ON GEOMETRY AND COMBINATORICS OF VAN KAMPEN DIAGRAMS

By

Alexey Yu. Muranov

Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August, 2006

Nashville, Tennessee

Approved:

Professor Alexander Olshanskiy,

Professor Thomas Kephart,

Professor Ralph McKenzie,

Professor Michael Mihalik,

Professor Mark Sapir.

To my parents, my sister, and my brother.

ACKNOWLEDGEMENTS

I would first like to thank my advisor Alexander Ol'shanskiy. He has been supervising my research and guiding me in the area of Combinatorial and Geometric Group Theory for more than seven years. Discussions I had with him were a source of ideas. His words of encouragement meant a lot to me. The classes I took from him during my undergraduate and graduate studies were an excellent opportunity to learn some of the classic and modern chapters of Algebra.

Second, I would like to express my gratitude to the professors at Mathematics Department of Vanderbilt University whose courses and seminars I have had privilege to attend. Among them are Dietmar Bisch, notes from his course are nearly camera-ready, Ralph McKenzie, he is willing to talk about good math any time, Mike Mihalik, he taught an unforgettable Topology course, Mark Sapir, he teaches that problems are easier than what they seem, Guoliang Yu, his classes are nice "appetizers" for various subjects.

I am grateful to Tom Kephart, who is a professor at Physics Department, for agreeing to be a member of my PhD defense committee.

Finally, I would like to mention my fellow graduate students, in the math department and beyond, some of whom already graduated, who made my stay in Nashville enjoyable, and with whom I could discuss my research or other more interesting topics. I am particularly thankful to Dr. Ashot Minasyan, Dr. Dima Sonkin, Dr. Petar Marković, Dr. Miklós Maróti, Dr. Nick Galatos, Dr. Chris Stephens, Jana Višňovská, Dr. Marcin Kozik, Tivadar Szemethy, Braňo Kusý, Ján Špakula, Bogdan Nica, Iva Kozáková.

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

The subject of this work is application of methods of combinatorial group theory to the problem of constructing groups with prescribed properties. It is shown how groups with certain properties can be presented by generators and defining relations, thus proving their existence. Several existence theorems proved in this paper are based on the same approach: van Kampen diagrams over group presentations are used to derive algebraic properties of the groups from combinatorial properties of their presentations.

The focus of this paper is on *boundnely generated* and *boundedly simple* groups.

0.a Boundedly generated groups

Definition. A group G is *m-boundedly generated* if it has (not necessarily normal) cyclic subgroups $C_1, \ldots, C_m$ such that $G = C_1 \cdots C_m$. A group G is called *boundedly generated* if it is m-boundedly generated for some natural m.

The definition of bounded generation was motivated by the work [CK83] of Carter and Keller. They proved that any matrix in $SL_n(\mathcal{O})$, where $n \geq 3$ and $\mathcal{O}$ is the ring of integers of a finite extension of the field of rational numbers, is the product of a bounded number of elementary matrices (bounded depending only on n and $\mathcal{O}$), which implies bounded generation in the defined above sense. Since then bounded generation has been studied in connection with the *congruence subgroup property* (*CSP*) (see [PR93]), *Kazhdan's property* (T) (see [Sha99]).

Because many linear groups are known to be boundedly generated, it was natural to ask if all boundedly generated groups are linear, or at least residually finite. The corollary of the following theorem provides the negative answer.

Definition. A set of group words is *symmetrized* if for every element of this set, its inverse and all its cyclic shifts are also in the set. A symmetrized set S of non-empty reduced group words satisfies the *small cancellation condition* $C'(\lambda)$, $\lambda > 0$, if for any two distinct elements W_1 and W_2 of S, the length of any common prefix of W_1 and W_2 is less than $\lambda \min\{|W_1|, |W_2|\}$ (see [LS01]).

Theorem 1. *Let F be a finite-rank free group with a fixed basis. Let S be an infinite subset of F whose all elements are cyclically reduced and are not proper powers. If the symmetrization of S satisfies the small cancellation condition $C'(\lambda)$ for some $\lambda < \frac{1}{13}$, then there exist an infinite simple 2-generated group P and a homomorphism $\phi\colon F \to P$ such that ϕ maps S onto P.*

Corollary 1.a. *There exists an infinite simple 2-generated group G and 27 elements $x_1, \dots, x_{27} \in G$ such that for every element g of G, there exists a natural number n such that $g = x_1^n \cdots x_{27}^n$. Such a group is, in particular, 27-boundedly generated.*

There are natural questions about the relation between the bounded generation property and the property of being *polycyclic*.

Definition. A group is *polycyclic* if it has a finite subnormal series with cyclic factors. A group is *virtually polycyclic* if it has a polycyclic subgroup of finite index.

Polycyclic groups are boundedly generated. Subgroups and homomorphic images of polycyclic and virtually polycyclic groups are polycyclic or, respectively, virtually polycyclic. Free groups of rank at least 2 are not virtually polycyclic.

Vasiliy Bludov posed the following question in The Kourovka Notebook (Problem 13.11 in [MK95], see also Question 6 in [Blu95]):

> If a torsion-free group G has a finite system of generators $a_1, \dots, a_n$ such that every element of G has a unique presentation in the form $a_1^{k_1} \dots a_n^{k_n}$ where $k_i \in \mathbb{Z}$, is it true then that G is virtually polycyclic?

In terms of *file bases* (see definitions in [Blu95]), Bludov's question is whether every torsion-free group with a *regular* file basis $(a_1, \dots, a_n)$ is virtually polycyclic.

Every group that has an n-element file basis is n-boundedly generated. The following two theorems (see [Itô55] and [LR80]) together yield that every 2-boundedly generated group is polycyclic:

Theorem (Noboru Itô, 1955). *If a group G has abelian subgroups A and B such that $G = AB$, then G is metabelian.*

Theorem (John Lennox and James Roseblade, 1980). *If a soluble group G has polycyclic subgroups A and B such that $G = AB$, then G is polycyclic.*

Some examples of non-polycyclic groups with finite regular file bases and with torsion are known. It is shown in [Blu95] (see Example 3 therein) that the semidirect product of the rank-n free abelian group $\mathbb{Z}^n$ with the symmetric group S_n defined via the action of S_n on $\mathbb{Z}^n$ by naturally permuting the standard generators has an n-element regular file basis of infinite-order elements. If $n \geq 5$, then such a semidirect product is virtually polycyclic but not polycyclic.

The following theorem answers Bludov's question negatively:

Theorem 2. *Provided $n \geq 63$, there exists a group G and pairwise distinct elements $a_1, \dots, a_n$ in G such that:*

(0) *G is generated by $\{a_1, \dots, a_n\}$;*

(1) *every $n - 21$ elements out of $\{a_1, \dots, a_n\}$ freely generate a free subgroup such that every two elements of this subgroup F are conjugate in G only if they are conjugate in F itself (in particular, G is not virtually polycyclic);*

(2) *for every element g of G, there is a unique n-tuple $(k_1, \dots, k_n) \in \mathbb{Z}^n$ such that $g = a_1^{k_1} \dots a_n^{k_n}$;*

(3) *G is torsion-free*;

(4) *G is the direct limit*[1] *of a sequence of hyperbolic groups with respect to a family of surjective homomorphisms*;

(5) *G is recursively presented and has decidable word and conjugacy problems.*

0.b Boundedly simple groups

Definition. The *conjugate* of a group element g by a group element h, denoted g^h, is hgh^{-1}. A group G is called *m-boundedly simple* if for every two nontrivial elements $g, h \in G$, the element h is the product of m or fewer conjugates of $g^{\pm 1}$. A group G is called *boundedly simple* if it is m-boundedly simple for some natural m.

Remark 0.1. Every boundedly simple group is simple; however, infinite alternating groups are simple but not boundedly simple.

Remark 0.2. For each natural m, the class of m-boundedly simple groups is definable by a formula of the restricted predicate calculus. The class of all simple groups is not.

Remark 0.3. A group is boundedly simple if and only if each of its ultrapowers is simple. If a group is m-boundedly simple, then all its ultrapowers are m-boundedly simple.

Definition. The *commutator* of two group elements x and y, denoted $[x, y]$, is $xyx^{-1}y^{-1}$. The *commutator length* of an element g of the derived subgroup of a group G is the minimal n such that there exist elements $x_1, \ldots, x_n, y_1, \ldots, y_n$ in G such that $g = [x_1, y_1] \ldots [x_n, y_n]$. The commutator length of the identity element is zero. The *commutator width* of a group G is the maximum of the commutator lengths of the elements of its derived subgroup $[G, G]$.

[1] *Direct limit* in the sense of Bourbaki (see [Bou98]).

Every non-abelian simple group coincides with its derived subgroup. In a non-abelian m-boundedly simple group, the derived subgroup in addition has finite commutator width: each element of the derived subgroup may be presented as the product of m or fewer commutators (take an arbitrary nontrivial commutator, every element is the product of m or fewer conjugates of this commutator).

It is not obvious that infinite boundedly simple finitely generated groups exist. It is also a complicated question what kinds of infinite groups can be 1-boundedly simple. (Every finite nontrivial 1-boundedly simple group is cyclic of order 2, for it has to be a p-group for some prime p, and thus must have a nontrivial center whose order is a power of p.) Note that a group is 1-boundedly simple if and only all nontrivial elements in this group are conjugate.

The following theorem appeared in [Ol'91]:

Theorem (Sergei Ivanov, 1989)**.** *For every big enough prime p, there exists a 2-generated infinite group of exponent p, in which there are exactly p distinct conjugacy classes, and therefore, every subgroup of order p has elements from all of these classes.*

It is easy to see that the group whose existence is stated in this theorem is $(p-1)$-boundedly simple. The original proof of Ivanov's Theorem works only for big p, say, $p \geq 10^{78}$. It heavily uses techniques of graded diagrams.

