density, i.e. $\mathscr{E} \propto a_t^{-3(w+1)}$ such that $\mathscr{E}^{(\mathrm{r})} \propto a_t^{-4}$ corresponding to $w = 3^{-1}$ when $\mathscr{E}$ is dominated by massless relativistic (r) degrees of

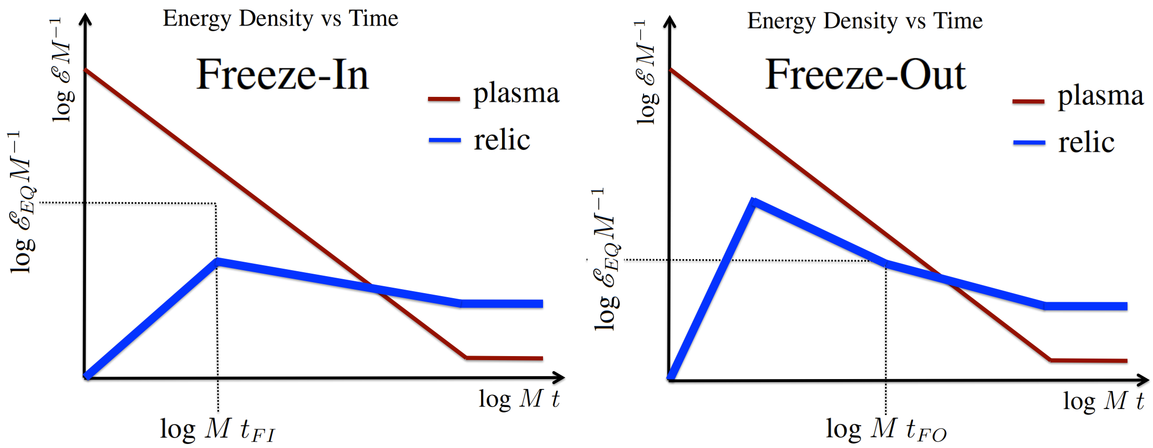

Figure 3.1: Conceptual representation of the evolution of the cosmological energy density $\mathscr{E}$ over cosmological time t with M an arbitrary mass parameter. **Left:** The freeze-in of a quantum field as a cosmological relic (solid blue curve) in which it fails to equilibrate with the dense environment of quantum fields comprising the primordial plasma (solid red curve), i.e. it does not attain the energy density $\mathscr{E}_{EQ}$ prior to its interactions becoming kinematically forbidden at a time t_{FI} **Right:** The freeze-out of a quantum field as a cosmological relic (solid blue curve) in which it equilibrates with the dense environment of quantum fields comprising the primordial plasma (solid red curve), i.e. attains an energy density $\mathscr{E}_{EQ}$ prior to its interactions becoming kinematically forbidden at a time t_{FO}

freedom and $\mathscr{E}^{(\mathrm{m})} \propto a_t^{-3}$ corresponding to $w = 0$ when $\mathscr{E}$ is dominated by massive non-relativistic matter (m).

3.1.2 Cosmological Relics

The so called "freeze-in" of a cosmological relic density is the kinetic process in which a massive quantum field with at least one interaction of feeble strength, i.e. the coupling parameter is of order 10^{-10}, and a negligible initial abundance of asymptotic single-particle states fails to equilibrate with the dense environment of quantum fields comprising the primordial plasma prior to its interactions becoming kinematically forbidden. An alternative, "freeze-out", is the process in which the field of interest first arrives at and then departs from kinetic equilibrium via considerably stronger, yet still perturbative, interactions. Fig. 3.1. provides a conceptual representation of the time evolution of the cosmological energy density associated with quantum interactions in the early universe.

We consider, as a toy model, a quantum field theory of the massive, neutral scalars Φ, χ, and φ, with hierarchy $M_\Phi > 2m_\chi \gg 2m_\varphi$, such that the nearly massless φ acts as a proxy for the kinetically equilibrated primordial plasma with Boltzmann parameter β_t. In addition, we restrict consideration of the quantum interactions to an epoch dominated by relativistic degrees of freedom, i.e.

$$\mathscr{E}_\varphi \gg \mathscr{E}_\Phi + \mathscr{E}_\chi \tag{3.11}$$

such that $a_t \propto (t - t_0)^{1/2}$. Given this toy model of scalars we examine the kinematics arising from the fundamental interactions

$$\mathscr{V}[\hat{\Phi}, \hat{\chi}, \hat{\varphi}] \;\; = \;\; \lambda\hat{\Phi}\hat{\chi}^2 + \lambda\hat{\Phi}\hat{\varphi}^2 + \lambda\hat{\Phi}^3 + \lambda\hat{\chi}^3 + g\hat{\varphi}^3 \tag{3.12}$$

where both Φ and χ have a negligible initial abundance. The field Φ may now be interpreted as a portal between two sectors of the quantum model that simultaneously contribute to the classical gravitational curvature of the FRW spacetime; one containing the χ-system as the relic of interest and another containing the φ-environment driving the cosmological expansion. Here, $\lambda \sim 10^{-10}$ is the feeble coupling and $g \gg \lambda$ is a substantively stronger, yet still perturbative, coupling given $[\lambda] = [g] = [M_\Phi]$.

There are then two quantum scattering processes that contribute to the abundance of the relic χ. These are of course the familiar one-body to two-body decay and the two-body scattering processes of the preceding chapter. The decay and annihilation processes,

$$|\Phi_{\vec{p}}\rangle \rightarrow |\chi_{\vec{q}_1}\,,\chi_{\vec{q}_2}\rangle \quad \text{and} \quad |\chi_{\vec{q}_1}\,,\chi_{\vec{q}_2}\rangle \rightarrow |\Phi_{\vec{p}}\rangle \tag{3.13}$$

$$|\Phi_{\vec{p}}\rangle \rightarrow |\varphi_{\vec{k}_1}\,,\varphi_{\vec{k}_2}\rangle \quad \text{and} \quad |\varphi_{\vec{k}_1}\,,\varphi_{\vec{k}_2}\rangle \rightarrow |\Phi_{\vec{p}}\rangle \tag{3.14}$$

with transition rates $\Gamma \propto \lambda^2$ dominate the two-body scattering processes

$$|\varphi_{\vec{k}_1}\,,\varphi_{\vec{k}_2}\rangle \rightarrow |\chi_{\vec{q}_1}\,,\chi_{\vec{q}_2}\rangle \text{ and } |\chi_{\vec{q}_1}\,,\chi_{\vec{q}_2}\rangle \rightarrow |\varphi_{\vec{k}_1}\,,\varphi_{\vec{k}_2}\rangle \tag{3.15}$$

with cross sections $\sigma \propto \lambda^4$. Hence, the frozen-in abundance of the relic χ via one to two-body decays, or annihilations, will dominate the frozen-out abundance due to two-body scattering. The plasma is assumed to maintain kinetic equilibrium through the two body scattering process

$$|\varphi_{\vec{k}_1}\,,\varphi_{\vec{k}_2}\rangle \rightarrow |\varphi_{\vec{k}_3}\,,\varphi_{\vec{k}_4}\rangle\,, \tag{3.16}$$

with cross section $\sigma(\varphi\varphi \rightarrow \varphi\varphi) \propto g^4$ regardless of portal interactions given $g^4 \gg \lambda^2$. In this model both the portal Φ and the relic χ are initially far-from-equilibrium such that the frozen-in abundance of the relic χ is derived first from the processes in Eq. (3.14) where the eventual decoupling of Φ from the plasma then results in both the decay process of Eq. (3.14) and of Eq. (3.13) at late times. The dominant frozen-in energy density via late decays and the subdominant frozen-in density from the two-body scattering in Eq. (3.15) are illustrated in Fig 3.2.

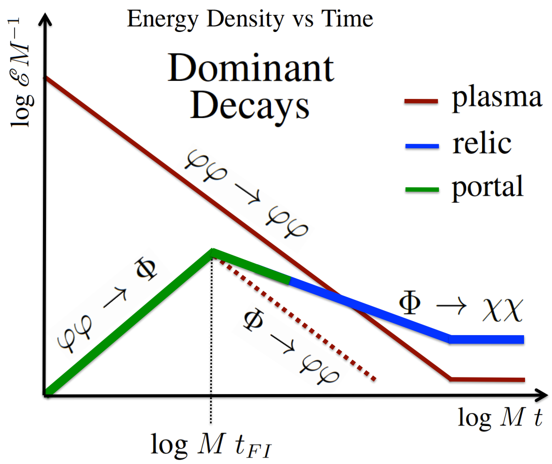

Figure 3.2: Conceptual representation of the evolution of the cosmological energy density $\mathscr{E}$ over cosmological time t with M an arbitrary mass parameter. **Left:** The dominant contribution to the energy density of the relic χ proceeds via the portal-plasma interaction $\Phi \to \varphi\varphi$ (solid green curve), with $\Gamma_{\varphi^2,\Phi} \propto \lambda^2$, such that the plasma maintains equilibrium through $\varphi\varphi \to \varphi\varphi$ (solid red curve), with $\sigma_{\varphi^2,\varphi^2} \propto g^4$, leading to the freeze-in of the portal Φ at a time t_{FI} (dashed line) and the late time decays $\Phi \to \varphi\varphi$ (dashed red curve) and $\Phi \to \chi\chi$ (solid blue curve), with $\Gamma_{\Phi,\chi^2} \propto \lambda^2$. **Right:** The subdominant frozen-in yield arises from the two-body scattering process $\varphi\varphi \to \chi\chi$ (solid blue curve) with $\sigma_{\varphi^2,\chi^2} \propto \lambda^4$ where the plasma maintains equilibrium via the scattering process $\varphi\varphi \to \varphi\varphi$ (solid red curve), with $\sigma_{\varphi^2,\varphi^2} \propto g^4$.

3.2 Quantum Field Theory in Curved Spacetime

In this section we review the standard operator formalism of quantum field theory extended to cosmological spacetime backgrounds (see Refs. [50, 51] for a detailed treatment of the strengths and weaknesses of this formalism in curved spacetime).

