Annular representation theory with applications to approximation and rigidity properties for rigid C*-tensor categories

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CHAPTER I

INTRODUCTION

Rigid C^*-tensor categories provide a unifying language for a variety of phenomena encoding "quantum symmetries". For example, they appear as the representation categories of Woronowicz' compact quantum groups, and as "gauge symmetries" in the algebraic quantum field theory of Haag and Kastler. Perhaps most prominently, they arise as categories of finite index bimodules over operator algebras, taking center stage in Jones' theory of subfactors. The study of these categories is a very active field of research. Categories with infinitely many isomorphism classes of simple objects may exhibit interesting analytic properties including amenability, the Haagerup property, or property (T), analogous to infinite discrete groups.

Recall that approximation and rigidity properties such as amenability, the Haagerup property, and property (T) can be defined for discrete groups in terms of the behavior of sequences of positive definite functions converging to the trivial representation, or equivalently through the properties of the Fell topology on the space of irreducible unitary representations near the trivial representation. In particular, approximation properties guarantee the existence of "small" representations converging to the trivial representation, while property (T) asserts that the trivial representation is isolated in the Fell topology.

Following the analogy with groups in the subfactor context, Popa introduced concepts of analytic properties for standard invariants of finite index inclusions of II_1 factors [40], [41], [43], [44]. For a finite index subfactor $N \subseteq M$, Popa introduced the symmetric enveloping inclusion $T \subseteq S$ (see [44]). One can view S as a sort of crossed product of T by the category of $M-M$ bimodules appearing in the standard invariant of $N \subseteq M$. Then one can use sequences of UCP maps $\psi_n : S \to S$ which are T-bimodular in place of positive definite functions to define approximation and rigidity properties, with the identity map replacing the trivial representation. Alternatively, one can use $S-S$ bimodules generated by T central vectors in place of unitary representations. While these definitions a-priori depend on the subfactor $N \subseteq M$, Popa showed that in fact these definitions depend only on the standard invariant of the subfactor. If the subfactor comes from a group either through the group diagonal construction or the Bisch-Haagerup construction, Popa ([43], [44]) and Bisch-Popa ([4]), Bisch-Haagerup ([6]) respectively, showed that the subfactor has an analytical property if and only if the group does, ensuring that these are in fact the right definitions for these properties in the subfactor setting.

In a remarkable paper, Popa and Vaes show how to extend these definitions to arbitrary rigid C*-tensor categories without reference to an ambient subfactor [47]. The *fusion algebra* of a category is the complex linear span of isomorphism classes of simple objects, with multiplication given by the fusion rules. Popa and Vaes define a class of *admissible* representations of the fusion algebra, which take the place of unitary representations of groups. The admissible representation theory of the fusion algebra of a category satisfies a number of important properties. First, there exists a universal and a trivial admissible representation. Second, the "point-wise product" of admissible states (after normalization) is again admissible. Thus approximation and rigidity properties have natural definitions in this setting, and many familiar equivalent characterizations of these properties are possible. Most importantly, in the subfactor setting, admissible representations of the fusion algebra are in one-to-one correspondence with S-S bimodules generated by T-central vectors (in fact, this was the motivation for admissible representations). In particular, in the case $\mathscr{C}$ is the category of $M-M$ bimodules for a finite index subfactor $N \subseteq M$, the category has an analytic property if and only if the subfactor standard invariant does, in Popa's sense.

The definition for admissible representation (or admissible state) of the fusion algebra seems at first glance to be a bit mysterious from the purely cateorical perspective. One of the goals of this thesis is to understand the admissible representation theory of the fusion algebra as a piece of the ordinary representation theory of another algebra, Ocneanu's tube algebra.

The *tube algebra* $\mathscr{A}$ is an associative $*$-algebra associated to a rigid C*-tensor category $\mathscr{C}$, introduced by Ocneanu [39]. In the fusion case (finite isomorphism classes of simple objects) this is a finite dimensional semi-simple algebra. This algebra's significance stems from the fact that irreducible representations of this algebra are in 1-1 correspondence with simple objects in the Drinfeld center $Z(\mathscr{C})$ (see [18], [31]). $Z(\mathscr{C})$ is always a modular tensor category, making it of great interest for applications in topological quantum field theory. Understanding the tube algebra provides a concrete (and sometimes practical) approach to finding the combinatorial data for $Z(\mathscr{C})$ from the combinatorial data of $\mathscr{C}$.

One approach to studying tensor categories is the *planar algebra* formalism, introduced by Jones in [22]. A planar algebra packages all the data of a rigid C*-tensor category into another algebraic object, given by vector spaces represented by planar pictures drawn in disks, along with a compatible action of the operad of planar tangles. This approach has been very useful, both technically and conceptually, leading to significant progress in both the classification and construction of new examples, particularly in the subfactor context [25]. Jones introduced the *annular category* of a planar algebra in [23], with the intention of providing obstructions to the existence of planar algebras with certain principal graphs. This has been quite successful and is a fundamental technique in the classification of subfactor planar algebras of small index. A much bigger category, the *affine annular category* of the planar algebra, was introduced and studied in [24]. The affine annular category of a planar algebra is "obtained" by drawing pictures in the interior of annuli rather than disks and applying only local relations. It was shown in [10] that the tensor category of finite dimensional Hilbert space representations of the affine annular category is braided monoidal equivalent to the Drinfeld center of the projection category of the planar algebra. A similar result in the TQFT setting was shown by Walker [49].

It is therefore not surprising that the affine annular category of a planar algebra and the tube algebra of the underlying category have equivalent representation theories, since the category of finite dimensional representations of both algebras are equivalent to the Drinfeld center. In this thesis, we introduce *annular algebras* $\mathscr{A}\Lambda$, with weight set $\Lambda \subseteq [Obj(\mathscr{C})]$, which are mild common generalizations of both the tube algebra and the affine annular category. Choosing $\Lambda := \mathrm{Irr}(\mathscr{C})$ yields the tube algebra of Ocneanu, denoted $\mathscr{A}$, while choosing Λ based on a planar algebra description yields the affine annular category $\mathscr{A}\mathscr{P}$ of Jones. We show that all sufficiently large (full) annular algebras are isomorphic after tensoring with the $*$-algebra of matrix units with countable index set (a strong form of "algebraic Morita equivalence"'). Thus any annular algebras have equivalent representation theories, unifying the two perspectives and providing a means of translating results from planar algebras to the tube algebra in a direct way.

With a unified perspective in hand, we investigate the representation theory of annular algebras associtaed to a rigid C*-tensor category. We show the existence of a universal C*-algebraic completion for annular algebras, whose representations are in 1-1 correspondence with representations of the underlying algebra. For each object $k \in \Lambda$, there is a corner of the annular algebra, denoted $\mathscr{A}\Lambda_{k,k}$, which is a unital $*$-algebra. Denoting the equivalence class of the tensor identity object by 0, then $\mathscr{A}\Lambda_{0,0}$ is canonically $*$-isomorphic to the fusion algebra of $\mathscr{C}$. We show that admissible representations of the fusion algebra in the sense of Popa and Vaes are precisely representations of the fusion algebra which are restrictions of $*$-representations of the tube algebra (or any full annular algebra). This allows us to put context to the admissible representations of [47] in a natural way.

After establishing the initial theory and the connection with Popa-Vaes, we use our perspective to investigate examples. For a discrete group G, we study the category $Vec(G)$ of G-graded vector spaces. The tube algebra here is easy to identify. We then turn our attention to the unshaded Temperley-Lieb-Jones categories

$TLJ(\delta)$ for $\delta \geq 2$, also realized as $Rep_{-q}(SU(2))$ for $q > 0$ where $\delta = q + q^{-1}$. We apply the theory and results developed by Jones and Jones-Reznikoff to identify the universal C*-algebras of the corners $\mathscr{A}_{k,k}$ of the tube algebra. We then turn our attention to the categories $Rep_q(G_2)$, and show that these categories have property (T) for positive $q \neq 1$. Finally, we provide some concluding remarks.

CHAPTER II

RIGID C*-TENSOR CATEGORIES AND ANNULAR ALGEBRAS

Most of the content of this chapter is taken from our joint paper with Shamindra Kumar Ghosh, "Annular representation theory for rigid C*-tensor categories", [14].

II.1 Rigid C*-tensor categories

In this paper we will be concerned with **rigid C*-tensor categories**, which we define as *semi-simple, C*-categories with strict tensor functor, simple unit and duals*. We also assume that $\mathscr{C}$ has at most countably many isomorphism classes of simple objects, which we can interpret as being analogous to "countable discrete groups". We refer the reader to [28] or [33] for a detailed treatment of the general theory, but we will briefly elaborate on the meaning of the terms appearing in our "definition".

A C*-category is a $\mathbb{C}$-linear category $\mathscr{C}$, and each morphism space $Mor(X,Y)$ has the structure of a Banach space satisfying $\|fg\| \leq \|f\|\|g\|$, together with a conjugate-linear, involutive, contravariant functor $* : \mathscr{C} \to \mathscr{C}$ which fixes objects and satisfies the C*-property, $||f^*f|| = ||ff^*|| = ||f||^2$ for all morphisms f. This makes each endomorphism algebra $Mor(X,X)$ into a C*-algebra. We also require that for all $f \in Mor(X,Y)$, f^*f is positive in $Mor(X,X)$ for all objects X,Y. We say the category is *semi-simple* if the category has direct sums, sub-objects, and each $Mor(X,Y)$ is finite dimensional.

A strict tensor functor is a bi-linear functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, which is associative, and has a distinguished unit $id \in Obj(\mathscr{C})$ such that $X \otimes id = X = id \otimes X$. In general, the strictness assumption is too strong, and most tensor categories arising naturally in mathematics do not satisfy this condition, but rather the more complicated pentagon and triangle axioms (see, for example, [33], Chapter 2). However, every tensor category is equivalent in the appropriate sense to a strict one by MacLane's strictness theorem, so it is convenient when studying categories up to equivalence to include this condition.

The category is *rigid*, (or *has duals*) if for each $X \in Obj(\mathscr{C})$, there exists $\overline{X} \in Obj(\mathscr{C})$ and morphisms $R \in Mor(id, \overline{X} \otimes X)$ and $\overline{R} \in Mor(id, X \otimes \overline{X})$ satisfying the so-called conjugate equations:

$$(1_{\overline{X}} \otimes \overline{R}^*)(R \otimes 1_{\overline{X}}) = 1_{\overline{X}} \text{ and } (1_X \otimes R^*)(\overline{R} \otimes 1_X) = 1_X$$

We say two objects X,Y are *isomorphic* if there exists $f \in Mor(X,Y)$ such that $f^*f = 1_X$ and $ff^* = 1_Y$. We call an object X *simple*, or *irreducible* if $Mor(X,X) \cong \mathbb{C}$. We note that for any simple objects X and Y, $Mor(X,Y)$ is either isomorphic to $\mathbb{C}$ or 0. Two simple objects are isomorphic if and only if $Mor(X,Y) \cong \mathbb{C}$. Isomorphism defines an equivalence relation on the collection of all objects and we denote the equivalence class of an object by $[X]$, and the set of isomorphism classes of simple objects $\mathrm{Irr}(\mathscr{C})$.

The semi-simplicity axiom means that our category has direct sums, subobjects, and every object is isomorphic to a direct sum of finitely many simple objects. This implies that for any object X, $Mor(X,X)$ is a finite dimensional C*-algebra over $\mathbb{C}$, hence a multi-matrix algebra. Each summand of the matrix algebra corresponds to an equivalence class of simple objects, and the dimension of the matrix algebra corresponding to a simple object Y is the square of the multiplicity with which Y occurs in X. In general for a simple object Y and any object X, we denote by N_X^Y the natural number describing the multiplicity with which $[Y]$ appears in the simple object decomposition of X. If X is equivalent to a subobject of Y, we write $X \prec Y$. We often write $X \otimes Y$ simply as XY for objects X and Y.

For two simple objects X and Y, we have that $[X \otimes Y] \cong \oplus_Z N_{XY}^Z [Z]$. This means that the tensor product of X and Y decomposes as a direct sum of simple objects of which N_{XY}^Z are equivalent to the simple object Z. The N_{XY}^Z specify the *fusion rules* of the tensor category and are a critical piece of data.

The **fusion algebra** is the complex linear span of isomorphism classes of simple objects $\mathbb{C}[\mathrm{Irr}(\mathscr{C})]$, with multiplication given by linear extension of the fusion rules. This algebra has a $*$-involution defined by $[X]^* = [\overline{X}]$ and extended conjugate-linearly. This algebra is a central object of study in approximation and rigidity theory for rigid C*-tensor categories.

Again, for a more detailed discussion and analysis of the axioms of a rigid C*-tensor category, see the paper of Longo and Roberts [28] and Chapter 2 of the book by Neshveyev and Tuset [33]. For the discussion of C*-tensor categories and their relationship with other notions of duality in tensor categories see the paper of Mueger [31].

In a rigid C*-tensor category, we can define the dimension of an object $d(X) = inf_{(R,\overline{R})} ||R|| ||\overline{R}||$, where the infimum is taken over all solutions to the conjugate equations for an object X. The function $d(\ .\ ) : Obj(\mathscr{C}) \to \mathbb{R}_+$ depends on objects only up to unitary isomorphism. It is multiplicative and additive and satisfies $d(X) = d(\overline{X})$ for any dual of X. We call solutions to the conjugate equations *standard* if $||R|| = ||\overline{R}|| = d(X)^{\frac{1}{2}}$, and such solutions are unique up to a norm 1 scalar. For a given standard solution, we have a trace Tr_X on endomorphism spaces $Mor(X,X)$ given by

$$Tr_X(f) = R^*(1_{\overline{X}} \otimes f)R = \overline{R}^*(f \otimes 1_{\overline{X}})\overline{R} \in Mor(id, id) \cong \mathbb{C}$$

This trace does not depend on the choice of dual for X, or on the choice of standard solutions. We note that $Tr(1_X) = d(X)$. See [28] for details.

We will frequently use the well known *graphical calculus* for tensor categories. See, for example, Section 2.5 of [31] or [51]. We refer the reader to [7] for the closely related planar algebra perspective.

Remark. We note that the assumption that Irr($\mathscr{C}$) is countable is not strictly necessary. In most cases, simple replacing sequences with nets provides a sufficient generalization.

II.2 Annular algebras

The tube algebra $\mathscr{A}$ of a rigid C*-tensor category $\mathscr{C}$ was introduced by Ocneanu in [39] in the subfactor context. This algebra has proved to be useful for computing the Drinfeld center $Z(\mathscr{C})$, since finite dimensional irreducible representations of $\mathscr{A}$ are in one-to-one correspondence with simple objects of $Z(\mathscr{C})$ (see [18], [19]). In general, arbitrary representations of $\mathscr{A}$ are in one-to-one correspondence with objects in Z(ind-$\mathscr{C}$) studied by Neshveyev and Yamashita in [35], an observation due to Stefaan Vaes, with a detailed proof appearing in [46].

The (affine) annular category of a planar algebra was introduced by Jones in [23], [24], with the purpose of providing obstructions to the existence of subfactor planar algebras with certain principal graphs. Since every planar algebra $\mathscr{P}$ with index δ contains the Temperley-Lieb-Jones planar algebra $TLJ(\delta)$, one can decompose $\mathscr{P}$ as a direct sum of irreducible representations of the annular $TLJ(\delta)$ category, and sometimes this information can be deduced purely fom the principle graph of the planar algebra, providing obstructions [23]. The irreducible representations of $TLJ(\delta)$ were completely determined by Jones [23] and Jones-Reznikoff [24], yielding a useful tool for the classification program of subfactors. The affine annular category of a planar algebra was further studied in [10], and they provide useful tools for the analysis of the affine annular category in the infinite depth setting.

Here we introduce a mild generalization of both the algebraic structures described above, which we call an **annular algebra** of the category. It depends on a choice of objects in the category, and is flexible enough to include both Ocneanu's tube algebra and Jones' affine annular categories as special cases. The tube algebra is in some sense a minimal example, while the affine annular category of a planar algebra is particularly suitable in the case when the category arises as the projection category of a planar algebra with a nice skein theoretic presentation. II.2.7 below shows that any two "sufficiently large" annular algebras (a class which include both the above mentioned examples) have equivalent representation theories in a strong sense. This result allows us to translate results of Jones-Reznikoff on the affine annular $TLJ(\delta)$ to the tube algebra setting in the examples chapter.

First we introduce a Hilbert space structure on certain morphism spaces which we will use frequently.

DEFINITION II.2.1. For a simple object α and and arbitrary object β, $Mor(\alpha,\beta)$ has a Hilbert space structure with inner product defined by $\eta^*\xi = \langle \xi, \eta\rangle 1_\alpha$.

Note that this inner product differs from the tracial inner product by a factor of $d(\alpha)$.

For a rigid C*-tensor category $\mathscr{C}$, choose a set of representatives $X_k \in k$ for each $k \in \mathrm{Irr}(\mathscr{C})$. Let $0 \in \mathrm{Irr}(\mathscr{C})$ denote the equivalence class of the tensor unit, and choose X_0 to be the strict tensor unit.

Let $[Obj(\mathscr{C})]$ be the set of equivalence classes of objects in $\mathscr{C}$. Let Λ be a subset of $[Obj(\mathscr{C})]$. For each $i \in \Lambda$, we choose a representative $Y_i \in i$. Then we define the **annular algebra with weight set** Λ as the vector space

$$\mathscr{A}\Lambda := \bigoplus_{i,j\in\Lambda,\ k\in \mathrm{Irr}(\mathscr{C})} Mor(X_k \otimes Y_i, Y_j \otimes X_k)$$

An element $x \in \mathscr{A}\Lambda$ is given by a sequence $x^k_{i,j} \in Mor(X_k \otimes Y_i, Y_j \otimes X_k)$ with only finitely many terms non-zero.

$\mathscr{A}\Lambda$ carries the structure of an associative $*$-algebra, with associative product $\cdot$ and $*$-involution (denoted #) defined as

$$(x\cdot y)^k_{i,j} = \sum_{s\in\Lambda, m,l\in \mathrm{Irr}(\mathscr{C})}\ \sum_{V\in onb(X_k,\ X_m\otimes X_l)} (1_j \otimes V^*)(x^m_{s,j}\otimes 1_l)(1_m \otimes y^l_{i,s})(V\otimes 1_i)$$

$$(x^\#)^k_{i,j} = (\overline{R}^*_k \otimes 1_j \otimes 1_k)(1_k \otimes (x^{\overline{k}}_{j,i})^* \otimes 1_k)(1_k \otimes 1_i \otimes R_k)$$

where $R_k \in Mor(id, \overline{X}_k \otimes X_k)$ and $\overline{R}_k \in Mor(id, X_k \otimes \overline{X}_k)$ are standard solutions to the conjugate equations for X_k. In the first sum, *onb* denotes an orthonormal basis with respect to the inner product from II.2.1, and we may have $onb(X_k,\ X_m \otimes X_l) = \varnothing$ if X_k is not equivalent to a sub-object of $X_m \otimes X_l$. It is clear that the isomorphism class of this algebra does not depend on the choices of representatives X_k. We often write the sequence of morphisms as a sum $x = \sum_{i,j\in\Lambda,\ k\in\mathrm{Irr}(\mathscr{C})} x^k_{i,j}$, where only finitely many terms are non-zero.

We denote the subspaces

$$\mathscr{A}\Lambda^k_{i,j} := Mor(X_k \otimes Y_i, Y_j \otimes X_k) \subset \mathscr{A}\Lambda$$

$$\text{and } \mathscr{A}\Lambda_{i,j} = \bigoplus_{k\in\mathrm{Irr}(\mathscr{C})} \mathscr{A}\Lambda^k_{i,j}.$$

For each $m \in \Lambda$, there is a projection $p_m \in \mathscr{A}\Lambda^0_{m,m}$ given by $p_m := 1_m \in Mor(id \otimes Y_m, Y_m \otimes id) \in \mathscr{A}\Lambda$. In particular $(p_m)^k_{i,j} = \delta_{k,0}\delta_{i,j}\delta_{j,m}1_m$. We see that $\mathscr{A}\Lambda_{i,j} = p_j\mathscr{A}\Lambda p_i$.