It was proved by Graham Higman, Bernhard Neumann, and Hanna Neumann in 1949 (see [HNN49]) that every torsion-free group can be embedded into a 1-boundedly simple group of the same cardinality. Their construction yielded non-finitely-generated groups.

The following outstanding result of Osin (see [Osi04]) shows that every countable torsion-free group can be embedded into a 2-generated 1-boundedly simple group, which in particular implies that there exist uncountably many pair-wise non-isomorphic 2-generated 1-boundedly simple groups (using the fact that the number of isomorphism classes of all torsion-free finitely generated groups is uncountable):

Theorem (Denis Osin, 2004)**.** *Any countable group G can be embedded into a 2-generated group C such that any two elements of the same order are conjugate in C and every finite-order element of C is conjugate to an element of G.*

In his proof, Osin developed and used theory of *relatively hyperbolic* groups, first defined by Gromov.

The following theorem, even though being weaker than Osin's results, was obtained earlier, and it serves as another illustration of the techniques used in the proofs of Theorems 1 and 2.

Theorem 3. *There exists a 14-boundedly simple 2-generated group G that has a free non-cyclic subgroup, and such that the word problem in G is decidable.*

0.c Summary

The goal of this work is to prove Theorems 1, 2, 3.

The groups in question, or rather their presentations, are constructed by imposing relations that force the group to be boundedly simple, or boundedly generated, or have a "regular file basis," accordingly, while in the same time choosing those relations so that certain small-cancellation-type conditions are satisfied. These conditions are more general (weaker) than the classical condition C', and are formulated not in terms of the defining relations of a presentation, but in terms of van Kampen diagrams over the presentation. Similar and even more general conditions on diagrams appeared before in [Ol'91] and in [San97]. The diagrams satisfying such conditions are sometimes called *diagrams with partitioned boundaries of cells.* This reflects the form of the imposed relations which are in a sense partitioned into subwords, some of which are almost arbitrary, while the others are chosen in a special way.

CHAPTER I

MAPS AND DIAGRAMS

1 Definitions

If X is a set, then $\|X\|$ shall denote the cardinality of X.

1.a Combinatorial complexes

In the context of this paper, *graph* is a synonym of 1-*complex* (multiple edges and loops are admissible). If Γ is a graph, then $\Gamma(0)$ denotes its vertex set, and $\Gamma(1)$ denotes its edge set; $\Gamma(1)$ may be empty, but $\Gamma(0)$ may not. The graph itself is an ordered pair: $\Gamma = (\Gamma(0), \Gamma(1))$. This paper largely deals with 2-*complexes*. If Φ is a 2-complex, then $\Phi(0)$, $\Phi(1)$, and $\Phi(2)$ denote its vertex set, its edge set, and its face set, respectively. The 2-complex Φ itself is the ordered triple $(\Phi(0), \Phi(1), \Phi(2))$. Below is a formal definition of combinatorial 0-, 1-, and 2-complexes and their morphisms.

A (combinatorial) 0-*complex* A is a 1-tuple $(A(0))$ where $A(0)$ is an arbitrary non-empty set. Elements of $A(0)$ are called *vertices* of the complex A.

A 0-complex with exactly 2 vertices will be called a *combinatorial* 0-*sphere*.

A *morphism* ϕ of a 0-complex A to a 0-complex B is a 1-tuple $(\phi(0))$ where $\phi(0)$ is an arbitrary function $A(0) \to B(0)$. If A, B, and C are 0-complexes, and $\phi\colon A \to B$ and $\psi\colon B \to C$ are morphisms, then the product $\psi\phi\colon A \to C$ is defined naturally: $(\psi\phi)(0) = \psi(0) \circ \phi(0)$. A morphism $\phi\colon A \to B$ is called an *isomorphism* of A with B if there exists a morphism $\psi\colon B \to A$ such that $\psi\phi$ is the *identity morphism* of the complex A and $\phi\psi$ is the identity morphism of the complex B.

A (combinatorial) 1-*complex* A is a 2-tuple (i.e., ordered pair) $(A(0), A(1))$ such that:

(1) $A(0)$ is an arbitrary non-empty set;

(2) $A(1)$ is a set of ordered pairs of the form (E, α) where E is a combinatorial 0-sphere and α is a morphism of E to the 0-complex $(A(0))$.

Elements of $A(0)$ are called *vertices* of the complex A, elements of $A(1)$ are called *edges*. The 0-complex $(A(0))$ is called the 0-*skeleton* of A. All 1-complexes will also be called *graphs*.

A 1-complex representing a circle (e.g., a 1-complex consisting of one vertex and one edge) will be called a *combinatorial* 1-*sphere*, or a *combinatorial circle*. (The definition of a 1-sphere could be made precise, but then it would become unreasonably long.)

If A and B are 1-complexes, then a *morphism* $\phi\colon A \to B$ is a 2-tuple $(\phi(0), \phi(1))$ such that:

(1) $\phi^0 = (\phi(0))$ is a morphism of the 0-skeleton A^0 of A to the 0-skeleton B^0 of B;

(2) $\phi(1)$ is a function on $A(1)$ such that the image of every $e = (E, \alpha) \in A(1)$ under $\phi(1)$ is an ordered pair (e', ξ) where $e' = (E', \alpha') \in B(1)$, ξ is an isomorphism of E with E', and $\phi^0\alpha = \alpha'\xi$.

Multiplication of morphisms of 1-complexes is defined naturally. For example, if A, B, C are 1-complexes, ϕ and ψ are morphisms, $\phi\colon A \to B$, $\psi\colon B \to C$, $e = (E, \alpha)$ is an edge of A, $e' = (E', \alpha')$ is an edge of B, $e'' = (E'', \alpha'')$ is an edge of C, $\phi(1)(e) = (e', \xi)$, $\psi(1)(e') = (e'', \zeta)$, then $(\psi\phi)(1)(e) = (e'', \zeta\xi)$ (note that $\zeta\xi$ is an isomorphism of the 0-complex E with the 0-complex E''). *Isomorphisms* of 1-complexes are defined in the usual way.

A (combinatorial) 2-*complex* A is a 3-tuple $(A(0), A(1), A(2))$ such that:

(1) $(A(0), A(1))$ is a 1-complex, called the 1-*skeleton* of A;

(2) $A(2)$ is a set of ordered pairs of the form (F, α) where F is a combinatorial 1-sphere and α is a morphism of F to the 1-skeleton of A.

Elements of $A(0)$ are called *vertices* of the complex A, elements of $A(1)$ are called *edges*, elements of $A(2)$ are called *faces*.

If A and B are 2-complexes, then a *morphism* $\phi\colon A \to B$ is defined as a 3-tuple $(\phi(0), \phi(1), \phi(2))$ such that:

(1) $\phi^1 = (\phi(0), \phi(1))$ is a morphism of the 1-skeleton A^1 of A to the 1-skeleton B^1 of B;

(2) $\phi(2)$ is a function on $A(2)$ such that the image of every $f = (F, \beta) \in A(2)$ under $\phi(2)$ is an ordered pair (f', ξ) where $f' = (F', \beta') \in B(2)$, ξ is an isomorphism of F with F', and $\phi^1\beta = \beta'\xi$.

Products of morphisms of 2-complexes are defined analogously to the case of 1-complexes. The notion of *isomorphism* for 2-complexes is the natural one.

A combinatorial n-complex A is called *finite* if all of the sets $A(0), \ldots, A(n)$ are finite. An n-complex A is a *subcomplex* of an m-complex B, $0 \le n \le m$, if $A(0) \subset B(0), \ldots, A(n) \subset B(n)$. Every morphism ϕ of a combinatorial n-complex A to a combinatorial n-complex B, $n \le 2$, naturally defines functions $\bar{\phi}^i\colon A(i) \to B(i)$, $0 \le i \le n$. This notation for these functions associated with a given morphism ϕ shall be used in this section for brevity.

If $e = (E, \alpha)$ is an edge of a graph Γ, then the vertices of Γ that are the images of the vertices of E under $\bar{\alpha}^0$ are called the *end-vertices* of e. An edge is *incident* to its end-vertices. A *loop* is an edge that has only one end-vertex. Two vertices are called *adjacent* if they form the set of end-vertices of some edge. The end-vertex of a loop is adjacent to itself. If v is a vertex of a graph Γ, then the number of all edges of

Γ incident to v plus the number of all loops incident to v is called the *degree* of the vertex v and is denoted by $d(v)$ or $d_\Gamma(v)$.

If $f = (F, \beta)$ is a face of a 2-complex Ψ, then f is said to be *incident* to the images of the vertices and edges of F under $\bar\beta^0$ and $\bar\beta^1$ respectively.

The *Euler characteristic* of a combinatorial 2-complex Ψ is denoted by χ_Ψ and is defined by

$$\chi_\Psi = \|\Phi(0)\| - \|\Phi(1)\| + \|\Phi(2)\|.$$

It is somewhat complicated to talk about combinatorial complexes in purely combinatorial terms. The geometrical intuition may help if together with every combinatorial complex consider some corresponding topological space.