3.2.1 Adiabatic States

We consider the quantum field theory of a neutral scalar Φ on $(\mathcal{M}_\Sigma, \mathbf{g})$ where the free field equation of motion is

$$(\Box_{\mathbf{g}} + M_\Phi^2 + \xi R)\hat{\Phi}(\vec{x}, t) = 0 \qquad (3.17)$$

with ξ as the coupling to the scalar curvature. Given Φ as a massive field with $\xi = 0$ as the minimal coupling to gravity the lack of Poincaré invariance in our non-stationary FRW spacetime makes the irreducible representations of the Minkowski space formalism, i.e. the unique vacuum state defined globally via the action of the operator $\hat{a}_{\vec{k}}\,|0\rangle_{-\infty} = 0$ no longer applicable at later times as $\vec{k} \to \vec{k}\, a_t^{-1}$ and $\omega_{\vec{k}} \to \omega_{\vec{k}}(t)$ via the definition of $\Box_{\mathbf{g}}$ in Eq. (3.2). If we posit that our spacetime is asymptotically stationary, meaning $(\mathcal{M}_\Sigma, \mathrm{g}) \to (\mathcal{M}_0, \eta)$ as $t \to \pm\infty$, the general Heisenberg field operator may be expressed

$$\hat{\phi}(\vec{x}, t) = \frac{1}{\sqrt{a_t^3 V}} \sum_{\vec{k}} \Big[T_{\vec{k}}(t)\; \hat{b}_{\vec{k}}(t) + \overline{T_{\vec{k}}(t)}\; \hat{b}^\dagger_{\vec{k}}(t) \Big] \exp(i\vec{k}\cdot\vec{x}) \tag{3.18}$$

where mode function $T_{\vec{k}}(t)$ is defined as

$$T_{\vec{k}}(t) := \frac{1}{\sqrt{2\,\omega^{(n)}_{\vec{k}}(t)}} \exp\Big(-i \int_{t_0}^{t} dt'\; \omega^{(n)}_{\vec{k}}(t') \Big) \tag{3.19}$$

given the specification of $\omega^{(n)}_{\vec{k}}(t)$ to order n via the recursive relation

$$[\omega^{(n+1)}_{\vec{k}}(t)]^2 = [\omega^{(0)}_{\vec{k}}(t)]^2 - \frac{3}{4}\frac{\dot{a}_t^2}{a_t^2} - \frac{3}{2}\frac{\ddot{a}_t}{a_t} + \frac{3}{4}\Bigg[\frac{\dot{\omega}^{(n)}_{\vec{k}}(t)}{\omega^{(n)}_{\vec{k}}(t)}\Bigg]^2 - \frac{1}{2}\frac{\ddot{\omega}^{(n)}_{\vec{k}}(t)}{\omega^{(n)}_{\vec{k}}(t)} \tag{3.20}$$

for

$$\omega^{(0)}_{\vec{k}}(t) = \sqrt{\frac{\vec{k}^2}{a_t^2} + M_\Phi^2 - \frac{R}{6}} \tag{3.21}$$

per the *ansatz* of adiabatic vacuum states [16]. The operator $\hat{b}_{\vec{k}}(t)$ is then defined by the Bogoliubov transformation

$$\hat{b}_{\vec{k}}(t) := \mu_{\vec{k}}(t)\; \hat{a}_{\vec{k}} + \overline{\nu}_{\vec{k}}(t)\; \hat{a}^\dagger_{\vec{k}} \tag{3.22}$$

with canonical condition $|\mu_{\vec{k}}(t)|^2 - |\nu_{\vec{k}}(t)|^2 = 1$ where

$$[\hat{b}_{\vec{k}}(t)\,,\, \hat{b}_{\vec{p}}(t)] = [\hat{b}^\dagger_{\vec{k}}(t)\,,\, \hat{b}^\dagger_{\vec{p}}(t)] = 0 \;\text{ and }\; [\hat{b}_{\vec{k}}(t)\,,\, \hat{b}^\dagger_{\vec{p}}(t)] = \delta^3(\vec{k} - \vec{p}) \tag{3.23}$$

provides for a unitarily equivalent theory in which $\hat{b}_{\vec{k}}(t)$ annihilates single-particle states with momentum $\vec{k}\, a_t^{-1}$ and energy $\omega_{\vec{k}}(t)$.

The time-dependent number operator

$$\hat{N}_{\vec{k}}(t) := \hat{b}^\dagger_{\vec{k}}(t)\; \hat{b}_{\vec{k}}(t) \tag{3.24}$$

is now an adiabatic invariant and given $\hat{\rho}_\Phi$ as the density matrix on the Φ field Hilbert space

$\mathcal{H}_\Phi$ at $t \to -\infty$, *i.e.* diagonal in the $\hat{a}^\dagger_{\vec{k}}\hat{a}_{\vec{k}}$ basis, and we may write its expectation value as

$$\langle \hat{N}_{\vec{k}}(t)\rangle = \text{Tr}\,[\hat{a}^\dagger_{\vec{k}}\,\hat{a}_{\vec{k}}\,\hat{\rho}_\Phi] + |\nu_{\vec{k}}(t)|^2\bigg(1 + \text{Tr}\,[\hat{a}^\dagger_{\vec{k}}\,\hat{a}_{\vec{k}}\,\hat{\rho}_\Phi]\bigg). \tag{3.25}$$

We see that the function $\nu_{\vec{k}}(t)$ controls the time-dependent gravitationally induced excitation of the field Φ; however, when summing over modes we find a divergence in the term

$$\int_0^\infty \frac{d^3k}{(2\pi)^3}\,\nu_{\vec{k}}(t) \tag{3.26}$$

for a $\nu_{\vec{k}}(t)$ that doesn't vanish sufficiently rapidly as $k \to \infty$. We must now choose a unitarily equivalent representation that does not result in a diverging particle number in the limit $t \to \infty$, i.e. the adiabatic limit $\dot{a}_\infty = 0$. This is accomplished with the relation

$$\nu_{\vec{k}}(t) = -\dot{\mu}_{\vec{k}}(t)\,\exp\bigg(2i\int_{t_0}^{t} dt'\,\omega^{(n)}_{\vec{k}}(t')\bigg) \tag{3.27}$$

solved via a convergent series for $n \geq 1$ that holds for all $t \in I_t$ a finite time interval. We emphasize that this formulation is predicated on the assumption of an asymptotically stationary "in" and "out" spacetime which is not physically motivated. In addition, the adiabatic vacuum state depends on the order of iteration n where the full iterative procedure does not prevent the possibility of negative values for $[\omega^{(n+1)}_{\vec{k}}(t)]^2$ [17, 18]. Hence, we postpone a derivation of an expression for the abundance of a cosmological relic, based on the operator formalism of quantum field theory, until the semiclassical approximation of the next section.

3.2.2 Semiclassical Approximation

In this section, semiclassical refers to treating the evolution of classical distribution functions $f(x_\mu, p_\mu)$, given a classical N particle state space $\mathcal{M}^N_\Omega$, with transition rates derived from quantum field theoretic probabilities. We define the semiclassical number density $n(t)$ in terms of a homogeneous and isotropic FRW, phase space density $f(\omega_{\vec{p}}, t)$ such that

$$n(t) \;\;:=\;\; \int \frac{d^3p}{(2\pi)^3}\, f(\omega_{\vec{p}}, t). \tag{3.28}$$

Boltzmann kinetics, i.e.

$$\widehat{\mathbf{L}}_{\mathbf{g}}[f(\omega_{\vec{p}},t)] \;\; = \;\; \widehat{\mathbf{C}}_{\mathscr{M}}[f(\omega_{\vec{p}},t)] \tag{3.29}$$

where

$$\widehat{\mathbf{L}}_{\mathbf{g}} \;\; = \;\; p^{\mu}\frac{\partial}{\partial x^{\mu}} - \Gamma^{\mu}_{\;\nu\rho}\, p^{\nu}p^{\rho}\frac{\partial}{\partial p^{\mu}} \tag{3.30}$$

is the FRW Liouville operator such that

$$\widehat{\mathbf{L}}_{\mathbf{g}}[f(\omega_{\vec{p}},t)] \;\; = \;\; \omega_{\vec{p}}\,\frac{\partial}{\partial t}f(\omega_{\vec{p}},t) - H_t\,\vec{p}^{\,2}\frac{\partial}{\partial \omega_{\vec{p}}}f(\omega_{\vec{p}},t) \tag{3.31}$$

and $\widehat{\mathbf{C}}_{\mathscr{M}}$ is the quantum collision operator, are now expressed

$$\dot{n}(t) + 3H_t\, n(t) \;\; = \;\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega_{\vec{p}}}\,\widehat{\mathbf{C}}_{\mathscr{M}}[f(\omega_{\vec{p}},t)]. \tag{3.32}$$

In this context the semiclassical Boltzmann equation approximates the rate of change in the number density of asymptotic single-particle states of a quantum field, using classical N particle distribution functions, given the external gravitational force of cosmological expansion entering via the connection of general relativity $\Gamma^{\mu}_{\nu\rho}$ and the transition rates of quantum field theory $\Gamma^{>(<)}_{\mathscr{M}}$ derived from the "in-out" $\mathbf{S}$-matrix formalism.

We continue with the decay dominated scenario for the cosmological freeze-in of the stable relic χ via late time decays of the portal Φ, i.e. the decay and annihilation processes of Eqs. (3.13) and (3.14), with the nearly massless φ acting as proxy for the primordial plasma. The Boltzmann equation describing the far-from-equilibrium evolution of Φ is written as

$$
\begin{aligned}
\dot{n}_\Phi(t) + 3H_t\, n_\Phi(t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big\{[1+f_\Phi(\omega_{\vec{p}},t)]\,\Gamma^{<}_{\mathscr{M}} - f_\Phi(\omega_{\vec{p}},t)\,\Gamma^{>}_{\mathscr{M}}\Big\} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Big\{ \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3}\frac{(2\pi)^4\delta^4(p-k_1-k_2)}{4\,\omega_{\vec{k}_1}\,\omega_{\vec{k}_2}} \\
&\times \Big[|\mathscr{M}_{\varphi\varphi\to\Phi}|^2 f^{EQ}_{\vec{k}_1} f^{EQ}_{\vec{k}_2}(1+f_\Phi) - |\mathscr{M}_{\Phi\to\varphi\varphi}|^2(1+f^{EQ}_{\vec{k}_1})(1+f^{EQ}_{\vec{k}_2})f_\Phi \Big] \\
&+ \int \frac{d^3q_1}{(2\pi)^3}\int \frac{d^3q_2}{(2\pi)^3}\frac{(2\pi)^4\delta^4(p-q_1-q_2)}{4\;\omega_{\vec{q}_1}\,\omega_{\vec{q}_2}} \\
&\times \Big[|\mathscr{M}_{\chi\chi\to\Phi}|^2 f_{\vec{q}_1} f_{\vec{q}_2}(1+f_\Phi) - |\mathscr{M}_{\Phi\to\chi\chi}|^2(1+f_{\vec{q}_1})(1+f_{\vec{q}_2})f_\Phi \Big]\Big\}
\end{aligned}
\tag{3.33}
$$

given

$$\Gamma^{<}_{\mathscr{M}} := \Gamma[\mathscr{M}(\varphi\varphi\to\Phi)] + \Gamma[\mathscr{M}(\chi\chi\to\Phi)] \tag{3.34}$$

$$\Gamma^{>}_{\mathscr{M}} := \Gamma[\mathscr{M}(\Phi\to\varphi\varphi)] + \Gamma[\mathscr{M}(\Phi\to\chi\chi)] \tag{3.35}$$

via Eq. (2.123). Crucially, effects of the φ-environment are statistically encoded in the phase space factor

$$f^{EQ}_{\vec{k}_1} := \frac{1}{\exp(\beta_t\omega_{\vec{k}_1})-1} \;\text{ and }\; f^{EQ}_{\vec{k}_2} := \frac{1}{\exp(\beta_t\omega_{\vec{k}_2})-1}. \tag{3.36}$$

In conventional calculations the following simplifying assumptions are often employed in order to approximate the full statistical treatment of Eq. (3.33).

