Balian-Low Type Theorems for Shift-Invariant Spaces

By

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To my family,

for neverending support

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Chapter 1

Introduction

In this chapter, we introduce and provide examples of shift-invariant spaces in $L^2(\mathbb{R}^d)$. In Section 1.1 we present the main questions of this thesis. In Section 1.2 we state and discuss our main theorems and the related Balian-Low Theorem.

1.1 Shift-Invariant Spaces

Definition 1.1.1. Let $V \subset L^2(\mathbb{R}^d)$ be a closed subspace. For a fixed $t \in \mathbb{R}^d$ define the shift operator $T_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $T_t f(x) = f(x-t)$.

- Given a set $\Gamma \subset \mathbb{R}^d$, V is said to be Γ-*invariant* if $T_\gamma V \subset V$ for each $\gamma \in \Gamma$.
- V is called *shift-invariant* if V is $\mathbb{Z}^d$-invariant.
- V is called *translation-invariant* if V is $\mathbb{R}^d$ invariant.

Examples of shift-invariant spaces can be produced through the following procedure.

1. Fix $F \subset L^2(\mathbb{R}^d)$. We say F is *nontrivial* if F contains a nonzero element.
2. Let $\mathscr{T}(F)$ denote the *set of integer translates of* F. That is, $\mathscr{T}(F) = \{T_l f : l \in \mathbb{Z}^d, f \in F\}$.
3. Let $V(F)$ be the $L^2(\mathbb{R}^d)$-closure of the linear span of $\mathscr{T}(F)$. Then, $V(F)$ is a shift-invariant space, and we shall call $V(F)$ the *shift-invariant space generated by* F. We call F a set of *generators for* $V(F)$.

Let $W \subset L^2(\mathbb{R}^d)$ be a shift-invariant space. Note that $V(W) = W$. Thus, every shift-invariant space can be generated through this procedure. However, this thesis will be focused on shift-invarant spaces which are finitely-generated. The shift-invariant space, W, is *finitely-generated* if there exists a finite set $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ such that $W = V(F) = V(f_1, ..., f_K)$. Similary, if there is a singleton $F = \{f\} \subset L^2(\mathbb{R}^d)$ such that $W = V(F) = V(f)$, we say that W is *principle or singly generated.*

We will now present three examples of principle shift-invariant spaces. The first two examples, the Paley-Wiener space, PW, and the Cardinal Spline spaces, S_n, are commonly used in sampling and approximation theory. We use these examples to emphasize three properties which we will be interested in throughout this thesis. First, in each case, the space is generated by integer translates of a single function f, and $\mathscr{T}(f)$ forms a strong type of basis (either an orthonomal or Riesz basis, see Section 2.2) for $V(f)$. Second, in two of the examples the resulting space $V(f)$ has *additional or extra-invariance*, that is, $V(f)$ is invariant under some non-integer shift, while the other case has no extra-invariance. Third, some of the spaces considered have a localized generator, while others have a generator with slow decay.

Example 1.1.2 (Paley-Wiener Space)**.** With the Fourier transform of $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$ and extended to $L^2(\mathbb{R}^d)$ by unitarity, the Paley-Wiener space is defined as

$$PW = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(\widehat{f}) \subset [-\frac{1}{2}, \frac{1}{2}] \right\}.$$

Since, $\{e^{-2\pi i l \xi}\}_{l \in \mathbb{Z}}$, forms an orthonormal basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$, we have that $\{e^{-2\pi i l \xi} \chi_{[-\frac{1}{2}, \frac{1}{2}]}\}_{l \in \mathbb{Z}}$ forms an orthonormal basis for $\widehat{PW} = \{\widehat{f} : f \in PW\}$. By the unitarity of the Fourier transform on $L^2(\mathbb{R})$, the set $\{T_l \operatorname{sinc}\}_{l \in \mathbb{Z}}$ must form an orthonormal basis for PW, where $\operatorname{sinc}(x) = \widehat{\chi_{[-\frac{1}{2}, \frac{1}{2}]}}(x) = \frac{\sin(\pi x)}{\pi x}$. Thus, $PW = V(\operatorname{sinc})$. A graph of sinc is given in figure 1.1. Note the slow decay of sinc: sinc is not integrable and $\int_{\mathbb{R}} |x| \operatorname{sinc}(x) dx = \infty$. Note also that PW is translation invariant due to the fact that for any $\gamma \in \mathbb{R}$ and any $f \in L^2(\mathbb{R})$ we have $\widehat{T_\gamma f}(\xi) = e^{2\pi i \xi \gamma} \widehat{f}(\xi)$.

Example 1.1.3 (Cardinal Splines)**.** The Cardinal Spline Space, S_n, consist of all $f \in C^{n-1}(\mathbb{R}) \cap L^2(\mathbb{R})$ such that for each integer $j \in \mathbb{Z}$, f restricted to the interval $[j, j+1]$ agrees with a polynomial of degree at most n. Let $b_0 = \chi_{[0,1]}$ and iteratively define

$$b_n(x) = b_0 * b_{n-1}(x) = \int_{\mathbb{R}} b_0(y) b_{n-1}(x-y) dx.$$

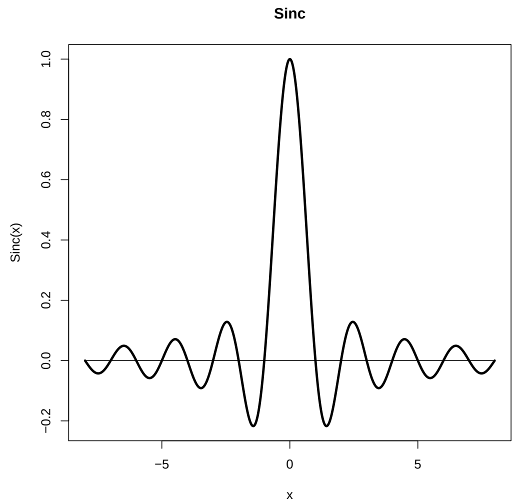

Figure 1.1: Graph of the sinc function

It is known that for each $n \in \mathbb{N}$, $S_n = V(b_n)$, and $\mathscr{T}(b_n)$ forms a Riesz basis for S_n. Graphs of the first few b_n functions are given in figure 1.2. Note that although the support of b_n increases in size with n, each b_n is compactly supported. Also, due to the fact that S_n functions have less regularity at the integers than in intervals of the form $(j, j+1)$ for $j \in \mathbb{Z}$, S_n cannot be invariant under any non-integer shift.

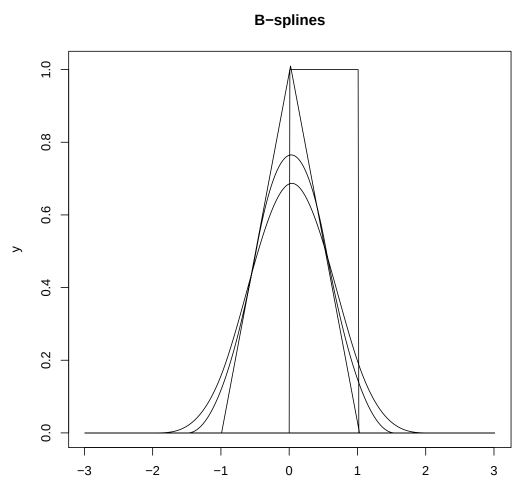

Figure 1.2: Graph of b_n

Example 1.1.4 (Strict $\frac{1}{2}\mathbb{Z}$-invariance)**.** Here we construct an example which shows that it is possible to have a principle shift-invariant space which has extra-invariance but which is not translation-invariant like the Paley-Wiener space. We start by defining the Fourier transform of the generator, f, by

$$\widehat{f}(\xi) = \begin{cases} c_k & \xi \in [2k - \frac{1}{2}, 2k + \frac{1}{2}], k \in \mathbb{Z} \\ 0 & \text{otherwise}, \end{cases}$$

where $c_0 = 0$ and $c_k = c_{-k} = \sqrt{3}^{-k}$ for $k \in \mathbb{N}$. A graph of $\widehat{f}$ is given in figure 1.3. The coefficients c_k were chosen such that $\|f\|_{L^2(\mathbb{R}^d)} = \|\widehat{f}\|_{L^2(\mathbb{R}^d)} = \sum_{k\in\mathbb{Z}} c_k^2 = 1$. Let m be the $\mathbb{Z}$-periodic function with Fourier coefficients (see Section 2.3) $\widehat{m}(k) = c_k$. Then, a brief calculation shows that $f(x) = m(2x)\mathrm{sinc}(x)$. Note that f has similar localization to the sinc function since m is periodic. A graph of f is given in figure 1.3 with the graph of $\frac{1}{|x|}$ superimposed.

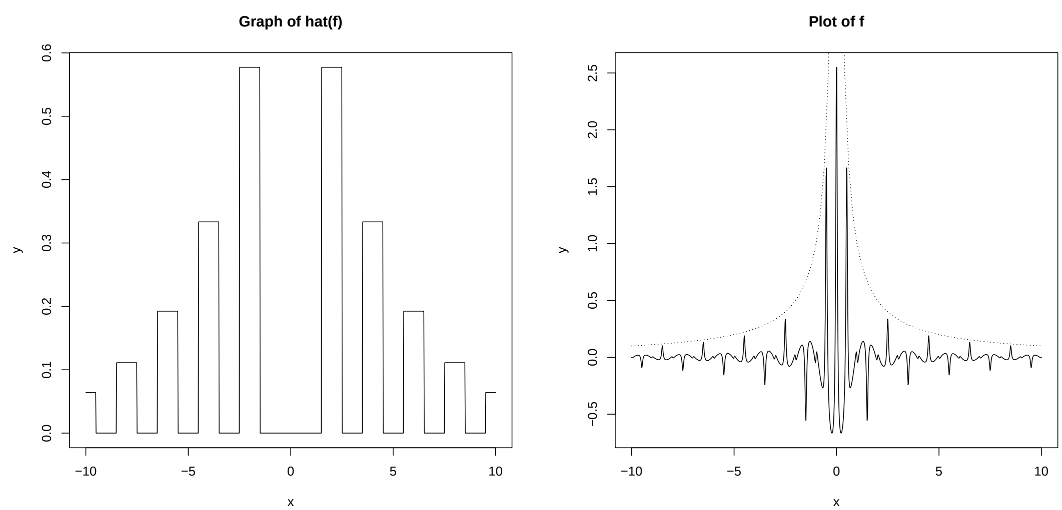

Figure 1.3: Graph of $\widehat{f}$ and f

We introduce tools (for example Equation (3.5) and Theorem 3.3.1) later in the thesis which allow us to prove more rigorously that $V(f)$ has the following properties. For any $n \in \mathbb{Z}$, $\widehat{T_n f}(\xi) = e^{2\pi i n\xi}\widehat{f}(\xi)$. Then,

$$\langle f, T_n f\rangle = \langle \widehat{f}, e^{2\pi i n\xi}\widehat{f}\rangle = \left(\sum_{k\in\mathbb{Z}} c_k^2\right)\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i n\xi} d\xi = \delta_{0,n}.$$

Thus, $\mathscr{T}(f)$ forms an orthonormal basis for $V(f)$. Also, we see that any $g \in \mathrm{span}(\mathscr{T}(f))$ satisfies,

$\widehat{g}(\xi) = w(\xi)\widehat{f}(\xi)$ where w is some $\mathbb{Z}$-periodic function. It can be shown (see Proposition 3.1.1) that this is also true for any $g \in V(f)$. Note, that $\widehat{T_{1/2}f}(\xi) = e^{\pi i \xi}\widehat{f}(\xi)$, and $e^{\pi i \xi}$ is $2\mathbb{Z}$-periodic. However, due to the support of $\widehat{f}$, $e^{\pi i \xi}\widehat{f}(\xi) = w(\xi)\widehat{f}(\xi)$ for all ξ where $w(\xi)$ is the $\mathbb{Z}$-periodic function which satisfies $w(\xi) = e^{\pi i \xi}$ for all $|\xi| \leq 1/2$. Thus, $T_{\frac{1}{2}}f \in V(f)$. The same is not true for $T_\tau f$ for any $\tau \notin \frac{1}{2}\mathbb{Z}$, and thus, $V(f)$ is $\frac{1}{2}\mathbb{Z}$-invariant, but is not invariant under any other shifts.

In summary, each example is a principle shift-invariant space, and the integer translates of the generator form a strong basis for the resulting space. However, these spaces differ greatly when considering extra-invariance properties and generator localization. The Paley-Wiener space is translation-invariant, and Example 1.1.4 is $\frac{1}{2}\mathbb{Z}$-invariant, but they both have poorly localized generators, whereas the Cardinal Splines have compactly supported generators but no extra-invariance. It turns out, that this negative relationship between extra-invariance and generator localization extends past these two examples, and understanding this relationship is the main focus of this thesis.

Finitely generated shift-invariant spaces are often used as approximation spaces. In this setting, it is often desirable that such a space have extra-invariance. Assuming that a shift-invariant space V is endowed with translation-invariance implies that if one can approximate a function g well by a function in V then one can also approximate $T_t g$ for any $t \in \mathbb{R}^d$ to the same accuracy. Shift-invariant spaces with high levels of extra-invariance, mimick this property in that $T_t g$ can be well approximated for all t in some fine lattice. In [37, 38], the authors examine the difficulties of using non-translation-invariant shift invariant spaces as approximation spaces. In this setting, much care has to be taken to "synchronize" the approxmiated function g with the non-translation-invariant space V.

Localized generators are also advantageous when trying to approximate a function using a shift-invariant space. Consider a principle shift-invariant space $V(f)$. For a function g, we would like to say that

$$\|g - \sum_{k \in \mathbb{Z}^d} c_k T_k f\|$$

is small for some collection $\{c_k\}_{k \in \mathbb{Z}^d}$ which is well-behaved (i.e. $\mathscr{T}(f)$ has a basis-type property which relates a norm of $\{c_k\}$ to some norm of g), perhaps for a variety of norms. In particular, if we would like to estimate, for some $x \in \mathbb{R}^d$,

$$|g(x) - \sum_{k\in\mathbb{Z}^d} c_k T_k f(x)|,$$

then the knowledge that f is well-localized implies that only a few of the terms $c_k T_k f(x)$ actually contribute to the sum. If f is not localized (like the sinc function) then many of these terms can contribute to the sum, and it is much harder to analyze this error. This also leads to problems when trying to approximate g by a truncted series such as $S_n(x) = \sum_{|k|\le n} c_k T_k f(x)$, which is desirable in any real world application.

This discussion leads directly to the main questions of the thesis.

Question 1.1.5. *Is it possible to construct finitely-generated shift-invariant spaces with extra invariance which also have localized generators?*

Extra-invariance properties of shift-invariant spaces were studied in [1, 3, 4], and in [2, 49] theorems are proven which show that if $\mathscr{T}(F)$ forms certain types of bases or generalizations of bases for $V(F)$, there are restrictions on the localization of the functions in F. We will prove sharp versions of these results, and generalize the results to new settings.

As Example 1.1.4 shows, it is possible for a finitely-generated shift-invariant space $V(F)$ to be invariant under some non-integer shift, but to also not be fully translation-invariant. Some of the existing results in the literature find different conclusions when the assumption of translation-invariance is replaced with the weaker assumption of extra-invariance. This leads to our second main question.

Question 1.1.6. *Is there truly a difference in the restrictions on the localization of generators if $V(F)$ is only assumed to be invariant under some non-integer shift as compared to if $V(F)$ is assumed to be fully translation-invariant? Are the previous results sharp or best possible?*

Our first main result, Theorem 1.2.1, is a sharp result proven in the setting of extra-invariance. Existing results in the literature either reached a weaker conclusion or had to assume translation-invariance to reach the same conclusion. Thus, in the setting of Theorem 1.2.1, the best possible results are the same in the extra-invariant case as in the translation-invariant case. However, we also present an example in Section 6.3, which proves that the known result, Theorem 1.2.6, is sharp. With the extra-invariance assumption in Theorem 1.2.6 replaced with translation-invariance, a stronger result is possible as seen in 1.2.7. Thus, in this setting, the best possible results differ based on the two assumptions.

Our third main question is the following.

Question 1.1.7. *How does the relative "strength" of the basis property affect the localization of the generators?*

In Theorem 1.2.9, we consider frame-like properties called (C_q)-systems which depend on a continuous parameter q and are such that as q increases, the (C_q)-property weakens. Theorem 1.2.9 is sharp and shows that if $\mathscr{T}(F)$ is a (C_q)-system for $V(F)$, the amount of localization allowed by the elements of F is increased as q increases. Thus, as the basis-type property considered is weakened, the best possible localization result is also weakened.

Our final main question is the following.

Question 1.1.8. *Are there other properties besides extra-invariance for which similar localization restrictions can be proven for the generators of a shift-invariant space possessing these properties?*

It is not possible to prove localization type results for the generators of shift-invariant spaces without some additional assumption on the space, such as the extra-invariance or translation-invariance considered in many of our theorems. We will see in Theorem 1.2.4 that replacing an extra-invariance assumption for $V(F)$ with the assumption that $\mathscr{T}(F)$ forms a redundant (non-minimal) frame for $V(F)$ can lead to similar results.

1.2 Main Theorems

Several of the results in this section appear in [32], while the unreferenced results will appear in a future paper with co-authors Shahaf Nitzan and Alexander Powell. Our results will show that under certain conditions on the basis properties of the generators, extra-invariance in finitely-generated shift-invariant spaces is incompatible with the spaces posessing well-localized generators. Consider, however, the following example. Let $f \in C^\infty(\mathbb{R})$ be supported in $\left[0, \frac{1}{4}\right]$ and have $\|f\|_2 = 1$. Let $f_k = T_{\frac{k}{4}} f$ for $k = 0, 1, 2, 3$, and $F = \{f_0, f_1, f_2, f_3\}$. Then, $\mathscr{T}(F)$ is an orthonormal system for $V(F)$, $V(F)$ is $\frac{1}{2}\mathbb{Z}$-invariant (in fact, the space is $\frac{1}{4}\mathbb{Z}$-invariant), and the generators of $V(F)$ are compactly supported.

For an arbitrary shift-invariant space V, let $\rho(V)$ denote the *minimal number of generators of* V. In other words,

$$\rho(V) = \min\{\#G : \exists G \subset L^2(\mathbb{R}^d) \text{ such that } V = V(G)\}. \tag{1.1}$$

For the example above, it can be shown that $\rho(V(F)) = 4$. Note that in this case, $[\frac{1}{2}\mathbb{Z}, \mathbb{Z}] = 2$ where $[\frac{1}{2}\mathbb{Z}, \mathbb{Z}]$ is the index of $\mathbb{Z}$ in the extra-invariance lattice $\frac{1}{2}\mathbb{Z}$ (see Section 2.1). Thus, $[\frac{1}{2}\mathbb{Z}, \mathbb{Z}]$ divides $\rho(V(F))$. In many of our theorems involving finitely-generated shift-invariant spaces, we must exclude cases in which this divisibility occurs to avoid such examples.

1.2.1 Frame and Riesz Basis Results and the Balian-Low Theorem

We first recall the definition of frames and Riesz bases for Hilbert spaces. We refer the reader to Section 2.2 for a more detailed discussion of basis and frame-type properties in Hilbert spaces. A sequence $\{h_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathscr{H}$ is a *frame* for $\mathscr{H}$ if there exist $0 < A \leq B < \infty$ such that for each $g \in \mathscr{H}$,

$$A\|g\|^2 \leq \sum_{n\in\mathbb{N}} |\langle g, h_n\rangle|^2 \leq B\|g\|^2. \tag{1.2}$$

A Riesz basis for $\mathscr{H}$ is the image of an orthonormal basis for $\mathscr{H}$ under a bounded, invertible linear operator on $\mathscr{H}$. Every Riesz basis is a frame, but there exist frames which are not Riesz bases. We refer to such frames as *redundant frames*.

Our first main result is the following. This result addresses Questions 1.1.5 and 1.1.6, and it resolves a question posed in [49].

Theorem 1.2.1 (Frame SIS BLT, Theorem 1.3, [32]). *Fix a lattice $\Gamma \subset \mathbb{R}^d$ with $\mathbb{Z}^d \subsetneq \Gamma$ and $[\Gamma : \mathbb{Z}^d] > 1$. Suppose that $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ is nontrivial and that $\mathscr{T}(F)$ is a frame (or Riesz Basis) for $V(F)$. If $[\Gamma : \mathbb{Z}^d]$ is not a divisor of $\rho(V(F))$ and $V(F)$ is Γ-invariant, then*

$$\exists\, 1 \le k \le K \ \textit{ such that } \ \int_{\mathbb{R}^d} |x|\,|f_k(x)|^2 dx = \infty.$$

For singly generated shift-invariant spaces the divisibility condition is unnecessary, and Theorem 1.2.1 takes the following form.

Corollary 1.2.2. *Fix a lattice $\Gamma \subset \mathbb{R}^d$ with $\mathbb{Z}^d \subsetneq \Gamma$ and $[\Gamma : \mathbb{Z}^d] > 1$. Suppose $f \in L^2(\mathbb{R}^d)$, $\|f\|_2 \neq 0$, and $\mathscr{T}(f)$ forms a frame for $V(f)$. If $V(f)$ is Γ-invariant, then $\int_{\mathbb{R}^d} |x|\,|f(x)|^2 dx = \infty$.*

The conclusion of Theorem 1.2.1 (and similarly Corollary 1.2.2) can be restated in terms of the Fractional Sobolev Spaces or Bessel Potential Spaces,

$$H^s(\mathbb{R}^d) = \{g \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{g}(\xi)|^2 d\xi < \infty\},$$

which we introduce in detail in Section 2.3. In particular, the conclusion of the Theorem 1.2.1 could be written as: *at least one of the generators satisfies $\widehat{f_k} \notin H^{1/2}(\mathbb{R}^d)$.*

To put Theorem 1.2.1 in perspective, note that existing results in the literature, see [2, 49], either give a weaker conclusion or require stronger hypotheses. In particular, the foundational Theorem 1.2 in [2] addresses singly generated shift-invariant spaces in dimension $d = 1$ which posess extra-invariance and gives the weaker conclusion that the generator $f \in L^2(\mathbb{R})$ satisfies $\widehat{f} \notin H^{1/2+\varepsilon}(\mathbb{R})$ whenever $\varepsilon > 0$. In higher dimensions and finitely many generators, the previous best results were given in [49]. Theorem 1.3 in [49] gives the weaker conclusion that at least one generator satisfies $\widehat{f_k} \notin H^{d/2+\varepsilon}(\mathbb{R}^d)$. On the other hand, Theorem 1.2 in [49] shows if the hypothesis of Γ-invariance is replaced by the notably stronger hypothesis of translation-invariance, then at least one generator satisfies $\widehat{f_k} \notin H^{1/2}(\mathbb{R}^d)$.

It should be noted that the spaces considered in [49] are constructed in the following way. For a generating set $F \subset L^2(\mathbb{R}^d)$, take a full rank lattice $\Lambda \subset \mathbb{R}^d$, let $\mathscr{T}^\Lambda(F) = \{T_\lambda f : \lambda \in \Lambda, f \in F\}$, and let $V^\Lambda(F)$ be the span closure of $\mathscr{T}^\Lambda(F)$. Then, it is assumed that $V^\Lambda(F)$ is invariant under a larger lattice Γ. Note, however, that $\Lambda = D\mathbb{Z}^d$ for some invertible matrix D, (see Section 2.1) and if we let $F_D = \{f(D\cdot) : f \in F\}$, then $V(F_D)$ is $D^{-1}\Gamma$-invariant, and all the localization and basis properties of F are transferred to F_D. Thus, our results are no less general than the results in [49].

Theorem 1.2.1 is sharp in the sense that $H^{1/2}(\mathbb{R}^d)$ cannot be replaced by $H^s(\mathbb{R}^d)$ when $s < 1/2$. For example, if χ_I is the characteristic function of the set $I = [-1/2, 1/2]^d$ and $f(x) = \widehat{\chi}_I(x)$, then the space $V(f)$ is translation invariant and $\widehat{f} \in H^s(\mathbb{R}^d)$ for all $0 < s < 1/2$, cf. Proposition 1.5 in [2] and Proposition 1.5 in [49].

Theorem 1.2.1 is also precise in the sense that it is possible for only one generator in a finitely-generated system to suffer from the localization constraint $f_k \notin H^{1/2}(\mathbb{R}^d)$. In particular, we construct examples of $F = \{f_k\}_{k=1}^K$ that satisfy the hypotheses of Theorem 1.2.1, and where $f_K \notin H^{1/2}(\mathbb{R}^d)$ but all other generators $f_1, \cdots, f_{K-1}$ are in $H^{1/2}(\mathbb{R}^d)$. This answers a question posed in [49] about the proportion of generators with good localization. See Section 6.2 for the examples.

Note that we have titled Theorem 1.2.1 as the Frame SIS BLT. Here SIS stands for shift-invariant space, and BLT stands for the Balian-Low Theorem. The Balian-Low Theorem is a famous result in time-frequncy analysis, which is related to Theorem 1.2.1. Given $f \in L^2(\mathbb{R})$ the associated *Gabor system* $\mathscr{G}(f, a, b) = \{f_{m,n}\}_{m,n \in \mathbb{Z}}$ is defined by $f_{m,n}(x) = e^{2\pi imbx} f(x - na)$. The original Balian-Low Theorem states that if $\mathscr{G}(f, a, b)$ is an orthonormal basis for $L^2(\mathbb{R})$ then f must be poorly localized in either time or frequency.

Theorem 1.2.3 (Balian-Low Theorem)**.** *Let $f \in L^2(\mathbb{R})$ and suppose that $\mathscr{G}(f, a, b)$ with $ab = 1$ is an orthonormal basis for $L^2(\mathbb{R})$.*

$$\left(\int_{\mathbb{R}} |x|^2 |f(x)|^2 dx \right) \left(\int_{\mathbb{R}} |\xi|^2 |\widehat{f}(\xi)|^2 d\xi \right) = \infty.$$

Theorem 1.2.1 is structured similar to the Balian-Low Theorem, but we substitute $\mathscr{T}(F)$ for

$\mathscr{G}(f,a,b)$, frames for orthonormal bases, $V(F)$ for $L^2(\mathbb{R})$, and we get directly a result about the localization of F instead of a combined result about f and $\widehat{f}$. It is interesting to note that $V(F)$ can never be all of $L^2(\mathbb{R}^d)$ for any finite collection $F \subset L^2(\mathbb{R}^d)$, and $\mathscr{G}(f,a,b)$ is symmetric in f and $\widehat{f}$ in the sense that $\widehat{\mathscr{G}(f,a,b)} = \mathscr{G}(\widehat{f},b,a)$, but $\mathscr{T}(F)$ is not symmetric in this sense. Thus, it is reasonable that the conclusion of the Balian-Low Theorem should be symmetric in f and $\widehat{f}$, and the conclusion of Theorem 1.2.1 should not be.

The Balian-Low Theorem has had an interesting history. The above theorem was formulated independently by Balian [7] and Low [42]. The following excerpt is taken from [13].