These corner algebras $\mathscr{A}\Lambda_{m,m} = p_m\mathscr{A}\Lambda p_m$ are unital $*$-algebras. We call $\mathscr{A}\Lambda_{m,m}$ the **weight m centralizer algebra**. The motivation for the terminology comes from the case when $\mathscr{C}$ is $Vec(G)$ for a discrete group G. In this example $m \in \mathrm{Irr}(\mathscr{C})$ corresponds to an element of the group G, and $\mathscr{A}\Lambda_{m,m}$ is isomorphic to the group algebra of the centralizer subgroup of the element m (see section IV.).

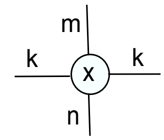
Suppose Λ contains the strict tensor identity, labeled as usual by X_0. Recall the fusion algebra of $\mathscr{C}$ is the complex linear span of isomorphism classes of simple objects $\mathbb{C}[\mathrm{Irr}(\mathscr{C})]$. Multiplication is the linear extension of fusion rules and $*$ is given on basis elements by the duality. From the definition of multiplication in $\mathscr{A}\Lambda$, one easily sees the following:

PROPOSITION II.2.2. *The fusion algebra $\mathbb{C}[Irr(\mathscr{C})]$ is $*$-isomorphic to $\mathscr{A}\Lambda_{0,0}$, via the map* $[X_k] \to 1_k \in (X_k \otimes id, id \otimes X_k) \in \mathscr{A}\Lambda^k_{0,0}$.

Proof. This is a straightforward application of the definitions of the tube algebra operations. □

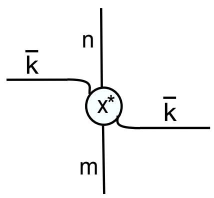
DEFINITION II.2.3. The *annular category with weight set* Λ is the category where Λ is the space of objects, and the morphism space from k to m is given by $\mathscr{A}\Lambda_{k,m} := \bigoplus_{j\in Irr(\mathscr{C})} \mathscr{A}\Lambda^j_{k,m}$. Composition is given by the restriction of annular multiplication.

The annular category and annular algebra basically contain the same information, so one can go between these two perspectives at leisure. We feel the algebra perspective is slightly more convenient for the purpose of representation theory, however, any analysis of the algebra seems to always reduce to studying the centralizer algebras first, so the two points of view are not actually distinct in practice. We remark that this category is *not* a tensor category in general.

We introduce a bit of graphical calculus for annular algebras, extending the well known graphical calculus for tensor categories. For $x \in Mor(X_k \otimes Y_n,\ X_m \otimes X_k)$, we draw the picture 

.

The normal bottom to top axis is given by the slanted arrow pointing from the lower left corner to the top right corner. Conversely, if we see such a picture labeled with an x, then x will represent a morphism in the space obtained by pulling the left side string down to the bottom left and the right side string up to the top right, or in other words, "rotating the diagram" by $\frac{\pi}{4}$. For example, the picture



represents the morphism

$$x^{\#} = (\overline{R}_{\bar{k}}^{*} \otimes 1_n \otimes 1_{\bar{k}})(1_{\bar{k}} \otimes (x)^{*} \otimes 1_{\bar{k}})(1_{\bar{k}} \otimes 1_m \otimes R_{\bar{k}}) \in Mor(X_{\bar{k}} \otimes Y_m, Y_n \otimes X_{\bar{k}}),$$

where $x \in Mor(X_k \otimes Y_n, Y_m \otimes X_k)$ is described above. As we shall see, this graphical calculus will be convenient for writing certain identities and equations that may take a large amount of space to write as compositions and tensor products of morphisms, but consist of a simple picture using this formalism. We remark that diagrams having no side strings can be interpreted as morphisms in the category, and our graphical calculus restricts to the standard graphical calculus for tensor categories.

DEFINITION II.2.4. Define the linear functionals on $\mathscr{A}\Lambda$

1. $\Omega(x) := \sum_{k \in \mathrm{Irr}(\mathscr{C})} Tr_k(x^0_{k,k})$, where Tr_k denotes the canonical (unnormalized) trace on $Mor(Y_k, Y_k)$, and we canonically identify $Mor(id \otimes Y_k, Y_k \otimes id) \cong Mor(Y_k, Y_k)$;

2. $\omega(x) := \sum_{k \in \mathrm{Irr}(\mathscr{C})} tr_k(x^0_{k,k})$, where $tr_k(\ .\ ) := \frac{1}{d(X_k)} Tr_k(\ .\ )$

Positive definiteness of both functionals can be deduced from the positive definiteness of Tr_k in $\mathscr{C}$, or following the same line of arguments used in the proof of [10, Proposition 3.7]. It is easy to see that Ω is a tracial functional on $\mathscr{A}\Lambda$, while ω is not due to the normalization factor. It will be convenient, however, to have both functionals at hand.

DEFINITION II.2.5. The **tube algebra**, $\mathscr{A}$, is the annular algebra with weight set $\mathrm{Irr}(\mathscr{C})$.

The tube algebra is the "smallest" annular algebra that contains all of the information of the annular representation theory of the category as described in the next section, and hence is the best for many purposes. In fact, a sufficiently large arbitrary annular algebra is "Morita equivalent" to the tube algebra in a strong sense. Our notion sufficiently large is given by the following definition:

DEFINITION II.2.6. A weight set $\Lambda \subseteq Obj(\mathscr{C})$ is *full* if every simply object is equivalent to a sub-object of some $X_k,\ k \in \Lambda$.

For a countable set I, let $F(I)$ denote the $*$-algebra spanned by the system of matrix units $\{E_{i,j} \in B(l^2(I)) : i,j \in I\}$ with respect to the orthonormal basis I in $l^2(I)$. Further, for sets I,J, we will denote the span of the system of matrix units $\{E_{i,j} \in B(l^2(I),l^2(J)) : i \in I, j \in J\}$ by $F(I,J)$.

PROPOSITION II.2.7. *If Λ is full, then $F(I)\otimes\mathscr{A} \cong F(I)\otimes\mathscr{A}\Lambda$ as $*$-algebras.*

Proof. We see abstractly that $\mathscr{A}\Lambda^k_{m,n} \cong \bigoplus_{s,t\in\mathrm{Irr}(\mathscr{C})} Mor(X_t,Y_n)\otimes\mathscr{A}^k_{s,t}\otimes\overline{Mor(X_s,Y_m)}$ since an arbitrary element $f \in \mathscr{A}\Lambda^k_{m,n}$ can be decomposed uniquely as:

$$f = \sum_{s,t\in\mathrm{Irr}(\mathscr{C})} \sum_{\substack{V\in onb(X_t,Y_n)\\ W\in onb(X_s,Y_m)}} [(VV^*\otimes 1_k)f(1_k\otimes WW^*)]$$

where $onb(X_s,Y_m)$ is an orthonormal basis for $Mor(X_s,Y_m)$ with respect to the inner product defined in the definition of annular algebras. We see this decomposition does not depend on the choice of such a basis. Thus, the isomorphism implemented by the decomposition is

$$f \mapsto \sum_{s,t\in\mathrm{Irr}(\mathscr{C})} \sum_{\substack{V\in onb(X_t,Y_n)\\ W\in onb(X_s,Y_m)}} V\otimes[(V^*\otimes 1_k)f(1_k\otimes W)]\otimes\overline{W};$$

This map has its inverse defined by taking $*$ in the third tensor component and then composing the morphisms in the obvious way.

If we let $B_{s,m}$ denote an orthonormal basis of $Mor(X_s,Y_m)$ for all $s\in Irr(\mathscr{C})$, $m\in\Lambda$, then we have a vector space isomorphism

$$\mathscr{A}\Lambda^j_{m,n} \cong \bigoplus_{s,t\in Irr(\mathscr{C})} M_{B_{t,n}\times B_{s,m}}(\mathbb{C})\otimes\mathscr{A}^j_{s,t}, \text{ namely } (V\otimes 1_j)\circ h\circ(1_j\otimes W^*) \leftrightarrow E_{V,W}\otimes h.$$

Moreover, multiplication and # on the whole algebra $\mathscr{A}\Lambda$ correspond exactly with those on the matrix and the tube algebra parts.

Next, for $s \in \mathrm{Irr}(\mathscr{C})$, we define the set $I_s := \bigsqcup_{m\in\Lambda} I \times B_{s,m}$. We see that as a $*$-algebra we can identify $F(I)\otimes\mathscr{A}\Lambda \cong \bigoplus_{m,n\in\Lambda}\bigoplus_{s,t\in\mathrm{Irr}(\mathscr{C})} F(I)\otimes M_{B_{t,n}\times B_{s,m}}(\mathbb{C})\otimes\mathscr{A}_{s,t} \cong \bigoplus_{s,t\in\mathrm{Irr}(\mathscr{C})} F(I_t,I_s)\otimes\mathscr{A}_{s,t}$. Since Λ is full, I_t is non-empty, and we can identify it with I for all $t\in\mathrm{Irr}(\mathscr{C})$. Hence, it follows that $F(I)\otimes\mathscr{A}\Lambda \cong F(I)\otimes\mathscr{A}$ as $*$-algebras. □

As we shall see in the next section, this correspondence allows us to pass between representations of $\mathscr{A}\Lambda$ and $\mathscr{A}$ for any full weight set Λ.

Before studying representation theory, we describe another useful way to realize annular algebras as the quotient of a much bigger graded algebra. For any weight set Λ, we define

$$\widetilde{\mathscr{A}\Lambda} := \bigoplus_{\alpha\in Obj(\mathscr{C}),\ i,j\in\Lambda} Mor(\alpha\otimes Y_i,\ Y_j\otimes\alpha)$$

Notice that the direct sum is taken over Λ and *all* objects in contrast with the definition for annular algebras. As with annular algebras, $x\in\widetilde{\mathscr{A}\Lambda}$ is described by a sequence $(x^{\alpha}_{i,j})$ where $\alpha\in Obj(\mathscr{C})$ and $i,j\in\Lambda$ with only finitely many non-zero term. $\widetilde{\mathscr{A}\Lambda}$ becomes an associative algebra with multiplication defined by:

$$(x\cdot y)^{\alpha}_{i,j} = \sum_{s\in\Lambda}\ \sum_{\beta,\gamma\in Obj(\mathscr{C}):\ \alpha=\beta\otimes\gamma)} (x^{\beta}_{s,j}\otimes 1_{\gamma})(1_{\beta}\otimes y^{\gamma}_{i,s}).$$

Note that associativity follows from strictness of our category. For a $*$-structure, we need duals and standard solutions to the conjugate equations for every $\alpha\in Obj(\mathscr{C})$ which are chosen once and for all in a consistent way. A convenient notion for this purpose is a *spherical structure* in the sense of [31, Definition 2.6]. Such a choice for any rigid C*-tensor category $\mathscr{C}$ is always possible by a result of Yamagami (see [50]). Thus we assume that we have chosen a spherical structure, which in particular picks a dual object (along with a standard solution to the conjugate equations) for each object in such a way that $\overline{\overline{\alpha}}=\alpha$. Since $\widetilde{\mathscr{A}\Lambda}$ is built out of morphism spaces which already have a $*$, we will denote the $*$-structure here by # as in the annular algebra case, which is defined as:

$$(x^{\#})^{\alpha}_{i,j} = (\overline{R}^{*}_{\alpha}\otimes 1_j\otimes 1_{\alpha})(1_{\alpha}\otimes(x^{\overline{\alpha}}_{j,i})^{*}\otimes 1_{\alpha})(1_{\alpha}\otimes 1_i\otimes R_{\alpha})$$

It is easy to check that # is a conjugate-linear, anti-isomorphic involution (by the definition of spherical structure).

We define the family of maps $\Psi^{\alpha} : Mor(\alpha \otimes Y_i, Y_j \otimes \alpha) \to \mathscr{A}\Lambda$ given by

$$\Psi^{\alpha}(f) = \sum_{k \prec \alpha} \sum_{V \in onb(k,\alpha)} (1_j \otimes V^*) f (V \otimes 1_i).$$

Then the family of Ψ^{α} extends linearly to a surjective map $\Psi : \widetilde{\mathscr{A}\Lambda} \to \mathscr{A}\Lambda$. It is also easy to see that Ψ is a $*$-homomorphism. Using basic linear algebra, one can see that $Ker(\Psi)$ is spanned by (not necessarily homogeneous) vectors of the form $f(s \otimes 1_i) - (1_j \otimes s) f \in \widetilde{\mathscr{A}\Lambda}$ for $f \in Mor(\alpha \otimes Y_i, Y_j \otimes \beta)$, and $s \in Mor(\beta, \alpha)$.

We remark that the graphical calculus for annular algebras makes perfect sense in this setting, we simply allow side strings to be labeled by arbitrary objects. In fact, we can now give a heuristic explanation for the words tube and annular associated to these algebras.

Take a diagram with top bottom and side strings as in our graphical calculus convention, and attach the bottom string to the inner disk of an annulus and the top strings to the boundary of the outer disk. Then attach the side strings to each other around the "bottom" of the inner disk. We allow isotopies in the interior of the annulus, so that the following pictures are equal:

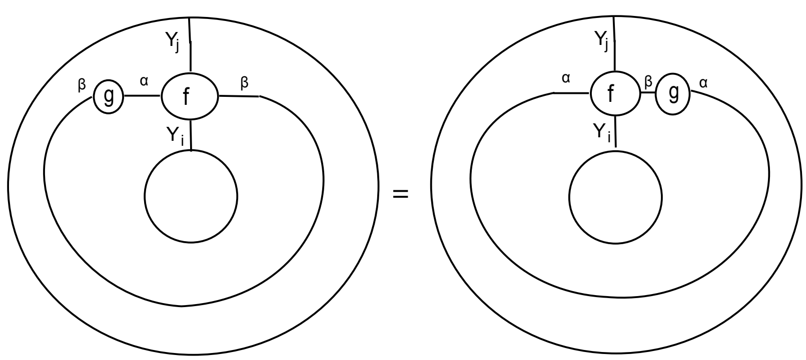


Figure II.1: Pictures drawn in the annulus

This picture explains the kernel of the map Ψ. Cutting the bottom string and returning to a rectangular picture, the difference of the resulting homs spans $Ker(\Psi)$. We also remark that composing such pictures and decomposing the identity on the side strings yields the multiplication structure we defined for annular algebras.

Such pictures can be formalized in the setting of Jones' *planar algebras*. The result is Jones' affine annular category of a planar algebra. If $\mathscr{P}$ is an (unshaded) planar algebra the affine annular category $\mathscr{A}\mathscr{P}$ is the category with objects given by $\mathbb{N}$, and morphisms all annular tangles labeled by $\mathscr{P}$ subject to local relations. For proper definitions, see [24], [23] and [10]. Composition of morphisms is given by composing annuli. This category can be made into an algebra in the obvious way, which we also call $\mathscr{A}\mathscr{P}$. If we let $\mathscr{C} := Proj(\mathscr{P})$ be the projection category of a planar algebra, choose the objects $\Lambda := \{1_k \in P_{k,k}\}_{k\in\mathbb{N}} \subseteq Obj(\mathscr{C})$. Then it follows from [10] that $\mathscr{A}\Lambda \cong \mathscr{A}\mathscr{P}$. We will see an example of this correspondence in IV in our analysis of $TLJ(\delta)$ categories. We refer the reader to [22] and [7] for the definitions of planar algebras and the second reference for the projection category of a planar algebra.

CHAPTER III

REPRESENTATION THEORY OF ANNULAR ALGEBRAS AND ANALYTIC PROPERTIES

In this chapter, we will develop the representation theory of the tube algebra, and construct a universal C*-algebra $C^*(\mathscr{A})$, analogous to group C*-algebras. We then show the equivalence of the "weight 0 piece" of this representation theory with the admissible representation theory of the fusion algebra introduced by Popa and Vaes [47]. This allows the approximation and rigidity properties introduced by Popa and Vaes to be reinterpreted in the annular context.

III.1 Representations of annular algebras

The representation category $Rep(\mathscr{A}\Lambda)$ is simply the category of (non-degenerate) star representations of $\mathscr{A}\Lambda$ as bounded operators on a Hilbert space. We begin this section by showing that for a full weight set, $Rep(\mathscr{A}\Lambda)$ is equivalent to $Rep(\mathscr{A})$, removing the ambiguity of choosing a weight set in our discussions of representation theory. The resulting representation category has interesting and useful applications. It comes equipped with a tensor functor making it into a braided monoidal category. It was shown in [10] that the category of finite dimensional representations is (contravariantly) monoidally equivalent to the Drinfeld center, $Z(\mathscr{C})$. In the case where $\mathrm{Irr}(\mathscr{C})$ is finite, the tube algebra $\mathscr{A}$ is finite dimensional. Thus understanding its representation theory becomes a computable way of determining the categorical data of the Drinfeld center, and as far as we know is the most commonly used method for understanding $Z(\mathscr{C})$ (see [18], [19]).

It is shown in [46] that $Rep(\mathscr{A})$ (forgetting the tensor structure) is equivalent to the category $Z(\text{ind-}\mathscr{C})$ introduced and studied by Neshveyev and Yamashita in [35]. The ind-category is basically the "direct sum completion" of $\mathscr{C}$, defined by allowing arbitrary direct sums in $\mathscr{C}$. It is still a tensor category (though no longer rigid), hence one can apply the usual definitions to obtain a Drinfeld center. Actually it is easy to see from the definitions in [10] that they are braided monoidal equivalent.

We will show that another application of the representation theory is to provide natural definitions for approximation and rigidity properties such as amenability, the Haagerup property, and property (T) for rigid C*-tensor categories. One simply generalizes the corresponding definitions for groups given in terms of representation theory, using the trivial representation of $\mathscr{A}$ (see Definition III.1.12) in place of the trivial representation for groups.

The main technical difficulty we have to face is a universal bound on the norm of $\mathscr{A}$ for non-degenerate $*$-representations. We will see that data from the category provides us with a satisfactory universal bound. With this in hand, we can take arbitrary direct sums of representations, and construct a universal C*-completion of $\mathscr{A}$. We begin with the formal definitions and immediate consequences.

DEFINITION III.1.1. A *non-degenerate representation* of an annular algebra $\mathscr{A}\Lambda$ is a star homomorphism $\pi : \mathscr{A}\Lambda \to B(H))$ for some Hilbert space H with the property that $\pi(\mathscr{A}\Lambda)\xi = 0$ for $\xi \in H$ implies $\xi = 0$. We denote the category of non-degenerate representations with bounded intertwiners $Rep(\mathscr{A}\Lambda)$

The non-degeneracy condition is minor. An arbitrary $*$ representation decomposes as a direct sum of a non-degenerate subspace and a degenerate space, so we can restrict our attention to the non-degenerate piece. For a non-degenerate representation (π, H) and for $k \in \Lambda$, we define $H_k := \pi(p_k)H \leq H$, where p_k is the identity projection in $\mathscr{A}\Lambda_{k,k}$ described above. We easily see that $H \cong \oplus_{k\in\Lambda} H_k$. In this way, π defines maps $\pi : \mathscr{A}\Lambda_{k,m} \to B(H_k, H_m)$. Conversely, if we have a sequence of Hilbert spaces $\{H_k\}_{k\in\Lambda}$ and a family of maps $\pi_{k,m} : \mathscr{A}\Lambda_{k,m} \to B(H_k, H_m)$ compatible with multiplication and the $*$-structure on $\mathscr{A}\Lambda$, we can define a unique representation $\pi : \mathscr{A}\Lambda \to B(H)$ where $H := \oplus_{k\in\Lambda} H_k$. It is often convenient to pass between these two pictures.

All representations we consider in this paper are non-degenerate.