Given a 2-complex Φ, put a 1-point topological space D^0_v into correspondence to every vertex v of Φ, a topological closed segment D^1_e to every edge e of Φ, and a topological closed disk D^2_f to every face f of Φ. Assume that D^m_x and D^n_y are disjoint unless $m = n$ and $x = y$. Consider the topological sum

$$\sum_{v\in\Phi(0)} D^0_v + \sum_{e\in\Phi(1)} D^1_e + \sum_{f\in\Phi(2)} D^2_f.$$

Take the quotient of it over an equivalence relation in accordance with the structure of Φ. For example, if $\{v_1, v_2\}$ is the set of end-vertices of an edge e of Φ, then one of the end-vertices of D^1_e should be identified with the only point of $D^0_{v_1}$, and the other end-vertex should be identified with the only point of $D^0_{v_2}$. Taking the quotient may be done in two steps: first, attach the end-vertices of the segments $\{D^1_e\}_{e\in\Phi(1)}$ to the points of the discrete topological space $\sum_{v\in\Phi(0)} D^0_v$; then, attach the boundaries of the discs $\{D^2_f\}_{f\in\Phi(2)}$ to the obtained "skeleton." The constructed topological quotient space is unique up to homeomorphism. This space or any homeomorphic one is called a *topological space of the complex* Φ. It may also be said that Φ *represents* this topological space. Note that the restriction of the quotient function to the interior (in the geometric sense) of every D^1_e, $e \in \Phi(1)$, and every D^2_f, $f \in \Phi(2)$, is a homeomorphism onto the image in the quotient space. Every morphism of combinatorial complexes defines some set of continuous functions from a given topological space of the first complex to a given topological space of the second.

From now on, combinatorial and topological languages shall be used together. Moreover, some facts intuitively clear from topological point of view shall be used without proofs.

It is a simple but tiresome task to formulate a combinatorial criterion in terms of the local structure of a 2-complex (*stars* at its vertices) that would determine if a topological space of this complex is a 2-dimensional surface (i.e., a 2-manifold or 2-manifold with a boundary). For an example of constructing surfaces combinatorially using simplicial complexes see [Ale56]. Keeping this in mind, the following definition may be viewed as combinatorial. A *combinatorial surface* is a 2-complex representing some surface. A *combinatorial sphere* and a *combinatorial disc* are 2-complexes representing a sphere and a disc respectively. They play an important role in this paper. Every combinatorial sphere and every combinatorial disc are finite (because of the compactness of spheres and discs). Combinatorial spheres and combinatorial discs may also be defined combinatorially.

A finite graph Γ is *planar* if it is a subgraph of the 1-skeleton of some combinatorial sphere. Actually, every *connected* finite planar graph having at least one edge is the 1-skeleton of some combinatorial sphere.

Using the Euler charcteristic of a sphere, it is easy to prove the following well-known

Proposition 1.1. *For any planar graph Γ without loops and without multiple edges, the number of edges of Γ is less than three times the number of its vertices*:

$$\|\Gamma(1)\| < 3\|\Gamma(0)\|.$$

Proof. Without loss of generality, assume that Γ is connected. (It is enough to prove the above inequality for every connected component of Γ.) If $\|\Gamma(0)\| \leq 2$, then the statement is obvious. If $\|\Gamma(0)\| \geq 3$, then Γ is the 1-skeleton of some combinatorial sphere Φ, and the degree of every face of Φ is at least 3. On one hand,

$$2\|\Phi(1)\| = \sum_{f\in\Phi(2)} |\partial f| \geq 3\|\Phi(2)\|.$$

On the other hand, since the Euler characteristic of every shere is 2,

$$\|\Phi(0)\| - \|\Phi(1)\| + \|\Phi(2)\| = 2.$$

Therefore,

$$\|\Gamma(1)\| = 3\|\Phi(0)\| - 2\|\Phi(1)\| + 3\|\Phi(2)\| - 6 \leq 3\|\Phi(0)\| - 6. \qquad \square$$

An *orientation* of an edge $e = (E, \alpha)$ of a complex Γ is a total order on the two-element set of vertices of E. There are two possible orientations of every edge, they are *opposite* to each other. An *oriented edge* is an edge together with one of its orientations. The set of all oriented edges of a complex Γ will be denoted by $\hat{\Gamma}(1)$. Every morphism ϕ of a combinatorial n-complex Γ_1 to a combinatorial n-complex Γ_2, $n \in \{1, 2\}$, naturally defines a function $\hat{\Gamma}_1(1) \to \hat{\Gamma}_2(1)$; denote this function by $\hat{\phi}^1$. Two oriented edges obtained from the same edge by picking opposite orientations are *inverse* to each other. The oriented edge inverse to an oriented edge e is denoted by e^{-1}. If e is an oriented edge, (E, α) is a corresponding non-oriented edge, $E(0) = \{v_1, v_2\}$, and v_1 precedes v_2 with respect to the chosen total order on $E(0)$, then $\bar{\alpha}^0(v_1)$ is called the *tail* of e and $\bar{\alpha}^0(v_2)$ is called the *head* of e. An oriented edge *leaves* its tail and *enters* its head. Clearly, the head of an oriented edge is the tail of the inverse oriented edge, and vice versa. For any vertex v in a complex Γ, the number of oriented edges leaving v equals the number of oriented edges entering v and equals the degree of v.

Here follow several definitions related to paths in combinatorial complexes.

A *path* is a finite sequence of alternating vertices and oriented edges such that the following conditions hold: it starts with a vertex and ends with a vertex; the vertex immediately preceding an oriented edge is its tail; the vertex immediately following an oriented edge is its head. The *initial vertex* of a path is its first vertex, the *terminal vertex* of a path is its last vertex, and an *end-vertex* of a path is either its initial or its terminal vertex. A path *starts* at its initial vertex and *ends* at its terminal vertex. The *length* of a path $p = (v_0, e_1, v_1, \ldots, e_n, v_n)$ is n; it is denoted by $|p|$. The vertices $v_1, \ldots, v_{n-1}$ of this path are called its *intermediate* vertices. A *trivial path* is a path of length zero. By abuse of notation, a path of the form (v_1, e, v_2), where e is an oriented edge from v_1 to v_2, shall be denoted by e, and a trivial path (v) shall be denoted by v.

The *inverse path* to a path p is defined naturally and is denoted by p^{-1}. If the terminal vertex of a path p_1 coincides with the initial vertex of a path p_2, then the *product* p_1p_2 is defined naturally. A path s is called an *initial subpath* of a path p if $p = sq$ for some path q. A path s is called a *terminal subpath* of a path p if $p = qs$ for some path q.

A *cyclic path* is a path such that its terminal vertex coincides with its initial vertex. A *cycle* is the set of all *cyclic shifts* of some cyclic path. The cycle represented by a cyclic path p shall be denoted by $\langle p \rangle$. The *length* of a cycle c, denoted by $|c|$, is the length of an arbitrary representative of c. A *trivial cycle* is a cycle of length zero. A path p is a *subpath of a cycle* c if for some representative r of c and for some natural n, p is a subpath of r^n (i.e., of the product of n copies of r). Let c be a cycle in which no oriented edge occurs more than once. Then a set of paths S is said to *cover* c if all the elements of S are nontrivial subpaths of c, and every oriented edge that occurs in c also occurs in some path from S.

A path is *reduced* if it does not have a subpath of the form ee^{-1} where e is an oriented edge. A cyclic path is *cyclically reduced* if it is reduced and its first oriented edge is not inverse to its last oriented edge. (For example, all trivial paths are cyclically reduced.) A cycle is *reduced* if it consists of cyclically reduced cyclic paths. A path is *simple* if it is nontrivial, reduced, and none of its intermediate vertices appears in it more than once. A cycle is *simple* if it consists of simple cyclic paths.

All paths in any graph are naturally partially ordered by the relation "is a subpath of." A path is called *maximal* in some set of paths if this path is not a proper subpath of any other path in this set.

Every morphism ϕ of combinatorial complexes naturally defines a function from the set of paths (respectively cycles) of the first complex to the set of paths (respectively cycles) of the second. The image of a given path or cycle under this function will be referred to as the *image relative to* ϕ, or ϕ*-image*.

An *oriented arc* in a complex Γ is a simple path all intermediate vertices of which have degree 2 in Γ. A *non-oriented arc*, or simply *arc*, may be defined as a pair of mutually inverse oriented arcs. The *length* of a non-oriented arc u is defined as the length of either of the two associated oriented arcs and is denoted by $|u|$. The concepts of a *subarc*, a *maximal arc*, and so on, are self-explanatory. Sometimes edges will be viewed as arcs, and oriented edges as oriented arcs. The *set of edges* of an oriented arc (respectively of an arc) is the set of all edges that with some orientation occur in that oriented arc (respectively in either of the two corresponding oriented arcs). An *intermediate vertex* of an arc is an intermediate vertex of either of the corresponding oriented arcs. Two arcs, or oriented arcs, are said to *overlap* if their sets of edges are not disjoint. A set of arcs, or oriented arcs, is called *non-overlapping* if any two distinct elements of this set have disjoint sets of edges. An arc (or an edge) u *lies* on a path p if at least one of the oriented arcs (respectively oriented edges) associated with u is a subpath of p. A set of arcs A *covers* a set of arcs (or edges) B if every element of B is a subarc of some element of A.

Combinatorial complexes representing orientable surfaces may be oriented in a combinatorial manner. To define the combinatorial notion of *orientation*, consider first an arbitrary combinatorial circle Γ. Consider a function that for every edge e of Γ chooses an orientation of e. Call an edge with a chosen orientation a *chosen oriented edge*. Say that such a function chooses *coherent orientations* if every vertex of Γ is the tail of exactly one of the chosen oriented edges and the head of exactly one of the chosen oriented edges. An *orientation* of a combinatorial circle is a function on the set of its edges that chooses coherent orientations of the edges. An *oriented combinatorial circle* is a combinatorial circle together with one of its two possible orientations. Note that in an oriented combinatorial circle Γ there is a unique simple cycle whose oriented edges are the chosen ones. Call this cycle the *chosen cycle* of Γ.