> The proofs given by Balian and Low each contained a gap.... This gap was independently addressed in two ways. Battle [8] provided an elegant and entirely new proof .... Daubechies, Coifman, and Semmes [23] retained the original approach of Balian and Low, filling the gap directly. In the process, they extended the result from Gabor systems which form orthonormal bases for $L^2(\mathbb{R})$ to Gabor systems which form exact frames [Riesz bases]- a natural generalization of orthonormal bases.

The excerpt mentions that the Balian-Low Theorem can be extended to Riesz bases. In fact, there have been many generalizations of the Balian-Low Theorem, e.g., see the surveys [13, 22] and articles [5, 6, 8, 9, 10, 11, 12, 24, 29, 31, 34, 35, 39, 41, 45, 46].

Our next main result is closely related to the work in [29], which considers Balian-Low type properties for Gabor frames of subspaces. It is the only main theorem which does not contain an extra invariance assumption as desired in Question 1.1.8. A redundant frame is a non-minimal frame, or a frame which is not a Riesz basis.

Theorem 1.2.4 (Redundant Frame SIS BLT, Theorem 1.5, [32])**.** *Suppose that $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ is nontrivial and that $\mathscr{T}(F)$ is a redundant frame for $V(F)$. If $K = \rho(V(F))$ then*

$$\exists\, 1 \le k \le K \ \textit{ such that } \quad \int_{\mathbb{R}^d} |x|\,|f_k(x)|^2 dx = \infty.$$

For singly generated shift-invariant spaces, Theorem 1.2.4 takes the following form.

Corollary 1.2.5. *Suppose $f \in L^2(\mathbb{R}^d)$ with $\|f\|_2 \neq 0$. If $\mathscr{T}(f)$ is a frame for $V(f)$, but is not a Riesz basis for $V(f)$, then $\int_{\mathbb{R}^d} |x|\,|f(x)|^2 dx = \infty$, i.e., $\widehat{f} \notin H^{1/2}(\mathbb{R}^d)$.*

Corollary 1.2.5 stated with the weaker conclusion $\widehat{f} \notin H^{d/2+\varepsilon}(\mathbb{R}^d)$ (or more generally that f is not integrable) may be considered folklore [30]. The conclusion of Corollary 1.2.5 with the condition $\widehat{f} \notin H^{1/2}(\mathbb{R}^d)$ provides a significant and sharp improvement of this.

This result is sharp, as can be seen by considering $V(f)$ with $f(x) = \widehat{\chi_J}(x)$ and $J = [0,1/2]^d$, cf. (3.7). Moreover, Example 6.2.3 shows that it is possible for only a single generator in Theorem 1.2.4 to have poor localization.

Theorem 1.2.1 is an example of a situation where Question 1.1.6 is answered in the negative. That is, the best possible result on the localization of F is the same with the assumption of extra-invariance as it is with the stronger assumption of translation-invariance. We now look a case, proven in [49], where translation-invariance actually gives a stronger result than extra-invariance.

Theorem 1.2.6 (Theorem 1.4, [49])**.** *Fix a lattice $\Gamma \subset \mathbb{R}^d$ with $\mathbb{Z}^d \subsetneq \Gamma$ and $[\Gamma : \mathbb{Z}^d] > 1$. Suppose that $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ is nontrivial, $\widehat{f_k}$ is continuous for each k, and $\mathscr{T}(F)$ is a frame (or Riesz Basis) for $V(F)$. If $[\Gamma : \mathbb{Z}^d]$ is not a divisor of $\rho(V(F))$ and $V(F)$ is Γ-invariant, then there exists $1 \leq k \leq K$ such that for any $\varepsilon > 0$,*

$$\operatorname{esssup}_{\xi\in\mathbb{R}^d} |\widehat{f_k}(\xi)||\xi|^{\frac{d}{2}+\varepsilon} = \infty.$$

In [49], the authors suspect that the exponent on the weight, $|\xi|^{\frac{d}{2}+\varepsilon}$, in this theorem should be independent of the dimension d, and that the result should hold with $\frac{d}{2}$ replaced by $\frac{1}{2}$ for all dimensions d. In Proposition 1.6 of [49], the authors construct a function $f \in L^2(\mathbb{R}^d)$ such that $V(f)$ is extra-invariant, $\widehat{f}$ is continuous, and $|\xi|^{\frac{1}{2}}|\widehat{f}(\xi)| \in L^\infty(\mathbb{R}^d)$.

We contribute to this result, in Section 6.3, by constructing a function which satisifes all of these properties, but also satisfies $|\xi|^{\frac{d}{2}}|\widehat{f}(\xi)| \in L^\infty(\mathbb{R}^d)$. Thus, we prove that Theorem 1.2.6 is actually sharp in its present form.

Contrary to this result, the following result for translation-invariant shift-invariant spaces is also known, see for example [28].

Theorem 1.2.7. *Suppose $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ is nontrivial, and $\mathscr{T}(F)$ is a frame for $V(F)$. If $V(F)$ is translation-invariant, then there exists $1 \leq k \leq K$ such that $\widehat{f_k}$ is not continuous.*

Therefore, in this case, translation-invariance leads to a stronger result than extra-invariance.

1.2.2 Minimal System Results

Theorems 1.2.1 is a sharp improvement of an existing theorem about frames and Riesz bases. It is natural to question whether similar results hold for other basis-type properties instead of frames. Our next main result is the analog of Theorem 1.2.1 in the setting of minimal systems. A sequence $\{h_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathscr{H}$ is a minimal system if for any $n \in \mathbb{N}$,

$$h_n \notin \overline{\operatorname{span}\{h_m : m \neq n\}}.$$

Every Riesz basis is minimal, and so the following addresses Question 1.1.7. In fact, a set is a Riesz basis if and only if it is a minimal frame. Thus, we would expect a weaker result than Theorem 1.2.1 when we consider minimal systems in the same setting.

Theorem 1.2.8 (Minimal SIS BLT). *Fix a lattice $\Gamma \subset \mathbb{R}^d$ with $\mathbb{Z}^d \subset \Gamma$ and $[\Gamma : \mathbb{Z}^d] > 1$. Suppose that $F = \{f_k\}_{k=1}^K \subset L^2(\mathbb{R}^d)$ is nontrivial and that $\mathscr{T}(F)$ is a minimal system for $V(F)$. If $[\Gamma : \mathbb{Z}^d]$ is not a divisor of $\rho(V(F))$ and $V(F)$ is Γ-invariant, then*

$$\exists\, 1 \leq k \leq K \ \ \textit{such that} \quad \int_{\mathbb{R}^d} |x|^2 \, |f_k(x)|^2 dx = \infty.$$

In other words, at least one of the generators satisfies $\widehat{f_k} \notin H^1(\mathbb{R}^d)$.

Theorem 1.2.8 is sharp, which is shown in Lemma 6.1.1.

1.2.3 (C_q)-system Results

Our final main result is also related to Question 1.1.7. It is proven under the assumption that $\mathscr{T}(F)$ forms a (C_q)-*system* for $V(F)$. These systems, introduced in [44, 47], are generalizations of frames. In [45], a generalization of the Balian-Low theorem was proven in the setting that $\mathscr{G}(f,a,b)$ forms a minimal (C_q)-system for $L^2(\mathbb{R})$.

Let $2 \leq q \leq \infty$. Then, a sequence $\{h_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathscr{H}$ is a (C_q)-system for $\mathscr{H}$ if there is a $C > 0$ such that for each $g \in \mathscr{H}$, g can be approximated to arbitrary accuracy by a finite sum $\sum c_n h_n$ such that

$$\|c_n\|_q \leq C\|g\|_{\mathscr{H}}.$$

The name (C_q)-system stands for completeness with l^q control on the coefficients. A Bessel (C_2)-system is exactly a frame, and if $\mathscr{T}(F)$ is a minimal system for $V(F)$ then $\mathscr{T}(F)$ is also a (C_∞)-system for $V(F)$. As q increases, the (C_q) property gets weaker in the sense that a (C_q)-system is also a $(C_{q'})$-system for all $q' \geq q$. Thus, in some sense, (C_q)-systems for shift invariant spaces form a continuous bridge between frames, which have l^2 control over the coefficients, and minimal systems, which have little control over the coefficients. Section 2.2 contains more information on (C_q)-systems.

Note that unlike Theorems 1.2.1 and 1.2.8, the following theorem is only proven for dimension $d = 1$ and with the restriction that our generating set is of minimal size.

Theorem 1.2.9 ((C_q)-system SIS BLT)**.** *Let $1 < N \in \mathbb{N}$ and $2 < q < \infty$. Suppose $F = \{f_1, ..., f_K\} \in L^2(\mathbb{R})$ is nontrivial, $V(F)$ is $\frac{1}{N}\mathbb{Z}$-invariant, and N does not divide $\rho(V(F))$. If $K = \rho(V(F))$, and $\mathscr{T}(F)$ is a (C_q)-system in $V(F)$, then there exists $1 \leq k \leq K$ such that $\widehat{f_k} \notin H^{\frac{q-1}{q}}(\mathbb{R})$. In other words,*

$$\int_{\mathbb{R}} |x|^{\frac{2(q-1)}{q}} |f_k(x)|^2 dx = \infty.$$

Lemma 6.1.1 also proves that this result is sharp. In the limit as $q \to 2$, the exponent on the weight in Theorem 1.2.9 tends to 1, the same exponent as in Theorem 1.2.1. Similarly, as $q \to \infty$, the exponent tends to 2, which agrees with Theorem 1.2.8. The proof of Theorem 1.2.9 does not directly extend to these cases, but those results can be viewed as limiting cases of Theorem 1.2.9.

Chapter 2

Background Information

In this chapter, we introduce and collect facts about several topics which will be used throughout this thesis. In Section 2.1, we define lattices and other topics related to periodic functions. Section 2.2 reviews the basis and basis-like properties which are used in our main theorems. Section 2.3 introduces fractional Sobolev spaces on $\mathbb{R}^d$ and $\mathbb{T}^d$, and Section 2.4 introduces the discrete Hilbert transform, the discrete conjugate transform, and other related Fourier multipliers, and proves their boundedness on certain sequence spaces.

2.1 Miscellaneous Background

A set $\Gamma \subset \mathbb{R}^d$ is a (full-rank) lattice if there exists a $d \times d$ nonsingular matrix D such that $\Gamma = D(\mathbb{Z}^d)$. The dual lattice associated to Γ is defined as $\Gamma^* = \{\xi \in \mathbb{R}^d : \forall x \in \Gamma, e^{2\pi i x\cdot\xi} = 1\}$. In terms of the matrix D, the *dual lattice* can equivalently be defined as $\Gamma^* = (D^*)^{-1}(\mathbb{Z}^d)$.

Given nested lattices $\Lambda \subset \Gamma$, the *index* of Λ in Γ is denoted by $[\Gamma : \Lambda]$, and is defined as the order of the quotient group Γ/Λ when Γ and Λ are viewed as discrete subgroups of $\mathbb{R}^d$. Moreover, $[\Gamma : \Lambda] > 1$ if and only if the inclusion $\Lambda \subset \Gamma$ is strict, i.e., $\Lambda \subsetneq \Gamma$.

A function f defined on $\mathbb{R}^d$ will be said to be Γ-periodic if $f(x+\gamma) = f(x)$ for all $x \in \mathbb{R}^d$ and $\gamma \in \Gamma$. We will typically only be interested in $\mathbb{Z}^d$-periodic functions, since an arbitrary lattice in $\mathbb{R}^d$ can be linearly mapped to $\mathbb{Z}^d$. We let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ denote the d-dimensional torus.

2.2 Basis-type Properties in Hilbert Spaces

Given a separable Hilbert space, $\mathscr{H}$, and a collection of vectors, $\{h_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$, $\{h_n\}_{n\in\mathbb{N}}$ is called *complete* in $\mathscr{H}$ if the span of $\{h_n\}_{n\in\mathbb{N}}$ is dense in $\mathscr{H}$. Thus, if $\{h_n\}_{n\in\mathbb{N}}$ is complete in $\mathscr{H}$, any vector $g \in \mathscr{H}$ can be approximated to arbitrary accuracy by a finite linear combination of elements of $\{h_n\}_{n\in\mathbb{N}}$. However, without stronger conditions imposed on $\{h_n\}_{n\in\mathbb{N}}$, the coefficients of these linear combinations can be erratic and not useful in applications.

A common property imposed on $\{h_n\}_{n\in\mathbb{N}}$ is *orthogonality*, i.e. for any $n \neq m$,

$$\langle h_n, h_m \rangle = \langle h_n, h_m \rangle_{\mathscr{H}} = 0.$$

When $\{h_n\}_{n\in\mathbb{N}}$ is complete, orthogonal, and for each $n \in \mathbb{N}$, $\|h_n\| = \|h_n\|_{\mathscr{H}} = 1$, then $\{h_n\}_{n\in\mathbb{N}}$ is called an *orthonormal basis* for $\mathscr{H}$. If $\{h_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathscr{H}$, then each $g \in \mathscr{H}$ satisfies,

$$g = \sum_{n\in\mathbb{N}} \langle g, h_n \rangle h_n,$$

where the convergence is in the norm of $\mathscr{H}$ and is unconditional. These coefficients $\{\langle g, h_n \rangle\}_{n\in\mathbb{N}}$ satisfy Parseval's equality,

$$\|g\|^2 = \sum_{n\in\mathbb{N}} |\langle g, h_n \rangle|^2. \tag{2.1}$$

Parseval's equality shows that small perturbations of a vector $g \in \mathscr{H}$ lead to small changes in the coefficients of the perturbation's expansion in the orthonormal basis.

2.2.1 Riesz Bases, Frames, and Minimality

The material in this subsection can be found in several sources. For example, see [21, 33].

A Riesz basis for $\mathscr{H}$ is the image of an orthonormal basis for $\mathscr{H}$ under a bounded, invertible linear operator on $\mathscr{H}$. Thus, if $\{h_n\}_{n\in\mathbb{N}}$ is a Riesz basis, then there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and a bounded, invertible linear operator T such that $Te_n = h_n$ for each n. For any square summable sequence, $\{c_n\}_{n\in\mathbb{N}}$, the vector $g = \sum_{n\in\mathbb{N}} c_n e_n \in \mathscr{H}$ and $\|g\| = \|c_n\|_{l^2(\mathbb{N})}$. Then $Tg = \sum_{n\in\mathbb{N}} c_n h_n \in \mathscr{H}$ and there exist constants $0 < A \leq B < \infty$ only depending on the operator T such that

$$A\|g\| \leq \|Tg\| \leq B\|g\|.$$

In other words, for any sequence $\{c_n\} \in l^2(\mathbb{N})$,

$$A\|c_n\|_{l^2(\mathbb{N})} \leq \| \sum_{n\in\mathbb{N}} c_n h_n\| \leq B\|c_n\|_{l^2(\mathbb{N})}. \tag{2.2}$$

It is straightforward to verify that $\{h_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathscr{H}$ if and only if Equation 2.2 holds for each $\{c_n\} \in l^2(\mathbb{N})$. Thus, Riesz bases are characterized by a weaker version of Parseval's equality (2.1).

A frame can be defined with a different weakening of Parseval's equality. A sequence $\{h_n\}_{n\in\mathbb{N}}$ is a *frame* for $\mathscr{H}$ if there exist $0 < A \leq B < \infty$ such that for each $g \in \mathscr{H}$,

$$A\|g\|^2 \leq \sum_{n\in\mathbb{N}} |\langle g, h_n\rangle|^2 \leq B\|g\|^2. \tag{2.3}$$

In Equations (2.2) and (2.3), A and B are referred to as the lower and upper frame bounds, respectively. If only the upper inequality in Equation (2.3) holds, then $\{h_n\}_{n\in\mathbb{N}}$ is called a *Bessel sequence* for $\mathscr{H}$.

Although frames need not even be finitely linearly independent, they have proven to be useful tools in signal processing and approximation theory due to the following. If $\{h_n\}_{n\in\mathbb{N}}$ is a frame for $\mathscr{H}$, then there exists a *dual frame* $\{\widetilde{h_n}\}_{n\in\mathbb{N}}$ such that any $g \in \mathscr{H}$ can be represented as

$$g = \sum_{n\in\mathbb{N}} < g, h_n > \widetilde{h_n} = \sum_{n\in\mathbb{N}} < g, \widetilde{h_n} > h_n.$$

Therefore, an arbitrary function can be reconstructed from its *frame coefficients*, $\{< g, h_n >\}_{n\in\mathbb{N}}$, and the $l^2(\mathbb{N})$ norms of the frame coefficients is equivalent to $\|g\|$.

Every Riesz basis is a frame, but there exist frames which are not Riesz bases. For example, let $\{e_n\}_{n\in\mathbb{N}}$ be an orthornormal basis for $\mathscr{H}$, and let $\{h_n\}_{n\in\mathbb{N}}$ be defined by $h_{2n} = h_{2n+1} = e_n$. Thus, $\{h_n\}_{n\in\mathbb{N}}$ consists of two copies of $\{e_n\}_{n\in\mathbb{N}}$. It is straightforward to see that $\{h_n\}_{n\in\mathbb{N}}$ forms a frame (A and B can be chosen to equal 2), but $\{h_n\}_{n\in\mathbb{N}}$ cannot be a Riesz basis since the elements are not finitely linearly independent. In fact, a strong version of linear independence is exactly what is needed to guarantee that a given frame is a Riesz basis.

The collection $\{h_n\}_{n\in\mathbb{Z}}$ is a *minimal system* for $\mathscr{H}$ if for each $j \in \mathbb{Z}$,

$$h_j \notin \overline{\operatorname{span}\{h_n : n \neq j\}},$$

where the closure is taken in the norm of $\mathscr{H}$. In a finite dimensional Hilbert space, this property is equivalent to linear independence, but in infinite dimensions it can happen that any finite subset of $\{h_n\}_{n\in\mathbb{Z}}$ is linearly independent but the collection is not minimal.

A system is called *exact* if it is both complete and minimal. We will be interested in cases where $\mathscr{T}(F)$ is a minimal system for $V(F)$, but by definition $\mathscr{T}(F)$ is complete in $V(F)$, so $\mathscr{T}(F)$ is minimial for $V(F)$ if and only if it is exact for $V(F)$.

Note that the definition of minimality implies that for each $n \in \mathbb{N}$, there exist a $\widetilde{h_n} \in \mathscr{H}$ such that

$$< h_m, \widetilde{h_n} >= \delta_{m,n}.$$

The collection $\{\widetilde{h_n}\}_{n\in\mathbb{N}}$ is called a *biorthogonal dual* of $\{h_n\}_{n\in\mathbb{N}}$. It is straightforward to show that if a collection $\{h_n\}_{n\in\mathbb{Z}} \subset \mathscr{H}$ has a biorthogonal dual, then $\{h_n\}_{n\in\mathbb{Z}}$ is minimal, and thus minimality is equivalent to the existence of a biorthogonal dual. It can be shown that if $\{h_n\}_{n\in\mathbb{N}}$ is exact, then this biorthogonal dual is unique. This is summarized in the following proposition.

Proposition 2.2.1. *Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $\mathscr{H}$.*

1. *$\{h_n\}_{n\in\mathbb{N}}$ is a minimal system in $\mathscr{H}$ if and only if there exists a biorthogonal dual of $\{h_n\}_{n\in\mathbb{N}}$.*

2. *$\{h_n\}_{n\in\mathbb{N}}$ is exact in $\mathscr{H}$ if and only if there exists a unique biorthognal dual of $\{h_n\}_{n\in\mathbb{N}}$.*

2.2.2 (C_q)-systems

In [44, 45, 47], a generalization of frames called (C_q)-systems were introduced and used to see if previous results which were known for frames could be extended to these systems. For $2 \leq q \leq \infty$, we say $\{h_n\}_{n\in\mathbb{N}}$ is a (C_q)-system for $\mathscr{H}$ if each $g \in \mathscr{H}$ can be approximated, with an arbitrarily small error, by a finite linear combination $\sum a_n n_n$ such that

$$\|a_n\|_{l^q(\mathbb{N})} \leq C\|g\|_{\mathscr{H}},$$

where C does not depend on g. It is clear from the definition that if $\{h_n\}_{n\in\mathbb{N}}$ is a (C_q)-system for some q, then $\{h_n\}_{n\in\mathbb{N}}$ is complete. The name (C_q)-system can thus be interpreted as complete with l^q control over the coefficients. Note also that if $q_1 > q_2$ then a (C_{q_2})-system is always a (C_{q_1})-system.

In [47], it is shown that $\{h_n\}_{n\in\mathbb{N}}$ is a (C_q)-system for $\mathscr{H}$ if and only if

$$c\|g\|_{\mathscr{H}} \leq \|\langle g, h_n\rangle_{\mathscr{H}}\|_p, \ \ \forall g \in \mathscr{H},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < c$ does not depend of g. Thus, (C_q)-systems obey a weaker version of the left hand side of the frame inequality 2.3, and a Bessel (C_2)-system is a frame. Similarly, an exact Bessel (C_q)-system can be thought of as a generalization of a Riesz basis, and an exact Bessel (C_2)-system is a Riesz basis.

In [45], the following theorem is proven related to minimal (C_q)-systems.

Theorem 2.2.2 (Theorem 3 in [45])**.** *Let $2 \leq q \leq \infty$, and let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $\mathscr{H}$. The following are equivalent.*

1. *The system $\{h_n\}_{n\in\mathbb{N}}$ is an exact (C_q)-system for $\mathscr{H}$.*

2. *The system $\{h_n\}_{n\in\mathbb{N}}$ is a exact, and for all $f \in \mathscr{H}$,*

$$\| < f, \widetilde{h_n} > \|_{l^q(\mathbb{N})} \leq C\|f\|,$$

where $\{\widetilde{h_n}\}_{n\in\mathbb{N}}$ is the biorthogonal dual of $\{h_n\}_{n\in\mathbb{N}}$.

3. The system $\{h_n\}_{n\in\mathbb{N}}$ is complete, and

$$\|a_n\|_{l^q(\mathbb{N})} \leq C\|\sum a_n h_n\|,$$

for every finite sequence of numbers $\{a_n\}$.

Suppose $\mathscr{T}(F)$ is a minimal system for $V(F)$. It is not hard to show that the biorthogonal dual of $\mathscr{T}(F)$ is a set of the form $\mathscr{T}(G)$ where g_j is the element biorthogonal to f_j. Then, for some $g \in \mathscr{H}$,

$$| < g, T_n g_j >_{L^2(\mathbb{R}^d)} | \leq \|g\|_{L^2(\mathbb{R}^d)} \|T_n g_j\|_{L^2(\mathbb{R}^d)} = \|g\|_{L^2(\mathbb{R}^d)} \|g_j\|_{L^2(\mathbb{R}^d)}.$$

Therefore,

$$\| < g, T_n g_j >_{L^2(\mathbb{R}^d)} \|_\infty \leq \max_j \{\|g_j\|_{L^2(\mathbb{R}^d)}\} \|g\|_{L^2(\mathbb{R}^d)}.$$

By part 2 of Theorem 2.2.2, we find that $\mathscr{T}(F)$ is a (C_∞)-system for $V(F)$.

In summary, we find that (C_2)-systems are closely related to frames and Riesz bases, and if $\mathscr{T}(F)$ is a minimal system for $V(F)$, then it is a (C_∞)-system for $V(F)$. Therefore, in some sense, (C_q) systems for a continuous interpolation between Riesz bases and minimal systems, and the (C_q)-property becomes weaker as q increases.

2.3 Fractional Sobolev Spaces

Recall that the Fourier coefficients of $f \in L^2(\mathbb{T}^d)$ are defined by

$$\forall k \in \mathbb{Z}^d, \quad \widehat{f}(k) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i x\cdot k} dx$$

where dx is Lebesgue measure on the torus $\mathbb{T}^d$. Also recall Parseval's theorem

$$\int_{\mathbb{T}^d} |f(x)|^2 dx = \sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^2, \tag{2.4}$$

and the translation property

$$\forall y \in \mathbb{R}^d, \forall k \in \mathbb{Z}^d, \quad \widehat{T_y f}(k) = \widehat{f}(k)e^{-2\pi i y\cdot k}. \tag{2.5}$$

We now define two classes of Sobolev functions which will be used throughout this thesis.

Definition 2.3.1. Given $s > 0$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of all measurable functions f defined on $\mathbb{R}^d$ such that $\|f\|_{H^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 d\xi\right)^{1/2} < \infty$. Equivalently, $f \in H^s(\mathbb{R}^d)$ if and only if $f \in L^2(\mathbb{R}^d)$ and

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi\right)^{1/2} < \infty. \tag{2.6}$$

The following is an equivalent characterization of (2.6) when $0 < s < 1$, e.g., [**?**],

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^d)} = C(d,s)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x+y)-f(x)|^2}{|y|^{d+2s}} dxdy. \tag{2.7}$$

We do not prove this here, but we prove a simlar result for periodic Sobolev spaces.

Similar to the above definition, we now define a Sobolev space $H^s(\mathbb{T}^d)$ of periodic functions.

Definition 2.3.2. Given $s > 0$, define the Sobolev space $H^s(\mathbb{T}^d) = \{f \in L^2(\mathbb{T}^d) : \|f\|_{\dot{H}^s(\mathbb{T}^d)} < \infty\}$, where $\|f\|_{\dot{H}^s(\mathbb{T}^d)} = \left(\sum_{k\in\mathbb{Z}^d} |k|^{2s}|\widehat{f}(k)|^2\right)^{1/2}$. Similar to above, we define

$$\|f\|_{H^s(\mathbb{T}^d)} = \left(\sum_{k\in\mathbb{Z}^d}(1+|k|^2)^s|\widehat{f}(k)|^2\right)^{1/2}.$$

The following proposition gives a useful equivalent characterization of $\|f\|_{\dot{H}^s(\mathbb{T}^d)}$ for $0 < s < 1$. Equation (2.8) was proven in Proposition 1.3 in [14], and equation (2.9) is an extension to $H^s(\mathbb{T}^d)$ of the equivalence on page 66 in [16]. We provide a proof of (2.9) below. We use the notation $X \asymp Y$ to indicate that there exist absolute constants $0 < C_1 \leq C_2$ such that $C_1 X \leq Y \leq C_2 X$.

