A Measure Theoretic Approach for the Recovery of Remanent Magnetizations

By

Cristóbal Villalobos Guillén

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Approved:

Douglas P. Hardin, Ph.D.

Akram Aldroubi, Ph.D.

Marcelo Disconzi, Ph.D.

Guilherme A. R. Gualda, Ph.D.

Edward Saff, Ph.D.

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Chapter 1

Introduction

The study of *(remanent) magnetizations* of ancient rocks has proven to be of great importance for paleogeology, in particular for paleomagnetism. Examples include igneous rocks formed by the cooling of lava after a volcanic eruption, which record the intensity and orientation of the Magnetic Field of the earth [1]; rocks from Mars or the Moon which can be used to study past Magnetic Fields generated by such bodies [2, 3, 4]; and rocks from meteorites used for studying the Magnetic Field of the early stages of the solar system [4].

Our work is motivated by the problem of recovering the magnetization $\mathbf{M}$ of a rock sample from a given set of measurements for $\mathbf{B}$, the magnetic field it generates. Here we will present a method for this recovery in the theoretical case when $\mathbf{M}$ is modeled by a vector valued measure and the given measurements are enough to determine the field at any point outside of the sample. We will also give sufficient conditions on the sample and the region where the measurements are taken that would ensure the convergence of our method to the original magnetization of the sample. Even though this document deals with idealized objects, computer reconstructions have given promising results strongly suggesting that our method is applicable for real-world recovery of Magnetizations. On Section 3.2 we will present some of this computations. Most of the work presented here comes from two research papers, one of which has been sent for publication, [5], and another that is still in development. More work is needed in order to fully understand the connection between theory and application, however it is left for future research.

Recall that the magnetization of an object is a density for the *magnetic moment*, a vector which in turn is a measurement for the strength and direction of objects that generate magnetic fields. $\mathbf{M}$ is usually represented by a vector field (a vector valued function), however we are interested in working with physical idealizations consisting of magnetized regions with no volume, such as points, lines or surfaces, whose densities cannot be represented with vector fields. The alternative we consider in this document is to use instead $\mathbb{R}^3$-valued measures for modeling magnetizations since these mathematical objects are capable of representing the aforementioned densities (for example using a vector times a Dirac delta to represent a point dipole). In particular Borel $\mathbb{R}^3$-valued measures are both enough for representing the objects we are interested in and form a Banach space with the total variation norm for measures (defined in (1.9)). This norm is of particular interest for us since we use its finite dimensional analog for the recovery of magnetizations from real-world data via a form of the group LASSO regularization method.

From the equations relating $\mathbf{B}$ and $\mathbf{M}$ it follows that the former depends on the latter by a linear operator (see (1.5)) with non-trivial kernel (e.g., see Example 1), which makes the recovery of $\mathbf{M}$ an ill-posed inverse problem. However, we show in this document that under certain assumptions on a set S containing our samples and the region Q in which we take the measurements, it is possible to recover $\mathbf{M}$ provided it is either *piecewise unidirectional* (i.e. each connected component is magnetized on a single direction) or *sparse* (in the sense of having a *purely 1-unrectifiable* support). The notion of purely 1-unrectifiable set is classical from geometric measure theory, and intuitively it means that the set contains no arc. What we show here is that a measure whose support has this property is the unique element of least total variation in its coset modulo the null space of the forward operator, at least when Q and S satisfy certain hypotheses. Whether the property of having purely 1-unrectifiable support qualifies a measure as being "sparse" is debatable: for instance the support could still disconnect the space (like the Koch curve does in 2-D). Nevertheless, it comprises standard notions of sparsity, such as being a finite sum of Dirac masses, which is why we consider purely 1-unrectifiability of the support as a generalized notion of sparsity in this context.

The conditions we put on S and Q are a generalization of the case when each is a rectangular subset of one of two parallel planes, one on each, as it is studied on [6, 7, 8]. More precisely, the specific conditions that we put on S, abbreviated by saying it is a *slender* set, are that it is a closed set of Lebesgue measure zero, the complement of which has no component of finite measure.

In the last chapter we will focus on the ideal case where the magnetized sample is contained in a subset of the horizontal plane. For this case we will show that all magnetizations which do not generate a magnetic field can be decomposed as a superposition of loops (see Section 3.1). The findings presented in this chapter rely on the theory of functions of Bounded Variation and sets of finite perimeter and give a characterization for magnetizations that do not generate a magnetic field.

1.1 Physical problem

The original motivation of the presented work was to look for a mathematical framework for measurements obtained from a scanning magnetic microscope (SMM) such as the instrument used by E.A. Lima and B.P. Weiss of the MIT Department of Earth, Atmospheric, and Planetary Sciences [6, 9]. The SMM uses an ultrasensitive magnetometer called a Super conducting Quantum Interference Device (SQUID) to measure one component of this magnetic flux density at a rectangular grid of points in a plane a certain distance above the rock sample.

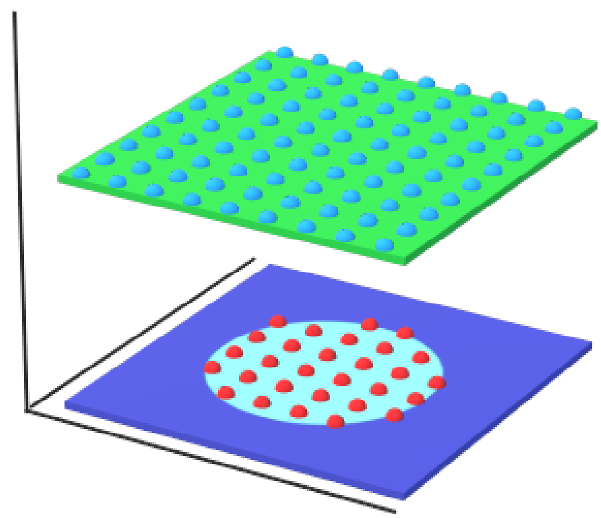

Figure 1: Setup for the SQUID.

We will now present a review of the quantities that we will model and the equations governing them (see for example [10]). Recall that the *magnetic field* $\mathbf{B}$ (classically referred to as *magnetic flux density*) is a vector field used to calculate the effect of electric currents or magnetized materials over their environment. The *magnetization* as we mentioned above represents the density of *(magnetic) net moment*.

In order to relate $\mathbf{B}$ and $\mathbf{M}$ we will use another vector field $\mathbf{H}$ called the *Magnetic Intensity* (but classically referred to as the *Magnetic Field*) which can be defined as

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}, \tag{1.1}$$

where $\mu_0 = 4\pi \times 10^{-7} Hm^{-1}$ which is called *the magnetic constant* or *the vacuum permeability.*

We are working with the Magnetization generated by a permanent Magnet, which means that there are no time dependent quantities and no *external current density.* Thus *Maxwell's equations* reduce to

$$\begin{aligned} \nabla \times \mathbf{H} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{1.2}$$

Since $\mathbf{H}$ is curl free it can then be expressed as the gradient of the *Magnetic (Scalar) Potential*,

$$\mathbf{H} = -\nabla\Phi,$$

and since $\mathbf{B}$ is divergence free, then Φ and $\mathbf{M}$ are related with the following Poisson's equation:

$$\Delta\Phi = \nabla \cdot \mathbf{M}. \tag{1.3}$$

Therefore, for a given $\mathbf{M}$, $\mathbf{B}$ is determined up to the gradient of a harmonic function. For finite magnetizations the magnetic field vanishes at infinity which, together with Liouville's theorem, implies that this gradient is zero and then the harmonic function itself is just a constant. The value of this constant is usually taken to be zero as well since it is desirable for points infinitely far away from the source to have zero potential. Hence the Magnetic Potential Φ generated by a magnetization distribution $\mathbf{M}$ at a point $\mathbf{x}$ not in the support of $\mathbf{M}$ is given by

$$\Phi(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \cdot \mathbf{M}(\mathbf{y})\, \mathrm{d}\mathbf{y} \tag{1.4}$$

and, for any such $\mathbf{x}$, the magnetic field $\mathbf{B}$ is equal to

$$\mathbf{B}(\mathbf{x}) = -\frac{\mu_0}{4\pi} \left(\int \frac{\mathbf{M}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} \mathrm{d}\mathbf{y} - 3 \int (\mathbf{x} - \mathbf{y}) \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{M}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^5} \mathrm{d}\mathbf{y} \right). \tag{1.5}$$

Here $\mathrm{d}\mathbf{y}$ denotes integration by the Lebesgue measure on $\mathbb{R}^3$ and, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, we use $\mathbf{x} \cdot \mathbf{y}$ and $|\mathbf{x}|$ for the Euclidean scalar product and norm respectively.

As we mentioned earlier, we will model magnetizations by $\mathbb{R}^3$-valued measures. Parting from this choice we will use equations (1.4) and (1.5) to define Φ and $\mathbf{B}$ outside of the sample. Later we will show that such definitions can be extended to distributions on the whole space that still satisfy (1.2) and (1.3).

1.2 Mathematical context of our work

This document investigates a connection between geometric measure theory and regularization theory for inverse problems. This connection essentially rests on the structure of the null-space of the forward operator: in fact, the conditions on Q and S set up in this work are designed so that this null-space consists exactly of divergence-free measures. They generalize in the case of $\mathbb{R}^3$-valued measures those given in [6] for more general distributions. This characterization of the null-space is central to the present approach, and requires restrictions on the global geometry of the situation which are implemented by the conditions we put on S and Q. Dwelling on them, we rely on classical tools from geometric measure theory, and on material from [11] and [12], to proceed with the proof of our main result, namely the minimality of the total variation of a sparse measure in its coset modulo the null-space. More general situations, notably the case where S is a closed surface or where it has positive Lebesgue measure in $\mathbb{R}^3$ (and thus is not slender), are left here for further research.

After early studies [13, 14] and the seminal work in [15, 16, 17, 18], approximately solving underdetermined systems of linear equations in $\mathbb{R}^n$ by minimizing the residuals while penalizing the l^1-norm has undergone a success story in identification. In fact, under appropriate assumptions on the matrix of the system, this kind of approximation favors the recovery of sparse solutions, *i.e.* solutions having a large number of zero components, when they exist. This has resulted in the theory of compressed sensing, which shows by and large that a sparse signal can be recovered from much less linear observations than is *a priori* needed, see for example [19] and the bibliography therein.

In recent years, natural analogs in infinite-dimensional settings have been investigated by several authors, but then the situation is much less understood. A Tikhonov-like regularization theory was developed in [20, 21, 22] for linear equations whose unknown is a (possibly $\mathbb{R}^n$-valued) measure, by minimizing the residuals while penalizing the total variation. As expected from the non-reflexive character of spaces of measures, consistency estimates generally hold in a rather weak sense, such as weak-$*$ convergence of subsequences to solutions of minimum total variation, or convergence in the Bregman distance when the so-called source condition holds. Algorithms and proofs typically rely on Fenchel duality, and reference [22] contains an extension of the soft thresholding algorithm to the case where the unknown gets parametrized as a finite linear combination of Dirac masses, whose location no longer lies on a fixed grid in contrast with the discrete case. References [23, 24], which deal with inverse source problems for elliptic operators, dwell on the same circle of ideas but suggest a different thresholding method, connected with a Newton step, or else a finite element discretization of the equation having a linear combination of Dirac masses amongst its solutions. These methods yield contructive algorithms to approximate a solution of minimum total variation to the initial equation by a sequence of discrete measures, which is always possible in theory since these are weak-$*$ dense in the space of measures supported on an open subset of $\mathbb{R}^n$. To obtain an asymptotic recovery result, in the weak-$*$ sense as the regularizing parameter goes to zero, it remains to identify conditions on a measure ensuring that it is the unique element of least total variation in its coset modulo the null-space of the forward operator.

1.3 Statement of problem and overview of results

Let us first describe in some detail the inverse potential problem in divergence form for $\mathbb{R}^3$-valued measures. Without loss of generality, we consider the issue of recovering a magnetization distribution from a collection of measurements of the magnetic field the magnetization generates. For a closed subset $S \subset \mathbb{R}^3$, let $\mathcal{M}(S)$ denote the space of finite signed Borel measures on $\mathbb{R}^3$ whose support lies in S. We model *magnetization distributions* supported in S as $\mathbb{R}^3$-valued measures $\mathbf{M} \in \mathcal{M}(S)^3$. Hereafter, we usually call a member of $\mathcal{M}(S)^3$ a magnetization supported on S, as this terminology is suggestive of the problems we address.

We will show in Lemma 2.1 that the following distribution is well defined. For a magnetization $\mathbf{M}$ we will define $\Phi(\mathbf{M}) \in L_{loc}(\mathbb{R}^3)$, the **scalar magnetic potential of M** as the unique distribution that satisfies

$$\Delta\Phi = \operatorname{div}\mathbf{M} \tag{1.6}$$

and for any point $\mathbf{x}$ not in the support of $\mathbf{M}$

$$\Phi(\mathbf{M})(\mathbf{x}) = \int (\operatorname{grad}\Gamma)(\mathbf{x}-\mathbf{y})\cdot \mathrm{d}\mathbf{M}(\mathbf{y}) = \frac{1}{4\pi}\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\cdot \mathrm{d}\mathbf{M}(\mathbf{y}), \tag{1.7}$$

where $\Gamma(\mathbf{x}) := -1/(4\pi|\mathbf{x}|)$ is the Newtonian kernel while grad denotes the gradient. Then we define the magnetic field $\mathbf{B}(\mathbf{M})$ generated by $\mathbf{M}$ as

$$\mathbf{B}(\mathbf{M}) := \mu_0 \left(\mathbf{M} - \operatorname{grad} \Phi(\mathbf{M})\right). \tag{1.8}$$

Remark 1.1. *Immediately we obtain the following:*

1. $\Phi(\mathbf{M})$ *and the components of* $\mathbf{B}(\mathbf{M})$ *are harmonic functions on* $\mathbb{R}^3 \setminus S$.

2. *For* $\mathbf{x} \in \mathbb{R}^3 \setminus S$

$$\begin{aligned}\mathbf{B}(\mathbf{M})(\mathbf{x}) &= -\mu_0 \operatorname{grad} \Phi(\mathbf{M})(\mathbf{x}) = -\frac{\mu_0}{4\pi} \operatorname{grad} \int (\operatorname{grad} \Gamma)(\mathbf{x}-\mathbf{y}) \cdot \mathrm{d}\mathbf{M}(\mathbf{y}) \\ &= -\frac{\mu_0}{4\pi} \left(\int \frac{1}{|\mathbf{x}-\mathbf{y}|^3} \, \mathrm{d}\mathbf{M}(\mathbf{y}) - 3 \int (\mathbf{x}-\mathbf{y}) \frac{(\mathbf{x}-\mathbf{y}) \cdot \mathrm{d}\mathbf{M}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^5} \right)\end{aligned}$$

where the last equality comes from the smoothness of Γ *on* $\mathbb{R}^3 \setminus \{0\}$ *and the equivalence*

$$\operatorname{grad}_{\mathbf{x}} \left(\operatorname{grad}_{\mathbf{y}} \frac{1}{|\mathbf{x}-\mathbf{y}|} \cdot \mathbf{a} \right) = \frac{\mathbf{a}}{|\mathbf{x}-\mathbf{y}|^3} - 3(\mathbf{x}-\mathbf{y}) \frac{(\mathbf{x}-\mathbf{y}) \cdot \mathbf{a}}{|\mathbf{x}-\mathbf{y}|^5},$$

for a fixed $\mathbf{a} \in \mathbb{R}^{\mathbf{3}}$ *(here* $\operatorname{grad}_{\mathbf{x}}$ *denotes the gradient with respect to the variable* $\mathbf{x}$*).*

3. *The difference of any two distributions satisfying* (1.6) *must be harmonic which implies that* $\Phi(\mathbf{M})$ *must be unique whenever it is well defined.*

4. *Finally,* (1.8) *implies that* $\mathbf{B}(\mathbf{M})$ *is divergence free.*

The mapping $\mathbf{M} \to \mathbf{B}(\mathbf{M})$ is generally not injective. We say that magnetizations $\mathbf{M}, \mathbf{N} \in \mathcal{M}(S)^3$ are **S-equivalent** if $\mathbf{B}(\mathbf{M})$ and $\mathbf{B}(\mathbf{N})$ agree on $\mathbb{R}^3 \setminus S$ in which case we write

$$\mathbf{M} \overset{S}{\equiv} \mathbf{N}.$$

A magnetization $\mathbf{M}$ is said to be **S-silent** (or **silent in** $\mathbb{R}^3 \setminus S$) if it is S-equivalent to the zero magnetization; i.e., if $\mathbf{B}(\mathbf{M})$ vanishes on $\mathbb{R}^3 \setminus S$. It is suggestive from (1.6) (and not difficult to verify, see Lemma 2.3) that a divergence-free magnetization $\mathbf{M} \in \mathcal{M}(S)^3$ is S-silent. Conversely, we show in Lemma 2.3 that if S is a slender set (see definition in Section 2.1.1), then any S-silent magnetization is divergence free. Smirnov [11] characterizes divergence-free measures on $\mathbb{R}^3$ in terms of measures that are absolutely continuous with respect to the 1-dimensional Hausdorff measure.

We shall assume that scalar data of the form $f = \mathbf{A}(\mathbf{M}) := \mathbf{v} \cdot \mathbf{B}(\mathbf{M})$, for some fixed nonzero vector $\mathbf{v} \in \mathbb{R}^3$, is given on a closed subset $Q \subset \mathbb{R}^3 \setminus S$, where $\mathbf{A}$ is the *forward operator* mapping $\mathbf{M}$ to the restriction on Q of $\mathbf{B}(\mathbf{M})$. We will consider the situation where

(a) $\mathbf{A} : \mathcal{M}(S)^3 \to L^2(Q)$ boundedly and,

(b) $\mathbf{A}(\mathbf{M}) = 0$ if and only if $\mathbf{M}$ is S-silent.

Since $\mathbf{B}(\mathbf{M})$ is harmonic on $\mathbb{R}^3 \setminus S$, condition (a) will hold if Q and S are positively separated, and if Q is compact (which is the case in practice) then $\mathbf{A}$ is a compact operator. Lemma 2.4 provides sufficient conditions for (b) to hold. Condition (b) means that the observation is "faithful", *i.e.* if the $\mathbf{v}$-component of the field on Q is zero then the field is indeed zero everywhere off S. In this case, the null space of $\mathbf{A}$ (which is a crucial ingredient of the inverse problem) coincides with S-silent magnetizations which depend solely on the geometry of S and can be studied using potential and measure-theoretic tools.

In general, $\mathbf{A}$ has a nontrivial null space and, if Q is "thin enough", it has dense range (see Lemma 2.14). Note that a typical magnetic sensor is a coil measuring the component of the field parallel to its axis, which is why we assume that measurements are of the form $\mathbf{v} \cdot \mathbf{B}(\mathbf{M})$. The fact that $\mathbf{v}$ is a constant vector means that the orientation of the sensor is kept fixed. In some cases, *e.g.* in Magneto-Encephalography, $\mathbf{v}$ would rather depend on the point where measurements are made. We do not consider this (more complicated) situation, as we are particularly motivated by applications to Scanning Magnetic Microscopy (SMM) where measurements of the vertical component of the magnetic field, namely $B_3(\mathbf{M}) = \mathbf{e}_3 \cdot \mathbf{B}(\mathbf{M})$ (with $\mathbf{e}_i$ to denote the i-th unit vector of the canonical basis of $\mathbb{R}^3$, for $i = 1, 2, 3$), are taken on a rectangle Q in a plane $x_3 = h$ for some $h > 0$, while S is contained in the half-space $x_3 \leq 0$.

In Chapter 3, we further concentrate, as in [6, 7, 8], on thin samples modeled as *planar magnetizations* with supports contained in some $S \subset \mathbb{R}^2$ (here we identify $\mathbb{R}^2$ with the $x_3 = 0$ plane in $\mathbb{R}^3$). In this case, elaborating on results from [12] concerning gradients of functions of bounded variation in the plane, we obtain a precise structure theorem for divergence-free magnetizations in $\mathbb{R}^2$ (this is a purely measure-theoretic result) which results in an accurate description of the kernel of the forward operator (see Theorem 3.13).

1.4 Notation

For a vector $\mathbf{x}$ in the Euclidean space $\mathbb{R}^n$ ($n = 2$ or 3), we denote the j-th component of $\mathbf{x}$ by x_j and the partial derivative with resp ect to x_j by ∂_{x_j}. By default, we consider vectors $\mathbf{x}$ as column vectors; e.g., for $\mathbf{x} \in \mathbb{R}^3$ we write $\mathbf{x} = (x_1, x_2, x_3)^T$ where "T" denotes "transpose". We also use bold symbols to represent vector-valued functions and measures, and the corresponding nonbold symbols with subscripts to denote the respective components; e.g., $\mathbf{M} = (\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3)$ or $\mathbf{B}(\mathbf{M}) = (B_1(\mathbf{M}), B_2(\mathbf{M}), B_3(\mathbf{M}))$. We let $\delta_{\mathbf{x}}$ stand for the Dirac delta measure at $\mathbf{x} \in \mathbb{R}^3$ and refer to a magnetization of the form $\mathbf{M} = \mathbf{m}\delta_{\mathbf{x}}$ for some $\mathbf{m} \in \mathbb{R}^3$ as the **point dipole at $\mathbf{x}$ with moment $\mathbf{m}$**. For $\mathbf{x} \in \mathbb{R}^3$ and $R > 0$, we let $\mathbb{B}(\mathbf{x}, R)$ denote the open ball centered at $\mathbf{x}$ with radius R, and $\mathbb{S}(\mathbf{x}, R)$ the boundary sphere. Given a finite measure $\mathrm{M} \in \mathcal{M}(\mathbb{R}^3)$ and a Borel set $E \subset \mathbb{R}^3$, we will denote by $\mathrm{M}\lfloor E$ the measure obtained by restricting M to E (*i.e.* for every Borel set $B \subset \mathbb{R}^3$, $\mathrm{M}\lfloor E(B) := \mathrm{M}(E \cap B)$).

For $\mathbf{M} \in \mathcal{M}(\mathbb{R}^k)$ (in what follows $k = 2$ or 3), the **total variation measure** $|\mathbf{M}|$ is defined on Borel sets $B \subset \mathbb{R}^k$ by

$$|\mathbf{M}|(B) := \sup_{\mathcal{P}} \sum_{P \in \mathcal{P}} |\mathbf{M}(P)|, \tag{1.9}$$

where the supremum is taken over all finite Borel partitions $\mathcal{P}$ of B. Since $|\mathbf{M}|$ is a Radon measure, the Radon-Nikodym derivative $\mathbf{u}_{\mathbf{M}} := \mathrm{d}\mathbf{M}/\mathrm{d}|\mathbf{M}|$ exists and satisfies $|\mathbf{u}_{\mathbf{M}}| = 1$ a.e. with respect to $|\mathbf{M}|$. The **total variation norm of $\mathbf{M}$** is then defined as

$$\|\mathbf{M}\|_{TV} := |\mathbf{M}|(\mathbb{R}^3). \tag{1.10}$$

We shall identify $\mathbf{M} \in \mathcal{M}(\mathbb{R}^k)^k$ with the linear form on $(C_c(\mathbb{R}^k))^k$ (the space of $\mathbb{R}^k$-valued continuous functions on $\mathbb{R}^k$ with compact support equipped with the sup-norm) given by

$$\langle \mathbf{f}, \mathbf{M} \rangle := \int \mathbf{f} \cdot \mathrm{d}\mathbf{M}, \qquad \mathbf{f} \in (C_c(\mathbb{R}^k))^k. \tag{1.11}$$

The norm of the functional (1.11), is $\|\mathbf{M}\|_{TV}$. It extends naturally with the same norm to the space $(C_0(\mathbb{R}^k))^k$ of $\mathbb{R}^k$-valued continuous functions on $\mathbb{R}^k$ vanishing at infinity.

At places, we also identify $\mathbf{M}$ with the restriction of (1.11) to $(C_c^\infty(\mathbb{R}^3))^3$, where $C_c^\infty(\mathbb{R}^3)$ is the space of C^∞-smooth functions with compact support, equiped with the usual topology [25]. We refer to a continuous linear functional on $(C_c^\infty(\mathbb{R}^m))^n$ as being a distribution.

We denote Lebesgue measure on $\mathbb{R}^n$ by $\mathcal{L}_n$ and d-dimensional Hausdorff measure by $\mathcal{H}_d$, see [26] for the definitions. We normalize $\mathcal{H}_d$ for $d = 1$ and 2 so that it coincides with arclength and surface area for smooth curves and surfaces, respectively. We denote the Hausdorff dimension of a set $E \subset \mathbb{R}^3$ by $\dim_{\mathcal{H}}(E)$.

Chapter 2

Recovery method and sufficient conditions for convergence

The method we consider for the magnetization recovery consist mainly in solving two extremal problems: The first is that of minimizing the total variation norm over magnetizations S-equivalent to a given one. To fix notation, for $\mathbf{M} \in \mathcal{M}(S)^3$, let

$$\mathfrak{M}(\mathbf{M}) := \inf\{\|\mathbf{N}\|_{TV} : \mathbf{N} \overset{S}{\equiv} \mathbf{M}\}. \tag{2.1}$$

Extremal Problem 1 (EP-1)**.** *Given* $\mathbf{M}_0 \in \mathcal{M}(S)^3$*, find* $\mathbf{M} \overset{S}{\equiv} \mathbf{M}_0$ *such that* $\|\mathbf{M}\|_{TV} = \mathfrak{M}(\mathbf{M}_0)$*.*

The second extremal problem involves minimizing the following functional defined for $\mathbf{M} \in \mathcal{M}(S)^3$, $f \in L^2(Q)$, and $\lambda > 0$, by

$$\mathcal{F}_{f,\lambda}(\mathbf{M}) := \|f - \mathbf{A}\mathbf{M}\|^2_{L^2(Q)} + \lambda\|\mathbf{M}\|_{TV}. \tag{2.2}$$

Extremal Problem 2 (EP-2)**.** *Given* $f \in L^2(Q)$*, find* $\mathbf{M}_\lambda \in \mathcal{M}(S)^3$ *such that*

$$\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda) = \inf_{\mathbf{M}\in\mathcal{M}(S)^3} \mathcal{F}_{f,\lambda}(\mathbf{M}). \tag{2.3}$$

We remark that the total variation norm is convex on $\mathcal{M}(S)^3$ but not strictly convex and so there may be multiple $\mathbf{M}$ that solve (EP-1) for a given $\mathbf{M}_0$. However, we show that EP-1 uniquely recovers magnetizations for two important cases: (a) *purely 1-unrectifiable* magnetizations (meaning that the support is purely 1-unrectifiable, see Theorem 2.8) and (b) *unidirectional* planar magnetizations (see Theorem 2.13). Purely 1-unrectifiable magnetizations include all magnetizations whose support has 1-dimensional Hausdorff measure zero and so EP-1 recovers a large class of 'sparse' magnetizations (see Section 2.1). As to *uni-directional* magnetizations, they are those $\mathbf{M}$ such that $\mathbf{u}_\mathbf{M}$ is constant $|\mathbf{M}|$-a.e. and form an important class for applications as they represent remanent magnetizations formed in a uniform external field.