1. Assume the absence of stimulated emission such that $(1+f)\simeq 1$
2. Assume a Maxwell–Boltzmann like distribution for all fields in kinetic equilibrium such that $f^{EQ}(\omega,t) := \exp(-\beta_t\omega)$
3. Assume quantum time-reversal symmetry such that $|\mathscr{M}_{\varphi\varphi\to\Phi}|^2 = |\mathscr{M}_{\Phi\to\varphi\varphi}|^2$

Following these simplifying assumptions, Eq (3.33) becomes

$$\begin{aligned}\dot{n}_\Phi(t) + 3H_t\, n_\Phi(t) \;\; &\simeq \;\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\Bigg\{ \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3}\frac{(2\pi)^4\delta^4(p-k_1-k_2)}{4\,\omega_{\vec{k}_1}\,\omega_{\vec{k}_2}} \\ &\times \;\; |\mathscr{M}_{\Phi\to\varphi\varphi}|^2\left[f^{EQ}_{\vec{k}_1} f^{EQ}_{\vec{k}_2} - f_\Phi \right] \\ &+ \;\; \int \frac{d^3q_1}{(2\pi)^3}\int \frac{d^3q_2}{(2\pi)^3}\frac{(2\pi)^4\delta^4(p-q_1-q_2)}{4\;\omega_{\vec{q}_1}\,\omega_{\vec{q}_2}} \\ &\times \;\; |\mathscr{M}_{\Phi\to\chi\chi}|^2\Big[f_{\vec{q}_1} f_{\vec{q}_2} - f_\Phi \Big]\Bigg\}. \end{aligned} \tag{3.37}$$

Applying the principle of detailed balance

$$f^{EQ}_{\vec{k}_1} f^{EQ}_{\vec{k}_2} = \exp(-\beta_t[\omega_{\vec{k}_1} \;+\; \omega_{\vec{k}_2}]) = \exp(-\beta_t\omega_{\vec{p}}) = f^{EQ}_\Phi, \tag{3.38}$$

as well as imposing the freeze-in condition $f^{EQ}_\Phi - f_\Phi \simeq f^{EQ}_\Phi$, and ignoring the subdominant buildup in χ until late time gives

$$\dot{n}_\Phi(t) + 3H_t\, n_\Phi(t) \;\; \simeq \;\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\,\Gamma[\mathscr{M}(\Phi\to\varphi\varphi)]\; f^{EQ}_\Phi(\omega_{\vec{p}},\beta_t). \tag{3.39}$$

We now scale out the cosmological expansion by defining the number of Φ particles in a comoving volume at a time t as $N_{\Phi,t} := n_\Phi(t)/\sigma(\beta_t)$ with $\sigma(\beta_t)$ the entropy density of our scalar proxy, where

$$\sigma(\beta_t) = \frac{2\pi^2}{45\beta_t^3} \tag{3.40}$$

for $\beta_t := \mu\, t^{1/2}$ given μ as a constant with $[\mu] = [M_\Phi^{-1/2}]$, such that $S := \sigma(\beta_t)\, a_t^3$ is the conserved entropy of the comoving volume. Here, $\dot{N}_{\Phi,t}\,\sigma(\beta_t) = \dot{n}_\Phi(t) + 3H_t\, n_\Phi(t)$ and

$$N_{\Phi,\infty} \;\; \simeq \;\; \int_0^\infty \frac{dt}{\sigma(t)}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\,\Gamma[\mathscr{M}(\Phi\to\varphi\varphi)]\; f^{EQ}_\Phi(\omega_{\vec{p}},t) \tag{3.41}$$

where we have restored the explicit time dependencies for clarity.

We now wish to compute the cosmological observable of interest, i.e. the relic abundance of the massive non-relativistic asymptotic single-particle states $|\chi_{\vec{q}}\rangle$ via the decay $|\Phi_{\vec{p}}\rangle \;\to\; |\chi_{\vec{q}_1}\,,\chi_{\vec{q}_2}\rangle$ of the frozen-in asymptotic states $|\Phi_{\vec{p}}\rangle$ at a time $t_f \to \infty$. Here, the relic abundance is given as

$$\Omega_{\chi,\infty} = s_\infty\, B_\chi\, 2m_\chi\, N_{\Phi,\infty} \tag{3.42}$$

with s_∞ a constant, for $[s_\infty] = [m_\chi^{-1}]$, accounting for both the present day entropy density σ_∞ and the present day critical energy density $\mathscr{E}_{C,\infty}$; B_χ is then the branching ratio accounting for the portion of late decays into the relic χ, i.e.

$$B_\chi := \frac{\Gamma[\mathscr{M}(\Phi \to \chi\chi)]}{\Gamma[\mathscr{M}(\Phi \to \varphi\varphi)] + \Gamma[\mathscr{M}(\Phi \to \chi\chi)]} \simeq \left(1 - \frac{4m_\chi^2}{M_\Phi^2}\right)^{1/2}\left[1 + \left(1 - \frac{4m_\chi^2}{M_\Phi^2}\right)^{1/2}\right]^{-1}. \tag{3.43}$$

The relic abundance of the stable asymptotic states of χ, in the semiclassical approximation, is then

$$\Omega_{\chi,\infty} \;\simeq\; s_\infty\, 2m_\chi B_\chi \int_0^\infty \frac{dt}{\sigma(t)} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}}\, \Gamma[\mathscr{M}(\Phi \to \varphi\varphi)]\, f_\Phi^{EQ}(\omega_{\vec{p}}, t). \tag{3.44}$$

We are now interested in determining if applying the open quantum system formalism of the previous chapter to this simple scenario of cosmological freeze-in, i.e. computing the relic abundance of χ via the full non-equilibrium quantum field theory solution to the Boltzmann equation as found in Eq. (2.209), results in a substantive correction. In this case Eq. (3.44) becomes

$$\Omega^{\Pi}_{\chi,\infty} = s_\infty\, 2m_\chi\, B_\chi \int_0^\infty \frac{dt}{\sigma(t)} \int \frac{d^3p}{(2\pi)^3}\, n_\Phi^{EQ}(\omega_{\vec{p}}) \Big[\, 1 - \, \exp(-\Gamma^{\Pi}_{\vec{p}}\, t)\, \Big]. \tag{3.45}$$

Here, $\Gamma^{\Pi}_{\vec{p}} := \Gamma^{\Pi<}_{\vec{p}} - \Gamma^{\Pi>}_{\vec{p}} = \omega_{\vec{p}}^{-1}\mathrm{Im}\,[\Pi(\omega_{\vec{p}}, \beta)]$ via Eq. (2.205) and is explicitly derived by Ho and Scherrer in Ref. [4] such that, e.g.

$$\Gamma^{\Pi>}_{\vec{p}} \;=\; \frac{1}{1 - \exp(-\beta_t\omega_{\vec{p}})}\left[\, \Gamma^{>}_{\vec{p}} + \frac{\lambda^2}{4\pi\, \beta_t\, \omega_{\vec{p}}\, p}\, \ln\!\left(\frac{1 - \exp(-\beta_t\omega^+)}{1 - \exp(-\beta_t\omega^-)}\right)\right] \theta(M_\phi^2 - 4m_\varphi^2) \tag{3.46}$$

given

$$\Gamma^{>}_{\vec{p}} = \frac{\lambda^2}{8\pi\omega_{\vec{p}}}\left(1 - \frac{4m_\varphi^2}{M_\Phi^2}\right)^{1/2} \quad \text{and} \quad \omega^\pm = \frac{1}{2}\left[\omega_{\vec{p}} \pm p\left(1 - \frac{4m_\varphi^2}{M_\Phi^2}\right)^{1/2}\right]. \tag{3.47}$$

We now restore the Λ-CDM parameters to their observed physical values such that

$$\Omega_{\chi,\infty} \to \Omega_{\chi,\infty} h = \frac{2889.2\ \text{cm}^{-3}\ 2m_\chi\, B_\chi\, N_{\Phi,\infty}}{1.05375 \times 10^{-5}\ \text{cm}^{-3}\ \text{GeV}} \tag{3.48}$$

and present our intermediate result $\Delta(\Omega_\Phi)$ as a correction to the semiclassical approximation by including the full dynamics of non-equilibrium quantum field theory to order λ^2 in

Φ and to all orders in φ:

$$\Delta(\Omega_\Phi) := (\Omega_{\chi,\infty}h)^{-1}(\Omega^{\Pi}_{\chi,\infty}h - \Omega_{\chi,\infty}h) \sim \mathcal{O}(1). \tag{3.49}$$

Numerical calculations were performed over a wide range of masses M_Φ and initial plasma conditions, i.e. "reheat" temperatures, such that for all observationally allowed ranges of $\beta^{-1}(t_0) > M_\phi$ the correction $\Delta(\Omega_\Phi) \sim \mathcal{O}(1)$ persisted. It is important to note that the renormalization term via the non-equilibrium formalism, $\delta\omega_{\vec{p}} \propto \text{Re}\,\Pi(\omega_{\vec{p}}, \beta)$, contributes a β_t dependent correction to the mass term M_Φ, i.e.

$$M_\Phi \to M_\Phi + \Delta M_\Phi(\beta_t), \tag{3.50}$$

where

$$\Delta M_\Phi(\beta_t) \approx \frac{\lambda^2}{24\,\beta_t^2\,M_\Phi^2} \tag{3.51}$$

is negligible for all $\beta_t M_\Phi > 10^{-10}$ and hence unimportant in our freeze-in scenario.

Emboldened by this intermediate result, we now wish to determine if carrying out a full non-equilibrium quantum field theory in curved spacetime calculation results in additional corrections. Given that the curved spacetime of interest is additionally FRW, i.e. non-stationary, and given the previously discussed failings of the operator formalism is this arena we must seek an alternate formulation. To this end we dedicate the remainder of this work to derivations in the framework of algebraic quantum field theory in curved spacetime.

3.3 Algebraic Quantum Field Theory in Curved Spacetime

In this section we review the algebraic formulation of quantum field theory pertinent to FRW spacetime (see Refs. [52, 53] for a general introduction to the algebraic approach in the context of curved spacetime and [8, 9] for an extension to cosmological spacetime).

3.3.1 Algebraic Canonical Quantization

We consider now the classical theory of a neutral scalar field $\phi(x_\mu)$ on the globally hyperbolic spacetime $(\mathcal{M}_\Sigma, \mathbf{g})$ via the free Lagrangian

$$\mathscr{L}_0 = -\frac{1}{2}\Big(\mathrm{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2 + \xi R\phi^2\Big). \tag{3.52}$$

Here, canonical quantization is realized by constructing the Borchers–Uhlmann algebra, a topological $*$-algebra (with unit) defined as

$$\mathcal{A}(\mathcal{M}_\Sigma, \mathbf{g}) := \mathcal{A}_0(\mathcal{M}_\Sigma, \mathbf{g})/\mathcal{I}(\mathcal{M}_\Sigma, \mathbf{g}) \tag{3.53}$$

where $\mathcal{A}_0(\mathcal{M}_\Sigma, \mathbf{g}) = \bigoplus_{n=0}^{\infty} D(\mathcal{M}_\Sigma^n)$ given $D(\mathcal{M}_\Sigma^0) = \mathbb{C}$, is the free tensor algebra over $\mathcal{D}(\mathcal{M}_\Sigma)$ as the space of smooth compactly supported densities $f(x_\mu)$ on $\mathcal{M}_\Sigma$ and $\mathcal{I}(\mathcal{M}_\Sigma, \mathbf{g})$ the $*$-ideal. The free field $\phi(x_\mu)$ is henceforth denoted by the formal symbol A_x. The "smeared" fields

$$A(f) = \int_{\mathcal{M}_\Sigma} d\mu_{\mathrm{g}}\, f(x_\mu) A_x, \tag{3.54}$$

where $d\mu_{\mathrm{g}}$ is the measure on $\mathcal{M}_\Sigma$, generate the algebra $\mathcal{A}(\mathcal{M}_\Sigma, \mathbf{g})$ such that $f \to A(f)$ is $\mathbb{R}$–linear and

$$A(f)^* = A(\overline{f}) \tag{3.55}$$

$$[A(f), A(g)] = iE(f, g) \tag{3.56}$$

$$A(\widehat{K} f) = 0 \tag{3.57}$$

$\forall f, g \in \mathcal{D}(\mathcal{M}_\Sigma)$ and $A(f), A(g) \in \mathcal{A}(\mathcal{M}_\Sigma, \mathbf{g})$; while $\mathcal{I}(\mathcal{M}_\Sigma, \mathbf{g})$ is generated by elements including $\widehat{K} f$ and the causal propagator $E := E^{>} - E^{<}$ defined via the unique advanced $(>)$ and retarded $(<)$ fundamental solutions of the Klein–Gordon operator

$$\widehat{K} = (\Box_{\mathrm{g}} + m^2 + \xi R). \tag{3.58}$$

The smearing of A_x in Eq. (3.54) is necessary to overcome the infinitely many degrees of freedom encoded in the traditional field operator and may be interpreted physically as contributing to a weighted measurement of the quantum observable, i.e. measurements require quantum interactions to occur in a region of finite spatial extent over an interval of finite time.