Now, consider a face $f = (F, \beta)$ of a 2-complex Φ. An *orientation* of f is an orientation of the combinatorial circle F. An *oriented face* is a face together with one of its orientations. The set of all oriented faces of a complex Φ will be denoted by $\hat{\Phi}(2)$. (Note that $\|\hat{\Phi}(2)\| = 2\|\Phi(2)\|$.) Every morphism ϕ of a combinatorial 2-complex Φ_1 to a combinatorial 2-complex Φ_2 naturally defines a function $\hat{\Phi}_1(2) \to \hat{\Phi}_2(2)$; denote this function by $\hat{\phi}^2$. Two oriented faces obtained from the same face by picking *opposite* orientations are *inverse* to each other. The oriented face inverse to an oriented face f is denoted by f^{-1}. Each oriented face has a uniquely defined *boundary cycle*: if f is an oriented face of Φ and (F, β) is its underlying non-oriented face, then the boundary cycle of f is the cycle in Φ which is the β-image of the chosen cycle of F. Boundary cycles of mutually inverse oriented faces are mutually inverse; they are also called *boundary cycles* of the corresponding non-oriented face.

Consider now a combinatorial surface Φ. Consider a function θ that for every face f of Φ chooses an orientation of f. Call a face with a chosen orientation a *chosen* *oriented face*. For every oriented edge e of Φ, count the number of all such ordered pairs (f, x) that $f = (F, \beta)$ is a face of Φ, x is an oriented edge of F, x is chosen with respect to the orientation $\theta(f)$ of F, and $e = \hat{\beta}^1(x)$ (in particular, f is incident to e). The function θ is said to choose *coherent orientations* if for every oriented edge of Φ the number defined above is either 1 or 0. An *orientation* of a combinatorial surface is a choice of coherent orientations of all of its faces. A combinatorial surface that admits orientation is called *orientable*. There exist exactly two orientations of any connected orientable combinatorial surface. The *contour cycle* of a (non-oriented) face f in an oriented combinatorial surface, or in any 2-complex with chosen orientations of faces, is the boundary cycle of the chosen orinted face associated with f. An *isomorphism* of oriented combinatorial surfaces A and B is an isomorphism of A and B as 2-complexes which additionally preserves the chosen orientations of the faces.

It is convenient to have defined the following two operations on combinatorial complexes:

(1) *removing a face*—this operation is self-explanatory;

(2) *removing an arc* not incident to any face: if an arc u is not incident to any face in a complex C, then *to remove* u *from* C shall mean to remove all edges and all *intermediate* vertices of u from C;

1.b Maps

Definition. A *nontrivial map* Δ consists of:

(1) a finite connected combinatorial complex C;

(2) a function that orients all faces of C (i.e., for each face it chooses one of its orientations);

(3) a function that for every face f of C chooses a representative of the contour cycle of f, called the *contour* of f; and

(4) a (possibly empty) indexed system of cyclic paths in C, which are called the *contours* of Δ.

It is required that a 2-complex obtained from C by attaching one new face along each of the contours of Δ is a closed orientable combinatorial surface. It is further required that the attached faces may be oriented so that the orientations of all faces are coherent, and the contours of Δ represent the contour cycles of the corresponding attached faces.

The cycle represented by a contour of a diagram Δ is called a *contour cycle* of Δ. The contour of a face f shall be denoted by ∂f, and the contours of a map Δ shall be denoted by $\partial_1\Delta$, $\partial_2\Delta$, $\partial_3\Delta$, et cetera. Following is the notation for contour cycles of faces and maps: $\bar{\partial} f = \langle \partial f \rangle$ and $\bar{\partial}_i\Delta = \langle \partial_i\Delta \rangle$.

Definition. A map with empty system of contours is called *closed*. If Δ in a nontrivial map, then a closed map obtained from it by attaching new faces along its contours, assigning the contours of Δ to be the contours of the new faces, and appropriately choosing orientations of the new faces, is called a *closure* of Δ.

Definition. A *trivial map* is a combinatorial complex consisting of a single vertex together with the trivial cyclic path in it called its *contour*.

Remark 1.1. It is not possible to define a closure of a trivial map similarly to a closure of a nontrivial map since no face in a combinatorial complex can have trivial boundary cycle.

Definition. A map is *simple* if all its contours are simple paths, and distinct contours do not have common vertices. A map is *semi-simple* if every edge in it is incident to a face. A map is *degenerate* if it has no faces.

Definition. If f is a face of a map Δ, then a *submap of* Δ *obtained by removing the face* f is a (not uniquely determined) map Ψ such that:

(1) the underlying combinatorial complex of Ψ is obtained from the underlying combinatorial complex of Δ by removing the face f,

(2) the chosen orientations of faces of Ψ are those inherited from Δ,

(3) the contours of faces of Ψ are those inherited from Δ, and

(4) the system of the contours of Ψ consists of (in an arbitrary order) all the contours of Δ together with a new one obtained from ∂f or $(\partial f)^{-1}$ by an arbitrary cyclic shift.

If u is an arc of Δ not incident to any face, then a *submap of* Δ *obtained by removing the arc* u is a (not uniquely determined) map Ψ such that:

(1) the underlying combinatorial complex of Ψ is obtained from the underlying combinatorial complex of Δ by removing the arc u and, if the obtained complex is not connected, by picking one of the connected components;

(2) the chosen orientations of faces of Ψ are those inherited from Δ;

(3) the contours of faces of Ψ are those inherited from Δ;

(4) the system of the contours of Ψ consists of (in an arbitrary order) all the contours of Δ that are paths in Ψ together with a new one obtained as follows:

   (a) if the subcomplex obtained by removing u is connected, then, first, take path p_1 and p_2, an oriented arc v, and indices i and j such that:

      (i) v and v^{-1} are the oriented arcs associated with u,

      (ii) $\langle vp_1\rangle = (\bar{\partial}_i\Delta)^{\pm 1}$,

      (iii) $\langle vp_2\rangle = (\bar{\partial}_j\Delta)^{\pm 1}$, and

(iv) $i \neq j$,

and second, take an arbitrary cyclic shift of $p_1 p_2^{-1}$ or of $p_2 p_1^{-1}$ as a (new) contour of Ψ;

(b) if the subcomplex obtained by removing u is not connected, then, first, take path p_1 and p_2, and an oriented arc v such that:

(i) v and v^{-1} are the oriented arcs associated with u,

(ii) either $\langle v p_1 v^{-1} p_2 \rangle$ or $\langle p_2^{-1} v p_1^{-1} v^{-1} \rangle$ is a contour cycle of Δ, and

(iii) p_1 is a path in Ψ,

and second, take an arbitrary cyclic shift of $p_1^{\pm 1}$ as a (new) contour of Ψ.

A map Ψ is called a *submap* of a map Δ if it can be obtained from Δ by an arbitrary sequence of operations of removing a face or removing an arc that is not incident to any face.

The next definition is equivalent to the previous one. It is given for convenience.

Definition. A map Ψ is a *submap* of a map Δ if all of the following conditions hold:

(1) the underlying combinatorial complex of Ψ is a subcomplex of the underlying combinatorial complex of Δ;

(2) the chosen orientations of faces of Ψ are those inherited from Δ;

(3) the contours of faces of Ψ are those inherited from Δ;

(4) for every contour q of Ψ that is not a contour of Δ, there is $n \in \mathbb{N}$ and there are paths $p_0, \ldots, p_n$ in Ψ and oriented arcs $v_1, \ldots, v_n$ in Δ such that:

(a) $p_0 p_1 \ldots p_n = q$,

(b) none of the edges or intermediate vertices of any of the oriented arcs v_1, $\ldots$, v_n is in Ψ,

(c) for every $i = 1, \ldots, n-1$, at least one of the paths $(v_i p_i v_{i+1}^{-1})^{\pm 1}$ is a subpath either of some contour cycle of Δ, or of the contour cycle of some face of Δ that is not in Ψ, and

(d) at least one of the paths $(v_n p_n p_0 v_1^{-1})^{\pm 1}$ is a subpath either of some contour cycle of Δ, or of the contour cycle of some face of Δ that is not in Ψ.

Note that every connected subcomplex of the underlying complex of any map Δ has a structure of a submap of Δ, which is unique up to permutation of contours and replacing some of the contours with their cyclic shifts or cyclic shifts of their inverses. Note also that a proper submap of any map cannot be closed.

Definition. A *disc map* is either a trivial map, or a map with exactly one contour whose closure is a sphere. An *annular map* is any map with exactly two contours whose closure is a sphere.

Definition. An *exceptional map* is a spherical map, the 1-skeleton of whose underlying 2-complex is a combinatorial circle (such a map must consist of two faces attached to each other along their boundary cycles).

In a non-exceptional spherical map, no two distinct maximal arcs can overlap.

The contours of faces in a disc map may be thought of as *oriented counterclockwise*, and the contour of the map itself may be thought of as *oriented clockwise* (this convention corresponds to the way a disc map is usually pictured).

An oriented arc u of a map Φ is *incident* to a face f of Φ if it is a subpath of one of the boundary cycles of f. An arc is *incident* to f if one/both of the corresponding oriented arcs are incident to f. An arc which is incident to some face is either *internal* (does not lie on any of the contour cycles of the map) or *external* (lies on some contour cycle of the map). Internal oriented arcs divide into *inter-facial* (incident to two different faces) and *intra-facial* (incident to only one face). An internal arc u is *between* faces f_1 and f_2 if $\{f_1, f_2\}$ is the set of all faces incident to u. Every edge may be considered as an arc. Therefore, it makes sense to say that an edge incident to some face is internal, external, inter-facial, or intra-facial. An oriented arc or an oriented edge is internal, external, inter-facial or intra-facial if the corresponding non-oriented arc or edge is such. Every intra-facial oriented arc u in a disc map is either *outward* (u is an initial subpath of some simple path with the terminal vertex on the contour of the map), or *inward* (u^{-1} is outward).