We will use the net moment of $\mathbf{M}$ while studying uni-directional magnetizations. Note that, since $\mathbf{M}$ is the density of net moment, the proper way of define the latter in this context is

$$\langle\mathbf{M}\rangle := \mathbf{M}(\mathbb{R}^3). \tag{2.4}$$

Under the assumption that S is compact or a slender set, Lemma 2.3 and Lemma 2.10 show that S-equivalent magnetizations must have the same net moment. In this case, $\langle\mathbf{M}\rangle$ is uniquely defined by the measurements. However, the remark after the proof of lemma 2.3 shows this needs not hold in general.

We turn to Extremal Problem 2 (EP-2). Its solutions connect to those of EP-1 as follows. If $f = \mathbf{A}(\mathbf{M}) + e$ with $e \in L^2(Q)$, any weak-* accumulation point of the solutions to EP-2 when $\lambda \to 0$ and $e/\sqrt{\lambda} \to 0$ must be a solution of EP-1. This Tikhonov-like regularization theory is by now essentially understood in a more general context [20, 21, 22]. In Section 2.3, we improve on some previous results by showing that the result holds not only for the $\mathbb{R}^3$-valued measures involved but also for their total variation measures, and that weak-* convergence can be upgraded to narrow convergence (see Theorem 2.18). Hence, "no mass is lost" in the limit. Another feature of the solutions to EP-2, which is more specific to the present situation, is that they are supported on "small" sets, of codimension at least 1 in S (see Corollary 2.16).

Altogether, when the "true" magnetization $\mathbf{M}$ is sparse in one of the senses mentioned above (that is: if it can be recovered by EP-1), we obtain in Theorem 2.19 an asymptotic recovery result which, from the strict point of view of inverse problems, recaps the main contributions of the document. Finally, observe that if the operator $\mathbf{A}$ has a nontrivial null space (the usual case considered in this document), then Extremal Problem 2 (EP-2) is *a priori* expected to have multiple solutions since the TV-norm is not strictly convex. Still, dwelling on Corollary 2.16 and Theorem 3.13, we show in Chapter 3 that its solution is unique when S is contained in a plane (see Theorem 3.16). This fact, which may come as a surprise, completes our set of results regarding the inverse potential problem in divergence form.

2.1 Equivalent magnetizations, net moments, and total variation

In this section, we discuss some regularity issues for magnetic fields and potentials, and we study the connection between S-silent sources and $\mathbb{R}^3$-valued measures which are distributionally divergence-free. This leads us to introduce the class of slender sets, and subsequently to solve Extremal Problem 1 for certain magnetizations when S is slender. Such magnetizations are "sparse", in the sense that either their support is purely 1-unrectifiable (*cf.* Theorem 2.8) or they assume a single direction on each piece of some finite partition of S (*cf.* Theorem 2.13). We also give conditions on S and Q ensuring that the forward operator has kernel the space of S-silent magnetizations (*cf.* Lemma 2.4).

2.1.1 Divergence-free and silent magnetizations

Equations (2) and (1.7) define harmonic functions pointwise off S and we will show in Lemma 2.1 that those functions extend uniquely to a locally integrable function and a distribution on $\mathbb{R}^3$ that satisfy (1.6) and (2). Therefore $\Phi(\mathbf{M})$ and $\mathbf{B}(\mathbf{M})$ are well defined at the beginning of Subsection 1.3.

Lemma 2.1. *Let S be a closed proper subset of $\mathbb{R}^3$ and $\mathbf{M} \in \mathcal{M}(S)^3$. Then, the integral in the right-hand side of (1.7) converges absolutely for a.e. $\mathbf{x} \in \mathbb{R}^3$. If we denote that resulting function by $\Phi(\mathbf{M})$, it holds for each p, q with $1 \le p < 3/2 < q \le \infty$ that $\Phi(\mathbf{M}) \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$, and that $\Delta\Phi(\mathbf{M}) = \operatorname{div}\mathbf{M}$ in the distributional sense.*

Furthermore, $\langle T_{\boldsymbol{\alpha}}\mathbf{f}, \mathbf{B}(\mathbf{M})\rangle \to 0$ as $|\boldsymbol{\alpha}| \to \infty$ for every $\mathbf{f} \in (C_c^\infty(\mathbb{R}^3))^3$ (here $T_{\boldsymbol{\alpha}}\mathbf{f}$ denotes the translation of the argument of $\mathbf{f}$ by $\boldsymbol{\alpha}$).

Remark 2.2. *The relation $\mathbf{M} = \mathbf{B}(\mathbf{M})/\mu_0 + \operatorname{grad}\Phi(\mathbf{M})$ is the Helmholtz-Hodge decomposition of the $\mathbb{R}^3$-valued measure $\mathbf{M}$ into the sum of a gradient and a divergence-free term. Although $\mathbf{M}$ is a distribution of order 0, note that the summands will generally have order -1.*

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^3)$ be valued in $[0,1]$, identically 1 on $\mathbb{B}(0,1)$ and 0 outside of $\mathbb{B}(0,2)$. Writing $\mathbf{f}_1 = \phi \operatorname{grad}\Gamma$ and $\mathbf{f}_2 = (1-\phi)\operatorname{grad}\Gamma$, we have that $|\mathbf{f}_1| \in L^p(\mathbb{R}^3)$ for $1 \le p < 3/2$ and $|\mathbf{f}_2| \in L^q(\mathbb{R}^3)$. For any $r \in [1,\infty]$, Jensen's inequality implies that the convolution of a finite signed measure with an L^r function is an L^r function with norm not exceeding the mass of the measure times the initial norm, and Fubini's theorem entails that the integrals converge absolutely a.e. Therefore $\mathbf{f}_1 * \mathbf{M} \in L^p(\mathbb{R}^3)$ and $\mathbf{f}_2 * \mathbf{M} \in L^q(\mathbb{R}^3)$, showing that $(\operatorname{grad}\Gamma) * \mathbf{M} \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3)$ (the product under the convolution integral is here the scalar product). Then we can define $\Phi(\mathbf{M}) = (\operatorname{grad}\Gamma) * \mathbf{M}$.

We next show that

$$\Delta\Phi(\mathbf{M}) = \Delta((\operatorname{grad}\Gamma) * \mathbf{M}) = \operatorname{div}\mathbf{M}. \tag{2.5}$$

Let $\psi \in C_c^\infty(\mathbb{R}^3)$ and recall that $G := \Gamma * \psi$ is a smooth function vanishing at infinity such that $\Delta G = \psi$ [27, Cor. 4.3.2&4.5.4]. Now, differentiating under the integral sign, we have that $(\operatorname{grad}\Gamma) * \psi = \operatorname{grad} G$, therefore

$$\begin{aligned}\langle \Delta\Phi(\mathbf{M}), \psi\rangle &= \langle (\operatorname{grad}\Gamma) * \mathbf{M}, \Delta\psi\rangle = -\langle \mathbf{M}, (\operatorname{grad}\Gamma) * \Delta\psi\rangle = -\langle \mathbf{M}, \Delta((\operatorname{grad}\Gamma) * \psi)\rangle \\ &= -\langle \mathbf{M}, \Delta(\operatorname{grad} G)\rangle = -\langle \mathbf{M}, \operatorname{grad}\Delta G\rangle = \langle \operatorname{div}\mathbf{M}, \Delta G\rangle = \langle \operatorname{div}\mathbf{M}, \psi\rangle\end{aligned}$$

which proves (2.5). Finally, the finiteness of $\mathbf{M}$ and the fact that $\Phi(\mathbf{M}) \in L^1(\mathbb{R}^3)^3 + L^2(\mathbb{R}^3)^3$ show:

$$\langle T_{\boldsymbol{\alpha}}\mathbf{f}, \mathbf{B}(\mathbf{M})\rangle = \mu_0\left(\langle T_{\boldsymbol{\alpha}}\mathbf{f}, \mathbf{M}\rangle - \langle \operatorname{div} T_{\boldsymbol{\alpha}}\mathbf{f}, \Phi(\mathbf{M})\rangle\right) \to 0$$

as $|\boldsymbol{\alpha}| \to \infty$, for every $\mathbf{f} \in (C_c^\infty(\mathbb{R}^3))^3$. □

The equation $\Delta\Phi(\mathbf{M}) = \operatorname{div}\mathbf{M}$ is suggestive of the existence of a relationship between S-silent and divergence free magnetizations. As will be seen from the next lemma, for any closed set S, all divergence free magnetization supported on S are S-silent but the converse is not always true as seen in the next construction.

Example 1. *Let S be the closed unit Euclidean ball centered at the origin, $M = \mathcal{L}_3\lfloor S$, and $\mathbf{M} \in \mathcal{M}(S)^3$ the $\mathbb{R}^3$-valued measure equal to $(4\pi/3)^{-1}M\mathbf{e}_1$. Then, by the mean value theorem, we get that*

$$\Phi(\mathbf{M})(\mathbf{x}) = \frac{1}{4\pi}\int (\operatorname{grad}\Gamma)(\mathbf{x}-\mathbf{y})\cdot \mathrm{d}\mathbf{M}(\mathbf{y}) = \frac{1}{4\pi}\frac{x_1}{|\mathbf{x}|^3}, \qquad \mathbf{x} \notin S,$$

since $\frac{1}{4\pi}\frac{x_1}{|\mathbf{x}|^3}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$. Note that $\frac{1}{4\pi}\frac{x_1}{|\mathbf{x}|^3}$ is also the magnetic potential generated by the dipole $\mathbf{N} := \delta_0\mathbf{e}_1$, therefore $\mathbf{M}$ and $\mathbf{N}$ are S-equivalent, that is $\mathbf{M}-\mathbf{N}$ is S-silent. However, this magnetization is not divergence free since, for every $f \in C_c^\infty(\mathbb{R}^3)$ supported in $\mathbb{B}(0,1)$, it holds that $\langle f, \operatorname{div}(\mathbf{M}-\mathbf{N})\rangle = -\langle f, \operatorname{div}\mathbf{N}\rangle = -\partial_{x_1}f(0)$.

An analogous argument shows that, for a_3 the area of $\mathbb{S}(0,1)$ and $\tilde{M} := \mathcal{H}_2\lfloor\mathbb{S}(0,1)$, the $\mathbb{R}^3$-valued measure $a_3^{-1}\tilde{M}\mathbf{e}_1$ is likewise S-equivalent to $\mathbf{N}$.

The following variant of this example is also instructive: there is a sequence $\mathbf{x}_n \in B(0,1)$ and a sequence c_n of real numbers with $\sum_n |c_n| < \infty$ such that $\boldsymbol{\alpha} = \mathbf{e}_1\sum_n c_n\delta_{\mathbf{x}_n}$ is S-equivalent to $a_3^{-1}\tilde{M}\mathbf{e}_1$, therefore $\boldsymbol{\alpha}-\mathbf{N}$ is a S-silent magnetization consisting of a sum of countably many point dipoles. To see that $\mathbf{x}_n$ and c_n exist, recall Bonsall's theorem (whose proof in the ball is the same as in the disk, cf. [28, Thms. 5.21&5.22]) that whenever $\mathbf{x}_n$ is a sequence in $B(0,1)$ which is nontangentially dense in $\mathbb{S}(0,1)$, each function $h \in L^1(\tilde{M})$ can be written as $h(\boldsymbol{\xi}) = \sum_n c_n P_{\mathbf{x}_n}(\boldsymbol{\xi})$ where $P_{\mathbf{x}_n}(\boldsymbol{\xi}) = (1/4\pi)(1-|\mathbf{x}_n|^2)/|\boldsymbol{\xi}-\mathbf{x}_n|^3$ is the familiar Poisson kernel of the unit ball at $\mathbf{x}_n$, and c_n is a sequence of real numbers with absolutely convergent sum. Choosing $h \equiv 1$ and observing that, for $y \notin S$,

$$\frac{y_i - (\mathbf{x}_n)_i}{|\mathbf{y}-\mathbf{x}_n|^3} = \int \frac{y_i - \xi_i}{|\mathbf{y}-\boldsymbol{\xi}|^3}P_{\mathbf{x}_n}(\boldsymbol{\xi})\,\mathrm{d}\tilde{M}(\boldsymbol{\xi}), \qquad i \in \{1,2,3\}$$

by the Poisson representation of harmonic functions, we easily check that $\boldsymbol{\alpha} = \mathbf{e}_1\sum_n c_n\delta_{\mathbf{x}_n}$ is S equivalent to $a_3^{-1}\tilde{M}\mathbf{e}_1$, as desired.

We now present a family of sets for which S-silent magnetizations are divergence-free. We will call a closed set $S \subset \mathbb{R}^3$ a **slender set** if $\mathcal{L}_3(S) = 0$ and each connected component C of $\mathbb{R}^3 \setminus S$ satisfies $\mathcal{L}_3(C) = \infty$. In particular, if $\mathcal{L}_3(S) = 0$ and $\mathbb{R}^3 \setminus S$ is connected, then S is a slender set. As well, any closed subset of a plane in $\mathbb{R}^3$ is slender. A closed surface, however, is not a slender set.

Lemma 2.3. *Let $S \in \mathbb{R}^3$ be closed and $\mathbf{M} \in \mathcal{M}(S)^3$. If $\operatorname{div}\mathbf{M} = 0$, then $\mathbf{M}$ is S-silent. Furthermore, if S is a slender set and $\mathbf{M}$ is S-silent, then $\operatorname{div}\mathbf{M} = 0$.*

Proof. Since $\Phi(\mathbf{M}) \in L^1(\mathbb{R}^3)^3 + L^2(\mathbb{R}^3)^3$ by Lemma 2.1, we get from the Schwarz inequality:

$$\int_E |\Phi(\mathbf{M})| \mathrm{d}\mathcal{L}_3 \leq C_1 + C_2 \left(\mathcal{L}_3(E)\right)^{1/2} \tag{2.6}$$

for some constants C_1, C_2 and each Borel set E of finite measure. If $\operatorname{div}\mathbf{M} = 0$, then $\Phi(\mathbf{M})$ is harmonic on $\mathbb{R}^3$ by the same lemma, therefore it is constant by (2.6) and the mean value theorem (*cf.* proof of [29, Thm. 2.1]). Consequently $\mathbf{M}$ is S-silent.

For the second statement, assume that S is a slender set and that $\mathbf{M} \in \mathcal{M}(S)^3$ is S-silent. Since $\mathbf{B}(\mathbf{M}) = \operatorname{grad}\Phi(\mathbf{M})$ and $\mathbf{M}$ is S-silent, then $\Phi(\mathbf{M})$ is constant on each connected component of $\mathbb{R}^3 \setminus S$. If $\mathfrak{C}$ is such a component, we can apply (2.6) with $E = \mathfrak{C} \cap B(0, n)$ and let $n \to \infty$ to conclude that the corresponding constant is zero, because $\mathcal{L}_3(\mathfrak{C}) = +\infty$. Hence, $\Phi(\mathbf{M})$ must be zero on $\mathbb{R}^3 \setminus S$, and since $\mathcal{L}_3(S) = 0$ it follows that $\Phi(\mathbf{M})$ is zero as a distribution, so that $\operatorname{div}\mathbf{M} = \Delta\Phi(\mathbf{M}) = 0$. □

In typical Scanning Magnetic Microscopy experiments, data consists of point-wise values of one component of the magnetic field taken on a plane not intersecting S. Of course, finitely many values do not characterize the field, but it is natural to ask how one can choose the measurement points to ensure that infinitely many of them would, in the limit, determine it uniquely. We next provide a sufficient condition that such data (more generally, data measured on an analytic surface which needs not be a plane) determines the field in the complement of S. The condition dwells on the remark that a nonzero real analytic function on a connected open subset of $\mathbb{R}^k$ has a zero set of Hausdorff dimension at most $k-1$. It is so because the zero set is locally a countable union of smooth (even real-analytic) embedded submanifolds of strictly positive codimension, see [30, thm 5.2.3]. This fact sharpens the property that the zero set of a nonzero real analytic function in $\mathbb{R}^k$ has Lebesgue measure zero, and will be used at places in the document. Using local coordinates, it is immediately checked that the previous bound on the Hausdorff dimension remains valid when $\mathbb{R}^k$ is replaced by a smooth real-analytic manifold embedded in $\mathbb{R}^m$ for some $m > k$.

We also need at this point a version of the Jordan-Brouwer separation theorem for a connected, properly embedded (*i.e.* complete but not necessarily compact) surface in $\mathbb{R}^3$, to the effect that the complement of such a surface has two connected components. In the smooth case which is our concern here, we give in Appendix A a short, differential topological argument for this result which we assume is known but for which we could not find an appropriate reference.

Lemma 2.4. *Let $S \subset \mathbb{R}^3$ be closed and suppose $\mathbb{R}^3 \setminus S$ is connected and contains a nonempty open half-cylinder of direction $\mathbf{v} \in \mathbb{R}^3 \setminus \{0\}$. Furthermore, let $\mathcal{A}$ be a smooth complete and connected real analytic surface in $\mathbb{R}^3 \setminus S$ that is positively separated from S and such that S lies entirely within one of the two connected components of $\mathbb{R}^3 \setminus \mathcal{A}$. Let also $Q \subset \mathbb{R}^3 \setminus S$ be such that the closure of $Q \cap \mathcal{A}$ has Hausdorff dimension strictly greater than 1. If $\mathbf{M} \in \mathcal{M}(S)^3$ is such that $v \cdot \mathbf{B}(\mathbf{M})$ vanishes on $Q \cap \mathcal{A}$, then $\mathbf{M}$ is S-silent.*

Proof. Suppose $\mathbf{M} \in \mathcal{M}(S)^3$ is such that $\mathbf{v} \cdot \mathbf{B}(\mathbf{M})$ vanishes on Q. As $\mathbf{v} \cdot \mathbf{B}(\mathbf{M})$ is harmonic in $\mathbb{R}^3 \setminus S$ it is real-analytic there, and since $\mathbf{v} \cdot \mathbf{B}(\mathbf{M})$ vanishes on the closure of $Q \cap \mathcal{A}$ which has Hausdorff dimension > 1 it must vanish identically on $\mathcal{A}$.

Observe now that $\mathbb{R}^3 \setminus \mathcal{A}$ has two connected components (see Theorem A.1), and let $\mathcal{U}$ be the one not containing S. Note, using (2), that if $\mathbf{N} \in \mathcal{M}(S)^3$ and $\mathbf{x} \notin \operatorname{supp}(\mathbf{N})$, then,

$$|\mathbf{B}(\mathbf{N})(\mathbf{x})| \leq 4c\,(\mathrm{dist}(\mathbf{x},\mathrm{supp}(\mathbf{N})))^{-3}\,\|\mathbf{N}\|_{TV}. \tag{2.7}$$

For $R>0$ let $\mathbf{M}_R := \mathbf{M}\lfloor B(0,R)$ and $\tilde{\mathbf{M}}_R := \mathbf{M}-\mathbf{M}_R$. Then

$$\mathbf{B}(\mathbf{M})(\mathbf{x}) = \mathbf{B}(\mathbf{M}_R)(\mathbf{x}) + \mathbf{B}(\tilde{\mathbf{M}}_R)(\mathbf{x}),$$

and applying (2.7) to $\mathbf{M}_R$ and $\tilde{\mathbf{M}}_R$ for R large enough, using that $\mathcal{A}$ is positively separated from S, we get that $\limsup_{\mathbf{x}\in U,|\mathbf{x}|\to\infty}|\mathbf{B}(\mathbf{M})(\mathbf{x})|<\varepsilon$ for any $\varepsilon>0$, hence $\mathbf{B}(\mathbf{M})(\mathbf{x})\to 0$ as $\mathbf{x}\to\infty$ in $\mathcal{U}$. Since $v\cdot\mathbf{B}(\mathbf{M})$ vanishes on the boundary of $\mathcal{U}$, we may use the maximum principle to conclude that $v\cdot\mathbf{B}(\mathbf{M})$ vanishes on $\mathcal{U}$ and therefore on $\mathbb{R}^3\setminus S$ as the complement of S is connected.

This implies that the magnetic potential $\Phi(\mathbf{M})$ is constant on every line segment parallel to v not intersecting S. Now, $\mathbb{R}^3\setminus S$ contains a half-cylinder $\mathcal{C}$ of direction v, and shrinking the latter if necessary we may assume it is positively separated from S. From (1.7) we get

$$|\Phi(\mathbf{N})(\mathbf{x})| \leq \frac{1}{4\pi}\,(\mathrm{dist}(\mathbf{x},\mathrm{supp}(\mathbf{N})))^{-2}\,\|\mathbf{N}\|_{TV} \tag{2.8}$$

for $\mathbf{N}\in\mathcal{M}(S)^3$ and $\mathbf{x}\notin\mathrm{supp}(\mathbf{N})$, and we conclude that $\Phi(\mathbf{M})(\mathbf{x})$ goes to zero as $\mathbf{x}\to\infty$ in $\mathcal{C}$. Hence its value on each half line contained in $\mathcal{C}$ is zero, so $\Phi(\mathbf{M})\equiv 0$ in $\mathcal{C}$. Consequently it vanishes identically (thus also $\mathbf{B}(\mathbf{M})$) in the connected open set $\mathbb{R}^3\setminus S$, by real analyticity. □

The following example shows that Lemma 2.4 needs not hold if S is not contained in a single component of $\mathbb{R}^3\setminus\mathcal{A}$ or if $\mathcal{A}$ fails to be analytic.

Example 2. *Let S be equal to $\{\mathbf{e}_3,-\mathbf{e}_3\}$, $\mathbf{M}=(\delta_{\mathbf{e}_3}+\delta_{-\mathbf{e}_3})\mathbf{e}_2$, and $\mathcal{A}=\{x_3=0\}$. Then $\mathbf{e}_3\cdot\boldsymbol{b}(\mathbf{M})$ is zero on $\mathcal{A}$ but $\mathbf{M}$ is not S-silent. Also, whenever Q is a bounded subset of $\{x_3=0\}$ with $\dim_{\mathcal{H}}Q>1$, there is a closed C^∞-smooth surface Z containing Q such that $\mathbf{e}_3$ and $-\mathbf{e}_3$ lie in the same component of $\mathbb{R}^3\setminus Z$; however, Z cannot be analytic.*

Remark 2.5. *If S is as Lemma 2.4 and $Q\subset\mathbb{R}^3$ is positively separated from S and has closure $\bar{Q}$ of Hausdorff dimension >2, then the conclusion of the lemma still holds and the proof is easier. In this case indeed, it follows directly from the hypothesis on Q that $v\cdot\mathbf{B}(\mathbf{M})$ is identically zero in $\mathbb{R}^3\setminus S$ as soon as it vanishes on Q, and the rest of the proof is as before. We shall not investigate this situation which, from the point of view of inverse problems, corresponds to the case where measurements of the field are taken in a volume rather than on a surface. Though more information can be gained this way, the experimental and computational burden often becomes discouraging.*

If $\mathbb{R}^3\setminus S$ is not connected but has a connected component $\mathcal{V}$ containing a half-cylinder, then by replacing S with $\tilde{S}:=\mathbb{R}^3\setminus\mathcal{V}$ and selecting an appropriate $\mathcal{A}$, Q and v, Lemma 2.4 may be applied to the effect that $\mathbf{M}$ is $\tilde{S}$-silent whenever $v\cdot\mathbf{B}(\mathbf{M})$ vanishes on Q. Thus, if each component $\mathcal{V}_i$ of $\mathbb{R}^3\setminus S$ contains a half-cylinder and can be associated with suitable $\mathcal{A}_i$, Q_i and v_i, and if $v_i\cdot\mathbf{B}(\mathbf{M})$ vanishes on Q_i for all i, then $\mathbf{M}$ is S-silent.