THEOREM III.1.2. *If* Λ *is full, then* $Rep(\mathscr{A}\Lambda) \cong Rep(\mathscr{A})$ *as additive categories.*

Proof. if (π, H) is a representation of $\mathscr{A}\Lambda$, then $(1 \otimes \pi, \ell^2(I) \otimes H)$ provides a representation of $F(I) \otimes \mathscr{A}\Lambda$ for any countable set I (see discussion preceding Proposition II.2.7). By II.2.7, this provide a representation of $F(I) \otimes \mathscr{A}$, and cutting down by the projection $E_{i,i}$ for any index set i, yields a representation of $\mathscr{A}$. It is easy to see that this yields an additive functor $F : Rep(\mathscr{A}\Lambda) \to Rep(\mathscr{A})$, with the obvious inverse. □

At this point, since we are mostly interested in representation theory, one might wonder why we bother considering annular algebras with arbitrary weight sets. Our motivation for doing so is that many categories have a nice description with respect to some particular weight set. For example, the planar algebras of Jones come equipped with a weight set indexed by the natural numbers and given by the number of strings on boundary components as discussed in the previous section. The resulting annular algebra is called the *affine annular category* of the planar algebra, which is typically viewed as an annular category instead of an algebra [23], [24], [10]. With this weight set, the structure of the annular algebra may become transparent via skein theory, and often has a simple description in terms of planar diagrams. This is clearly illustrated in the $TLJ(\delta)$ categories which we discuss in Chapter IV. For these categories, the tube algebra at first glance may seem daunting, but applying Theorem III.1.2, we can transport the classification of irreducible affine annular representations by Jones and Reznikoff (see [24]) from the planar algebra setting to the tube algebra setting. This allows us to analyze the tube algebras for these categories, which appears to be quite difficult without these techniques.

In light of the above theorem, however, we lose little generality by focusing our attention on the tube algebra $\mathscr{A}$. All of the following results and proofs will be made for $\mathscr{A}$, but can easily be translated to the more general setting of $\mathscr{A}\Lambda$ where Λ is full. The remainder of this section will focus on the demonstrating the existence of a universal C*-algebra, denoted $C^*(\mathscr{A})$, which encodes the representation theory of $\mathscr{A}$. This universal C*-algebra is directly analogous to and generalizes in some sense the universal C*-algebra for groups. In studying the algebra for groups, the notion of a positive definite function on the group is quite handy, and here we introduce a similar notion. As we will see in the next section, the true analogy with groups is not with $\mathscr{A}$ itself, but with the centralizer algebras $\mathscr{A}_{k,k}$. The corners are unital $*$-algebras with unit p_k, and hence have a positive cone. One of the key points is that to encode the representation theory of the whole tube algebra requires us to extend this positive cone to include positive elements coming from "outside" $\mathscr{A}_{k,k}$ itself. In particular, we want elements of the form $f^\# \cdot f$ with $f \in \mathscr{A}_{k,m}$ for arbitrary m to be considered positive. Thus any"local" notion of positive definite functions for the centralizer algebras needs to capture this kind of positivity.

DEFINITION III.1.3. For $k \in \mathrm{Irr}(\mathscr{C})$, a linear functional $\phi : \mathscr{A}_{k,k} \to \mathbb{C}$ is called a *weight k annular state* if

1. $\phi(p_k) = 1$.

2. $\phi(f^\# \cdot f) \geq 0$ for all $f \in \mathscr{A}_{k,m}$ and $m \in \mathrm{Irr}(\mathscr{C})$.

We denote the collection of weight k annular states Φ_k (for general Λ, we denote this set by $\Phi\Lambda_k$)

The goal now is to prove a *GNS* type theorem, which takes a weight k annular state and produces a unique "k-cyclic" representation of the whole tube algebra. If $(\pi, H) \in Rep(\mathscr{A})$ and $\xi \in \pi(p_k)H$ is a unit vector, then the functional $\langle \pi(\ .\ )\xi, \xi \rangle$ restricted to $\mathscr{A}_{k,k}$ is a weight k-annular state. We will show all weight

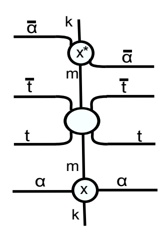
k annular states are of this form. The positivity condition in the definition assures that when constructing a Hilbert space, the natural inner product will be positive semidefinite. The only difficulty generalizing the usual *GNS* construction is that $\mathscr{A}$ does not already have a natural norm structure, so we cannot use positivity to assert boundedness of the tube algebra action as in the usual C*-algebra GNS construction. Our situation is analogous to groups, but even there, group elements must have norm 1, so the action of an arbitrary element in the group algebra is bounded in the L^1 norm.

The trick will be to take an annular state and reduce boundedness of the tube algebra action to the situation of a positive linear functional on a finite dimensional C*-algebra. Recall the functional $\omega : \mathscr{A} \to \mathbb{C}$ defined right before Definition II.2.5.

LEMMA III.1.4. *Let $y \in \mathscr{A}^t_{m,n}$ for $t \in Irr(\mathscr{C})$. Then, $\phi(x^\# \cdot y^\# \cdot y \cdot x) \leq d(X_t)^2 \omega(y \cdot y^\#)\phi(x^\# \cdot x)$ for all $\phi \in \Phi_k$ and $x \in \mathscr{A}_{k,m}$.*

Proof. Let $x = \sum_{j \in \mathrm{Irr}(\mathscr{C})} x_j \in \mathscr{A}_{k,m}$ where this sum is finite and each $x_j \in \mathscr{A}^j_{k,m}$. Then define the object $\alpha := \oplus X_j$, where the j here are the same j in the description of x. Then viewing $x \in Mor(\alpha \otimes X_k,\ X_m \otimes \alpha)$, we have $\Psi^\alpha(x) = x$. Notice that since each X_j has a chosen dual (the object chosen to represent the equivalence class of $\overline{X}_j$), this distinguishes a conjugate object $\overline{\alpha}$. Let $\phi \in \Phi_k$. We define a linear functional $\tilde{\phi}_x$ on the finite dimensional C*-algebra $End(X_t \otimes X_m \otimes \overline{X}_t)$ by

$$\tilde{\phi}_x(\ .\ ) := \phi \circ \Psi^{\bar{t}\bar{\alpha} \otimes t\alpha} \left(\quad \right).$$



To evaluate $\tilde{\phi}_x(\ .\ )$ on a morphism $f \in End(X_t \otimes X_m \otimes \overline{X}_t)$, we insert f into the unlabeled disc in the above diagram and evaluate. We claim that $\tilde{\phi}_x$ is a positive linear functional on the finite dimensional C*-algebra $End(X_t \otimes X_m \otimes \overline{X}_t)$. For positive w in this algebra, we see that

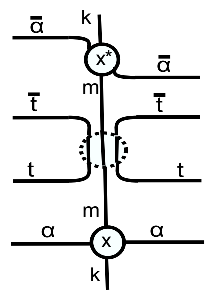
$$\tilde{\phi}_x(w) = \sum_{j \in \mathrm{Irr}(\mathscr{C})} \sum_{V \in onb(j, tm\bar{t})} \tilde{\phi}_x(w^{\frac{1}{2}} V V^* w^{\frac{1}{2}})$$

$$= \sum_{j \in \mathrm{Irr}(\mathscr{C})} \sum_{V \in onb(j, tm\bar{t})} \phi\left(\left[\Psi^t\left((V^* w^{\frac{1}{2}} \otimes 1_t)(1_t \otimes 1_m \otimes R_t)\right)\Psi^\alpha(x)\right]^{\#}\right.$$

$$\left. \cdot \left[\Psi^t\left((V^* w^{\frac{1}{2}} \otimes 1_t)(1_t \otimes 1_m \otimes R_t)\right)\Psi^\alpha(x)\right]\right), \quad \text{(III.1)}$$

which is non-negative by definition of annular state. Then by positivity of $\tilde{\phi}_x$,

$$\tilde{\phi}_x(w) \leq \|w\|\tilde{\phi}_x(1_{tm\bar{t}}) = \|w\|\phi \circ \Psi^{\bar{t}\bar{\alpha}\otimes t\alpha}\left(\quad \right) = \|w\| d(X_t)\phi(x^{\#} \cdot x).$$



In the last equality we use the "annular relation" describing the kernel of Ψ to pull the side t-cap from the left around to the right, yielding a closed t-circle hence a factor of $d(X_t)$. Now for $y \in (X_t \otimes X_m, X_n \otimes X_t)$, consider the morphism $\tilde{y} := (1_n \otimes \overline{R}_t^*)(y \otimes 1_{\bar{t}}) \in Mor(X_t \otimes X_m \otimes \overline{X}_t,\ X_n)$. Then $\tilde{y}^*\tilde{y} \in End(X_t \otimes X_m \otimes \overline{X}_t)$, and we see that

$$\phi(x^{\#} \cdot y^{\#} \cdot y \cdot x) = \tilde{\phi}_x(\tilde{y}^*\tilde{y}) \leq d(X_t)\|\tilde{y}^*\tilde{y}\|\phi(x^{\#} \cdot x) = d(X_t)\|\tilde{y}\tilde{y}^*\|\phi(x^{\#} \cdot x) = d(X_t)^2\omega(y \cdot y^{\#})\phi(x^{\#} \cdot x)$$

For the last inequality, note that $\tilde{y}\tilde{y}^*$ is a scalar times 1_n (X_n being simple), so to find that scalar we apply the categorical trace and compare with $\omega(y \cdot y^{\#})$, yielding the required result (we recommend the reader draw a picture here). □

We note the proof of this lemma has obvious modifications for general annular algebras associated to full weight sets.

Now, if $\phi \in \Phi_k$, we define a sesquilinear form on the vector space $\hat{H}_\phi := \bigoplus_{m \in \mathrm{Irr}(\mathscr{C})} \mathscr{A}_{k,m}$ by $\langle x, y\rangle_\phi := \phi(y^{\#} \cdot x)$. By definition this form is positive semi-definite. Furthermore, this vector space has a natural action of $\mathscr{A}$ by left multiplication. We construct a Hilbert space by taking the quotient by the kernel of this form and completing, which we denote H_ϕ. Recall an arbitrary $y \in \mathscr{A}$ can be written $y = \sum_{m,n,j \in \mathrm{Irr}(\mathscr{C})} y^j_{m,n}$ where this sum is finite and each $y^j_{m,n} \in \mathscr{A}^j_{m,n}$. By the previous lemma, each $y^j_{m,n}$ preserves the kernel of the form and is bounded, therefore we have $\pi_\phi(y^j_{m,n}) \in B(H_\phi)$. Extending linearly, $\pi_\phi : \mathscr{A} \to B(H)$ is a (non-degenerate) $*$-representation of the tube algebra.

COROLLARY III.1.5. *A functional $\phi : \mathscr{A}_{k,k} \to \mathbb{C}$ is in Φ_k if and only if there exists a non-degenerate $*$-representation (π, H) of $\mathscr{A}$, and a unit vector in $\xi \in \pi(p_k)H$, such that $\phi(x) = \langle \pi(x)\xi, \xi \rangle$. Furthermore the sub-representation on $H_\xi := [\pi(\mathscr{A})\xi] \subseteq H$ is unitarily equivalent to the representation H_ϕ described above.*

Continuing the analogy with groups, we notice that Lemma III.1.4 provides us with a bound similar to the L^1-norm for groups. Since an arbitrary element in the tube algebra will have its norm bounded by the constant in Lemma III.1.4 in any representation, we can take arbitrary direct sums of representations. This allows us to define a universal representation, and a corresponding universal C*-algebra.

DEFINITION III.1.6.

1. The *universal representation* of the tube algebra is given by $(\pi_u, H_u) := \bigoplus_{k \in \mathrm{Irr}(\mathscr{C}),\ \phi \in \Phi_k} (\pi_\phi, H_\phi)$.
2. The *universal norm* on $\mathscr{A}$ is given by $\|x\|_u := \|\pi_u(x)\|$.
3. The *universal C*-algebra* is the completion $C^*(\mathscr{A}) := \overline{\pi_u(\mathscr{A})}^{\|\|_u}$.

Note that non-degenerate $*$-representations of $\mathscr{A}$ are in 1-1 correspondence with non-degenerate, bounded $*$-representations of $C^*(\mathscr{A})$. Note that the universal norm is finite (so that such an infinite direct sum exists), follows from Lemma III.1.4. We record the consequences of Lemma III.1.4 for the universal norm in the following corollary:

COROLLARY III.1.7. *Let $\sum_{j,k,m \in Irr(\mathscr{C})} x^j_{k,m} \in \mathscr{A}$. Then $0 < ||x||_u \leq \sum_{j,m,n \in Irr(\mathscr{C})} d(X_j)\omega(x^j_{m,n} \cdot (x^j_{m,n})^\#)^{\frac{1}{2}}$.*

Proof. The bound on the right follows from Lemma III.1.4. The strict positivity of the universal norm follows from the fact that ω is a positive definite functional on $\mathscr{A}$ and $\omega|_{\mathscr{A}_{k,k}}$ is a weight k annular state for all $k \in \mathrm{Irr}(\mathscr{C})$. □

We now turn our attention back to the centralizer algebras $\mathscr{A}_{k,k}$. We want to study the representation theory of these unital $*$-algebras, under the restriction that the representations must "come from" a tube algebra representation. The reason for studying these representations is that while we are interested in the whole algebra $\mathscr{A}$ and its representation theory, often we are able to understand the centralizer algebras and their admissible representations with much greater ease. The following proposition is an easy corollary of the GNS construction:

COROLLARY III.1.8. *Let $k \in Irr(\mathscr{C})$, and let (π_k, H_k) be a non-degenerate $*$-representation of $\mathscr{A}_{k,k}$. The following are equivalent:*

1. *Every vector state in (π_k, H_k) is weight k annular state.*
2. $||\pi_k(x)|| \leq ||x||_u$ *for all $x \in \mathscr{A}_{k,k}$.*
3. *(π_k, H_k) extends to a continuous representation of the unital C*-algebra $p_k C^*(\mathscr{A}) p_k$.*
4. *There exists a representation (π, H) of $\mathscr{A}$ such that $(\pi, H)|_{\mathscr{A}_{k,k}}$ is unitarily equivalent (π_k, H_k).*

Proof. (1) implies (2) implies (3) follows from the above discussion. For (3) implies (4), we construct the representation (π, H) in a manner analogous to the *GNS* construction. We see that $p_m \mathscr{A}_{k,m} p_k$ provides a Hilbert C*-bimodule for the corner algebras $p_m C^*(\mathscr{A}) p_m$ and $p_k C^*(\mathscr{A}) p_k$ for all $m \in \mathrm{Irr}(\mathscr{C})$ with the obvious left and right inner products. By standard Hilbert C*-bimodule theory, we have an induced representation (π_m, H_m) of $p_m C^*(\mathscr{A}) p_m$, where H_m is the Hilbert space completion of $p_m \mathscr{A} p_k \otimes H_k$ with respect to the induced inner product $\langle f \otimes \xi, g \otimes \eta \rangle_m := \langle \pi_k(g^\# \cdot f)\xi, \eta \rangle_k$. By bimodule theory, $H := \bigoplus_{m \in \mathrm{Irr}(\mathscr{C})} H_m$ carries a $*$-representation, π, of $\mathscr{A}$. (4) implies (1) follows from the *GNS* reconstruction result.

□

DEFINITION III.1.9. A representation satisfying the equivalent conditions of the previous corollary is called a *weight k admissible representation*.

Admissible representations can be seen simply as representations of the centralizer algebras which are restrictions of representations of the whole tube algebra. Alternatively, they are representations of the corner algebras which induce representations of the whole tube algebra. Thus, understanding admissible representations for all weights allows us to understand representations of the whole tube algebra. Since the norm in weight k admissible representations is bounded by the universal norm for $\mathscr{A}_{k,k}$, one can construct a universal C*-algebra completion $C^*(\mathscr{A}_{k,k})$. From the above proposition, it is clear that $C^*(\mathscr{A}_{k,k}) \cong p_k C^*(\mathscr{A}) p_k$.

We remark that Proposition II.2.7 implies $C^*(\mathscr{A}) \otimes K \cong C^*(\mathscr{A}\Lambda) \otimes K$ where K is the C*-algebra of compact operators on a separable Hilbert space.

We end this section with two canonical examples of a non-degenerate $*$-representation of $\mathscr{A}$ that always exists for all categories. The first, the so-called left regular representation, is analogous to the left regular representation for groups (though not strictly analogous as we shall see!). The second, the so-called "trivial representation", is rather non-trivial, but serves a similar role to the trivial representation in group theory for approximation and rigidity properties.

DEFINITION III.1.10. The *left regular representation* has Hilbert space $L^2(\mathscr{A},\omega)$, and action π_ω given by left annular multiplication.

That the action here is bounded follows from the fact that $\omega|_{\mathscr{A}_{k,k}}$ is an annular weight k state, hence every vector state in $\pi_\omega(p_k)L^2(\mathscr{A},\omega)$ is in Φ_k. Applying Lemma III.1.4 yields the boundedness.

Recall in the previous section that we had a canonical isomorphism $\mathbb{C}[\mathrm{Irr}(\mathscr{C})] \cong \mathscr{A}_{0,0}$.

LEMMA III.1.11. *The one dimensional representation of $\mathscr{A}_{0,0}$ defined by the character $1_{\mathscr{C}}([X]) = d(X)$ for all $X \in Irr(\mathscr{C})$, is a weight 0 annular state.*

Proof. Let δ_α denote the map canonically identifying $Mor(\alpha\otimes id,\ id\otimes\alpha)$ with $Mor(\alpha,\alpha)$ for all objects α for all objects α. Since $\mathscr{A}_{0,0}^k := Mor(X_k\otimes id,\ id\otimes X_k)$, we have a map $\delta := \bigoplus_{k\in\mathrm{Irr}(\mathscr{C})} \delta_k : \mathscr{A}_{0,0} \to \bigoplus_{k\in\mathrm{Irr}(\mathscr{C})} Mor(X_k,X_k)$. Now we can see $1_{\mathscr{C}}(x) = Tr(\delta(x))$, where $Tr := \oplus_{k\in\mathrm{Irr}(\mathscr{C})} Tr_k$. Furthermore, one can check that for $x \in Mor(\alpha\otimes id,\ id\otimes\alpha)$, $1_{\mathscr{C}}(\Psi^\alpha(x)) = Tr_\alpha(\delta_\alpha(x))$.

For $x = \sum_{j\in\mathrm{Irr}(\mathscr{C})} x_{0,m}^j \in \mathscr{A}_{0,m}$, setting $\alpha := \oplus X_j$ where the j appear in the sum for x, we have $1_{\mathscr{C}}(x^\# \cdot x) = 1_{\mathscr{C}}(\Psi^{\overline{\alpha}\alpha}(x^\#\cdot x)) = Tr(\delta_{\overline{\alpha}\alpha}(x^\#\cdot x)) = 0$ for all $m \neq 0$ in $\mathrm{Irr}(\mathscr{C})$ by sphericality of the trace, since $Mor(id,\ X_m) = \{0\}$ for $m\neq 0$. Therefore it suffices to check $1_{\mathscr{C}}(x^\#\cdot x) \geq 0$ for $x \in \mathscr{A}_{0,0}$, which follows since $1_{\mathscr{C}}$ is a $*$-homomorphism.

□

We note that for $k \in \mathrm{Irr}(\mathscr{C})$, $k \neq 0$, $\pi_{1_{\mathscr{C}}}(p_k) = 0$. Thus all "higher weight" spaces in the trivial representation are 0, so that in fact $1_{\mathscr{C}}$ is a character on $\mathscr{A}$

DEFINITION III.1.12. The *trivial representation* of $\mathscr{A}$ is the one dimensional representation $1_{\mathscr{C}}$ of $\mathscr{A}$.

The trivial representation will play a similar role in our representation theory to the trivial representation in the theory of groups.