Lemma 2.3.3. *Fix* $0 < s < 1$*, and suppose that* $f \in L^2(\mathbb{T}^d)$*. Then*

$$\|f\|^2_{\dot{H}^s(\mathbb{T}^d)} \asymp \int_{\mathbb{T}^d} \int_{[-\frac{1}{2},\frac{1}{2})^d} \frac{|f(x+y)-f(x)|^2}{|y|^{d+2s}} dydx. \tag{2.8}$$

Moreover, if $\{e_j\}_{j=1}^d$ *is the canonical basis for* $\mathbb{R}^d$ *then*

$$\|f\|^2_{\dot{H}^s(\mathbb{T}^d)} \asymp \sum_{j=1}^d \int_{\mathbb{T}^d} \int_{[-\frac{1}{2},\frac{1}{2})} \frac{|f(x+te_j)-f(x)|^2}{|t|^{1+2s}} dtdx. \tag{2.9}$$

The implicit constants in (2.8) *and* (2.9) *depend only on s and d.*

Proof. We prove (2.9), and remark that (2.8) can be shown with a similar argument. By Parseval's theorem

$$\begin{aligned}\sum_{j=1}^d \int_{[-\frac{1}{2},\frac{1}{2})} \int_{\mathbb{T}^d} \frac{|f(x+te_j)-f(x)|^2}{|t|^{1+2s}} dxdt &= \sum_{j=1}^d \int_{[-\frac{1}{2},\frac{1}{2})} \frac{\sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^2 |e^{-2\pi i k\cdot te_j}-1|^2}{|t|^{1+2s}} dt \\ &= \sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^2 H(k), \end{aligned} \tag{2.10}$$

where

$$H(k) = \sum_{j=1}^d \int_{[-\frac{1}{2},\frac{1}{2})} \frac{|e^{-2\pi i tk\cdot e_j}-1|^2}{|t|^{1+2s}} dt = \sum_{j=1}^d \int_{[-\frac{1}{2},\frac{1}{2})} \frac{|e^{-2\pi i tk_j}-1|^2}{|t|^{1+2s}} dt. \tag{2.11}$$

If $k_j = 0$, then $\int_{[-\frac{1}{2},\frac{1}{2})} \frac{|e^{-2\pi i tk_j}-1|^2}{|t|^{1+2s}} dt = 0 = |k_j|^{2s}$. If $k_j \neq 0$, then,

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|e^{-2\pi i tk_j}-1|^2}{|t|^{1+2s}} dt &= |k_j|^{2s} \int_{-\frac{|k_j|}{2}}^{\frac{|k_j|}{2}} \frac{|e^{-2\pi i t}-1|^2}{|t|^{1+2s}} dt \\ &\leq |k_j|^{2s} \int_{-\infty}^{\infty} \frac{|e^{-2\pi i t}-1|^2}{|t|^{1+2s}} dt\end{aligned}$$

Using the fact that $|e^{2\pi i t}-1| \leq \min(2\pi|t|, 2)$, it is straigtforward to see that for all $0 < s < 1$, $\int_{-\infty}^{\infty} \frac{|e^{-2\pi i t}-1|^2}{|t|^{1+2s}} dt < \infty$. Thus, there is a constant $C > 0$ such that

$$H(k) \leq C \sum_{j=1}^d |k_j|^{2s} \leq Cd|k|^{2s}. \tag{2.12}$$

Next, we use the fact that $(1-\cos(2\pi\theta)) \geq \theta^2/2$ for all $|\theta| \leq 1/4$ to find a lower bound on $H(k)$. Note that for any nonzero component, k_j, of k, we must have $k_j \geq 1$. Then,

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{|e^{-2\pi i t k_j}-1|^2}{|t|^{1+2s}} dt &= |k_j|^{2s} \int_{-\frac{|k_j|}{2}}^{\frac{|k_j|}{2}} \frac{|e^{-2\pi i t}-1|^2}{|t|^{1+2s}} dt \\
&= 2|k_j|^{2s} \int_{-\frac{|k_j|}{2}}^{\frac{|k_j|}{2}} \frac{1-\cos(2\pi t)}{|t|^{1+2s}} dt \\
&= 2|k_j|^{2s} \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1-\cos(2\pi t)}{|t|^{1+2s}} dt \\
&= |k_j|^{2s} \int_{-\frac{1}{4}}^{\frac{1}{4}} |t|^{1-2s} dt \\
&= 2^{-3+4s} |k_j|^{2s}
\end{aligned}$$

Thus, using (2.11), (2.12), and the equivalence of the l^1 and $l^{\frac{1}{s}}$ norms, there exists constants $C_1, C_2 > 0$ such that

$$C_1|k|^{2s} \leq 2^{-3+4s} \sum_{j=1}^{d} |k_j|^{2s} \leq H(k) \leq C_2|k|^{2s}.$$

□

The following theorem shows that for large enough s, $H^s(\mathbb{T}^d)$ embeds into a space of Hölder continuous periodic functions $C^\alpha(\mathbb{T}^d)$. For $0 < \alpha \leq 1$, let $\|f\|_{\dot{C}^\alpha(\mathbb{T}^d)} = \sup_{x \neq t \in \mathbb{T}^d} \frac{|f(x+t)-f(x)|}{|t|^\alpha}$, and $\|f\|_{C^\alpha(\mathbb{T}^d)} = \|f\|_{L^\infty(\mathbb{T}^d)} + \|f\|_{\dot{C}^\alpha(\mathbb{T}^d)}$. Then, $C^\alpha(\mathbb{T}^d)$ is the collection of functions, f, on $\mathbb{T}^d$ such that $\|f\|_{C^\alpha(\mathbb{T}^d)} < \infty$, and $\|\cdot\|_{C^\alpha(\mathbb{T}^d)}$ is a norm on $C^\alpha(\mathbb{T}^d)$.

This theorem is well known when $\mathbb{T}^d$ is replaced by $\mathbb{R}^d$, (See Theorem 8.2 in [25]) and the following proof is a straightforward adaptation of a version of the $\mathbb{R}^d$ result. We will only use the periodic version in the remaining portion of the thesis.

Theorem 2.3.4 (Periodic Hölder Sobolev Embedding)**.** *Let $s \in (\frac{d}{2}, \frac{d}{2}+1)$. Then, there exists $C > 0$, depending only on s such that,*

$$\|f\|_{C^{s-d/2}(\mathbb{T}^d)} \leq C\|f\|_{H^s(\mathbb{T}^d)},$$

and

$$\|f\|_{\dot{C}^{s-d/2}(\mathbb{T}^d)} \leq C\|f\|_{\dot{H}^s(\mathbb{T}^d)}.$$

Proof. Fix $s \in (\frac{d}{2}, \frac{d}{2}+1)$, and suppose $f \in H^s(\mathbb{T}^d)$. Note that since $s > \frac{d}{2}$,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)| &\leq \left(\sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^2(1+|k|)^{2s}\right)^{1/2} \left(\sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2s}\right)^{1/2} \\
&= \|f\|_{H^s(\mathbb{T}^d)} \left(\sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2s}\right)^{1/2} < \infty. \qquad (2.13)
\end{aligned}$$

Thus, $\widehat{f} \in l^1(\mathbb{Z}^d)$, and we are guaranteed that the inversion formula $f(x) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{2\pi i x\cdot k}$ holds for almost every $x \in \mathbb{T}^d$. Again using (2.13), there exists a $C > 0$ such that for almost every $x \in \mathbb{T}^d$,

$$|f(x)| \leq \sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)| \leq C\|f\|_{H^s(\mathbb{T}^d)},$$

or $\|f\|_{L^\infty(\mathbb{T}^d)} \leq C\|f\|_{H^s(\mathbb{T}^d)}$.

Also,

$$\begin{aligned}
|f(x+t)-f(x)|^2 &= \left|\sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{2\pi i(x+t)\cdot k} - \sum_{k\in\mathbb{Z}^d} \widehat{f}(k)e^{2\pi i x\cdot k}\right|^2 \\
&= \left|\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \widehat{f}(k)e^{2\pi i x\cdot k}(e^{2\pi i t\cdot k}-1)\right|^2 \\
&\leq \left(\sum_{k\in\mathbb{Z}^d} |\widehat{f}(k)|^2|k|^{2s}\right)\left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{|e^{2\pi i t\cdot k}-1|^2}{|k|^{2s}}\right) \\
&\leq \|f\|^2_{\dot{H}^s(\mathbb{T}^d)} \left(\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{|e^{2\pi i t\cdot k}-1|^2}{|k|^{2s}}\right) \qquad (2.14)
\end{aligned}$$

The remaining sum can be bounded by splitting it into pieces as follows.

$$\begin{aligned}\sum_{|k|\le\frac{1}{|t|},k\neq 0}\frac{|e^{2\pi it\cdot k}-1|^2}{|k|^{2s}} &\le 4\pi^2|t|^2\sum_{|k|\le\frac{1}{|t|},k\neq 0}|k|^{2(1-s)}\\ &\le C|t|^2\int_{|\xi|\le\frac{1}{|t|}}|\xi|^{2(1-s)}d\xi\\ &\le C_1|t|^{2s-d}, \end{aligned}\tag{2.15}$$

which is finite since $s<\frac{d}{2}+1$, and similarly

$$\begin{aligned}\sum_{|k|>\frac{1}{|t|}}\frac{|e^{2\pi it\cdot k}-1|^2}{|k|^{2s}} &\le 2\sum_{|k|>\frac{1}{|t|}}|k|^{-2s}\\ &\le C_2|t|^{2s-d}. \end{aligned}\tag{2.16}$$

Combining, (2.14), (2.15), and (2.16) shows that there is a $C>0$ such that

$$\sup_{x\neq t\in\mathbb{T}^d}\frac{|f(x+t)-f(x)|}{|t|^{s-d/2}}\le C\|f\|_{\dot{H}^s(\mathbb{T}^d)}.$$

$\square$

2.4 Discrete Hilbert Transform

Note that $\widehat{H^s(\mathbb{T})}=\{\widehat{m}:m\in H^s(\mathbb{T})\}$ is given by the collection of sequences $\alpha=\{\alpha_l\}_{l\in\mathbb{Z}}$ of complex numbers such that $\sum_{l\in\mathbb{Z}}|\alpha_l|^2(1+|l|^{2s})<\infty$. We will be interested in studying when certain multiplication operators are bounded from $H^s(\mathbb{T})$ to itself. This leads to studying boundedness of certain convolution operators on $\widehat{H^s(\mathbb{T})}$.

In fact, for a particular collection of periodic functions $m\in H^s(\mathbb{T})$ for $1/2<s<1$, we would like to show that $\operatorname{sign}(x)m(x)\in H^s(\mathbb{T})$. Let ψ be the 1-periodic function satisfying

$$\psi(x)=\frac{\pi i}{2}\operatorname{sign}(x)\quad -\frac{1}{2}<x\le\frac{1}{2}.$$

Then,

$$\widehat{\psi}(l) = \begin{cases} \frac{1}{l} & l \in 2\mathbb{Z}+1 \\ 0 & l \in 2\mathbb{Z}. \end{cases}$$

For a sequence $\alpha = \{\alpha_l\}_{l\in\mathbb{Z}}$ define the *discrete Hilbert transform* of α, $H\alpha$ by

$$H\alpha = \alpha \star \widehat{\psi},$$

so that

$$[H\alpha]_n = \sum_{n-l\in 2\mathbb{Z}+1} \frac{\alpha_l}{n-l}.$$

Similarly, define the *discrete conjugate function* of α, $T\alpha$, by

$$[T\alpha]_n = \sum_{l\in\mathbb{Z}, l\neq n} \frac{\alpha_l}{n-l}.$$

We will show the following in a sequence of lemmas.

Proposition 2.4.1. *If $\sum_{l\in\mathbb{Z}} \alpha_l = 0 = \sum_{l\in\mathbb{Z}} (-1)^l \alpha_l$ and $1/2 < s < 3/2$ then there exists a $C > 0$ depending only on s such that,*

$$\sum_{l\in\mathbb{Z}} |(H\alpha)_l|^2 (1+|l|^{2s}) \leq C \sum_{l\in\mathbb{Z}} |\alpha_l|^2 (1+|l|^{2s}).$$

In particular, if $m \in H^s(\mathbb{T})$ for $1/2 < s < 3/2$ and $m(0) = m(1/2) = 0$, then $\tilde{m}(x) = \operatorname{sign}(x) m(x)$ satisfies $\tilde{m} \in H^s(\mathbb{T})$.

Theorem 10 in [40] gives conditions under which T is a bounded operator between weighted l^p spaces. Before stating the theorem, we need the following definition. A sequence $w = \{w_l\}_{l\in\mathbb{Z}}$ with $w_l \geq 0$ is a *discrete A_p weight* if there exists a $C > 0$ such that for all $m \leq n$ with $m, n \in \mathbb{Z}$ there holds

$$\left(\sum_{l=m}^{n} w_l\right)\left(\sum_{l=m}^{n} w_l^{-\frac{1}{p-1}}\right)^{p-1} \leq C(n-m+1)^p.$$

Theorem 2.4.2 (Theorem 10 in [40]). *If $\{w_l\}_{l\in\mathbb{Z}}$ is a discrete A_p weight then*

$$\sum_{l\in\mathbb{Z}} |(T\alpha)_l|^p w_l \leq C \sum_{l\in\mathbb{Z}} |\alpha_l|^p w_l.$$

Next we show that weights related to $\widehat{H^s(\mathbb{T})}$ are A_2 weights for certain values of s. Let w_l^λ be defined by

$$w_l^\lambda = \begin{cases} 1 & l=0 \\ |l|^\lambda & l\neq 0. \end{cases}$$

Lemma 2.4.3. *For $-1<\lambda<1$, the weight, $w^\lambda = \{w_l^\lambda\}_{l\in\mathbb{Z}}$, is a discrete A_2 weight.*

Proof. First, when $\lambda=0$, the claim is obvious. By symmetry it suffices to consider $0<\lambda<1$. Note that for any $n,m\in\mathbb{Z}$, we have

$$\left(\sum_{l=m}^{n} w_l^\lambda\right)\left(\sum_{l=m}^{n} w_l^{-\lambda}\right) \leq 2^\lambda \left(\int_{m-1}^{n+1}(1+|x|)^\lambda dx\right)\left(\int_{m-1}^{n+1}(1+|x|)^{-\lambda} dx\right). \tag{2.17}$$

If $0\leq a\leq b$, we have

$$\begin{aligned}
&\left(\int_a^b (1+x)^\lambda dx\right)\left(\int_a^b (1+x)^{-\lambda} dx\right) \\
&\quad= \frac{1}{1-\lambda^2}\left((1+b)^{1+\lambda}-(1+a)^{1+\lambda}\right)\left((1+b)^{1-\lambda}-(1+a)^{1-\lambda}\right) \\
&\quad= \frac{1}{1-\lambda^2}\left((1+b)^2+(1+a)^2-(1+a)^{1+\lambda}(1+b)^{1-\lambda}-(1+a)^{1-\lambda}(1+b)^{1+\lambda}\right) \\
&\quad\leq \frac{1}{1-\lambda^2}\left((1+b)^2+(1+a)^2-2(1+a)(1+b)\right) \\
&\quad= \frac{1}{1-\lambda^2}(b-a)^2,
\end{aligned}$$

where we use the fact that

$$2(1+a)(1+b) \leq (1+a)^{1+\lambda}(1+b)^{1-\lambda}+(1+a)^{1-\lambda}(1+b)^{1+\lambda}.$$

If $a < 0 \leq b$ and $|a| < b$, we have

$$\begin{aligned}
&\left(\int_a^b (1+|x|)^{\lambda} dx\right)\left(\int_a^b (1+|x|)^{-\lambda} dx\right) \\
&\quad = \left(\int_0^b (1+x)^{\lambda} dx + \int_0^{|a|} (1+x)^{\lambda} dx\right)\left(\int_0^b (1+x)^{-\lambda} dx + \int_0^{|a|} (1+x)^{-\lambda} dx\right) \\
&\quad \leq \frac{1}{1-\lambda^2}\left[b^2 + a^2 + \left(\int_0^b (1+x)^{\lambda} dx\right)\left(\int_0^{|a|} (1+x)^{-\lambda} dx\right)\right. \\
&\qquad\qquad \left. + \left(\int_0^{|a|} (1+x)^{\lambda} dx\right)\left(\int_0^b (1+x)^{-\lambda} dx\right)\right] \\
&\quad \leq \frac{1}{1-\lambda^2}\left[b^2 + a^2 + 2\left(\int_0^b (1+x)^{\lambda} dx\right)\left(\int_0^b (1+x)^{-\lambda} dx\right)\right] \\
&\quad \leq \frac{3}{1-\lambda^2}\left[b^2+a^2\right] \leq \frac{3}{1-\lambda^2}(b-a)^2.
\end{aligned}$$

The other cases for a and b follow similarly, and we find that for any $a, b \in \mathbb{R}$ with $a \leq b$,

$$\left(\int_a^b (1+|x|)^{\lambda} dx\right)\left(\int_a^b (1+|x|)^{-\lambda} dx\right) \leq \frac{3}{1-\lambda^2}(b-a)^2.$$

Combining this with equation 2.17, we find that

$$\begin{aligned}
\left(\sum_{l=m}^{n} w_l^{\lambda}\right)\left(\sum_{l=m}^{n} w_l^{-\lambda}\right) &\leq \frac{3(2)^{\lambda}}{1-\lambda^2}(n-m+2)^2 \\
&\leq \frac{12(2)^{\lambda}}{1-\lambda^2}(n-m+1)^2.
\end{aligned}$$

□

Lemma 2.4.3 and Theorem 2.4.2 immediately implies that for $-1/2 < s < 1/2$,

$$\begin{aligned}
\sum_{l \in \mathbb{Z}\setminus\{0\}} |(T\alpha)_l|^2 |l|^{2s} &\leq \sum_{l \in \mathbb{Z}} |(T\alpha)_l|^2 w_l^{2s} \\
&\leq C \sum_{l \in \mathbb{Z}} |\alpha_l|^2 w_l^{2s}
\end{aligned}$$

We would like to extend this result to higher values of s, but to do so we need an extra assumption.

Lemma 2.4.4. *For $1/2 < s < 3/2$, If $\sum_{l\in\mathbb{Z}} \alpha_l = 0$, then there exists a $C > 0$ only depending on s such that,*

$$\sum_{l\in\mathbb{Z}} |(T\alpha)_l|^2(1+|l|^{2s}) \leq C\sum_{l\in\mathbb{Z}} |\alpha_l|^2(1+|l|^{2s}).$$

Proof. First, since $w_l^0 = 1$ is a trivial A_2 weight, by Theorem 2.4.2, we have that

$$\sum_{l\in\mathbb{Z}} |(T\alpha)_l|^2 \leq C\sum_{l\in\mathbb{Z}} |\alpha_l|^2.$$

Also, by assumption we have

$$0 = \sum_{l\in\mathbb{Z}} \alpha_l = n\sum_{l\neq n} \frac{\alpha_l}{n-l} - \sum_{l\neq n} \frac{l\alpha_l}{n-l} + \alpha_n,$$

or, in other words,

$$n[T\alpha]_n = [T(\{l\alpha_l\}_{l\in\mathbb{Z}})]_n - \alpha_n.$$

By assumption, $-1 < 2s-2 < 1$. Thus,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}\setminus\{0\}} |(T\alpha)_n|^2|n|^{2s} &= \sum_{n\in\mathbb{Z}\setminus\{0\}} |n(T\alpha)_n|^2|n|^{2s-2} \\
&= \sum_{n\in\mathbb{Z}\setminus\{0\}} |[T(\{l\alpha_l\}_{l\in\mathbb{Z}})]_n - \alpha_n|^2|n|^{2s-2} \\
&\leq 2\sum_{n\in\mathbb{Z}\setminus\{0\}} |[T(\{l\alpha_l\}_{l\in\mathbb{Z}})]_n|^2|n|^{2s-2} + 2\sum_{n\in\mathbb{Z}\setminus\{0\}} |\alpha_n|^2|n|^{2s-2} \\
&\leq C\sum_{n\in\mathbb{Z}\setminus\{0\}} |n\alpha_n|^2 w_n^{2s-2} + 2\sum_{n\in\mathbb{Z}} |\alpha_n|^2(1+|n|^{2s}) \\
&\leq C\sum_{n\in\mathbb{Z}} |\alpha_n|^2(1+|n|^{2s}).
\end{aligned}$$

Therefore,

$$\sum_{l\in\mathbb{Z}} |(T\alpha)_l|^2(1+|l|^{2s}) \leq C\sum_{l\in\mathbb{Z}} |\alpha_l|^2(1+|l|^{2s}).$$

□

Now we can prove Proposition 2.4.1.

Proof of 2.4.1. First we decompose the sequence α into the even (E) and odd (O) indices,

$$\alpha_l^E = \frac{\alpha_l + (-1)^l \alpha_l}{2}, \qquad \alpha_l^O = \frac{\alpha_l - (-1)^l \alpha_l}{2}.$$

Note that $\sum_{l\in\mathbb{Z}} \alpha_l^E = 0$ and $\sum_{l\in\mathbb{Z}} \alpha_l^O = 0$.

If $n = 2m \in 2\mathbb{Z}$, then

$$\begin{aligned}
[H\alpha]_n &= \sum_{n-l\in 2\mathbb{Z}+1} \frac{\alpha_l}{n-l} \\
&= \sum_{l\in 2\mathbb{Z}+1} \frac{\alpha_l}{n-l} \\
&= \sum_{l\neq n} \frac{\alpha_l^O}{n-l} \\
&= (T\alpha^O)_n.
\end{aligned}$$

Similarly, if $n = 2m+1 \in 2\mathbb{Z}+1$, then

$$[H\alpha]_n = (T\alpha^E)_n.$$

We have,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} |[H\alpha]_n|^2(1+|n|^{2s}) &= \sum_{n\in 2\mathbb{Z}} |[H\alpha]_n|^2(1+|n|^{2s}) + \sum_{n\in 2\mathbb{Z}+1} |[H\alpha]_n|^2(1+|n|^{2s}) \\
&= \sum_{n\in 2\mathbb{Z}} |[T\alpha^O]_n|^2(1+|n|^{2s}) + \sum_{n\in 2\mathbb{Z}+1} |[T\alpha^E]_n|^2(1+|n|^{2s}) \\
&\leq \sum_{n\in\mathbb{Z}} |[T\alpha^O]_n|^2(1+|n|^{2s}) + \sum_{n\in\mathbb{Z}} |[T\alpha^E]_n|^2(1+|n|^{2s}) \\
&\leq C\sum_{n\in\mathbb{Z}} |\alpha_n^O|^2(1+|n|^{2s}) + C\sum_{n\in\mathbb{Z}} |\alpha_n^E|^2(1+|n|^{2s}) \\
&= C\sum_{n\in\mathbb{Z}} |\alpha_n|^2(1+|n|^{2s}).
\end{aligned}$$

Now, suppose $m \in H^s(\mathbb{T})$ with $m(0) = m(1/2) = 0$. This is equivalent to saying

$$\sum_{l\in\mathbb{Z}} |\widehat{m}(l)|^2(1+|l|^{2s}) < \infty,$$

$\sum_{l\in\mathbb{Z}} \widehat{m}(l) = m(0) = 0$, and $\sum_{l\in\mathbb{Z}} \widehat{m}(l)(-1)^l = m(1/2) = 0$. We also know $\widehat{\tilde{m}} = H\widehat{m}$. Thus, $\tilde{m} \in H^s(\mathbb{T})$. $\square$

2.4.1 Fourier Multipliers on l^p

The following important fact also follows from Theorem 10 in [40], with a similar argument to Theorem 2.4.1.

Theorem 2.4.5. *For $1 < p < \infty$,*

$$\|H\alpha\|_p \leq \|\alpha\|_p.$$

Let M be the operator such that

$$Mm(x) = \text{sign}(x)m(x)$$

for any measurable m defined on $\mathbb{T}$. Then, $\widehat{H\alpha} = \frac{\pi i}{2}M(\widehat{\alpha})$. Note then that

$$\widehat{(I - \frac{2}{\pi i}H)\alpha} = 2\chi_{[0,1/2]}\widehat{m}.$$

By Theorem 2.4.5, $S = \frac{1}{2}(I - \frac{2}{\pi i}H)$ is a bounded Fourier multiplier from l^p to l^p for $1 < p < \infty$ with symbol $\chi_{[0,1/2]}$. We will use this to show that for any interval $I \subset \mathbb{T}$ the Fourier multiplier with symbol χ_I is also a bounded operator from l^p to l^p.

Proposition 2.4.6. *Let $J \subset \mathbb{T}$ be an interval. For a sequence $\alpha = \{\alpha_l\}_{l\in\mathbb{Z}}$ let S_J be the operator such that*

$$\widehat{S_J\alpha} = \chi_J\widehat{\alpha}.$$

Then, S_J is a bounded operator from l^p to l^p.

Proof. Identify the torus $\mathbb{T}$ with $(-1/2, 1/2]$. For $J \subset \mathbb{T}$, we can identify J by two points $a, b \in [-1/2, 1/2]$ where a is the initial point of J and b is the terminal point of J, but a is not necessarily less than b. We say $J = [a, b]$. Suppose first that the length of J is $1/2$. Then,

$$S_J = M_a S_{[0,1/2]} M_{-a}$$

where $[M_a \alpha]_l = e^{-2\pi i l a} \alpha_l$, and so S_J is bounded. Now suppose the length of J is less than $1/2$. Then,

$$S_J = S_{[b-1/2,b]} S_{[a,a+1/2]}$$

where $b - 1/2$ and $a + 1/2$ are taken modulo $\mathbb{Z}$. Then, S_J is bounded. Finally, if the length of $J = [a, b]$ is greater than $1/2$, the complimentary interval $[b, a]$ has length less than $1/2$ and $S_{[b,a]}$ is bounded. But $S_J = I - S_{[b,a]}$, and thus S_J is bounded. $\square$

Chapter 3

Structure of Shift-Invariant Spaces

Shift-invariant spaces have been thoroughly studied in the literature, and many properties of finitely-generated shift-invariant spaces can be characterized in terms of a periodic, matrix-valued function called the Gramian. In this chapter, we explore the Gramian in detail. Section 3.1 shows that finitely-generated shift-invariant spaces in $L^2(\mathbb{R}^d)$ are isometrically isomorphic to vector-valued weighted $L^2(\mathbb{T}^d)$ spaces with the weight given by the Gramian. Section 3.2 characterizes several of the basis and frame-type properties of $\mathscr{T}(F)$ in $V(F)$ through properties of the Gramian associated to F. Section 3.3 collects other assorted facts about $V(F)$ which can be characterized by the Gramian matrix, and states an important result about finding generating sets of minimal size for $V(F)$ through linear combinations of elements of F.