Lemma 2.6. *Let $S=S_0\cup S_1\subset\mathbb{R}^3$ for some disjoint closed sets S_0 and S_1. If $\mathbf{M}\in\mathcal{M}(S)^3$ is S-silent, then for $i=0,1$ the restriction $\mathbf{M}\lfloor S_i$ is S_i-silent.*

Proof. Let $\mathbf{B}_0=\mathbf{B}(\mathbf{M}\lfloor S_0)$ and $\mathbf{B}_1=\mathbf{B}(\mathbf{M}\lfloor S_1)$. Note that $\mathbf{B}_0$ and $\mathbf{B}_1$ are harmonic in $\mathbb{R}^3\setminus S_0$ and $\mathbb{R}^3\setminus S_1$ respectively. Also, as $\mathbf{M}$ is S-silent, it holds that $\mathbf{B}_0(\mathbf{x})=-\mathbf{B}_1(\mathbf{x})$ for $\mathbf{x}\notin S$. Hence the function

$$\widetilde{\mathbf{B}}(\mathbf{x}) = \begin{cases} \mathbf{B}_0(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^3 \setminus S_0, \\ -\mathbf{B}_1(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^3 \setminus S_1, \end{cases}$$

is harmonic on $\mathbb{R}^3$ (note that the two definitions agree on $\mathbb{R}^3 \setminus S$). Moreover, Lemma 2.1 implies that for every $\mathbf{f} \in C_c^\infty(\mathbb{R}^3)$,

$$|\langle T_\alpha \mathbf{f}, \widetilde{\mathbf{B}} \rangle| \leq |\langle T_\alpha \mathbf{f}, \mathbf{B}_0 \rangle| + |\langle T_\alpha \mathbf{f}, \mathbf{B}_1 \rangle| \to 0 \qquad \text{as} \qquad |\alpha| \to \infty. \tag{2.9}$$

Since $\widetilde{\mathbf{B}}$ is harmonic, the mean value property applied to (2.9) with radially symmetric $\mathbf{f}$ implies that $\widetilde{\mathbf{B}}(\mathbf{x})$ vanishes as $\mathbf{x} \to \infty$ and therefore is identically 0 by Liouville's theorem. Thus both $\mathbf{B}_0$ and $\mathbf{B}_1$ are zero on $\mathbb{R}^3 \setminus S_0$ and $\mathbb{R}^3 \setminus S_1$ respectively and hence $\mathbf{M}\lfloor S_0$ is S_0 silent and $\mathbf{M}\lfloor S_1$ is S_1 silent. □

2.1.2 Decomposition of divergence free magnetizations and recovery of magnetizations with sparse support

A set $E \subset \mathbb{R}^2$ is said to be **1-rectifiable** (e.g., see [31, Def. 15.3]) if there exist Lipschitz maps $f_i : \mathbb{R} \to \mathbb{R}^n$, $i = 1, 2, ...$, such that

$$\mathcal{H}_1 \left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}) \right) = 0.$$

A set $B \subset \mathbb{R}^n$ is **purely 1-unrectifiable** if $\mathcal{H}_1(E \cap B) = 0$ for every 1-rectifiable set E. Clearly a set of $\mathcal{H}_1$-measure zero is purely 1-unrectifiable.

We call a Lipschitz mapping $\mathbf{c} : [0, \ell] \to \mathbb{R}^3$ a **rectifiable curve** and let $\mathbf{C} := \mathbf{c}([0, \ell])$ denote its image. If $\mathbf{c}$ is an arclength parametrization of $\mathbf{C}$; i.e., if $\mathbf{c}$ satisfies

$$\mathcal{H}_1(\mathbf{c}([\alpha, \beta])) = \beta - \alpha, \qquad \forall [\alpha, \beta] \subset [0, \ell], \tag{2.10}$$

then we call $\mathbf{c}$ an **oriented rectifiable curve**. By Rademacher's Theorem (see [26]), $\mathbf{c}$ is differentiable a.e. on $[0, \ell]$. Furthermore, it follows from (2.10) that $|\mathbf{c}'(t)| = 1$ a.e. on $[0, \ell]$. For a given oriented rectifiable curve $\mathbf{c}$ we define $\mathbf{R}_\mathbf{c} \in \mathcal{M}(S)^3$ through the relation

$$\langle \mathbf{R}_\mathbf{c}, \mathbf{f} \rangle = \int_0^\ell \mathbf{f}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \mathrm{d}t, \tag{2.11}$$

for $\mathbf{f} \in C_0(\mathbb{R}^3)^3$. Alternatively, since $\mathbf{R}_\mathbf{c}$ is absolutely continuous with respect to $\mathcal{H}_1$ we may consider the Radon-Nikodym derivative $\boldsymbol{\tau}$ of $\mathbf{R}_\mathbf{c}$ with respect to $\mathcal{H}_1$ and we remark that $\boldsymbol{\tau}(\mathbf{c}(t)) = \mathbf{c}'(t)$ for a.e. $t \in [0, \ell]$. Then, for a Borel set $B \subset \mathbb{R}^3$ we have

$$\mathbf{R}_\mathbf{c}(B) = \int_B \boldsymbol{\tau} \, \mathrm{d}(\mathcal{H}_1 \lfloor \mathbf{C}). \tag{2.12}$$

We remark that if B is purely 1-unrectifiable, then $|\mathbf{R}_\mathbf{c}|(B) = \mathcal{H}_1(B \cap \mathbf{C}) = 0$ and, furthermore, Fubini's Theorem implies $\mathcal{L}_3(B) = 0$.

Let $\mathcal{C} \subset \mathcal{M}(S)^3$ denote the collection of oriented rectifiable curves with topology inherited from $\mathcal{M}(S)^3$. Suppose $\operatorname{div} \mathbf{M} = 0$ (as a distribution). Smirnov [11, Theorem A] shows that $\mathbf{M}$ can be decomposed into elements from $\mathcal{C}$. In particular, it can be proven that there is a positive Borel measure ρ on $\mathcal{C}$ such that

$$\mathbf{M}(B) = \int \mathbf{R}(B)\ \mathrm{d}\rho(\mathbf{R}), \tag{2.13}$$

and

$$|\mathbf{M}|(B) = \int |\mathbf{R}|(B)\ \mathrm{d}\rho(\mathbf{R}), \tag{2.14}$$

for any Borel set $B \subset \mathbb{R}^3$. From the representation (2.13) of a divergence free magnetization, we immediately obtain the following lemma.

Lemma 2.7. *Suppose $S \subset \mathbb{R}^3$ is closed and purely 1-unrectifiable. If $\mathbf{M} \in \mathcal{M}(S)^3$ is divergence free, then $\mathbf{M} = 0$.*

Theorem 2.8. *Suppose $S \subset \mathbb{R}^3$ is a closed, slender set. If $\mathbf{M} \in \mathcal{M}(S)^3$ has support that is purely 1-unrectifiable and $\mathbf{N} \in \mathcal{M}(S)^3$ is S-equivalent to $\mathbf{M}$, then $\|\mathbf{N}\|_{TV} > \|\mathbf{M}\|_{TV}$ unless $\mathbf{N} = \mathbf{M}$.*

Proof. Since $\boldsymbol{\theta} := \mathbf{N} - \mathbf{M}$ is S-silent, Lemma 2.3 implies $\operatorname{div} \boldsymbol{\theta} = 0$ and so $\boldsymbol{\theta}$ can be represented in the form (2.13), where (2.14) holds. Since the support of $\mathbf{M}$ is purely 1-unrectifiable, it follows from (2.14) and the remark after (2.12) that the measures $\mathbf{M}$ and $\boldsymbol{\theta}$ are mutually singular. Thus, $\|\mathbf{N}\|_{TV} = \|\mathbf{M}\|_{TV} + \|\boldsymbol{\theta}\|_{TV} > \|\mathbf{M}\|_{TV}$ unless $\mathbf{N} = \mathbf{M}$. □

Example 3. *Recall from Example 1 that if $S = \mathbb{B}(0, 1)$, the magnetizations $\mathbf{M}$ modeling a uniformly magnetized ball and $\mathbf{N}$ which is a point dipole at 0, with same net moment as $\mathbf{M}$, are S-equivalent. Moreover, it is easy to verify that $\|\mathbf{M}\|_{TV} = \|\mathbf{N}\|_{TV} = 1$. Since the support of $\mathbf{N}$ is a single point, it is purely 1-unrectifiable, hence the assumption that S is a slender set cannot be eliminated from Theorem 2.8.*

The previous example entails that total variation minimization is not sufficient alone to distinguish magnetizations with purely 1-unrectifiable support among all equivalent magnetizations supported on S when S is not slender. However, as the following result shows, the recovery problem for general S has at most one solution when restricted to magnetizations whose support is purely 1-unrectifiable and has finite $\mathcal{H}_2$ measure.

To see this, we shall need a consequence of the Besicovitch-Federer Projection Theorem [31, Thm 18.1]; namely that the complement of a closed, purely 2-unrectifiable set with finite $\mathcal{H}_2$ measure is connected. We will also need the fact that a purely 1-unrectifiable set is purely 2-unrectifiable. We are confident these facts are known (e.g., see the introduction in [32]), but since we have not explicitly found proofs in the literature, we provide outlines of the arguments.

With regard to the first fact, let $F \subset \mathbb{R}^3$ be a closed, purely 2-unrectifiable set with finite $\mathcal{H}_2$ measure and suppose $\mathbb{B}(\mathbf{x}, r)$, $\mathbb{B}(y, r)$ are disjoint balls in $\mathbb{R}^3 \setminus F$. By the Besicovitch-Federer Projection Theorem there exists a plane P such that the intersection of the orthogonal projections of these balls onto P minus the orthogonal projection of F onto P is nonempty and therefore the balls can be joined by a line segment not intersecting F.

As to the second fact, it follows from [33, Lemma 3.2.18] that it is enough for a set F to be purely 2-unrectifiable that $\mathcal{H}_2(F \cap \psi(K)) = 0$ for any compact set $K \subset \mathbb{R}^2$ and any bi-Lipschitz mapping $\psi : K \to \mathbb{R}^3$. Since bi-Lipschitz maps preserve unrectifiability we restrict our considerations to $\mathbb{R}^2$ where the result follows easily from Fubini's theorem.

As a consequence of these facts, the complement of a closed, purely 1-unrectifiable set with finite $\mathcal{H}_2$ measure must be connected.

Corollary 2.9. *Suppose S is a closed, proper subset of $\mathbb{R}^3$ and that $\mathbf{N} \in \mathcal{M}(S)^3$ has purely 1-unrectifiable support of finite $\mathcal{H}_2$ measure. If $\mathbf{M} \in \mathcal{M}(S)^3$ is S-equivalent to $\mathbf{N}$ but not equal to $\mathbf{N}$, then the support of $\mathbf{M}$ is not a purely 1-unrectifiable set with finite $\mathcal{H}_2$ measure.*

Proof. Suppose $\mathbf{M} \in \mathcal{M}(S)^3$ is S-equivalent to $\mathbf{N}$ and has support that is purely 1-unrectifiable with finite $\mathcal{H}_2$ measure. Then the support $\tilde{S}$ of $\mathbf{M} - \mathbf{N}$ is also purely 1-unrectifiable with finite $\mathcal{H}_2$ measure. Therefore, its complement is connected and thus $\tilde{S}$ is slender.

Moreover, $\mathbf{M} - \mathbf{N}$ is S-silent, hence its field vanishes on the nonempty open set $\mathbb{R}^3 \setminus S$, and since $\tilde{S}$ is closed with $\mathcal{L}_3(\tilde{S}) = 0$ (because it is slender), the field must vanish on a nonempty open subset of $\mathbb{R}^3 \setminus \tilde{S}$. Since $\tilde{S}$ has connected complement, we conclude that $\mathbf{M} - \mathbf{N}$ is $\tilde{S}$-silent. Consequently, it is divergence free by Lemma 2.3 and hence, by Lemma 2.7, $\mathbf{M} - \mathbf{N}$ is the zero measure. □

Corollary 2.9 applies in particular if $\mathbf{N}$ is a finite sum of point dipoles. However, in view of Example 1, it does not apply in general to a convergent series of point dipoles.

2.1.3 The net moment of silent magnetizations

Our next result shows that, under certain assumptions on their support, silent measures have vanishing moment:

Lemma 2.10. *Let $S \subset \mathbb{R}^3$ be a closed set and $\mathbf{M} \in \mathcal{M}(S)^3$ be S-silent. Assume that one of the following conditions is satisfied:*

(a) S is compact,

(b) $\operatorname{div} \mathbf{M} = 0$.

Then the net moment $\langle \mathbf{M} \rangle = 0$.

Proof. Fix $i \in \{1, 2, 3\}$. Let $\phi \in C_0^\infty(\mathbb{R}^3)$ be supported on $\mathbb{B}(0, 2)$, $\phi(\mathbf{x}) = x_i$ on $\mathbb{B}(0, 1)$ and for any $n > 0$ let $\phi_n(\mathbf{x}) := n\phi(\mathbf{x}/n)$. Note that for any $n > 0$, $\|\operatorname{grad} \phi\|_\infty = \|\operatorname{grad} \phi_n\|_\infty$, ϕ_n is supported on $\mathbb{B}(0, 2n)$ and for $\mathbf{x} \in \overline{\mathbb{B}}(0, n)$, $\phi_n(\mathbf{x}) = x_i$, $\operatorname{grad} \phi_n(\mathbf{x}) = \mathbf{e}_i$ and $\Delta\phi_n(\mathbf{x}) = 0$.

If S is compact take $n > 0$ such that $S \subset \mathbb{B}(0, n)$. Then $\langle \operatorname{grad} \phi_n, \mathbf{M} \rangle = \langle \mathbf{M} \rangle_i$, the i-th component of the moment of $\mathbf{M}$. Since $\mathbf{M}$ is S-silent and $\Phi(\mathbf{M}) \in L^1(\mathbb{R}^3)^3 + L^2(\mathbb{R}^3)^3$, $\Phi(\mathbf{M})$ is zero on $\mathbb{R}^3 \setminus S$. Thus, $\Phi(\mathbf{M})$ is supported on $S \subset \mathbb{B}(0, n)$, and since $\Delta\Phi(\mathbf{M}) = \operatorname{div} \mathbf{M}$ by Lemma 2.1, we obtain:

$$\langle \mathbf{M} \rangle_i = \langle \operatorname{grad} \phi_n, \mathbf{M} \rangle = \langle \phi_n, \operatorname{div} \mathbf{M} \rangle = \langle \phi_n, \Delta\Phi(\mathbf{M}) \rangle = \langle \Delta\phi_n, \Phi(\mathbf{M}) \rangle = 0.$$

Therefore taking $i = 1, 2, 3$, we get that $\langle \mathbf{M} \rangle = 0$, as announced.

Assume next that $\operatorname{div} \mathbf{M} = 0$. For any integer $m > 0$, let $D_m := \mathbb{B}(0, 2^{m+1}) \setminus \mathbb{B}(0, 2^m)$ and $M_m := |\mathbf{M}|(D_m)$. Because $\sum_m M_m \leq \|\mathbf{M}\|_{TV} < \infty$, we have that

$$|\langle \operatorname{grad} \phi_{2^m}|_{D_m}, \mathbf{M} \rangle| \leq M_m \|\operatorname{grad} \phi\|_\infty \to 0 \quad \text{as} \quad m \to \infty. \tag{2.15}$$

Now, let $\mathbf{E}_i$ be the constant function equal to $\mathbf{e}_i$ on $\mathbb{R}^3$. By (2.15), we see that

$$\lim_{m\to\infty} \langle \operatorname{grad} \phi_{2^m}, \mathbf{M} \rangle = \lim_{m\to\infty} \langle \operatorname{grad} \phi_{2^m}|_{\mathbb{B}(0,2^m)}, \mathbf{M} \rangle = \langle \mathbf{E}_i, \mathbf{M} \rangle = \langle \mathbf{M} \rangle_i,$$

and since $\langle \operatorname{grad} \phi_n, \mathbf{M} \rangle = \langle \phi_n, \operatorname{div} \mathbf{M} \rangle = 0$ for each $n > 0$, by our assumption, we conclude that $\langle \mathbf{M} \rangle = 0$, as desired. □

Assumptions (a) or (b) cannot be dropped in Lemma 2.10: in fact, it is not sufficient that a magnetization be S-silent for its net moment to vanish, as shown by the following example.

Example 4. *Consider the case where* $S = \mathbb{R}^3 \backslash \mathbb{B}(0,R)$ *and let* $\mathbf{M} = \mathbf{v}\mathcal{H}_2 \lfloor \mathbb{S}(0,R)$ *where* $\mathbf{v} \in \mathbb{R}^3 \backslash \{0\}$*. The density of* $\mathbf{M}$ *with respect to* $\mathcal{H}_2 \lfloor \mathbb{S}(0,R)$ *is the constant map* $\mathbf{f_v} : \mathbb{S}(0,R) \to \mathbb{R}^3$ *given by* $\mathbf{f_v}(\mathbf{x}) = \mathbf{v}$*, which is the trace on* $\mathbb{S}(0,R)$ *of the gradient of the function* $x \mapsto \mathbf{v} \cdot \mathbf{x}$ *which is harmonic on a neighborhood of* $\overline{\mathbb{B}(0,R)}$*, hence* $\mathbf{f_v}$ *a fortiori belongs to the Hardy space* $\mathcal{H}^2_{+,R}$ *of harmonic gradients in* $\mathbb{B}(0,R)$*. Therefore* $\mathbf{M}$ *is silent in that ball [34, Lemma 4.2], and still* $\langle \mathbf{M} \rangle = 4\pi\mathbf{v}$*. Integrating this example over* $R \in [1,\infty)$ *against the weight* $1/R^4$ *further shows that the* $\mathbb{R}^3$*-valued measure* $\mathrm{d}\mathbf{N}(\mathbf{x}) = \mathbf{v}|\mathbf{x}|^{-4}\chi_{\{|\mathbf{x}|\geq 1\}}(\mathbf{x})d\mathcal{L}_3(\mathbf{x})$*, is silent in the ball* $\mathbb{B}(0,1)$ *but has* $\langle \mathbf{N} \rangle = 4\pi\mathbf{v}$*. This provides us with an example of a (non-compactly supported) measure with non-zero total moment which is silent in the complement of its support.*

2.1.4 Total variation and unidirectional magnetizations

For $\mathbf{M} \in \mathcal{M}(S)^3$ we can write $\mathrm{d}\mathbf{M} = \mathbf{u_M}\mathrm{d}|\mathbf{M}|$, therefore the Cauchy Schwarz inequality yields that

$$|\langle \mathbf{M} \rangle|^2 = \int \langle \mathbf{M} \rangle \cdot \mathbf{u_M}\, \mathrm{d}|\mathbf{M}| \leq |\langle \mathbf{M} \rangle|\, \|\mathbf{M}\|_{TV}, \tag{2.16}$$

where equality holds if and only if either $\langle \mathbf{M} \rangle = 0$ or $\mathbf{u_M} = \langle \mathbf{M} \rangle / |\langle \mathbf{M} \rangle|$ a. e. with respect to $|\mathbf{M}|$. We say that $\mathbf{M}$ is **uni-directional** if $\mathbf{u_M}$ is constant a.e. with respect to $|\mathbf{M}|$ (note that the zero magnetization is uni-directional). Thus, (2.16) implies the following:

Lemma 2.11. *If* $\mathbf{M} \in \mathcal{M}(S)^3$*, then* $|\langle \mathbf{M} \rangle| \leq \|\mathbf{M}\|_{TV}$ *with equality if and only if* $\mathbf{M}$ *is uni-directional.*

We call a magnetization **uni-dimensional** if it is the difference of two uni-directional magnetizations. The next lemma states that a uni-dimensional magnetization which is divergence free must be the zero magnetization.

Lemma 2.12. *If* $\mathbf{M} \in \mathcal{M}(\mathbb{R}^3)^3$ *is uni-dimensional and* $\operatorname{div} \mathbf{M} = 0$*, then* $\mathbf{M} = 0$.

Proof. Suppose $\mathbf{M} \in \mathcal{M}(\mathbb{R}^3)^3$ is uni-dimensional and divergence free. Then $\mathbf{M} = \mathrm{M}\mathbf{v}$ for some $v \in \mathbb{R}^3$ and $\mathrm{M} \in \mathcal{M}(\mathbb{R}^3)$ where $0 = \operatorname{div}(\mathrm{M}\mathbf{v}) = \mathbf{v} \cdot \operatorname{grad} \mathrm{M}$. A standard argument (see below) shows that M is translation invariant with respect to any vector parallel to $\mathbf{v}$ and therefore M is finite only if it is zero.

Without loss of generality we may assume that $\mathbf{v} = \mathbf{e}_1$. To see the translation invariance of M take any $f \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ and let $\tilde{f}$ be a translation of f in the x_1 direction. Then $f - \tilde{f} = \partial_{x_1} g$ with $g \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ defined by:

$$g(x_1, x_2, x_3) := \int_{-\infty}^{x_1} (f - \tilde{f})(y, x_2, x_3)\mathrm{d}\mathcal{L}_1(y),$$

and so $\mathrm{M}(f - \tilde{f}) = \langle \partial_{x_1} g, \mu \rangle = -\langle g, \partial_{x_1} \mu \rangle = 0$. □

Theorem 2.13. *Let* $S = \bigcup_{i=1}^n S_i$ *for some disjoint closed sets* $S_1, S_2, \ldots, S_n$ *in* $\mathbb{R}^3$ *and suppose that* S *is either compact or slender. Let* $\mathbf{M} \in \mathcal{M}(S)^3$ *be such that* $\mathbf{M}_i := \mathbf{M}\lfloor S_i$ *is uni-directional for* $i = 1, 2, \ldots, n$.

If $\mathbf{N} \in \mathcal{M}(S)^3$ *is* S*-equivalent to* $\mathbf{M}$*, then* $\mathbf{N}_i := \mathbf{N}\lfloor S_i$ *and* $\mathbf{M}_i$ *are* S_i*-equivalent for* $i = 1, 2, \ldots, n$*, moreover*

$$\|\mathbf{M}\|_{TV} \leq \|\mathbf{N}\|_{TV}, \tag{2.17}$$

with equality in (2.17) *if and only if* $\mathbf{N}_i$ *is uni-directional in the same direction as* $\mathbf{M}_i$ *for* $i = 1, 2, \ldots, n$. *Furthermore, if* S *is slender and equality holds in* (2.17), *then* $\mathbf{M} = \mathbf{N}$.

Proof. Since $\mathbf{M}$ and $\mathbf{N}$ are S-equivalent, their difference $\boldsymbol{\tau} := \mathbf{N} - \mathbf{M}$ is S-silent. In addition, it follows from Lemma 2.3 that if S is slender then $\operatorname{div} \boldsymbol{\tau} = 0$. By Lemma 2.6, the restriction $\boldsymbol{\tau}_i := \boldsymbol{\tau}\lfloor S_i$ is S_i-silent and thus $\mathbf{M}_i$ and $\mathbf{N}_i$ are S_i-equivalent. Since either S is compact or $\operatorname{div} \boldsymbol{\tau} = 0$, the same is true of each S_i, $\boldsymbol{\tau}_i$ and we can use Lemma 2.10 to obtain that $\langle \mathbf{M}_i \rangle = \langle \mathbf{N}_i \rangle$ for $i = 1, 2, \ldots, n$. Then

$$\|\mathbf{N}\|_{TV} = \sum_{i=1}^{n} \|\mathbf{N}_i\|_{TV} \geq \sum_{i=1}^{n} |\langle \mathbf{N}_i \rangle| = \sum_{i=1}^{n} |\langle \mathbf{M}_i \rangle| = \sum_{i=1}^{n} \|\mathbf{M}_i\|_{TV} = \|\mathbf{M}\|_{TV}, \tag{2.18}$$

where the next to last equality follows from the uni-directionality of $\mathbf{M}_i$. By Lemma 2.11, equality holds in (2.18) if and only if each $\mathbf{N}_i$ is uni-directional, and it must have the direction of $\mathbf{M}_i$ since their moments agree. In particular $\boldsymbol{\tau}$ is then unidimensional, hence if in addition S is slender, so that $\operatorname{div} \boldsymbol{\tau} = 0$, we get from Lemma 2.12 that equality holds in (2.17) only when $\mathbf{M} = \mathbf{N}$. □

Note that $\mathbf{M}$ and $\mathbf{N}$ from the previous theorem can be different, even when equality holds in (2.17). That is the case in Example 1, taking $\mathbf{M}$ and $\mathbf{N}$ as defined in that construction.

2.2 Magnetization-to-field operators

Let $S \subset \mathbb{R}^3$ be closed and $Q \subset \mathbb{R}^3 \setminus S$ be compact. For $\mathbf{M} \in \mathcal{M}(S)^3$ and $\mathbf{v}$ a unit vector in $\mathbb{R}^3$, the component of the magnetic field $\mathbf{B}(\mathbf{M})$ in the direction $\mathbf{v}$ at $\mathbf{x} \notin S$ is given, in view of (2), by

$$B_{\mathbf{v}}(\mathbf{M})(\mathbf{x}) := \mathbf{v} \cdot \mathbf{B}(\mathbf{M})(\mathbf{x}) = -\frac{\mu_0}{4\pi} \int \mathbf{K}_{\mathbf{v}}(\mathbf{x} - \mathbf{y}) \cdot \, \mathrm{d}\mathbf{M}(\mathbf{y}), \tag{2.19}$$

where

$$\mathbf{K}_{\mathbf{v}}(x) = \frac{\mathbf{v}}{|x|^3} - 3x \frac{\mathbf{v} \cdot x}{|x|^5} = \operatorname{grad}(\mathbf{v} \cdot \operatorname{grad} \Gamma). \tag{2.20}$$

Consider a finite, positive Borel measure ρ with support contained in Q and let $\mathbf{A} : \mathcal{M}(S)^3 \to L^2(Q, \rho)$ be the operator defined by

$$\mathbf{A}(\mathbf{M})(\mathbf{x}) := B_{\mathbf{v}}(\mathbf{M})(\mathbf{x}), \qquad \mathbf{x} \in Q. \tag{2.21}$$

Since $\mathbf{K}_{\mathbf{v}}$ is continuous on $\mathbb{R}^3 \setminus \{0\}$ and Q and S are positively separated, it follows that $B_{\mathbf{v}}$ is continuous on Q and consequently $\mathbf{A}$ does indeed map $\mathcal{M}(S)^3$ into $L^2(Q, \rho)$.

If $\Psi \in L^2(Q, \rho)$, then using Fubini's Theorem and (2.19) we have that

$$\langle \Psi, \mathbf{A}(\mathbf{M}) \rangle_{L^2(Q,\rho)} = -\frac{\mu_0}{4\pi} \iint \Psi(\mathbf{x}) \mathbf{K}_{\mathbf{v}}(\mathbf{x} - \mathbf{y}) \cdot \, \mathrm{d}\mathbf{M}(\mathbf{y}) \, \mathrm{d}\rho(\mathbf{x}) = \langle \mathbf{A}^*(\Psi), \mathbf{M} \rangle, \tag{2.22}$$

where for $\mathbf{x} \in S$ the adjoint operator $\mathbf{A}^*$ is given by

$$\mathbf{A}^*(\Psi)(\mathbf{x}) := -\frac{\mu_0}{4\pi} \int \Psi(\mathbf{y}) \mathbf{K}_{\mathbf{v}}(\mathbf{x} - \mathbf{y}) \, \mathrm{d}\rho(\mathbf{y}), \qquad \mathbf{x} \in S. \tag{2.23}$$

In view of (2.20), a compact way of re-writing (2.23) is

$$\mathbf{A}^*(\Psi)(\mathbf{x}) := -\mu_0 \operatorname{grad}(\operatorname{grad} U^{\rho,\psi} \cdot \mathbf{v})(\mathbf{x}), \qquad U^{\rho,\psi}(\mathbf{x}) = -\frac{1}{4\pi}\int \frac{\Psi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \mathrm{d}\rho(\mathbf{y}). \tag{2.24}$$

Since Q and S are positively separated it follows as in the proof of Lemma 2.4 that $\mathbf{A}^*(\Psi) \in C_0(S)^3$ and thus $\mathbf{A}^* : (L^2(Q,\rho))^* \sim L^2(Q,\rho) \to C_0(S)^3 \subset (\mathcal{M}(S)^3)^*$.