3.3.2 Homogeneous and Isotropic States

The algebraic states $\omega : \mathcal{A} \to \mathbb{C}$, where $\omega(A^* A) \geq 0$ and $\omega(\mathbb{1}) = 1\ \forall A \in \mathcal{A}$ define the $n-$point functions $\omega(A_1 A_2 ... A_n)$. In the case of quasifree states, i.e. the Gaussian states

$$\omega(A_1 A_2 ... A_n) = \begin{cases} \sum_X \prod_{\{i,j\} \in X} \omega(A_i A_j) & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{3.59}$$

$$X \equiv \textit{the set of all possible parings } \{i,j\} \textit{ where } i<j,$$

we require the two–point function $\omega(A_iA_j)$ be of the physically admissible Hadamard form

$$\omega(A_xA_y) = \lim_{\epsilon\downarrow 0}\frac{1}{8\pi^2}\left[\frac{U(x_\mu,y_\mu)}{\sigma_\epsilon(x_\mu,y_\mu)} + V(x_\mu,y_\mu)\log\left(\frac{\sigma_\epsilon(x_\mu,y_\mu)}{L^2}\right) + F(x_\mu,y_\mu)\right] \quad (3.60)$$

where the functions U, V, and F are smooth real–valued bi-distributions and

$$\sigma_\epsilon(x_\mu,y_\mu) := \sigma(x_\mu,y_\mu) + 2i\epsilon[\tau(x_\mu)-\tau(y_\mu)] + \epsilon^2, \quad (3.61)$$

with $\sigma(x_\mu,y_\mu)$ the signed squared geodesic distance; while $\tau:\mathcal{M}_\Sigma\to\mathbb{R}$ is an arbitrary time function, and L the length scale. This allows us to extend the factored $*$-algebra to $\mathcal{W}(\mathcal{M}_\Sigma,\mathbf{g})$ such that $\mathcal{A}(\mathcal{M}_\Sigma,\mathbf{g})\subset\mathcal{W}(\mathcal{M}_\Sigma,\mathbf{g})$ where renormalization up to mass and curvature ambiguities is carried out by local and covariant Hadamard point-splitting regularization, i.e. normal ordered Wick products $W\in\mathcal{W}(\mathcal{M}_\Sigma,\mathbf{g})$ are defined in the coincidence limit

$$\omega(:A_x^2:) := \lim_{y\to x}[\omega(A_xA_y) - \mathbb{H}(x_\mu,y_\mu)] \quad (3.62)$$

given the purely geometric Hadamard parametrix $\mathbb{H}(x_\mu,y_\mu)$ as the first two terms in Eq. (3.60) and

$$\omega(:A^n(f):) := \int_{\mathcal{M}_\Sigma^n}\prod_{i=1}^{n} d\mu_{\mathrm{g}}(x_i)\; f(x_1)\delta(x_1,x_2,...x_n) : A_1A_2...A_n : . \quad (3.63)$$

Hence, the time ordered products necessary to define perturbative interactions as well as prove the spin-statistics and CPT theorems allow for reliable cosmological observables [13, 54–57].

We now pass to conformal time η such that the Klein–Gordon operator of Eq. (3.58) is rewritten as

$$\widehat{K}_\eta = \frac{1}{a_t^2}\left[\partial_\eta^2 - \vec{\nabla}^2 + a_t^2m^2 + a_t^2\left(\xi-\frac{1}{6}\right)R\right]. \quad (3.64)$$

Following the formulation of Ref. [17] with explicit constructions found in Ref. [58] the symmetric part,

$$\omega^s(A(f)A(g)) := \frac{1}{2}\Big[\omega(A(f)A(g)) + \omega(A(g)A(f))\Big], \quad (3.65)$$

of a quasifree homogeneous and isotropic states in FRW spacetimes is expressed

$$\omega^s(A(f)A(g)) = \int d^3k \int d\eta_x \int d\eta_y \; \mathfrak{X}_{\vec{k}} \left\{ \overline{X_{\vec{k}}(\eta_x)} X_{\vec{k}}(\eta_y) + X_{\vec{k}}(\eta_x) \overline{X_{\vec{k}}(\eta_y)} \right\} \overline{\hat{f}_{\vec{k}}(\eta_x)} \hat{g}_{\vec{k}}(\eta_y) \tag{3.66}$$

with, for example,

$$\hat{f}_{\vec{k}}(\eta_x) = \int \frac{d^3x}{(2\pi)^{3/2}} \, f(\eta_x, \vec{x}) \, \exp(-i\vec{k} \cdot \vec{x}) \tag{3.67}$$

as the spatial Fourier transform. The mode functions $X_{\vec{k}}(\eta)$ satisfy

$$\overline{X_{\vec{k}}(\eta)} X_{\vec{k}}(\eta)' - \overline{X_{\vec{k}}(\eta)}' X_{\vec{k}}(\eta) = i \tag{3.68}$$

given $X_{\vec{k}}(\eta)'$ as the derivative with respect to η. Members of the set of unitarily equivalent mode functions satisfying Eq. (3.68) are expressed as a Bogoliubov transformation such that

$$X_{\vec{k}}(\eta) = \mathfrak{p}_{\vec{k}} \, T_{\vec{k}}(\eta) + \mathfrak{q}_{\vec{k}} \, \overline{T_{\vec{k}}(\eta)} \tag{3.69}$$

with $|\mathfrak{p}_{\vec{k}}|^2 - |\mathfrak{q}_{\vec{k}}|^2 = 1$ and $T_{\vec{k}}(\eta)$ an arbitrary reference mode that satisfies the time portion of $\widehat{K}_\eta \, T_{\vec{k}}(\eta) = 0$. Here, $\mathfrak{X}_{\vec{k}} \geq 1/2$ is polynomially bounded in k such that equality obtains the pure state while inequality corresponds to the generic mixed state, i.e. the convex combination

$$\omega^s(A(f)A(g)) = \sum_n \lambda_n \, \omega_n^s; \;\; \lambda_n \geq 0, \; \sum_n \lambda_n = 1 \tag{3.70}$$

of at least two other mixed states ω_i^s and ω_j^s such that $\omega_i^s \neq \omega_j^s$. Crucially, in the sense of distributions, we may restrict the free field state to a Cauchy surface of constant conformal time η such that Eq. (3.66) becomes

$$\omega_\eta^s(A(f)A(g)) = 2 \int d^3k \; \mathfrak{X}_{\vec{k}} \, |X_{\vec{k}}(\eta)|^2 \, \overline{\hat{f}}_{\vec{k}} \, \hat{g}_{\vec{k}}. \tag{3.71}$$

Given the fields in our model are real scalers, there is a Gel'fand–Naimark–Segal (GNS)–representation $\pi_\omega : \mathcal{A} \to \mathcal{T}(\mathcal{D})$, where $\mathcal{T}(\mathcal{D})$ is the Banach space of linear operators on a dense domain $\mathcal{D}$ of the Hilbert space $\mathcal{H}_\omega$, with cyclic vector $\Omega_\omega \in \mathcal{D} \subset \mathcal{H}_\omega$ such that

$$\omega(A) = \langle \Omega | \pi_\omega(A) | \Omega \rangle \,, \tag{3.72}$$

where the irreducible representations $\pi_\omega(A)$ are in one-to-one correspondence with the pure algebraic states and contain the usual annihilation and creation operators over $\mathcal{D}$ as the bosonic Fock space over the one–particle space $\mathcal{H}_\omega^{(1)}$. However, for more robust models that include interacting fields of perturbative Yang–Mills theory in a general non-stationary spacetime; an equivalent correspondence with $\pm$-helicity one-particle states of the electromagnetic field is not possible [53, 55]. Hence, we continue in the algebraic framework without regard to a Hilbert space representation.

3.3.3 Ground States as States of Low Energy

We now propose generalized ground states from states of low energy (SLE) as put forward in Ref. [20] with explicit constructions in FRW spacetimes found in Refs. [8, 9]. Here, we focus on a massive minimally coupled, i.e. $\xi = 0$, free scalar field. We remind the reader that the cosmological observables of interest is the expectation value of the smeared quantum energy density

$$\mathscr{E}_{A(f)} := \omega(\, \mathrm{T}_{00}(:A^2(f):)\,) \tag{3.73}$$

consistent with the local and covariant semiclassical Einstein equation

$$\mathrm{R}_{\mu\nu}(x_\mu) - \frac{1}{2} R\, \mathrm{g}_{\mu\nu}(x_\mu) = 8\pi G\, \omega\Big(\mathrm{T}_{\mu\nu}(:A_x^2:)\Big) \tag{3.74}$$

where the RHS is interpreted as the expectation value of the free field stress–energy tensor $T_{\mu\nu}$ corresponding to the quantum matter field A_x. Here, semiclassical refers to the LHS of Eq. (3.74) as those of the classical Einstein field equations. Though quantum energy densities restricted to a point are not bound from below [18], those smeared along the worldline of an isotropic observer in FRW spacetimes do have a lower bound when Hadamard states are consider [19].

SLE are then the quasifree pure homogeneous and isotropic states specified by mode functions that minimize the energy density per mode

$$\begin{aligned} \mathscr{E}_{\vec{k}}(\eta) &= \frac{1}{2a_t^4(2\pi)^3}\Big(|X'_{\vec{k}}(\eta)|^2 - a_t H_t(|X_{\vec{k}}(\eta)|^2)' \\ &\quad + (k^2 + a_t^2 m^2 + a_t^2 H_t^2)|X_{\vec{k}}(\eta)|^2\Big) \end{aligned} \tag{3.75}$$

via the Bogoliubov coefficients of Eq. (3.69) such that

$$\mathfrak{p}_{\vec{k}} = \exp\Big(i[\pi - \arg c_2(\vec{k})]\Big)\sqrt{\frac{c_1(\vec{k})}{2\sqrt{c_1^2(\vec{k}) - |c_2(\vec{k})|^2}} + \frac{1}{2}} \tag{3.76}$$

$$\mathfrak{q}_{\vec{k}} = \sqrt{\frac{c_1(\vec{k})}{2\sqrt{c_1^2(\vec{k}) - |c_2(\vec{k})|^2}} - \frac{1}{2}} \tag{3.77}$$

where, for a comoving observer,

$$c_1(\vec{k}) \;:=\; \frac{1}{2}\int_{t_i}^{t_f} dt\, f^2(t)\bigg\{|X'_{\vec{k}}(\eta)|^2 - a_t H_t(|X_{\vec{k}}(\eta)|^2)' + (k^2 + a_t^2 m^2 + a_t^2 H_t^2)|X_{\vec{k}}(\eta)|^2\bigg\} \tag{3.78}$$

and

$$c_2(\vec{k}) \;:=\; \frac{1}{2}\int_{t_i}^{t_f} dt\, f^2(t)\bigg\{X'^2_{\vec{k}}(\eta) - a_t H_t[X^2_{\vec{k}}(\eta)]' + (k^2 + a_t^2 m^2 + a_t^2 H_t^2)X_{\vec{k}}(\eta)^2\bigg\}. \tag{3.79}$$

Convolution with the compactly supported function $f(t)$ is then taken over a finite interval of cosmological time, i.e. $t_i, t_f \in I_t \subset \mathbb{R}$. In what follows we take as our reference

$$T_{\vec{k}}(\eta) := \frac{1}{\sqrt{2\Omega_{\vec{k}}(\eta)}}\exp\bigg(-i\int_{\eta_0}^{\eta} d\bar{\eta}\,\Omega_{\vec{k}}(\bar{\eta})\bigg). \tag{3.80}$$

where

$$\Omega_{\vec{k}}(\bar{\eta}) = \sqrt{\vec{k}^2 + a_t^2 m^2 - a_t^2 R/6} \tag{3.81}$$

such that as we approach the asymptotically dS spacetime $\widetilde{\mathcal{M}_\Sigma}$

$$\lim_{\eta\to-\infty} T_{\vec{k}}(\eta) = \frac{1}{\sqrt{2k}}\exp(-ik\eta) \tag{3.82}$$

gives the Bunch–Davies vacuum. This is consistent with a bulk–to–boundary correspondence via the injective $*-$homomorphism $\alpha_f : \mathcal{A}(\widetilde{\mathcal{M}_\Sigma}) \to \mathcal{A}(\mathfrak{I}^-)$ in order to construct an induced Hadamard ground state, i.e. Bunch–Davies, on the bulk FRW spacetime [59, 60].