Definition. Let Δ be a map, c be a nontrivial cyclic path in Δ, Δ' be a simple disc map. Say that c *cuts* Δ' out of Δ if there exists a morphism ζ of the underlying 2-complex of Δ' to the underlying 2-complex of Δ preserving the chosen orientations of the faces such that $\bar{\zeta}^2\colon \Delta'(2) \to \Delta(2)$ is injective and some ciclic shift of c is the ζ-image of the contour of Δ'. Call such a morphism ζ a *pasting morphism*.

Remark 1.2. If a cyclic path c cuts a simple disc map Δ' out of a map Δ, then Δ' is essentially determined by Δ and c.

A simple disc map that is cut out of a given map Δ is commonly called a "submap" of Δ, but it is not compatible with the definition of a submap in this paper.

If Δ_0 is a simple disc submap of a map Δ, then the contour of Δ_0 cuts Δ_0 out of Δ, and the natural morphism of Δ_0 to Δ is the corresponding pasting morphism. In a disc map, any simple cyclic path oriented clockwise is the contour of some simple disc submap; therefore, it cuts out that simple disc submap.

Let Δ be a combinatorial surface or a subcomplex of a combinatorial surface. Call two distinct faces of Δ *contiguous* if there exist an edge incident with both of them. (The notion of contiguity will only be used for distinct faces.) Let F be a non-empty set of faces of Δ. Let S be the set of all (non-ordered) pairs of distinct contiguous faces in the set F. Clearly, there exist a graph Γ and bijections

$$\alpha_0\colon F \to \Gamma(0), \quad \alpha_1\colon S \to \Gamma(1),$$

such that a face $f \in F$ belongs to a pair $P \in S$ if and only if the vertex $\alpha_0(f)$ is incident to the edge $\alpha_1(P)$. Indeed, choose vertices of Γ in bijective correspondence with faces of the set F, and connect any two vertices that correspond to distinct contiguous faces with an edge.

Definition. Such a graph Γ is called a *contiguity graph* for the set F in Δ.

Remark 1.3. Every contiguity graph is unique up to graph isomorphism. It cannot have loops or multiple edges.

If Δ is a spherical map, then the contiguity graph for any set of faces of Δ is planar.

1.c S-maps

Definition. A *selection* on a face f of a map Δ is a set of nontrivial reduced subpaths of $\bar{\partial} f$ such that for each path in this set, all of its nontrivial subpaths belong to the set as well (i.e., the set is closed under taking nontrivial subpaths). A *selection* on a map Δ is a set of nontrivial reduced subpaths of the contour cycles of faces of Δ which is closed under taking nontrivial subpaths. An *S-map*, or *map with selection*, is a map together with a selection on it.

A path in an S-map is *selected* if it belongs to the selection. An oriented edge in an S-map is *selected* if it is selected as a path. A path or an oriented edge is *double-selected* if it is selected along with its inverse. An edge or an arc is *double-selected* if it is internal and both of the corresponding oriented edges or arcs are selected. An external edge is *selected* if one of the associated oriented edges is selected.

If ζ is a morphism of the underlying 2-complex of a map Δ_1 to the underlying 2-complex of a map Δ_2, and ζ preserves the chosen orientations of the faces, then any selection on Δ_2 naturally induces a selection on Δ_1 via ζ: a path p in Δ_1 is selected if and only if its ζ-image in Δ_2 is selected (note that if the ζ-image of a path is reduced, then the path itself is reduced). Say that a cycle c *cuts* a simple disc S-map Δ' out of an S-map Δ if c cuts the underlying map of Δ' out of the underlying map of Δ, and the selection on Δ' is the one induced from Δ via some pasting morphism. Only those pasting morphisms of the underlying map of Δ' that induce the given selection on Δ' are called *pasting morphisms* of the S-map Δ' to Δ.

Any submap of an S-map has a naturally induced selection. Say that an S-map Δ_0 is an *S-submap* of an S-map Δ if Δ_0 is a submap of Δ together with the induced selection.

1.d Group presentations and van Kampen diagrams

Recall that an (abstract) *group presentation* is an ordered pair $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ where $\mathfrak{A}$ is an arbitrary set, called *alphabet*, and $\mathcal{R}$ is a set of words in the *group alphabet* $\mathfrak{A}^{\pm 1}$. The elements of $\mathcal{R}$ are called *defining words*. The same group presentation may be also written as $\langle \mathfrak{A} \,\|\, R = 1,\ R \in \mathcal{R} \rangle$, here the relations $R = 1$, $R \in \mathcal{R}$, are called *defining relations*. It will be assumed in the rest of this paper that $\mathcal{R}$ does not contain the empty word.

A *group word* in the alphabet $\mathfrak{A}$ is a word in the alphabet $\mathfrak{A}^{\pm 1}$. The *inverse* word to a group word W is denoted by W^{-1}. The mth *power* of a group word W for an integer m is denoted by W^m and defined as follows. If m is a positive integer, then W^m is the result of concatenation of m copies of W. If m is a negative integer, then W^m is the result of concatenation of $-m$ copies of W^{-1}. By definition, W^0 is the empty word. If A and B are two group words, then define $A^B = BAB^{-1}$. A group word is called *reduced* if it has no subwords of the form xx^{-1} where $x \in \mathfrak{A}^{\pm 1}$. The reduced word obtained from a group word W by cancelling one-by-one all subwords of the form xx^{-1}, $x \in \mathfrak{A}^{\pm 1}$, is called the *reduced form* of W (it is well-defined). A group word is *cyclically reduced* if it is reduced and its first letter is not inverse to its last one. The empty word is cyclically reduced by definition. Two word are *freely equal* if their reduced forms coincide. A set of words (or relations) is called *symmetrized* if along with every word W it contains all cyclic shifts of W and W^{-1}.

Sometimes it is convenient to distinguish between a letter x of the alphabet $\mathfrak{A}$ and the corresponding letter of the group alphabet $\mathfrak{A}^{\pm 1}$ denoted by x^{+1} or simply by x. Call letters of the alphabet $\mathfrak{A}$ *basic letters* and letters of the group alphabet $\mathfrak{A}^{\pm 1}$ *group letters*. Thus, a basic letter x_1, group letter x_1, and one-letter group word x_1 are three different things.

Every presentation $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ defines a group G, and there is a natural function from the set of all group words in the alphabet $\mathfrak{A}$ onto this group. The image of a group word W under this function shall be denoted $[W]_G$, or $[W]_{\mathcal{R}}$, or simply $[W]$. Two group words W_1 and W_2 are said to be *equal in the group* G if $[W_1]_G = [W_2]_G$, or, equivalently, if the relation $W_1 = W_2$ is a consequence of the relations $R = 1$, $R \in \mathcal{R}$.

Definition. If $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ is a group presentation, then a *van Kampen diagram*, or simply *diagram*, over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ is a map together with a labelling of its oriented edges such that every pair of mutually inverse oriented edges are labelled with mutually inverse group letters from $\mathfrak{A}^{\pm 1}$, and the group word that "reads" on the contour of each face (in the direction from the initial to the terminal vertex) belongs to $\mathcal{R}^{\pm 1}$.

If p is a path in a diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$, then let the *label* of p be the group word (over $\mathfrak{A}^{\pm 1}$) that reads on this path. In any van Kampen diagram, let the label of an oriented edge e be denoted by $\ell(e)$, and the label of a path p be denoted by $\ell(p)$. If e is a non-oriented edge in a diagram, then let the *label* of e, denoted also by $\ell(e)$, be the basic letter x such that the group letters x and x^{-1} label the oriented edges associated with e.

Definition. A pair of distinct faces $\{f_1, f_2\}$ in a diagram Δ is called *strongly cancellable* if there are paths p_1 and p_2 in Δ such that

(1) either $\langle p_1 \rangle = \bar{\partial} f_1$, or $\langle p_1^{-1} \rangle = \bar{\partial} f_1$,

(2) either $\langle p_2 \rangle = \bar{\partial} f_2$, or $\langle p_2^{-1} \rangle = \bar{\partial} f_2$,

(3) p_1 and p_2 have a common nontrivial initial subpath, and

(4) $\ell(p_1) = \ell(p_2)$.

A diagram Δ is called *weakly reduced* if it does not have strongly cancellable pairs of faces.

According to van Kampen's Lemma (see [Ol'91]), a relation $W = 1$ is a consequence of a system of relations $\{\, R = 1 \mid R \in \mathcal{R} \,\}$ if and only if there exists a disc diagram Δ (which can be picked weakly reduced) such that the label of $\partial_1 \Delta$ is W, and for every face f of Δ, the label of ∂f either is an element of $\mathcal{R}$, or is inverse to some element of $\mathcal{R}$. Such a diagram is called a *deduction diagram* for the word W or relation $W = 1$.

Definition. A symmetrized set S of group words in a given alphabet satisfies the *small cancellation condition* $C'(\lambda)$, $\lambda > 0$, if every element of S is a non-empty reduced word, and for any two distinct words $W_1, W_2 \in S$, the length of any common prefix of W_1 and W_2 is less than $\lambda \min\{|W_1|, |W_2|\}$. A symmetrized group presentation $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ is said to satisfy the condition $C'(\lambda)$, $\lambda > 0$, if the set of defining words $\mathcal{R}$ satisfies $C'(\lambda)$.