We shall also be concerned with two sets of additional assumptions on (S,Q), namely:

(I) $\mathbb{R}^3 \setminus Q$ is connected, $\mathcal{L}_3(Q) = 0$ and $\mathcal{H}_d(S) > 2$.

(II) $\mathbb{R}^3 \setminus Q$ is connected, $\mathcal{L}_3(Q) = 0$ and there is a smooth complete real analytic surface $\mathcal{B}$ such that Q lies in a single connected component of $\mathbb{R}^3 \setminus \mathcal{B}$, while $\mathcal{H}_d(S \cap \mathcal{B}) > 1$.

Lemma 2.14. *Let $S \subset \mathbb{R}^3$ be closed, $Q \subset \mathbb{R}^3 \setminus S$ be compact, ρ be a finite, positive Borel measure with support contained in Q, and $\mathbf{v}$ a unit vector in $\mathbb{R}^3$.*

(a) The operator $\mathbf{A} : \mathcal{M}(S)^3 \to L^2(Q,\rho)$ defined in (2.21) *is compact.*

(b) Each function in the range of $\mathbf{A}^*$ is the restriction to S of a real-analytic $\mathbb{R}^3$-valued function on $\mathbb{R}^3 \setminus Q$.

(c) If either assumption (I) *or* (II) *holds, then $\mathbf{A}^*$ is injective, hence $\mathbf{A}$ has dense range.*

(d) If $Q, S, \mathbf{v}$ satisfy the assumptions of Lemma 2.4 and the support of ρ contains $Q \cap \mathcal{A}$, then every element in the kernel of $\mathbf{A}$ is S-silent.

Proof. Let $h := \operatorname{dist}(S,Q) > 0$. Outside an open ball of radius h, the kernel $\mathbf{K}_\mathbf{v}$ and its first order derivatives are bounded, say by some constant C. Thus, if $\mathbf{M}_n$ is a sequence in the unit ball of $\mathcal{M}(S)^3$, then $|B_\mathbf{v}(\mathbf{M}_n)|$ and its partial derivatives are bounded by C on Q. Therefore $B_\mathbf{v}(\mathbf{M}_n)$ is a uniformly bounded family of equicontinuous functions on the compact set Q, hence it is relatively compact in the uniform topology by Ascoli's Theorem. Therefore, this family is relatively compact in $L^2(Q,\rho)$. Besides, since $\mathbf{K}_\mathbf{v}$ is a harmonic vector field in $\mathbb{R}^3 \setminus \{0\}$, differentiating (2.23) under the integral sign shows that the components of $\mathbf{A}^*(\Psi)$ are harmonic in $\mathbb{R}^3 \setminus Q$, thus, real analytic.

To see that $\mathbf{A}$ has dense range if either (I) or (II) is satisfied, we prove that $\mathbf{A}^*$ is injective in this case. For this, assume that $\mathbf{A}^*\Psi = 0$ for some $\Psi \in L^2(Q,\rho)$ and let us show that Ψ is zero ρ-a.e. Assume first that (II) holds, and consider the $\mathbb{R}^3$-valued function

$$\mathbf{D}(\mathbf{x}) = \frac{1}{4\pi}\int \Psi(\mathbf{y})\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\,\mathrm{d}\rho(\mathbf{y}), \qquad \mathbf{x} \in \mathbb{R}^3 \setminus Q.$$

Arguing as we did to get (2.5) and observing that $L^2(Q,\rho) \subset L^1(Q,\rho)$ since ρ is finite, we find that $\mathbf{D}$ extends to a locally integrable function on $\mathbb{R}^3$ with $\operatorname{div} \mathbf{D} = \Psi \mathrm{d}\rho$ as distributions. Note that $\operatorname{grad}(\mathbf{D} \cdot \mathbf{v})$ is a harmonic vector field on $\mathbb{R}^3 \setminus Q$ which is equal to $-\mathbf{A}^*\Psi/\mu_0$ on S, hence it vanishes there. Since $S \cap \mathcal{B}$ has Hausdorff dimension strictly greater than 1 and $\mathcal{B}$ is real analytic, it holds that $\operatorname{grad}(\mathbf{D} \cdot \mathbf{v})$ vanishes on $\mathcal{B}$. Moreover, as $\mathbf{K}_\mathbf{v}$ vanishes at infinity and Ψ has compact support, $\operatorname{grad}(\mathbf{D} \cdot \mathbf{v})$ vanishes at infinity as well. Thus, by the maximum principle, it must vanish in the component of $\mathbb{R}^3 \setminus \mathcal{B}$ which does not contain Q, therefore also in $\mathbb{R}^3 \setminus Q$ by real analyticity and since it is connected. This means that $\mathbf{D} \cdot \mathbf{v}$ is constant in $\mathbb{R}^3 \setminus Q$, and it is in fact identically zero because it is clear from the compactness of Q that $\mathbf{D}$ vanishes at infinity. Now, it holds that $\mathbf{D} = \operatorname{grad}(\Gamma * (\Psi \mathrm{d}\rho))$. Hence, $\Gamma * (\Psi \mathrm{d}\rho)$ must be constant on half lines parallel to v contained in

$\mathbb{R}^3 \setminus Q$, and since it vanishes at infinity while Q is compact we find that $\Gamma * (\Psi d\rho)$ is identically zero in $\mathbb{R}^3 \setminus Q$. Now, being the Newton potential of a finite measure, $\Gamma * (\Psi \mathrm{d}\rho)$ is a locally integrable function and, since $\mathcal{L}_3(Q) = 0$, we just showed that it is zero almost everywhere. Hence it is the zero distribution, and so is its weak Laplacian $\Psi \mathrm{d}\rho$. Consequently Ψ is zero ρ-a.e., as desired. If (I) holds instead of (II) the proof of (c) is similar but easier, because we conclude directly that $\operatorname{grad}(\mathbf{D} \cdot \mathbf{v}) = 0$ on $\mathbb{R}^3 \setminus Q$, since it is harmonic there and vanishes on S which has Hausdorff dimension strictly greater than 2. Finally, to prove (d), observe that if $\mathbf{A}(\mathbf{M}) = 0$ a.e. with respect to ρ, then by continuity $\mathbf{A}(\mathbf{M}) = 0$ on the support of ρ and so on $Q \cap \mathcal{A}$. Thus, by Lemma 2.4 $\mathbf{M}$ is S-silent whenever $\mathbf{M}$ is in the kernel of $\mathbf{A}$. □

From the point of view of inverse problems, S will be a set which is known to contain the support of the sources to be recovered, and Q is the set on which the component of the field in the direction v is measured. It may look strange to assume that $\mathcal{L}_3(Q) = 0$, for it seems that the bigger Q the more information we gain from the measurements. We commented on this assumption, in the remark after Lemma 2.4 already. Let us add here that it teams up with (I) or (II) to make for a dense image of $\mathbf{A}$. This simplifies somewhat the derivation of consistency results like Theorem 2.18, which typically require determining the closure of the image of the forward operator. For instance, in paleomagnetism, Q would be a planar domain and S a rock sample which is either volumic (then (I) is met) or sanded down to a thin slab (then (II) is met, with $\mathcal{B}$ a plane).

2.3 Regularization by penalizing the total variation

In this section, we consider the inverse magnetization problem of recovering $\mathbf{M} \in \mathcal{M}(S)^3$ from the knowledge of $\mathbf{A}(\mathbf{M})$, where $\mathbf{A}$ is the operator defined in (2.21). We will study the regularization scheme EP-2, based on (2.3), that penalizes the total variation of the candidate approximant, and prove that solutions to EP-2 exist and are necessarily "localized", in the sense that their support has dimension at most 2 if S has nonempty interior in $\mathbb{R}^3$ (which falls under assumption (I) in Section 2.2), and dimension at most 1 if S is contained in some unbounded analytic surface where it has nonempty interior (which falls under assumption (II) in Section 2.2). The existence of a solution to EP-2, as well as the optimality condition given in Theorem 2.15, fall under the scope of [22, prop. 3.6] and could just have been referenced. We nevertheless provide a proof, partly because it may be interesting in its own right as it is independent from the Fenchel duality used in [22], but mainly because we want to discuss non-uniqueness in a specific manner which is used to establish Theorem 3.16. We conclude this section with a 'consistency' result showing that solutions to EP-2 approach those of EP-1, in the limit that the regularization parameter λ and the (additive) perturbation on the data vanish in a controlled manner. Our account of this regularization theory is new inasmuch as it includes the asymptotic behavior of total variation measures of the solutions, and deals with narrow convergence (not just weak-*).

Hereafter, as in Section 2.2, we let $S \subset \mathbb{R}^3$ be closed, $Q \subset \mathbb{R}^3 \setminus S$ be compact, ρ be a finite, positive Borel measure supported in Q, and v a unit vector in $\mathbb{R}^3$. The operator $\mathbf{A}$ is then defined by (2.21). For $\mathbf{M} \in \mathcal{M}(S)^3$, $f \in L^2(Q, \rho)$, and $\lambda > 0$, we recall from (2.2) the definition of $\mathcal{F}_{f,\lambda}$:

$$\mathcal{F}_{f,\lambda}(\mathbf{M}) := \|f - \mathbf{A}\mathbf{M}\|^2_{L^2(Q,\rho)} + \lambda \|\mathbf{M}\|_{TV}, \tag{2.25}$$

and from (2.3) that $\mathbf{M}_\lambda \in \mathcal{M}(S)^3$ denotes a minimizer of $\mathcal{F}_{f,\lambda}$ whose existence is proved in Theorem 2.15 below; i.e.,

$$\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda) = \inf_{\mathbf{M} \in \mathcal{M}(S)^3} \mathcal{F}_{f,\lambda}(\mathbf{M}). \tag{2.26}$$

Theorem 2.15. *Given $f \in L^2(Q,\rho)$, notations being as above, a solution to (2.26) does exist. A $\mathbb{R}^3$-valued measure $\mathbf{M}_\lambda \in \mathcal{M}(S)^3$ is such a solution if and only if:*

$$\begin{array}{lll} \mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda) & = \frac{\lambda}{2}\mathbf{u}_{\mathbf{M}_\lambda} & |\mathbf{M}_\lambda|\text{-a.e. and} \\ |\mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda)| & \leq \frac{\lambda}{2} & \text{everywhere on } S. \end{array} \tag{2.27}$$

Moreover, $\mathbf{M}'_\lambda \in \mathcal{M}(S)^3$ is another solution if and only if:

(a) $\mathbf{A}(\mathbf{M}'_\lambda - \mathbf{M}_\lambda) = 0$,

(b) there is a $|\mathbf{M}_\lambda|$-measurable non-negative function g and a positive measure $N_s \in \mathcal{M}(S)$, singular to $|\mathbf{M}_\lambda|$ and supported on the set $\{\mathbf{x} \in S\colon\ |\mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda)(\mathbf{x})| = \lambda/2\}$, such that

$$\mathrm{d}\mathbf{M}'_\lambda = g\mathrm{d}\mathbf{M}_\lambda + 2\frac{\mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda)}{\lambda}\mathrm{d}N_s. \tag{2.28}$$

Proof. Fix $\lambda > 0$ and let $\mathbf{M}_n$ a minimizing sequence for the right hand side of (2.26). By construction $\|\mathbf{M}_n\|_{TV}$ is bounded, hence we can find a subsequence that converges weak-$*$ to some $\mathbf{M}_\lambda$, by the Banach-Alaoglu Theorem. Renumbering if necessary, let us denote this subsequence by $\mathbf{M}_n$ again. The Banach-Alaoglu Theorem also entails that

$$\|\mathbf{M}_\lambda\|_{TV} \leq \liminf_n \|\mathbf{M}_n\|_{TV}. \tag{2.29}$$

Moreover, since $\mathbf{A}$ is compact, $f - \mathbf{A}(\mathbf{M}_n)$ converges to $f - \mathbf{A}(\mathbf{M}_\lambda)$ in $L^2(Q,\rho)$, hence

$$\|f - \mathbf{A}(\mathbf{M}_\lambda)\|_{L^2(Q,\rho)} = \lim_n \|f - \mathbf{A}(\mathbf{M}_n)\|_{L^2(Q,\rho)}. \tag{2.30}$$

Because $\mathbf{M}_n$ is minimizing, it now follows from (2.29) and (2.30) that $\mathbf{M}_\lambda$ meets (2.26) and that (2.29) is both an equality and a true limit.

Let now $\mathbf{N} \in \mathcal{M}(S)^3$ be absolutely continuous with respect to $|\mathbf{M}_\lambda|$ with Radon-Nykodim derivative $\mathbf{N}_a \in (L^1(\mathbf{M}_\lambda))^3$; that is to say: $\mathrm{d}\mathbf{N} = \mathbf{N}_a\mathrm{d}|\mathbf{M}_\lambda|$.

We evaluate $\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda + t\mathbf{N})$ for small t. On the one hand,

$$\|f-\mathbf{A}(\mathbf{M}_\lambda+t\mathbf{N})\|^2_{L^2(Q,\rho)} = \|f-\mathbf{A}(\mathbf{M}_\lambda)\|^2_{L^2(Q,\rho)} - 2t\langle f-\mathbf{A}(\mathbf{M}_\lambda)\,,\,\mathbf{A}(\mathbf{N})\rangle + t^2\|\mathbf{A}(\mathbf{N})\|^2_{L^2(Q,\rho)}. \tag{2.31}$$

On the other hand, since it has unit norm $|\mathbf{M}_\lambda|$-a.e., the Radon-Nykodim derivative $\mathbf{u}_{\mathbf{M}_\lambda}$ has a unique norming functional when viewed as an element of $(L^1(\mathbf{M}_\lambda))^3$, given by

$$\boldsymbol{\Psi} \mapsto \int \boldsymbol{\Psi} \cdot \mathbf{u}_{\mathbf{M}_\lambda}\ \mathrm{d}|\mathbf{M}_\lambda|, \qquad \boldsymbol{\Psi} \in (L^1(\mathbf{M}_\lambda))^3.$$

Hence, the $(L^1(\mathbf{M}_\lambda))^3$-norm is Gâteaux differentiable at $\mathbf{u}_{\mathbf{M}_\lambda}$ [35, Part 3, Ch. 1, Prop. 2, Remark 1] and we get that

$$\|\mathbf{M}_\lambda + t\mathbf{N}\|_{TV} = \int |\mathbf{u}_{\mathbf{M}_\lambda} + t\mathbf{N}_a|\ \mathrm{d}|\mathbf{M}_\lambda| = \|\mathbf{M}_\lambda\|_{TV} + t\int \mathbf{N}_a \cdot \mathbf{u}_{\mathbf{M}_\lambda}\mathrm{d}|\mathbf{M}_\lambda| + t\varepsilon(t), \tag{2.32}$$

where $\varepsilon(t) \to 0$ when $t \to 0$. From (2.31) and (2.32), we gather that

$$\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda + t\mathbf{N}) - \mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda) = -2t\langle \mathbf{A}^*(f - \mathbf{A}(\mathbf{M}_\lambda))\,,\,\mathbf{N}\rangle + t\lambda\int \mathbf{N}_a \cdot \mathbf{u}_{\mathbf{M}_\lambda}\ \mathrm{d}|\mathbf{M}_\lambda| + o(t).$$

The left hand side is nonnegative by definition of $\mathbf{M}_\lambda$, so the coefficient of t in the right hand side is zero otherwise we could adjust the sign for small $|t|$. Consequently

$$\int (-2\mathbf{A}^*(f - \mathbf{A}(\mathbf{M}_\lambda)) + \lambda \mathbf{u}_{\mathbf{M}_\lambda}) \cdot \mathbf{N}_a \, \mathrm{d}|\mathbf{M}_\lambda| = 0, \qquad \mathbf{N}_a \in L^1(|\mathbf{M}_\lambda|),$$

which implies the first equation in (2.27).

Assume next that the second inequality in (2.27) is violated:

$$|\mathbf{A}^*(f - \mathbf{A}\mathbf{M}_\lambda)|(\mathbf{x}) > \lambda/2 \tag{2.33}$$

for some $\mathbf{x} \in S$. Then $|\mathbf{M}_\lambda|(\{\mathbf{x}\}) = 0$ by the first part of (2.27) just proven, and the measure

$$\mathbf{N} = \frac{\mathbf{A}^*(f - \mathbf{A}\mathbf{M}_\lambda)(\mathbf{x})}{|\mathbf{A}^*(f - \mathbf{A}\mathbf{M}_\lambda)|(\mathbf{x})} \delta_{\mathbf{x}}, \tag{2.34}$$

is singular with respect to $\mathbf{M}_\lambda$. Hence, for $t > 0$,

$$\|\mathbf{M}_\lambda + t\mathbf{N}\|_{TV} = \|\mathbf{M}_\lambda\|_{TV} + t\|\mathbf{N}\|_{TV} = \|\mathbf{M}_\lambda\|_{TV} + t, \tag{2.35}$$

and it follows from (2.31), (2.34) and (2.35) that

$$\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda + t\mathbf{N}) - \mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda) = -2t|\mathbf{A}^*(f - \mathbf{A}\mathbf{M}_\lambda)|(\mathbf{x}) + t\lambda + O(t^2)$$

which is strictly negative for $t > 0$ small enough, in view of (2.33). But this cannot hold since $\mathbf{M}_\lambda$ is a minimizer of (2.25), thereby proving the second inequality in (2.27) by contradiction.

Conversely, assume that (2.27) holds. Let $\mathbf{N} \in \mathcal{M}(S)^3$ and write the Radon-Nykodim decomposition of $\mathbf{N}$ with respect to $\mathbf{M}_\lambda$ as $d\mathbf{N} = \mathbf{N}_a d|\mathbf{M}_\lambda| + d\mathbf{N}_s$, where $\mathbf{N}_a \in L^1(|\mathbf{M}_\lambda|)$ and $\mathbf{N}_s$ is singular with respect to $|\mathbf{M}_\lambda|$. Setting $t = 1$ in (2.31), we get that

$$\begin{aligned} \|f - \mathbf{A}(\mathbf{M}_\lambda + \mathbf{N})\|^2_{L^2(Q,\rho)} \quad - \quad & \|f - \mathbf{A}(\mathbf{M}_\lambda)\|^2_{L^2(Q,\rho)} \geq -2\langle f - \mathbf{A}(\mathbf{M}_\lambda) \, , \, \mathbf{A}(\mathbf{N})\rangle \\ &= -2 \int \mathbf{A}^*(f - \mathbf{A}(\mathbf{M}_\lambda)) \cdot \mathbf{N}_a \, \mathrm{d}|\mathbf{M}_\lambda| - 2\langle \mathbf{A}^*(f - \mathbf{A}(\mathbf{M}_\lambda)) \, , \, \mathbf{N}_s\rangle \\ &= -\lambda \int (\mathbf{N}_a \cdot \mathbf{u}_{\mathbf{M}_\lambda}) \mathrm{d}|\mathbf{M}_\lambda| - 2\langle \mathbf{A}^*(f - \mathbf{A}(\mathbf{M}_\lambda)) \, , \, \mathbf{N}_s\rangle \\ &\geq -\lambda \int (\mathbf{N}_a \cdot \mathbf{u}_{\mathbf{M}_\lambda}) \mathrm{d}|\mathbf{M}_\lambda| - \lambda \|\mathbf{N}_s\|_{TV}. \end{aligned} \tag{2.36}$$

In another connection, we have that

$$\|\mathbf{M}_\lambda + \mathbf{N}\|_{TV} = \int |\mathbf{u}_{\mathbf{M}_\lambda} + \mathbf{N}_a| \, \mathrm{d}|\mathbf{M}_\lambda| + \|\mathbf{N}_s\|_{TV} = \int (1 + 2\mathbf{N}_a \cdot \mathbf{u}_{\mathbf{M}_\lambda} + |\mathbf{N}_a|^2)^{1/2} \, \mathrm{d}|\mathbf{M}_\lambda| + \|\mathbf{N}_s\|_{TV},$$

and since $|\mathbf{u}_{\mathbf{M}_\lambda}| = 1$ a.e. with respect to $|\mathbf{M}_\lambda|$, we obtain:

$$(1 + 2\mathbf{N}_a \cdot \mathbf{u}_{\mathbf{M}_\lambda} + |\mathbf{N}_a|^2)^{1/2} \geq |1 + \mathbf{N}_a \cdot \mathbf{u}_{\mathbf{M}_\lambda}|, \qquad |\mathbf{M}_\lambda|\text{-a.e.} \tag{2.37}$$

Thus, if we let E_+ (resp. E_-) be the subset of $\operatorname{supp}|\mathbf{M}_\lambda|$ where $\mathbf{u}_{\mathbf{M}_\lambda} \cdot \mathbf{N}_a > -1$ (resp. ≤ -1), we obtain:

$$\lambda\|\mathbf{M}_\lambda+\mathbf{N}\|_{TV}\geq \lambda\int_{E_+}(1+\mathbf{N}_a\cdot\mathbf{u}_{\mathbf{M}_\lambda})\mathrm{d}|\mathbf{M}_\lambda|+\lambda\|\mathbf{N}_s\|_{TV}, \tag{2.38}$$

Besides, it follows from (2.36) that

$$\|f-\mathbf{A}(\mathbf{M}_\lambda+\mathbf{N})\|^2_{L^2(Q,\rho)}-\|f-\mathbf{A}(\mathbf{M}_\lambda)\|^2_{L^2(Q,\rho)}\geq -\lambda\int_{E_+}(\mathbf{N}_a\cdot\mathbf{u}_{\mathbf{M}_\lambda})\mathrm{d}|\mathbf{M}_\lambda|+\lambda\int_{E_-}\mathrm{d}|\mathbf{M}_\lambda|-\lambda\|\mathbf{N}_s\|_{TV}. \tag{2.39}$$

Adding up (2.38) and (2.39), using that $\|\mathbf{M}_\lambda\|_{TV}=\int_{E_+}\mathrm{d}|\mathbf{M}_\lambda|+\int_{E_-}\mathrm{d}|\mathbf{M}_\lambda|$, we obtain:

$$\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda+\mathbf{N})-\mathcal{F}_{f,\lambda}(\mathbf{M}_\lambda)\geq 0, \tag{2.40}$$

thereby showing that $\mathbf{M}_\lambda$ indeed meets (2.26).

Finally, observe that in the previous estimates we neglected the term $t^2\|\mathbf{A}\mathbf{N}\|^2_{L^2(Q,\rho)}$ in (2.31) and the term $|\mathbf{N}_a|^2-(\mathbf{N}_a\cdot\mathbf{u}_{\mathbf{M}_\lambda})^2$ in (2.37), as well as the term $\lambda\int_{E_-}(|\mathbf{N}_a\cdot\mathbf{u}_{\mathbf{M}_\lambda}|-1)\mathrm{d}|\mathbf{M}_\lambda|$ in (2.38) and (2.39), along with the term $\lambda\|\mathbf{N}_s\|_{TV}-2\langle\mathbf{A}^*(f-\mathbf{A}(\mathbf{M}_\lambda))\,,\,\mathbf{N}_s\rangle$ in (2.36). Hence, equality holds in (2.40) if and only if they are all zero. Thus, for $\mathbf{M}'_\lambda=\mathbf{M}_\lambda+\mathbf{N}$ to be another solution to (2.26), it is necessary and sufficient that $\mathbf{A}\mathbf{N}=0$ and $\mathbf{N}_a=h\mathbf{u}_{\mathbf{M}_\lambda}$ with h a real-valued function such that $h\geq -1$, a.e. with respect to $|\mathbf{M}_\lambda|$, while $\mathbf{N}_s$ is supported on the subset of S where $|\mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda)|=\lambda/2$ and $\mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda)=(\lambda/2)\mathbf{u}_{\mathbf{N}_s}$ at $|\mathbf{N}_s|$-a.e. point. Thus, $\mathbf{M}'_\lambda=g\mathbf{M}_\lambda+\mathbf{N}_s$ with $g:=1+h\geq 0$, which gives us (a) and (b). □

That any two minimizers of (2.25) must differ by a member of the kernel of $\mathbf{A}$ is but a simple consequence of the strict convexity of the $L^2(Q,\rho)$-norm. In particular, if the assumptions on $Q,S,\mathcal{A}$ and v of Lemma 2.4 hold and the support of ρ contains $Q\cap\mathcal{A}$, then any two minimizers are S-equivalent, by (d) of Lemma 2.14. The second assertion of Theorem 2.15 means that when (a) holds, then $\|\mathbf{M}'_\lambda\|_{TV}=\|\mathbf{M}_\lambda\|_{TV}$ if and only if (b) holds.

Corollary 2.16. *Assumptions and notation being as in Theorem 2.15, the union of the supports of all minimizers of (2.25), for fixed f and $\lambda>0$, is contained in a finite union of points, embedded curves and surfaces, each of which is real-analytic and bounded. Furthermore, if there is a unbounded real analytic surface $\mathcal{B}$ such that $\dim_{\mathcal{H}}(S\cap\mathcal{B})>1$, then the aforementioned union of supports has an intersection with $\mathcal{B}$ which is contained in a finite union of points and embedded real analytic curves.*

Proof. Recall from (b) in Lemma 2.14 that $\mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda)$ is the restriction to S of a $\mathbb{R}^3$-valued real analytic vector field on $\mathbb{R}^3\setminus Q$ that vanishes at infinity. Set $g=|\mathbf{A}^*(f-\mathbf{A}\mathbf{M}_\lambda)|^2$ which vanishes at infinity and is a real analytic function $\mathbb{R}^3\setminus Q\to\mathbb{R}$. Theorem 2.15 implies that the support of $|\mathbf{M}_\lambda|$, and also of any other minimizer of (2.25), is included in the zero set of the real analytic function $h:=g-\lambda^2/4$. Note that h is independent of the minimizer $\mathbf{M}_\lambda$ under consideration, since any two have the same image under $\mathbf{A}$ by Theorem 2.15. Now, since h is not the zero function because g vanishes at infinity, its zero set is a finite union of points and real analytic embedded curves and surfaces, see the discussion after Lemma 2.3.