3.3.4 Excited States as Generalized Hadamard States

The formulation of a generalized free field state in the algebraic framework,

$$\omega^B(A_x A_y) := \frac{\omega(B_x A_x A_y B_y)}{\omega(B_x B_y)} \tag{3.83}$$

follows from a generalized Hadamard condition such that any finite excitation of a free field Hadamard state is itself a Hadamard state [61]. For example, in Minkowski spacetime the free field KMS state is indeed Hadamard and invariant under the $*-$automorphisms α_t

such that, given the global temperature parameter β^{-1},

$$\omega(\ \alpha_t(A(f))A(g)\ ) = \omega(\ A(g)\alpha_{t-i\beta}(A(f))\ ) \tag{3.84}$$

where

$$\alpha_t(A(f)) := A(\ f(\tau_0^{-1}(x_\mu))\ ) \tag{3.85}$$

for $\tau_0 : x_\mu \mapsto x_\mu + t\ \vec{e}_0$ with $\vec{e}_0$ a timelike unit vector. We direct the reader to Refs. [62, 63] for a rigorous and extensive treatment of both the vacuum and the thermal KMS state, constructed at a finite time in a Hamiltonian approach to perturbative algebraic quantum field theory in Minkowski spacetime via a distinguished time-direction using a one-parameter group of automorphisms α_t, where the interacting dynamics are related to free dynamics by a co-cycle in the algebra of the free field; for a similar treatment of non-equilibrium steady states see Ref. [64]. However, in FRW spacetimes there is no time translation invariance and hence no abelian one-parameter group of automorphisms α_t implemented as unitary operators on a corresponding Fock space [18], i.e. there is no well defined Hamiltonian as the generator of time translations and no strict notion of local thermal equilibrium in non-stationary spacetimes. This has led to several innovative and interesting frameworks, e.g. the Almost Equilibrium States of Ref. [65], Local S_x Thermal Equilibrium States found in Refs. [23], and the Bulk-to-Boundary Approximate KMS States in Ref. [66].

In this work, we invoke the notion of a propagator-family [14, 15] as a non-commutative two-parameter family of automorphisms $\alpha_{t,s}$ such that $\alpha_{t,r} = \alpha_{t,s} \circ \alpha_{s,r}$ and the following group automorphism properties are imposed to ensure the dynamics are consistent with a causal propagator:

$$\alpha_{t,t} = \mathbb{1} \tag{3.86}$$

$$\alpha_{t,s}^{-1} = \alpha_{s,t} \tag{3.87}$$

$$\alpha_{t,s}(A_s B_s) = \alpha_{t,s}(A_s)\ \alpha_{t,s}(B_s). \tag{3.88}$$

We define the evolution of the state via the composition

$$\omega_t(A) := \omega_s(A) \circ \beta_{s,t} = \omega(\ \alpha_{t,s}(A_s)\ ) \tag{3.89}$$

where $\beta_{r,t} = \beta_{r,s} \circ \beta_{s,t}$. The infinitesimal generators of time shifts are then defined via the relations

$$\dot{\alpha}_{t,s} \;=\; d_t \circ \alpha_{t,s} \tag{3.90}$$

$$\dot{\beta}_{s,t} \;=\; \beta_{s,t} \circ \delta_t \tag{3.91}$$

where

$$d_t : \;=\; \lim_{\Delta t \to 0} \frac{\alpha_{t+\Delta t,t} - \alpha_{t,t}}{\Delta t} \tag{3.92}$$

$$\delta_t : \;=\; \lim_{\Delta t \to 0} \frac{\beta_{t,t+\Delta t} - \beta_{t,t}}{\Delta t} \tag{3.93}$$

such that

$$\dot{\alpha}_{t,s}(A_s B_s) = \dot{\alpha}_{t,s}(A_s) B_t + A_t \, \dot{\alpha}_{t,s}(B_s). \tag{3.94}$$

We may not equate Eq. (3.92) with the Heisenberg equation of motion in non-stationary spacetimes; however, we may define a generator of a perturbed time shift via the relation

$$\delta_t^P(A) := [iP_t, A] \tag{3.95}$$

given

$$\dot{\beta}_{s,t}^P = \beta_{s,t}^P \circ (\delta_t + \delta_t^P). \tag{3.96}$$

with $\beta_{t,t}^P = \mathbb{1}$ and time dependent perturbation P_t. Hence, we let

$$\beta_{t_i,t_f}^P(A_{t_i}) := \mathfrak{U}(t_f, t_i)^{-1} \beta_{t_i,t_f}(A_{t_i}) \, \mathfrak{U}(t_f, t_i) \tag{3.97}$$

where

$$\mathfrak{U}(t_f, t_i) := \mathrm{T}\Big[\exp\Big(- i \int_{t_i}^{t_f} dt \; \beta_{s,t}(P_s) \Big)\Big] \tag{3.98}$$

with T[...] as the time ordered product and $\mathfrak{U}(t_f, t_i)^{-1} := \mathfrak{U}(t_i, t_f)$ such that

$$\omega_{t_f}^P(A) := \omega(\, \beta_{t_i,t_f}^P(A_{t_i}) \,) = \omega(\, \alpha_{t_f,t_i}(A_{t_i}) \,) \circ \gamma_{t_i,t_f} \tag{3.99}$$

given $\gamma_{t_i,t_f} := \mathrm{Ad}\; \mathfrak{U}(t_f, t_i)^{-1}$. The generalized excited state may now be written as

$$\omega_{t_f}^P(AA) = \frac{\omega(\, \beta_{t_f,t_i}^P(A_{t_i} A_{t_i}) \,)}{\omega(\, \beta_{t_f,t_i}^P(\mathbb{1}) \,)} = \frac{\omega(\, \alpha_{t_f,t_i}(A_{t_i}) \, \alpha_{t_f,t_i}(A_{t_i}) \,) \circ \gamma_{t_i,t_f}}{\omega(\mathbb{1}) \circ \gamma_{t_i,t_f}}. \tag{3.100}$$

3.3.5 Excited States via Generalized Perturbative Interactions

We begin with the classical Lagrangian $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_I$ given an interaction term of the general form

$$\mathscr{L}_I := -\sum_i \kappa_i \Phi_i, \tag{3.101}$$

where κ_i as a perturbative coupling parameter and Φ_i as any polynomial in the field ϕ. Interacting time ordered products as elements of the free field algebra are in general expressed via Bogoliubov's formula

$$\mathrm{T}\left[\prod_{i=1}^{m}\int d\mu_i\, f_i\Phi_i\right] = \sum_n \frac{i^n}{n!}\mathrm{R}_n\left[\prod_{i=1}^{m}\int d\mu_i\, f_i\Phi_i;\ \int d\mu\, \theta\mathscr{L}_I^{\otimes n}\right], \tag{3.102}$$

where $\mathrm{R}_n[...]$ is the retarded product to order n, as defined in Sec. 4.1 of ref [13] and $\theta \in \mathcal{D}(\mathcal{M}_{\mathbf{g}})$ a smooth function of compact support. This is of course a well studied perturbative power series with no expectation of convergence and we do not rigorously prove the existence of P_t here. Instead, we invoke the axioms and analysis of Ref. [13] such that if the perturbative quantum field theory satisfies the field equations in the presence of an arbitrary classical current source $J(x_\mu)$ we may at least rely on Wick polynomials $W^J \in \mathcal{W}(\mathcal{M}_\Sigma, \mathbf{g}, J)$ as self-interactions in the form of an arbitrary but finite n^{th} order perturbative correction to ϕ and more generally on the existence of time ordered products and a conserved stress-energy tensor. Hence, we follow Ref. [13] in constructing an interacting theory with $\mathscr{L}_I$ given by the very general, yet nontrivial, classical interaction Lagrangian

$$\mathscr{L}_I = -J(x_\mu)\phi(x_\mu). \tag{3.103}$$

The interacting quantum theory, now generated by elements of $\mathcal{W}(\mathcal{M}_\Sigma, \mathbf{g}, J)$, is constructed such that eqs. (3.55) and (3.56) remain satisfied by the sourced $A^J(f)$ and $\mathcal{W}(\mathcal{M}_\Sigma, \mathbf{g}, J) \to \mathcal{W}(\mathcal{M}_\Sigma, \mathbf{g})$ via the relation

$$A^J(\widehat{K}f) = \int d\mu_{\mathbf{g}}\, f(x_\mu)J(x_\mu)\cdot \mathbb{1} \tag{3.104}$$

where Eq. (3.57) is recovered in the case of a vanishing source.

In order to derive a general expression for the excited state $\omega^P_{t_f}(A^J(f)A^J(g))$ we express the time averaged perturbation as

$$P_{t_f} = \kappa\int_{t_i}^{t_f} dt_u \int d^3u\, \theta(t_u, \vec{u})W^J_u \tag{3.105}$$

with $\theta(t_u, \vec{u}) = h(t_u)\psi(\vec{u})$ such that the adiabatic limit corresponds to the constant function

$$\psi(\vec{u}) = 1 \text{ on supp } f \subset \mathcal{M}_\Sigma^{(I_t)} = \{(t_u, \vec{u}) |\, t_i < t_u < t_f\}. \tag{3.106}$$

Hence,

$$\begin{aligned} \mathfrak{U}(t_f, t_i) &= \mathbb{1} - i\kappa \int_{t_i}^{t_f} dt_u \int d^3u\, h(t_u) W_u^J \\ &\quad - \frac{\kappa^2}{2} \int_{t_i}^{t_f} dt_u \int_{t_i}^{t_f} dt_v \int d^3u \int d^3v\, h(t_u) h(t_v)\, \mathrm{T}\Big[W_u^J W_v^J\Big] \end{aligned} \tag{3.107}$$

truncated to second order in κ. The perturbed state $\omega_{t_f}^P(AA)$ of Eq. (3.100), rewritten as

$$\omega_{t_f}^P(A^J(f)A^J(g)) = \frac{\omega\Big(\, \mathfrak{U}^{-1}(t_i, t_f)\; A^J(f)^J A(g)\; \mathfrak{U}(t_i, t_f)\,\Big)}{\omega\Big(\, \mathfrak{U}^{-1}(t_i, t_f)\; \mathbb{1}\; \mathfrak{U}(t_i, t_f)\,\Big)}, \tag{3.108}$$

may now be expressed

$$\begin{aligned} \omega_{t_f}^P(A^J(f)A^J(g)) &= \Big\{ \omega(A^J(f)A^J(g)) + \omega\Big(\frac{\kappa^2}{2} \int_{t_i}^{t_f} dt_u \int d^3u \int_{t_i}^{t_f} dt_v \int d^3v\, h(t_u)h(t_v)\Big[\\ &\quad \overline{\mathrm{T}}[W_u^J]\; A^J(f)\; A^J(g)\; \mathrm{T}[W_v^J] - A^J(f)\; A^J(g)\; \mathrm{T}[W_u^J\; W_v^J] \\ &\quad + \mathrm{T}[W_u^J]\; A^J(f)\; A^J(g)\; \overline{\mathrm{T}}[W_v^J] - \overline{\mathrm{T}}[W_u^J\; W_v^J]\; A^J(f)\; A^J(g)\, \Big]\Big) \Big\} \\ &\times \quad \Big\{ \omega(\mathbb{1}) + \omega\Big(\frac{\kappa^2}{2} \int_{t_i}^{t_f} dt_u \int d^3u \int_{t_i}^{t_f} dt_v \int d^3v\, h(t_u)h(t_v)\Big[\\ &\quad \overline{\mathrm{T}}[W_u^J]\; \mathrm{T}[W_v^J] - \mathrm{T}[W_u^J\; W_v^J] + \mathrm{T}[W_u^J]\; \overline{\mathrm{T}}[W_v^J] - \overline{\mathrm{T}}[W_u^J\; W_v^J]\,\Big]\Big)\Big\}^{-1} \end{aligned} \tag{3.109}$$