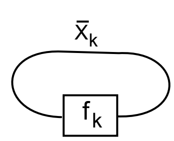
III.2 Analytic properties

In a remarkable paper [47], Popa and Vaes introduced a representation theory for rigid C*-tensor categories. They introduce the concept of cp-multipliers for $\mathscr{C}$, which are a class of functions in $\ell^\infty(\mathrm{Irr}(\mathscr{C}))$. Normalizing these functions provide positive linear functionals on the fusion algebra $\mathbb{C}[\mathrm{Irr}(\mathscr{C})]$. An *admissible representation* of $\mathbb{C}[\mathrm{Irr}(\mathscr{C})]$ is a $*$-representation such that every vector state is a certain normalization of a cp-multiplier (see Definition III.2.4 and the following discussion). The class of admissible representations of the fusion algebra provides a good notion for the representation theory for $\mathscr{C}$, generalizing unitary representations of a discrete group if $\mathscr{C}$ is equivalent to $Vec(G)$. In this context, they define approximation and rigidity properties, generalizing the definitions from the world of discrete groups. They show that if $\mathscr{C}$ is equivalent to the category of M-M bimodules in the standard invariant of a finite index inclusion $N \subseteq M$ of II_1 factors, then the definitions of approximation and rigidity properties given via cp-multipliers are equivalent to the definitions for the standard invariant of the subfactor defined via the symmetric enveloping algebra for the subfactor $N \subseteq M$ given by Popa.

We will show in this section that admissible representations of the fusion algebra in the sense of Popa and Vaes exactly coincide with weight 0 admissible representations of $\mathscr{A}$. Thus the admissible representation theory of Popa and Vaes is the restriction of ordinary representation theory of the tube algebra. In a recent paper of Neshveyev and Yamashita [35], given an object of $Z(\text{ind-}\mathscr{C})$ they construct a representation of the fusion algebra. They then show that the class of representations of the fusion algebra that arises in this way is exactly the class identified by Popa and Vaes. Thus the equivalence of $Z(\text{ind-}\mathscr{C})$ and $Rep(\mathscr{A})$ observed by Vaes following the release of our paper [14] (written in [46]) provides an alternate, though indirect, proof of this result.

We now assume that Λ contains the strict tensor unit indexed by 0, so that $X_0 = id$. From Proposition II.2.2 we see that $\mathscr{A}\Lambda_{0,0}$ is $*$-isomorphic to the fusion algebra of $\mathscr{C}$. If ϕ is a function on $\mathrm{Irr}(\mathscr{C})$, it defines a functional on $\mathbb{C}[\mathrm{Irr}(\mathscr{C})]$ by sending $f = \sum_k f_k \in \mathscr{A}\Lambda_{0,0}$ (where $f_k \in \mathscr{A}\Lambda^k_{0,0}$) to

$$\phi(f) = \sum_k \frac{\phi(X_k)}{d(X_k)}$$



This is because f_k is really a scalar times the single string labeled X_k. Now since any annular algebra has $\mathscr{A}\Lambda_{0,0} \cong \mathbb{C}[\mathrm{Irr}(\mathscr{C})]$, we can naturally identify the algebraic duals $\widehat{(\mathscr{A}_{0,0})}$ and $\widehat{(\mathscr{A}\Lambda_{0,0})}$, both as functions $\phi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$. Recall from Definition III.1.3 that Φ_k denotes the set of weight k annular states on $\mathscr{A}_{k,k}$, while $\Phi\Lambda_k$ denotes the weight k annular states on $\mathscr{A}\Lambda_{k,k}$. We have the following lemma:

LEMMA III.2.1. *If $\phi : Irr(\mathscr{C}) \to \mathbb{C}$, then for any full Λ, $\phi \in \Phi\Lambda_0$ if and only if $\phi \in \Phi_0$.*

Proof. Since we can embed $\mathscr{A}$ as a sub-algebra of $\mathscr{A}\Lambda$ as in the proof of Proposition II.2.7, it is clear that $\phi \in \Phi\Lambda_0 \Rightarrow \phi \in \Phi_0$. For the converse, suppose $\phi \in \Phi_0$. Let $f = \sum_k f^k_{0,m} \in \mathscr{A}\Lambda_{0,m}$, where $f^k_{0,m} \in \mathscr{A}\Lambda^k_{0,m}$. Then we have

$$\phi(f^\# \cdot f) = \sum_{j \in \mathrm{Irr}(\mathscr{C})} \sum_{V \in onb(m,X_j)} \phi\big(\big(f^\#(1 \otimes V^*)\big) \cdot ((V \otimes 1)f)\big)$$

But each $(V \otimes 1)f \in \mathscr{A}_{0,j}$, and thus each term in the right hand sum is positive. Therefore $\phi(f^\# \cdot f) \geq 0$ for all $m \in \Lambda$.

□

LEMMA III.2.2. *If $\phi : Irr(\mathscr{C}) \to \mathbb{C}$, define $\phi^{op} : Irr(\mathscr{C}) \to \mathbb{C}$ by $\phi^{op}(X_k) = \phi(\overline{X}_k)$. Then ϕ is an annular state if and only if ϕ^{op} is an annular state.*

Proof. We only need to check the positivity condition. Suppose $\phi \in \mathscr{A}_{0,0}$. Define the map $r : \mathscr{A} \to \mathscr{A}$, given for $f \in \mathscr{A}^k_{i,j} := Mor(X_k \otimes X_i, X_j \otimes X_k$ by $r(f) = \overline{f} \in \mathscr{A}^{\bar{k}}_{\bar{j},\bar{i}}$. r is an anti-isomorphism with respect to annular multiplication. Then if $f \in \mathscr{A}\Lambda_{0,m}$, $\phi^{op}(f^\# \cdot f) = \phi(r(f^\# \cdot f)) = \phi(r(f) \cdot r(f)^\#)$. □

Now we recall several definitions from [47]. Let $\mathscr{C}$ be a rigid C*-tensor category and let $Irr(\mathscr{C})$ be the set of simple objects.

DEFINITION III.2.3. A *multiplier* on a rigid C*-tensor category is a family of linear maps $\Theta_{\alpha,\beta} : End(\alpha \otimes \beta) \to End(\alpha \otimes \beta)$ for all $\alpha, \beta \in Obj(\mathscr{C})$ such that

1. Each $\Theta_{\alpha,\beta}$ is $End(\alpha) \otimes End(\beta)$-bimodular
2. $\Theta_{\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2}(1 \otimes X \otimes 1) = 1 \otimes \Theta_{\alpha_2,\beta_1}(X) \otimes 1$ for all $\alpha_i, \beta_i \in \mathscr{C}, X \in End(\alpha_2 \otimes \beta_2)$

A multiplier is a *cp-multiplier* if each $\Theta_{\alpha,\beta}$ is completely positive.

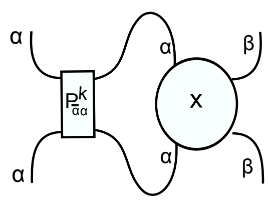
In [47, Proposition 3.6], it is shown that multipliers are in one-one correspondence with functions $\phi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$. For such a ϕ, we define a multiplier $\Theta^{\phi}_{\alpha,\beta}$ as follows:

For an object $\alpha \in \mathscr{C}$, and for $k \in \mathrm{Irr}(\mathscr{C})$ with $X_k \prec \alpha\overline{\alpha}$, define the central projection in $End(\alpha \otimes \overline{\alpha})$

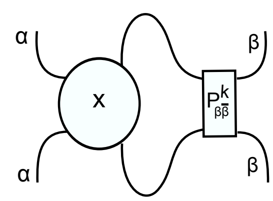
$$P^k_{\alpha\overline{\alpha}} := \sum_{W \in onb(\alpha\overline{\alpha}, X_k)} W^* W$$

Then for $x \in End(\alpha \otimes \beta)$,

$$\Theta^{\phi}_{\alpha,\beta}(x) = \sum_{k \in \mathrm{Irr}(\mathscr{C})} \phi(X_k)$$



$$= \sum_{k \in \mathrm{Irr}(\mathscr{C})} \phi(X_k)$$



We note this sum is finite. In the above pictures, we apply our conventions for horizontal strings locally. Popa and Vaes show every multiplier is of this form.

DEFINITION III.2.4. A function $\phi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ is a *cp-multiplier* if Θ^{ϕ} is a cp-multiplier in the sense of Definition III.2.3.

It is shown in [47] that if $\phi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ is a cp-multiplier, then $d(.)\phi(.) : \mathbb{C}[\mathrm{Irr}(\mathscr{C})] \to \mathbb{C}$ is a state on the fusion algebra.

DEFINITION III.2.5.

1. A function $\phi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ is called an *admissible state* if $\frac{\phi(.)}{d(.)}$ is a cp-multiplier.

2. A (non-degenerate) $*$-representation π of $\mathscr{A}\Lambda_{0,0} \cong \mathbb{C}[\mathrm{Irr}(\mathscr{C})]$ is called *admissible* if every vector state in the representation is admissible.

3. Define $\| \|_u := \sup_{\pi \text{ admissible}} \| \|_\pi$ on $\mathscr{A}\Lambda_{0,0} \cong \mathbb{C}[\mathrm{Irr}(\mathscr{C})]$. $C^*(\mathscr{C})$ is defined as the completion of $\mathscr{A}\Lambda_{0,0} \cong \mathbb{C}[\mathrm{Irr}(\mathscr{C})]$ with respect to this universal norm. It is shown in [47] that this is finite and a C*-norm.

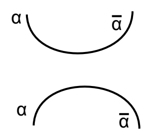
We will show that admissible states are exactly the same as weight 0 annular states. First, a lemma due to Popa and Vaes:

LEMMA III.2.6. *([47, Lemma 3.7]) Let $\mathscr{C}$ be a rigid C*-tensor category and Θ a multiplier on $\mathscr{C}$. Then the following are equivalent:*

1. *For all $\alpha,\beta \in \mathscr{C}$, the map $\Theta_{\alpha,\beta} : End(\alpha\otimes\beta) \to End(\alpha\otimes\beta)$ is completely positive.*

2. *For all $\alpha,\beta \in \mathscr{C}$, the map $\Theta_{\alpha,\beta} : End(\alpha\otimes\beta) \to End(\alpha\otimes\beta)$ is positive.*

3. *For all $\alpha \in \mathscr{C}$ we have that $\Theta_{\alpha,\overline{\alpha}}(\overline{R}_\alpha\overline{R}_\alpha^*)$ is positive.*

Note that $\overline{R}_\alpha\overline{R}_\alpha^*$ is given in pictures by 

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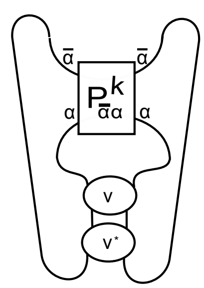
THEOREM III.2.7. *ϕ is a weight 0 annular state if and only if ϕ is admissible in the sense of Definition III.2.5.*

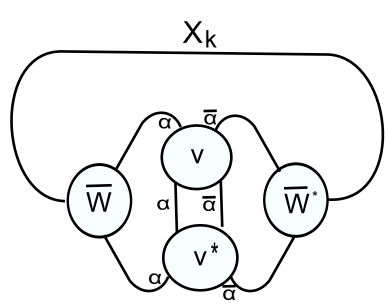
Proof. First let $\phi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ be an arbitrary function. We define the multiplier Θ^ψ associated to the function $\psi(.) := \frac{\phi(.)}{d(.)}$ as above. Take any vector $v \in End(\alpha\otimes\overline{\alpha})$. Then we have

$$\langle\Theta^\psi_{\alpha,\overline{\alpha}}(\overline{R}_\alpha\overline{R}_\alpha^*)v, v\rangle = Tr(\Theta^\psi_{\alpha,\overline{\alpha}}(\overline{R}_\alpha\overline{R}_\alpha^*)vv^*)$$

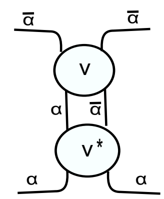
$$= \sum_{k\in\mathrm{Irr}(\mathscr{C})} \frac{\phi(X_k)}{d(X_k)}$$



$$= \sum_{k\in\mathrm{Irr}(\mathscr{C})} \sum_{W\in onb(\overline{\alpha}\alpha, X_k)} \frac{\phi(X_k)}{d(X_k)}$$



$$= \phi^{op} \circ \Psi^{\overline{\alpha}\alpha} \left(\quad \right)$$



Here we use our graphical calculus conventions for side strings, and $\overline{W}$ represents the image of of the morphism W under the appropriate duality functor. In the last equality, we view ϕ^{op} as a functional on $\mathbb{C}[\mathrm{Irr}(\mathscr{C})]$.

Now, since weight 0 annular states are the same for all full annular categories, without loss of generality we set $\Lambda = [Obj(\mathscr{C})]$. Let $x := (v^* \otimes 1_\alpha) \circ (1_\alpha \otimes R_\alpha) \in \widetilde{\mathscr{A}\Lambda}^{\alpha}_{0,\alpha\overline{\alpha}}$. Then the last term in the above equality can be interpreted as

$$\phi^{op} \circ \Psi^{\overline{\alpha}\alpha}(x^{\#} \cdot x) = \phi^{op}(\Psi^{\alpha}(x)^{\#} \cdot \Psi^{\alpha}(x)).$$

If $\phi \in \Phi\Lambda_0$, then by Lemma III.2.2 the above expression is non-negative for all v, α, hence Θ^{ψ} is a cp-multiplier by Lemma III.2.6 (3).

Conversely, if Θ^{ψ} is a cp-multiplier we need to show that ϕ is an annular state, and it suffices to show $\phi^{op} \in \Phi\Lambda_0$. But by Lemma III.2.1 it suffices to check this for the tube algebra. Let $f = \sum_{k \in \mathrm{Irr}(\mathscr{C})} f^k_{0,m} \in \mathscr{A}_{0,m}$. Set $\alpha := \oplus X_k$ where k appears in the description of f. Then since $f^k_{0,m} \neq 0$, $X_m \prec \alpha\overline{\alpha}$. Then set

$$v^* := \sum_{k \in \mathrm{Irr}(\mathscr{C})} \sum_{W \in onb(X_m, \alpha\overline{\alpha})} (1_{\alpha\overline{\alpha}} \otimes \overline{R}^*_k)(W \otimes 1_k \otimes 1_{\overline{k}})(f^k_{0,m} \otimes 1_{\overline{k}}) \in End(\alpha\overline{\alpha})$$

Then since Θ^{ψ} is a cp-multiplier,

$$\phi^{op}(f^{\#} \cdot f) = \langle \Theta^{\psi}_{\alpha,\overline{\alpha}}(\overline{R}_\alpha \overline{R}^*_\alpha)v, v\rangle \geq 0$$

Thus ϕ is an annular state.

□

COROLLARY III.2.8. *(π, H) be a $*$ representation of the fusion algebra $\mathbb{C}[Irr(\mathscr{C})]$. Then the following are equivalent:*

1. (π, H) *is admissible in the sense of Definition III.1.9, namely, there exists a non-degenerate* $*$*-representation of* $\mathscr{A}$ *which restricted to* $\mathscr{A}_{0,0}$ *is unitarily equivalent to* (π, H)*.*

2. (π, H) *is admissible in the sense of Popa and Vaes, Definition III.2.5.*

COROLLARY III.2.9. $C^*(\mathscr{C}) \cong C^*(\mathscr{A}_{0,0})$

We consider the affine state $1_{\mathscr{C}}$ corresponding to the trivial representation, given by $1_{\mathscr{C}}(X) = d(X)$ for each $X \in Irr(\mathscr{C})$. We note that if $\phi \in \Phi\Lambda_0$, then ϕ is a state on $C^*(A\Lambda_{0,0})$. Furthermore, for each simple object $X \in Irr(\mathscr{C})$, $||X||_u = d(X)$, since we have $||X|| \leq d(X)$ by Lemma III.1.4, and this value is realized in the trivial representation. Furthermore, $C^*(A\Lambda_{0,0})$ contains the one dimensional subspace $\hat{X} \cong \mathbb{C}[X]$ for $X \in Irr(\mathscr{C})$. Hence for an annular state $\phi \in \Phi\Lambda_0$, when viewed as a state on $C^*(A\Lambda_{0,0})$, $||\phi|_{\hat{X}}|| = |\frac{\phi(X)}{d(X)}|$. Hence the numbers $|\frac{\phi(X)}{d(X)}|$ are "local norms" of the state ϕ. Now, we recall the definitions of approximation and rigidity properties given by Popa and Vaes, but present them translated into our annular language.

DEFINITION III.2.10. [47] A rigid C*-tensor category (with Irr($\mathscr{C}$) countable) is said

1. to be *amenable* if there exists a sequence of finitely supported weight 0 annular states ϕ_n that converges to $1_{\mathscr{C}}$ pointwise on $Irr(\mathscr{C})$.

2. to have *property (T)* if for every sequence of annular states ϕ_n which converges pointwise to $1_{\mathscr{C}}$, the sequence of functions $\frac{\phi_n(\ .\ )}{d(\ .\ )}$ converges uniformly to 1 on $Irr(\mathscr{C})$.

3. to have the *Haagerup property* if there exists a sequence of annular states ϕ_n each of which vanish at ∞ (for every ε, there exists a finite subset $K \subseteq Irr(\mathscr{C})$ such that $|\frac{\phi(X)}{d(X)}| < \varepsilon$ for all $X \in K^c$), which converge to $1_{\mathscr{C}}$ pointwise.

There are many familiar equivalent characterizations of these properties, many of which are proved in [47]. We record one of these which we will use for property (T):

PROPOSITION III.2.11. *([47, Proposition 5.5])* $\mathscr{C}$ *has property (T) if and only if there exists a projection* $p \in C^*(\mathscr{C})$ *such that* $\alpha p = d(\alpha)p$ *for all* $\alpha \in Irr(\mathscr{C})$*.*

For categories with abelian fusion rules (for example, all braided categories), $C^*(\mathscr{C}) \cong C(Z)$ for some compact Hausdorff space Z. Points in Z correspond to one-dimensional representations of the fusion algebra, so $1_{\mathscr{C}} \in Z$. We have the following easy consequence of the above proposition:

COROLLARY III.2.12. *If $\mathscr{C}$ has abelian fusion rules so that $C^*(\mathscr{C}) \cong C(Z)$ for some compact Hausdorff space Z, then $\mathscr{C}$ has property (T) if and only if the trivial representation $1_{\mathscr{C}}$ is isolated in Z*

Proof. If $\mathscr{C}$ has abelian fusion rules $C^*(\mathscr{C}) \cong C(Z)$, where Z is the spectrum of $C^*(\mathscr{C})$. If $1_{\mathscr{C}}$ is isolated in the spectrum, then the characteristic function $\delta_{\{1_{\mathscr{C}}\}} \in C(Z) \cong C^*(\mathscr{C})$ is a projection satisfying the required property. Conversely, if we had such a projection p, then it could be represented by the characteristic function of some clopen set $Y \subseteq Z$. Since $\alpha p = d(\alpha)p$, this implies that when viewing an object α as a function on Z, $\alpha|_Y = d(\alpha) = \alpha(1_{\mathscr{C}})$. Extending by linearity, we see that for an arbitrary element in the fusion algebra β, $\beta|_Y = 1_{\mathscr{C}}(\beta)$. This equality extends to the C^*-closure $C^*(\mathscr{C}) \cong C(Z)$. Since the points of Y are not separated by $C(Z)$ from $1_{\mathscr{C}}$, by the Stone-Weierstrass theorem we have $Y = \{1_{\mathscr{C}}\}$, hence $\{1_{\mathscr{C}}\}$ is clopen, hence $1_{\mathscr{C}}$ is isolated in Z. □

This corrolary seems a bit strange, since infinite abelian discrete groups can never have property (T) (they are always amenable!). However, as we shall see in Chapter 4, there are examples of categories with abelian fusion rules having property (T), and this corollary comes in handy.

CHAPTER IV

EXAMPLES

The first two sections of this chapter are based on my joint paper with Shamindra Kumar Ghosh, "Annular representation theory for rigid C*-tensor categories" [14], while the third section is based on my paper "Quantum G_2 categories have property (T)" [20]

IV.1 Vec(G)

Let G be a discrete group and let $\mathscr{C}$ be the category of G-graded vector spaces (with trivial associator). The tube algebra of this example is known, and is one of the earliest examples of a tube algebra, though we were unable to track down the earliest description in the literature. The tube algebra in this case is essentially the Drinfeld double of the Hopf algebra $\mathbb{C}[G]$, which was one of the motivating examples in the definition of the Drinfeld center. This example is typically presented in the case of finite groups, while here we consider discrete groups in general.