3.1 The Gramian and Weighted $L^2(\mathbb{T}^d)$ Spaces

Let $F = \{f_1, f_2, ..., f_k\}$ for some $f_j \in L^2(\mathbb{R}^d)$. Suppose

$$g = \sum_{k=1}^{K} \sum_{n} c_{n,k} T_n f_k,$$

is a finite sum. Then, $g \in V(F)$, and we have

$$\begin{aligned}\widehat{g}(\xi) &= \sum_{k=1}^{K} \sum_{n} c_{n,k} e^{-2\pi i n \cdot \xi} \widehat{f_k}(\xi) \\ &= \sum_{k=1}^{K} m_k(\xi) \widehat{f_k}(\xi),\end{aligned}$$

where each m_k is $\mathbb{Z}^d$ periodic, and $\sum_{k=1}^{K} m_k \widehat{f_k} \in L^2(\mathbb{R}^d)$. With slight abuse of notation, for each $\xi \in \mathbb{R}^d$, let $\widehat{F}(\xi)$ be the column vector $\widehat{F}(\xi) = (\widehat{f_1}(\xi), \widehat{f_2}(\xi), ..., \widehat{f_K}(\xi))^T$, and define the row vector

$M(\xi) = (m_1(\xi), m_2(\xi), ..., m_K(\xi))$. Then $\widehat{g} = M\widehat{F}$, and

$$\begin{aligned} \|g\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |M(\xi)\widehat{F}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^d} M(\xi)\widehat{F}(\xi)\widehat{F}(\xi)^* M(\xi)^* d\xi \\ &= \int_{\mathbb{T}^d} M(\xi) \sum_{l\in\mathbb{Z}^d} \widehat{F}(\xi+l)\widehat{F}(\xi+l)^* M(\xi)^* d\xi. \end{aligned} \tag{3.1}$$

The matrix $P(\widehat{F})(\xi) = \sum_{l\in\mathbb{Z}^d} \widehat{F}(\xi+l)\widehat{F}(\xi+l)^*$ is called the *Gramian* matrix of $\widehat{F}$. Note that for almost every ξ, $P(\widehat{f})(\xi)$ is a positive semi-definite Hermitian matrix.

It will be useful to consider the individual entries of $P(\widehat{F})$. Define the *bracket product* of two $L^2(\mathbb{R}^d)$ functions f and g to be

$$[f,g](x) = \sum_{l\in\mathbb{Z}^d} f(x+l)\overline{g(x+l)}, \tag{3.2}$$

See [20] for more on bracket products. Note that $[f,g] \in L^1(\mathbb{T}^d)$, and $P(\widehat{F})(\xi)$ is the matrix of bracket products of elements of $\widehat{F} = \{\widehat{f_k} : f_k \in F\}$.

$$P(\widehat{F})(x) = \sum_{k\in\mathbb{Z}^d} \widehat{F}(x+k)\widehat{F}^*(x+k) = \begin{pmatrix} \left[\widehat{f_1},\widehat{f_1}\right](x) & \left[\widehat{f_1},\widehat{f_2}\right](x) & \cdots & \left[\widehat{f_1},\widehat{f_K}\right](x) \\ \left[\widehat{f_2},\widehat{f_1}\right](x) & \left[\widehat{f_2},\widehat{f_2}\right](x) & \cdots & \left[\widehat{f_2},\widehat{f_K}\right](x) \\ \vdots & \vdots & \ddots & \vdots \\ \left[\widehat{f_K},\widehat{f_1}\right](x) & \left[\widehat{f_K},\widehat{f_2}\right](x) & \cdots & \left[\widehat{f_K},\widehat{f_K}\right](x). \end{pmatrix}$$

A straightforward calculation shows that for any $f,g \in L^2(\mathbb{R}^d)$ and any $k \in \mathbb{Z}^d$, we have

$$\widehat{\left[\widehat{g},\widehat{f}\right]}(k) = \langle g, T_k f\rangle.$$

Therefore, the Grammian encodes all of the inner products of shifts of the generators as Fourier series coefficients of its entries. Note that if $F = \{f\}$, then $P(F)(x) = P(f)(x) = [f,f](x) = \sum_{k\in\mathbb{Z}^d} |f(x+k)|^2$ which is a scalar-valued, nonnegative, $\mathbb{Z}^d$-periodic function, and $P(f)$ is often called the *periodization* of f.

Equation (3.1) can be rewritten as

$$\|g\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{T}^d} M(\xi)P(\widehat{F})(\xi)M(\xi)^* d\xi. \tag{3.3}$$

We see an immediate connection between the span of $\mathscr{T}(F)$ and the collection of measurable, $\mathbb{Z}^d$-periodic, vector-valued functions $M = (m_1, ..., m_K)$ such that

$$\|M\|_{L^2_{P(\widehat{F})}(\mathbb{T}^d)} = \int_{\mathbb{T}^d} M(\xi)P(\widehat{F})(\xi)M(\xi)^* d\xi < \infty.$$

Define the space $L^2_{P(\widehat{F})}(\mathbb{T}^d)$ to be the collection of equivalence classes of such functions under $\|\cdot\|_{L^2_{P(\widehat{F})}(\mathbb{T}^d)}$. Then, $L^2_P(\mathbb{T}^d)$ is a Hilbert space with inner product,

$$< M, N >= \int_{\mathbb{T}^d} M(x)P(x)N(x)dx.$$

For $g \in \mathrm{span}\{\mathscr{T}(F)\}$, we find that $\|g\|_{L^2(\mathbb{R}^d)} = \|M\|_{L^2_{P(\widehat{F})}(\mathbb{T}^d)}$ for any M such that $\widehat{g} = M\widehat{F}$ almost everywhere. The following proposition shows that this relationship extends to $V(F) = \overline{\mathrm{span}\{\mathscr{T}(F)\}}$.

The following proof is an adaptation of the single-generator proof in [36]. Let $P = P(\widehat{F})$, and define the operator $J : L^2_P(\mathbb{T}^d) \to L^2(\mathbb{R}^d)$ for any $M \in L^2_P(\mathbb{T}^d)$,

$$\widehat{JM} = M\widehat{F} = m_1\widehat{f}_1 + m_2\widehat{f}_2 + \cdots + m_k\widehat{f}_k.$$

Proposition 3.1.1. *J is an isometry between $L^2_P(\mathbb{T}^d)$ and $V(F)$.*

Proof. The calculation in Equation 3.1 remains valid for any $M \in L^2_P(\mathbb{T}^d)$. Thus, J is an isometry on $L^2_P(\mathbb{T}^d)$, and $JM \in L^2(\mathbb{R}^d)$. Suppose $h \in V(F)^\perp$. Then, h is orthogonal to shifts of all generators which is equivalent to $[\widehat{h}, \widehat{f}_j] = 0$ for all j. Then, for any $M \in L^2_P(\mathbb{T}^d)$,

$$\left[\widehat{h}, \widehat{Jm}\right] = m_1\left[\widehat{h}, \widehat{f}_1\right] + \cdots + m_k\left[\widehat{h}, \widehat{f}_k\right] = 0.$$

Thus, $Jm \in (V(F)^{\perp})^{\perp} = V(F)$.

Since J is an isometry, its range R is a closed subspace of $V(F)$. If there exists a function $g \in V(F) \setminus R$, then

$$0 = \langle g, Jm \rangle = \int_{\mathbb{T}^d} m_1[\widehat{g}, \widehat{f}_1] + \cdots + m_k[\widehat{g}, \widehat{f}_k] dx$$

for any choice of $m \in L^2_P(\mathbb{T}^d)$. Choose m' such that $m'_j[\widehat{g}, \widehat{f}_j] = |[\widehat{g}, \widehat{f}_j]|$. Clearly, $m' \in L^2_P(\mathbb{T}^d)$. Then,

$$0 = \langle g, Jm' \rangle = \int_{\mathbb{T}^d} |[\widehat{g}, \widehat{f}_1]| + \cdots + |[\widehat{g}, \widehat{f}_k]| dx$$

and so $[\widehat{g}, \widehat{f}_j] = 0$ for all j. This implies $g \in V(F)^{\perp}$. This contradiction shows that J is surjective onto $V(F)$. $\square$

Fix $F = \{f_1, ..., f_k\} \subset L^2(\mathbb{R}^d)$. Then, for almost every $x \in \mathbb{T}^d$, $P(x) = P(\widehat{F})(x)$ is positive semidefinite, and if $\lambda_1(x) \geq \lambda_2(x) \geq ... \geq \lambda_K(x)$ are the eigenvalues of $P(x)$, there exists a unitary matrix $U(x)$ such that

$$P(x) = U(x)\Lambda(x)U(x)^*, \tag{3.4}$$

where $\Lambda(x)$ is the diagonal matrix with the eigenvalues as entries. Note that $U(x)$ is not unique. Fortunately, Lemma 2.3.5 in [48] shows that the eigenvalue functions $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_K$ are measurable, and there exists a matrix valued function $U(x)$ such that $u_{j,k}$ is measurable for each $j, k \in \{1, ..., K\}$, and for almost every $x \in \mathbb{T}^d$, we have $U(x)$ is unitary and (3.4) holds.

For almost every $x \in \mathbb{T}^d$, the trace of $P(\widehat{F})(x)$ is

$$\operatorname{tr}\left[P(\widehat{F})(x)\right] = \sum_{k=1}^{K} [\widehat{f}_k, \widehat{f}_k](x) = \sum_{k=1}^{K} \lambda_k(x).$$

Note that $[\widehat{f}_k, \widehat{f}_k] \in L^1(\mathbb{T}^d)$ since $\widehat{f}_k \in L^2(\mathbb{R}^d)$, and thus for each k, we have $\lambda_k \in L^1(\mathbb{T}^d)$.

3.2 Basis and Frame Type Properties for Shift-Invariant Spaces

In this section, we find necessary and sufficient conditions for $\mathscr{T}(F)$ to have basis or frame type properties for $V(F)$.

3.2.1 Orthonormal Bases, Frames, and Riesz Bases for $V(F)$

The following result can be found in various forms in several sources including [17, 26, 36, 48]. For $K \times K$ matrices M, N, we use $M \leq N$ to denote that for all row vectors $y \in \mathbb{R}^K$, it holds that $yMy^* \leq yNy^*$. We also denote the $K \times K$ identity matrix by I_K or just I if the dimension is clear.

Theorem 3.2.1. *Let $F = \{f_1, ..., f_K\}$ for some $f_j \in L^2(\mathbb{R}^d)$, and let $P = P(\widehat{F})$.*

1. *$\mathscr{T}(F)$ forms an orthonormal basis for $V(F)$ if and only if*

$$P(x) = I \quad a.e.\ x \in \mathbb{T}^d. \tag{3.5}$$

2. *$\mathscr{T}(F)$ forms a Riesz basis for $V(F)$ if and only if there exists $0 < A \leq B < \infty$ such that*

$$AI \leq P(x) \leq BI, \quad a.e.\ x \in \mathbb{T}^d, \tag{3.6}$$

3. *$\mathscr{T}(F)$ forms a frame for $V(F)$ if and only if there exists $0 < A \leq B < \infty$ such that*

$$AP(x) \leq P(x)^2 \leq BP(x), \quad a.e.\ x \in \mathbb{T}^d. \tag{3.7}$$

In the case of a single generator, $F = \{f\}$, Theorem 3.2.1 says that $\mathscr{T}(F)$ forms

1. an orthonormal basis for $V(F)$ if $P(x) = 1$ almost everywhere,
2. a Riesz basis for $V(F)$ if $P, P^{-1} \in L^\infty(\mathbb{T}^d)$,
3. a frame for $V(F)$ if $P \in L^\infty(\mathbb{T}^d)$ and P is bounded away from zero whenever it is non-zero.

For multiple generators, Theorem 3.2.1 says the same for each of the eigenvalue functions $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_K$ of P.

To give perspective on these results and to show the usefulness of Proposition 3.1.1, we provide a proof of the Riesz basis result below.

Proof. Based on Proposition 3.1.1, it suffices to prove that $E = \{e^{2\pi i n\cdot x} e_j\}_{1\leq j\leq K, n\in\mathbb{Z}^d}$, where e_j is a canonical basis row vector in $\mathbb{R}^K$, is a Riesz basis for $L^2_P(\mathbb{T}^d)$ if and only if the stated condition holds. The Riesz basis property is equivalent to the existance of constants $0 < C_1 \leq C_2 < \infty$ such that for any $c_{j,n} \in l^2(\mathbb{Z}^d)^K$, we have

$$C_1\|c_{j,n}\|^2_{l^2(\mathbb{Z}^d)^K} \leq \|\sum_{j=1}^{K}\sum_{n\in\mathbb{Z}^d} c_{j,n}e^{2\pi i n\cdot x}e_j\|^2_{L^2_P(\mathbb{T}^d)} \leq C_2\|c_{j,n}\|^2_{l^2(\mathbb{Z}^d)^K}. \tag{3.8}$$

Note that for each j, $\{c_{j,n}\}_{n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$. Thus, there exist $m_j \in L^2(\mathbb{T}^d)$ such that $\widehat{m_j}(n) = c_{j,n}$. Letting $M(x) = (m_1(x), ..., m_K(x))$ and using Parseval's equality, we see that the Riesz basis property (3.8) is equivalent to saying, for all $M = (m_1, ..., m_K) \in L^2(\mathbb{T}^d)^K$ we must have

$$C_1\|M\|^2_{L^2(\mathbb{T}^d)^K} \leq \|M\|^2_{L^2_P(\mathbb{T}^d)} \leq C_2\|M\|^2_{L^2(\mathbb{T}^d)^K}. \tag{3.9}$$

Let U and Λ be a measurable diagonalization of P as in (3.4). For any $M \in L^2_P(\mathbb{T}^d)$, note that

$$\int_{\mathbb{T}^d} M(x)P(x)M(x)^*dx = \int_{\mathbb{T}^d} \widetilde{M}(x)\Lambda(x)\widetilde{M}^*(x)dx \tag{3.10}$$

where $\widetilde{M}(x) = M(x)U(x)$, and since $U(x)$ is unitary for almost every $x \in \mathbb{T}^d$,

$$\|\widetilde{M}\|_{L^2(\mathbb{T}^d)^K} = \|M\|_{L^2(\mathbb{T}^d)^K}. \tag{3.11}$$

Combining (3.9), (3.10), and (3.11) we see that E is a Riesz basis for $L^2_P(\mathbb{T}^d)$ if and only if E is a Riesz basis for $L^2_\Lambda(\mathbb{T}^d)$.

If $AI \leq P(x) \leq BI$, then for each $j \in \{1, ..., K\}$, we have $A \leq \lambda_j(x) \leq B$ for almost every $x \in \mathbb{T}^d$.

This gives,

$$A\int_{\mathbb{T}^d}\widetilde{M}(x)\widetilde{M}(x)^*dx \le \int_{\mathbb{T}^d}\widetilde{M}(x)\Lambda(x)\widetilde{M}(x)^*dx \le B\int_{\mathbb{T}^d}\widetilde{M}(x)\widetilde{M}(x)^*dx, \tag{3.12}$$

and so E is a Riesz basis for $V(F)$.

Now, if the lower condition is not satisfied in (3.6), then for any $l \in \mathbb{N}$, there exists a set $S_l \subset \mathbb{T}^d$ of positive measure such that the smallest eigenvalue, λ_K, of P, satisfies $\lambda_K(x) \le \frac{1}{l}$ for almost every $x \in S_l$.

For each l, let $M_l(x) = \frac{1}{|S_l|}\chi_{S_l}(x)e_K$. Then, $\|M_l\|_{L^2(\mathbb{T}^d)^K} = 1$, but

$$\|M_l\|_{L^2_\Lambda(\mathbb{T}^d)} = \frac{1}{|S_l|}\int_{S_l}\lambda_K(x)dx \le \frac{1}{l}.$$

Thus, the Riesz Basis condition cannot be satisfied. A similar argument shows the same result if the upper condition is not satisfied in (3.6). □

3.2.2 Minimal systems for $V(F)$

We now prove a characterization for $\mathscr{T}(F)$ forming a minimal system for $V(F)$. A single-generator, dimension 1 version of the following result is given in [36].

Proposition 3.2.2. *Let $F = \{f_1, ..., f_K\} \subset L^2(\mathbb{R}^d)$, and let $P = P(\widehat{F})$. The following are equivalent.*

1. *$\mathscr{T}(F)$ is a minimal system for $V(F)$.*

2. *$P(x)$ is invertible for almost every x, and the diagonal elements of P^{-1} are integrable on $\mathbb{T}^d$.*

3. *The eigenvalue functions $\lambda_1(x) \ge \lambda_2(x) \ge ... \ge \lambda_K(x)$ of $P(x)$ satisfy*

$$\frac{1}{\lambda_j} \in L^1(\mathbb{T}^d).$$

for each j.

Note that if $P(x)$ was diagonal for almost every x, this proposition shows that $\frac{1}{P(\widehat{f_j})} \in L^1(\mathbb{T}^d)$ for each j, which agrees with the singly-generated characterization of minimality.

Proof. It suffices to replace *(1)* with the statement, $E = \{e^{2\pi i n\cdot x}e_j\}_{1\leq j\leq K, n\in\mathbb{Z}^d}$ is a minimal system for $L^2_P(\mathbb{T}^d)$.

First we show *(1)* implies *(2)*. Assume E is a minimal system for $L^2_P(\mathbb{T}^d)$. From Propositon 2.2.1, E must have a biorthogonal dual system in $L^2_P(\mathbb{T}^d)$. Let $g_{j,n}$ be the dual element for $e^{2\pi i n\cdot x}e_j$. Then $g_{j,0}$ satisfies

$$\begin{aligned}
\delta_{l,0}\delta_{k,j} &= \langle g_{j,0}, e^{2\pi i l\cdot x}e_k\rangle_{L^2_P(\mathbb{T}^d)} \\
&= \int_{\mathbb{T}^d} g_{j,0}P(x)e_k^* e^{-2\pi i l\cdot x}dx \\
&= \int_{\mathbb{T}^d} e^{-2\pi i l\cdot x} g_{j,0}(x)P_k(x)dx \\
&= \widehat{g_{j,0}P_k}(l),
\end{aligned}$$

where $P_k(x)$ is the k^{th} column of $P(x)$. Then, for any $k \neq j$, we must have $g_{j,0}(x)P_k(x) = 0$ for almost every x, and we also have $g_{j,0}(x)P_j(x) = 1$ almost everywhere. Equivalently, $g_{j,0}(x)P(x) = e_j$. If we let W be the matrix with j^{th} row, $g_{j,0}$, then we find $W(x) = P(x)^{-1}$ almost everywhere. Also, since $g_{j,0} = (w_{j,1}, ...w_{j,K})$ is in $L^2_P(\mathbb{T}^d)$, we have

$$\int_{\mathbb{T}^d} w_{j,j}(x)dx = \int_{\mathbb{T}^d} g_{j,0}(x)P(x)g^*_{j,0}(x)dx < \infty. \tag{3.13}$$

Note that $w_{j,j}(x)$ must be nonnegative almost everywhere since it can be written as $w_j(x)P(x)w_j^*(x)$. Thus, $w_{j,j} \in L^1(\mathbb{T}^d)$.

Next, we show *(2)* implies *(1)*. Suppose $P(x)$ is invertible for almost every x, and the diagonal entries of the inverse are integrable. Let $W(x) = P(x)^{-1}$. Let

$$w_j(x) = (w_{j,1}(x), w_{j,2}(x), ..., w_{j,K}(x)),$$

be the j^{th} row of $W(x)$. Define $g_{j,n}(x) = e^{2\pi i n\cdot x} w_j(x)$. Equation 3.13 shows that $g_{j,0} = w_j \in L^2_P(\mathbb{T}^d)$. It is a straightforward calculation to show that $g_{j,n}$ is biorthogonal to $e^{2\pi i n\cdot x} e_j$, and therefore E is minimal for $L^2_P(\mathbb{T}^d)$.

Finally, the equivalence of *(2)* and *(3)* follows from the trace formula,

$$\operatorname{tr}(P(x)^{-1}) = \sum_{j=1}^{K} w_{j,j}(x) = \sum_{j=1}^{K} \frac{1}{\lambda_j(x)}.$$

Since, $w_{j,j}$ and $\frac{1}{\lambda_j}$ are nonnegative almost everywhere for each j, the trace formula gives both

$$\begin{aligned} w_{j,j}(x) &\leq \sum_{j=1}^{K} \frac{1}{\lambda_j(x)} \\ \frac{1}{\lambda_j(x)} &\leq \sum_{j=1}^{K} w_{j,j}(x). \end{aligned}$$

Therefore, if either $\{w_{j,j}\}_{j=1}^K \subset L^1(\mathbb{T}^d)$ or $\{\frac{1}{\lambda_j}\}_{j=1}^K \subset L^1(\mathbb{T}^d)$, then both sets are subsets of $L^1(\mathbb{T}^d)$.

□

3.2.3 Minimal (C_q)-systems for $V(F)$

Lemma 1 in [45] gives necessary and sufficient conditions for $E = \{e^{2\pi i n\cdot x}\}_{n\in\mathbb{Z}^2}$ to form a minimal (C_q)-system in $L^2_W(\mathbb{T}^2)$, where W is a scalar-valued, almost everywhere positive weight. This proof directly carries over to $L^2_W(\mathbb{T}^d)$ for any dimension d. We will now extend this to the case of matrix-valued, almost everywhere positive weights to find necessary and sufficient conditions for $\mathscr{T}(F)$ to form a minimal (C_q)-system for $V(F)$.

Proposition 3.2.3. *Fix $q > 2$. Let $F = \{f_1, ..., f_K\} \subset L^2(\mathbb{R}^d)$, and let $P = P(\widehat{F})$.*

a) $\mathscr{T}(F)$ forms a minimal (C_q)-system for $V(F)$ if and only if for all $M = (m_1, ..., m_k) \in L^2_P(\mathbb{T}^d)$, we have

$$\left(\sum_{j=1}^{k} \sum_{l\in\mathbb{Z}^d} |\widehat{m_j}(l)|^q\right)^{\frac{1}{q}} < C\|M\|_{L^2_P(\mathbb{T}^d)}. \tag{3.14}$$

b) Let $W(x) = P(\widehat{F})(x)^{-1}$ for almost every x if such an inverse exits. Let $w_j = (w_{j,1}, w_{j,2}, ..., w_{j,k})$

be the j^{th} row of W. If

$$w_{j,j} \in L^{\frac{q}{q-2}}[0,1]^d$$

for all $j \in \{1,..,k\}$, then $\mathscr{T}(F)$ is a minimal (C_q)-system in $V(F)$.

c) If the eigenvalue functions $\lambda_1(x) \geq \lambda_2(x) \geq ... \geq \lambda_k(x)$ of $P(\widehat{F})(x)$ satisfy

$$\frac{1}{\lambda_j} \in L^{\frac{q}{q-2}}[0,1]^d$$

for each j, then $\mathscr{T}(F)$ is a minimal (C_q)-system in $V(F)$.

Proof. Note that if we prove part *b)*, then part *c)* follows from the trace formula calculation at the end of Proposition 3.2.2. As before, it suffices to prove these results with $\mathscr{T}(F)$ replaced by $E = \{e^{2\pi in\cdot x}e_j\}_{1\leq j\leq K, n\in\mathbb{Z}^d}$ and $V(F)$ replaced with $L^2_P(\mathbb{T}^d)$.

We start with a calculation which will be used for proving parts *a)* and *b)* of the proposition. Assume, $\mathscr{T}(F)$ is minimal for $V(F)$. By Proposition 3.2.2, $W(x) = P(x)^{-1}$ is defined almost everywhere, and by the proof of the same proposition, the biorthogonal dual element corresponding to $g_{j,n}(x) = e^{2\pi in\cdot x}e_j$ is given by $\widetilde{g_{j,n}}(x) = e^{2\pi in\cdot x}e_jP(x)^{-1} = e^{2\pi in\cdot x}w_j(x)$. For any $M = (m_1, ..., m_K) \in L^2_P(\mathbb{T}^d)$, We have

$$\begin{aligned} \langle M, \widetilde{g_{j,n}}\rangle_{L^2_P(\mathbb{T}^d)} &= \int_{[0,1]^d} M(x)P(x)w_j(x)^* e^{-2\pi in\cdot x}dx \\ &= \int_{[0,1]^d} m_j(x)e^{-2\pi in\cdot x}dx = \widehat{m_j}(n) \end{aligned} \tag{3.15}$$

a) Theorem 2.2.2 shows that E is a minimal (C_q)-system for $L^2_P(\mathbb{T}^d)$ if and only if $\mathscr{T}(F)$ is minimal, and the biorthogonal dual system $\{\widetilde{g_{j,n}}\}_{1\leq j\leq K, n\in\mathbb{Z}^d}$, satisfies

$$\left(\sum_{j=1}^{k}\sum_{n\in\mathbb{Z}^d} |\langle M, \widetilde{g_{j,n}}\rangle_{L^2_P(\mathbb{T}^d)}|^q\right)^{1/q} \leq C\|M\|_{L^2_P(\mathbb{T}^d)}, \qquad \forall M \in L^2_P(\mathbb{T}^d)$$

However, Equation 3.15 shows the following.

$$\left(\sum_{j=1}^{k}\sum_{n\in\mathbb{Z}^d}|\langle M,\widetilde{g_{j,n}}\rangle_{L^2_P(\mathbb{T}^d)}|^q\right)^{1/q}=\left(\sum_{j=1}^{k}\sum_{n\in\mathbb{Z}^d}|\widehat{m_j}(n)|^q\right)^{1/q} \tag{3.16}$$

b) We will show that (3.14) holds under these assumptions. For any $M\in L^2_P(\mathbb{T}^d)$ and using the Hausdorff-Young inequality, we have

$$\begin{aligned}\sum_{j=1}^{k}\sum_{l\in\mathbb{Z}^d}|\widehat{m_j}(l)|^q &\leq \sum_{j=1}^{k}\left(\int_{\mathbb{T}^d}|m_j(x)|^p dx\right)^{q/p}\\ &=\sum_{j=1}^{k}\left(\int_{\mathbb{T}^d}\left|M(x)P(x)w_j(x)^*\right|^p dx\right)^{q/p}\end{aligned}$$

Now, by applying Cauchy-Schwartz and then Hölder's inequality (with $t=\frac{2}{p}=\frac{2(q-1)}{q}$ and $r=\frac{2}{2-p}=\frac{2(q-1)}{q-2}$), we have

$$\begin{aligned}\sum_{j=1}^{k}\sum_{l\in\mathbb{Z}^d}|\widehat{m_j}(l)|^q &\leq \sum_{j=1}^{k}\left(\int_{\mathbb{T}^d}\left|M(x)P(x)w_j(x)^*\right|^p dx\right)^{\frac{q}{p}}\\ &\leq \sum_{j=1}^{k}\left(\int_{\mathbb{T}^d}\left|M(x)\sqrt{P(x)}\right|^p\left|\sqrt{P(x)}w_j(x)^*\right|^p dx\right)^{\frac{q}{p}}\\ &\leq \left(\int_{\mathbb{T}^d}|M(x)\sqrt{P(x)}|^2 dx\right)^{q/2}\sum_{j=1}^{k}\left(\int_{\mathbb{T}^d}\left|w_j(x)\sqrt{P(x)}\right|^{\frac{2q}{q-2}}dx\right)^{\frac{q-2}{2}}\\ &=\|M\|^q_{L^2_P(\mathbb{T}^d)}\sum_{j=1}^{k}\left(\int_{\mathbb{T}^d}\left(w_j(x)P(x)w_j(x)^*\right)^{\frac{q}{q-2}}dx\right)^{\frac{q-2}{2}}\\ &=\|M\|^q_{L^2_P(\mathbb{T}^d)}\sum_{j=1}^{k}\left(\int_{\mathbb{T}^d}w_{j,j}(x)^{\frac{q}{q-2}}dx\right)^{\frac{q-2}{2}}.\end{aligned}$$

Since $w_{j,j}\in L^{\frac{q}{q-2}}(\mathbb{T}^d)$, Equation (3.14) holds. □

3.2.4 Nonminimal (C_q)-systems for $V(f)$

Without the assumption of minimality, the situation is more difficult. In this setting we do not have the luxury of using Propositon 3.2.2. Still, we have the following result which is similar to Proposition 3.2.3 part *(1)*, but which only holds for a generating set of minimal size in dimension 1 and with extra assumptions on $P = P(\widehat{F})$.