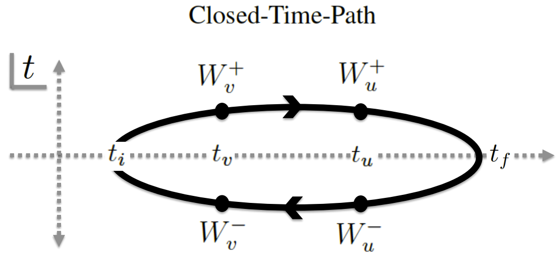

Figure 3.3: Closed–time–path evolution for the finite macroscopic cosmological time interval $t_i, t_f \in I_t$, given $t_i < t_v < t_u < t_f$ on the forward(+) branch $t_f < t_u < t_v < t_i$ on the backward(-) branch.

or

$$\begin{aligned}\omega^P_{t_f}(A(f)A(g)) &= Z_\omega\Big\{\omega_{t_f}(A(f)A(g)) \\ &+ \omega\Big(\frac{\kappa^2}{2}\int_{t_i}^{t_f} dt_x \int d^3x \int_{t_i}^{t_f} dt_u \int d^3u \int_{t_i}^{t_f} dt_v \int d^3v \int_{t_i}^{t_f} dt_y \int d^3y \Big[\\ &\quad f(t_x,\vec{x})h(t_u)h(t_v)g(t_y,\vec{y})\Big(\\ &\quad W_u^- A_x^- A_y^+ W_v^+ - A_x^- A_y^+ W_u^+ W_v^+ + W_v^- A_x^- A_y^+ W_u^+ - W_v^- W_u^- A_x^- A_y^+\Big)\Big)\Big]\Big\}\end{aligned} \tag{3.110}$$

where

$$\begin{aligned}Z_\omega := 1 \;-\; \omega_{t_f}\Big(\frac{\kappa^2}{2}\int_{t_i}^{t_f} dt_u \int d^3u \int_{t_i}^{t_f} dt_v \int d^3v \, h(t_u)g(t_v)\Big[\\ W_u^- W_v^+ - W_u^+ W_v^+ + W_v^- W_u^+ - W_v^- W_u^-\Big]\Big)\end{aligned} \tag{3.111}$$

is a state dependent normalization factor. Here, we replace the superscript J of the sourced field with the time-ordering index $\pm$ corresponding to the forward(+) and backward(-) branch of the closed-time-path depicted in Fig. 3.3. In addition, we relabel A_x^J as A_x^- and A_y^J as A_y^+ to denote that no point $x_\mu \in$ supp $f(x_\mu)$ is in the past of $y_\mu \in$ supp $g(y_\mu)$. This is equivalent to the Schwinger–Keldysh "in–in" formalism and hence appropriate for far-from-equilibrium interactions. Notably, the CPT theorem in FRW spacetimes relates an in-state in an expanding universe to an in-state in the corresponding contracting universe [57]. We also note that both the excitation and the perturbative portion of the normalization factor, i.e. the terms proportional to κ^2, are finite via the properties of Hadamard states.

Equivalently, we may write the symmetric part of the perturbed state as a homogeneous and isotropic quasifree state restricted to the Cauchy surface Σ_t

$$\omega_t^{S,P}(A(f)A(g)) \;=\; 8\pi \int dk\, k^2\, \mathfrak{X}_{\vec{k}}^{P_t}\, |X_{\vec{k}}(t)|^2\, \overline{\hat{\hat{f}}}_{\vec{k}}\, \hat{g}_{\vec{k}} \tag{3.112}$$

for all $t > t_f$ where, for example, the source $J(x_\mu)$ vanishes for $t \notin I_t$. Here, $\mathfrak{X}_{\vec{k}}^{P_t}$ is now the polynomialy bounded function perturbed via the generator P_t and $X_{\vec{k}}(t)$ the mode functions defined in Eq. (3.69), such that

$$\begin{aligned}\mathfrak{X}_{\vec{k}}^{P_t} \;=\; & Z_\omega\Bigg[\mathfrak{X}_{\vec{k}}^{0} + \kappa^2\,\omega_t\Bigg(\widehat{\mathfrak{D}}_k\Bigg[\int_{t_i}^{t} dt_x \int d^3x \int_{t_i}^{t_f} dt_u \int d^3u \int_{t_i}^{t_f} dt_v \int d^3v \int_{t_i}^{t} dt_y \int d^3y \Bigg\{ \\ & f(t_x,\vec{x})g(t_y,\vec{y})h(t_u)h(t_v) \\ & \times\Big[W_u^- A_x^- A_y^+ W_v^+ - A_x^- A_y^+ W_u^+ W_v^+ + W_v^- A_x^- A_y^+ W_u^+ - W_v^- W_u^- A_x^- A_y^+\Big]\Bigg\}\Bigg)\Bigg]\end{aligned} \tag{3.113}$$

given $\mathfrak{X}_{\vec{k}}^{0}$ as the polynomially bounded function, as defined in Eq. (3.66), for the state $\omega_0(A(f)A(g))$ specified at the time t_i and the differential operator $\widehat{\mathfrak{D}}_k$ defined as

$$\widehat{\mathfrak{D}}_k := \left(8\pi\, k^2\, |X_{\vec{k}}(t)|^2\, \overline{\hat{\hat{f}}}_{\vec{k}}\, \hat{g}_{\vec{k}}\right)^{-1} \frac{d}{dk}. \tag{3.114}$$

3.4 Renormalized Energy Density from the Algebraic State

In this section we derive Eq. (3.132) as the general form of the renormalized, perturbed energy density via interacting quantum fields in cosmological spacetimes. This constitutes the main result of this work and was derived for the first time by this author in Ref. [28].

3.4.1 General Form from the Perturbed Stated

Following the formulations and results in Refs. [58, 8, 9], we now review the expectation value of the energy density of a minimally coupled, i.e. $\xi = 0$, free scalar field in an arbitrary mixed state propagating in a non-stationary FRW spacetime background. We begin with the expectation value of the renormalized stress-energy tensor, taken in the free field limit at a finite cosmological time $t > t_f$ and restricted to the total diagonal such that

$$\begin{aligned}&\omega(\,\mathbf{T}_{\mu\nu}(:A_x^2:)\,) = \\ &\left.\left\{\omega^{S,P}\Big(\widehat{D}_{x,y}[A_x A_y]\Big) - \widehat{D}_{x,y}\mathbb{H}_1^s(x_\mu,y_\mu) + \frac{1}{3}\widehat{K}_x\,\mathbb{H}_1^s(x_\mu,y_\mu) + C_{\mu\nu}(x_\mu)\right\}\right|_{x_\mu = y_\mu}.\end{aligned} \tag{3.115}$$

Here, the bi–differential operator $\widehat{D}$ is defined

$$\widehat{D}_{a,b} := \frac{1}{2}\bigg(\frac{\partial}{\partial_0^a}\frac{\partial}{\partial_0^b} + \frac{1}{a_t^2}\nabla^a\nabla^b + m^2\bigg) \tag{3.116}$$

and the purely geometric Hadamard parametrix is expressed

$$\mathbb{H}_n(x_\mu, y_\mu) = \lim_{\epsilon\downarrow 0}\frac{1}{4\pi^2}\bigg[\frac{1}{\sigma_\epsilon(x_\mu, y_\mu)} + \frac{1}{L^2}\sum_{m=1}^{n} V_m\bigg(\frac{\sigma(x_\mu, y_\mu)}{L^2}\bigg)^m \log\bigg(\frac{\sigma_\epsilon(x_\mu, y_\mu)}{L^2}\bigg)\bigg], \tag{3.117}$$

where V_m satisfies the so called Hadamard recursion relations (see e.g. Ref. [58]).

$$\mathbb{H}_1^s(x_\mu, y_\mu) := \frac{1}{2}\bigg(\mathbb{H}_1(x_\mu, y_\mu) + \mathbb{H}_1(y_\mu, x_\mu)\bigg) \tag{3.118}$$

is then the symmetric Hadamard bi–distribution truncated to order $n = 1$ where

$$V_1 = -\frac{1}{3}\widehat{K}_x\mathbb{H}_1^s(x_\mu, y_\mu) \tag{3.119}$$

and $C_{\mu\nu}(x_\mu)$ carries the renormalization freedom of Wick products contained in a conserved stress-energy tensor.

The renormalized energy density of the perturbed state, taken in the free field limit for $\eta(t) > \eta_f$, is found via the restriction of the stress-energy tensor; first to the partial diagonal $\eta_x = \eta_y = \eta(t)$ then in the coincidence limit $\vec{x} = \vec{y}$ such that

$$\mathscr{E}_A^{P_{\eta(t)}} := \omega^P_{\eta(t)}(\ \mathrm{T}_{00}(:A^2(f):)\ ). \tag{3.120}$$

This is a nuanced expression that we briefly explain term by term. Here,

$$\begin{aligned}\omega^P\bigg(\widehat{D}_{x,y}[A(f)A(g)]\bigg)\bigg|_{x=y} \quad = \quad & \frac{1}{2\pi^2}\int_0^\infty dk\,\bigg\{k^2\ \mathfrak{X}_{\vec{k}}^{P_{\eta(t)}}\ \frac{1}{a_t^4}\bigg[|X'_{\vec{k}}(\eta)|^2 - a_t H_t(|X_{\vec{k}}(\eta)|^2)' \\ & + (k^2 + a_t^2 m^2 + a_t^2 H_t^2)|X_{\vec{k}}(\eta)|^2\bigg]\bigg\}\end{aligned} \tag{3.121}$$

is the divergent mode integral with mode functions $X_{\vec{k}}(\eta)$ found via the SLE minimized energy density of the ground state and the polynomially bounded function $\mathfrak{X}_{\vec{k}}^{P_\eta(t)}$ determined by the perturbation $P_{\eta(t)}$ via Eq. (3.113).

$$\begin{aligned}\widehat{D}_{x,y}\mathbb{H}^s_1(f,g)|_{x=y} &= \frac{1}{4\pi^2}\Bigg[-\frac{1}{a_t^4}\frac{2}{r_+^4}(f)+\frac{m^2+H_t^2}{2a_t^2}\frac{1}{r_+^2}(f) \\ &+ \left(\frac{m^4}{16}-\frac{2m^2H_t^2}{16}+\frac{2\ddot{H}_tH_t}{16}+\frac{6\dot{H}_tH_t^2}{16}-\frac{\dot{H}_t^2}{16}\right)\left(\mathrm{lo}_0(f)+\log(a_t^2)\right) \\ &+ \frac{\Box_{\mathrm{g}}R}{120}+m^2\left(\frac{7H_t^2}{24}+\frac{\dot{H}_t}{4}\right)-\frac{m^4}{8}+\frac{H_t^4}{80}-\frac{11H_t\ddot{H}_t}{120} \\ &- \frac{61H_t^2\dot{H}_t}{120}-\frac{19\dot{H}_t^2}{240}\Bigg], \end{aligned} \tag{3.122}$$

where the singular counterterms given by the symmetric distributions r_+^4, r_+^2, and lo_0 are defined via the convolutions

$$\frac{2}{r_+^4}(f) := \lim_{\epsilon\to+0}\int_{\mathbb{R}^3} d^3x\,\frac{\nabla f(\vec{x})}{\vec{x}^2+\epsilon^2} \tag{3.123}$$

$$\frac{1}{r_+^2}(f) := \lim_{\epsilon\to+0}\int_{\mathbb{R}^3} d^3x\,\frac{f(\vec{x})}{\vec{x}^2+\epsilon^2} \tag{3.124}$$

$$\mathrm{lo}_0(f) := \int_{\mathbb{R}^3} d^3x\, f(\vec{x})\log(\vec{x}^2), \tag{3.125}$$

for a fixed $f\in C_0^\infty(\mathbb{R}^3)$ are the geometric contribution of the parametrix. Here, the sum of the singular terms may then be rewritten as a mode integral, i.e.