Simple objects in $\mathscr{C}$ are one-dimensional vector spaces indexed by elements of a group, and we identify Irr($\mathscr{C}$) with the group G. The tensor product corresponds to group multiplication, and duality corresponds to inverses of group elements. To be clear, we are actually using a "strictified" version of the category, where $X \otimes Y = XY$ for $X, Y \in G$, with equality instead of isomorphism of objects.

For $X, Y, Z \in G$, by Frobenius reciprocity $\mathscr{A}^Z_{X,Y} \cong Mor(X, \overline{Z}YZ)$ which is 1 dimensional if $X = Z^{-1}YZ$ as group elements, and 0 otherwise. Thus in the tube category language, there is a non-zero hom between X, Y iff X is conjugate to Y. If we set $Conj(G) := \{$conjugacy classes of $G\}$, then we have a first decomposition $\mathscr{A} \cong \bigoplus_{\Gamma \in Conj(G)} \mathscr{A}_\Gamma$, where $\mathscr{A}_\Gamma := \bigoplus_{X,Y \in \Gamma} \mathscr{A}_{X,Y}$.

Thus to understand the whole tube algebra, it suffices to determine the structure of $\mathscr{A}_\Gamma$ for each conjugacy class Γ. For $X \in G$, $\mathscr{A}_{X,X} := \bigoplus_{Y \in Z_G(X)} \mathscr{A}^Y_{X,X}$, where $Z_G(X)$ is the centralizer subgroup of X in G. Since each $\mathscr{A}^Y_{X,X} = Mor(YX, XY)$ is non-zero if and only if $XY = YX$, we can identify this space with $Mor(YX, YX)$ which in turn is isomorphic to $\mathbb{C}$. Thus we have a natural vector space isomorphism $\alpha : \mathscr{A}_{X,X} \cong \mathbb{C}[Z_G(X)]$. Furthermore, it is easy to check that this is a $*$-algebra isomorphism. More specifically for $Y \in Z_G(X)$, we can choose $f^Y_X \in \mathscr{A}^Y_{X,X} = Mor(YX, YX)$ to be the identity in the later morphism space. Then we have from the tube algebra multiplication $f_X^Y \cdot f_X^Z = f_X^{YZ} \in \mathscr{A}_{X,X}$, and # corresponds to inverses. Now, for each $X, Y \in \Gamma$, $Z_G(X) \cong Z_G(Y)$. In fact these are conjugate by any group element that conjugates Y to X. The number of possible conjugators from X to Y is $|Z_G(X)|$. It is now easy to see that $\mathscr{A}_\Gamma \cong \mathbb{C}[Z_{G(X)}] \otimes F(\Gamma)$, where $F(\Gamma)$ is the algebra of finite rank operators on the Hilbert space $\ell^2(\Gamma)$ with respect to the obvious basis (see discussion preceeding Proposition II.2.7). The diagonal copies of $Z_G(X)$ are the $\mathscr{A}_{X,X}$, and the matrix unit copies are given by $\mathscr{A}_{X,Y}$.

We have the following claim: Let $X \in \mathrm{Irr}(\mathscr{C}) \cong G$, and let $Z_G(X)$ be the centralizer subgroup of X in G. Then if (π, H) is a unitary representation of $Z_G(X)$, then (π, H) extends to a representation of $\mathscr{A}_\Gamma$, where Γ is the conjugacy class of X. To see this we simply note that since $\mathscr{A}_\Gamma \cong \mathbb{C}[Z_G(X)] \otimes F(\Gamma))$, we can define the Hilbert space $H_\Gamma := H \otimes \ell^2(\Gamma)$, with the obvious action. It is clear that this is a $*$ representation by bounded operators of $\mathscr{A}_\Gamma$. Therefore

$$C_u^*(\mathscr{A}_{X,X}) \cong C_u^*(Z_G(X))$$

In particular $C_u^*(\mathscr{A}_{0,0}) \cong C_u^*(G)$. Our discussion gives us another proof of the following, originally due to Popa:

PROPOSITION IV.1.1. *If $\mathscr{C}$ is $Vec(G)$, the $\mathscr{C}$ has an approximation or rigidity property if and only if the group does*

Proof. Since every representation of G is an admissible weight 0 representation of the tube algebra, cp-multipliers in our context correspond precisely to cp-multipliers for groups. Thus the definitions presented in III.2.10 agree precisely with the common definitions for groups.

□

IV.2 Temperley-Lieb-Jones categories

The Temperley-Lieb-Jones categories $TLJ(\delta)$ for $\delta \geq 2$ are equivalent to the categories $Rep_{-q}(SU(2))$, where $\delta = q + q^{-1}$ for q a positive real number. They provide a fundamental class of rigid C*-tensor categories with infiniteley many isomorphism classes of simple objects. They satisfy the universal property that for any rigid C*-tensor category $\mathscr{C}$ generated by a symmetrically self-dual object X with $d(X) = \delta$, there exists a unique, dimension presenving, dominant tensor functor $F : TLJ(\delta) \to \mathscr{C}$. This categories have a nice planar algebra description and a nice categorical description simultaneously. To describe them, fix a positive real number $\delta \geq 2$. Then there is a unique $q \in \mathbb{R}$ such that $q + q^{-1} = \delta$. We can then define for $n \in \mathbb{N}$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ if $q \neq 1$, and $[n]_1 = n$.

The rigid C*-tensor category $TLJ(\delta)$ has:

1. Self dual simple objects indexed by natural numbers, with 0 indexing the identity.

2. $d(k) = [k+1]_q$

3. $k \otimes m \cong (k+m) \oplus (k+m-2) \oplus \cdots \oplus |k-m|$

For the rest of this section, we use $[n]$ to denote $[n]_q$, assuming q is fixed. The above properties are merely a summary of some relevant categorical data. These categories have much more structure than this, for example there are complicated 6-j symbols, and these categories naturally have a braiding (non-unitary unless $q = 1$). These categories also can be realized as the projection categories of of the following planar algebras:

Define the unoriented, unshaded planar algebra $TL(\delta)$ as follows:

1. $P_0 \cong \mathbb{C}$

2. $P_{2n+1} = 0$

3. $P_{2n} :=$ Linear span of disks with $2n$ boundary points with strings connecting boundary points

4. strings do not cross

5. All boundary points are connected to some other boundary point with a string

6. Closed circles multiply the diagram by a factor of δ

We note that in our generic case $\delta \geq 2$, this is a spherical C*-planar algebra (see [7], [23] for definitions of spherical C*-planar algebras). We have $dim(P_{2n}) = \frac{1}{n+1}\binom{2n}{n}$. We remark that this is one of the most important example of a planar algebra since the universal property shows that an arbitrary planar algebra is a "quotient" of one of these. It is usually presented as a shaded planar algebra in the subfactor context, and there exists many detailed expositions, see [22],[23]. We can realize the category described above as $TLJ(\delta) = Proj(TL(\delta))$ (see [7] for definition of the projection category of a planar algebra). The object k in $TLJ(\delta)$ corresponds to the k^{th} **Jones-Wenzl idempotent** in the planar algebra $TL(\delta)$, denoted f_k. These projections satisfy the property that applying a cap or cup to the top or bottom of f_k results in 0, called uncapability. f_k is a minimal projection in $TL_{k,k}$ and can be defined by an inductive formula, see [29] or [22] for details. We remark that f_k corresponds to the $k+1$ dimensional irreducible (co)-representation of the compact quantum group $SU_{-q}(2)$.

The affine annular representations of this planar algebra have been studied in detail by Jones, Jones-Reznikoff, and Reznikoff (see [24], [23], and [48] respectively). We will make use of these results to analyze the universal C*-algebra structure on the centralizer algebras of the tube algebra of this category. The beginning of this section can be deduced in its entirety from the work of Jones and Reznikoff, which is in turn inspired by **??**. We include these results here for the purpose of self-containment, and due to the slight differences in our setting. We remark here that we use the "annular category" picture for ATL to fit with the perspective of Jones and Jones-Reznikoff.

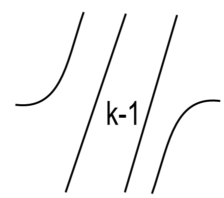
As discussed in section 2.2, a planar algebra $\mathscr{P}$ naturally provides an annular algebra $\mathscr{A}\mathscr{P}$. The affine annular category ATL is easy to describe. The weights will simply be natural numbers, and they will signify the number of strings on the boundaries of disks. The object in $Proj(TL(\delta))$ corresponding to $k \in \mathbb{N}$ is $1_k \in TL_{k,k}$. Then $ATL_{k,m}$ will consist of all TL diagrams in an annulus with k boundary points on the internal circle and m on the external circle. This means there are $\frac{k+m}{2}$ non-intersecting strings in the annulus, and each string touches precisely one boundary point (on either the inner or outer disk). We consider these diagrams only up to affine annular isotopy. That the set of affine annular pictures described here (isotopy classes of non-intersecting string diagrams) is really a basis for the annular category of the planar algebra follows from the analysis of [10] and the fact that $TL(\delta)$ for $\delta \geq 2$ has no local skein relations except removing closed circles. Composition is the obvious one, and homologicaly trivial circles in the annulus multiply the diagram by a factor of δ. For more details on this annular category in particular see [24].

We consider here a subcategory of $Rep(ATL)$ consisting of all locally finite representations. By this we mean the set of Hilbert representations of ATL, (π, V_k) such that each V_k is a finite dimensional Hilbert space, and $\pi : ATL_{k,m} \to B(V_k, V_m)$ is a $*$-homomorphism. This category is closed under finite direct sums. In the literature, Hilbert representations of $\mathscr{A}\mathscr{P}$ are called Hilbert $\mathscr{P}$-modules, and so we use these terms interchangeably in the planar algebra setting.

DEFINITION IV.2.1. A lowest weight k Hilbert TL-module is a (non-degenerate) representation (π, V_m) such that $V_m = 0$ for all $m < k$, and $V_k \neq 0$.

Irreducible representations of ATL are representations which are irreducible as representations of the corresponding annular algebra. It is straightforward to check that this implies each V_k is irreducible as a representation of $ATL_{k,k}$. Following the proof in [23], one can show that every locally finite Hilbert TL-module is isomorphic to the direct sum of irreducible lowest weight k modules. It then becomes our task to classify and construct these.

To do so we start by noting that $ATL_{0,0}$ is isomorphic to the fusion algebra $\mathbb{C}[\text{Irr}(TLJ(\delta)]$, which is abelian. Thus an irreducible lowest weight 0 module will be a 1 dimensional representation of the fusion rules. Let v_0 be a non-zero vector in the one dimensional space normalized so that $\langle v_0, v_0 \rangle = 1$. We notice the identity object (f_0) must go to the identity and we may identify $\pi(f_k)$ with some number (its eigenvalue on v_0). But from the fusion rules, all these numbers are determined by $\pi(f_1)$. Since f_1 in $ATL_{0,0}$ is self dual and this must be a $*$-representation, we see that $\pi(f_1)$ (hence $\pi(f_k)$ for all k) must be a real number. Furthermore, by the bounds on the universal norm for the weight 0 case (III.1.4), we must have $|\pi(f_1)| \leq \delta$. Let $t := \pi(f_1) \in [-\delta, \delta]$. Then this parameter determines π completely. We still must see which of these extend to Hilbert TL-modules, but we will see that all of them will.

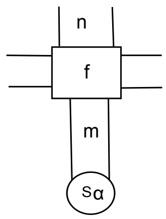
Now, consider $k > 0$. Let $ATL_{k,k}^{<k}$ be the ideal in $ATL_{k,k}$ spanned by diagrams with less than k through strings. We see that in a lowest weight k representation, this ideal must act by 0. An irreducible lowest weight k representation will then neccessarily be an irreducible representation of the algebra $ATL_{k,k}/ATL_{k,k}^{<k}$. We define the element $\rho_k \in ATL_{k,k}$, known informally as "rotation by one", with the picture 

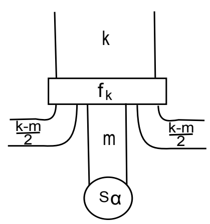
.

Here we use the graphical calculus conventions of annular algebras. We see that this element is invertible in $ATL_{k,k}$, and we call its inverse $\rho_k^{-1} = \rho_k^*$ the "left rotation by one". The powers of ρ_k form a subgroup of the algebra $ATL_{k,k}$ isomorphic to $\mathbb{Z}$, hence $ATL_{k,k}/ATL_{k,k}^{<k} \cong \mathbb{C}[\mathbb{Z}]$, which is abelian. Therefore an irreducible lowest weight k $*$-representation will be an irreducible unitary representation of $\mathbb{Z}$, hence determined by some $\omega \in S^1$.

We have now found all candidates for irreducible lowest weight m representations of ATL for all m. The question that remains is which of these representations of the fusion algebra and $\mathbb{Z}$ extend to a representation of the entire annular category, i.e. have a canonical extension. Since all the spaces are finite dimensional (as we shall see), the annular actions are bounded, hence it suffices to demonstrate that the inner products of the canonical extension are positive semi-definite. If we have a representation of $ATL_{m,m}/ATL_{m,m}^{<m}$ (or $ATL_{0,0}$)

representation determined by the parameter $\alpha \in S^1$ (or in $[-\delta, \delta]$) on the one dimensional vector space V_m^α, define $\hat{V}_n^\alpha := ATL_{m,n} \otimes_{ATL_{m,m}} V_m^\alpha$. If we let $s_\alpha \in V_m^\alpha$ be normalized, we can represent simple tensors in the vector space $\hat{V}_n^\alpha$ by

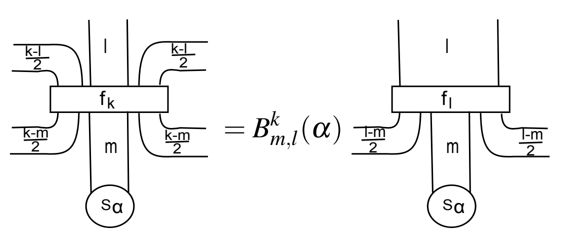
$f \otimes s_\alpha :=$ 

Connecting the bottom m strings to the rotation eigenvector signifies that we are taking a relative tensor product over $ATL_{m,m}$. Now, we can easily see that $dim(f_k ATL_{m,k} \otimes_{ATL_{m,m}} V_m^\alpha)$ is at most one. To see this, we note that all the strings emanating from s_α must enter the f_k consecutively, since apply a cap to f_k results in 0. The remaining $k-m$ strings coming from f_k that are not attached to s_α must be connected to each other somehow, but by uncapability of f_k, they must be connected "around the bottom of the annulus". If $m-k$ is even, there is precisely one way to do this, and if $m-k$ is odd this is impossible. In particular, $f_k \hat{V}_k^\alpha$ is spanned by the vector $g_{m,k}^\alpha :=$ 

. We note that $g_{m,k}^\alpha = 0$ for $k < m$.

To understand $\hat{V}_n^\alpha$, for each Jones-Wenzl idempotent $f_{n-2j} \prec n$, let (f_{n-2j}, n) denote the linear space of planar algebra elements $x \in P_{n-2j,n}$ such that $x f_{n-2j} = x$, for $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$. (f_{n-2j}, n) is precisely the space of morphisms in the projection category of TL from f_{n-2j} to 1_n (see [7]). It is clear that $\hat{V}_n^\alpha \cong \bigoplus_{0 \leq j \leq \lfloor \frac{n}{2} \rfloor} (f_{n-2j}, n) \otimes g_{m,n-2j}^\alpha$.

With this nice decomposition, we want to see how the canonical inner product behaves. First, we will do some diagrammatics that will allow us to clearly see the canonical inner product is positive semi-definite. We closely follow the work of [24]. Let α be the parameter of a lowest weight m representation. We define the numbers $B_{m,l}^k(\alpha)$ by the following:

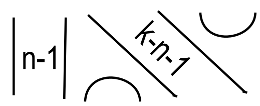


Note that $B_{m,k}^k = 1$ for all $k \geq m$. As a matter of convention, we use α to represent an arbitrary irreducible representation parameter, while we use t to represent a weight parameter (so that $t \in [-\delta, \delta]$) while we use $\omega \in S^1$ to represent a weight > 0 parameter.

LEMMA IV.2.2. $[k]^2 - \left[\frac{k-m}{2}\right]^2 - \left[\frac{m+k}{2}\right]^2 = (q^k + q^{-k}) \left[\frac{k-m}{2}\right] \left[\frac{m+k}{2}\right]$

Proof. Direct computation. □

Recall that $TL_{k,k}$ as a vector space is the linear span of all isotopy classes of rectangular diagrams and k non-intersecting strings, with k boundary points on the top and bottom of the rectangle, and each string is attached to exactly two of these boundary points. Thus $f_k \in TL_{k,k}$ can be written as a linear combination of such diagrams. In general it is difficult to compute the coefficient of an arbitrary diagram in f_k, however there are several types of diagrams which have relatively easy coefficients. First, the coefficient of the identity diagram $1_k \in TL_{k,k}$ is one. The coefficient of the diagram

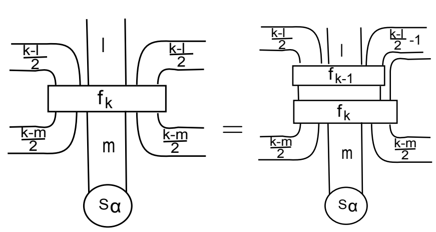


in f_k is $(-1)^{n-k}\frac{[n]}{[k]}$. For a proof of these formulas we refer the reader to Morrison's paper [29]. We note that the f_k are invariant under vertical and horizontal reflection. This implies diagrams obtained from one another by horizontal or vertical reflection will have the same coefficients in f_k. With these formulas in hand, we have the following proposition:

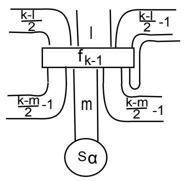
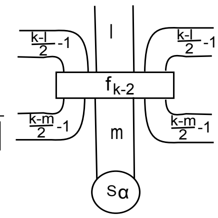
PROPOSITION IV.2.3.

1. *For $m > 0$, $\omega \in S^1$ and k even,* $B^k_{m,l}(\omega) = \frac{\left[\frac{k-m}{2}\right]\left[\frac{m+k}{2}\right]}{[k][k-1]} \left(q^k + q^{-k} - \omega^2 - \omega^{-2}\right) B^{k-2}_{m,l}(\omega)$
2. $B^k_{0,l}(t) = \frac{1}{[k][k-1]} \left([k]^2 - t^2\left[\frac{k}{2}\right]^2\right) B^{k-2}_{0,l}$
3. *For $m > 0$, $\omega \in S^1$ and k odd, we have* $B^k_{m,l}(\omega) = \frac{\left[\frac{k-m}{2}\right]\left[\frac{m+k}{2}\right]}{[k][k-1]} \left(q^k + q^{-k} - (i\omega)^2 - (i\omega)^{-2}\right) B^{k-2}_{m,l}(\omega)$

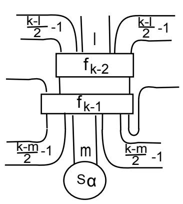
Proof. First assume $m > 0$. Then we have

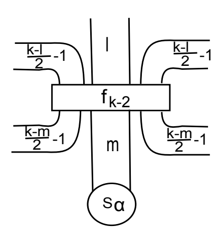


By uncapability of f_k, we see that there are precisely 3 diagrams that can be inserted into the bottom f_k. The identity diagram 1_k, with no cups or caps on either the top or bottom, is the first. There can only be one cup in the top, which must be on the top right. Such a diagram must have exactly one cap on the bottom, and it can be either at position $\frac{k-m}{2}$ or $\frac{k+m}{2}$. As mentioned above, the coefficient of such a diagram is $(-1)^{\frac{k+m}{2}}\frac{[\frac{k-m}{2}]}{[k]}$ for the former and $(-1)^{\frac{k-m}{2}}\frac{[\frac{k+m}{2}]}{[k]}$ for the latter. We see then pick up a value of ω^{-1} for the first diagram, and an ω for the second diagram. Then the above is equal to

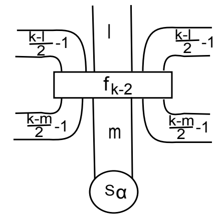
$$(-1)^{\frac{k-m}{2}}\left(\frac{(-1)^m[\frac{k-m}{2}]\omega^{-1}+[\frac{k+m}{2}]\omega}{[k]}\right)$$  $$+\frac{[k]}{[k-1]}$$ 

The diagram on the right and its coefficient is obtained from plugging in 1_k in for f_k. We see that applying annular relations introduces a copy of f_{k-1}, where the top left most string is attached to the bottom left most string around the left side. This is nothing other than the left trace preserving conditional expectation $E_L : TL_{k-1,k-1} \to TL_{k-2,k-2}$ applied to f_{k-1}. Since $E_L(f_{k-1})$ is uncapable on both the top and bottom, we have $E_L(f_{k-1}) = cf_{k-2}$ for some scalar c. Taking the trace on both sides givese us $Tr(f_{k-1}) = cTr(f_{k-2})$, hence $c = \frac{[k]}{[k-1]}$.