This and other classical small cancellation conditions and their applications may be found in [LS01].

2 Auxilliary diagrammatic conditions

2.a Condition $\mathscr{Z}$

Definition. An S-map Δ is said to satisfy the *condition* $\mathscr{Z}(n)$, $n \in \mathbb{N} \cup \{0\}$, relative to its simple disc submap Φ if for every set S of selected paths that covers $\bar{\partial}_1\Phi$, the set S has more than n maximal elements; in particular, S has at least $n+1$ element.

Proposition 2.1. *Let Φ be a non-degenerate disc map. Suppose a set S consists of n reduced paths and covers $\bar{\partial}_1\Phi$. Then there exists a maximal simple disc submap Φ_1 of Φ whose contour is a product of n or fewer subpaths of the paths comprising S.*

Proof. Let c be the result of a cyclic reduction of the contour of Φ. Since Φ is a non-degenerate disc map, the cyclic path c is the contour of some non-degenerate disc map Φ'; in particular, c is nontrivial. Let $q_1, \ldots, q_{n'}$ be such paths that c is a cyclic shift of the product $q_1 \cdots q_{n'}$, each q_i is a nontrivial subpath of some element of S, and each element of S has at most one of the paths $q_1, \ldots, q_{n'}$ as a subpath. It is not hard to prove that such paths $q_1, \ldots, q_{n'}$ exist. Note that $n' \le n$. Without loss of generality, assume that $c = q_1 \cdots q_{n'}$.

If Φ' is simple, then $\Phi_1 = \Phi'$ is a desired submap. Consider the case when Φ' is not simple. Since the map Φ' is non-degenerate, it has two maximal simple disc submaps Φ_1 and Φ_2 with disjoint sets of faces, whose contours are subpaths of $\bar{\partial}_1\Phi' = \langle c \rangle$. Let v_i be the initial vertex of q_i, $i = 1, \ldots, n'$. Let w_i be the initial (and the terminal) vertex of $\partial_1\Phi_i$, $i = 1, 2$. Let k_i be the number of vertices from the set $\{v_1, \ldots, v_{n'}, w_1, w_2\}$ which occur in $\partial_1\Phi_i$, $i = 1, 2$. It is easy to see that either $k_1 \le n'$ or $k_2 \le n'$. For the sake of definiteness, assume that $k_1 \le n'$. Then $\partial_1\Phi_1 = q_1^{(1)} \cdots q_{k_1}^{(1)}$ where every $q_i^{(1)}$ is a nontrivial subpath of some q_j. ☐

Corollary 2.1.a. *Let Δ be an S-map. Suppose the contour cycle of some non-degenerate disc submap Ψ of Δ is covered by a set of n or fewer selected paths. Then*

Δ *does not satisfy the condition* $\mathcal{Z}(n)$ *relative to some maximal simple disc submap of* Ψ.

2.b Conditions $\mathcal{A}$ and $\mathcal{B}$

Roughly speaking, the condition $\mathcal{A}$ defined in this section is a generalization of the small cancellation condition C' formulated in terms of the underlying maps of reduced van Kampen diagrams, rather than in terms of defining relations. (About the condition C' and van Kampen diagrams, see subsection 1.d.)

Definition. Let Δ be an S-map. Let $\bar{a} = (k; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where

$$k\colon \Delta(2) \to \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4\colon \Delta(2) \to [0, 1].$$

The S-map Δ is said to satisfy the *condition* $\mathcal{A}(\bar{a})$ if it satisfies the following five conditions:

$\mathcal{A}_1(k)$ For each face f of Δ, there exists at least one selected subpath of $\bar{\partial} f$, and the number of maximal selected subpaths of $\bar{\partial} f$ does not exceed $k(f)$ (note that if all nontrivial subpaths are selected, then there is no maximal selected subpath).

$\mathcal{A}_2(\lambda_1)$ For each face f of Δ, if S is the number of non-selected oriented edges in ∂f, then

$$S \leq \lambda_1(f)|\partial f|.$$

$\mathcal{A}_3(\lambda_2)$ For each face f of Δ, if u is a double-selected intra-facial arc incident to f, then

$$|u| \leq \lambda_2(f)|\partial f|.$$

$\mathcal{A}_4(\lambda_2, \lambda_3)$ For every two distinct faces f_1, f_2 of Δ, if U is a non-overlapping set of double-selected arcs between f_1 and f_2 which covers the set of all double-selected edges between f_1 and f_2, then

$$\sum_{u \in U} |u| \leq \|U\| \cdot \min\Big\{\lambda_2(f_1)|\partial f_1|, \lambda_2(f_2)|\partial f_2|\Big\} + \min\Big\{\lambda_3(f_1)|\partial f_1|, \lambda_3(f_2)|\partial f_2|\Big\}.$$

$\mathcal{A}_5(\lambda_2, \lambda_4)$ For each face f of Δ, if p is a simple selected subpath of the contour cycle of a distinct face of Δ, U is a non-overlapping set of double-selected arcs incident to f and lying on p, and U covers the set of all double-selected edges that are incident to f and lie on p, then

$$\sum_{u \in U} |u| \leq \Big(\|U\|\lambda_2(f) + \lambda_4(f)\Big)|\partial f|.$$

In particular, if a double-selected inter-facial arc u is incident to a face f, then

$$|u| \leq \Big(\lambda_2(f) + \lambda_4(f)\Big)|\partial f|.$$

A map Δ is said to satisfy the condition $\mathcal{A}(\bar{a})$ if there exists a selection on Δ such that Δ with this selection satisfies $\mathcal{A}(\bar{a})$.

Remark 2.1. If for each face f of Δ, the length of every double-selected arc incident to f is at most $\lambda_2(f)|\partial f|$, then Δ automatically satisfies the conditions $\mathcal{A}_3(\lambda_2)$, $\mathcal{A}_4(\lambda_2, \lambda_3)$, $\mathcal{A}_5(\lambda_2, \lambda_4)$.

If Δ_1 and Δ_2 are two maps, ζ is a morphism from Δ_1 to Δ_2 which preserves the chosen orientations of the faces, and

$$k\colon \Delta_2(2) \to \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4\colon \Delta_2(2) \to [0, 1],$$
$$\bar{a} = (k; \lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

then define

$$k.\zeta = k \circ \bar{\zeta}^2, \quad \lambda_1.\zeta = \lambda_1 \circ \bar{\zeta}^2, \quad \lambda_2.\zeta = \lambda_2 \circ \bar{\zeta}^2, \quad \lambda_3.\zeta = \lambda_3 \circ \bar{\zeta}^2, \quad \lambda_4.\zeta = \lambda_4 \circ \bar{\zeta}^2,$$

$$\bar{a}.\zeta = (k.\zeta; \lambda_1.\zeta, \lambda_2.\zeta, \lambda_3.\zeta, \lambda_4.\zeta),$$

where $\bar{\zeta}^2$ is the function $\Delta_1(2) \to \Delta_2(2)$ associated with ζ.

Remark 2.2. If an S-map Δ satisfies the condition $\mathcal{A}(\bar{a})$, a simple disc S-map Δ' is cut out of Δ, and ζ is a pasting morphism, then the S-map Δ' satisfies the condition $\mathcal{A}(\bar{a}.\zeta)$. A similar statement is true for each of the conditions $\mathcal{A}_1$–$\mathcal{A}_5$ separately.

Remark 2.3. A symmetrized group presentation $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ satisfies the condition $C'(\lambda)$, $\lambda > 0$, if and only if for every reduced simple disc diagram Δ over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$, there exists $\lambda_2 < \lambda$ such that Δ satisfies the condition $\mathcal{A}(0; 0, \lambda_2, 0, 0)$ (all the constants here are regarded as constant functions on $\Delta(2)$).

Definition. Let Δ be an S-map. Let $\lambda_1, \lambda_2 \in [0, 1]$. The S-map Δ is said to satisfy the *condition* $\mathcal{B}(\lambda_1, \lambda_2)$ if it satisfies the following three conditions:

$\mathcal{B}_0$ For each face f of Δ, the contour cycle of f has at least one selected subpath and at most one maximal selected subpath (note that if all nontrivial subpaths are selected, then there is no maximal selected subpath).

$\mathcal{B}_1(\lambda_1)$ For each face f of Δ, there is a selected subpath of ∂f of length at least $(1 - \lambda_1)|\partial f|$.

$\mathcal{B}_2(\lambda_2)$ For each face f of Δ, the length of every double-selected arc incident to f is at most $\lambda_2|\partial f|$.

A map Δ is said to satisfy the condition $\mathcal{B}(\lambda_1, \lambda_2)$ if there exists a selection on Δ such that Δ with this selection satisfies $\mathcal{B}(\lambda_1, \lambda_2)$.

For all admissible values of λ_1 and λ_2, the condition $\mathcal{B}(\lambda_1, \lambda_2)$ is equivalent to the condition $\mathcal{A}(1; \lambda_1, \lambda_2, 0, 0)$.

3 Estimating lemmas

If X is a set, then $\mathscr{P}(X)$ shall denote the set of all subsets of X (the power set of X).

The following lemma first appeared in [Hal35]:

Lemma (Philip Hall, 1935)**.** *Let A and B be two finite sets. Let f be a function from A to $\mathscr{P}(B)$. Let a function $F\colon \mathscr{P}(A) \to \mathscr{P}(B)$ be defined by $F(X) = \bigcup_{x\in X} f(x)$. Then the following are equivalent*:

(I) *There exists an injection $h\colon A \to B$ such that for all $x \in A$, $h(x) \in f(x)$.*

(II) *For each subset X of A, $\|X\| \leq \|F(X)\|$.*

The proof of this fact given here seems to be more concise then the original one:

Proof. The implication (I)$\Rightarrow$(II) is obvious. To prove the converse implication, induct on $\|A\|$.