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\frac{1}{2\pi^2}\int dk\,k^2 I(k)\exp(i[\vec{k}\cdot\vec{x}+ik\epsilon]) &= \\ \frac{1}{2\pi^2}\Bigg\{ &- \mathfrak{C}_{-1}\frac{2}{r_+^4}(f)+\mathfrak{C}_0\frac{1}{r_+^2}(f)+\mathfrak{C}_1\mathrm{lo}_0(f) \\ &+ 4\pi\int_{\mathbb{R}^3} d^3x\,f(\vec{x})\lim_{M\to\infty}\Bigg[\int_0^M dk\,k\Bigg(k\,I(k) \\ &- \mathfrak{C}_{-1}k^2-\mathfrak{C}_0\Bigg)-\mathfrak{C}_1\Bigg(\log(ML)-1+\gamma_{EM}\Bigg)\Bigg]\Bigg\}\end{aligned} \tag{3.126}$$

where γ_{EM} is the Euler–Mascheroni constant and the integrand $I(k)$ has asymptotic behavior

$$I(k\to\infty)=\sum_{m=-1}^{1}\frac{\mathfrak{C}_m}{k^{2m+1}}+\mathcal{O}(k^{-5}), \tag{3.127}$$

such that the subtraction of singular terms may occur inside the mode integral of Eq.

(3.121); and

$$\frac{1}{3}\widehat{K}_\eta \mathbb{H}_1^s(f,g)|_{x=y} = \frac{1}{4\pi^2}\left(\frac{3\dot{H}_t^2}{40} + \frac{\dddot{H}_t}{20} + \frac{7H_t^2\dot{H}_t}{60} + \frac{7H_t\ddot{H}_t}{20} - \frac{29H_t^4}{60} - \frac{m^4}{8} + \frac{m^2H_t^2}{2} + \frac{m^2\dot{H}_t}{4}\right). \tag{3.128}$$

$$C_{00}(\,\eta(t)\,) \;:=\; \mathfrak{c}_1 m^4 \mathrm{g}_{00} + \mathfrak{c}_2 m^2 \mathrm{G}_{00} + \;(3\mathfrak{c}_3 + \mathfrak{c}_4)(6\dot{H}_t^2 - 12\ddot{H}_t H_t - 36\dot{H}_t H_t^2) \tag{3.129}$$

allows for a renormalization freedom via the coefficients $\mathfrak{c}_{\{1,2,3,4\}}$, which are not fixed *a priori* in the theory. However, they may be constrained either by experiment or physical arguments. This is to say that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ correspond to a renormalization of the cosmological constant Λ and Newton's constant G respectively, as quantities appearing in Einstein's equation, while the sum $(3\mathfrak{c}_3 + \mathfrak{c}_4)$ is constrained by higher order derivative corrections to the semiclassical field equations. In this work we take the position that $\mathfrak{c}_{\{2,3,4\}}$ are not free parameters at the length scale, L of Eq. (3.60), probed by current experiments that support the Λ-CDM model and we omit the afforded freedom. However, we do embrace renormalization of the vacuum energy density where the requirement that this scheme reduces to normal ordering [8, 13, 56], i.e. subtraction of $\mathscr{E}_A^0$ as the reference state in Minkowski spacetime where

$$\mathscr{E}^0_{A,t_i} = \frac{1}{2}\int_0^\infty \frac{d^3k}{(2\pi)^3}\left\{|T_{\vec{k}}(t_i)'|^2 + \Omega_{\vec{k}}|T_{\vec{k}}(t_i)|^2\right\} \tag{3.130}$$

fixes $\mathfrak{c}_1$ as a function of L such that

$$\mathfrak{c}_1(L)\; m^4 \mathrm{g}_{00} = -\frac{m^4}{32\pi^2}\left(\log(mL) - \log(2) - \frac{3}{4} + \gamma_{EM}\right)\mathrm{g}_{00}. \tag{3.131}$$

Hence, we find as our main result the general expression for $\mathscr{E}_A^{P_{\eta(t)}}$ as the renormalized, perturbed energy density of a massive, minimally coupled scalar field in the free field limit to be

$$
\begin{aligned}
\mathscr{E}_A^{P_{\eta(t)}} &= \frac{1}{2\pi^2}\int_0^\infty dk \Bigg\{ \frac{k^2}{a_t^4}\Big[|X'_{\vec{k}}(\eta)|^2 - a_t H_t(|X_{\vec{k}}(\eta)|^2)' + (k^2 + a_t^2 m^2 + a_t^2 H_t^2)|X_{\vec{k}}(\eta)|^2\Big] \\
&\times \; Z_\omega \Bigg[\mathfrak{X}_{\vec{k}}^0 + \omega_{\eta(t)}\Bigg(\int_{\eta_i}^{\eta(t)} d\eta_x \int d^3x \int_{\eta_i}^{\eta(t)} d\eta_y \int d^3y \; f(\eta_x,\vec{x})g(\eta_y,\vec{y}) \\
&\quad \times \int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \int_{\eta_i}^{\eta_f} d\eta_v \int d^3v \; h(\eta_u)h(\eta_v) \Bigg\{ \\
&\quad \widehat{\mathfrak{D}}_k \Big[W_u^- A_x^- A_y^+ W_v^+ - A_x^- A_y^+ W_u^+ W_v^+ \\
&\qquad + W_v^- A_x^- A_y^+ W_u^+ - W_v^- W_u^- A_x^- A_y^+ \Big] \Bigg\} \Bigg)\Bigg|_{x=y} \Bigg] \\
&- \; k\frac{1}{2a_t^4} - \frac{1}{k}\frac{H_t^2 + m^2}{4a_t^2} - \frac{1}{k^3}\left(\frac{m^4 - 2m^2 H_t^2 + 2\ddot{H}_t H_t + 6\dot{H}_t H_t^2 - \dot{H}_t^2}{16}\right) \Bigg\} \\
&- \; \frac{m^2 H_t^2}{96\pi^2} - \frac{m^4(1 - 4\log(2))}{128\pi^2} + \frac{12H_t^4 + 48H_t^2\dot{H}_t + 36\dot{H}_t^2}{96\pi^2}.
\end{aligned}
\tag{3.132}
$$

3.4.2 Trilinear Interaction Example

As an example of a concrete realization of Eq. (3.132) that is in principal amenable to a numerical calculation we choose here the perturbation P_t to be the product

$$
P_\eta = \kappa \int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \; \theta(\eta_u, \vec{u}) A_u B_u C_c \tag{3.133}
$$

corresponding to a trilinear scalar interaction with classical Lagrangian

$$
\mathscr{L}_I = -\kappa \phi_1 \phi_2 \phi_3. \tag{3.134}
$$

We concede that such a product is not the self-interacting Wick polynomial W_u employed in the previous section, however we believe this example highlights key features of the perturbed energy density and is thus useful. In this instance, the perturbed stated may be written

$$\omega^P_{\eta(t)}(A(f)A(g)) = Z_\omega\Big\{\omega^0_{\eta(t)}(A(f)A(g)) + \omega\Big(\frac{\kappa^2}{2}\int_{\eta_i}^{\eta(t)} d\eta_x \int d^3x \int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \int_{\eta_i}^{\eta_f} d\eta_v \int d^3v \int_{\eta_i}^{\eta(t)} d\eta_y \int d^3y \Big[f(\eta_x,\vec{x})g(\eta_y,\vec{y})h(\eta_u)\psi(\vec{u})h(\eta_v)\psi(\vec{v}) \times \Big(A^-_u B^-_u C^-_u A^-_x A^+_y A^+_v B^+_v C^+_v - A^-_x A^+_y A^+_u B^+_u C^+_u A^+_v B^+_v C^+_v + A^-_v B^-_v C^-_v A^-_x A^+_y A^+_u B^+_u C^+_u - A^-_v B^-_v C^-_v A^-_u B^-_u C^-_u A^-_x A^+_y\Big)\Big]\Big)\Big\} \tag{3.135}$$

where

$$Z_\omega = 1 - \omega\Big(\frac{\kappa^2}{2}\int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \int_{\eta_i}^{\eta_f} d\eta_v \int d^3v \int d^3y \Big\{h(\eta_u)\psi(\vec{u})h(\eta_v)\psi(\vec{v}) \times \Big[A^-_u B^-_u C^-_u A^+_v B^+_v C^+_v - A^+_u B^+_u C^+_u A^+_v B^+_v C^+_v + A^-_v B^-_v C^-_v A^+_u B^+_u C^+_u - A^-_v B^-_v C^-_v A^-_u B^-_u C^-_u\Big]\Big\}\Big). \tag{3.136}$$

Taking, as an example, the first term of order κ^2 in Eq. (3.135), defined as $\omega(u^-x^-y^+v^+)$, we may expand it as a homogeneous and isotropic quasifree state such that

$$\omega(u^-x^-y^+v^+) := Z_\omega\omega\Big(\frac{\kappa^2}{2}\int_{\eta_i}^{\eta(t)} d\eta_x \int d^3x \int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \int_{\eta_i}^{\eta_f} d\eta_v \int d^3v \int_{\eta_i}^{\eta(t)} d\eta_y \int d^3y \Big[f(\eta_x,\vec{x})g(\eta_y,\vec{y})h(\eta_u)\psi(\vec{u})h(\eta_v)\psi(\vec{v})\ A^-_u B^-_u C^-_u A^-_x A^+_y A^+_v B^+_v C^+_v\Big\}\Big) \tag{3.137}$$

becomes

$$\omega(u^-x^-y^+v^+) = Z_\omega\,\omega\Big(\frac{\kappa^2}{2}\int_{\eta_i}^{\eta(t)} d\eta_x \int d^3x \int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \int_{\eta_i}^{\eta_f} d\eta_v \int d^3v \int_{\eta_i}^{\eta(t)} d\eta_y \int d^3y \Big[f(\eta_x,\vec{x})g(\eta_y,\vec{y})h(\eta_u)\psi(\vec{u})h(\eta_v)\psi(\vec{v}) \times \Big[\omega(A^-_u A^-_x)\omega(A^+_y A^+_v)\omega(B^-_u B^+_v)\omega(C^-_u C^+_v) + \omega(A^-_x A^+_v)\omega(A^-_u A^+_y)\omega(B^-_u B^+_v)\omega(C^-_u C^+_v) + \omega(A^-_x A^+_y)\omega(A^-_u A^+_v)\omega(B^-_u B^+_v)\omega(C^-_u C^+_v)\Big]\Big\} \tag{3.138}$$

given the cluster property $\omega(A_u B_v) = \omega(A_u)\omega(B_v)\ \forall t \notin I_t$. In addition, we may simplify Eq. (3.138) via a cancellation of terms of the form

$$
\begin{aligned}
& - \omega(A(f)A(g))\,\frac{\kappa^2}{2}\int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \int_{\eta_i}^{\eta_f} d\eta_v \int d^3v \Big\{ h(\eta_u)\psi(\vec{u})h(\eta_v)\psi(\vec{v}) \\
& \times \Big[\omega(A_u^- A_v^+)\omega(B_u^- B_v^+)\omega(C_u^- C_v^+)\Big]\Big\}
\end{aligned}
$$

by a similar expansion of Z_ω where we now write

$$
\begin{aligned}
\omega(u^- x^- y^+ v^+) \;=\; & \frac{\kappa^2}{2}\int_{\eta_i}^{\eta(t)} d\eta_x \int d^3x \int_{\eta_i}^{\eta_f} d\eta_u \int d^3u \int_{\eta_i}^{\eta_f} d\eta_v \int d^3v \int_{\eta_i}^{\eta(t)} d\eta_y \int d^3y \Big[\\
& f(\eta_x,\vec{x})g(\eta_y,\vec{y})h(\eta_u)\psi(\vec{u})h(\eta_v)\psi(\vec{v}) \\
& \times \Big[\omega(A_u^- A_x^-)\omega(A_y^+ A_v^+)\omega(B_u^- B_v^+)\omega(C_u^- C_v^+) \\
& \;+\; \omega(A_x^- A_v^+)\omega(A_u^- A_y^+)\omega(B_u^- B_v^+)\omega(C_u^- C_v^+)\Big]\Big\}.
\end{aligned} \tag{3.139}
$$