We want an expression just involving the diagram to the right in the sum, so we consider the left diagram in the sum and apply the same sort of argument. We notice the diagram on the left is in fact equal to . Now here we see that the identity gives 0. There is only one possible place for a cup, and that is on the top left. A careful consideration of caps and through strings shows a cap at the bottom right gives 0. There are precisely 2 places for a bottom cap that give non-zero contributions, namely with positions at $\frac{k-m}{2}-1$ and $\frac{k+m}{2}-1$. The coefficients in f_{k-1} of these in diagrams are $(-1)^{\frac{m-k}{2}+1}\frac{[\frac{k+m}{2}]}{[k-1]}$ and $(-1)^{\frac{m+k}{2}+1}\frac{[\frac{k-m}{2}]}{[k-1]}$ respectively. Again the first coefficient picks up an ω^{-1} and the second picks up an ω. Thus this diagram is equal to

$$-(-1)^{\frac{m-k}{2}}\left(\frac{[\frac{k+m}{2}]\omega^{-1}+(-1)^k[\frac{k-m}{2}]\omega}{[k-1]}\right)$$ 

Putting everything together we end up with

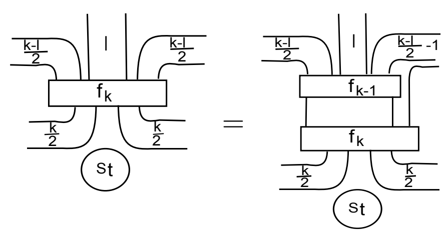
$$\frac{1}{[k][k-1]}\left([k]^2-\left[\frac{k-m}{2}\right]^2-\left[\frac{k+m}{2}\right]^2-(-1)^k(\omega^2+\omega^{-2})\left[\frac{k-m}{2}\right]\left[\frac{k+m}{2}\right]\right)$$ 

By the quantum number identity of Lemma IV.2.2, the above coefficient is

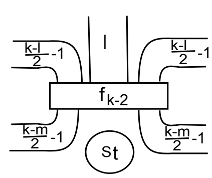
$$\frac{\left[\frac{k-m}{2}\right]\left[\frac{m+k}{2}\right]}{[k][k-1]}\left(q^k+q^{-k}-(-1)^k(\omega^2+\omega^{-2})\right).$$

We immediately see the desired formulas for k even and k odd (for k odd the i comes from the $(-1)^k=-1$, which we then bring inside the (ω^2) as an i).

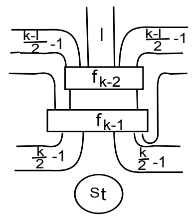
Now for $m=0$, we must have that k,l are even. We perform the same analysis:



Evaluating the bottom f_k with Temperley-Lieb diagrams, we see the identity 1_k yields

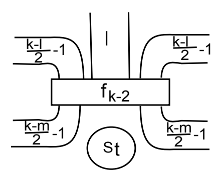
$$\frac{[k]}{[k-1]}$$ 

Now there is only one possible non-zero cap location in the top (in the top right), and one possible cap on the bottom, at position $\frac{k}{2}$. This diagram has coefficient $(-1)^{\frac{k}{2}}\frac{[\frac{k}{2}]}{[k]}$. The cap at the bottom yields a factor of t since it produces a homologically non-trivial circle around s_t, resulting in

$$(-1)^{\frac{k}{2}} t \frac{\left[\frac{k}{2}\right]}{[k]}$$ 

.

As in the case $m > 0$, the identity 1_{k-1} yields 0 at this step, and thus there is precisely one diagram which gives a non-zero contribution, with a cup in the upper left hand corner, and a cap on the bottom at position $\frac{k}{2} - 1$. The coefficient of this diagram in f_{k-1} is $(-1)^{\frac{k}{2}-1} \frac{[\frac{k}{2}]}{[k-1]}$. Again a factor of t pops out. Combining all the terms, we end up with our original expression equal to

$$\frac{1}{[k][k-1]} \left([k]^2 - t^2 \left[\frac{k}{2} \right]^2 \right)$$ 

This gives us the desired formula. □

As a corollary of Proposition IV.2.3, we can analyze the inner products on the spaces $\hat{V}_n^\alpha$ following Jones and Reznikoff [24]. Let α be the parameter of a lowest weight m representation.

$$\hat{V}_n^\alpha \cong \bigoplus_{0 \leq j \leq \lfloor \frac{n}{2} \rfloor} (f_{n-2j}, n) \otimes g^\alpha_{m,n-2j}.$$

We see that this decomposition is orthogonal with respect to the sesquilinear form defined by our lowest weight m representation. If $x \otimes g^\alpha_{m,n-2j}, y \otimes g^\alpha_{m,n-2j} \in (f_{n-2j}, n) \otimes g^\alpha_{m,n-2j}$, we see that

$$\langle x \otimes g^\alpha_{m,n-2j}, y \otimes g^\alpha_{m,n-2j} \rangle_\alpha = \langle x, y \rangle \langle g^\alpha_{m,n-2j}, g^\alpha_{m,n-2j} \rangle_\alpha = \langle x, y \rangle B^{n-2j}_{m,m}(\alpha),$$

where $\langle x, y \rangle$ denotes the positive definite inner product in the planar algebra. An inspection of the formulas shows that $B^{n-2j}_{m,m}(\alpha) \geq 0$. Thus our inner product is positive semidefinite, hence, taking the quotient by the kernel of our form, we obtain a sequence of finite dimensional Hilbert spaces $\{V_k^\alpha\}$ with $V_k^\alpha = 0$ for $k < m$ (and 0 if the parity of k is distinct from the parity of m). We notice also that this inner product is uniquely determined by α, thus for a given lowest weight k and parameter α, there is a unique Hilbert TL-module constructed as above.

In some cases, however, even with $k \geq m$, it may be that $g^{\alpha}_{m,k} = 0$ in the quotient with respect to the positive semi-definite inner product. This happens precisely when $B^k_{m,m}(\alpha) = 0$. Inspecting the coefficients as in [24], we can determine when this happens. For the weight 0 case, we see that this happens precisely when $k > 0$ and $t = \pm\delta$. For $\delta > 2$, all other $B^k_{m,m}(\alpha)$ are strictly positive for all m, $k \geq m$ and α . When $\delta = 2$ and hence $q = 1$, the weight 0 story is the same, but for higher weights we see that we run in to a problem in two places: For m even, $\omega = \pm 1$, $B^k_{m,m}(\pm 1) = 0$ for all $k > m$. For m odd, we see that the problem occurs at $\omega = \pm i$, and $B^k_{m,m}(\pm i) = 0$ for all $k > m$. This will be relevant when we analyze the tube algebra representations of $TLJ(\delta)$, so we record the results in the following proposition.

PROPOSITION IV.2.4. *[24], [48]: Irreducible lowest weight m representations are classified as follows: For a lowest weight m representation with parameter α, let $g^{\alpha}_{m,k}$ be the vector described above. Recall that $g^{\alpha}_{m,k} = 0$ if $k < m$.*

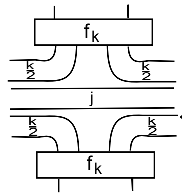
1. *For $t \in [-\delta, \delta]$, there exists a unique irreducible lowest weight 0 Hilbert TL-module $V^{t,0} := \{V^t_k :$ k is even$\}$. For $t \in (-\delta, \delta)$, $g^t_{0,k} \neq 0$ for all even k. $g^{\pm\delta}_{0,k} = 0$ for all $k > 0$.*

2. *For $m > 0$, $\omega \in S^1$, there exists a unique irreducible lowest weight m Hilbert TL-module $V^{\omega,m} :=$ $\{V^{\omega,m}_k$: $m - k$ is even$\}$. For $\delta > 2$, $g^{\omega}_{m,k} \neq 0$ for all $k \geq m$ with $k - m$ even. For $\delta = 2$, k even, $g^{\pm 1}_{m,m} = 1$ and $g^{\pm 1}_{m,k} = 0$ for all $k > m$. For $\omega \neq \pm 1$, $g^{\omega}_{m,k} \neq 0$ for all even $k \geq m$. If m is odd, then $g^{\pm i}_{m,m} = 1$ and $g^{\pm i}_{m,k} = 0$ for all $k > m$, and for $\omega \neq \pm i$, $g^{\omega}_{m,k} \neq 0$ for all odd $k \geq m$.*

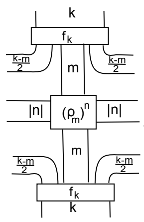
3. *Define the space $X^+_\infty := [-\delta, \delta] \sqcup S_1 \sqcup S_1 \sqcup \ldots$, with infinitely many copies of S_1, and $X^-_\infty := S^1 \sqcup S^1 \sqcup$ $\ldots$. Then irreducible representations in $Rep(ATL)$ are parameterized (as a set) by $X^+_\infty \sqcup X^-_\infty$.*

We thank Makoto Yamashita for pointing out that the parametrization (3) coincides with the parameterization of irreducible representations of the quantum Lorentz group $SL_q(2,\mathbb{C})$, the Drinfeld double of $SU_q(2)$, determined by Pusz in [45]. However, as pointed out by the reviewer, Pusz only considers $q > 0$, and we should expect (3) to parameterize irreducible representations of $SL_{-q}(2,\mathbb{C})$.

We proceed to analyze the corners of the tube algebra of $TLJ(\delta)$. We will denote the tube algebra for $TLJ(\delta)$ by $\mathscr{A}$. Since the simple objects in our category are indexed by k, (namely, the k^{th} Jones-Wenzl idempotent f_k), we let k denote the equivalence class of f_k as opposed to the identity in $TL_{k,k}$ from the planar algebra. To study the tube algebra we construct a nice basis for $\mathscr{A}_{k,k}$ which will allow us to exploit the planar algebra description of this category. From the proof of Proposition 3.5, we see that $\mathscr{A}_{k,k} \cong f_k ATL_{k,k} f_k$. In other words $\mathscr{A}_{k,k}$ is the cut down of the affine Temperley-Lieb $ATL_{k,k}$ space by the rectangular k^{th} Jones-Wenzl projection f_k. Thus we can construct a basis of $\mathscr{A}_{k,k}$ which consist of diagrams as follows:

For k even and $j \in \mathbb{N}$, set $x^k_{0,j} :=$ 

. For $n \in \mathbb{Z}$ and $0 < m \leq k$ with $k - m$ even, define

$x^k_{m,n} :=$ 

. Again, these pictures can and should be interpreted as representing annular tangles, with the strings on the left connecting to strings on the right around the bottom of an annulus. In the center of the diagram $x^k_{m,n}$ is the n^{th} power of the rotation ρ_m. We define the rank of the diagram as $Rank(x^k_{m,n}) = m$. We see that the rank of a diagram in $\mathscr{A}_{k,k}$ must be the same parity as k. The rank corresponds to the number of strings starting from the bottom f_k and going all the way to the top f_k.

PROPOSITION IV.2.5. *Let* $B := \{x^k_{m,n} \,:\, m \in \mathbb{N},\, 0 \leq m \leq k,\, k - m = 0 \,mod\, 2,\, n \in \mathbb{Z} \text{ or } n \in \mathbb{N} \text{ for } m = 0\}$. *Then B is a basis for $\mathscr{A}_{k,k}$.*

Proof. Since $\mathscr{A}_{k,k} \cong f_k ATL_{k,k} f_k$, we see that the only diagrams that are not zero are in B, hence B is a spanning set. To see that these are linearly independent, we note that the diagrams listed above without the f_k (replacing each f_k by $1_k \in ATL_{k,k}$) are linearly independent in $ATL_{k,k}$, since they correspond to distinct isotopy classes of diagrams. We also note that these diagrams have no rectangular caps on their boundaries, which means that any cap on the top or bottom has to go "around the bottom of the annulus". We have a bijective correspondence between B and $ATL_{k,k}$ diagrams with no rectangular caps on the top and bottom boundaries, given by replacing the Jones-Wenzl idempotents in $x^k_{n,m}$ with the $1_k \in TL_{k,k}$. We also note that by definition, the diagrams in $ATL_{k,k}$ with no rectangular caps on their boundaries must be linearly independent from the set of diagrams with some rectangular caps on their boundaries. Suppose there exists some $\{b_i\}_{1 \leq i \leq n} \subseteq B$ and $\lambda_i \in \mathbb{C}$ such that $\sum_i \lambda_i b_i = 0$. Let $\hat{b}_i \in ATL_{k,k}$ be the diagram obtained by replacing the top and bottom Jones-Wenzl idempotents in b_i with the identity. Then evaluating the Jones-Wenzl idempotents at the top and the bottom of the diagrams in terms of TL diagrams, we see that the only terms in both the top and bottom Jones-Wenzls that give no rectangular caps on the boundary are the identity diagrams $1_k \in TL$, and these have coefficient 1 in f_k. Since these diagrams are independent from the diagrams with caps, we notice that our equation implies $\sum_i \lambda_i \hat{b}_i = 0$. But our correspondence is bijective, and these are independent in $ATL_{k,k}$, hence there is no such collection of λ_i. □

PROPOSITION IV.2.6. *For every k, $\mathscr{A}_{k,k}$ is abelian.*

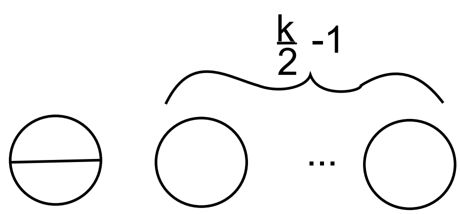
Proof. Define the map $r : \mathscr{A} \to \mathscr{A}$ given for $f \in \mathscr{A}_{i,j}^{k} := Mor(X_k \otimes X_i, X_j \otimes X_k)$ by $r(f) = \overline{f} \in \mathscr{A}_{\bar{j},\bar{i}}^{\bar{k}}$. r is an anti-isomorphism with respect to annular multiplication. Then since $f_k = \overline{f_k}$, by the symmetry of our basis diagrams it is easy to see that $r : \mathscr{A}_{k,k} \to \mathscr{A}_{k,k}$ given by a global rotation by π is in fact the identity map on B, hence on all of $\mathscr{A}_{k,k}$. Then we have for any $x, y \in \mathscr{A}_{k,k}$, $x \cdot y = r(x \cdot y) = r(y) \cdot r(x) = y \cdot x$. Thus $\mathscr{A}_{k,k}$ is abelian.

□

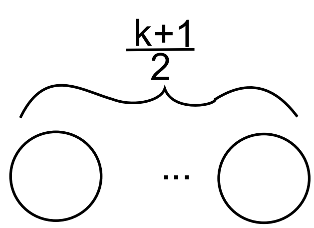
This means $C^*(\mathscr{A}_{k,k})$ will be a unital, abelian C*-algebra, hence isomorphic to the algebra of continuous complex valued functions on some compact Hausdorff space. We describe these spaces below.

1. Define the space $X_0 := [-\delta, \delta]$.

2. For k even, $k > 0$, we define the space

$X_k :=$ 

3. For k odd, define the space

$X_k :=$ 

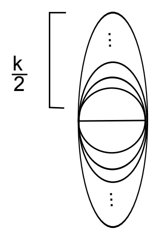
We will demonstrate the following:

THEOREM IV.2.7. *If $\delta > 2$ then $C^*(\mathscr{A}_{k,k}) \cong C(X_k)$.*

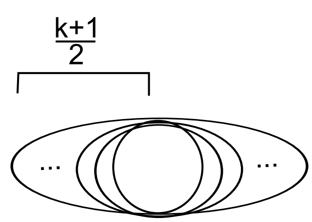
For $\delta = 2$, the situation is different. As discovered in [24], the annular representation theory of $ATL(2)$ is not generic. In particular, there are some "missing" one dimensional representations. This will force us to identify points, resulting in some interesting topological spaces.

1. Define the space $Y_0 := [-2,2]$.

2. For k even, $k > 0$ define the space

$Y_k :=$ 

3. For k odd, define the space

$Y_k :=$ 

THEOREM IV.2.8. *If $\delta = 2$ then $C^*(\mathscr{A}_{k,k}) \cong C(Y_k)$.*

We note that in the case $k = 0$, this essentially recovers a result of Popa and Vaes. The only difference is that they use the even part of the $TLJ(\delta)$ category while we take the category as a whole, thus they have the "square" of this interval, namely $[0, \delta^2]$ (see [47]).

To understand the one dimensional representations of $\mathscr{A}_{k,k}$ (which we often call characters) we note that (almost) every lowest weight m representation with parameter α and $k-m$ even gives a one dimensional representation representation of $\mathscr{A}_{k,k}$. We simply take the vector $g^{\alpha}_{k,m}$. Then this will be an eigenvector of $\mathscr{A}_{k,k}$ viewed as a sub-algebra of $ATL_{k,k}$. Thus if we understand the action of $\mathscr{A}_{k,k}$ on the vector $g^{\alpha}_{m,k}$ we will understand the characters. There is a snag, however. From the above proposition, some of these $g^{\alpha}_{m,k}$ are 0 in the semi-simple quotient, hence do not produce characters on $\mathscr{A}_{k,k}$. Furthermore, it is not a priori clear that every admissible representation of $\mathscr{A}_{k,k}$ comes from ATL in the manner described here. For example, it seems feasible that a one dimensional representation of $\mathscr{A}_{k,k}$ may have its canonical extension infinite dimensional in other weight spaces. We will show that this is not the case.

LEMMA IV.2.9. *For $\delta > 2$, one dimensional representations of $\mathscr{A}_{k,k}$ are parameterized as a set by*

1. *If k is even, $k > 0$, the space $X_k := (-\delta, \delta) \sqcup S^1 \sqcup \cdots \sqcup S^1$ if with $\frac{k}{2}$ copies of S^1*
2. *If k is odd, the space $X_k := S^1 \sqcup \cdots \sqcup S^1$ with $\frac{k+1}{2}$ copies of S^1*
3. *If $k = 0$, the space $X_0 := [-\delta, \delta]$.*

LEMMA IV.2.10. *For $\delta = 2$, one dimensional representations of $\mathscr{A}_{k,k}$ are parameterized by:*

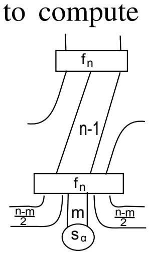
1. *If $k > 0$ is even, the space $Y_k := (-2,2) \sqcup \left(S^1 - \{-1,1\}\right) \sqcup \left(S^1 - \{-1,1\}\right) \sqcup \cdots \sqcup S^1$ with $\frac{k}{2} - 1$ copies of $S^1 - \{-1,1\}$ and one copy of S^1.*
2. *If k is odd, $Y_k := \left(S^1 - \{-i,i\}\right) \sqcup \cdots \sqcup S^1$ with $\frac{k+1}{2} - 1$ copies of $S^1 - \{-i,i\}$ and one copy of S^1.*
3. *$Y_0 := [-\delta, \delta]$.*

Proof. This set produces characters by evaluating the action of $\mathscr{A}_{k,k}$ on the vectors $g^{\alpha}_{m,k}$ for $k - m \geq 0$ and $k - m$ even. Now, the reason that $\pm\delta$ is missing in the interval $(-\delta, \delta)$ from all but $k = 0$ is that the trivial representation of $\mathscr{A}_{0,0}$, does not extend to higher weight spaces by Proposition IV.2.4 (1), i.e. $g^{\pm\delta}_{0,k} = 0$ in the semi-simple quotient of the canonical extension. In the case $\delta = 2$, we have from Proposition IV.2.4 (2) that the characters corresponding to the lowest weight k representations are "missing", meaning that the corresponding $g^{\alpha}_{m,k}$ are 0 for the parameters $\omega = \pm 1$ for even $m > 0$ and $\omega = \pm i$ for odd $m > 0$. Thus the sets listed describes all possible characters on $\mathscr{A}_{k,k}$ coming from ATL, by [24]. Applying Proposition II.2.7, we see that this yields all possible characters. In particular, suppose we have a character α on $\mathscr{A}_{k,k}$. Let m be the smallest m such that the canonical extension to $\mathscr{A}_{m,m}$ is non-zero. It is straightforward to check that since α is irreducible, the canonical extension to $\mathscr{A}_{m,m}$ is one dimensional. Then when extended to an ATL representation, this extends to an irreducible lowest weight m representation, and we apply the classification of these described in the beginning of this section.