Theorem 3.2.4. *Fix $2 \leq q < \infty$. Suppose $F = \{f_1, ..., f_K\} \subset L^2(\mathbb{R})$ is such that $P = P(\widehat{F})$ has continuous eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_K$, and such that for some set of positive measure $\lambda_K > 0$. Suppose that λ_K has at least one zero in $\mathbb{T}$ and let $I := [a,b] \subset \mathbb{T}$ be such that, $\lambda_K(a) = \lambda_K(b) = 0$, and $\lambda_K > 0$ on (a,b). If $\mathscr{T}(F)$ is a (C_q)-system for $V(F)$, then for every $\mathbb{Z}$-periodic $M = (m_1, ..., m_K)$ supported on I such that $\int_I M(x)P(x)M(x)^* dx < \infty$ we have*

$$\left(\sum_{n \in \mathbb{Z}} |\hat{M}(n)|^q \right)^{\frac{1}{q}} < C \|M\|_{L^2_P(I)}$$

where C depends only on I, q, and P.

Proof. Recall that $\mathscr{T}(F)$ is a (C_q)-systems for $V(F)$ if and only if $E = \{e^{2\pi i n x} e_j\}_{1 \leq j \leq K, n \in \mathbb{Z}}$ is a (C_q)-system for $L^2_P(\mathbb{T})$.

Step I. **For any interval $I := [a,b] \subset \mathbb{T}$ such that, $\lambda_K(a) = \lambda_K(b) = 0$, $\lambda_K(x) > 0$ on (a,b), we have $\lambda_K^{-1} \in L^1(I)$.**

Let h_ε be the indicator function of $I_\varepsilon := [a + \varepsilon, b - \varepsilon]$ (when thinking of $\mathbb{T}$ as an interval of length 1 which contains $[a,b]$ as an interval). For $x \in I_\varepsilon$, $W(x) = P(x)^{-1}$ exists almost everywhere, and since λ_K is bounded away from zero on I_ε, the elements of $W(x)$ are bounded on this set as well. Let w_j be the j^{th} row of $W(x)$, and define $g_{\varepsilon,j} = h_\varepsilon w_j \in L^\infty(\mathbb{T})^K \subset L^2_P(\mathbb{T})$.

Since, $E = \{e^{2\pi i n x} e_j\}_{1 \leq j \leq K, n \in \mathbb{Z}}$ is a (C_q)-system in $L^2_P(\mathbb{T})$ it is also a (C_q)-system in $L^2_P(I)$ so

$$\|g_{\varepsilon,j}\|_{L^2_P(I)} \leq \|\{\langle g_{\varepsilon,j}, e_k e^{2\pi i l x} \rangle_{L^2_P(I)}\}_{1 \leq k \leq K, l \in \mathbb{Z}}\|_p \qquad 1/p + 1/q = 1 \tag{3.17}$$

Note,

$$\langle g_{\varepsilon,j}, e_l\rangle_{L^2_P(I)} = \int_{\mathbb{T}^d} h_\varepsilon(x) w_j(x) P(x) e_k^* e^{-2\pi i l x} dx = \delta_{j,k}\widehat{h_\varepsilon}(l).$$

So (3.17) implies that $\|g_{\varepsilon,j}\|_{L^2_P(I)} < C$ for some constant not depending on ε. Since

$$\|g_{\varepsilon,j}\|^2_{L^2_P(I)} = \int_{a+\varepsilon}^{b-\varepsilon} w_{j,j}(x),$$

the lemma follows from the trace formula.

Step II. $L^2_P(I) \subset L^1(\mathbb{T})^K$ **and for** M_k **and** M **supported on** I**,** $M_k \to M$ **in** $L^2_P(I)$ **implies** $M_k \to M$ **in** $L^1(\mathbb{T})^K$**.**

Note that the operator bound $\lambda_K(x) I \leq P(x)$ holds for the identity matrix I. Then, for any $M \in L^2_P(\mathbb{T})$,

$$\begin{aligned}\int_I |M(x)| dx &\leq \left(\int_I |M(x)|^2 \lambda_K(x) dx\right)^{1/2} \left(\int_I \frac{1}{\lambda_K(x)} dx\right)^{1/2} \\ &\leq \left(\int_I M(x) P(x) M(x)^* dx\right)^{1/2} \left(\int_I \frac{1}{\lambda_K(x)} dx\right)^{1/2}.\end{aligned}$$

The result follows easily from this observation.

Step III. **Suppose** $M_n, M \in L^1(\mathbb{T})^K$ **are such that** M_n **converges to** M **in** $L^1(\mathbb{T})^K$**. If there exists a** $C > 0$ **such that for all** n**,** $\|\widehat{M_n}\|_q \leq C$**, then** $\|\widehat{M}\|_q \leq C$**.**

Consider each $\widehat{M_n} \in l^q(\mathbb{Z})^K$ as a bounded linear functional on $l^p(\mathbb{Z})^K$. The Banach Alaoglu Theorem states that the ball of radius C in $l^q(\mathbb{Z})^K$ is weak* compact, and thus there exists a subsequence $\{\widehat{M_{n_j}}\}$ which converges to some $\widehat{N}$ in the weak* topology. In particular, $\widehat{M_{n_j}}(l) \to \widehat{N}(l)$ for each l. However, since M_{n_j} converges to M in $L^1(\mathbb{T})^K$, $\widehat{M_{n_j}}(l) \to \widehat{M}(l)$ for each l, and thus, $\widehat{M_{n_j}}$ converges to $\widehat{M}$ in the weak* topology. Also, for any $x \in l^p(\mathbb{Z})^K$,

$$|\langle \widehat{M}, x\rangle| \leq \sup_j |\langle \widehat{M_{n_j}}, x\rangle| \leq C\|x\|_{l^p(\mathbb{Z})^K}.$$

Therefore, $\|\widehat{M}\|_{l^q(\mathbb{Z})^K} \leq C$.

Step IV. **Completing the proof.**

Let I be an interval as in Step I. Clearly, if $\{e^{2\pi inx}e_j\}_{1\leq j\leq K, n\in\mathbb{Z}}$ is a (C_q)–system in $L^2_P(\mathbb{T})$ then it is a (C_q)–system in $L^2_P(I)$.

Let $M \in L^2_P(I)$. Then, there exist finite sums, P_k, of terms in E which satisfy $P_k \to M$ in $L^2_P(I)$ and $\|\widehat{P_k}\|_q < C\|M\|_{L^2_P(I)}$. By Proposition 2.4.6, it follows that $\|\widehat{P_k\chi_I}\|_q < C_1\|\widehat{P_k}\|_q < C'\|M\|_{L^2_P(I)}$. By Step II we have $P_k\chi_I \to M$ in $L^1(I)$, and therefore also in $L^1(\mathbb{T})$, and hence $\|\widehat{M}\|_q < C'\|M\|_{L^2_P(I)}$ by Step III. $\square$

3.3 Extra Invariance and Minimal Number of Generators

The first result in this section shows that extra-invariance can also be characterized in terms of Gramians. We sketch the proof below and refer the reader to [49] for a more rigorous proof. If Γ is a lattice such that $\Gamma \supsetneq \mathbb{Z}^d$, and $R \subset \mathbb{Z}^d$ is a set of representatives of the quotient $\mathbb{Z}^d/\Gamma^*$, then by rearranging terms in the sum we can always write the Grammian as

$$\begin{aligned} P(\widehat{F})(x) &= \sum_{l\in\mathbb{Z}^d} \widehat{F}(x+l)\widehat{F}(x+l)^* \\ &= \sum_{k\in R}\sum_{\gamma\in\Gamma^*} \widehat{F}(x+\gamma+k)\widehat{F}(x+\gamma+k)^* \\ &= \sum_{k\in R} P_{\Gamma^*}(\widehat{F})(x+k), \end{aligned}$$

where we define $P_{\Gamma^*}(\widehat{F})(x) = \sum_{\gamma\in\Gamma^*}\widehat{F}(x+\gamma)\widehat{F}(x+\gamma)^*$.

Theorem 3.3.1 ([1, 3, 49])**.** *Let $\Gamma \subset \mathbb{R}^d$ be a lattice with $\mathbb{Z}^d \subsetneq \Gamma$. Let $R \subset \mathbb{Z}^d$ be a collection of representatives of the quotient $\mathbb{Z}^d/\Gamma^*$. The space $V(F)$ is Γ-invariant if and only if*

$$\operatorname{rank}\left[P(\widehat{F})(x)\right] = \sum_{k\in R} \operatorname{rank}\left[P_{\Gamma^*}(\widehat{F})(x+k)\right],\ a.e.\ x\in\mathbb{R}^d.$$

Sketch of proof. Consider the operator

$$A(x) : \mathbb{C}^k \mapsto l^2(\mathbb{Z}^d)$$

defined by

$$(A(x)u)_l = u_1\overline{f_1(x-l)} + u_2\overline{f_2(x-l)} + \ldots + u_k\overline{f_k(x-l)}$$

for any $l \in \mathbb{Z}^d$ and $u = (u_1, \ldots, u_k)^T \in \mathbb{C}^k$. We may think of $A(x)$ as an $\infty \times k$ matrix where the i^{th} column of $A(x)$ consists of f_i evaluated at $x - l$. Then

$$P(x) = P(\widehat{F})(x) = A(x)^* A(x).$$

Similarly, we can define an A_{Γ^*} so that $P_{\Gamma^*} = P_{\Gamma^*}(\widehat{F}) = A_{\Gamma^*}(x)A_{\Gamma^*}(x)$. It is straightforward to show that $\text{rank}\,[P(x)] = \text{rank}\,[A]$, and similarly for P_{Γ^*}. Note that

$$\text{rank}\,[A(x)] = \dim\left(\text{span}\left\{(\widehat{f}_j(x+k))_{k\in\mathbb{Z}^d} : j \in \{1, \ldots, K\}\right\}\right).$$

Let $V(x) = \text{span}\left\{(\widehat{f}_j(x+k))_{k\in\mathbb{Z}^d} : j \in \{1, \ldots, K\}\right\}$ and

$$W(x) = \text{span}\left\{(m(x+k)\widehat{f}_j(x+k))_{k\in\mathbb{Z}^d} : j \in \{1, \ldots, K\}, m \text{ is } \Gamma^* - \text{periodic}\right\}.$$

Clearly, $V(x) \subset W(x)$, and using Proposition 3.1.1 it is possible to show that $W(x) \subset V(x)$ for almost every $x \in \mathbb{R}^d$ if and only if $V(F)$ is Γ-invariant. However, the space $W(x)$ orthogonally decomposes into the spaces $W_r(x) = \text{span}\left\{(e_r(k)\widehat{f}_j(x+k))_{k\in\mathbb{Z}^d} : j \in \{1, \ldots, K\}\right\}$ where $e_r(k) = 1$ for $k \in r + \Gamma^*$ and $e_r(k) = 0$ for other values of k. For a collection of representatives, R, as in the statement of the theorem,

$$W(x) = \bigoplus_{r\in R} W_r(x).$$

Then,

$$\dim\left[V(x)\right] = \sum_{r\in R} \dim\left[W_r(x)\right],$$

for almost every $x \in \mathbb{R}^d$, and this is equivalent to the rank condition of the theorem. $\square$

Note that $|R| = [\mathbb{Z}^d : \Gamma^*] = [\Gamma : \mathbb{Z}^d]$. Further, if it was known that the rank of P_{Γ^*} were constant almost everywhere, say $\operatorname{rank}[P_{\Gamma^*}] = J$, then for almost every x we would have,

$$\operatorname{rank}[P(x)] = |R|J = [\Gamma : \mathbb{Z}^d]J.$$

The following proposition shows that in this case $\operatorname{rank}[P(x)]$ will in fact be equal to, $\rho(V(F))$, the minimal number of generators of $V(F)$. Thus, if P_{Γ^*} has constant rank, we find that $[\Gamma : \mathbb{Z}^d]$ divides $\rho(V(F))$. In light of this, several of the proofs of main results derive a contradiction by showing that the assumptions from the theorem imply that the rank of P_{Γ^*} is constant.

Proposition 3.3.2 (Proposition 4.1, [49])**.** *Let $F \subset L^2(\mathbb{R}^d)$. The minimal number of generators of $V(F)$ is given by*

$$\rho(V(F)) = \operatorname{ess\,sup}_{x\in\mathbb{R}^d}\left(\operatorname{rank}\left[P(\widehat{F})(x)\right]\right).$$

We provide a short proof of this result.

Proof. Let $P = P(\widehat{F})$, and let $P(x) = U(x)\Lambda(x)U(x)^*$ be a measurable diagonalization. Let $u_j(x)$ be the j^{th} column of $U(x)$. Then, $u_j^* \in L^2_{P(\widehat{F})}(\mathbb{T}^d)$ since

$$\int_{\mathbb{T}^d} u_j^*(x)P(x)u_j dx = \int_{\mathbb{T}^d} \lambda_j(x)dx < \infty.$$

Then $\widehat{g_j} = u_j^*\widehat{F} \in L^2(\mathbb{R}^d)$, and $g_j \in V(F)$. Let G be the column vector with entries g_j. Then, $\widehat{G} = U^*\widehat{F}$. Note that $P(\widehat{G})(x) = \Lambda(x)$ almost everywhere.

We now show $V(G) = V(F)$. We have that $h \in V(F)$ if and only if there exists an $M \in L^2_P(\mathbb{T}^d)$ such that $\widehat{h} = M\widehat{F} = MU\widehat{G}$. It is straightforward to check that $MU \in L^2_{P(\widehat{G})}(\mathbb{T}^d)$, and so $h \in V(G)$. Thus $V(F) \subset V(G)$. The other inclusion follows similarly. Thus, $V(G) = V(F)$.

Note that g_j is only nonzero if λ_j is nonzero, and thus G can be reduced to a set of size

$$D(F) = \operatorname{ess\,sup}_{x \in \mathbb{R}^d} \left(\operatorname{rank} \left[P(\widehat{F})(x) \right] \right).$$

This shows that $\rho(V(F)) \leq D(F)$. For the other direction note that for almost every $x \in \mathbb{T}^d$,

$$\operatorname{rank} \left[P(\widehat{F})(x) \right] = \dim \left[\operatorname{span}\{(\widehat{f}_j(x+l))_{1 \leq j \leq K, l \in \mathbb{Z}^d}\} \right]$$

and the right hand side is independent of the particular set of generators for the space $V(F)$. Thus, $D(F) = D(H)$ for any other set of generators $H \subset L^2(\mathbb{R}^d)$ with $V(F) = V(H)$, and no generating set of size less than $D(F)$ exists. Therefore, $\rho(V(F)) = D(F)$. $\square$

Chapter 4

Intermediate Results

In this chaper, we prove several results which will be used in the proofs of our main theorems. In particular, in Section 4.1, we prove a strong embedding theorem for the Gramian matrices, and in Section 4.2 we prove results about the restriction of $H^s(\mathbb{T}^d)$ functions to lines and about $H^s(\mathbb{T}^d)$ functions which have a negative power of integrability.

4.1 Periodization and Gramian Embeddings

Theorem 4.1.1 (Periodization and Grammian Embedding)**.** *Fix $0 < s \leq 1$. Suppose $F = \{f_1, f_2, ..., f_k\} \subset H^s(\mathbb{R}^d)$ and $\lambda_1(x) \geq \cdots \geq \lambda_k(x) \geq 0$ are the eigenvalues of $P = P(F)(x)$, then $\sqrt{\lambda_i} \in H^s(\mathbb{R}^d)$.*

Proof. Let A be the operator defined by

$$(A(x)u)_l = u_1\overline{f_1(x-l)} + u_2\overline{f_2(x-l)} + ... + u_k\overline{f_k(x-l)}$$

for any $l \in \mathbb{Z}^d$ and $u = (u_1, ..., u_k)^T \in \mathbb{C}^k$, as in Theorem 3.3.1, so that

$$P(x) = A(x)^*A(x).$$

The Courant-Fischer-Weyl min-max theorem, e.g., Corollary III.1.2 in [15], says that

$$\lambda_k(x) = \max\{\min\{\langle u, P(x)u\rangle : u \in U, |u| = 1\} : \dim(U) = k\},$$

Then, for almost every $x, y \in \mathbb{R}^d$,

$$\left|\sqrt{\lambda_k(x)} - \sqrt{\lambda_k(y)}\right| \leq |\sqrt{\max\{\min\{\langle u, P(x)u\rangle : u \in U, |u| = 1\} : \dim(U) = k\}} \\ - \sqrt{\max\{\min\{\langle v, P(y)v\rangle : v \in V, |v| = 1\} : \dim(V) = k\}}|.$$

Without loss of generality, $\lambda_k(x) \geq \lambda_k(y)$. Choose a subspace U_0 that realizes the maximum $\lambda_k(x)$. We have

$$\begin{aligned} \left|\sqrt{\lambda_k(x)} - \sqrt{\lambda_k(y)}\right| &\leq \sqrt{\min\{\langle u, P(x)u\rangle : u \in U_0, |u| = 1\}} \\ &\quad - \sqrt{\max\{\min\{\langle v, P(y)v\rangle : v \in V, |v| = 1\} : \dim(V) = k\}}| \\ &\leq \sqrt{\min\{\langle u, P(x)u\rangle : u \in U_0, |u| = 1\}} - \sqrt{\min\{\langle v, P(y)v\rangle : v \in U_0, |v| = 1\}}. \end{aligned}$$

Next, choose $u_0 \in U_0$ with $\|u_0\| = 1$ such that the minimum in the right term is achieved at u_0. Then,

$$\begin{aligned} \left|\sqrt{\lambda_k(x)} - \sqrt{\lambda_k(y)}\right| &\leq \sqrt{\min\{\langle u, P(x)u\rangle : u \in U_0, |u| = 1\}} - \sqrt{\langle u_0, P(y)u_0\rangle} \\ &\leq \sqrt{\langle u_0, P(x)u_0\rangle} - \sqrt{\langle u_0, P(y)u_0\rangle} \\ &= \|A(x)u_0\| - \|A(y)u_0\| \end{aligned}$$

From the triangle inequality, we hvae

$$\begin{aligned} \left|\sqrt{\lambda_k(x)} - \sqrt{\lambda_k(y)}\right| &\leq \|(A(x) - A(y))u_0\| \\ &\leq \|A(x) - A(y)\|_{op} \\ &\leq \|A(x) - A(y)\|_{frob}. \end{aligned}$$

Therefore, we may conclude that

$$\begin{aligned}\left|\sqrt{\lambda_k(x)}-\sqrt{\lambda_k(y)}\right|^2 &\le \|A(x)-A(y)\|_{frob}^2\\ &=\sum_{j=1}^{K}\sum_{l\in\mathbb{Z}^d}\left|f_j(x-l)-f_j(y-l)\right|^2. \end{aligned} \tag{4.1}$$

<u>*Case 1:* $s<1$</u> Using Equations (2.8) and (4.1), we find

$$\begin{aligned}\|\sqrt{\lambda_k}\|_{\dot{H}^s(\mathbb{T}^d)}^2 &\le C\int_{\mathbb{T}^d}\int_{[-\frac{1}{2},\frac{1}{2})^d}\frac{|\sqrt{\lambda_k(x)}-\sqrt{\lambda_k(y)}|^2}{|y|^{d+2s}}dydx\\ &\le C\int_{\mathbb{T}^d}\int_{[-\frac{1}{2},\frac{1}{2})^d}\frac{\sum_{j=1}^{k}\sum_{l\in\mathbb{Z}^d}|f_j(x-l)-f_j(y-l)|^2}{|y|^{d+2s}}dydx\\ &=C\sum_{j=1}^{k}\int_{\mathbb{R}^d}\int_{[-\frac{1}{2},\frac{1}{2})^d}\frac{|f_j(x)-f_j(y)|^2}{|y|^{d+2s}}dydx\\ &\le C\sum_{j=1}^{k}\|f_j\|_{\dot{H}^s(\mathbb{R}^d)}^2<\infty.\end{aligned}$$

<u>*Case 2:* $s=1$</u> For notational simplicity, let $g=\sqrt{\lambda_k}$. Equation (4.1) implies for any $i\in\{1,\dots,K\}$

$$\begin{aligned}\int_{\mathbb{T}^d}|g(x+te_i)-g(x)|^2dx &\le \int_{\mathbb{T}^d}\sum_{j=1}^{K}\sum_{l\in\mathbb{Z}^d}\left|f_j(x+te_i-l)-f_j(x-l)\right|^2dx\\ &=\sum_{j=1}^{K}\int_{\mathbb{R}^d}\left|f_j(x+te_i)-f_j(x)\right|^2dx.\end{aligned}$$

Using Parseval's equality for $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{T}^d)$ we have

$$\sum_{k\in\mathbb{Z}^d}|\widehat{g}(k)|^2|e^{2\pi ik_it}-1|^2\le\sum_{j=1}^{K}\int_{\mathbb{R}^d}|\widehat{f}_j(\xi)|^2|e^{2\pi i\xi_it}-1|^2d\xi.$$

Using the fact that $|e^{2\pi i\xi_it}-1|\le 2\pi|\xi_i||t|$, for any nonzero t, we find,

$$\sum_{k\in\mathbb{Z}^d}|\widehat{g}(k)|^2\left|\frac{e^{2\pi ik_it}-1}{t}\right|^2\le 2\pi\sum_{j=1}^{K}\int_{\mathbb{R}^d}|\widehat{f}_j(\xi)|^2|\xi_i|^2d\xi<\infty$$

Note that $|e^{2\pi i k_i t}-1|^2 = 2(1-\cos(2\pi k_i t))$ and for any $|\theta| \leq \frac{1}{4}$, $1-\cos(2\pi\theta) \geq \frac{\theta^2}{2}$. Thus, for any $|t|>0$ we have

$$\sum_{|k_i| \leq \frac{1}{4|t|}} |\widehat{g}(k)|^2 |k_i|^2 \leq \sum_{k \in \mathbb{Z}^d} |\widehat{g}(k)|^2 \left| \frac{e^{2\pi i k_i t}-1}{t} \right|^2,$$

and the right hand side is uniformly bounded in t. Thus, taking the limit as $t \to 0$, we find

$$\sum_{k \in \mathbb{Z}^d} |\widehat{g}(k)|^2 |k_i|^2 \leq 2\pi \sum_{j=1}^{K} \int_{\mathbb{R}^d} |\widehat{f}_j(\xi)|^2 |\xi_i|^2 d\xi.$$

Summing this expression over each $i \in \{1,...,K\}$ we find that there is a constant $C>0$ such that

$$\|g\|^2_{\dot{H}^1(\mathbb{T}^d)} \leq C \sum_{j=1}^{K} \|f_j\|^2_{\dot{H}^1(\mathbb{R}^d)} < \infty.$$

□

4.2 Sobolev Space Properties

4.2.1 Restriction of $H^s(\mathbb{T}^d)$ Functions to Lines

The following shows that the restriction of a periodic Sobolev function to lines typically has the same degree of Sobolev smoothness. For $x' \in [0,1]^{d-1}$, let $L_j(x') = \{(x_1,...x_{j-1},t,x_j,...,x_d) : t \in [0,1]\}$.

Proposition 4.2.1. *Fix $0<s\leq 1$, and suppose $g \in H^s(\mathbb{T}^d)$. There exists a representative of g such that for all j and almost every $x' \in [0,1]^{d-1}$, the function $g_{x',j} = g|_{L_j(x')}$ satisfies $g_{x',j} \in H^s(\mathbb{T})$.*

Proof. Case 1: $0<s<1$ This portion of the proof will make use of the equivalent norm (2.9). For

$x' \in [0,1]^{d-1}$, let $x'_j(t) = (x'_1, ..., x'_{j-1}, t, x'_j, ..., x'_{d+1})$. We have

$$\|g\|^2_{\dot{H}^s(\mathbb{T}^d)} \asymp \sum_{j=1}^{d} \int_{\mathbb{T}^d} \int_{[0,1]} \frac{|g(x+te_j)-g(x)|^2}{|t|^{1+2s}} dt dx \tag{4.2}$$

$$= \sum_{j=1}^{d} \int_{[0,1]^{d-1}} \int_{[0,1]} \int_{[0,1]} \frac{|g(x'_j(r+t)) - g(x'_j(r))|^2}{|t|^{1+2s}} dt dr dx' \tag{4.3}$$

$$= \sum_{j=1}^{d} \int_{[0,1]^{d-1}} \|g_{x',j}\|^2_{\dot{H}^s(\mathbb{T})} dx'. \tag{4.4}$$

Since $\|g\|^2_{\dot{H}^s(\mathbb{T}^d)} < \infty$, we must have

$$\|g_{x',j}\|_{\dot{H}^s(\mathbb{T})} < \infty,$$

for almost every $x' \in [0,1]^d$ and for each $j \in \{1, ..., d\}$.

Case 2: $s = 1$ This follows from the absolute continuity on lines characterization of $H^1(\mathbb{R}^d)$ (see for example [43]) which is easily carried over to the space $H^1(\mathbb{T}^d)$. The result says that if $g \in H^1(\mathbb{T}^d)$, then there exist a $\tilde{g}$ which is equal to g almost everywhere such that $\tilde{g}$ is absolutely continuous on almost every line parallel to any coordinate axis, and the classical derivative of $\tilde{g}$ on, say, $L_1(x')$ agrees with the distributional partial derivative $D_{x_1} g$ restricted to $L_1(x')$. Since these partial derivatives must be in $L^2(\mathbb{T}^d)$, we must have that $g_{x',j}$ is in $H^1(\mathbb{T})$ for almost every $x' \in [0,1]^d$. $\square$

4.2.2 $H^s(\mathbb{T}^d)$ Functions with a Negative Power of Integrability

It is known that if g is the characteristic function of a measurable set $S \subset \mathbb{R}^d$ with positive finite Lebesgue measure then $g \notin H^{1/2}(\mathbb{R}^d)$, e.g., see [16]. Here, we show that this result holds for the periodic Sobolev space $H^{1/2}(\mathbb{T}^d)$ for S with measure strictly between 0 and 1, and we prove a generalization of this result for $H^s(T^d)$ for $1/2 < s \leq 1$. We start with a lemma which will be used for the $1/2 < s \leq 1$ portion of the proof.

Lemma 4.2.2. *Suppose $S \subset [0,1]^d$ is measurable. Further, suppose that for each axis in $\mathbb{R}^d$, the* *intersection of S with almost every line parallel to the axis has 1-dimensional Lebesgue measure either zero or one. Then, the d-dimensional Lebesgue measure of S is either zero or one.*

Proof. We will prove this by induction on d. For $d = 1$, the statement is trivially true. For this proof, we use $|\cdot|_d$ to denote the d-dimensional Lebesgue measure.