Taking the limit $\psi \to 1$ and carrying out the spatial integrals we find

$$
\begin{aligned}
\omega(u^- x^- y^+ v^+) \;=\; & \kappa^2 \int d^3k \int d^3p \int_{\eta_i}^{\eta(t)} d\eta_x \int_{\eta_i}^{\eta(t)} d\eta_y \int_{\eta_i}^{\eta_f} d\eta_u \int_{\eta_i}^{\eta_f} d\eta_v \,\Big\{ \\
& \overline{\hat{f}}_{\vec{k}}(\eta_x)\hat{g}_{\vec{k}}(\eta_y)h(\eta_u)h(\eta_v) \\
& \times \mathfrak{X}_{\vec{k}}^{\eta(t)}\Big(\overline{X_{\vec{k}}(\eta_x)}X_{\vec{k}}(\eta_u) + X_{\vec{k}}(\eta_x)\overline{X_{\vec{k}}(\eta_u)}\Big) \\
& \times \mathscr{X}_{\vec{k}}^{-+}\Big(\overline{X_{\vec{k}}(\eta_v)}X_{\vec{k}}(\eta_y) + X_{\vec{k}}(\eta_v)\overline{X_{\vec{k}}(\eta_y)}\Big) \\
& \times \mathscr{Y}_{\vec{p}}^{-+}\Big(\overline{Y_{\vec{p}}(\eta_u)}Y_{\vec{p}}(\eta_v) + Y_{\vec{p}}(\eta_u)\overline{Y_{\vec{p}}(\eta_v)}\Big) \\
& \times \mathscr{Z}_{\vec{k}-\vec{p}}^{-+}\Big(\overline{Z_{\vec{k}-\vec{p}}(\eta_u)}Z_{\vec{k}-\vec{p}}(\eta_v) + Z_{\vec{k}-\vec{p}}(\eta_u)\overline{Z_{\vec{k}-\vec{p}}(\eta_v)}\Big)\Big\},
\end{aligned} \tag{3.140}
$$

via the construction of the homogeneous and isotropic Hadamard states of Eq. (3.66). Here, we introduce a more compact notation with the expression

$$\begin{aligned}\omega_{\eta(t)}(u^- x^- y^+ v^+) \quad = \quad & \kappa^2 \int d^3k \int_{\eta_i}^{\eta(t)} d\eta_x \int_{\eta_i}^{\eta(t)} d\eta_y \int_{\eta_i}^{\eta_f} d\eta_u \int_{\eta_i}^{\eta_f} d\eta_v \, \Bigg\{ \\ & \overline{\hat{\overline{f}}}_{\vec{k}}(\eta_x) \hat{g}_{\vec{k}}(\eta_y) h(\eta_u) h(\eta_v) \\ & \times \mathfrak{X}^{\eta(t)}_{\vec{k}} \Big(\overline{X_{\vec{k}}(\eta_x)} X_{\vec{k}}(\eta_u) + X_{\vec{k}}(\eta_x) \overline{X_{\vec{k}}(\eta_u)} \Big) \\ & \times \mathscr{X}^{-+}_{\vec{k}} \Big(\overline{X_{\vec{k}}(\eta_v)} X_{\vec{k}}(\eta_y) + X_{\vec{k}}(\eta_v) \overline{X_{\vec{k}}(\eta_y)} \Big) \, D^{-+}_{\vec{k}}(\eta_u, \eta_v) \Bigg\}\end{aligned} \tag{3.141}$$

where

$$\begin{aligned}D^{-+}_{\vec{k}}(\eta_u, \eta_v) \quad := \quad & \int d^3p \, \Bigg\{ \mathscr{Y}^{-+}_{\vec{p}} \Big(\overline{Y_{\vec{p}}(\eta_u)} Y_{\vec{p}}(\eta_v) + Y_{\vec{p}}(\eta_u) \overline{Y_{\vec{p}}(\eta_v)} \Big) \\ & \quad \times \, \mathscr{Z}^{-+}_{\vec{k}-\vec{p}} \Big(\overline{Z_{\vec{k}-\vec{p}}(\eta_u)} Z_{\vec{k}-\vec{p}}(\eta_v) + Z_{\vec{k}-\vec{p}}(\eta_u) \overline{Z_{\vec{k}-\vec{p}}(\eta_v)} \Big) \Bigg\}.\end{aligned} \tag{3.142}$$

A similar treatment of the remaining terms in Eq. (3.135) allows the function $\mathfrak{X}^{P_{\eta(t)}}_{\vec{k}}$ in Eq. (3.113) to be written as

$$\begin{aligned}\mathfrak{X}^{P_{\eta(t)}}_{\vec{k}} = \mathfrak{X}^0_{\vec{k}} \quad + \quad & \frac{\kappa^2}{|X_{\vec{k}}(\eta)|^2} \int_{\eta_i}^{\eta_f} d\eta_u \int_{\eta_i}^{\eta_f} d\eta_v \, \Big\{ h(\eta_u) h(\eta_v) \\ & \times \mathfrak{X}^{\eta(t)}_{\vec{k}} \Big(\overline{X_{\vec{k}}(\eta)} X_{\vec{k}}(\eta_u) + X_{\vec{k}}(\eta) \overline{X_{\vec{k}}(\eta_u)} \Big) \\ & \times \Big[\, \mathscr{X}^{-+}_{\vec{k}} \Big(\overline{X_{\vec{k}}(\eta_v)} X_{\vec{k}}(\eta) + X_{\vec{k}}(\eta_v) \overline{X_{\vec{k}}(\eta)} \Big) D^{-+}_{\vec{k}}(\eta_u, \eta_v) \\ & - \, \mathscr{X}^{++}_{\vec{k}} \Big(\overline{X_{\vec{k}}(\eta_v)} X_{\vec{k}}(\eta) + X_{\vec{k}}(\eta_v) \overline{X_{\vec{k}}(\eta)} \Big) D^{++}_{\vec{k}}(\eta_u, \eta_v) \\ & + \, \mathscr{X}^{+-}_{\vec{k}} \Big(\overline{X_{\vec{k}}(\eta_v)} X_{\vec{k}}(\eta) + X_{\vec{k}}(\eta_v) \overline{X_{\vec{k}}(\eta)} \Big) D^{+-}_{\vec{k}}(\eta_v, \eta_u) \\ & - \, \mathscr{X}^{--}_{\vec{k}} \Big(\overline{X_{\vec{k}}(\eta_v)} X_{\vec{k}}(\eta) + X_{\vec{k}}(\eta_v) \overline{X_{\vec{k}}(\eta)} \Big) D^{--}_{\vec{k}}(\eta_v, \eta_u) \Big] \Big\}.\end{aligned} \tag{3.143}$$

Here, the exact form of the perturbed state first requires the form of $\mathfrak{X}^0_{\vec{k}}$, $\mathscr{Y}^{\pm\pm}_{\vec{p}}$, and $\mathscr{Z}^{\pm\pm}_{\vec{k}-\vec{p}}$. In order to carry out a numerical calculation these functions may simply specify that of the SLE vacuum or, for example, a Bulk-to-Boundary Approximate KMS state. On the other hand, additional interactions may be considered such that a system of coupled equations for

$\mathfrak{X}^0_{\vec{k}}$, $\mathscr{Y}^{\pm\pm}_{\vec{p}}$, and $\mathscr{Z}^{\pm\pm}_{\vec{k}-\vec{p}}$ may be employed. In addition, $\mathscr{E}_A^{P_{\eta(t)}}$, as a function of H_t, is of course subject to the so called back-reaction problem via the semiclassical Friedmann equation

$$H_t^2 = \frac{8\pi G}{3}\,\omega_t^P\bigg(\mathrm{T}_{00}(:A(f)^2:)\bigg). \tag{3.144}$$

However, we may still in principal carry out a concrete numerical calculation by imposing the solution for a_t, i.e. we may take a_t to maintain a fixed form of $a_t^{(\Lambda)} \propto \exp(H_\Lambda t)$, $a_t^{(r)} \propto (t-t_i)^{1/2}$, or $a_t^{(m)} \propto (t-t_i)^{3/2}$ during an epoch dominated by a constant vacuum energy (Λ), radiation (r), or matter (m) respectively.

Chapter 4

Conclusion

In this work we began with first principles of algebraic quantum field theory in curved spacetime where we employed both the SLE construction of renormalizable ground states and a two-parameter family of automorphisms, including a time averaged perturbation, in describing the dynamics of a dense environment of interacting quantum fields in FRW spacetimes. We then derived for the first time Eq. (3.132) as an expression that is in principal amenable to a numerical calculations for the renormalized energy density of a massive, minimally coupled free scalar field perturbed during a finite time interval via quantum interactions, including those far-from-equilibrium, while propagating in a non-stationary spacetime background. This algebraic expression is thus appropriate for computing cosmological observables, i.e. relic abundance calculations associated with common proposals for quantum matter production in the early universe, in order to determine if there are disparities between the algebraic approach and the general approximation, that are in principle experimentally verifiable by future high-precision electromagnetic and/or gravitational-wave detectors. If there are indeed discernible disparities they may serve to illuminate the interplay between quantum interactions and the dynamics of classical spacetime.

An additional application of the algebraic state containing the perturbation derived in Eq. (3.113) is a search for finite time and density corrections to the standard calculation of the observable power spectrum of super-Hubble fluctuations of the proposed quantum field responsible for inflation. Beginning with the linearized Einstein–Klein–Gordon system these fluctuations may be quantized according to the algebraic framework. The gauge invariant perturbations of the field, and hence the comoving curvature perturbations, may then be given the standard treatment via the Bardeen potentials and the Mukhanov–Sasaki variable, *i.e.* a Klein–Gordon field with time-dependent mass. An examination of the spectrum found via the perturbed two-point function may then be compared to that of the spectrum computed in the Bunch–Davies vacuum state. Furthermore, corrections arising from this perturbed algebraic calculation may be probed by a direct comparison with existing calculations carried out in an effective field theory approach to the operator framework. We leave this for future work.

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