Assume next that there is an unbounded real analytic surface $\mathcal{B}$ with $\dim_{\mathcal{H}}(S\cap\mathcal{B})>1$. If the zero set of h intersected with $S\cap\mathcal{B}$ had Hausdorff dimension strictly greater than 1, then h would be identically zero on $\mathcal{B}$ since it is real analytic, which is impossible because g vanishes at infinity and $\mathcal{B}$ is unbounded. Therefore, the restriction of h to $\mathcal{B}$ is a nonzero real analytic function with compact zero set, consisting necessarily of a finite union of points and embedded real analytic curves. □

Remark 2.17. *For instance in paleomagnetism, when trying to recover magnetizations on thin slabs of rock via the regularization scheme* (2.26)*, a case in which $\mathcal{B}$ is a plane. It is in particular crucial to the proof of Theorem 3.16. Note that if we omit the assumption that $\mathcal{B}$ is unbounded in Corollary 2.16, then we can only conclude that the support of $\mathbf{M}_\lambda$ consists of finitely many points and arcs, or else that $h = 0$ on $\mathcal{B}$. This remark applies, e.g. in MEG inverse problems, where $\mathcal{B}$ is typically a closed surface.*

Even if $f \in \operatorname{range} \mathbf{A}$, say $f = \mathbf{A}(\mathbf{M}_0)$ for some $\mathbf{M}_0 \in \mathcal{M}(S)^3$, it is clear from (2.27) that $\mathbf{M}_\lambda \neq \mathbf{M}_0$ when $\lambda > 0$, unless $\mathbf{M}_0 = 0$. The purpose of the regularizing term $\lambda\|\mathbf{M}\|_{TV}$ in (2.25) is rather to get a $\mathbf{M}_\lambda$ which is not too far from $\mathbf{M}_0$ when f gets replaced by $f_e = f + e$ in (2.26). Here, e is some error (e.g. due to measurements) and f_e represents the actual data. To clarify the matter, whenever $f, e \in L^2(Q, \rho)$ we set $f_e := f + e$ and, for $\lambda > 0$, we let $\mathbf{M}_{\lambda,e}$ be a minimizer of (2.25) when f gets replaced by f_e. Thus, with the notation of (2.26), we have that $\mathbf{M}_\lambda = \mathbf{M}_{\lambda,0}$. Typical results to warrant a regularization approach based on approximating $\mathbf{M}_0$ by $\mathbf{M}_{\lambda,e}$ are of "consistency" type, namely they assert that $\mathbf{M}_{\lambda,e}$ yields information on $\mathbf{M}_0$ as $\|e\|_{L^2(Q,\rho)}$ and λ go to 0 in a combined fashion, see for example [20, thms. 2&5] or [21, thm. 3.5&4.4]. We give below a theorem of this type, which goes beyond [21, thm. 3.5] in that we deal not just with weak-$*$ convergence of subsequences μ_{λ_n,e_n}, but more generally with narrow convergence of both μ_{λ_n,e_n} and $|\mu_{\lambda_n,e_n}|$. We will not consider quantitative convergence properties involving the Bregman distance, that require an additional source condition which needs not be generally satisfied here.

As an extra piece of notation, we define for $\mathbf{M}_0 \in \mathcal{M}(S)^3$:

$$\mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0) := \min\{\|\mathbf{M}\|_{TV} \colon \mathbf{A}(\mathbf{M}) = \mathbf{A}(\mathbf{M}_0)\}. \tag{2.41}$$

The infimum in the right-hand side of (2.41) is indeed attained, by the Banach-Alaoglu theorem and since the kernel of $\mathbf{A}$ is weak-$*$ closed. When S and Q satisfy the conditions of Lemma 2.4, then this kernel consists of S-silent magnetizations and $\mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0)$ is just $\mathfrak{M}(\mathbf{M}_0)$ defined in (2.1). But when these conditions are not satisfied (for instance if S is smooth compact surface), then the two quantities may not coincide.

Recall that a sequence $\mathbf{M}_n \in \mathcal{M}(S)^3$ converges in the narrow sense to $\mathbf{M} \in \mathcal{M}(S)^3$ if $\int \varphi \cdot \mathrm{d}\mathbf{M}_n \to \int \varphi \cdot \mathrm{d}\mathbf{M}$ as $n \to \infty$, whenever $\varphi : S \to \mathbb{R}^3$ is continuous and bounded. When S is compact this is equivalent to weak-$*$ convergence, but if S is unbounded it means that $\mathbf{M}_n$ does not "loose mass at infinity".

Theorem 2.18. *Assumptions and notation being as in Theorem 2.15, given $f \in L^2(Q, \rho)$, the following hold.*

(a) If $f = \mathbf{A}(\mathbf{M}_0)$ with $\mathbf{M}_0 \in \mathcal{M}(S)^3$, while $e \in L^2(Q, \rho)$ and $\lambda > 0$, then

$$\|\mathbf{M}_{\lambda,e}\|_{TV} \leq \frac{\|e\|^2_{L^2(Q,\rho)}}{\lambda} + \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0) \tag{2.42}$$

and

$$\lim_{\lambda \to 0^+,\, \|e\|_{L^2(Q,\rho)}/\sqrt{\lambda} \to 0} \|\mathbf{M}_{\lambda,e}\|_{TV} = \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0). \tag{2.43}$$

As $\lambda \to 0$ and $\|e\|_{L^2(Q,\rho)}/\sqrt{\lambda} \to 0$, any weak-$*$ cluster point $\mathbf{M}^*$ of $\mathbf{M}_{\lambda,e}$ (there must be at least one since $\|\mathbf{M}_{\lambda,e}\|_{TV}$ is bounded) meets $\mathbf{A}(\mathbf{M}^*) = \mathbf{A}(\mathbf{M}_0) = f$ and satisfies:

$$\|\mathbf{M}^*\|_{TV} = \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0). \tag{2.44}$$

Moreover, if $\lambda_n \to 0^+$ and $\|e_n\|_{L^2(Q,\rho)}/\sqrt{\lambda_n} \to 0$, with λ_n, e_n such that $\mathbf{M}_{\lambda_n,e_n}$ converges weak-$*$ to $\mathbf{M}^*$, we have that

$$\lim_{n\to\infty} \int \left| \frac{2\mathbf{A}^*(f_{e_n} - \mathbf{A}\mathbf{M}_{\lambda_n,e_n})}{\lambda_n} - \mathbf{u}_{\mathbf{M}^*} \right| \mathrm{d}|\mathbf{M}^*| = 0, \tag{2.45}$$

also $\mathbf{M}_{\lambda_n,e_n}$ and $|\mathbf{M}_{\lambda_n,e_n}|$ converge respectively to $\mathbf{M}^*$ and $|\mathbf{M}^*|$ in the narrow sense.

(b) If $f \notin \operatorname{range}\mathbf{A}$ and either assumption (I) or (II) in Section 2.2 holds, then $\|\mathbf{M}_{\lambda,e}\|_{TV} \to \infty$ as $\lambda \to 0$ and $e \to 0$.

(c) If $\lambda \geq 2\sup_{\mathbf{x}\in S} |(\mathbf{A}^* f)(\mathbf{x})|$, then the unique minimizer of the right-hand side of (2.1) is the zero magnetization.

Proof. If $f \in \operatorname{range}\mathbf{A}$, or if $\operatorname{range}\mathbf{A}$ is dense in $L^2(Q,\rho)$, it is clear that $\mathcal{F}_{f_e,\lambda}(\mathbf{M}_{\lambda,e}) \to 0$ as $\lambda \to 0$ and $e \to 0$, hence $\|f - \mathbf{A}(\mathbf{M}_{\lambda,e})\|_{L^2(Q,\rho)} \to 0$ in this case. In particular, if $f \notin \operatorname{range}\mathbf{A}$ but either assumption (I) or (II) in Section 2.2 holds, then $\operatorname{range}\mathbf{A}$ is dense by Lemma 2.14 (c) and so $\|\mathbf{M}_{\lambda,e}\|_{TV} \to \infty$ otherwise a subsequence would converge weak-$*$ to some $\mathbf{M}_0 \in \mathcal{M}(S)^3$ implying in the limit that $f = \mathbf{A}(\mathbf{M}_0)$, a contradiction which proves b.

Next, let $\widetilde{\mathbf{M}}_0$ be a minimizer of the right hand side of (2.1), so that $\mathbf{A}(\widetilde{\mathbf{M}}_0) = f$ and $\|\widetilde{\mathbf{M}}_0\|_{TV} = \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0)$. By the optimality of $\mathbf{M}_{\lambda,e}$, we have that

$$\begin{aligned}\|f_e - \mathbf{A}(\mathbf{M}_{\lambda,e})\|^2_{L^2(Q)} + \lambda\|\mathbf{M}_{\lambda,e}\|_{TV} &= \mathcal{F}_{f_e,\lambda}(\mathbf{M}_{\lambda,e}) \leq \mathcal{F}_{f_e,\lambda}(\widetilde{\mathbf{M}}_0) \\ &= \|e\|^2_{L^2(Q,\rho)} + \lambda\|\widetilde{\mathbf{M}}_0\|_{TV} \\ &= \|e\|^2_{L^2(Q,\rho)} + \lambda\mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0),\end{aligned} \tag{2.46}$$

implying that (2.42) holds. Thus, if $\mathbf{M}^*$ is a weak-$*$ cluster point of $\{\mathbf{M}_{\lambda,e}\}$ as $\lambda \to 0^+$ with $\|e\|_{L^2(Q,\rho)} = o(\sqrt{\lambda})$, and if λ_n, e_n are sequences with these limiting properties such that $\mathbf{M}_{\lambda_n,e_n}$ converges weak-$*$ to $\mathbf{M}^*$, we deduce from the Banach-Alaoglu Theorem that

$$\|\mathbf{M}^*\|_{TV} \leq \liminf_n \|\mathbf{M}_{\lambda,e}\|_{TV} \leq \limsup_n \|\mathbf{M}_{\lambda,e}\|_{TV} \leq \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0). \tag{2.47}$$

Also, since $\mathbf{A}$ is weak-$*$ to weak continuous (it is even compact), we get that

$$\|f - \mathbf{A}(\mathbf{M}^*)\|_{L^2(Q,\rho)} \leq \lim_n \|f - \mathbf{A}(\mathbf{M}_{\lambda,e})\|_{L^2(Q,\rho)} = 0, \tag{2.48}$$

where the last equality was obtained in the proof of (b). From (2.48) it follows that $\mathbf{A}(\mathbf{M}^*) = f$, and from (2.47) we now see that (2.44) holds, by definition of $\mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0)$. Moreover, since a weak-$*$ convergent subsequence can be extracted from any subsequence of $\mathbf{M}_{\lambda,e}$, we deduce from what precedes that (2.43) takes place. Next, if λ_n, e_n are as before, we get in view of (2.43) and (2.46) that $\|f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n})\|^2_{L^2(Q,\rho)} = o(\lambda_n)$, which is equivalent to

$$0 = \lim_n \frac{2}{\lambda_n}\langle f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n}), f - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n})\rangle + \frac{2}{\lambda_n}\langle f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n}), e_n\rangle,$$

and since $\|e_n\|_{L^2(Q,\rho)} = o(\sqrt{\lambda_n})$ while $f = \mathbf{A}(\mathbf{M}^*)$, we obtain:

$$\begin{aligned}
&0 = \lim_n \left\langle \tfrac{2\mathbf{A}^*(f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n}))}{\lambda_n}, \mathbf{M}_{\lambda_n,e_n} - \mathbf{M}^* \right\rangle = \lim_n \left\langle \tfrac{2\mathbf{A}^*(f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n}))}{\lambda_n}, \mathbf{M}_{\lambda_n,e_n} \right\rangle \\
&- \lim_n \left\langle \tfrac{2\mathbf{A}^*(f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n}))}{\lambda_n}, \mathbf{M}^* \right\rangle = \lim_n \left(\|\mathbf{M}_{\lambda_n,e_n}\|_{TV} - \int \tfrac{2\mathbf{A}^*(f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n}))}{\lambda_n} \cdot \mathrm{d}\mathbf{M}^* \right) \\
&= \|\mathbf{M}^*\|_{TV} - \lim_n \int \tfrac{2\mathbf{A}^*(f_{e_n} - \mathbf{A}(\mathbf{M}_{\lambda_n,e_n}))}{\lambda_n} \cdot \mathbf{u}_{\mathbf{M}^*} \mathrm{d}|\mathbf{M}^*|,
\end{aligned} \tag{2.49}$$

where we used the first relation in (2.27) to get the third equality and (2.43), (2.44) to get the last one. By the second relation in (2.27), we know that $|2\mathbf{A}^*(f_{e_n} - \mathbf{A}\mathbf{M}_{\lambda_n})/\lambda_n| \le 1$ everywhere on S, hence (2.49) implies that for any $\varepsilon > 0$

$$\limsup_n |\mathbf{M}^*| \left\{ \mathbf{x} \in S : \; \frac{2\mathbf{A}^*(f_{e_n} - \mathbf{A}\mathbf{M}_{\lambda_n,e_n})}{\lambda_n} \cdot \mathbf{u}_{\mathbf{M}^*} < 1 - \varepsilon \right\} = 0.$$

Therefore, using the Borel-Cantelli Lemma and a diagonal argument, we may extract a subsequence λ_{k_n} for which $2\mathbf{A}^*(f_{\mathbf{e}_{k_n}} - \mathbf{A}\mathbf{M}_{\lambda_{k_n},\mathbf{e}_{k_n}})/\lambda_{k_n}$ converges pointwise $|\mathbf{M}^*|$-a.e. to $\mathbf{u}_{\mathbf{M}^*}$. So, by dominated convergence, it follows that

$$\lim_{n\to\infty} \int \left| \frac{2\mathbf{A}^*(f_{\mathbf{e}_{k_n}} - \mathbf{A}(\mathbf{M}_{\lambda_{k_n},\mathbf{e}_{k_n}}))}{\lambda_{k_n}} - \mathbf{u}_{\mathbf{M}^*} \right| \mathrm{d}|\mathbf{M}^*| = 0,$$

and since the reasoning can be applied to any subsequence of λ_n, we obtain (2.45).

We now prove that $|\mathbf{M}_{\lambda_n,e_n}|$ converges weak-$*$ to $|\mathbf{M}^*|$. For this, it is enough to show that if $|\mathbf{M}_{\lambda_n,e_n}|$ converges weak-$*$ to $\mathbf{N} \ge 0 \in \mathcal{M}(S)$, then $\mathbf{N} = |\mathbf{M}^*|$. For this, let $\psi : S \to [0,1]$ be a continuous function with compact support. Pick $\varepsilon > 0$, and then n_ε such that the integral in the left-hand side of (2.45) is less than ε for $n \ge n_\varepsilon$. As $|2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))/\lambda_{n_\varepsilon}| \le 1$ everywhere on S by (2.27), we obtain from the definition of n_ε and (2.44) that

$$\begin{aligned}
&\int \psi \, \mathrm{d}\mathbf{N} = \lim_n \int \psi \, \mathrm{d}|\mathbf{M}_{\lambda_n,e_n}| \ge \lim_n \left| \int \psi \tfrac{2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))}{\lambda_{n_\varepsilon}} \cdot \mathrm{d}\mathbf{M}_{\lambda_n,e_n} \right| \\
&= \left| \int \psi \tfrac{2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))}{\lambda_{n_\varepsilon}} \cdot \mathrm{d}\mathbf{M}^* \right| \ge \int \psi \mathrm{d}|\mathbf{M}^*| - \int \left| \tfrac{2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))}{\lambda_{n_\varepsilon}} - \mathbf{u}_{\mathbf{M}^*} \right| \cdot \mathrm{d}|\mathbf{M}^*| \\
&\ge \int \psi \mathrm{d}|\mathbf{M}^*| - \varepsilon,
\end{aligned}$$

where we used in the equality above that $\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))$ is continuous on S, by Lemma 2.14 (b). Since $\varepsilon > 0$ was arbitrary, we conclude that $\mathbf{N} - |\mathbf{M}^*| \ge 0$. However, since $\||\mathbf{M}^*|\|_{TV} = \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0)$ by (2.44), whereas $\|\mathbf{N}\|_{TV} \le \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0)$ by the Banach Alaoglu theorem, we conclude that $\mathbf{N} - |\mathbf{M}^*|$ is the zero measure, as desired.

To establish that $\mathbf{M}_{\lambda_n,e_n}$ converges to $\mathbf{M}^*$ in the narrow sense, pick $\varepsilon > 0$ and n_ε as before. Fix R_ε so large that $|\mathbf{M}^*|(S \cap \overline{B}(0, R_\varepsilon) > \|\mathbf{M}^*\|_{TV} - \varepsilon$ and for each R let $\psi_R : S \to [0,1]$ be continuous, identically 1 on $S \cap B(0,R)$ and 0 on $S \setminus B(0,2R)$. Reasoning as before, we get that

$$
\begin{aligned}
\|\mathbf{M}^*\|_{TV} &\geq \limsup_n |\mathbf{M}_{\lambda_n,e_n}|(S\cap \overline{B}(0,2R_\varepsilon)) \geq \limsup_n \int \psi_{R_\varepsilon}\, \mathrm{d}|\mathbf{M}_{\lambda_n,e_n}| \\
&\geq \lim_n \left| \int \psi_{R_\varepsilon} \frac{2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))}{\lambda_{n_\varepsilon}} \cdot \mathrm{d}\mathbf{M}_{\lambda_n,e_n} \right| \\
&= \left| \int \psi_{R_\varepsilon} \frac{2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))}{\lambda_{n_\varepsilon}} \cdot \mathrm{d}\mathbf{M}^* \right| \\
&\geq \left| \int \psi_{R_\varepsilon} \mathbf{u}_{\mathbf{M}^*} \cdot \mathrm{d}\mathbf{M}^* \right| - \left| \int \psi_{R_\varepsilon} \left(\frac{2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))}{\lambda_{n_\varepsilon}} - \mathbf{u}_{\mathbf{M}^*} \right) \cdot \mathrm{d}\mathbf{M}^* \right| \\
&\geq \int \psi_{R_\varepsilon} \mathrm{d}|\mathbf{M}^*| - \int \left| \frac{2\mathbf{A}^*(f_{\mathbf{e}_{n_\varepsilon}} - \mathbf{A}(\mathbf{M}_{\lambda_{n_\varepsilon},\mathbf{e}_{n_\varepsilon}}))}{\lambda_{n_\varepsilon}} - \mathbf{u}_{\mathbf{M}^*} \right| \cdot \mathrm{d}|\mathbf{M}^*| \geq |\mathbf{M}^*|(S\cap \overline{B}(0,R)) - \varepsilon \\
&\geq \|\mathbf{M}^*\|_{TV} - 2\varepsilon.
\end{aligned}
$$

Hence, in view of (2.43) and (2.44), we see from what precedes that for n large enough $|\mathbf{M}_{\lambda_n,e_n}|(S\setminus \overline{B}(0,2R_\varepsilon)) \leq 3\varepsilon$, say. Therefore, if we fix a bounded and continuous $\varphi : S \to \mathbb{R}^3$ with $|\varphi| \leq M$, we have since $\mathbf{M}_{\lambda_n,e_n}$ converges weak-$*$ to $\mathbf{M}^*$ that

$$
\begin{aligned}
\limsup_n \left| \int \varphi . d(\mathbf{M}_{\lambda_n,e_n} - \mathbf{M}^*) \right| &\leq \limsup_n \left| \int \psi_{2R_\varepsilon}\varphi \cdot \mathrm{d}(\mathbf{M}_{\lambda_n,e_n} - \mathbf{M}^*) \right| + \\
&\qquad \limsup_n \left| \int (1-\psi_{2R_\varepsilon})\varphi \cdot \mathrm{d}(\mathbf{M}_{\lambda_n,e_n} - \mathbf{M}^*) \right| \\
&\leq 0 + 6M\varepsilon.
\end{aligned}
$$

Because $\varepsilon > 0$ was arbitrary, we deduce that $\mathbf{M}_{\lambda_n,e_n}$ converges to $\mathbf{M}^*$ in the narrow sense, and the fact that $|\mathbf{M}_{\lambda_n,e_n}|$ converges to $|\mathbf{M}^*|$ in the narrow sense as well can be shown in a similar way. This proves (a).

Finally, suppose $\lambda \geq 2\sup_{\mathbf{x}\in S}|(\mathbf{A}^* f)(\mathbf{x})|$. Theorem 2.15 shows that the zero magnetization is a minimizer of $\mathcal{F}_{f,\lambda}$ and that any other minimizer $\mathbf{M}$ must be silent, but then $\mathcal{F}_{f,\lambda}(\mathbf{M}) = \|f\| + \lambda\|\mathbf{M}\|_{TV}$ showing that in fact the zero magnetization is the unique minimizer of $\mathcal{F}_{f,\lambda}$. □

From (a) of Theorem 2.18, any sequence $\mathbf{M}_{\lambda_n,e_n}$ with $\lambda_n = o(1)$ and $\|e_n\|_{L^2(Q)} = o(\sqrt{\lambda_n})$ has a subsequence converging in the narrow sense to some $\mathbf{M}^*$ such that $\mathbf{A}(\mathbf{M}^*) = \mathbf{A}(\mathbf{M}_0) = f$ and $\|\mathbf{M}^*\|_{TV} = \mathfrak{M}_{\mathbf{A}}(\mathbf{M}_0)$. If such a $\mathbf{M}^*$ is unique, we get narrow convergence of $\mathbf{M}_{\lambda,e}$ to $\mathbf{M}^*$ when $\lambda \to 0$ and $\|e\|_{L^2(Q)}/\sqrt{\lambda} \to 0$. Thus Theorems 2.8 and 2.13 give a "sparse recovery" result as follows.

Theorem 2.19. *Let $S, Q \subset \mathbb{R}^3$ satisfy the assumptions of Lemma 2.4 with Q compact, and assume in addition that S is a slender set with $S = \bigcup_{i=1}^n S_i$ for some finite collection of disjoint closed sets $S_1, S_2, \ldots, S_n$. Suppose $\mathbf{M}_0 \in \mathcal{M}(S)^3$ and set $f = \mathbf{A}\mathbf{M}_0$. If either*

(a) $\mathbf{M}_0 \lfloor S_i$ is uni-directional for $i = 1, 2, \ldots, n$,

(b) or supp $\mathbf{M}_0$ *is purely 1-unrectifiable*

then $\mathbf{M}_{\lambda,e}$ converges narrowly to $\mathbf{M}_0$ as $\lambda \to 0$ and $\|e\|_{L^2(Q)}/\sqrt{\lambda} \to 0$.

Remark 2.20. *In the setting of Theorem 2.18 (a), it is generally not true that $\|\mathbf{M}^* - \mathbf{M}_{\lambda_n,e_n}\|_{TV} \to 0$. For instance, let $S \subset \mathbb{R}^2 \times \{0\}$ be compact, assume that $Q \subset \mathbb{R}^2 \times \{h\}$ for some $h > 0$, let ρ be*

2-dimensional Hausdorff measure and $\mathbf{M}_0 = \chi_S v$*, where* $v \in \mathbb{R}^3$*. Then* $\mathbf{M}_0$ *is unidirectional, and we know from Theorem 2.19 that* $\mathbf{M}_{\lambda,e}$ *converges narrowly to* $\mathbf{M}_0$ *as* $\lambda \to 0$ *and* $\|e\|_{L^2(Q)}/\sqrt{\lambda} \to 0$*. Still, the support of* $\mathbf{M}_{\lambda,e}$ *has Hausdorff dimension at most 1, by Corollary 2.16, therefore* $\mathbf{M}_{\lambda,e}$ *and* $\mathbf{M}_0$ *are mutually singular. Hence* $\|\mathbf{M}_{\lambda,e} - \mathbf{M}_0\|_{TV} = \|\mathbf{M}_{\lambda,e}\|_{TV} + \|\mathbf{M}_0\|_{TV}$ *cannot go to zero when* λ *goes to zero.*

Chapter 3

Thin plate case

In this section we will only consider magnetizations supported on $\mathbb{R}^2 \times \{0\}$ and hence, with a slight abuse of notation, given $S \subset \mathbb{R}^2$ and any $\mathrm{M} \in \mathcal{M}(S \times \{0\})$, we shall identify S with $S \times \{0\} \subset \mathbb{R}^3$ and M with $\mathrm{M}\lfloor\mathbb{R}^2$. We also identify $\mathcal{H}_2\lfloor(\mathbb{R}^2 \times \{0\})$ with the 2 dimensional Lebesgue measure on $\mathbb{R}^2$, denoted by $\mathcal{L}_2$.

For any $\mathbf{M} = (\mathrm{M}_1, \mathrm{M}_2, \mathrm{M}_3)^T \in \mathcal{M}(S)^3$, we single out the **tangential component** $\mathbf{M}_T := (\mathrm{M}_1, \mathrm{M}_2)^T \in \mathcal{M}(S)^2$ so that $\mathbf{M} = (\mathbf{M}_T, \mathrm{M}_3)$. In addition, we let $\mathfrak{R}$ denote the rotation by $\pi/2$ in $\mathbb{R}^2$; i.e., $\mathfrak{R}((x_1, x_2)^T) = (-x_2, x_1)^T$.