□

We can identify the points of the circles with characters of various weights, with each distinct circle corresponding to distinct weights. The interior of the interval $(-\delta,\delta)$ corresponds to weight 0 characters. We know now that all characters must be given by X_k, but we do not yet know that distinct points in X_k yield truly distinct characters on $\mathscr{A}_{k,k}$. They yield distinct representations for ATL, but independent characters might become the same when restricted to the tube algebra. In fact, we will show that they are distinct, but first we see how to evaluate characters on a special subset of our basis, namely elements in $\mathscr{A}_{k,k}$ of the form $x^k_{m,1}$.

Let $t \in (-\delta,\delta)$, and k even. Then we see that $t(x^k_{0,j}) = t^j B^k_{0,0}(t)$. This is non-zero for $t \in (-\delta,\delta)$. For $m > 0$, and ω the eigenvalue for a lowest weight m representation, we see that for $n \geq m$, $\omega(x^k_{n,1})g^{\omega}_{m,k} = x^k_{n,1}g^{\omega}_{m,k} = B^k_{m,n}(\omega)\omega(\rho_n)g^{\omega}_{m,k}$, where here, we identify $\rho_n \in \mathscr{A}_{n,n}$ and the ω as a character on $\mathscr{A}_{n,n}$. Thus to compute the value of $\omega(x^k_{n,1})$, we simply need to determine the value of $\omega(\rho_n)$. In pictures, we have to compute the scalar that pops out when we substitute TL diagrams in the bottom f_n of the picture



. If $n > m$, we see that there are precisely two diagrams which give non-zero contributions, a cup in the upper right hand corner, and a bottom cap at positions $\frac{n-m}{2}$ and $\frac{n+m}{2}$. The coefficients of these diagrams in f_n are given by $(-1)^{\frac{m+n}{2}}[\frac{n-m}{2}]/[n]$ and $(-1)^{\frac{m-n}{2}}[\frac{n+m}{2}]/[n]$ respectively. The first diagram gives an eigenvalue of ω^{-1} and the second gives and eigenvalue of ω, and thus we get $\omega(\rho_n) = (-1)^{\frac{m-n}{2}}\left(\omega^{-1}(-1)^n[\frac{n-m}{2}] + \omega[\frac{n+m}{2}]\right)/[n]$. If $n = m$, we simply get ω. We apply the same procedure for $m = 0$, which is even easier since there is only one TL diagram to evaluate.

We also notice that applying this same procedure to arbitrary basis diagrams, we see that an element of $\mathscr{A}_{k,k}$ evaluated at a character α will depend on α only as polynomial either in α and α^{-1} if $\alpha \in S^1$ or just in α if $\alpha \in (-\delta,\delta)$. We record these results in the following lemma, which expresses our knowledge of how to evaluate characters:

LEMMA IV.2.11. *Let $k > 0$.*

1. *For k even, $t \in (-\delta,\delta)$, k even, we have $t(x^k_{0,j}) = t^j B^k_{0,0}(t)$.*

2. *For k even, $t \in (-\delta,\delta)$, $t(x^k_{n,0}) = B^k_{0,n}(t)$. $t(x^k_{n,1}) = (-1)^{\frac{n}{2}} t \frac{[\frac{n}{2}]}{[n]} B^k_{0,n}(t)$.*

3. *For $\omega \in S^1$ of lowest weight $m > 0$, for $k, n \geq m$,*

$$\omega(x^k_{n,1}) = \frac{(-1)^{\frac{m-n}{2}}}{[n]} \left((-1)^n [\frac{n-m}{2}]\omega^{-1} + [\frac{n+m}{2}]\omega \right) B^k_{m,n}(\omega),$$

where here $[0] = 0$.

4. *If $\omega \in S^1$, then $\omega(x^k_{j,m}) \in \mathbb{C}[\omega, \omega^{-1}]$, and if $t \in (-\delta, \delta)$, then $t(x^k_{j,m}) \in \mathbb{C}[t]$.*

LEMMA IV.2.12. *For $\delta \geq 2$, and X_k as above, $\mathscr{A}_{k,k}$ separates the points of X_k.*

Proof. For each pair of distinct characters $\alpha_1, \alpha_2 \in X_k$, we must show that there exists $f \in \mathscr{A}_{k,k}$ such that $\alpha_1(f) \neq \alpha_2(f)$.

First consider the $k = 0$ case. Then $t(x^0_{0,1}) = t$ separates all points in $[-\delta, \delta]$.

Now suppose $k > 0$. If α_1 and α_2 correspond to different weights, assume without loss of generality that the weight of α_1 is strictly less than the weight of α_2. Then suppose the weight of α_1 is m. Then we pick the diagram $x^k_{m,0}$. Then from the above proposition, we have that $\alpha_1(x^k_{m,0}) = B^k_{m,m}(\alpha_1) \neq 0$, while $\alpha_2(x^k_{m,0}) = 0$ since $x^k_{m,0}$ has rank m. Thus we can separate characters of different weights, and only need to show that we can separate characters of the same weight.

Consider the case when $\delta > 2$, and k even.

Suppose $\alpha_1, \alpha_2 \in (-\delta, \delta)$. Then we have $B^k_{0,0}(\alpha_1), B^k_{0,0}(\alpha_2) \neq 0$, and thus if $\alpha_1(x^k_{0,0}) = B^k_{0,0}(\alpha_1) \neq B^k_{\alpha_2} = \alpha_2(x^k_{0,0})$ we are done. If $B^k_{0,0}(\alpha_1) = B^k_{0,0}(\alpha_2) \neq 0$, then $\alpha_1(x^k_{0,1}) = \alpha_1 B^k_{0,0}(\alpha_1) \neq \alpha_2 B^k_{0,0}(\alpha_2) = \alpha_2 B^k_{0,0}(\alpha_2)$. Thus we can separate the weight 0 characters with $\mathscr{A}_{k,k}$.

Now suppose $\alpha_1, \alpha_2 \in X_k$ are of the same weight $m > 0$ but $\alpha_1 \neq \alpha_2$. Then $\alpha_1(x^k_{m,0}) = B^k_{m,m}(\alpha_1)$, and $\alpha_2(x^k_{m,0}) = B^k_{m,m}(\alpha_2)$. If $B^k_{m,m}(\alpha_1) \neq B^k_{m,m}(\alpha_2)$ we are done. Suppose these are equal. They are not 0 by Proposition IV.2.4 (2). Then $\alpha_1(x^k_{m,1}) = \alpha_1 B^k_{m,m}(\alpha_1)$ while $\alpha_2(x^k_{m,1}) = \alpha_2 B^k_{m,m}(\alpha_2)$. Since $\alpha_1 \neq \alpha_2$ we are finished.

The other cases are the same. For $\delta = 2$, we simply remove the points in the domain where $B^k_{m,m} = 0$, and the above proof applies.

□

Now, we know that $C^*(\mathscr{A}_{k,k})$ will be a unital (f_k is the unit) abelian C*-algebra thus it must be isomorphic to the continuous functions on some compact Hausdorff space. Since the characters evaluated on $\mathscr{A}_{k,k}$ are simply polynomials in the parameters of X_k (Lemma IV.2.11 (4)), away from the "missing" points ($\pm\delta$, and when $\delta = 2$, the points corresponding to ± 1 on the even circles and $\pm i$ on the odd circles), the topology on the set of characters precisely agrees with the natural topology on the spaces. Let us now consider the case when $\delta > 2$. The only "missing" points are $t = \pm\delta$. In other words, since our character space is compact and the topology on X_k as characters agrees with the natural topology on $(-\delta, \delta)$, if we have a sequence of characters $t_n \subseteq (-\delta, \delta)$, such that $t_n \to \pm\delta$, this sequence must be converging to some other character in X_k. Thus to identify the topology on X_k as the space of characters, we must identify which character such a sequence t_n converges to. It must live in X_k since X_k contains all characters.

LEMMA IV.2.13. *Let $\delta > 2$, and let $k = 2n$ be even. Let ω_{-1} be the point $-1 \in S^1 \subseteq X_k$ corresponding to the weight 2 copy of S^1 and similarly, ω_1 the point in the same circle corresponding to 1. Then for any $f \in \mathscr{A}_{k,k}$, if $\{t_n\} \subseteq (-\delta, \delta)$ is a sequence such that $t_n \to \delta$, $t_n(f) \to \omega_{-1}(f)$. If $t_n \to -\delta$, then $t_n(f) \to \omega_1(f)$.*

Proof. First from the list of coefficients above $B^k_{0,0} \to 0$ as $t_j \to \pm\delta$, , and thus $t_n(x^k_{0,j}) = t^j B^k_{0,0} \to 0$ $t_j \to \pm\delta$, $t_j(x_l) \to 0$ for all l. Thus the limit of t_j must be some higher weight character. We see that

$$B^k_{0,2}(t_j) = \prod_{1 \le i \le \frac{k}{2}} \frac{[2i]^2 - t_j^2[i]^2}{[2i-1][2i]} \to \prod_{1 \le i \le \frac{k}{2}} \frac{[2i]^2 - [2]^2[i]^2}{[2i-1][2i]}$$

On the other hand using our formula for the B's and Lemma IV.2.2,

$$B^k_{2,2}(\pm 1) = \prod_{i=2}^{n} \frac{[2i]^2 - [i-1]^2 - [i+1]^2 - 2[i+1][i-1]}{[2i][2i-1]}$$

Using the fact that $[2][i] = [i+1] + [i-1]$, and comparing each term in the product with the same denominator, we see that the term in the limit of the t_j is $[2i] - [2]^2[i]^2 = [2i] - ([i+1] + [i-1])^2 = [2i] - [i+1]^2 - [i-1]^2 - 2[i+1][i-1]$, which is precisely the term in $B^k_{2,2}(\pm 1)$. Therefore we see that lim t_n must be a lowest weight 2 character, and it must be $\omega_{\pm}$. The problem is, we do not know which it is. To determine this, we notice that $\alpha(x^k_{2,1}) = \alpha B^k_{0,2}$.

Therefore, as $t_n \to \delta$, $t_n(x^k_{2,1}) \to -B^k_{0,2}(-1) = \omega_{-1}(x^k_{2,1})$. Since $x^k_{2,1}$ separates points, we see that $\lim_{t \to \delta} t = \omega_{-1}$. Similarly, $\lim_{t \to -\delta} t = \omega_1$.

□

LEMMA IV.2.14. *Let $k > 0$, $\delta = 2$.*

1. *Suppose k is even. Let $\omega_{\pm 1}$ be the characters on the weight k circle. If ω_n is a subset of the lowest weight m circle for some $m \leq k$ such that $\omega_n \to \pm 1$, then $\omega_n(f) \to \omega_{\pm(-1)^{\frac{k-m}{2}}}(f)$.*

2. *Let k be odd and $\omega_{\pm i}$ be the characters on the weight k circle corresponding to $\pm i$. If ω_n is a subset of a weight m circle for some $m \leq k$ such that $\omega_n \to \pm i$, then $\omega_n(f) \to \omega_{\mp(-1)^{\frac{m-2}{2}} i}$.*

Proof. If $\omega_n \to \pm 1$ by examining coefficients, we see that $B^k_{m,n}(\omega_n) \to 0$. Since this coefficient occurs in the evaluation of $\omega_n(x)$ for all diagrams of rank $< k$, we see that ω_n must be converging to a lowest weight k character. To determine which one, we note that $x^k_{k,1} = \rho_n$, and compute

$$\omega_n(x^k_{k,1}) = (-1)^{\frac{k-m}{2}} \frac{1}{k}\left((-1)^k \frac{k-m}{2}\omega_n^{-1} + \frac{k+m}{2}\omega_n\right).$$

If k is even, then as $\omega_n \to \pm 1$, $\omega_n(x^k_{k,1}) \to \pm(-1)^{\frac{k-m}{2}}$. Since $x^k_{k,1}$ separates lowest weight k representations, we are done.

If k is odd, then as $\omega_n \to \pm i$, $\omega_n(x^k_{k,1}) \to \mp(-1)^{\frac{k-m}{2}} i$. Since $x^k_{k,1}$ separates lowest weight k representations, we are done. □

Proof of Theorems IV.2.7 and IV.2.8 The above lemmas have identified the appropriate topology on the sets $X_k$and Y_k, and it agrees with the pictures we have drawn.

In particular, consider $\delta > 2$. For k odd, we don't even need the lemmas, since there are no "missing" points. For k even, we must identify the points $\pm\delta$ on the interval $[-\delta, \delta]$ with the points ∓ 1 respectively, on the weight 2 circle. Thus we have that $C^*(\mathscr{A}_{k,k})$ is an abelian C*-algebra whose spectrum is the compact Hausdorff space X_k .

Now assume $\delta = 2$. For k even, by the above lemma, the weight m circle for $m > 0$ will be glued on to the weight k circle at the points ± 1, and it alternates which endpoint goes to which endpoint on the circle as $\frac{k-m}{2}$ changes parity. We know by the above lemma that the interval is glued with its endpoints attached to the points ± 1 on the weight 2 circle which in turn is glued to the points ± 1 on the weight k circle, resulting in the space pictured as Y_k. For k odd, we glue the points $\pm i$ to the highest weight circle in an alternating fashion as described in the above lemma. Topologically, we obtain the space Y_k pictured □.

We conclude this section with a corollary of this analysis. This theorem (discussing a slightly different form of the $TLJ(\delta)$) is due to Popa -Vaes [47] and Brothier-Jones [5].

PROPOSITION IV.2.15. *The categories $TLJ(\delta)$ have the Haagerup property for all $\delta \geq 2$.*

Proof. Consider the admissible characters t, for $|t| \leq \delta$. If $t = h + h^{-1}$, then $t([k]) = [k+1]_h$. The claim is that the cp-multiplier $\hat{t} := \frac{t(\cdot)}{d(\cdot)} \in \ell^\infty(TLJ(\delta))$ is c_0. First consider the case $q \neq 1$ (or in other words $\delta > 2$). Note that $\hat{t}([k]) = \frac{[k+1]_h}{[k+1]_q} = \frac{q-q^{-1}}{h-h^{-1}} \frac{h^{k+1}-h^{-k-1}}{q^{k+1}-q^{-k-1}}$. Since $h < q$, the limit of this expression as k goes to infinity is clearly 0. If $q = 1$, then $TLJ(\delta)$ is well known to be amenable, since it is equivalent to Rep(SU(2)) (see, for example, [47]), hence has the Haagerup property.

□

IV.3 Quantum G_2 categories

In this section we prove that the rigid C*-tensor categories associated to G_2-type quantum groups have property (T). This section is based on our paper "Quantum G_2 categories have property (T)", [20].

There are many ways to describe rigid C*-tensor categories. As we have seen in the $TLJ(\delta)$ case, one of the most useful is the planar algebra approach introduced by Jones [22]. The idea is to use formal linear combinations of planar diagrams to represent morphisms in your category. These diagrams satisfy some linear dependences called skein relations in modern parlance.

The $(G_2)_q$ categories we describe are a particularly nice type of planar algebra called a trivalent category. These were introduced in their current form by Morrison, Peters, and Snyder [30]. Using dimension restrictions on morphism spaces as a notion of "small", they were able to classify the "smallest" examples. The $(G_2)_q$ categories appear in their classification list.

DEFINITION IV.3.1. ([30, Definition 2.4]) A *trivalent category* $\mathscr{C}$ is a non-degenerate, evaluable, pivotal category over $\mathbb{C}$, with a tensor generating object X satisfying dim $Mor(id, X) = 0$, dim $Mor(id, X \otimes X) = 1$, dim $Mor(id, X \otimes X \otimes X) = 1$, generated (as a planar algebra) by a trivalent vertex for X .

We summarize the basic properties of trivalent categories:

1. Objects in the category can be represented by $\mathbb{N} \cup \{0\}$, and correspond to tensor powers of a generating object X.

2. $Mor(k, m)$ is the complex linear span of isotopy classes of planar trivalent graphs embedded in a rectangle, with m boundary points on the top of the rectangle, k boundary points on the bottom, and no boundary points on the sides of the rectangle. These diagrams are subject to *skein relations*, which are linear dependences among the trivalent graphs which make $Mor(k, m)$ finite dimensional.

(Note: We consider graphs with no vertices at all, namely line segments attached to the boundaries, as trivalent graphs)

3. $Mor(0,0) \cong \mathbb{C}$. In other words, our skein relations reduce every closed trivalent graph to a scalar multiple of the empty trivalent graph. Identifying the empty graph with $1 \in \mathbb{C}$, this means we have associated to every closed trivalent graph a complex number.

4. Composition of morphisms is vertical stacking of rectangles.

5. Tensor product on objects is addition of natural numbers, on morphisms it is horizontal stacking of rectangles.

6. Duality is given by rotation by π or $-\pi$ (these manifestly agree in our setting).

7. The linear functional $Tr : Mor(k,k) \to \mathbb{C}$ given by connecting the top strings of the rectangle to the bottom is non-degenerate.

DEFINITION IV.3.2. A trivalent category is a *C*-trivalent category* if the maps $* : Mor(k,m) \to Mor(m,k)$ given by reflecting graphs across a horizontal line and conjugating complex coefficients are well-defined modulo the skein relations, and $Tr(x^*x) \geq 0$ for every $x \in Mor(k,m)$, and $k,m \in \mathbb{N}\bigcup\{0\}$.