If $\|A\| = 0$, then $A = \emptyset$ and the conclusion of this lemma is obvious (take $h = \emptyset$).

Let n be a natural number. Suppose the implication (I)$\Leftarrow$(II) holds under the additional assumption that $\|A\| < n$. Now, assume that $\|A\| = n$ and suppose that (II) holds.

Case 1: there exists a proper non-empty subset A_1 of the set A such that

$$\|F(A_1)\| = \|A_1\|.$$

Then let $B_1 = F(A_1)$, $A_2 = A \setminus A_1$, $B_2 = B \setminus B_1$. Note that $\|A_1\| < n$ and $\|A_2\| < n$. Let $f_1 = f|_{A_1}$, $F_1 = F|_{A_1}$. Note that $f_1 : A_1 \to \mathscr{P}(B_1)$ and $F_1 : \mathscr{P}(A_1) \to \mathscr{P}(B_1)$. Let $f_2\colon A_2 \to \mathscr{P}(B_2)$, $F_2\colon \mathscr{P}(A_2) \to \mathscr{P}(B_2)$ be defined by

$$f_2(x) = f(x) \cap B_2, \qquad F_2(X) = F(X) \cap B_2.$$

Obviously,

$$\Big(\forall X \subset A_1\Big)\Big(\|X\| \le \|F_1(X)\|\Big).$$

It is also easy to see that

$$\Big(\forall X \subset A_2\Big)\Big(\|X\| \le \|F_2(X)\|\Big).$$

Therefore, by the inductive assumption, there exist injections $h_1 : A_1 \to B_1$ and $h_2 \colon A_2 \to B_2$ such that

$$\Big(\forall x \in A_1\Big)\Big(h_1(x) \in f_1(x)\Big) \quad \text{and} \quad \Big(\forall x \in A_2\Big)\Big(h_2(x) \in f_2(x)\Big).$$

Clearly, $h = h_1 \cup h_2$ is an injection $A \to B$ such that

$$\Big(\forall x \in A\Big)\Big(h(x) \in f(x)\Big).$$

Case 2: for each proper non-empty subset A_1 of the set A,

$$\|F(A_1)\| \ge \|A_1\| + 1.$$

Then take an arbitrary $x_0 \in A$ and an arbitrary $y_0 \in f(x_0)$. Let $A_1 = A \setminus \{x_0\}$, $B_1 = B \setminus \{y_0\}$. Note that $\|A_1\| = n - 1$. Let $f_1 \colon A_1 \to \mathscr{P}(B_1)$, $F_1 \colon \mathscr{P}(A_1) \to \mathscr{P}(B_1)$ be defined by

$$f_1(x) = f(x) \setminus \{y_0\}, \qquad F_1(X) = F(X) \setminus \{y_0\}.$$

It is obvious that

$$\Big(\forall X \subset A_1\Big)\Big(\|X\| \le \|F_1(X)\|\Big).$$

Therefore, by the inductive assumption, there exists an injection $h_1 \colon A_1 \to B_1$ such that

$$\Big(\forall x \in A_1\Big)\Big(h_1(x) \in f_1(x)\Big).$$

Let $h : A \to B$ be the extension of h_1 to the whole of A by putting $h(x_0) = y_0$. Clearly, h is an injection satisfying

$$\Big(\forall x \in A\Big)\Big(h(x) \in f(x)\Big).$$

Since either Case 1 or Case 2 must take place, the implication (I)$\Leftarrow$(II) holds when $\|A\| = n$. The inductive step is done. $\square$

Corollary. *Let A and B be two finite sets. Let f be a function from A to $\mathscr{P}(B)$ and let w be a function from B to $\mathbb{N} \cup \{0\}$. Let a function $F \colon \mathscr{P}(A) \to \mathscr{P}(B)$ be defined by $F(X) = \bigcup_{x \in X} f(x)$. Then the following are equivalent:*

(I) *There exists a function $h \colon A \to B$ such that:*

   (1) *for all $x \in A$, $h(x) \in f(x)$, and*

   (2) *for each $y \in B$, the full pre-image of y under h consists of at most $w(y)$ elements.*

(II) *For each subset X of A, $\|X\| \le \sum\limits_{y \in F(X)} w(y)$.*

Proof. The implication (I)$\Rightarrow$(II) is obvious. To prove the converse implication, consider the set B' and functions $f' \colon A \to \mathscr{P}(B')$ and $F' \colon \mathscr{P}(A) \to \mathscr{P}(B')$ defined by the formulae

$$B' = \Big\{\, (b, n) \;\Big|\; b \in B,\; n \in \mathbb{N} \cup \{0\},\; n < w(b) \,\Big\},$$
$$f'(x) = \Big\{\, (b, n) \;\Big|\; b \in f(x),\; n \in \mathbb{N} \cup \{0\},\; n < w(b) \,\Big\},$$
$$F'(X) = \bigcup \{\, f'(x) \mid x \in X \,\}.$$

Since

$$\Big(\forall X \subset A\Big)\Big(\|F'(X)\| = \sum_{y \in F(X)} w(y)\Big),$$

it follows from the previous lemma that there exists an injection $h'\colon A \to B'$ such that

$$\Big(\forall x \in A\Big)\Big(h'(x) \in f'(x)\Big).$$

Let $h = p_1 \circ h'$ where p_1 is the first projection from $B \times \Big(\mathbb{N} \cup \{0\}\Big)$ onto B. The function h is a desired one. $\square$

Lemma 3.1. *The maximal possible Euler characteristic of a closed connected combinatorial surface is* 2, *and among all closed connected surfaces, only spheres have Euler characteristic* 2. *The maximal possible Euler characteristic of a proper connected subcomplex of a combinatorial surface is* 1, *and every such complex either consists of a single vertex, or can be turned into a combinatorial sphere by attaching* 1 *face.*

Proof. This lemma follows from the classification of compact (or finite combinatorial) surfaces. $\square$

Estimating Lemma 1. *Let Δ be an S-map satisfying the condition $\mathcal{Z}(2)$ relative to every proper simple disc submap. Let c be the number of contours of Δ. Let U be a set of selected arcs of Δ such that no two distinct elements of U are subarcs of a same double-selected arc of Δ. Let k be a function $\Delta(2) \to \mathbb{N} \cup \{0\}$. Suppose Δ satisfies the condition $\mathcal{A}_1(k)$. Then either U is empty, or*

$$\|U\| \leq \sum_{f \in \Delta(2)} \Big(3 + 2k(f)\Big) - 3\chi_\Delta - c.$$

Furthermore, there exist a set L and a function $h\colon U \setminus L \to \Delta(2)$ such that:

(1) *either L is empty, or $\|L\| \leq -3\chi_\Delta - c$,*

(2) *each arc $u \in U \setminus L$ is incident to the face $h(u)$, and*

(3) *the full pre-image of each face $f \in \Delta(2)$ under h consists of at most $3 + 2k(f)$ arcs.*

Proof. For each $x \in U$, let $g(x)$ be the set of all faces of Δ incident to u. For each $X \subset U$, let $G(X) = \bigcup_{x \in X} g(x)$. Take an arbitrary non-empty subset E of U. It is to be proved that

$$\|E\| \leq \sum_{f \in G(E)} \Big(3 + 2k(f)\Big) - 3\chi_\Delta - c.$$

Let $\bar{\Delta}$ be a closure of Δ. Note that $\chi_{\bar{\Delta}} = \chi_\Delta + c$.

Denote $G(E)$ by F. Let V be the set of all connected components of the 2-complex obtained from $\bar{\Delta}$ by removing all the faces that belong to F, and all the edges and intermediate vertices of all the arcs that belong to E. Then

$$\chi_{\bar{\Delta}} = \sum_{\Psi \in V} \chi_\Psi - \|E\| + \|F\|.$$

Endow each element of V with a structure of a submap of $\bar{\Delta}$.

Let $\hat{E}$ be the set of all oriented arcs associated with arcs from E. For every element Ψ of V, let $d(\Psi)$ denote the number of elements of $\hat{E}$ whose terminal vertex is in Ψ. Then

$$\sum_{\Psi \in V} d(\Psi) = 2\|E\|.$$

Combining this and the previous equality, have

$$\begin{aligned} \|E\| &= 3 \sum_{\Psi \in V} \chi_\Psi - 2\|E\| + 3\|F\| - 3\chi_{\bar{\Delta}} \\ &= 3\|F\| + \sum_{\Psi \in V} \Big(3\chi_\Psi - d(\Psi)\Big) - 3\chi_{\bar{\Delta}}. \end{aligned}$$

By Lemma 3.1, each element of V has Euler characteristic at most 1, and if the Euler characteristic of $\Psi \in V$ is 1, then Ψ is a disc map. Let

$$V'_i = \{\, \Psi \in V \mid d(\Psi) = i \text{ and } \chi_\Psi = 1 \,\} \quad \text{for} \quad i = 0, 1, 2, \ldots.$$

Note that each V'_i contains only disc maps. Clearly, $V'_0 = \emptyset$. Therefore,

$$\|E\| \le 3\|F\| + 2\|V'_1\| + \|V'_2\| - 3\chi_{\bar{\Delta}}.$$

To complete the proof, essentially, it is only left to prove that

$$\|V'_1\| + \|V'_2\| \le \sum_{f \in F} k(f) + c,$$

and then to apply the corollary of Hall's Lemma.