Remark 3.1. *Any $S \subset \mathbb{R}^2 \times \{0\}$ is a slender set and hence it satisfies the hypothesis of Lemma 2.3, Theorem 2.8 and Theorem 2.19. Also, thanks to Lemma 2.10, any S-silent magnetization has zero moment for such a S.*

3.1 Loop decomposition of silent sources and tree-like magnetizations

For an open set $\Omega \subset \mathbb{R}^2$, recall the space $BV(\Omega)$ of functions of *bounded variation* consists of integrable functions whose first order distributional derivatives are signed measures on Ω (see, [36]). We let $BV_{loc}(\Omega)$ denote the space of functions whose restriction to any relatively compact open subset Ω_1 of Ω lies in $BV(\Omega_1)$.

If $\phi \in BV(\Omega)$, it follows at once by mollification of continuous functions compactly supported in Ω that

$$\|\operatorname{grad}\phi\|_{TV} = \sup_{\varphi \in C_c^1(\Omega), |\varphi| \leq 1} \int \varphi \cdot \mathrm{d}(\operatorname{grad}\phi) = \sup_{\varphi \in C_c^1(\Omega), |\varphi| \leq 1} \int \phi \operatorname{div}\varphi \, \mathrm{d}\mathcal{L}_2, \tag{3.1}$$

where $C_c^1(\Omega)$ denotes the space of continuously differentiable functions with compact support in Ω and TV refers here to the total variation on Ω.

Lemma 3.2. *If $\phi \in BV_{loc}(\mathbb{R}^2)$ and $\operatorname{grad}\phi \in \mathcal{M}(\mathbb{R}^2)^2$, then there exists $p \in \mathbb{R}$ such that $\phi - p \in L^2(\mathbb{R}^2)$.*

Proof. Assume that $\operatorname{grad}\phi \in \mathcal{M}(\mathbb{R}^2)^2$. Thanks to Poincaré's inequality for BV functions in the plane (see [26, theorem 1 section 5.6.1]) there is a constant $\mathcal{K}$ such that

$$\|\phi - (\phi)_{\mathbb{B}}\|_{L^2(\mathbb{B})} \leq \mathcal{K}\, \|\operatorname{grad}\phi\|_{TV},$$

for all open balls $\mathbb{B} \subset \mathbb{R}^2$, where $(\phi)_{\mathbb{B}} = \left(\int_{\mathbb{B}} \phi \, \mathrm{d}\mathcal{L}_2\right)/\mathcal{L}_2(\mathbb{B})$. Note that $\|\operatorname{grad}\phi\|_{TV} < \infty$ since $\operatorname{grad}\phi \in \mathcal{M}(\mathbb{R}^2)^2$. Given $n \in \mathbb{N}$, let $N_n := \|\phi - (\phi)_{\mathbb{B}(0,n)}\|_{L^2(\mathbb{B}(0,n))}$. Examining the function $c \mapsto \|\phi - c\|^2_{L^2(\mathbb{B})}$ we see that its minimum is attained when $c = (\phi)_{\mathbb{B}}$ from which we may conclude that N_n is an increasing sequence in n. Thus, for $m, n \in \mathbb{N}$ such that $n < m$, we have $N_n \leq N_m \leq \mathcal{K}\, \|\operatorname{grad}\phi\|_{TV}$ and

$$\begin{aligned}\|(\phi)_{\mathbb{B}(0,m)} - (\phi)_{\mathbb{B}(0,n)}\|_{L^2(\mathbb{B}(0,n))} &\leq \|\phi - (\phi)_{\mathbb{B}(0,m)}\|_{L^2(\mathbb{B}(0,n))} + N_n \\ &\leq 2N_m \leq 2\mathcal{K}\, \|\operatorname{grad}\phi\|_{TV} < \infty.\end{aligned} \tag{3.2}$$

Thus $\{N_n\}_{n\in\mathbb{N}}$ converges to a number $N \le \mathcal{K}\ \|\operatorname{grad}\phi\|_{TV}$ and, since the left hand side of (3.2) is $\sqrt{\pi}n|(\phi)_{\mathbb{B}(0,m)} - (\phi)_{\mathbb{B}(0,n)}|$, it follows that $\{(\phi)_{\mathbb{B}(0,n)}\}_{n\in\mathbb{N}}$ is a Cauchy sequence, converging to some number p. Hence, for every $n \in \mathbb{N}$,

$$\|\phi - p\|_{L^2(\mathbb{B}(0,n))} \le N_n + \|p - (\phi)_{\mathbb{B}(0,n)}\|_{L^2(\mathbb{B}(0,n))} \le N + 2\mathcal{K}\ \|\operatorname{grad}\phi\|_{TV},$$

therefore $\phi - p \in L^2(\mathbb{R}^2)$. ☐

Lemma 3.3. *Let S be a closed subset of $\mathbb{R}^2$ and $\mathbf{M} \in \mathcal{M}(S)^3$. The following are equivalent:*

(a) $\mathbf{M}$ is S-silent.

(b) $M_3 = 0$ and $\operatorname{div}\mathbf{M}_T = 0$ in the distributional sense on $\mathbb{R}^2$.

(c) $M_3 = 0$ and $\mathbf{M}_T = \mathfrak{R}\operatorname{grad}\phi = (-\partial_{x_2}\phi, \partial_{x_1}\phi, 0)^T$ for some $\phi \in BV_{loc}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.

Proof. Considering $\mathbf{M}$ as an element of $\mathcal{M}(\mathbb{R}^3)^3$ and observing that $\mathbf{M}$ can be written in tensor product form as $\mathbf{M} = (\mathbf{M}\lfloor\mathbb{R}^2) \otimes \delta_{x_3=0}$, it follows that $\operatorname{div}\mathbf{M} = (\operatorname{div}\mathbf{M}_T) \otimes \delta_{x_3=0} + \mathrm{M}_3 \otimes \delta'_{x_3=0}$, where $\delta_{x_3=0}$ is the Dirac point mass at zero on $\mathbb{R}$ in the variable x_3 and $\delta'_{x_3=0}$ is its distributional derivative. From this and Lemma 2.3, it follows that (a)⇒(b). Next, for any $\phi \in C_c^\infty(\mathbb{R}^3)$, let $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^2)$ be given by $\phi_0(x_1, x_2) = \phi(x_1, x_2, 0)$ and $\phi_1(x_1, x_2) = \partial_{x_3}\phi(x_1, x_2, 0)$. By the definition of distributional derivatives, we get that

$$\langle \operatorname{div}\mathbf{M}, \phi\rangle = -\langle \mathrm{M}_1, \partial_{x_1}\phi_0\rangle - \langle \mathrm{M}_2, \partial_{x_2}\phi_0\rangle - \langle \mathrm{M}_3, \phi_1\rangle, \tag{3.3}$$

which makes it clear, using Lemma 2.3 again, that (b)⇒(a).

Finally we are left to prove (b)⇔(c). Suppose (b) holds. Then $(-\mathrm{M}_2, \mathrm{M}_1)^T$ satisfies the Schwartz rule when viewed as an $\mathbb{R}^2$ valued distribution on $\mathbb{R}^2$; i.e, $\partial_{x_2}(-\mathrm{M}_2) = \partial_{x_1}\mathrm{M}_1$. Therefore, $(-\mathrm{M}_2, \mathrm{M}_1)^T$ is the gradient of a scalar valued distribution ϕ (see, [25]). Moreover, since the components of $\operatorname{grad}\phi$ are finite signed measures, $\phi \in BV_{\mathrm{loc}}$ [37, Theorem 6.7.7] and thanks to Lemma 3.2 there exists a constant p for which $\phi - p \in L^2$. Replacing ϕ by $\phi - p$, we can take $p = 0$ so that (b)⇒(c). In the other direction if $\mathbf{M}_T = (-\partial_{x_2}\phi, \partial_{x_1}\phi)^T$ for some $\phi \in BV_{\mathrm{loc}}(\mathbb{R}^2)$, then $\operatorname{div}\mathbf{M}_T = -\partial_{x_1}\partial_{x_2}\phi + \partial_{x_2}\partial_{x_1}\phi = 0$ so that (c)⇒(b). ☐

We next collect several definitions and properties connected to BV-functions that are central to what follows. For $E \subset \mathbb{R}^2$ a Borel set, the **measure theoretical boundary** of E is the set denoted by $\partial_M E$ defined by

$$\partial_M E := \left\{ \mathbf{x} \in \mathbb{R}^2 : \limsup_{\rho\to 0} \frac{\mathcal{L}_2(\mathbb{B}(\mathbf{x}, \rho) \cap E)}{\mathcal{L}_2(\mathbb{B}(\mathbf{x}, \rho))} > 0 \text{ and } \limsup_{\rho\to 0} \frac{\mathcal{L}_2(\mathbb{B}(\mathbf{x}, \rho) \setminus E)}{\mathcal{L}_2(\mathbb{B}(\mathbf{x}, \rho))} > 0 \right\}.$$

A measurable set $E \subset \mathbb{R}^2$ such that $\operatorname{grad}\chi_E \in \mathcal{M}(\mathbb{R}^2)$ is said to be **of finite perimeter** (In [26, 36], the definition is that $\chi_E \in BV(\mathbb{R}^2)$. The present definition means, in view of Lemma 3.2, that either χ_E or $\chi_{\mathbb{R}^2\setminus E}$ lies in $BV(\mathbb{R}^2)$). For such a set it holds that

$$|\operatorname{grad}\chi_E| = \mathcal{H}_1\lfloor\partial_M E. \tag{3.4}$$

The identity (3.4) can be obtained by combining [26, Section 5.7.3 Theorem 5.1.5 (iii)], which says that (3.4) holds if $\partial_M E$ gets replaced by the so-called reduced boundary of E, with [26, Section 5.8 Lemma 5.5], asserting that $\partial_M E$ differs from the reduced boundary by a set of $\mathcal{H}^1$-measure zero.

This results implies that a set of finite perimeter has a reduced boundary of finite $\mathcal{H}_1$-measure. Nevertheless, as the following example shows, such a set may have a Euclidean boundary of positive $\mathcal{L}_2$ measure.

Example 5. *Let $E_1 = \overline{\mathbb{B}}(0,1) \subset \mathbb{R}^2$ and $\{q_j\}_{j\in\mathbb{N}}$ be a sequence of all points in E_1 with rational coordinates. Having defined inductively a closed set $E_n \subset E_1$, let j_n be the smallest integer such that q_{j_n} belongs to E_n and B_n the largest open ball around q_{n_j}, contained in E_n, with radius $r_n \leq 2^{-n}$ (this open ball could logically be empty, if either E_n contains no q_j –the argument shows that in fact this cannot happen– or if q_{n_j} is a boundary point of E_n). Now define the closed set $E_{n+1} = E_n \setminus B_n$, and let $E = \bigcap E_n$, which is closed.*

Note that E has no interior, hence its Euclidean boundary is E itself. Moreover, we have that $\mathcal{L}_2(E) \geq \pi - \pi \sum_{n=1}^{\infty} r_n^2 \geq \pi(1 - \sum_{n=1}^{\infty} 4^{-n}) > 0$.

Note that each E_n is of finite perimeter. Hence $\{\chi_{E_n}\}$ is a nonincreasing sequence of integrable functions and thus χ_E, their point-wise limit, is integrable. Also, these functions are such that $\|\operatorname{grad}\chi_{E_n}\| \leq 2\pi \sum_{n=0}^{\infty} r_n \leq 4\pi$, therefore we can use [36, Theorem 5.2.1] to the effect that $\chi_E \in BV(\mathbb{R}^2)$, i.e. E is a set of finite perimeter.

We then define the **generalized unit inner normal vector** $\boldsymbol{\nu}_E$ to be the Radon-Nikodym derivative $\mathbf{u}_{\operatorname{grad}\chi_E}$ to be the Radon-Nikodym derivative $\mathbf{u}_{\operatorname{grad}\chi_E}$. The Radon Nikodym Theorem now gives us the following version of the *Gauss-Green formula*:

Lemma 3.4. *Let $E \subset \mathbb{R}^2$ be a set of finite perimeter. Then for each Borel set $B \subset \mathbb{R}^2$:*

$$\operatorname{grad}\chi_E(B) = \int_B \boldsymbol{\nu}_E \, \mathrm{d}\,(\mathcal{H}_1 \lfloor \partial_M E)\,, \tag{3.5}$$

or, equivalently, $\mathrm{d}\operatorname{grad}\chi_E = \boldsymbol{\nu}_E \mathrm{d}\mathcal{H}_1 \lfloor \partial_M E$ *as measures.*

The connection with the classical Gauss-Green formula becomes transparent from the distributional version of (3.5), namely:

$$\int \chi_E \operatorname{div} \,\boldsymbol{\varphi}\, \mathrm{d}\mathcal{L}_2 = -\int \boldsymbol{\varphi} \cdot \boldsymbol{\nu}_E \, \mathrm{d}\,(\mathcal{H}_1 \lfloor \partial_M E)\,, \quad \boldsymbol{\varphi} \in (C_c^1(\mathbb{R}^2))^2, \tag{3.6}$$

where $C_c^1(\mathbb{R}^2)$ is the space of C^1-smooth functions with compact support. The identity (3.6) was initially proven in the works [38, 39] and [40, 41], see also [26, Section 5.8 Theorem 5.16] and [42, Theorem 10.3.2]).

Whenever $\phi \in BV(\mathbb{R}^2)$, the sup-level sets

$$E_t := \{\mathbf{x} \in \mathbb{R}^2 \mid \phi(\mathbf{x}) > t\} \tag{3.7}$$

have finite perimeter for a.e. $t \in \mathbb{R}$ [26, Section 5.5 Theorem 5.9]. These sup-level sets are the key ingredient of the co-area formula for BV-functions. In Theorem 3.8 to come, we will give a version of this formula for "homogeneous" BV-functions, namely locally integrable functions whose distributional derivatives are finite signed measures, though the function itself needs not be integrable. First, we need a couple of lemmas that will be used for the proof of that theorem. We mention that these lemmas and theorem seem difficult to find in the literature.

If ψ is an integrable function on a real interval (a, b), its essential variation is defined as

$$\operatorname{ess}V_a^b(\psi) := \sup\{\sum_{i=1}^{k} |\psi(t_i) - \psi(t_{i-1})|\}, \tag{3.8}$$

where the supremum is taken over all finite partitions $a < t_0 < t_1 < \cdots < t_k < b$ such that each t_i is a point of approximate continuity of ψ, i.e. a point $\mathbf{x}$ where ψ is continuous with respect to a set of density 1 at $\mathbf{x}$. For instance, Lebesgue points are approximate continuity points [36, Rem. 4.4.5].

For ϕ a locally integrable function on $\mathbb{R}^2$, let us denote by ϕ_{x_1} and ϕ_{x_2} the partial functions of a single variable, i.e. $\phi(x_1, x_2) = \phi_{x_1}(x_2) = \phi_{x_2}(x_1)$. It follows from [36, Thm. 5.3.5] that $\phi \in BV_{loc}(\mathbb{R}^2)$ if and only if, for every bounded rectangle $\mathfrak{Q} = (a, b) \times (c, d) \subset \mathbb{R}^2$:

$$\int_c^d \mathrm{ess}V_a^b(\phi_{x_i})\,\mathrm{d}x_j < \infty \qquad \text{for } (i,j) = (1,2) \quad \text{and} \quad (i,j) = (2,1). \tag{3.9}$$

Moreover, if we let $\mathfrak{Q}' = (a', b') \times (c', d')$ be another bounded open rectangle such that $\overline{\mathfrak{Q}} \subset \mathfrak{Q}'$, the proof shows (This is a slight improvement of what is proven there, based on sharpening a little [36, Thm. 5.3.1].) that

$$|\partial_{x_i}\phi|(\mathfrak{Q}) \leq \int_{c'}^{d'} \mathrm{ess}V_{a'}^{b'}(\phi_{x_i})\,\mathrm{d}x_j \leq |\partial_{x_i}\phi|(\mathfrak{Q}'). \tag{3.10}$$

Lemma 3.5. *If $\phi \in BV_{loc}(\mathbb{R}^2)$, then $\phi^+ = \max\{\phi, 0\}$ and $\phi^- = \max\{-\phi, 0\}$ belong to $BV_{loc}(\mathbb{R}^2)$. Furthermore, if $\mathfrak{Q}$, $\mathfrak{Q}'$, are two bounded open rectangles such that $\overline{\mathfrak{Q}} \subset \mathfrak{Q}'$, it holds that*

$$|\operatorname{grad}\phi^{\pm}|(\mathfrak{Q}) \leq \sqrt{2}|\operatorname{grad}\phi|(\mathfrak{Q}'). \tag{3.11}$$

Proof. By (3.9) and (3.10), it is enough to prove that if ψ is a real integrable function on a real interval (a, b), then $\mathrm{ess}V_a^b(\psi) \geq \mathrm{ess}V_a^b(\psi^+)$. Consider a sum $\sum_{i=1}^k |\psi^+(t_i) - \psi^+(t_{i-1})|$ where the t_i are approximate continuity points of ψ^+, and assume without loss of generality that ψ^+ does not vanish at two consecutive t_i. If $\psi^+(t_i) > 0$, then t_i is an approximate continuity point of ψ and $\psi^+(t_i) = \psi(t_i)$. If on the contrary $\psi^+(t_i) = 0$, then either we can find a Lebesgue point τ_i of ψ in (t_{i-1}, t_{i+1}) with $\psi(\tau_i) < 0$ (we set $t_{-1} = a$ and $t_{k+1} = b$), in which case $|\psi(\tau_i) - \psi(t_{i+1})| > |\psi^+(t_i) - \psi^+(t_{i+1})|$ and $|\psi(\tau_i) - \psi(t_{i-1})| > |\psi^+(t_i) - \psi^+(t_{i-1})|$ (if $i = 1$ or k we ignore the inequality involving a or b), or else $\psi = \psi^+$ a.e. in (t_{i-1}, t_{i+1}) and in particular t_i is an approximate continuity point of ψ with $\psi(t_i) = 0$. Altogether, replacing $\psi^+(t_i)$ by $\psi(t_i)$ or by $\psi(\tau_i)$ at those i such that $\psi^+(t_i) = 0$ and τ_i can be found as above, we form a sum of the type indicated in (3.8) which is no less that $\sum_{i=1}^k |\psi^+(t_i) - \psi^+(t_{i-1})|$. This achieves the proof. □

Lemma 3.6. *If $\phi \in BV_{loc}(\mathbb{R}^2)$ is such that $\operatorname{grad}\phi \in \mathcal{M}(\mathbb{R}^2)^2$, then for a.e. $t \in \mathbb{R}$ the set E_t defined in (3.7) has finite perimeter.*

Proof. By Lemma 3.2, we may assume that $\phi \in L^2(\mathbb{R}^2)$. Then, for any $s > 0$, we have that $\mathcal{L}_2(E_s) < \infty$. By Lemma 3.5, the function $\tilde{\phi}$ which is $\phi - s$ on E_s and 0 elsewhere lies in $BV_{loc}(\mathbb{R}^2)$, and since $\tilde{\phi}$ is integrable it belongs in fact to $BV(\mathbb{R}^2)$. For every $t > s$, E_t is the sup-level set of $\tilde{\phi}$ at level $t - s$, and thus for a.e. $t > s$ it has finite perimeter. If we now consider a sequence $s_n \to 0$, we find by countable additivity of sets of measure zero that E_t has finite perimeter for a.e. $t > 0$.

Analogously, for any $s < 0$, the function $\check{\phi}$ which $\phi - s$ on $\mathbb{R}^2 \setminus E_s$ and zero elsewhere lies in $BV(\mathbb{R}^2)$, and its sup-level set at level $t - s$ coincides with E_t for any $t < s$. Hence, for a.e. $t < 0$, E_t is of finite perimeter. □

Lemma 3.6 implies that for a.e. $t \in \mathbb{R}$, $\operatorname{grad}\chi_{E_t}$ and $|\operatorname{grad}\chi_{E_t}|$ are well defined and hence the integrals in the following lemma and theorem make sense. Lemma 3.7 is proven for BV functions in [42, Theorem 10.3.3]. In the statement below, $\mathbb{R}^+$ refers to the non-negative real numbers.

Lemma 3.7. *If $\phi \in BV_{loc}(\mathbb{R}^2)$ is such that* $\operatorname{grad}\phi \in \mathcal{M}(\mathbb{R}^2)^2$ *and E_t for every $t \in \mathbb{R}$ is as in* (3.7), *then*

$$\int f \mathrm{d}|\operatorname{grad}\phi| = \int_{-\infty}^{\infty} \int f \mathrm{d}|\operatorname{grad}\chi_{E_t}|\,\mathrm{d}t \quad \textit{for each Borel function} \quad f : \mathbb{R}^2 \to \mathbb{R}^+ \tag{3.12}$$

and

$$\int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\phi) = \int_{-\infty}^{\infty} \int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})\,\mathrm{d}t \quad \textit{for each} \quad \boldsymbol{\varphi} \in C_c^1(\mathbb{R}^2)^2 \tag{3.13}$$

(here $C_c^1(\mathbb{R}^2)$ denotes the space of continuously differentiable functions with compact support in $\mathbb{R}^2$.).

Proof. By Lemma 3.2 we may assume that $\phi \in L^2(\mathbb{R}^2)$. Argueing as in the proof of Lemma 3.6, the function ϕ_n equal to $\phi - 1/n$ on $E_{1/n}$, to $\phi + 1/n$ on $\mathbb{R}^2 \setminus E_{-1/n}$, and to zero elsewhere is seen to lie in $BV(\mathbb{R}^2)$ for each integer $n \geq 1$. Denoting by E_t^n the sup-level sets of ϕ_n and applying [42, Theorem 10.3.3] to the latter, we get that (3.12) and (3.13) hold with ϕ_n instead of ϕ and E_t replaced by E_t^n. By inspection, these equalities can be rewritten as

$$\int f d|\operatorname{grad}\phi_n| = \int_{-\infty}^{-1/n} \int f \mathrm{d}|\operatorname{grad}\chi_{E_t}|\,\mathrm{d}t + \int_{1/n}^{\infty} \int f \mathrm{d}|\operatorname{grad}\chi_{E_t}|\,\mathrm{d}t \tag{3.14}$$

and

$$\int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\phi_n) = \int_{-\infty}^{-1/n} \int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})\,\mathrm{d}t + \int_{1/n}^{\infty} \int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})\,\mathrm{d}t. \tag{3.15}$$

On the one hand, since ϕ_n converges pointwise to ϕ and $|\phi_n| \leq 2|\phi| + 2/n$, it converges to ϕ in $L^1_{loc}(\mathbb{R}^2)$ so that, for each $\boldsymbol{\varphi} \in (C_c^1(\mathbb{R}^2))^2$, we have by dominated convergence that

$$\lim_{n\to\infty} \int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\phi_n) = -\lim_{n\to\infty} \int \phi_n \operatorname{div}\boldsymbol{\varphi} = -\int \phi \operatorname{div}\boldsymbol{\varphi} = \int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\phi). \tag{3.16}$$

On the other hand, by (3.11) it holds that $|\operatorname{grad}\phi_n|(\mathfrak{Q}) \leq 2\sqrt{2}\|\operatorname{grad}\phi\|_{TV}$ for each bounded open rectangle $\mathfrak{Q}$, hence $\|\operatorname{grad}\phi_n\|_{TV} \leq 2\sqrt{2}\|\operatorname{grad}\phi\|_{TV}$ for all n. Thus, choosing $f \equiv 1$ in (3.14), we get that $t \mapsto \|\operatorname{grad}\chi_{E_t}\|_{TV}$ is integrable over $\mathbb{R}$. Now, as $|\int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})| \leq \sup|\boldsymbol{\varphi}|\|\operatorname{grad}\chi_{E_t}\|_{TV}$, we obtain (3.13) upon applying the dominated convergence theorem to the right hand side of (3.15) while taking (3.16) into account.

Next, pick $\varepsilon > 0$ and n_ε so large that $\int_{-1/n}^{1/n} \|\operatorname{grad}\chi_{E_t}\|_{TV} < \varepsilon$ for $n \geq n_\varepsilon$. Let $\Omega \subset \mathbb{R}^2$ be open and $\boldsymbol{\varphi}_n \in (C_c^1(\Omega))^2$ such that $|\boldsymbol{\varphi}_n| \leq 1$ and (see (3.1))

$$\int \boldsymbol{\varphi}_n \cdot \mathrm{d}\operatorname{grad}\phi_n \geq \|(\operatorname{grad}\phi_n)\lfloor\Omega\|_{TV} - \varepsilon.$$

Then, if we fix $n \geq n_\varepsilon$, we get from (3.16), (3.15) and the dominated convergence theorem:

$$
\begin{aligned}
\|(\operatorname{grad}\phi)\lfloor\Omega\|_{TV} &\geq \int \boldsymbol{\varphi}_n \cdot \mathrm{d}\operatorname{grad}\phi = \lim_m \int \boldsymbol{\varphi}_n \cdot \mathrm{d}\operatorname{grad}\phi_m \\
&= \lim_m \int_{-\infty}^{-1/m} \int \boldsymbol{\varphi}_n \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})\,\mathrm{d}t + \int_{1/m}^{\infty} \int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})\,\mathrm{d}t \\
&\geq \int_{-\infty}^{-1/n} \int \boldsymbol{\varphi}_n \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})\,\mathrm{d}t + \int_{1/n}^{\infty} \int \boldsymbol{\varphi} \cdot \mathrm{d}(\operatorname{grad}\chi_{E_t})\,\mathrm{d}t - \varepsilon \\
&\geq \|(\operatorname{grad}\phi_n)\lfloor\Omega\|_{TV(\Omega)} - 2\varepsilon.
\end{aligned}
$$

Consequently $\|(\operatorname{grad}\phi)\lfloor\Omega\|_{TV(\Omega)} \geq \limsup_n \|(\operatorname{grad}\phi_n)\lfloor\Omega\|_{TV(\Omega)}$, but as $\phi_n \to \phi$ in $L^1_{loc}(\mathbb{R}^2)$ we also know from [36, Thm. 5.2.1] that $\|\operatorname{grad}\phi\lfloor\Omega\|_{TV} \leq \liminf_n \|(\operatorname{grad}\phi_n)\lfloor\Omega\|_{TV(\Omega)}$. Hence, for any open set $\Omega \subset \mathbb{R}^2$, we get that

$$
\lim_{n\to\infty} |\operatorname{grad}\phi_n|(\Omega) = |\operatorname{grad}\phi|(\Omega) \tag{3.17}
$$

which implies, by dominated convergence on the right hand side of (3.14), that (3.12) holds when $f = \chi_\Omega$. Observe now that if we restrict to $f \in C_c(\mathbb{R}^2)$, then the two sides of (3.12) define finite positive Borel measures and we just showed they coincide on open sets, therefore they coincide on all Borel sets. That is, (3.12) in fact holds for each $f \in C_c(\mathbb{R}^2)$. Finally, because every bounded Borel function is the bounded pointwise limit a.e. of a sequence of continuous functions, by Lusin's theorem, we get by dominated convergence that (3.12) holds for any such f, and the case of a nonnegative Borel function f follows by monotone convergence. □

Theorem 3.8. *Suppose $\phi \in BV_{loc}(\mathbb{R}^2)$, $\operatorname{grad}\phi \in \mathcal{M}(\mathbb{R}^2)^2$ and let E_t be as in (3.7). Then, for any Borel set B:*

(a) $|\operatorname{grad}\phi|(B) = \int_{-\infty}^{\infty} |\operatorname{grad}\chi_{E_t}|(B)\,\mathrm{d}t = \int_{-\infty}^{\infty} \mathcal{H}^1(\partial_M E_t \cap B)\,\mathrm{d}t,$ *and*

(b) $\operatorname{grad}\phi(B) = \int_{-\infty}^{\infty} \operatorname{grad}\chi_{E_t}(B)\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_B \boldsymbol{\nu}_{E_t}\,\mathrm{d}\left(\mathcal{H}_1\lfloor\partial_M E_t\right)\,\mathrm{d}t.$

Proof. Taking $f = \chi_B$ on (3.12) implies the first equality in (a), and the second one is just the combination with (3.4).