Suppose the result holds for all $d < k$ for some $k > 1$. Consider now $d = k$. Let $e_1, ..., e_d$ be the canonical basis vectors, we can label all lines in the e_d direction by $L(x) = \{(x,t) : t \in [0,1]\}$ for some $x \in [0,1]^{d-1}$. The condition that $|S \cap L(x)|_1$ is either zero or one for almost every $x \in [0,1]^{d-1}$, is equivalent to saying that S differs by a set of measure zero from the set $T = S' \times [0,1]$ where $S' = \{x \in [0,1]^{d-1} : |S \cap L(x)|_1 = 1\}$ is measurable in $[0,1]^{d-1}$. Note, $|S'|_{d-1} = |T|_d = |S|_d > 0$, and intersecting S' with lines parallel to the axes in $\mathbb{R}^{d-1}$ is equivalent to intersecting T with lines parallel to the same axes in $\mathbb{R}^d$. Thus, S' satisfies the property that its intersections with lines parallel to the axes in $\mathbb{R}^{d-1}$ have measure either zero or one. By induction, $|S|_d = |S'|_{d-1}$ is either zero or one. $\square$

For the main results in this thesis, we only need the $s = 1/2$ and $s = 1$ part of the following theroem. The result for the other values of s can be used to prove weak results for (C_q)-systems and might also be useful outside of this thesis. The proof must be split into two cases, which results from the fact that the one-dimensional periodic Sobolev embedding embeds $H^s(\mathbb{T})$ into $C^{s-1/2}(\mathbb{T})$ for $1/2 < s \leq 1$, but does not hold for $s = 1/2$.

Proposition 4.2.3. *Suppose $1/2 \leq s \leq 1$. Fix a nonnegative, nonzero $g \in H^s(\mathbb{T}^d)$, and let $S = \{x \in [0,1]^d : g(x) > 0\}$. If $\frac{1}{g} \in L^{\frac{2}{2s-1}}(S)$ (we set $\frac{2}{2s-1} = \infty$ for $s = 1/2$), then $|S| = 1$. In other words, g is nonzero almost everywhere.*

Proof. **Case 1:** $s = 1/2$ Without loss of generality, we can assume that $\|g^{-1}\|_{L^\infty(S)} \leq 1$. Notice that for a.e. $x, y \in \mathbb{T}^d$,

$$|g(x+y) - g(x)| \geq |\chi_S(x+y) - \chi_S(x)|,$$

and so, by equation (2.8), $\chi_S \in H^{1/2}(\mathbb{T}^d)$ since $g \in H^{1/2}(\mathbb{T}^d)$. Therefore, it suffices to prove the lemma in the case that $g = \chi_S$ for some $S \subset \mathbb{T}^d$.

Step I: We begin by addressing the case $d = 1$. For the sake of contradiction, suppose there exists a set $S \subset [-\frac{1}{2}, \frac{1}{2})$ with $0 < |S| < 1$ such that g is the $\mathbb{Z}$-periodic extension of χ_S to $\mathbb{R}$ and that $g \in H^{1/2}(\mathbb{T})$.

For any interval $I \subset [-\frac{1}{2}, \frac{1}{2})$, we have

$$\begin{aligned}
\frac{1}{|I|^2}\int_I\int_I |g(x)-g(y)|dxdy &= \frac{1}{|I|^2}\int_I\int_{I-x}|g(x+y)-g(x)|dydx \\
&\le \frac{1}{|I|^2}\left(\int_I\int_{I-x}\frac{|g(x+y)-g(x)|^2}{|y|^2}dydx\right)^{1/2}\left(\int_I\int_{I-x}|y|^2dydx\right)^{1/2} \\
&\le \left(\int_I\int_{I-x}\frac{|g(x+y)-g(x)|^2}{|y|^2}dydx\right)^{1/2}. \qquad (4.5)
\end{aligned}$$

Since g is the indicator function of a set, we have

$$\int_I\int_{1/2\le|y|\le 1}\frac{|g(x+y)-g(x)|^2}{|y|^2}dydx \le \int_I\int_{1/2\le|y|\le 1}\frac{1}{|1/2|^2}\,dydx \le 4|I|. \qquad (4.6)$$

If $x \in I \subset [-\frac{1}{2}, \frac{1}{2})$ then $I - x \subset [-1, 1]$. This, together with (4.6), implies that

$$\begin{aligned}
\int_I\int_{I-x}\frac{|g(x+y)-g(x)|^2}{|y|^2}dydx &\le \int_I\int_{-1}^{1}\frac{|g(x+y)-g(x)|^2}{|y|^2}dydx \\
&\le \int_I\int_{-1/2}^{1/2}\frac{|g(x+y)-g(x)|^2}{|y|^2}dydx + 4|I|. \qquad (4.7)
\end{aligned}$$

Using $g \in H^{1/2}(\mathbb{T})$, (2.8), (4.5), (4.7), and absolute continuity of the Lebesgue integral, it follows that

$$\lim_{|I|\to 0}\frac{1}{|I|^2}\int_I\int_I|g(x)-g(y)|dxdy = 0. \qquad (4.8)$$

Since $0 < |S| < 1$, for every sufficiently small $\varepsilon > 0$, there exists an interval $Q_\varepsilon \subset [-1/2, 1/2)$ such that $|Q_\varepsilon| < \varepsilon$ and $|Q_\varepsilon \cap S| = |Q_\varepsilon \cap S^c| = |Q_\varepsilon|/2$ (for example, this follows from the Lebesgue differentiation theorem). So, for every sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
\frac{1}{|Q_\varepsilon|^2}\int_{Q_\varepsilon}\int_{Q_\varepsilon}|g(x)-g(y)|dxdy &\geq \frac{1}{|Q_\varepsilon|^2}\int_{Q_\varepsilon\cap S}\int_{Q_\varepsilon\cap S^c}|g(x)-g(y)|dxdy \\
&= \frac{1}{|Q_\varepsilon|^2}\int_{Q_\varepsilon\cap S}\int_{Q_\varepsilon\cap S^c} 1\ dxdy \\
&= \frac{|Q_\varepsilon\cap S|\,|Q_\varepsilon\cap S^c|}{|Q_\varepsilon|^2} = 1/4.
\end{aligned} \tag{4.9}
$$

On the other hand, by (4.8),

$$
\lim_{\varepsilon\to 0}\frac{1}{|Q_\varepsilon|^2}\int_{Q_\varepsilon}\int_{Q_\varepsilon}|g(x)-g(y)|dxdy = 0. \tag{4.10}
$$

Since (4.9) and (4.10) form a contradiction, it follows that either $|S| = 0$ or $|S| = 1$.

Step II: Next, we address the case where $d \geq 2$. Suppose that $S \subset [-\frac{1}{2}, \frac{1}{2})^d$ and that g is the $\mathbb{Z}^d$-periodic extension of χ_S to $\mathbb{R}^d$. Let $\{e_j\}_{j=1}^d$ be a the canonical basis vectors for $\mathbb{R}^d$. Define for $t \in [-1/2, 1/2)$

$$
\psi_{x,k}(t) = g(x + te_k).
$$

Note that for a.e. $x \in [-\frac{1}{2}, \frac{1}{2})^d$, $\psi_{x,k}$ is 1-periodic and $\psi_{x,k} \in L^2(\mathbb{T})$. Also, by Proposition 4.2.1, for all k and almost every $x \in \mathbb{T}^d$, $\psi_{x,k} \in H^{1/2}(\mathbb{T})$. However, since $g(x) \in \{0, 1\}$ for almost every x, we also have that for each $1 \leq k \leq d$ and almost every $x \in \mathbb{T}^d$,

$$
\psi_{x,k}(t) \in \{0, 1\}, \quad \text{for } \ a.e.\ t \in \mathbb{R}. \tag{4.11}
$$

It follows from Case 1 that for each $1 \leq k \leq d$ and almost every $x \in \mathbb{T}^d$

$$
g(x + te_k) = 0 \text{ for } a.e.\ t \in \mathbb{R}, \quad \text{or} \quad g(x + te_k)(t) = 1 \text{ for } a.e.\ t \in \mathbb{R}. \tag{4.12}
$$

To complete the proof it now suffices to show that $g(x) = g(y)$ for a.e. $x, y \in \mathbb{T}^d$. For this, it suffices to show that $g(x) = g(y)$ for almost every $x, y \in [-1/2, 1/2)^d$. Similarly to Lemma 2 in [19], and using (4.12), one has

$$
\begin{aligned}
\int_{[-1/2,1/2)^d} \int_{[-1/2,1/2)^d} & |g(x_1,...,x_d) - g(y_1,...,y_d)| dx dy \\
& \leq \int_{[-1/2,1/2)^d} \int_{[-1/2,1/2)^d} |g(x_1,...,x_d) - g(y_1,x_2,...,x_d)| dx dy \\
& + \int_{[-1/2,1/2)^d} \int_{[-1/2,1/2)^d} |g(y_1,x_2,...,x_d) - g(y_1,y_2,x_3...,x_d)| dx dy \\
& \vdots \\
& + \int_{[-1/2,1/2)^d} \int_{[-1/2,1/2)^d} |g(y_1,y_2,...,y_{d-1},x_d) - g(y_1,...,y_d)| dx dy \\
& = 0.
\end{aligned}
$$

Thus, $g(x) = g(y)$ for almost every $x, y \in \mathbb{T}^d$.

Case 2: $s > 1/2$ *Step 1:* As in Case 1, we first consider the $d = 1$ case. By the Hölder embedding, we have $g \in C^{s-1/2}(\mathbb{T})$. Thus, S is an open set. If $S \neq [0,1]$, then g must have a zero. Without loss of generality, assume $g(0) = 0$ and g is nonzero for all $x \in (0, t_0)$ for some $t_0 > 0$. From the Hölder conditon, we have

$$|g(x)| = |g(x) - g(0)| \leq C|x|^{s-1/2}.$$

Then,

$$
\begin{aligned}
\int_S \left(\frac{1}{g(x)}\right)^{2/(2s-1)} dx &\geq \int_0^{t_0} \left(\frac{1}{g(x)}\right)^{2/(2s-1)} dx \\
&\geq \int_0^{t_0} \frac{1}{|x|} dx = \infty.
\end{aligned}
$$

Thus, for $d = 1$, we actually find the stronger result that under the assumptions, $S = [0,1]$.

Step 2: Now we prove the result for $d > 1$. Assume $|S| < 1$. Lemma 4.2.2 shows that in some axis direction (without loss of generality, say in the direction of e_d), for a set of positive measure $A \subset [0,1]^{d-1}$, $0 < |S \cap L(x)| < 1$ for each $x \in A$.

Since $g \in H^s(\mathbb{T}^d)$, then for almost every $x \in [0,1]^{d-1}$, the function $g_x(t) = g(x,t)$ satisfies $g_x \in H^s(\mathbb{T})$. Since $H^s(\mathbb{T})$ embeds into $C^{s-1/2}(\mathbb{T})$, we can find a representative of g for which $g_x \in C^{s-1/2}(\mathbb{T})$ for almost every $x \in [0,1]^{d-1}$. We denote this representative of g by g', and we denote $S' = \{x \in [0,1]^d : g'(x) > 0\}$. Note that the symmetric difference of S and S' satisfies $|S \bigtriangleup S'| = 0$. By Fubini's Theorem, we must have for almost every $x \in A$, $0 < |S' \cap L(x)| = |S \cap L(x)| < 1$.

Then, for almsot every $x \in A$, we have $g'_x \in H^s(\mathbb{T})$, and $\frac{1}{g'_x} \in L^{\frac{2}{2s-1}}(S' \cap L(x))$ (again by Fubini's Theorem), and $S' \cap L(x) = \{t \in [0,1] : g'_x(t) > 0\}$. By the $d = 1$ case, we must have that $|S' \cap L(x)|$ is either zero or one for almost every $x \in A$, which is a contradiction. Therefore, $|S| = 1$.

$\square$

Chapter 5

Proofs of Main Theorems

In this chapter, we give proofs of the main theorems of this thesis. Section 5.1 contains an important lemma about Gramians of $H^{1/2}(\mathbb{R}^d)$ functions which is then used in the proof of Theorem 1.2.1 and Theorem 1.2.4. Section 5.2 proves a generalization of Theorem 1.2.8 which can be used to provide a weak version of the (C_q)-system result which holds in all dimensions. In Section 5.3 we prove Theorem 1.2.9.

5.1 Proof of Theorems 1.2.1 and 1.2.4

We first prove a lemma about Gramian's of $H^{1/2}(\mathbb{R}^d)$ functions, which is crucial to both Theorems 1.2.1 and 1.2.4.

Lemma 5.1.1. *Suppose $F = \{f_1, ..., f_K\} \subset H^{1/2}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$, and there exist a constant $0 < A < \infty$ such that for almost every $x \in \mathbb{T}^d$, $P(x) = P(F)(x)$ satisfies*

$$AP(x) \leq (P(x))^2.$$

Then, there exists an integer $1 \leq J \leq K$ such that the rank of $P(x)$ equals J almost everywhere. (In short, we say that the rank of P is constant almost everywhere.)

Proof. Let $\lambda_1(x) \geq \lambda_2(x) \geq ... \geq \lambda_K(x)$ be the eigenvalues of $P(x)$, and for some $1 \leq k \leq K$, let $U_k(x)$ be a unit-norm eigenvector of $P(x)$ corresponding to $\lambda_k(x)$. Then, for almost every $x \in \mathbb{T}^d$, we have

$$A\langle U_k(x), P(x)U_k(x)\rangle \leq \langle U_k(x), (P(x))^2 U_k(x)\rangle,$$

which is equivalent to

$$A\lambda_k(x) \leq \lambda_k(x)^2.$$

However, this bound shows that when $\lambda_k(x)$ is nonzero, $0 < A \leq \lambda_k$.

By Theorem 4.1.1, $\sqrt{\lambda_k} \in H^{1/2}(\mathbb{T}^d)$ and by Proposition 4.2.3, λ_k must be either zero almost everywhere in $\mathbb{T}^d$ or λ_k must be positive almost everywhere on $\mathbb{T}^d$. Since this holds for all k, the rank of P must be constant almost everywhere in $\mathbb{T}^d$. $\square$

We also use the following lemma from [49] in all proofs of theorems involving extra-invariance. For a Hermitian postive semidefinite matrix M, let $\mu^-(M)$ be the smallest non-zero eigenvalue of M.

Lemma 5.1.2 (Lemma 3.1 in [49]). *Suppose $C, A_1, \ldots, A_l$ are all Hermitian positive semidefinite matrices such that $C = A_1 + \ldots + A_l$ and $\operatorname{rank}[C] = \sum_{j=1}^{l} \operatorname{rank}\left[A_j\right]$. Then,*

$$\mu^-(C) \leq \min_j [\mu^-(A_j)].$$

Now we are ready to prove the theorem.

Proof of Theorem 1.2.1. Assume, for the sake of contradiction, that $\widehat{f_k} \in H^{1/2}(\mathbb{R}^d)$ for all $1 \leq k \leq K$.

Recall the extra-invariance condition from Theorem 3.3.1 shows that for almost every $x \in \mathbb{R}^d$,

$$\operatorname{rank}[P(x)] = \sum_{k \in R} \operatorname{rank}[P_{\Gamma^*}(x+k)],$$

where $P_{\Gamma^*}(x) = \sum_{\gamma \in \Gamma^*} \widehat{F}(x+\gamma)\widehat{F}(x+\gamma)^*$, and $R \subset \mathbb{Z}^d$ is a collection of representatives of the quotient group $\mathbb{Z}^d/\Gamma^*$. Later in the proof we will use the fact that $|R| = [\mathbb{Z}^d : \Gamma^*] = [\Gamma : \mathbb{Z}^d]$.

We first show that P and P_{Γ^*} have constant rank almost everywhere. The frame characterization (3.7) shows that there exist constants $0 < A \leq B < \infty$ such that for almost every $x \in \mathbb{T}^d$, $P(x) = P(\widehat{F})(x)$ satisfies

$$AP(x) \leq (P(x))^2 \leq BP(x).$$

Then, Lemma 5.1.1 implies that P has constant rank almost everywhere.

Let D be an invertible matrix such that $\Gamma^* = D\mathbb{Z}^d$. Then,

$$\begin{aligned} T(x) &= P_{\Gamma^*}(Dx) \\ &= \sum_{\gamma \in \Gamma^*} \widehat{F}(Dx+\gamma))\widehat{F}(Dx+\gamma)^* \\ &= \sum_{l \in \mathbb{Z}^d} \widehat{F}(D(x+l)))\widehat{F}(D(x+l))^* \end{aligned}$$

is $\mathbb{Z}^d$ invariant, and is exactly the Gramian matrix of $\widehat{F}(Dx) = (\widehat{f_1}(Dx), ..., \widehat{f_K}(Dx))^T$. Since D is invertible, $\widehat{F}(D\cdot) \subset H^{1/2}(\mathbb{R}^d)$.

Also, Lemma 5.1.2 shows that for almost every x, the minimum nonzero eigenvalue (if one exists) of $P_{\Gamma^*}(x)$ is greater than or equal to the minimum nonzero eigenvalue of $P(x)$, which we know from the frame condition is bounded below by $A > 0$. Since P_{Γ^*} is positive semi-definite, this is equivalent to saying that for almost every $x \in \mathbb{R}^d$,

$$AP_{\Gamma^*}(x) \leq (P_{\Gamma^*}(x))^2.$$

This property is uneffected by rescaling by D, and thus for almost every $x \in \mathbb{T}^d$, we have

$$AT(x) \leq (T(x))^2.$$

Lemma 5.1.1 shows that T, and thus P_{Γ^*} must have constant rank almost everywhere.

Note that Proposition 3.3.2, combined with the fact that $P(x)$ has constant rank almost everywhere, implies that for almost every $x \in \mathbb{T}^d$,

$$\rho(V(F)) = \operatorname{rank}[P(x)].$$

Let J be the integer such that $\operatorname{rank}[P_{\Gamma^*}] = J$ almost everywhere. Then, the extra-invariance condition gives

$$\rho(V(F)) = \operatorname{rank}[P(x)] = \sum_{k\in R} \operatorname{rank}[P_{\Gamma^*}(x+k)] = |R|J = [\Gamma:\mathbb{Z}^d]J.$$

We have reached a contradiction since we assumed that $[\Gamma:\mathbb{Z}^d]$ does not divide $\rho(V(F))$. Therefore, at least one generator must not be in $H^{1/2}(\mathbb{R}^d)$.

$\square$

Theorem 1.2.4 can be proven using similar techniques to the proof of Theorem 1.2.1.

Proof of Theorem 1.2.4. Assume, for the sake of contradiction, that $\widehat{f_k} \in H^{1/2}(\mathbb{R}^d)$ for all $1 \le k \le K$. Using Lemma 5.1.1 and similar reasoning as in the proof of Theorem 1.2.1 we find that the rank of $P(x)$ must be constant almost everywhere. The assumption that $K = \rho(V(F))$, along with Proposition 3.3.2 implies that P is full rank almost everywhere. This forces the eigenvalue functions, λ_k, of P to be nonzero almost everywhere for all $1 \le k \le K = \rho(V(F))$. However, equation (3.7) shows that the eigenvalue functions are then bounded below by $A > 0$ almost everywhere. This is equivalent to P satisfying the lower bound in equation (3.6). The upper bound also follows from (3.7). Thus, $\mathscr{T}(F)$ forms a Riesz basis for $V(F)$ which gives a contradiction. $\square$

5.2 Proof of Theorem 1.2.8

We will actually prove a slightly more general theorem, which gives a weak result for (C_q)-systems as well as the full result for minimal systems.

Theorem 5.2.1. *Fix $\frac{1}{2} < s \le 1$, a lattice $\Gamma \supsetneq \mathbb{Z}^d$, and some nontrivial $F = \{f_1, ..., f_K\} \subset H^s(\mathbb{R}^d)$. Suppose $V(F)$ is Γ-invariant and $[\Gamma:\mathbb{Z}^d]$ does not divide $\rho(V(F))$. If $\lambda_1(x) \ge ... \ge \lambda_K(x)$ denote the eigenvalues of $P(x)$, then for some $1 \le k \le K$,*

$$\frac{1}{\lambda_k} \notin L^{\frac{1}{2s-1}}(\mathbb{T}^d).$$

Note that Theorem 5.2.1 implies Theorem 1.2.8 since part 3 of Proposition 3.2.2 shows that if $\mathscr{T}(F)$ forms a minimal system for $V(F)$, we must have $\lambda_k^{-1} \in L^1(\mathbb{T}^d)$ for all $1 \le k \le K$.

Part c of Theorem 3.2.3 shows that if $\lambda_k^{-1} \in L^{\frac{q}{q-2}}(\mathbb{T}^d)$, then $\mathscr{T}(F)$ is a minimal (C_q)-system for $V(F)$. Theorem 5.2.1 shows that under the assumptions of the theorem, the eigenvalues cannot satisfy this property for $s = \frac{q-1}{q}$. Theorem 5.2.1 is not enough to say that $\mathscr{T}(F)$ cannot form a minimal (C_q)-system for $V(F)$ under these assumptions, but it can be seen as a partial result in that direction.

Proof of Theorem 5.2.1. We assume for contradiction that for all $1 \leq k \leq K$, $\lambda_k^{-1} \in L^{\frac{1}{2s-1}}(\mathbb{T}^d)$. Theorem 3.3.1 gives

$$\operatorname{rank}[P(x)] = \sum_{k \in R} \operatorname{rank}[P_{\Gamma^*}(x+k)]\,,\ a.e.\ x \in \mathbb{R}^d$$

Applying Lemma 5.1.2, for almost every x we have,

$$\lambda_K(x) = \mu^-(P(x)) \leq \min_k \mu^-(P_{\Gamma^*}(x+k)) \leq \mu^-(P_{\Gamma^*}(x)),$$

where $\mu^-(A)$ is the smallest nonzero eigenvalue of A. Denote the eigenvalues of $P_{\Gamma^*}(x)$ by $\gamma_1(x) \geq \gamma_2(x) \geq \cdots \geq \gamma_K(x)$. We have,

$$\lambda_K(x) \leq \max_{i:\gamma_i(x)>0} \gamma_i(x).$$

Let J be the largest value such that γ_J is not equal to zero almost everywhere. As in the proof of Theorem 1.2.1, let D be an inveritble matrix such that $\Gamma^* = D\mathbb{Z}^d$, and let $g(x) = \sqrt{\gamma_J(Dx)}$. Then, on $S = \{x \in \mathbb{T}^d : g(x) > 0\}$, we have

$$\lambda_K(Dx) \leq g(x),$$

Thus, we must have

$$\int_S \left|\frac{1}{g(x)}\right|^{\frac{1}{2s-1}} dx \leq \int_S \left|\frac{1}{\lambda_K(Dx)}\right|^{\frac{1}{2s-1}} dx \leq \frac{1}{\det D} \int_{D^{-1}\mathbb{T}^d} \left|\frac{1}{\lambda_K(x)}\right|^{\frac{1}{2s-1}} dx < \infty.$$

This shows that $g^{-1} \in L^{\frac{1}{2s-1}}(S)$, and Theorem 4.1.1 shows that $g \in H^s(\mathbb{T}^d)$. Combining these conditions on g, Proposition 4.2.3 shows that g must be positive almost everywhere. Then, γ_J must be positive almost everywhere. This is equivalent to saying rank $[\mathrm{P}_{\Gamma^*}] = \mathrm{J}$ almost everywhere.

Finally, the extra invariance rank formula gives

$$\rho(V(F)) = \text{rank } P(x) = \sum_{k \in R} \text{rank } P_{\Gamma^*}(x+k) = J|R| = J[\Gamma : \mathbb{Z}^d],$$

which contradicts the assumptions of the theorem. □

5.3 Proof of Theorem 1.2.9

In this section, we will prove Theorem 1.2.9 and we will discuss some of the difficulties with extending this result to higher dimensions, non-minimal (C_q)-systems, and multiple generators. We start with a lemma which concerns $H^s(\mathbb{T})$ functions which have a zero.

Lemma 5.3.1. *Let $\frac{1}{2} < s < 1$. Suppose λ is a nonnegative function with $\sqrt{\lambda} \in H^s(\mathbb{T})$ and $\lambda(0) = 0$. Then, the $\mathbb{Z}$-periodic function which is defined by $m(x) = |x|^{-s}$ for $x \in [-\frac{1}{2}, \frac{1}{2})$, is such that*

$$\int_{\mathbb{T}} |m(x)|^2 \lambda(x) dx < \infty.$$

Proof. Step I: **Special case $\lambda(x) = 0$ for $x < 0$.** Suppose that in addition to having $\lambda(0) = 0$ we also know $\lambda(x) = 0$ for all $x \in [-1/2, 0]$. Since $\sqrt{\lambda} \in H^s(\mathbb{T})$, we have

$$\begin{aligned}
\infty &> \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \frac{|\sqrt{\lambda(x+t)} - \sqrt{\lambda(x)}|^2}{|t|^{2s+1}} dx dt \\
&\geq \int_0^{1/2} \int_{-t}^{0} \frac{|\sqrt{\lambda(x+t)} - \sqrt{\lambda(x)}|^2}{|t|^{2s+1}} dx dt \\
&= \int_0^{1/2} \frac{1}{t^{2s+1}} \int_0^t \lambda(x) dx dt.
\end{aligned}$$

Next, using integration by parts, we see

$$\begin{aligned}
&\int_0^{1/2} \frac{1}{t^{2s+1}} \int_0^t \lambda(x)dxdt \\
&\qquad = \left[-\frac{t^{-2s}}{2s} \int_0^t \lambda(x)dx \right]_{t=0}^{t=1/2} + \frac{1}{2s} \int_0^{1/2} \frac{\lambda(t)}{t^{2s}} dt
\end{aligned} \tag{5.1}$$

By Theorem 2.3.4, $\sqrt{\lambda} \in C^{s-1/2}(\mathbb{T})$, and since $\lambda(0) = 0$, we have $\lambda(x) < C|x|^{2s-1}$. Thus,

$$\frac{t^{-2s}}{2s} \int_0^t \lambda(x)dx \leq \frac{C}{4s^2}$$

for all $t \in [0, 1/2]$. This shows that the left term in equation (5.1) is finite, and we must have that

$$\int_0^{1/2} \frac{\lambda(t)}{t^{2s}} dt = \int_{\mathbb{T}} \frac{\lambda(t)}{|t|^{2s}} dt < \infty.$$

Step II: **General case.** Now, we will show the result for a general λ by a reduction to the case above.