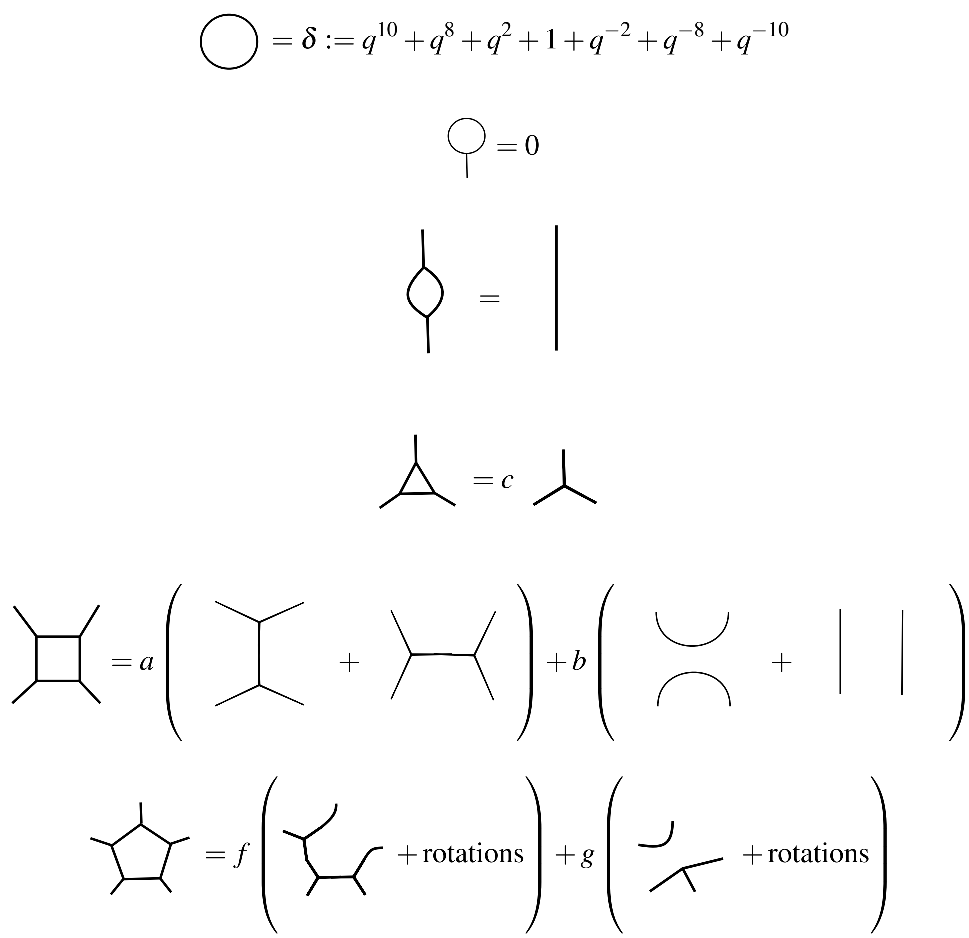
From a C*-trivalent category, we can construct a rigid C*-tensor category as follows: First, it can be shown that a category satisfying all these conditions has a negligible category ideal, generated by diagrams with $Tr(x^*x)$=0. Quotienting by this produces a trivalent category with condition (7) replaced by $Tr(x^*x) > 0$. Next, we take the projection completion. Objects in this category will be projections living in some $Mor(k,k)$. For two projections $P \in Mor(k,k),\ Q \in Mor(m,m)$, $Mor(P,Q) = \{f \in Mor(k,m)\ :\ QfP = f\}$. Now we formally add direct sums to the category. The resulting category will have objects direct sums of projections, and morphisms matrices of the morphisms between projections. The result is a rigid C*-tensor category, which we also call $\mathscr{C}$.

Notice the duality map we have defined is automatically pivotal. Also the strict tensor identity *id* is given by the empty diagram. Another consequence of the definitions is that the generating object X is symmetrically self-dual (see [7, Definition 2.10]).

The $(G_2)_q$ trivalent categories which we describe below were introduced by Kuperberg in [26] and [27]. Kuperberg showed that these categories are equivalent to the category of (type 1) finite dimensional representations of the Drinfeld-Jimbo quantum groups $U_q(\mathfrak{g}_2)$.

To define a trivalent category, it suffices to specify a set of skein relations. In general it is a difficult problem to determine whether a set of skein relations produces a trivalent category. In particular, one has to verify that your relations are consistent and evaluable. Otherwise you may end up with $Mor(0,0)$ being 0 dimensional, or with infinite dimensional morphism spaces. Kuperberg showed the following skein relations are indeed consistent and evaluable, resulting in a (non-zero) trivalent category. The skein theory we present for $(G_2)_q$ can be found in [30, Definition 5.21]. It differs from Kuperberg's description in two ways: The trivalent vertex is normalized, and the q^2 here is Kuperberg's q.

DEFINITION IV.3.3. $(G_2)_q$ for strictly positive q is the trivalent category defined by the following skein relations:



Where

$$a = \frac{q^2+q^{-2}}{(q+1+q^{-1})(q-1+q^{-1})(q^4+q^{-4})}$$

$$b = \frac{1}{(q+1+q^{-1})(q-1+q^{-1})(q^4+q^{-4})^2}$$

$$c = -\frac{q^2-1+q^{-2}}{q^4+q^{-4}}$$

$$f = -\frac{1}{(q+1+q^{-1})(q-1+q^{-1})(q^4+q^{-4})}$$

$$g = -\frac{1}{(q+1+q^{-1})^2(q-1+q^{-1})^2(q^4+q^{-4})^2}$$

[26], [27], and [30] shows this category is actually spherical. The duality maps $\cup$ and $\cap$ provide standard solutions for the simple object (minimal projection) spanning $Mor(1,1)$. This is the object X, which tensor generates our category.

Kuperberg showed that this category is isomorphic (not just equivalent) to the spherical category generated by the 7-dimensional fundamental representation (which we also call X) of $U_q(\mathfrak{g}_2)$ (see [27, Theorem 5.1]). A single string corresponds to the object X in $Rep(U_q(\mathfrak{g}_2))$, hence the natural number k an an object in $(G_2)_q$ corresponds to the object $X^{\otimes k}$ in $Rep(U_q(\mathfrak{g}_2))$. Since X tensor generates $Rep(U_q(\mathfrak{g}_2))$, we have the whole category appearing.

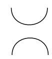
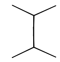
In both Kuperberg's work and Morrison, Peters, and Snyder's no $*$-structure is considered. However, $U_q(\mathfrak{g}_2)$ has a natural $*$-structure for positive $q \neq 1$ (along with all Drinfeld-Jimbo quantum groups), and it is shown, for example, in [33], Chapter 2.4, that the category of finite dimensional $*$-representations is a rigid C*-tensor category. Every type 1 finite dimensional representation of $U_q(\mathfrak{g}_2)$ for $q > 0$ is unitarizable and thus the rigid C*-tensor category of finite dimensional unitary type 1 representations $\mathcal{C}_q(\mathfrak{g}_2)$ is monoidally equivalent to $Rep(U_q(\mathfrak{g}_2))$. By [20, Proposition 5.1], the C*-category structure from $\mathcal{C}_q(\mathfrak{g}_2)$ transports to $(G_2)_q$ as the trivalent $*$-structure defined by reflecting a diagram across a horizontal line. Thus $(G_2)_q$ is a C*-trivalent category.

To prove $(G_2)_q$ has property (T), we need to know the structure of $Mor(2,2)$. $Mor(2,2)$ is a 4-dimensional abelian C*-algebra. To determine the minimal projections, we set

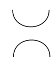
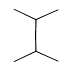
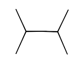
$$\xi := \sqrt{\delta^2c^4+2\delta(c^4-2c^3-c^2+4c+2)+(c^2-2c-1)^2} = \frac{(1+q^2)^2(1-q^2+q^6-q^8+q^{10}-q^{14}+q^{16})}{q^6(1+q^8)}.$$

This is manifestly non-zero for $q \neq 1$ and $q > 0$.

PROPOSITION IV.3.4. *([30, Proposition 4.16]) The minimal idempotents in the finite dimensional abelian algebra $Mor(2,2)$ are given by*

$\frac{1}{\delta}$  , 

and the two idempotents

$y_{\pm} = \frac{-(\delta+1)c^2 \pm \xi + 1}{\pm 2\xi}$  $+ \frac{\delta(c^2-2c-2) \mp \xi + c^2 - 2c - 1}{\pm 2\delta\xi}$  $- \frac{\delta(c+2)c \pm \xi + c^2 + 1}{\pm 2\xi}$  $+ \frac{\delta c + \delta + c}{\pm \xi}$  .

In our setting, we see that the idempotents are in fact projections, since our basis is self-adjoint and all coefficients are real numbers. These projections correspond to simple objects in the rigid C*-tensor category underlying $(G_2)_q$.

By Kuperberg's isomorphism, the fusion algebra of the underlying projection category of $(G_2)_q$ is isomorphic to the fusion algebra of the category $\mathcal{C}q(\mathfrak{g}_2)$ for positive $q \neq 1$, which in turn is isomorphic to the complexification of the representation ring $R(G_2)$. It is well known that for compact, simply connected, simple Lie groups G, the representation ring $R(G)$ is isomorphic to the ring of polynomials in the fundamental representations. For a specific reference, see [1, Theorem 6.41]. This implies the fusion algebra of $(G_2)_q$ is the (commutative) complex polynomial algebra in 2 self-adjoint variables $\mathbb{C}[Z_1, Z_2]$, where Z_1 and Z_2 correspond to the 14 and 7 dimensional fundamental representations of the quantum group $U_q(\mathfrak{g}_2)$ respectively. (Note that self-adjointness of the variables follows from self-duality of the corresponding representations).

The fusion graph with respect to X is given by the following figure

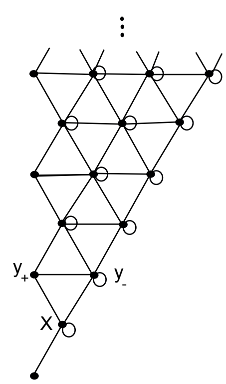


Figure IV.1: Fusion graph for $(G_2)_q$

Here, the vertex at the bottom corresponds to the identity, the next highest vertex corresponds to X itself, etc.

Now we consider the tube algebra $\mathscr{A}$ of the categories $(G_2)_q$ for some positive $q \neq 1$. In our analysis all such q will yield the same results, so we supress the dependence of the tube algebra $\mathscr{A}$ on q for notational convenience. Recall simple objects in the category correspond to minimal projections in some $Mor(k,k)$. Let us choose our set of representatives of projections so that it contains the empty diagram id, the single string X, and the two projections y_+ and y_-, representing their equivalence classes. For $x \in Mor(k,k)$, we let $i : Mor(k,k) \to Mor(k \otimes id, id \otimes k)$ be the canonical identification. Then define

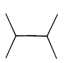
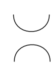
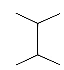
$$\Delta(x) := \Psi(i(x)) \in \mathscr{A}_{0,0}.$$

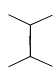
where Ψ is defined in the discussion of the tube algebra. In our setting we see that the map in Proposition 2.1 is defined by applying Δ to a projection.

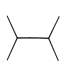
Translating the fusion algebra description to our setting, the variable Z_2 is represented by $\Delta(X)$, while Z_1 is represented by the projection $\Delta(y_+) \in Mor(2,2)$. We see that

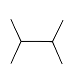
$$\mathscr{A}_{0,0} \cong \mathbb{C}[\Delta(y_+), \Delta(X)].$$

Going back to our expression for y_+, we see that $\frac{\delta c+\delta+c}{\xi} = \frac{(1+q^2+q^4)(1+q^8)}{q^4(1+q^2)^2} \neq 0$. Thus

 $= \frac{q^4(1+q^2)^2}{(1+q^2+q^4)(1+q^8)} \Big((y_+) - \frac{-(\delta+1)c^2+\xi+1}{2\xi}$  $- \frac{\delta(c^2-2c-2)-\xi+c^2-2c-1}{2\delta\xi}$  $+ \frac{\delta(c+2)c+\xi+c^2+1}{2\xi}$  $\Big).$

Then since $\Delta($  $) = \Delta(X)$, we have

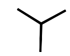
$\Delta($  $) = \frac{q^4(1+q^2)^2}{(1+q^2+q^4)(1+q^8)} \left(\Delta(y_+) - \frac{-(\delta+1)c^2+\xi+1}{2\xi} \Delta(X)^2 - \frac{\delta(c^2-2c-2)-\xi+c^2-2c-1}{2\xi} 1 + \frac{\delta(c+2)c+\xi+c^2+1}{2\xi} \Delta(X) \right).$

We denote $H :=$ . Since our polynomial expression for $\Delta(H)$ is linear in $\Delta(y_+)$ and the terms with powers of $\Delta(X)$ contain no $\Delta(y_+)$ terms, we can perform an invertible transformation implementing a change of basis, and write an arbitrary polynomial in $\Delta(y_+)$ and $\Delta(X)$ as a polynomial in $\Delta(H)$ and $\Delta(X)$, so that

$$\mathscr{A}_{0,0} \cong \mathbb{C}[\Delta(H), \Delta(X)].$$

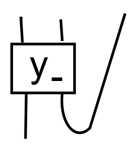
Therefore irreducible representations of $\mathscr{A}_{0,0}$ are 1-dimensional, and they are defined by assigning numbers to $\Delta(H)$ and $\Delta(X)$. Let us denote by α the value assigned to $\Delta(H)$ and t the value assigned to $\Delta(X)$ in our 1-dimensional representation. Let $\gamma_{\alpha,t} : \mathscr{A}_{0,0} \to \mathbb{C}$ denote the 1-dimensional representation viewed as a functional, given by evaluating polynomials in $\mathbb{C}[\Delta(H),\Delta(X)]$ at the point (α,t).

The key point is that while arbitrary values of α and t determine a representation of $\mathscr{A}_{0,0}$, not all are annular states (hence admissible representations). Recall that $\gamma_{\alpha,t}$ is admissible if and only if $\gamma_{\alpha,t}(x^{\#} \cdot x) \geq 0$ for all $x \in \mathscr{A}_{0,k}$ and for all $k \in \mathrm{Irr}((G_2)_q)$.

As a first restriction, for our representation to be admissible, $t \in \mathbb{R}$ since the object corresponding to a single string is self-dual and our representation must be a $*$-representation. We also must have $\alpha \geq 0$, since $\Delta(H) = T^{\#} \cdot T$, where $T \in Mor(X \otimes id, X \otimes X) \subseteq \mathscr{A}_{0,X}^{X}$ is given by the trivalent vertex $T :=$ 

We know also that $|t| \leq \delta$ by Lemma III.1.4. Here δ is the value of the closed circle defined in terms of q in the description of the skein theory. This restricts the possible one dimensional admissible representations to some subset $Z \subseteq \{(\alpha,t) \subseteq \mathbb{R}^2 \ : \ \alpha \geq 0,\ t \in [-\delta,\delta]\}$.

Since the fusion algebra is isomorphic to the polynomial algebra in two self adjoint variables, and irreducible representations correspond to evaluation at points $Z \subseteq \mathbb{R}^2$, the weak-$*$ topology on Z as linear functionals on $C^*((G_2)_q)$ agrees with the (subspace) topology on the plane. The trivial representation corresponds to the point $(0,\delta)$. We will show that for positive $q \neq 1$, there is a neighborhood of the point $(0,\delta)$ in the rectangle $\mathbb{R}^+ \times [-\delta,\delta]$ such that the functional $\gamma_{\alpha,t}$ is not an annular state.

To see this, let $s :=$ 

$\in Mor(X \otimes id,\ y_- \otimes X)$. We view $s \in \mathscr{A}_{0,y_-}^{X} \subset \mathscr{A}$.

For each pair (α,t), define the function $f(\alpha,t) := \gamma_{\alpha,t}(s^{\#} \cdot s)$. This can be directly computed from the representation of y_- in terms of our planar algebra basis, and we obtain

$$f(\alpha,t) = \delta\frac{-(\delta+1)c^2-\xi+1}{-2\xi} + t^2\frac{\delta(c^2-2c-2)+\xi+c^2-2c-1}{-2\delta\xi} - \alpha\frac{\delta(c+2)c-\xi+c^2+1}{-2\xi} + t\frac{\delta c+\delta+c}{-\xi}.$$

By construction, if the functional corresponding to (α,t) is an annular state, $f(\alpha,t)$ must be non-negative.

PROPOSITION IV.3.5. *For all positive $q \neq 1$, $(G_2)_q$ has property (T).*

Proof. Since y_- is a minimal projection in $Mor(2,2)$ and is not equivalent to id in $(G_2)_q$, $f(0,\delta) = 0$. This can also be seen by direct computation. Let $v := (x,y) \in \mathbb{R}^2$ be a non-zero vector in the fourth quadrant of the plane (including the axes), or in other words $x \geq 0$ and $y \leq 0$, but $(x,y) \neq (0,0)$. We wish to show that $f(x,\delta+y) < 0$ for sufficiently small $||v||$. This will demonstrate that in a neighborhood of $(0,\delta)$ in

$\mathbb{R}^+ \times [-\delta, \delta]$, the function $f(\alpha, t)$ will be strictly negative, hence the representation corresponding to $\gamma_{\alpha,t}$ is not admissible.

We see that for positive $q \neq 1$,

$$\frac{\partial f}{\partial \alpha}|_{(0,\delta)} = \frac{\delta(c+2)c - \xi + c^2 + 1}{2\xi} = -\frac{1+q^2+2q^4+q^6+q^8}{(q+q^3)^2} < 0.$$

This is always strictly negative (for $q \neq 0$). We compute

$$\frac{\partial f}{\partial t}|_{(0,\delta)} = \frac{(\delta+1)(c^2-c-1)}{-\xi} - 1 = \frac{(-1+q^2)^2(1+q^2+q^4)}{q^4} > 0$$

This expression is strictly positive for all $q \neq 1, q \neq 0$. Therefore we have that the directional derivative $\frac{\partial f}{\partial v}|_{(0,\delta)} < 0$ for v in the prescribed range. We remark that for $q = 1$, $\frac{\partial f}{\partial t}|_{(0,\delta)} = 0$, hence this part of our proof breaks down as expected, since $Rep(G_2)$ is amenable.

Letting B denote the compact set of unit vectors in the fourth quadrant, since $\frac{\partial f}{\partial v}|_{(0,\delta)}$ is a continuous function of v, there exists some $M < 0$ such that $\frac{\partial f}{\partial v}|_{(0,\delta)} \leq M < 0$ for $v \in B$.

Now, it is straightforward to compute

$$\frac{\partial^2 f}{\partial^2 t}|_{(0,\delta)} = \frac{2q^6(1+q^2+q^4)}{(1+q^2)^2(1+q^2+q^4+q^6+q^8+q^{10}+q^{12})} > 0,$$

and it is easy to see that all other second order partial derivatives are 0, and all higher order derivatives with respect to both variables are 0. Then by Taylor's theorem, we have

$$f(x, \delta+y) = x\frac{\partial f}{\partial \alpha}|_{(0,\delta)} + y\frac{\partial f}{\partial t}|_{(0,\delta)} + \frac{y^2}{2}\frac{\partial^2 f}{\partial^2 t}|_{(0,\delta)}$$

for arbitrary $v = (x,y)$ in the fourth quadrant. Let $v' = \frac{1}{\|v\|}v$. Since $y^2 \leq \|v\|^2$, setting $\lambda := \frac{1}{2}\frac{\partial^2 f}{\partial^2 t}|_{(0,\delta)}$ gives

$$f(x, \delta+y) = \|v\|\frac{\partial f}{\partial v'}|_{(0,\delta)} + y^2\lambda \leq \|v\|M + \|v\|^2\lambda = \|v\|(M + \|v\|\lambda).$$

If we set $\varepsilon = \frac{|M|}{\lambda}$, then since $M < 0$, for $0 < \|v\| < \varepsilon$, we see that $f(x, y+\delta) < 0$. Therefore $(G_2)_q$ has property (T) for positive $q \neq 1$. □

CHAPTER V

CONCLUDING REMARKS

As mentioned above, most of the research in this dissertation has been published in the papers [14] and [20]. Subsequent to their publication several authors have studied and applied the tube algebra and its representation theory.

First, in [36], the Neshveyev and Yamashita show that if $\mathbb{G}$ is a compact quantum group and $\mathscr{C} \cong Rep(\mathbb{G})$, then the Drinfeld double of the discrete dual $D(\hat{\mathbb{G}})$ is a full annular algebra. This explains the earlier result of Popa and Vaes that properties of $Rep(\mathbb{G})$ are the same as *central* properties of $\hat{\mathbb{G}}$, studied in [11] and [2]. They also show that if two categories are weakly morita equivalent, their tube algebra are strongly morita equivalent, hence approximation properties are the same for morita equivalent categories.

In [46], Popa, Shlyakhtenko and Vaes define generalized tube algebras for an arbitrary irreducibe quasi-regular inclusion of Von Neumann algebras. Using this tube algebra, they define a cohomology theory and L^2-Betti numbers for the inclusion, as a sort of Hoschild cohomology of this tube algebra. They also show the canonical bijection between $Rep(\mathscr{A})$ and the category $Z(\mathrm{Ind}-\mathscr{C})$ introduced in [35].

Finally, in [3], Arano and Vaes use the tube algebra to study analytic properties of categories associated to a totally disconnected, locally compact group.

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