Since ϕ is measurable, the mapping $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by $(\mathbf{x}, t) \to \chi_{E_t}(\mathbf{x})$ is measurable in view of (3.7). Thus, for each $\boldsymbol{\varphi} \in (C^1_c(\mathbb{R}^2))^2$, the function $I[\boldsymbol{\varphi}] : \mathbb{R} \to \mathbb{R}$ defined by

$$
t \mapsto \int_{E_t} \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\mathcal{L}_2(\mathbf{x}) = \int_{\mathbb{R}^2} \chi_{E_t} \operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\mathcal{L}_2(\mathbf{x}) = \int_{\mathbb{R}^2} \boldsymbol{\varphi}\cdot\mathrm{d}(\operatorname{grad}\chi_{E_t})
$$

is measurable. Now, we can find a sequence φ_n in $C^1_c(\mathbb{R}^2)$ with $|\varphi_n| \leq 1$ that converges boundedly pointwise a.e. to χ_B: for instance, by Lusin's Theorem, we may first construct a sequence ψ_n in $C_c(\mathbb{R}^2)$ that converges boundedly pointwise a.e. to χ_B, and then by mollification we can approximate ψ_n uniformly within $1/n$ by a smooth compactly supported function φ_n. Thus, $t \mapsto \operatorname{grad}\chi_{E_t}(B)$ is the a.e. pointwise limit of the sequence of measurable maps $I[\varphi_n\mathbf{e}_1]\mathbf{e}_1 + I[\varphi_n\mathbf{e}_2]\mathbf{e}_2$ which are all majorized in absolute value by $t \mapsto \|\chi_{E_t}\|_{TV}$ which integrable in view of (3.12). Thus, the dominated convergence theorem together with (3.13) implies the first equality of (b) and the second one comes from (3.5). □

For every $E \subset \mathbb{R}^2$ of finite perimeter, $\partial_M E$ is 1-rectifiable (see e.g. [42, theorem 10.3.2]). We say that a set $\mathbf{C} \subset \mathbb{R}^2$ is a **Jordan curve** if it is the image of the unit circle T under an injective continuous map $\mathbf{c}$. If there exists such a $\mathbf{c}$ which is Lipschitz, then $\mathbf{C}$ is called a **rectifiable Jordan curve**.

Remark 3.9. *If $\mathbf{c} : T \to \mathbb{R}^2$ is Lipschitz with constant λ, then $\mathcal{H}_1(\mathbf{C}) = \mathcal{H}_1(\mathbf{c}(T)) \leq \lambda 2\pi < \infty$. Conversely if $\mathcal{H}_1(\mathbf{C}) < \infty$ then choosing an arclength parametrization $\mathbf{c} : T \to \mathbf{C}$ gives a Lipschitz mapping with constant $\lambda = \mathcal{H}_1(\mathbf{C})/(2\pi)$. Thus, a Jordan curve $\mathbf{C}$ is rectifiable if and only if $\mathcal{H}_1(\mathbf{C}) < \infty$.*

Given a Jordan curve $\mathbf{C}$ we will denote by $\mathrm{int}(\mathbf{C})$ the bounded connected component of the complement of $\mathbf{C}$.

Let $E \subset \mathbb{R}^2$ be a set of finite perimeter and let $\mathbf{C} \subset \mathbb{R}^2$ be a Jordan curve such that $\mathbf{C} \subset \partial_M E$ up to a set of $\mathcal{H}_1$ measure zero. We say that $\mathbf{C}$ has **positive orientation** with respect to E if $\operatorname{grad} \chi_E \lfloor \mathbf{C} = \operatorname{grad} \chi_{\mathrm{int}(\mathbf{C})}$ and we say that $\mathbf{C}$ has **negative orientation** with respect to E if $\operatorname{grad} \chi_E \lfloor \mathbf{C} = -\operatorname{grad} \chi_{\mathrm{int}(\mathbf{C})}$. It may happen that a Jordan curve is included in $\partial_M E$ up to a set of $\mathcal{H}_1$-measure zero, and still the curve is neither oriented positively nor negatively with respect to E. Below, we show that there is a decomposition of $\partial_M E$ into countably many Jordan curves having a definite orientation with respect to E.

Lemma 3.10. *Let $E \subset F \subset \mathbb{R}^2$ where the sets E, F have finite perimeter. Then for $\mathcal{H}_1$-a.e. $\mathbf{x} \in \partial_M E \cap \partial_M F$, $\nu_F(\mathbf{x}) = \nu_E(\mathbf{x})$.*

Proof. Given $\epsilon > 0$, $\mathbf{x}, \mathbf{v} \in \mathbb{R}^2$ and $G \subset \mathbb{R}^2$, define the half-disk

$$H_\epsilon(\mathbf{x}, \mathbf{v}) := \{\mathbf{y} \in \mathbb{B}(\epsilon, \mathbf{x}) \ : \ (\mathbf{y} - \mathbf{x}) \cdot \mathbf{v} > 0\},$$

and let

$$L_G(\mathbf{x}, \mathbf{v}) := \lim_{\epsilon \to 0} \frac{\mathcal{L}_2(H_\epsilon(\mathbf{x}, \mathbf{v}) \cap G)}{\mathcal{L}_2(H_\epsilon(\mathbf{x}, \mathbf{v}))}$$

whenever the limit exists. Assume G has finite perimeter. Then, for $\mathcal{H}_1$-a.e. $\mathbf{x} \in \partial_M G$, $\nu_G(\mathbf{x})$ is the unique unit vector that satisfies

$$L_G(\mathbf{x}, \nu_G(\mathbf{x})) = 1 \qquad \text{and} \qquad L_G(\mathbf{x}, -\nu_G(\mathbf{x})) = 0,$$

(see [42, Proposition 10.3.4 and Theorem 10.3.2]). Since $E \subset F \subset \mathbb{R}^2$ then for $\mathcal{H}_1$-a.e. $\mathbf{x} \in \partial_M E \cap \partial_M F$ we get that $L_E(\mathbf{x}, -\nu_F(\mathbf{x})) = 0$. Moreover,

$$1 = L_E(\mathbf{x}, \nu_E(\mathbf{x})) + L_E(\mathbf{x}, -\nu_E(\mathbf{x})) = \lim_{\epsilon \to 0} \frac{\mathcal{L}_2(\mathbb{B}(\epsilon, \mathbf{x}) \cap E)}{\mathcal{L}_2(H_\epsilon(\mathbf{x}, \mathbf{v}))} \tag{3.18}$$

$$= L_E(\mathbf{x}, \nu_F(\mathbf{x})) + L_E(\mathbf{x}, -\nu_F(\mathbf{x})) = L_E(\mathbf{x}, \nu_F(\mathbf{x})). \tag{3.19}$$

Hence our lemma follows by uniqueness of $\nu_E(\mathbf{x})$. □

Lemma 3.11. *Let $\mathbf{C} \subset \mathbb{R}^2$ be a rectifiable Jordan curve, then $\partial_M(\mathrm{int}(\mathbf{C})) = \mathbf{C}$ up to a set of $\mathcal{H}_1$-measure zero.*

Proof. By [12, Theorem 7] $\partial_M(\mathrm{int}(\mathbf{C}))$ is equal to a Jordan curve $\tilde{\mathbf{C}}$ up to a set of $\mathcal{H}_1$-measure zero. Clearly $\partial_M(\mathrm{int}(\mathbf{C}))$ is a subset of the topological boundary of $\mathrm{int}(\mathbf{C})$ which is $\mathbf{C}$. Thus, $\mathcal{H}_1(\tilde{\mathbf{C}} \setminus \mathbf{C}) = 0$ whence $\tilde{\mathbf{C}} \cap \mathbf{C}$ is dense in $\tilde{\mathbf{C}}$, and so $\tilde{\mathbf{C}} \subset \mathbf{C}$ by compactness of $\mathbf{C}$. Therefore, by the Jordan curve theorem, $\tilde{\mathbf{C}} = \mathbf{C}$ which implies our lemma. □

Lemma 3.12. *The measure theoretic boundary of a set $E \subset \mathbb{R}^2$ of finite perimeter decomposes, up to a set of $\mathcal{H}_1$ measure zero, as a countable union of rectifiable Jordan curves with either positive or negative orientation with respect to E.*

Proof. By [12, Corollary 1], there exists two families, $\{\mathbf{C}_k^+\}_{k\in K}$ and $\{\mathbf{C}_j^-\}_{j\in J}$, of countably many rectifiable Jordan curves that satisfy the following (Minor differences with respect to the decomposition in [12, Corollary 1] are due to the fact that, to us, a set of finite perimeter could be the complement of a set of finite perimeter as defined in there):

(a) $\partial_M E = \bigcup_k \mathbf{C}_k^+ \cup \bigcup_j \mathbf{C}_j^-$ up to a set of $\mathcal{H}_1$-measure zero,

(b) For any two $\mathrm{int}(\mathbf{C}_k^+)$ and $\mathrm{int}(\mathbf{C}_l^+)$ either one is contained in the other or they are disjoint. Similarly, for any two $\mathrm{int}(\mathbf{C}_j^-)$ and $\mathrm{int}(\mathbf{C}_i^-)$ either one is contained in the other or they are disjoint.

(c) $\mathcal{H}_1(\partial_M E) = \sum_k \mathcal{H}_1(\mathbf{C}_k^+) + \sum_j \mathcal{H}_1(\mathbf{C}_j^-)$, in particular the curves are disjoint up to a set of $\mathcal{H}_1$-measure zero.

(d) If $l \neq k$ and $\mathrm{int}(\mathbf{C}_k^+) \subset \mathrm{int}(\mathbf{C}_l^+)$ then there exists a $\mathrm{int}(\mathbf{C}_j^-)$ such that $\mathrm{int}(\mathbf{C}_k^+) \subset \mathrm{int}(\mathbf{C}_j^-) \subset \mathrm{int}(\mathbf{C}_l^+)$. Analogously, if $j \neq i$ and $\mathrm{int}(\mathbf{C}_j^-) \subset \mathrm{int}(\mathbf{C}_i^-)$ then there exists a $\mathrm{int}(\mathbf{C}_k^+)$ such that $\mathrm{int}(\mathbf{C}_j^-) \subset \mathrm{int}(\mathbf{C}_k^+) \subset \mathrm{int}(\mathbf{C}_i^-)$.

(e) Let $I_k := \{j \in J : \mathrm{int}(\mathbf{C}_j^-) \subset \mathrm{int}(\mathbf{C}_k^+)\}$. If there exists a $\mathrm{int}(\mathbf{C}_j^-)$ not contained in any $\mathrm{int}(\mathbf{C}_k^+)$, let $Y := \mathbb{R}^2 \setminus \bigcup_{j\in J} \mathrm{int}(\mathbf{C}_j^-)$, otherwise let $Y := \emptyset$. Then the sets $Y_k := \mathrm{int}(\mathbf{C}_k^+) \setminus \bigcup_{j\in I_k} \mathrm{int}(\mathbf{C}_j^-)$ together with Y are pairwise disjoint and $E = Y \cup \bigcup_k Y_k$ up to a set of $\mathcal{L}_2$ measure zero (It is asserted in [12, Cor. 1] that $E = \bigcup_k Y_k$. Examination of the proof, however, shows that this is the case up to a set of planar measure zero).

What is left to show is that every $\mathbf{C}_k^+$ is positively oriented with respect to E and each $\mathbf{C}_j^-$ is negatively oriented with respect to E.

Fix $\ell \in K$. We will prove that for $\mathcal{H}_1$-a.e. $\mathbf{x} \in \mathbf{C}_\ell^+$, $\nu_E(\mathbf{x}) = \nu_{\mathrm{int}\mathbf{C}_\ell^+}(\mathbf{x})$ which will show that $\mathbf{C}_\ell^+$ is positively oriented with respect to E.

Let $F := \mathrm{int}(\mathbf{C}_\ell^+) \cap E$, $\tilde{K} := \{k \in K : \mathrm{int}(\mathbf{C}_k^+) \subset \mathrm{int}(\mathbf{C}_\ell^+)\}$ and $\tilde{J} := \bigcup_{k\in\tilde{K}} I_k$. Then $F = \bigcup_{k\in\tilde{K}} Y_k$ up to a set of $\mathcal{L}_2$ measure zero, and $\{\mathbf{C}_k^+\}_{k\in K}$ and $\{\mathbf{C}_j^-\}_{j\in\tilde{J}}$ satisfy (a), (d),

(c') each two different Jordan curves are disjoint up to a set of $\mathcal{H}_1$-measure zero,

(e') $\sum_k \mathcal{H}_1(\mathbf{C}_k) + \sum_j \mathcal{H}_1(\mathbf{C}_j^-) < \infty$, $k \in \tilde{K}$, $j \in \tilde{J}$.

By [12, Theorem 5], F is a set of finite perimeter and $\partial_M F = \bigcup_{k\in\tilde{K}} \mathbf{C}_k^+ \cup \bigcup_{j\in\tilde{J}} \mathbf{C}_j^-$ up to a set of $\mathcal{H}_1$ measure zero. Thus we can use Lemma 3.10 and then $\nu_E(\mathbf{x}) = \nu_F(\mathbf{x}) = \nu_{\mathrm{int}(\mathbf{C}_k^+)}(\mathbf{x})$ for $\mathcal{H}_1$-a.e. $\mathbf{x} \in (\partial_M F \cap \partial_M E \cap \partial_M \mathrm{int}(\mathbf{C}_k^+))$ which is $\mathbf{C}_k^+$ up to a set of $\mathcal{H}_1$-measure zero thanks to Lemma 3.11.

Take any $j \in J$. If there is a $k \in K$ such that $j \in I_k$, let $B \subset \mathbb{R}^2$ be an open ball containing $\overline{\mathrm{int}(\mathbf{C}_k^+)}$ and $G := B \setminus Y_k$, otherwise, let $B := \mathbb{R}^2$ and $G := \mathbb{R}^2 \setminus Y$. Thus $\partial_M G$ is equal, up to a set of $\mathcal{H}_1$ measure zero, to the euclidean boundary of B union $\partial_M Y_k$ and for $\mathcal{H}_1$-a.e. $\mathbf{x} \in \partial_M Y_k$, $\nu_E(\mathbf{x}) = \nu_{Y_k}(\mathbf{x}) = -\nu_G(\mathbf{x})$. Then, to show that a particular $\mathbf{C}_j^-$ is negatively oriented with respect to E is equivalent to showing that $\mathbf{C}_j^-$ is positively oriented with respect to G which follows from the argument presented above. $\square$

Note that, as can be seen in Example 5, the union of the $\mathbf{C}_n$ may not be all of the Euclidean boundary of Ω_n and that the extra part may have positive Lebesgue measure.

Recall the definition of R given in section 3. For a rectifiable Jordan curve $\mathbf{C} \subset \mathbb{R}^2$ we will refer to $\mathfrak{R}\boldsymbol{\nu}_{\mathrm{int}(\mathbf{C})}$ by the **unit tangent vector field** of $\mathbf{C}$. Note that its orientation is opposite to the usual one.

The previous result teams up with Theorem 3.8 and Lemma 3.3 to produce the following representation of planar measures which are silent magnetization distributions.

Theorem 3.13. *Let S be a closed subset of $\mathbb{R}^2$ and $\mathbf{M} \in \mathcal{M}(S)^3$ a S-silent magnetization. Then, for a.e. $t \in \mathbb{R}$ there is a sequence of sets $\mathbf{C}_n(t)$ such that:*

(i) each $\mathbf{C}_n(t)$ is either empty or a oriented rectifiable Jordan curve with unit tangent vector field $\boldsymbol{\tau}_n(t)$,

(ii) for a.e. $t \in \mathbb{R}$, the union $\cup_n \mathbf{C}_n(t)$ is (up to set of $\mathcal{H}_1$-measure zero) the reduced boundary of some set $\Omega(t) \subset \mathbb{R}^2$ of finite perimeter,

(iii) $\int_{\mathbb{R}} (\Sigma_{n=1}^{\infty} \mathcal{H}^1(\mathbf{C}_n(t)))\,\mathrm{d}t = \int_{\mathbb{R}} \mathcal{H}^1(\partial_M \Omega(t)) = \|\mathbf{M}\|_{TV} < +\infty$,

(iv) $\Omega(t_1) \supset \Omega(t_2)$ if $t_1 < t_2$,

(v) For any Borel set $B \subset \mathbb{R}^2$, it holds that

$$\mathbf{M}(B) = \int_{\mathbb{R}} \sum_{n=1}^{\infty} \left(\int_B \boldsymbol{\tau}_n(t)\,\mathrm{d}\left(\mathcal{H}_1 \lfloor \mathbf{C}_n(t)\right) \right) \mathrm{d}t.$$

The $\Omega(t)$ could have all different topology as it can be seen in the following example. (Here we just work with reals between 0 and 1 but a similar construction can be used to create a finite measure such that for every t and s, $\Omega(t)$ is topologically equivalent to $\Omega(s)$ if and only if $t = s$)

Example 6. *We will first generate a BV function, φ_∞, and show that its gradient is a finite measure for which the aforementioned property holds. We will construct this function as a the limit of a bounded increasing sequence of BV functions, ϕ_m's.*

Let us first define a family of sets of finite perimeter that we will use for the construction of the ϕ_m's. For any two integers m and n such that $1 \leq n \leq 2^m$, define the set $b(n,m) \subset \mathbb{R}^2$ to be the closed ball around the point (n,m) with perimeter 2^{-2m-1}, (with radius $2^{-2m-2}/\pi$) minus 2^m pairwise disjoint nonempty open balls contain within this closed ball. Further assume that the sum of the perimeters of this 2^m open balls is strictly lower than 2^{-2m-1}. Note that the $b(n,m)$'s are pairwise disjoint as well. Define the functions in $\mathbb{R}^2$, $\varphi_0 := \frac{1}{2}\chi_{b(1,0)}$ and for $m > 0$, $\varphi_m := \varphi_{m-1} + \sum_{k=1}^{2^m} \frac{2k-1}{2^{m+1}} \chi_{b(k,m)}$. Then $\|\operatorname{grad} \varphi_0\|_{TV} < 1/2$, for $m > 0$

$$\begin{aligned}
\|\operatorname{grad} \varphi_m\|_{TV} &= \|\operatorname{grad} \varphi_{m-1}\|_{TV} + \sum_{k=1}^{2^m} \frac{2k-1}{2^{m+1}} \|\operatorname{grad} \chi_{b(k,m)}\|_{TV} \\
&< \|\operatorname{grad} \varphi_{m-1}\|_{TV} + \sum_{k=1}^{2^m} \frac{2k-1}{2^{m+1}} (2^{-2m-1} + 2^{-2m-1}) \\
&= \|\operatorname{grad} \varphi_{m-1}\|_{TV} + \frac{2^{2m}}{2^{3m+1}},
\end{aligned}$$

and hence $\|\operatorname{grad}\varphi_m\|_{TV} < 1$ *for every* m*. Thus* φ_∞*, the pointwise limit of* $\{\varphi_m\}_m$*, is a BV function (see [36, Theorem 5.2.1]). Let* $S := b(1,0) \cup \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{2^m} b(n,m)$*. Then as a consequence of Lemma 3.3,* $\boldsymbol{\mu} := R \operatorname{grad} \varphi_\infty$ *belongs to* $\mathcal{M}(S)^3$ *and is* S*-silent.*

For the integers m*,* n*,* p *and* q *such that* $1 \leq n \leq 2^m$ *and* $1 \leq p \leq 2^q$*,* $b(n,m)$ *is topologically equivalent to* $b(p,q)$ *if and only if* $q = m$*. Hence given* $s,t \in (0,1)$*, for* $\Omega(t)$ *and* $\Omega(s)$ *to be topologically equivalent they must contain for a fix* m *the same number of sets from the family* $\{b(n,m)\}_{n=1}^{2^m}$*. However if* $s < t$ *then there exist two positive integers* m *and* n *such that* $s < \frac{2n-1}{2^m} < t$*, thus* $b(n,m) \subset \Omega(s) \setminus \Omega(t)$ *and therefore* $\Omega(t)$ *is not topologically equivalent to* $\Omega(s)$*.*

We say that a closed set $S \subset \mathbb{R}^2$ is **tree-like** if S contains no rectifiable Jordan curve.

Corollary 3.14. *Let* S *be a closed subset of* $\mathbb{R}^2$ *and* $\mathbf{M} \in \mathcal{M}(S)^3$ *be* S*-silent. Then, the support of* $\mathbf{M}$ *contains a rectifiable Jordan curve. Hence, if* S *is tree-like the only* S*-silent magnetization is the zero magnetization.*

Recalling the definitions at the beginning of section 5 and reasoning as in Theorem 2.18, we obtain:

Corollary 3.15. *Assumptions being as in Lemma 2.14 (d), assume in addition that* $S \subset \mathbb{R}^2$ *is tree-like. For* $\mathbf{M}_0$ *a magnetization supported on* S*, set* $f = \mathbf{A}\mathbf{M}_0$ *and, for* $\lambda > 0$ *and* $e \in L^2(Q)$*, let* $\mathbf{M}_{\lambda,e}$ *satisfy* (2.26) *where* f *is replaced by* $f+e$*. Then* $\mathbf{M}_{\lambda,e}$ *(resp* $|\mathbf{M}_{\lambda,e}|$*) converges to* $\mathbf{M}_0$ *(resp.* $|\mathbf{M}_0|$*) in the narrow sense as* $\lambda \to 0$ *and* $\|e\|_{L^2(Q)}/\sqrt{\lambda} \to 0$*.*

As regards the optimization problem (2.26) when $S \subset \mathbb{R}^2$, a noteworthy consequence of Theorem 3.13 and Corollary 2.16 is:

Theorem 3.16. *Let* S *be a proper closed subset of* $\mathbb{R}^2$ *and* $Q, S, \mathcal{A}, v$ *satisfy the assumptions of Lemma 2.4. Assume moreover that* ρ *is a finite, positive Borel measure with support contained in* Q *and containing* $Q \cap \mathcal{A}$*. Then for every* $f \in L^2(Q)$ *and* $\lambda > 0$*, the solution to* (2.3) *is unique.*

Proof. Assume for a contradiction that $\mathbf{M}_\lambda$ and $\mathbf{M}'_\lambda$ are two distinct minimizers in (2.3) and let $\mathbf{M} := \mathbf{M}'_\lambda - \mathbf{M}_\lambda$. Since $\mathbf{M}'_\lambda - \mathbf{M}_\lambda = \mathbf{M} \ll |\mathbf{M}|$ then the Radon-Nykodim decompositions of $\mathbf{M}_\lambda$ and $\mathbf{M}'_\lambda$ with respect to $|\mathbf{M}|$ are:

$$\mathrm{d}\mathbf{M}_\lambda = \boldsymbol{\gamma}\mathrm{d}|\mathbf{M}| + \mathrm{d}\mathbf{N}, \qquad \mathrm{d}\mathbf{M}'_\lambda = \boldsymbol{\gamma}'\mathrm{d}|\mathbf{M}| + \mathrm{d}\mathbf{N}, \tag{3.20}$$

where $|\mathbf{N}|$ is singular with respect to $|\mathbf{M}|$ and $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}'$ are $|\mathbf{M}|$-integrable R^3-valued functions.

Put for simplicity $\psi = (2/\lambda)(f - A(\mathbf{M}_\lambda)) = (2/\lambda)(f - A(\mathbf{M}'_\lambda))$. Thanks to (2.27) and (2.28) we know that $\mathbf{u}_{\mathbf{M}_\lambda} = A^*\psi$ and $\mathbf{u}_{\mathbf{M}'_\lambda} = A^*\psi$, $\mathbf{M}_\lambda$ and $\mathbf{M}'_\lambda$-a.e. respectively. Now, since $\mathrm{d}|\mathbf{M}_\lambda| = |\boldsymbol{\gamma}|\mathrm{d}|\mathbf{M}| + \mathrm{d}|\mathbf{N}|$ and $\mathrm{d}|\mathbf{M}'_\lambda| = |\boldsymbol{\gamma}'|\mathrm{d}|\mathbf{M}| + \mathrm{d}|\mathbf{N}|$, then

$$\begin{aligned}\mathbf{u}_{\mathbf{M}}\mathrm{d}|\mathbf{M}| = \mathrm{d}\mathbf{M} &= \mathbf{u}_{\mathbf{M}'_\lambda}\mathrm{d}|\mathbf{M}'_\lambda| - \mathbf{u}_{\mathbf{M}_\lambda}\mathrm{d}|\mathbf{M}_\lambda| \\ &= A^*\psi\mathrm{d}|\mathbf{M}'_\lambda| - A^*\psi\mathrm{d}|\mathbf{M}_\lambda| = A^*\psi(|\boldsymbol{\gamma}'| - |\boldsymbol{\gamma}|)\mathrm{d}|\mathbf{M}|.\end{aligned}$$

Therefore $\mathbf{u}_{\mathbf{M}} = A^*\psi(|\boldsymbol{\gamma}'| - |\boldsymbol{\gamma}|)$, $|\mathbf{M}|$-a.e. and since $|A^*\psi| = 1$ on the supports of $\mathbf{M}_\lambda$ and $\mathbf{M}'_\lambda$, then $\mathbf{u}_{\mathbf{M}} = \pm A^*\psi$, $|\mathbf{M}|$-a.e..