Let $g = \sqrt{\lambda} \in H^s(\mathbb{T})$. Define h to be the 1-periodic function satisfying

$$h(x) = \begin{cases} g(2x) & 0 \leq x \leq 1/2 \\ g(2x+1) & -1/2 \leq x \leq 0. \end{cases}$$

Then, h satisfies $h(0) = h(1/2) = 0$, and the Fourier coefficients of h are

$$\widehat{h}(l) = \begin{cases} 0 & l \in 2\mathbb{Z}+1 \\ \widehat{g}(l/2) & l \in 2\mathbb{Z}, \end{cases}$$

which implies $h \in H^s(\mathbb{T})$. By Theorem 2.4.2, we have that $\tilde{h}(x) = \operatorname{sign}(x)h(x) \in H^s(\mathbb{T})$. Note that $b = \frac{h+\tilde{h}}{2} \in H^s(\mathbb{T})$, but

$$b(x) = \begin{cases} \sqrt{\lambda(2x)} & 0 \leq x \leq 1/2 \\ 0 & -1/2 \leq x \leq 0. \end{cases}$$

By Step I of this proof,

$$2^{2s-1}\int_0^{1/2}\frac{\lambda(t)}{|t|^{2s}}dt \le \int_0^{1/2}\frac{\lambda(2t)}{|t|^{2s}}dt < \infty.$$

A similar argument shows that

$$\int_{-1/2}^{0}\frac{\lambda(t)}{|t|^{2s}}dx < \infty.$$

We find that

$$\int_{\mathbb{T}}\frac{\lambda(t)}{|t|^{2s}}dx < \infty.$$

□

Next we show that functions like m from the previous lemma have slowly decaying Fourier series. Based on Proposition 3.2.3 and Theorem 3.2.4, to prove that $\mathscr{T}(F)$ is not a (C_q)-system for $V(F)$, it suffices to find a $\mathbb{Z}$-periodic function $M = (m_1,...,m_K) \in L^2_P(\mathbb{T})$ such that $\|\widehat{M}\|_q = \infty$ and such that M is supported on some interval. In the proof of Theorem 1.2.9, we will build such a function using the following class of examples.

For $0 < \beta < 1$, and for some $0 < a \le 1/2$, let $m_{a,\beta}$ be the $\mathbb{Z}$-periodic function whose values on $[-\frac{1}{2},\frac{1}{2})$ are given by

$$m_{a,\beta}(\xi) = \begin{cases} \frac{1}{\xi^\beta} & : 0 \le \xi < a \\ 0 & : \text{otherwise} \end{cases}$$

Lemma 5.3.2. *Fix $\frac{1}{2} < \beta \le 1$. Then, there exist $C > 0$ (depending on a) such that $Ck^{\beta-1} \le |\widehat{m_{a,\beta}}(k)|$ for all such that $k \ge \frac{1}{2a}$. In particular, if $\beta = \frac{q-1}{q}$ for some $2 < q < \infty$, then $\|\widehat{m_{a,\beta}}\|_q = \infty$.*

Proof. For any $k > 0$, we have

$$\begin{aligned}
|\widehat{m_{a,\beta}}(k)| &= \left|\int_0^a e^{-2\pi i k\xi}\frac{1}{\xi^\beta}d\xi\right| \\
&= (2k)^{\beta-1}\left|\int_0^{2ka}\frac{\cos(\pi\xi) - i\sin(\pi\xi)}{\xi^\beta}d\xi\right| \\
&\ge (2k)^{\beta-1}\left|\int_0^{2ka}\frac{\sin(\pi\xi)}{\xi^\beta}d\xi\right|
\end{aligned}$$

It suffices to show that there exists a $C > 0$ such that for all $x \geq 1$

$$C \leq \int_0^x \frac{\sin(\pi\xi)}{\xi^\beta} d\xi.$$

For any integer $j \geq 1$, let

$$a_j = \left| \int_{j-1}^{j} \frac{\sin(\pi\xi)}{\xi^\beta} d\xi \right| = \int_{j-1}^{j} \frac{|\sin(\pi\xi)|}{\xi^\beta} d\xi,$$

and note that for any integer $k \geq 1$,

$$\int_0^k \frac{\sin(\pi\xi)}{\xi^\beta} d\xi = a_1 - a_2 + a_3 - ... + (-1)^{k-1} a_k.$$

It's straightforward to check that each $0 < a_j < \infty$, and $a_{j+1} > a_j$. Then, we have $a_1 - a_2 \leq \int_0^k \frac{\sin(\pi\xi)}{\xi^\beta} d\xi \leq a_1$ for all $k > 0$. Now, when $\lfloor x \rfloor$ is even we have,

$$\int_0^{\lfloor x \rfloor} \frac{\sin(\pi\xi)}{\xi^\beta} d\xi \leq \int_0^x \frac{\sin(\pi\xi)}{\xi^\beta} d\xi \leq \int_0^{\lceil x \rceil} \frac{\sin(\pi\xi)}{\xi^\beta} d\xi.$$

This inequality is reversed with $\lfloor x \rfloor$ is odd. In either case, we find that

$$0 < \int_0^2 \frac{\sin(\pi\xi)}{\xi^\beta} d\xi \leq \int_0^x \frac{\sin(\pi\xi)}{\xi^\beta} d\xi,$$

for any $x \geq 1$. $\square$

We now prove the theorem.

Proof of Theorem 1.2.9. Let $s = \frac{q-1}{q}$ and suppose for contradiction that $F \subset H^s(\mathbb{R})$. Since $K = \rho(V(F))$, by Proposition 3.3.2 we have that the smallest eigenvalue of $P(x) = P(\widehat{F})(x)$ satisfies $\lambda_K(x) > 0$ on some set of positive measure. By Theorem 4.1.1, for each k, $\sqrt{\lambda_k} \subset H^s(\mathbb{T}) \subset C^{s-1/2}(\mathbb{T})$, and so λ_k is continuous. We will show that λ_K must have a zero, and thus Theorem 3.2.4 will apply.

In this case, the lattice of extra-invariance is $\Gamma = \frac{1}{N}\mathbb{Z}$, and $\Gamma^* = N\mathbb{Z}$. From the proofs of Theorem 1.2.1 and Theorem 1.2.8, we have seen that to derive a contradiction, it is enough to prove that the rank of $P_{N\mathbb{Z}} = P_{N\mathbb{Z}}(\widehat{F})$ is constant almost everywhere. For the rest of the proof, we assume that rank $[P_{N\mathbb{Z}}]$ is non-constant, and show that this leads to contradictions. Let $\gamma_1(x) \geq \cdots \geq \gamma_K(x) \geq 0$ be the eigenvalues of $P_{N\mathbb{Z}}(\widehat{F})(x)$ and $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_K(x) \geq 0$ be the eigenvalues of $P(x)$.

Step 1: λ_K **has a zero. Without loss of generality, we may assume** $\lambda_K(0) = 0$**, and** $\lambda_K(x) > 0$ **for** $x \in I = (0,a)$**, where** $0 < x < a \leq 1/2$**.**

Using the scaling argument from the proof of Theorem 1.2.1, we can find a representative for $\sqrt{\gamma_i}(N\cdot) \in C^{s-1/2}(\mathbb{T})$ for all i (and similarly for $\sqrt{\lambda_i}$). Let J be the smallest index such that γ_J is not equal to zero almost everywhere. Since rank $[P_{N\mathbb{Z}}]$ is nonconstant, γ_J must have a zero.

Let $O = \{x \in [0,N] : \gamma_J(x) > 0\}$. Then, O is an open proper subset of $[0,N]$, and is non-empty by the choice of index J. Note that by the ordering of eigenvalues $\lambda_K(x)$ is always less than the smallest non-zero eigenvalue of $P(x)$. Then, Lemma 5.1.2 shows that $\lambda_K(x) \leq \gamma_J(x)$ for almost every $x \in O$. Since both functions are continuous, the inequality holds on all of O and extends to $\overline{O}$, the closure of O.

By the continuity of γ_J and the definition of O, γ_J must have a zero in $\overline{O}$. However, this implies λ_K also has a zero. By shifting all of the generators, we can ensure $\lambda_K(0) = 0$, and $\lambda_K(x) > 0$ for $0 < x < a \leq 1/2$ for some $a > 0$.

Note that now all of the assumptions of Theorem 3.2.4 are satisfied. Thus, Step 2 below is sufficient to prove the theorem.

Step 2: **There is an** $m \in L^2_P(\mathbb{T})$ **supported on** $I = [0,a]$**, such that** $\|\widehat{m}\|_q = \infty$**.** Let h be the $\mathbb{Z}$-periodic function such that $h(x) = x^{-s}$ for $0 < x < a$ and $h(x) = 0$ otherwise for $\frac{1}{2} \leq x < \frac{1}{2}$. Lemma 5.3.1 shows that $\int_{\mathbb{T}} |h(x)|^2 \lambda_K(x) dx < \infty$. Lemma 5.3.2 shows that $\|\widehat{h}\|_q = \infty$.

Consider a measurable diagonalization of $P = P(\widehat{F})$,

$$P(x) = U(x)\Lambda(x)U(x)^*,$$

where $\Lambda(x)$ is the diagonal matrix with entries $\{\lambda_1, \lambda_2, ..., \lambda_K\}$. By the unitarity of $U(x)$, for the function h above we have,

$$h(x) = |u_{1,K}(x)|^2 h(x) + |u_{2,K}(x)|^2 h(x) + \cdots + |u_{K,K}(x)|^2 h(x).$$

The triangle inequality shows that for some $1 \leq l \leq K$, we have

$$\| |u_{l,K}|^2 h \|_q = \infty.$$

Define $m(x)$ by

$$\begin{aligned} m(x) &= (0, 0, ..., u_{l,K}(x) h(x)) U(x)^* \\ &= (\overline{u_{1,K}(x)} u_{l,K}(x) h(x), ..., |u_{l,K}(x)|^2 h(x), ..., \overline{u_{K,K}(x)} u_{l,K}(x) h(x)). \end{aligned}$$

Note, $\|\widehat{m}\|_q = \infty$. Also,

$$\int_{\mathbb{T}} m(x) P(x) m(x)^* dx = \int_{\mathbb{T}} |u_{K,l}(x)|^2 |h(x)|^2 \lambda_K(x) dx \leq \int_{\mathbb{T}} |h(x)|^2 \lambda_K(x) dx < \infty$$

This contradicts the (C_q)-system necessary condition in Theorem 3.2.4. Therefore, we must have $F \not\subset H^s(\mathbb{R})$.

$\square$

Chapter 6

Examples

In this chapter, we present several examples which show that the main theorems are sharp in various senses. We start by proving, in Section 6.1, that the exponent on the weight in Theorems 1.2.8, and 1.2.9 are the best possible exponents. Then, in Section 6.2, we construct examples showing that shift-invariant spaces exist which satisfy all the assumptions of Theorem 1.2.1, and for which only a single generator has poor localization. Similar results are proven for Theorem 1.2.4. Finally, in Section 6.3, we show that the exponent in Theorem 1.2.6 is sharp.

6.1 Sharpness of Exponent in Theorems 1.2.8, and 1.2.9

Recall from the discussion in Section 1.2, that Theorem 1.2.1 is sharp and one such example showing this is given by $f_0 = \widehat{\chi_{[-\frac{1}{2},\frac{1}{2}]}} = \mathrm{sinc}(x)$. We will see that smoother versions of f_0 will provide counterexamples which show sharpness for the minimal and (C_q)-system results. Higher dimesional counterexamples can be constructed by taking tensor products of copies of the one dimensional examples in each direction. Define f_α by its Fourier Transform.

$$\widehat{f_\alpha}(\xi) = \begin{cases} 0 & : \xi \geq \frac{1}{2} \\ (\frac{1}{2} - \xi)^{\alpha/2} & : 0 \leq \xi \leq \frac{1}{2} \\ (\frac{1}{2} + \xi)^{\alpha/2} & : -\frac{1}{2} \leq \xi \leq 0 \end{cases}$$

Then, f_α has the following properties.

Lemma 6.1.1. *Fix $0 < \alpha < 1$.*

1) $V(f)$ is translation invariant.

2) $\frac{1}{P(\widehat{f_\alpha})} \in L^{\frac{q}{q-2}}(\mathbb{T})$ for all $q > \frac{2}{1-\alpha}$, and so $\mathscr{T}(f)$ is a minimal (C_q)-system for $V(f)$ for all $q > \frac{2}{1-\alpha}$.

3) $\widehat{f_\alpha} \in H^s(\mathbb{R})$ for all $s < \frac{1+\alpha}{2}$.

To see that this implies sharpness in Theorem 1.2.8, Lemma 6.1.1 gives an example of a minimal system $\mathscr{T}(f_\alpha)$ for $V(f_\alpha)$, each of which generates a translation-invariant space, and such that $\widehat{f_\alpha} \in H^s(\mathbb{R}^d)$ for s arbitrarily close to 1 as α goes to 1. Similarly, to see that Theorem 1.2.9 is sharp, for a fixed q, Lemma 6.1.1 gives examples of translation-invariant, shift-invariant spaces with a minimal (C_q)-system generator, f_α, for any $\alpha < 1 - \frac{2}{q}$, and $\widehat{f_\alpha} \in H^s(\mathbb{R})$ for any $\alpha > 2s - 1$. Let $\alpha = 1 - \frac{2}{q} - \varepsilon$ so that $\mathscr{T}(f_\alpha)$ is a minimal (C_q)-system for $V(f_\alpha)$. Then, $\widehat{f_\alpha} \in H^s(\mathbb{R})$ for any $1 - \frac{2}{q} - \varepsilon > 2s - 1$, which is equivalent to $\frac{q-1}{q} - \frac{\varepsilon}{2} > s$. Letting $\varepsilon \to 0$, we find examples of minimal (C_q)-systems with generators in $H^s(\mathbb{R})$ for all $s < \frac{q-1}{q}$.

Proof. 1) This follows from the fact that $\mathrm{supp}(\widehat{f_\alpha}) = [-1/2, 1/2]$.

2) Note, $P(\widehat{f_\alpha})$ restricted to $[-\frac{1}{2}, \frac{1}{2}]$ is simply given by $\widehat{f_\alpha}^2$ restricted to this interval. Then,

$$\int_{\mathbb{T}} \frac{1}{P(\widehat{f_\alpha})(\xi)^{\frac{q}{q-2}}} dx = 2 \int_0^{1/2} \frac{1}{(\frac{1}{2} - \xi)^{\frac{\alpha q}{q-2}}} dx,$$

and this integral is finite only when $\frac{\alpha q}{q-2} < 1$, which is equivalent to $q > \frac{2}{1-\alpha}$.

3) Since $\widehat{f_\alpha} \in L^1(\mathbb{R})$, we have

$$\begin{aligned} f_\alpha(x) &= \int_0^{\frac{1}{2}} e^{2\pi i x \xi} (\frac{1}{2} - \xi)^{\alpha/2} d\xi + \int_{-\frac{1}{2}}^0 e^{2\pi i x \xi} (\frac{1}{2} + \xi)^{\alpha/2} d\xi \\ &= \int_0^{\frac{1}{2}} (e^{2\pi i x \xi} + e^{-2\pi i x \xi})(\frac{1}{2} - \xi)^{\alpha/2} d\xi \\ &= 2 \int_0^{\frac{1}{2}} \cos(2\pi x \xi)(\frac{1}{2} - \xi)^{\alpha/2} d\xi \end{aligned}$$

Note that f_α must be an even function. Then, for $x > 0$ we have .

$$\begin{aligned} f_\alpha(x) &= 2\int_0^{\frac{1}{2}} \cos(2\pi x\xi)(\frac{1}{2}-\xi)^{\alpha/2}d\xi \\ &= \frac{2}{x}\int_0^{\frac{x}{2}} \cos(2\pi w)(\frac{1}{2}-\frac{w}{x})^{\alpha/2}dw \\ &= \frac{2^{1-\alpha/2}}{x^{1+\alpha/2}}\int_0^{\frac{x}{2}} \cos(2\pi w)(x-2w)^{\alpha/2}dw \\ &= \frac{1}{x^{1+\alpha/2}2^{\alpha/2}}\int_0^{x} \cos(\pi y)(x-y)^{\alpha/2}dy \end{aligned}$$

Now we show that $|\int_0^x \cos(\pi y)(x-y)^{\alpha/2}dy|$ is bounded by a constant (which depends on α) for all $x > 1$. First,

$$\begin{aligned} |\int_{x-1}^{x} \cos(\pi y)(x-y)^{\alpha/2}dy| &\le \int_{x-1}^{x} (x-y)^{\alpha/2}dy \\ &= \frac{2}{2+\alpha} \end{aligned}$$

Second, using integration by parts twice,

$$\begin{aligned} \int_0^{x-1} \cos(\pi y)(x-y)^{\alpha/2}dy &= \frac{\sin(\pi(x-1))}{\pi} + \frac{\alpha}{2\pi}\int_0^{x-1} \sin(\pi y)(x-y)^{\alpha/2-1}dy \\ &= \frac{\sin(\pi(x-1))}{\pi} + \frac{\alpha}{2\pi}\left[-\frac{\cos(\pi(x-1))}{\pi} + \frac{1}{\pi x^{1-\alpha/2}}\right. \\ &\qquad \left. + \frac{\alpha-2}{2\pi}\int_0^{x-1} \cos(\pi y)(x-y)^{\alpha/2-2}dy\right] \\ &\le \frac{\pi+\alpha}{\pi^2} + \frac{\alpha(\alpha-2)}{4\pi^2}\int_0^{x-1} \cos(\pi y)(x-y)^{\alpha/2-2}dy \end{aligned}$$

Note,

$$\begin{aligned} |\int_0^{x-1} \cos(\pi y)(x-y)^{\alpha/2-2}dy| &\le \int_0^{x-1} (x-y)^{\alpha/2-2}dy \\ &= \frac{2}{\alpha-2} - \frac{2}{(\alpha-2)x^{\alpha/2-1}} \le \frac{2}{\alpha-2} \end{aligned}$$

Finally, we have the bound for $x > 1$

$$|\int_0^x \cos(\pi y)(x-y)^{\alpha/2}dy| \leq \frac{2}{2+\alpha} + \frac{\pi+\alpha}{\pi^2} + \frac{\alpha}{2\pi^2}$$

Then,

$$|f_\alpha(x)|^2 \leq \frac{C(\alpha)}{|x|^{2+\alpha}}, \qquad |x| > 1$$

We have

$$\int_{\mathbb{R}} |x|^{2s}|f_\alpha(x)|^2 dx < \infty$$

whenever $2+\alpha-2s > 1 \implies s < \frac{1+\alpha}{2}$. $\square$

6.2 Finitely Generated Examples

The examples in this section first appeared in [32]. The first two examples show that there are multiply generated shift-invariant spaces for which the hypotheses of Theorem 1.2.1 hold, but for which the conclusion of the theorem only holds for a single generator. The collection of smooth compactly supported functions on $\mathbb{R}^d$ will be denoted by $C_c^\infty(\mathbb{R}^d)$.

Example 6.2.1. Let $I = [-1/2, 1/2)^d$. Define $f_1 \in L^2(\mathbb{R}^d)$ by $\widehat{f_1} = \chi_I$. Take any $g \in C_c^\infty(\mathbb{R}^d)$ that is supported on I and satisfies $\|g\|_2 = 1$, and define $f_2 \in L^2(\mathbb{R}^d)$ by $\widehat{f_2} = g$.

Let $F = \{f_1, f_2\}$ and $\Gamma = (\frac{1}{2}\mathbb{Z}) \times \mathbb{Z}^{d-1}$. The space $V(F) = V(f_1, f_2)$ has the following properties:

- $\mathscr{T}(F)$ is a frame for $V(F)$;
- $V(F)$ is Γ-invariant (it is actually translation invariant);
- $\widehat{f_1} \notin H^{1/2}(\mathbb{R}^d)$ and $\widehat{f_2} \in C_c^\infty(\mathbb{R}^d) \subset H^{1/2}(\mathbb{R}^d)$;

- $\rho(V(F)) = 1$ and $[\Gamma : \mathbb{Z}^d] = 2$, so that $[\Gamma : \mathbb{Z}^d]$ does not divide $\rho(V(F))$.

This can be verified by computing the Gramian $P(\widehat{F})(x)$. Since $P(\widehat{F})(x)$ is $\mathbb{Z}$-periodic, it suffices to only consider $x \in I$ in the subsequent discussion. A computation shows that for $x \in I$

$$P(\widehat{F})(x) = \begin{pmatrix} [\widehat{f_1}, \widehat{f_1}](x) & [\widehat{f_1}, \widehat{f_2}](x) \\ [\widehat{f_2}, \widehat{f_1}](x) & [\widehat{f_2}, \widehat{f_2}](x) \end{pmatrix} = \begin{pmatrix} 1 & \overline{g(x)} \\ g(x) & |g(x)|^2 \end{pmatrix}.$$

A further computation shows that

$$\left(P(\widehat{F})(x)\right)^2 = (1 + |g(x)|^2) \begin{pmatrix} 1 & \overline{g(x)} \\ g(x) & |g(x)|^2 \end{pmatrix} = (1 + |g(x)|^2)\, P(\widehat{F})(x)$$

Since $g \in C_c^\infty(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, we have the operator inequality

$$P(\widehat{F})(x) \le \left(P(\widehat{F})(x)\right)^2 \le (1 + \|g\|_\infty^2) P(\widehat{F})(x).$$

So, by (3.7), $\mathscr{T}(F)$ is a frame for $V(F)$.

The remaining properties can also be checked easily. Similar computations as above, together with Theorem 3.3.1, show that $V(F)$ is Γ-invariant. A direct computation shows that $\widehat{f_1} \notin H^{1/2}(\mathbb{R}^d)$. The condition $\rho(V(F)) = 1$ can be seen by using Proposition 3.3.2 and noting that $P(\widehat{F})(x)$ has rank 1 for all $x \in I$. Finally, it is easily verified that $[\Lambda : \mathbb{Z}^d] = 2$.

Example 6.2.2. Fix any integer $N \ge 2$. Let $I = [-1/2, 1/2)$ and define $f_{N+1} \in L^2(\mathbb{R})$ by $\widehat{f_{N+1}} = \chi_I$. Fix $0 < \varepsilon < \frac{1}{2N}$. Select $f \in C_c^\infty(\mathbb{R})$ with $\|f\|_2 = 1$ such that f is supported on $[0, 1/N]$, and such that $|\widehat{f}(x)| \le \varepsilon$ for all $x \in I$. For example, such an f can be constructed by suitably dilating and translating a given smooth compactly supported function. For $1 \le n \le N$, define $f_n(x) = f(x - n/N)$.

Define $F = \{f_n\}_{n=1}^{N+1} \subset L^2(\mathbb{R})$ and $\Gamma = \frac{1}{N}\mathbb{Z}$. The space $V(F)$ satisfies the following properties

- $\mathscr{T}(F)$ is a Riesz basis for $V(F)$;

- $V(F)$ is invariant under Γ;
- $\rho(V(F)) = N+1$ and $[\Gamma : \mathbb{Z}] = N$, so that $[\Gamma : \mathbb{Z}]$ does not divide $\rho(V(F))$;
- $\widehat{f_n} \in H^{1/2}(\mathbb{R})$ for each $1 \leq n \leq N$;
- $\widehat{f_{N+1}} \notin H^{1/2}(\mathbb{R})$.

The singly generated system $V(f_{N+1})$ is easily seen to be $\frac{1}{N}\mathbb{Z}$-invariant by Theorem 3.3.1 (in fact, $V(f_{N+1})$ is translation invariant). Moreover, the space $V(f_1, \cdots, f_N)$ is $\frac{1}{N}\mathbb{Z}$-invariant by construction. It follows that $V(F) = V(f_1, \cdots, f_{N+1})$ is $\frac{1}{N}\mathbb{Z}$-invariant. Also, $\widehat{f_n} \in H^{1/2}(\mathbb{R})$ for each $1 \leq n \leq N$ since $f_n \in C_c^\infty(\mathbb{R})$.

Since $\mathscr{T}(\{f_n\}_{n=1}^N)$ is an orthonormal basis for $V(\{f_n\}_{n=1}^N)$, one has for $1 \leq j,k \leq N$ that $[\widehat{f_j}, \widehat{f_k}](x) = \delta_{j,k}$ for almost every $x \in \mathbb{R}$. Also, if $1 \leq n \leq N$, then for $x \in I$,

$$[\widehat{f_n}, \widehat{f_{N+1}}](x) = \sum_{j \in \mathbb{Z}} \widehat{f_n}(x-j)\chi_I(x-j) = \widehat{f_n}(x) = e^{-2\pi i n x/N}\widehat{f}(x).$$

By our assumptions on f, we have that for $1 \leq n \leq N$, and $x \in I$,

$$\left|[\widehat{f_{N+1}}, \widehat{f_n}](x)\right| = \left|[\widehat{f_n}, \widehat{f_{N+1}}](x)\right| = |\widehat{f}(x)| \leq \varepsilon. \tag{6.1}$$

Recalling that $P(\widehat{F})(x)$ is $\mathbb{Z}$-periodic, we have that for all $x \in I$,

$$P_{\mathbb{Z}}(\widehat{F})(x) = \begin{pmatrix} 1 & 0 & \cdots & 0 & [\widehat{f_1}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) \\ 0 & 1 & \cdots & 0 & [\widehat{f_2}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & [\widehat{f_N}, \widehat{f_{N+1}}]_{\mathbb{Z}}(x) \\ [\widehat{f_{N+1}}, \widehat{f_1}]_{\mathbb{Z}}(x) & [\widehat{f_{N+1}}, \widehat{f_2}]_{\mathbb{Z}}(x) & \cdots & [\widehat{f_{N+1}}, \widehat{f_N}]_{\mathbb{Z}}(x) & 1 \end{pmatrix}.$$

The Gershgorin circle theorem, together with (6.1), shows that all eigenvalues of $P(\widehat{F})(x)$ lie in the interval $[1-N\varepsilon, 1+N\varepsilon]$. Since $0 < \varepsilon < \frac{1}{2N}$, the condition (3.6) holds, and hence $\mathscr{T}(F)$ is a Riesz basis for $V(F)$. Moreover, since $P(\widehat{F})(x)$ is full rank for a.e. $x \in I$, Proposition 3.3.2 shows that $\rho(V(F)) = N+1$.

The next example shows that there are multiply generated shift-invariant spaces for which the hypotheses of Theorem 1.2.4 hold and for which the conclusion only holds for a single generator.

Example 6.2.3. Let $J = [-1/4, 1/4]$. Define $f_1 \in L^2(\mathbb{R})$ by $\widehat{f_1} = \chi_J$. Select $f_2 \in C_c^\infty(\mathbb{R})$ such that f_2 is supported in $[-1/2, 1/2]$, $\|f_2\|_2 = 1$, and $|\widehat{f_2}(x)| < 1/2$ for all $x \in J$.

Define $F = \{f_1, f_2\}$. The space $V(F)$ satisfies the following properties:

- $\mathscr{T}(F)$ is a frame, but not a Riesz basis, for $V(F)$;
- The minimal number of generators $\rho(V(F)) = 2$;
- $\widehat{f_1} \notin H^{1/2}(\mathbb{R})$ and $\widehat{f_2} \in H^{1/2}(\mathbb{R})$.