From Theorem 2.15 point (a) and Lemma 2.14 point (d), we know that $\mathbf{M}$ is S-silent. Also, by the remark after Corollary 2.16, the supports of $\mathbf{M}_\lambda$ and $\mathbf{M}'_\lambda$ are contained in a finite collection of points and analytic arcs. Thus, applying Theorem 3.13 with $\mathbf{M}$ and the remark mentioned afterwards, we find there are finitely many piecewise analytic oriented Jordan curves $C_1, \cdots, C_N$ with respective unit tangent vector fields $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_n$, and positive real numbers $a_1, \cdots, a_N$ such that $\boldsymbol{\tau}_m = \boldsymbol{\tau}_n$ on $C_m \cap C_n$ and

$$\mathrm{d}\mathbf{M} = \sum_{n=1}^{N} a_n \boldsymbol{\tau}_n \mathrm{d}\left(\mathcal{H}_1 \lfloor C_n\right).$$

In particular, $\mathrm{d}|\mathbf{M}| = \sum_{n=1}^{N} a_n \mathrm{d}\left(\mathcal{H}_1 \lfloor C_n\right)$ and $\boldsymbol{\tau}_n = \mathbf{u}_{\mathbf{M}} = \pm A^*\psi$, $|\mathbf{M}|$-a.e., hence $\mathcal{H}_1$-a.e., on C_n.

Fix an n and let E be an analytic sub-arc of C_n. Then $\boldsymbol{\tau}_n$ must be analytic on E, and hence either $\boldsymbol{\tau}_n = A^*\psi$ or $\boldsymbol{\tau}_n = -A^*\psi$ everywhere on E. Therefore E is a subset of a trajectory of the autonomous differential equation $\dot{x} = A^*\psi(x)$. Moreover, since E is bounded and percursed at unit speed, the corresponding trajectory extends beyond the endpoints of E, and since two distinct trajectories cannot intersect we conclude that C_n is smooth and constitutes a single, periodic trajectory. This, however, is impossible because $A^*\psi$ is a gradient vector field, by (2.24). □

3.2 Numerical examples

In this section we present two examples of numerical reconstructions illustrating Theorem 2.19. In both cases, we are considering continuous problems with $v = \mathbf{e}_3$ (the measured field is $B_3(\mathbf{M}_0)$), $e = 0$ (no noise), and S and Q compact subsets of the $z = 0$ and $z = h$ planes, respectively. We discretize the continuous problems by restricting to magnetizations $\mathbf{M}$'s consisting of a finite number of dipoles located on a rectangular grid in the $z = 0$ plane intersected with S and samples $B_3(\mathbf{M})$ evaluated at points in a rectangular grid in the parallel $z = h$ plane intersected with a rectangle Q. Numerical solutions to the discretized problems are then obtained using the Fast Iterative Shrinkage-Thresholding algorithm (FISTA) from [43] together with the In-Crowd algorithm from [44]. We will consider in more detail in forthcoming work the relations between the solutions of such discretized problems and the associated continuous problems of the type addressed in this document and we will also address the algorithmic and computational details for obtaining solutions to these discrete problems. The examples provided here are only intended for illustrative purposes.

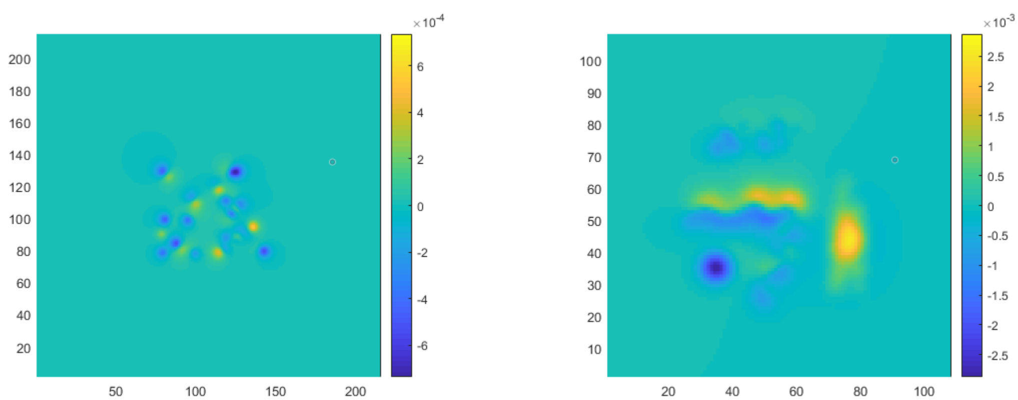

Figure 2: The fields $B_3(\mathbf{M})$ for (left) a sparse magnetization consisting of 20 dipoles (see Figure 3 for $\mathbf{M}$ and reconstructions $\mathbf{M}_\lambda$) and (right) a piecewise unidirectional magnetization with S consisting of four connected components (see Figure 4).

The continuous problem for the first example is designed to illustrate the recovery of magnetizations with sparse support as in part (b) of Theorem 2.19. In this example S and Q are squares in the planes $z = 0$ and $z = h = .1$, respectively, and $\mathbf{M}_0$ consists of 20 dipoles in S with moments of differing directions. The source and measurement grids both have $.0187 \times .0187$ spacing. The

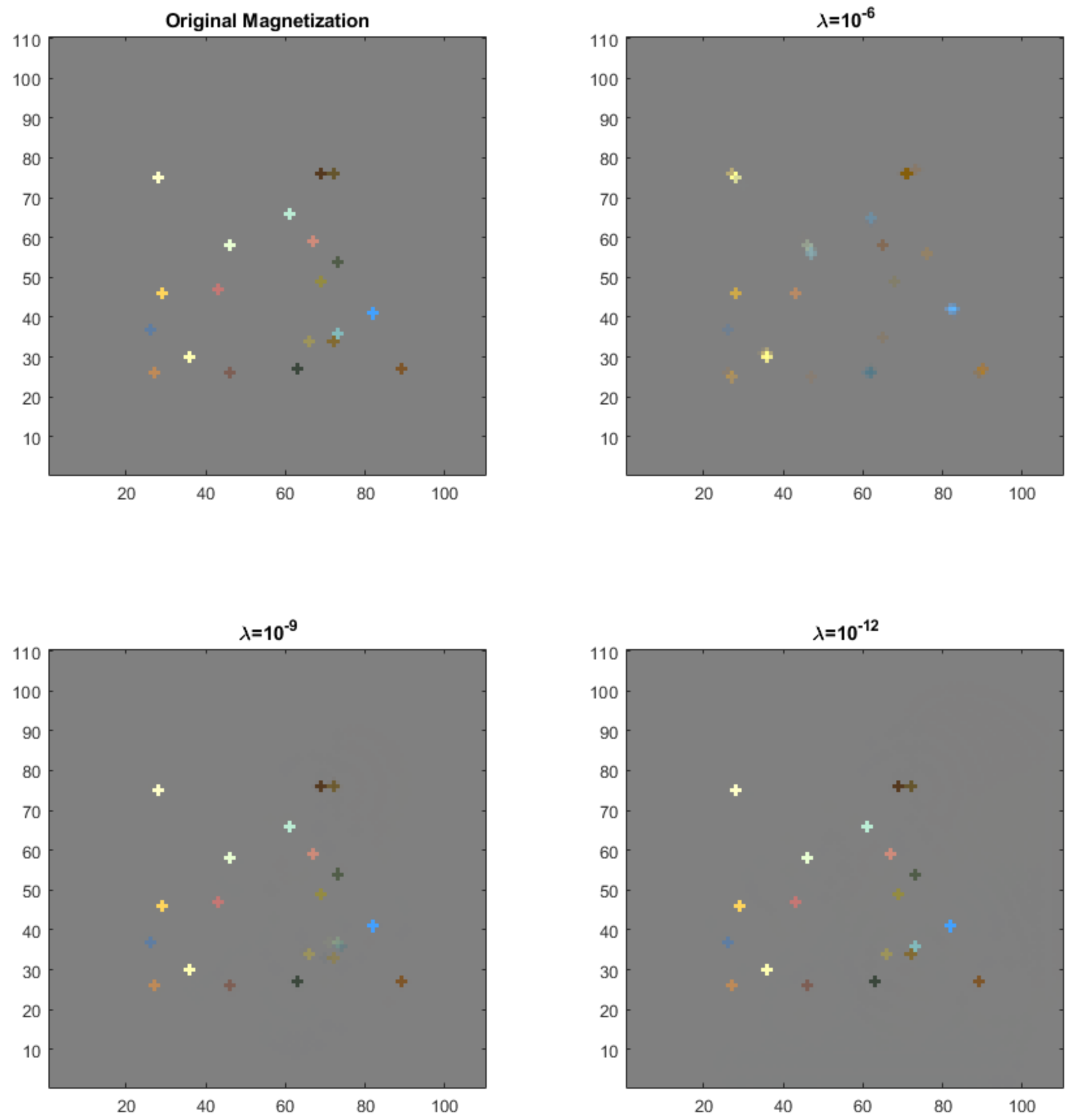

Figure 3: Example for a Sparse Magnetization. Each figure shows for its respective magnetization the magnitude and direction of its dipoles by representing different vectors with diferent colors. The closer a dipole is to zero the closer its assigned color is to grey. The relative distances in total variation to $\mathbf{M}_0$ are 1.294, 0.207 and 0.004 respectively, confirming convergence as expected. (Convolution with a 3 by 3 cross matrix was used to increase visibility)

source grid consists of 108×108 points and the moments of each dipole are allowed to take on any values in $\mathbb{R}^3$. The measurement grid consists of 215×215 points. Reconstructions are computed for three values of λ: 10^{-6}, 10^{-9} and 10^{-12}.

In the second case, S consists of the union of four disjoint compact regions as shown in Figure 4. The restriction of the magnetization $\mathbf{M}_0$ to each component is unidirectional as in part (a) of Theorem 2.19. The source grid now consists of the grid points that are contained in the set S and

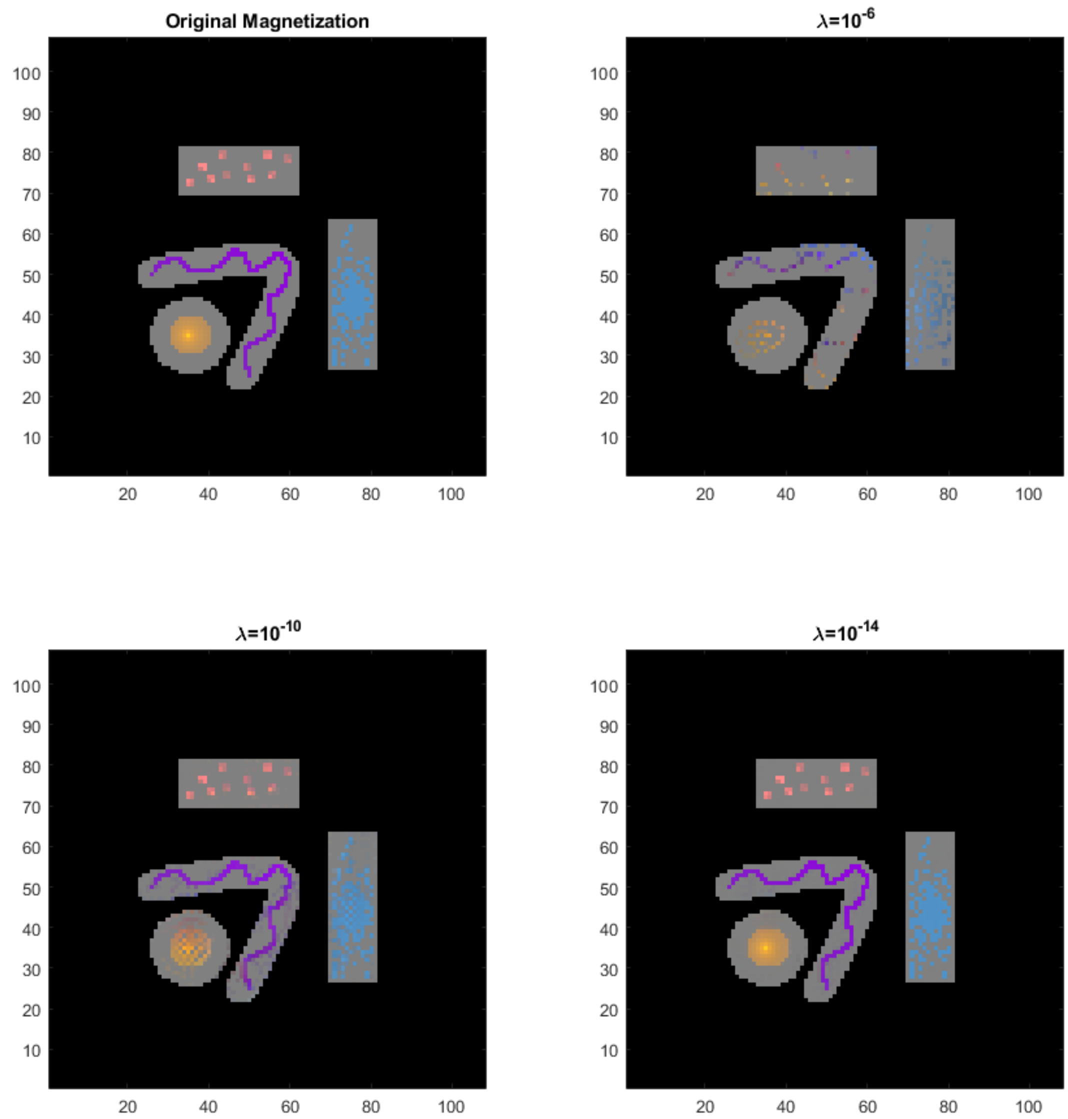

Figure 4: Example for a piece-wise Unidirectional Magnetization. The figures are made similarly to the ones of Figure 3 with black representing the complement of S. Here the relative distances are 1.167, 0.247 and 0.015

again the moments of these dipoles are unconstrained; i.e., there is no uni-directionality assumption when solving for a reconstruction. The $\mathbf{M}_\lambda$ are taken among all magnetizations supported in those regions for λ equal to 10^{-6}, 10^{-10} and 10^{-14}.

Appendix A

Jordan-Brouwer separation theorem in the non compact case

In this section, we record a proof of the Jordan-Brouwer separation theorem for smooth and connected, complete but not necessarily compact surfaces in $\mathbb{R}^3$. The argument applies in any dimension. We are confident the result is known, but we could not find a published reference. More general proofs, valid for non-smooth manifolds as well, could be given using deeper facts from algebraic topology. For instance, one based on Alexander-Lefschetz duality can be modeled after Theorem 14.13 in [45] (which deals with compact topological manifolds). More precisely, using Alexander-Spanier cohomology with compact support and appealing to [46, ch. 6, sec. 6 cor. 12 and sec. 9 thm. 10], one can generalize the proof just mentioned to handle the case of non-compact manifolds. Herafter, we merely deal with smooth surfaces and rely on basic notions from differential-topology, namely intersection theory modulo 2.

Recall that a smooth manifold X of dimension k embedded in $\mathbb{R}^n$ is a subset of the latter, each point of which has a neighborhood V such that $V \cap X = \phi(U)$ where U is an open subset of $\mathbb{R}^k$ and $\phi : U \to \mathbb{R}^n$ a C^∞-smooth injective map with injective derivative at every point. The map ϕ is called a parametrization of V with domain U, and the image of its derivative $D\phi(u)$ at u is the tangent space to X at $\phi(u)$, hereafter denoted by $T_{\phi(u)}X$. Then, by the constant rank theorem, there is an open set $W \subset \mathbb{R}^n$ with $W \cap X = V$ and a C^∞-smooth map $\psi : W \to U$ such that $\psi \circ \phi = \mathrm{id}$, the identity map of U. The restriction $\psi_{|V}$ is called a chart with domain V. This allows one to carry over to X local tools from differential calculus, see [47, ch. 1]. We say that X is closed if it is a closed subset of $\mathbb{R}^n$.

If X, Y are smooth manifolds embedded in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and if $Z \subset Y$ is a smooth embedded submanifold, a smooth map $f : X \to Y$ is said to be transversal to Z if $\mathrm{Im}Df(\mathbf{x}) + T_{f(\mathbf{x})}Z = T_{f(\mathbf{x})}Y$ at every $\mathbf{x} \in X$ such that $f(\mathbf{x}) \in Z$. If f is transversal to Z, then $f^{-1}(Z)$ is an embedded submanifold of X whose codimension is the same as the codimension of Z in Y. In particular, if X is compact and $\mathrm{dim}X + \mathrm{dim}Z = \mathrm{dim}Y$, then $f^{-1}(Z)$ consists of finitely many points. The residue class modulo 2 of the cardinality of such points is the intersection number of f with Z modulo 2, denoted by $I_2(f, Z)$. If in addition Z is closed in Y, then $I_2(f, Z)$ is invariant under small homotopic deformations of f, and this allows one to define $I_2(f, Z)$ even when f is not transversal to Z, because a suitable but arbitrary small homotopic deformation of f will guarantee transversality, see [47, ch. 2].

Theorem A.1. *If $\mathcal{A}$ is a C^∞-smooth complete and connected surface embedded in $\mathbb{R}^3$, then $\mathbb{R}^3 \setminus \mathcal{A}$ has two connected components.*

Proof. Let W be a tubular neighborhoud of $\mathcal{A}$ in $\mathbb{R}^3$ [47, Ch. 2, Sec. 3, ex. 3 & 16]. That is, W is an open neighborhood of $\mathcal{A}$ in $\mathbb{R}^3$ comprised of points $\mathbf{y}$ having a unique closest point from X, say $\mathbf{x}$, such that $|\mathbf{y} - \mathbf{x}| < \varepsilon(x)$ where ε is a suitable smooth and strictly positive function on $\mathcal{A}$. Thus, we can write $W = \{\mathbf{x} + t\mathbf{n}(\mathbf{x}), \mathbf{x} \in \mathcal{A}, |t| < \varepsilon(\mathbf{x})\}$, where $\mathbf{n}(\mathbf{x})$ is a normal vector to $\mathcal{A}$ at $\mathbf{x}$ of unit length. Note that, for each $\mathbf{x} \in \mathcal{A}$, there are two possible (opposite) choices of $\mathbf{n}(\mathbf{x})$, but the definition of W makes it irrelevant which one we make. Moreover, if we fix $\mathbf{n}(\mathbf{x})$ and $\eta \in (0, 1)$, we can find a neighborhood V of $\mathbf{x}$ in $\mathcal{A}$ such that, to each $\mathbf{y} \in V$, there is a unique choice of $\mathbf{n}(\mathbf{y})$ with $|\mathbf{n}(\mathbf{y}) - \mathbf{n}(\mathbf{x})| < \eta$. Indeed, if $\phi : U \to V$ is a parametrization with inverse ψ such that $\mathbf{x} \in V$, and if we set $\mathbf{n_y} := \partial_{x_1}\phi(\psi(\mathbf{y})) \wedge \partial_{x_2}\phi(\psi(\mathbf{y}))/\|\partial_{x_1}\phi(\psi(\mathbf{y})) \wedge \partial_{x_2}\phi(\psi(\mathbf{y}))\|$ where x_1, x_2 are Euclidean coordinates on $U \subset \mathbb{R}^2$ while ∂_{x_j} denotes the partial derivative with respect to x_j and the wedge indicates the vector product, then the two possible choices for $\mathbf{n}(\mathbf{y})$ when $\mathbf{y} \in V$ are $\pm\mathbf{n}_{\mathbf{y}}$. Thus, if we select for instance $\mathbf{n}(\mathbf{x}) = \mathbf{n}_{\mathbf{x}}$ and subsequently set $\mathbf{n}(\mathbf{y}) = \mathbf{n}_{\mathbf{y}}$, we get upon shrinking V if necessary that $|\mathbf{n}(\mathbf{x}) - \mathbf{n}(\mathbf{y})| < \eta$ and $|\mathbf{n}(\mathbf{x}) + \mathbf{n}(\mathbf{y})| > 2 - \eta$ for $\mathbf{y} \in V$. As a consequence, if $\Upsilon : [a, b] \to \mathcal{A}$ is a continuous path, and if $\mathbf{n}_b$ is a unit normal vector to $\mathcal{A}$ at $\Upsilon(b)$, there is a continuous choice of $\mathbf{n}(\Upsilon(\tau))$ for $\tau \in [a, b]$ such that $\mathbf{n}(\Upsilon(b)) = \mathbf{n}_b$.

Fix $\mathbf{x}_0 \in \mathcal{A}$ and let $\mathbf{n}_0$ be an arbitrary choice for $\mathbf{n}(\mathbf{x}_0)$. Pick t_0 with $0 < t_0 < \varepsilon(\mathbf{x}_0)$, and define two points in W by $\mathbf{x}_0^{\pm} = x_0 \pm t_0\mathbf{n}_0$. We claim that each $\mathbf{y} \in \mathbb{R}^3 \setminus \mathcal{A}$ can be joined either to $\mathbf{x}_0^+$ or to $\mathbf{x}_0^-$ by a continuous arc contained in $\mathbb{R}^3 \setminus \mathcal{A}$. Indeed, let $\mathbf{c} : [0, 1] \to \mathbb{R}^3$ be a continuous path with $\mathbf{c}(0) = \mathbf{y}$ and $\mathbf{c}(1) = \mathbf{x}_0$. Let $\tau_0 \in (0, 1]$ be smallest such that $\mathbf{c}(\tau_0) \in \mathcal{A}$; such a τ_0 exists since $\mathcal{A}$ is closed. Pick $0 < \tau_1 < \tau_0$ close enough to τ_0 that $\mathbf{c}(\tau_1) \in W$, say $\mathbf{c}(\tau_1) = \mathbf{x}_1 + t_1\mathbf{n}_1$ where $\mathbf{x}_1 \in \mathcal{A}$, $|t_1| < \varepsilon(\mathbf{x}_1)$, and $\mathbf{n}_1$ is a unit vector normal to $\mathcal{A}$ at $\mathbf{x}_1$. Since $\mathcal{A}$ is connected, there is a continuous path $\Upsilon : [\tau_1, 1] \to \mathcal{A}$ such that $\Upsilon(\tau_1) = \mathbf{x}_1$ and $\Upsilon(1) = \mathbf{x}_0$. Along the path Υ, there is a continuous choice of $\tau \to \mathbf{n}(\Upsilon(\tau))$ such that $\mathbf{n}(\mathbf{x}_0) = \mathbf{n}_0$; this follows from a previous remark. Changing the sign of t_1 if necessary, we may assume that $\mathbf{n}_1 = \mathbf{n}(\mathbf{x}_1)$. Let $\eta : [\tau_1, 1] \to \mathbb{R}_+$ be a continuous function such that $0 < |\eta(\tau)| < \varepsilon(\Upsilon(\tau))$ with $\eta(\tau_1) = t_1$ and $\eta(1) = \operatorname{sgn} t_1 |t_0|$. Such an η exists, since ε is continuous and strictly positive while $|t_1| < \varepsilon(\mathbf{x}_1)$ and $|t_0| < \varepsilon(\mathbf{x}_0)$. Now the concatenation of $\mathbf{c}$ restricted to $[0, \tau_1]$ and $\mathbf{c}_1 : [\tau_1, 1] \to \mathbb{R}^3$ given by $\mathbf{c}_1(\tau) = \Upsilon(\tau) + \eta(\tau)\mathbf{n}(\Upsilon(\tau))$ is a continuous path from y to either $\mathbf{x}_0^+$ or $\mathbf{x}_0^-$ (depending on the sign of t_1) which is entirely contained in $\mathbb{R}^3 \setminus \mathcal{A}$. This proves the claim, showing that $\mathbb{R}^3 \setminus \mathcal{A}$ has at most two components. To see that it has at least two, it is enough to know that any smooth cycle $\varphi : \mathbb{S}^1 \to \mathbb{R}^3$ has intersection number $I_2(\varphi, \mathcal{A}) = 0$ modulo 2. Indeed, if this is the case and if $\mathbf{x}_0^+$ and $\mathbf{x}_0^-$ could be joined by a continuous arc $\mathbf{c} : [0, 1] \to \mathbb{R}^3$ not intersecting $\mathcal{A}$, then $\mathbf{c}$ could be chosen C^∞-smooth (see [47, Ch.1, Sec. 6, Ex. 3]) and we could complete it into a cycle $\varphi : \mathbb{S}^1 \to \mathbb{R}^3$ by concatenation with the segment $[\mathbf{x}_0^-, \mathbf{x}_0^+]$ which intersects $\mathcal{A}$ exactly once (at $\mathbf{x}_0$), in a transversal manner. Elementary modifications at $\mathbf{x}_0^-$ and $\mathbf{x}_0^+$ will arrange things so that φ becomes C^∞-smooth, and this would contradict the fact that the number of intersection points with $\mathcal{A}$ must be even. Now, if $\mathbb{D}$ is the unit disk, any smooth map $\varphi : \mathbb{S}^1 \to \mathbb{R}^3$ extends to a smooth map $f : \overline{\mathbb{D}} \to \mathbb{R}^3$ (take for example $f(re^{i\theta}) = e^{1-1/r}\varphi(e^{i\theta})$). Thus, by the boundary theorem [47, p. 80], the intersection number modulo 2 of φ with any smooth and complete embedded submanifold of dimension 2 in $\mathbb{R}^3$ (in particular with $\mathcal{A}$) must be zero. This achieves the proof. $\square$

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