Recall that $P(\widehat{F})(x)$ is $\mathbb{Z}$-periodic. A computation shows that for $x \in [-1/2, 1/2]$

$$P(\widehat{F})(x) = \begin{pmatrix} \chi_J(x) & \chi_J(x)\widehat{f_2}(x) \\ \chi_J(x)\overline{\widehat{f_2}(x)} & 1 \end{pmatrix}.$$

For $1/4 < |x| < 1/2$, we have

$$P(\widehat{F})(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

so that $\lambda_1(x) = 1$ and $\lambda_2(x) = 0$, and for $|x| < 1/4$, we have

$$P(\widehat{F})(x) = \begin{pmatrix} 1 & \widehat{f_2}(x) \\ \overline{\widehat{f_2}(x)} & 1 \end{pmatrix}$$

so that $\lambda_1(x) = 1 + |\widehat{f_2}(x)|$ and $\lambda_2(x) = 1 - |\widehat{f_2}(x)|$.

By (3.6), $\mathscr{T}(F)$ is not a Riesz basis for $V(F)$. However Proposition 3.3.2, (3.7), and $|\widehat{f_2}(x)| < 1/2$ for $x \in J$, show that $\mathscr{T}(F)$ is a frame for $V(F)$ and $\rho(V(F)) = 2$.

6.3 Sharpness for Theorem 1.2.6

For this example, we will consider $\Gamma = \frac{1}{2}\mathbb{Z}^d$, but by considering maps from the unit cube to the fundamental domains of the lattice Γ, we could produce examples for arbitrary lattices.

6.3.1 Construction for $d = 1$

For clarity in understanding the construction, we first consider $d = 1$, although this case has already been shown to be sharp in [2, 49]. Let $I = [-\frac{1}{2}, \frac{1}{2})$. We will construct a partition of unity for χ_I, and then we will shift the pieces of this partition and scale them to construct a function $h \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. We will do this in such a way that $\widehat{f} = \sqrt{h} \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ will be a function which satisfies

$$\operatorname*{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)||\xi|^{1/2} < \infty,$$

$\mathscr{T}(f)$ forms an orthonormal basis for $V(f)$, and $V(f)$ is $\frac{1}{2}\mathbb{Z}$-invariant. (i.e. we outline the construction of $\widehat{f}^2$ for the generator f which we are looking for.)

We start with a simple symmetric trapezoid function with height 1, top width $\frac{1}{2}$, and bottom width $\frac{3}{4}$. Next, we consider a similar trapezoid with height 1, top width $\frac{3}{4}$, and bottom width $\frac{7}{8}$. In fact, we will consider the sequence of trapezoid functions $\{T_n\}$ with height 1, top width $\frac{2^n-1}{2^n}$ and bottom width $\frac{2^{n+1}-1}{2^{n+1}}$. A graph of the first five T_n functions is given in Figure 6.1.

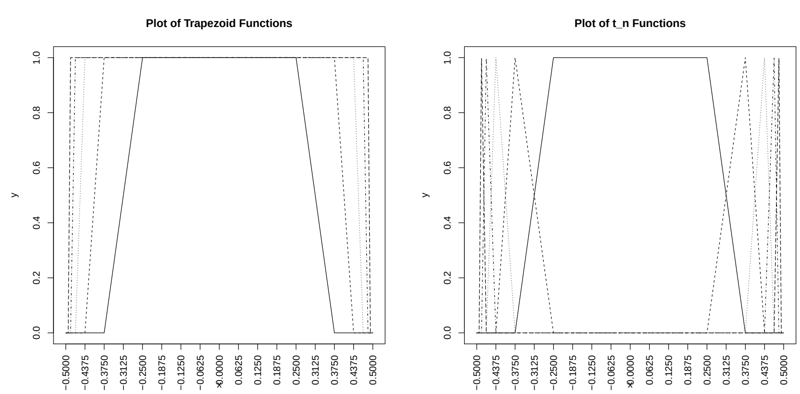

Figure 6.1: T_n and t_n Functions

It's clear that $T_n \to \chi_{I^o}$ pointwise on $\mathbb{R}$ (By I^o I mean the interior of I). Thus, the telescoping sum $\sum_{n=0}^{\infty}(T_{n+1}-T_n)$, where we define $T_0(x)=0$ for all $x\in\mathbb{R}$, converges pointwise to χ_{I^o}. The functions $t_n=T_{n+1}-T_n$ form an infinite partition of unity for χ_I, and note that each of these is continuous and for any $x\in I$, $t_n(x)>0$ for at most two values of n.

We can write T_n explicitly as

$$T_n(x)=\begin{cases} 1 & : |x|\le \frac{2^n-1}{2^{n+1}} \\ -2^{n+2}|x|+2^{n+1}-1 & : \frac{2^n-1}{2^{n+1}}<|x|<\frac{2^{n+1}-1}{2^{n+2}} \\ 0 & : |x|\ge \frac{2^{n+1}-1}{2^{n+2}}. \end{cases}$$

From this we can find that t_n for $n>0$ is given by

$$t_n(x)=\begin{cases} 0 & : |x|\le \frac{2^n-1}{2^{n+1}} \\ 2^{n+2}|x|-2^{n+1}+2 & : \frac{2^n-1}{2^{n+1}}<|x|<\frac{2^{n+1}-1}{2^{n+2}} \\ -2^{n+3}|x|+2^{n+2}-1 & : \frac{2^{n+1}-1}{2^{n+2}}<|x|<\frac{2^{n+2}-1}{2^{n+3}} \\ 0 & : |x|\ge \frac{2^{n+2}-1}{2^{n+3}}. \end{cases}$$

Figure 6.1 shows the first 5 t_n functions.

Our function $\widehat{f}^2$ is defined as

$$\widehat{f}^2(\xi)=\sum_{n=0}^{\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}t_n(\xi-2j).$$

A graph is given in Figure 6.2. This function essentially consists of taking $2n+1$ copies of t_n, dividing each of these by $2n+1$, and then shifting those pieces by the even integers which are less than or equal to $2n$ in absolute value.

We have already seen that $\sum_{n=0}^{\infty}t_n(x)=1$ on $(-1/2,1/2)$ and $\sum_{n=0}^{\infty}t_n(x)=0$ outside of this interval. Then, using Monotone Convergence Theoerm,

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}}t_n(x)dx=\int_{\mathbb{R}}\sum_{n=0}^{\infty}t_n(x)dx=1.$$

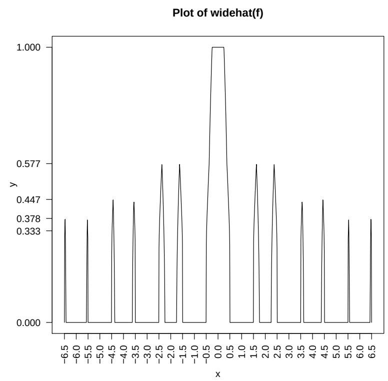

Figure 6.2: Graph of $\widehat{f}$

This immediately shows that $\widehat{f} \in L^2(\mathbb{R})$ since

$$\begin{aligned}
\int_{\mathbb{R}} |\widehat{f}(\xi)|^2 d\xi &= \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} t_n(\xi - 2j) d\xi \\
&= \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} \int_{\mathbb{R}} t_n(\xi - 2j) d\xi \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{R}} t_n(\xi) d\xi = 1.
\end{aligned}$$

By Theorem 3.3.1, $\widehat{f}$ is such that $V(f)$ is $\frac{1}{2}\mathbb{Z}$-invariant. Also, for almost every $\xi \in \mathbb{T}$,

$$\begin{aligned}
P(\widehat{f})(\xi) &= \sum_{k \in \mathbb{Z}} |\widehat{f}(\xi - k)|^2 \\
&= \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} t_n(\xi - 2j - k) \\
&= \sum_{k \in \mathbb{Z}} \sum_{n=0}^{\infty} t_n(\xi - k) = 1
\end{aligned}$$

The only piece left to check is that $\operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)||\xi|^{1/2} < \infty$. Note that on intervals of the form $[2k+1-\frac{1}{2}, 2k+1+\frac{1}{2}]$ for any $k \in \mathbb{Z}$, $\widehat{f} = 0$, and on intervals of the form $[2k - \frac{1}{2}, 2k + \frac{1}{2}]$ for any $k \in \mathbb{Z}$, $\widehat{f}$ is bounded above by $\sqrt{\frac{1}{2|k|+1}}$. Let $\xi \in \mathbb{R}$ be such that there exist a $k \in \mathbb{Z}$ such that

$\xi = 2k + \beta$ where $-1/2 < \beta \leq 1/2$. Then,

$$|\widehat{f}(\xi)||\xi|^{1/2} \leq (\frac{1}{2|k|+1})^{1/2}(2|k| + \frac{1}{2})^{1/2}$$
$$\leq 1.$$

Therefore, $\operatorname{ess\,sup}_{\xi \in \mathbb{R}} |\widehat{f}(\xi)||\xi|^{1/2} < \infty$.

6.3.2 Construction for $d > 1$

Fix a dimension $d > 1$. We define a sequence $\{\mathscr{T}_n\}_{n=0}^{\infty}$ which will play a role analogous to the role of T_n in the one dimensional case. In fact,

$$\mathscr{T}_n(x_1, x_2, ..., x_d) = T_n(x_1)T_n(x_2)\cdots T_n(x_d).$$

A graph of the two dimensional $\mathscr{T}_1$ is given in Figure 6.4. Note, that for any d, $\mathscr{T}_n$ will be equal to 1 on the symmetric hypercube of width $\frac{2^n-1}{2^n}$, and will be supported in the symmetric hypercube of width $\frac{2^{n+1}-1}{2^{n+1}}$. Just like the dimension 1 case, $\mathscr{T}_n \to \chi_{(I^d)^o}$ pointwise in $\mathbb{R}$. Next, we construct the sequence $\{\tau_n\}_{n=0}^{\infty}$ which will be similar to t_n. Specifically, we define

$$\tau_n(x) = \mathscr{T}_{n+1}(x) - \mathscr{T}_n(x).$$

A graph of the two-dimensional version of τ_1 is given in Figure 6.3. The support of τ_n is $H(\frac{2^{n+2}-1}{2^{n+2}}) \setminus H(\frac{2^n-1}{2^n})$, where $H(t)$ is the symmetric hypercube of width t. Any $x \in I^d$ is such that at most 2 values of $n \in \mathbb{N}$ satisfy $t_n(x) > 0$.

Now we define our function $\widehat{f}$ in similar fashion to the one dimensional case. We let $\alpha = (\alpha_1, ..., \alpha_d)$ denote a vector in $\mathbb{Z}^d$, and $||\alpha||_\infty = \max_{1 \leq i \leq d} |\alpha_i|$. Then, we define $\widehat{f}$ to be the positive function satisfying

$$\widehat{f}^2(\xi) = \sum_{n=0}^{\infty} \left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n} \tau_n(\xi - 2\alpha).$$

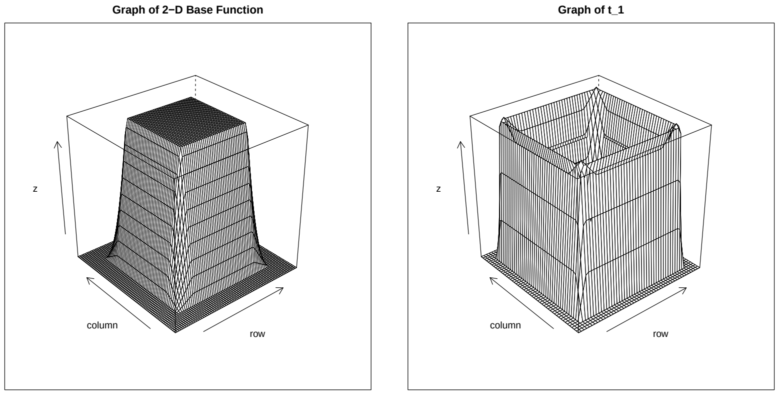

Figure 6.3: Graph of $\mathscr{T}_1$ and τ_1 Function in $\mathbb{R}^2$

A graph of $\widehat{f}$ in the two dimensional case is given in Figure 6.4.

Next we show that f is truely a counterexample.

Proposition 6.3.1. *Let $\widehat{f}$ be defined as above. Then, the following statements hold.*

1. $\widehat{f} \in L^2(\mathbb{R}^d) \cap C(\mathbb{R}^d)$.

2. *$V(f)$ is $\frac{1}{2}\mathbb{Z}^d$-invariant.*

3. *$\mathscr{T}(f)$ forms an orthonormal basis for $V(f)$.*

4. $\operatorname{ess\,sup}_{\xi \in \mathbb{R}^d} |\widehat{f}(\xi)||\xi|^{d/2} < \infty$

Proof. 1. First, we show that $\widehat{f} \in L^2(\mathbb{R}^d)$. By construction, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \tau_n(x) &= \sum_{n=0}^{\infty} (\mathscr{T}_{n+1}(x) - \mathscr{T}_n(x)) \\
&= \lim_{n\to\infty} \mathscr{T}_n(x) \\
&= \chi_{(I^d)^o}(x)
\end{aligned}$$

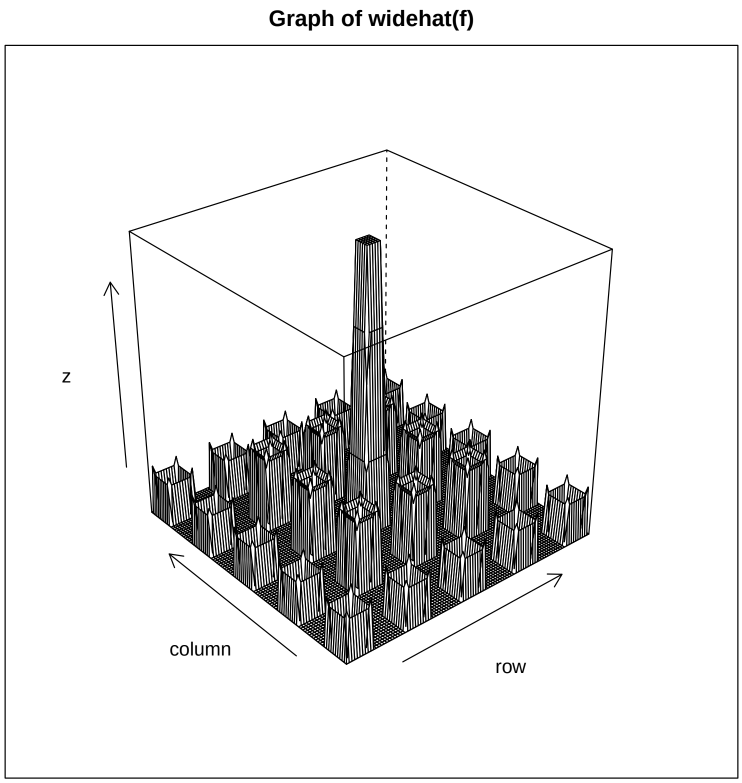

Figure 6.4: Graph of $\widehat{f}$ in $\mathbb{R}^2$

By the Monotone Convergence Theorem, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\int_{\mathbb{R}^d}\tau_n(x)dx &= \int_{\mathbb{R}^d}\sum_{n=0}^{\infty}\tau_n(x)dx \\ &= 1\end{aligned}$$

Now, we can calculate $\|\widehat{f}\|_2$.

$$\begin{aligned}\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2 d\xi &= \sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n}\int_{\mathbb{R}^d}\tau_n(\xi-2\alpha)d\xi \\ &= \sum_{n=0}^{\infty}\int_{\mathbb{R}^d}\tau_n(\xi)d\xi = 1\end{aligned}$$

Next, we show that $\widehat{f}$ is continuous. From the definition of $\widehat{f}^2$, we have see that

$$|\widehat{f}^2(\xi) - \sum_{n=0}^{N-1} \left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n} \tau_n(\xi - 2\alpha)| \leq \sum_{n=N}^{\infty} \left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n} \tau_n(\xi - 2\alpha).$$

each of the functions $\left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n} \tau_n(\xi - 2\alpha)$ is continuous, and it is an easy consequence of the definition of τ_n that

$$|| \sum_{n=N}^{\infty} \left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n} \tau_n(\xi - 2\alpha)||_\infty = (\frac{1}{2N+1})^d$$

Thus, $\sum_{n=0}^{N-1} \left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n} \tau_n(\xi - 2\alpha)$ converges uniformly to $\widehat{f}^2(\xi)$. Therefore, $\widehat{f}^2$ and $\widehat{f}$ are continuous.

2. This follows from Theorem 3.3.1.

3. Equation (3.5) shows that $\mathscr{T}(f)$ forms an orthonormal basis for $V(f)$ if and only if $P(\widehat{f})$ is equal to 1 almost everywhere. We have,

$$\begin{aligned}
P(\widehat{f})(\xi) &= \sum_{k \in \mathbb{Z}^d} |\widehat{f}(\xi - k)|^2 \\
&= \sum_{k \in \mathbb{Z}^d} \sum_{n=0}^{\infty} \left(\frac{1}{2n+1}\right)^d \sum_{||\alpha||_\infty \leq n} \tau_n(\xi - 2\alpha - k) \\
&= \sum_{k \in \mathbb{Z}^d} \sum_{n=0}^{\infty} \tau_n(\xi - k) \\
&= \sum_{k \in \mathbb{Z}^d} \chi_{I^d}(\xi - k) \\
&= 1
\end{aligned}$$

and this holds almost everywhere in $\mathbb{R}^d$.

4. Due to the fact that all norms on finite dimensional vector spaces are equivalent, it is equiv-

alent to show that

$$\operatorname*{ess\,sup}_{\xi \in \mathbb{R}^d} |\widehat{f}(\xi)| \|\xi\|_\infty^{d/2} < \infty.$$

In the definition of $\widehat{f}$, there are $(2n+1)^d$ copies of τ_n which are shifted to hypercubes of width less than 1 centered at each of the vectors 2α where $\alpha \in \mathbb{Z}^d$ satisfies $\|\alpha\|_\infty \leq n$. The maximum of $\widehat{f}$ restricted to any of the cubes with $\|\alpha\|_\infty = n$ is $(\frac{1}{2n+1})^{d/2}$. For any ξ in a cube of width one centered at an α with $|\alpha| = n$, we have

$$\begin{aligned} |\widehat{f}(\xi)| \|\xi\|_\infty^2 &\leq \left(\frac{1}{2n+1}\right)^{d/2} (2n+1/2)^{d/2} \\ &< 1 \end{aligned}$$

Therefore,

$$\operatorname*{ess\,sup}_{\xi \in \mathbb{R}^d} |\widehat{f}(\xi)| \|\xi\|_\infty^{d/2} \leq 1.$$

$\square$

BIBLIOGRAPHY

[1] A. Aldroubi, C. Cabrelli, C. Heil, K. Kornelson and U. Molter, Invariance of a shift-invariant space, Journal of Fourier Analysis and Applications 16 (2010), no.1, 60–75.

[2] A. Aldroubi, Q. Sun, H. Wang, Uncertainty principles and Balian-Low type theorems in principal shift-invariant spaces, Applied and Computational Harmonic Analysis, 30 (2011), no. 3, 337–347.

[3] M. Anastasio, C. Cabrelli,V. Paternostro, Invariance of a shift-invariant space in several variables, Complex Analysis and Operator Theory, 5 (2011), no. 4, 1031–1050.

[4] M. Anastasio, C. Cabrelli,V. Paternostro, Extra invariance of shift-invariant spaces on LCA groups,

[5] G. Ascensi, H. Feichtinger, N. Kaiblinger, Dilation of the Weyl symbol and Balian-Low theorem, Transactions of the AMS, 366 (2014), no. 7, 3865–3880.

[6] R. Balan, I. Daubechies, Optimal stochastic encoding and approximation schemes using Weyl-Heisenberg sets, Advances in Gabor analysis, Applied and Numerical Harmonic Analysis, 259–320, Birkhäuser, Boston, MA, 2003.

[7] R. Balian, Un principe d'incertitude fort en théorie du signal ou en mécanique quantique, Comptes Rendus de l'Académie des Sciences, 292 (1981), no. 20, 1357–1362.

[8] G. Battle, Heisenberg proof of the Balian–Low theorem, Letters in Mathematical Physics, 15 (1988),175–177.

[9] J.J. Benedetto, W. Czaja, P. Gadziński, A.M. Powell, The Balian–Low Theorem and regularity of Gabor systems, Journal of Geometric Analysis, 13 (2003), no. 2, 239–254.

[10] J.J. Benedetto, W. Czaja, A. Maltsev, The Balian-Low theorem for the symplectic form on $\mathbb{R}^{2d}$, Journal of Mathematical Physics, 44 (2003), no. 4, 1735–1750.

[11] J.J. Benedetto, W. Czaja, A.M. Powell, An optimal example for the Balian-low uncertainty principle, SIAM Journal of Mathematical Analysis, 38 (2006), no. 1, 333–345.

[12] J.J. Benedetto, W. Czaja, A.M. Powell, and J. Sterbenz, An endpoint $(1,\infty)$ Balian–Low Theorem, Mathematical Research Letters, 13 (2006), no. 2–4, 467–474.

[13] J.J. Benedetto, C. Heil, D. Walnut, Differentiation and the Balian-Low theorem, Journal of Fourier Analysis and Applications, 1 (1995), no. 4, 355–402.

[14] A. Benyi, T. Oh, The Sobolev inequality on the torus revisited, Publicationes Mathematicae Debrecen, 83.3 (2013), 359-374.

[15] R. Bhatia, Matrix Analysis, Springer, New York, 1997.

[16] J. Bourgain, H. Brezis, P. Mironescu, Lifting in Sobolev spaces, Journal d'Analyse Mathématique, 80 (2000), 37–86.

[17] M. Bownik, The structure of shift-invariant subspaces of $L^2(\mathbb{R}^n)$, Journal of Functional Analysis, 177 (2000) 282–309.

[18] M. Bownik, N. Kaiblinger, Minimal generator sets for finitely generated shift-invariant subspaces of $L^2(\mathbb{R}^n)$, Journal of Mathematical Analysis and Applications, 313 (2006) 342–352.

[19] H. Brezis, Y. Li, P. Mironescu, L. Nirenberg, Degree and Sobolev spaces, Topological Methods in Nonlinear Analysis, 13 (1999), 181–190.

[20] P. Casazza, M. Lammers, Bracket products for Weyl-Heisenberg frames, in "Advances in Gabor Analysis", Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2003, pages 71–98.

[21] O. Christensen, An Introduction to frames and Riesz bases, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2003.

[22] W. Czaja, A.M. Powell, Recent developments in the Balian-Low theorem, in “Harmonic Analysis and Applications”, Applied and Numerical Harmonic Analysis, Birkäuser, Boston, 2006, pages 79–100.

[23] I. Daubechies, The wavelet transform, time-frequency localization and signal analysis, IEEE Transactions on Information Theory, 39 (1990), 961-1005.

[24] I. Daubechies, A.J.E.M. Janssen, Two theorems on lattice expansions, IEEE Transactions on Information Theory, 39 (1993), no. 1, 3–6.

[25] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, Bull. Sci. Math, 136 (2012), no.5, 521-573 .

[26] C. de Boor, R.A. DeVore, A. Ron, The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$, Journal of Functional Analysis, 119 (1994), no. 1, 37–78.

[27] G.B. Folland, A. Sitaram, The uncertainty principle: a mathematical survey, Journal of Fourier Analysis and Applications, 3 (1997), no. 3, 207–238.

[28] H. Führ, J. Xian, Quantifying invariance properties of shift-invariant spaces, Applied and Computational Harmonic Analysis, 36 (2014), 514–521.

[29] J.-P. Gabardo, D. Han, Balian-Low phenomenon for subspace Gabor frames, Journal of Mathematical Physics, 45 (2004), no. 8, 3362–3378.

[30] K. Gröchenig, personal communication.

[31] K. Gröchenig, D. Han, C. Heil, G. Kutyniok, The Balian-Low theorem for symplectic lattices in higher dimensions, Applied and Computational Harmonic Analysis, 13 (2002), no. 2, 169–176.

[32] D. Hardin, M. Northington, A.M. Powell, A sharp Balian-Low uncertainty principle for shift-invariant spaces, Submitted. Available at `http://arxiv.org/pdf/1510.04855.pdf`.

[33] C. Heil, A Basis Theory Primer, Expanded Edition, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2011.

[34] C. Heil, A.M. Powell, Gabor Schauder bases and the Balian-Low theorem, Journal of Mathematical Physics, 47 (2006), no. 11, 113506, 21 pp.

[35] C. Heil, A.M. Powell, Regularity for complete and minimal Gabor systems on a lattice, Illinois Journal of Mathematics, 53 (2009), no. 4, 1077–1094.

[36] E. Hernández, H. Šikić, G. Weiss, E. Wilson, On the properties of the integer translates of a square integrable function, Contemporary Mathematics, 505 (2010), 233–249.

[37] J. Hogan, J. Lakey, Non-translation-invariance and the synchronization problem in wavelet sampling, Acta Applicandae Mathematicae, 107 (2009), 373–398.

[38] J. Hogan, J. Lakey, Non-translation-invariance in principal shift-invariant spaces, Advances in analysis, 471–485, World Scientific Publishing, Hackensack, NJ, 2005.

[39] A.J.E.M. Janssen, A decay result for certain windows generating orthogonal Gabor bases, Journal of Fourier Analysis and Applications, 14 (2008), 1–15.

[40] R. Hunt, B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Transactions of the American Mathematical Society, Vol 176, Feb. 1973, 227-251.

[41] M. Lammers, The finite Balian-Low conjecture, Proceedings of the 11th International Conference on Sampling Theory and Applications 2015 (Washington DC, May 25–29, 2015), 4 pages.

[42] F. Low, Complete sets of wave packets, in: "A Passion for Physics—Essays in Honor of Geoffrey Chew," C. DeTar et al., eds., World Scientific, Singapore, 1985, pages 17–22.

[43] V. Maz'ja, Sobolev Spaces, Springer-Verlang, New York, 1985.

[44] S. Nitzan, A. Olevskii, Quasi-frames of translates, C. R. Acad. Sci. Paris, 347 (2009), no. 13-14, 739-742.

[45] S. Nitzan, J.-F. Olsen, From exact systems to Riesz bases in the Balian-Low theorem, Journal of Fourier Analysis and Applications, 17 (2011), no. 4, 567–603.

[46] S. Nitzan, J.-F. Olsen, A quantitative Balian-Low theorem, Journal of Fourier Analysis and Applications, 19 (2013), no. 5, 1078–1092.

[47] S. Nitzan-Hahamov, A. Olevskii, Sparse exponential systems: completeness with estimates. Israel J. Math. 158, (2007), 201-215.

[48] A. Ron, Z. Shen, Frames and stable bases for shift invariant subspaces of $L^2(\mathbb{R}^d)$, Canadian Journal of Mathematics, 47 (1995), 1051-1094.

[49] R. Tessera, H. Wang, Uncertainty principles in finitely generated shift-invariant spaces with additional invariance, Journal of Mathematical Analysis and Applications, 410 (2014), 134–143.