Let W denote the set of those elements of V that contain the "improper" faces of $\bar{\Delta}$—the faces that are in $\bar{\Delta}(2) \setminus \Delta(2)$. Note that $\|W\| \le c$. For $i = 1, 2$, let $V''_i = V'_i \setminus W$.

For every face f of Δ, let $M(f)$ be the set of all maximal selected subpaths of $\bar{\partial} f$. According to the condition $\mathcal{A}_1(k)$, $\|M(f)\| \le k(f)$ for every $f \in \Delta(2)$.

Let $N = \bigcup_{f \in F} M(f)$. Define a function $s \colon N \to V$ by the following rule: $s(p)$ is the last element of V that the path p meets, i.e., such an element of V that some terminal subpath of p has a vertex in $s(p)$ and has no vertices in any other element of V. It is easy to see that s is well-defined. It is to be proved now that every element of $V''_1 \sqcup V''_2$ is in the range of s (is the image of some element of N under s).

Consider an arbitrary $\Psi \in V''_1 \sqcup V''_2$. Then Ψ is a disc submap of Δ. Suppose Ψ is not in the range of s.

Consider the case $d(\Psi) = 2$. Let u_1 and u_2 be those oriented arcs from $\hat{E}$ whose terminal vertices are in Ψ (there are exactly 2 such oriented arcs). Let f_i be the face (from F) whose contour cycle has u_i as a subpath, $i = 1, 2$. Then there exist paths q_1 and q_2 such that for $i = 1, 2$, the path $u_i q_i u_{3-i}^{-1}$ is a subpath of $\bar{\partial} f_i$. Let q_1 and q_2 be the shortest such paths. Note that $\langle q_1 q_2 \rangle = \bar{\partial}_1 \Psi$. Since Ψ is not in the range of s, both paths $u_1 q_1 u_2^{-1}$ and $u_2 q_2 u_1^{-1}$ are selected. Indeed, suppose that, for example, the path $u_1 q_1 u_2^{-1}$ is not selected. Then there exists a maximal selected subpath p of $\bar{\partial} f_1$ that contains u_1 as a subpath. The path p is an element of $M(f_1)$. Since p cannot contain $u_1 q_1 u_2^{-1}$ as a subpath, it follows that $s(p) = \Psi$, which contradicts to the above assumption. Hence, the paths $u_1 q_1 u_2^{-1}$ and $u_2 q_2 u_1^{-1}$ are selected; therefore, they are reduced.

Suppose the map Ψ is degenerate. Then the paths $u_1 q_1 u_2^{-1}$ and $u_2 q_2 u_1^{-1}$ are mutually inverse. They cannot be oriented arcs because that would contradict to the assumption that no two distinct elements of U are subarcs of a same double-selected arc. Therefore, $u_1 = u_2^{-1}$, $V = \{\Psi\}$, and $\Delta = \bar{\Delta}$ is an exceptional spherical map. Therefore N is the full pre-image of Ψ under s, which is empty. Therefore, all subpaths of $\bar{\partial} f_1$ and $\bar{\partial} f_2$ are selected. Hence, Δ does not satisfy the condition $\mathcal{Z}(2)$—not even $\mathcal{Z}(0)$—relative to any of its simple disc submaps (the submaps obtained from Δ by removing either the face f_1 or f_2). This gives a contradiction.

Suppose the map Ψ is non-degenerate. Let S be the set of all nontrivial paths in the set $\{q_1, q_2\}$. The set S consists of selected paths and covers $\bar{\partial}_1 \Psi$. Hence, by the corollary of Proposition 2.1, Δ does not satisfy the condition $\mathcal{Z}(2)$ relative to some maximal simple disc submap of Ψ. This gives a contradiction.

Consider the case $d(\Psi) = 1$. Let u be the oriented arcs from $\hat{E}$ whose terminal vertex is in Ψ (there is exactly 1 such oriented arc). Let f be the face (from F) whose contour cycle has u as a subpath. Then there exists a path q such that uqu^{-1} is a subpath of $\bar{\partial} f$. Let q be the shortest such path. Note that $\langle q \rangle = \bar{\partial}_1 \Psi$.

Suppose the map Ψ is degenerate. Then the path uqu^{-1} is not reduced. Therefore, it is not selected.

Suppose the map Ψ is non-degenerate. Then q is nontrivial. Since Δ satisfies the condition $\mathcal{Z}(2)$ relative to every maximal simple disc submap of Ψ, it follows from the corollary of Proposition 2.1 that q is not selected.

Consider the maximal selected subpath of $\bar{\partial} f$ that contains u as a subpath. The image of this selected path under s is Ψ. This gives a contradiction.

It is proved that $V_1'' \sqcup V_2''$ is in the range of s. Therefore,

$$\|V_1'' \sqcup V_2''\| \leq \|N\| \leq \sum_{f \in F} k(f),$$

and

$$\begin{aligned}
\|E\| &\leq 3\|F\| + 2(\|V_1' \sqcup V_2'\|) - 3\chi_{\bar{\Delta}} \\
&\leq 3\|F\| + 2\Big(\sum_{f \in F} k(f) + c\Big) - 3\chi_\Delta - 3c = \sum_{f \in F}(3 + 2k(f)) - 3\chi_\Delta - c.
\end{aligned}$$

Thus, for an arbitrary non-empty subset E of U,

$$\begin{aligned}
\|E\| &\leq 3\|G(E)\| + \sum_{f \in G(E)} \Big(3 + 2k(f)\Big) - 3\chi_\Delta - c \\
&= \sum_{f \in G(E)} \Big(3 + 2k(f)\Big) - 3\chi_\Delta - c.
\end{aligned}$$

In particular, if $U \neq \emptyset$,

$$\|U\| \leq \sum_{f \in \Delta(2)} \Big(3 + 2k(f)\Big) - 3\chi_\Delta - c.$$

Let ω be anything which is not a face of Δ. Let $\tilde{h}$ be a function $U \to \Delta(2) \sqcup \{\omega\}$ such that:

(1) for each $u \in U$, either $\tilde{h}(u) = \omega$, or $\tilde{h}(u)$ is a face of Δ incident to the arc u,

(2) the full pre-image of each face f of Δ under $\tilde{h}$ consists of at most $3 + 2k(f)$ arcs, and

(3) the full pre-image of ω under $\tilde{h}$ consists of at most $\max\{0, -3\chi_\Delta - c\}$ arcs.

Such a function $\tilde{h}$ exists by the corollary of Hall's Lemma. Let L be the full pre-image of ω under $\tilde{h}$. Let h be the restriction of $\tilde{h}$ to $\Delta(2) \setminus L$. The set L and the function h are the desired ones. $\square$

Estimating Lemma 2. *Let Δ be a non-degenerate map. Let S be the set of all (non-ordered) pairs of distinct contiguous faces of Δ. Then $\|S\| < 3\|\Delta(2)\|$, and there exists a function $h\colon S \to \Delta(2)$ such that:*

(1) *each pair $P \in S$ contains the face $h(P)$, and*

(2) *the full pre-image of each face $f \in \Delta(2)$ under h consists of at most 3 pairs.*

Proof. Define functions $g\colon S \to \mathscr{P}(\Delta(2))$ and $G\colon \mathscr{P}(S) \to \mathscr{P}(\Delta(2))$ by the formulae:

$$g(P) = P, \qquad G(X) = \bigcup X.$$

Take an arbitrary non-empty subset X of S. Let $F = \bigcup X = G(X)$. The set F is non-empty. Let Γ be the contiguity graph for F in Δ. Graph Γ is planar, has no loops and no multiple edges. Obviously, $\|\Gamma(0)\| = \|F\|$ and $\|\Gamma(1)\| \geq \|X\|$. By Proposition 1.1, $\|X\| < 3\|F\|$.

Thus, for any non-empty subset X of S, $\|X\| < 3\|G(X)\|$. In particular,

$$\|S\| < 3\|\Delta(2)\|.$$

Now, the conclusion easily follows from the corollary of Hall's lemma. $\square$

4 Main Theorem

Definition. Let Δ be a semisimple S-map. Let n_1 be the number of edges of Δ, $n_1^{(e)}$ be the number of external edges of Δ, and S be the number of selected external edges of Δ. Let γ be a real number. The S-map Δ is said to satisfy the *condition* $\mathcal{X}(\gamma)$ if

$$S \geq n_1 - \gamma(2n_1 - n_1^{(e)}) = \|\Delta(1)\| - \gamma \sum_{f \in \Delta(2)} |\partial f|.$$

Remark 4.1. In the above notation, if $\gamma < 1$ and Δ satisfies the condition $\mathcal{X}(\gamma)$, then

$$S \geq n_1 - \gamma\Big(2n_1 - n_1^{(e)}\Big) \geq \max\left\{\Big(1 - \frac{\gamma}{1-\gamma}\Big)n_1,\ \Big(1 - 2\gamma\Big)\Big(2n_1 - n_1^{(e)}\Big)\right\}.$$

Indeed, since $n_1^{(e)} \geq S$ and $\gamma < 1$,

$$\begin{aligned}
n_1^{(e)} &\geq n_1 - \gamma\Big(2n_1 - n_1^{(e)}\Big), \\
n_1 &\geq 2n_1 - n_1^{(e)} - \gamma\Big(2n_1 - n_1^{(e)}\Big), \\
n_1 &\geq (1-\gamma)\Big(2n_1 - n_1^{(e)}\Big), \\
\frac{n_1}{1-\gamma} &\geq 2n_1 - n_1^{(e)}.
\end{aligned}$$

Therefore,

$$n_1 - \gamma\Big(2n_1 - n_1^{(e)}\Big) \geq (1-2\gamma)\Big(2n_1 - n_1^{(e)}\Big)$$

and

$$n_1 - \gamma\Big(2n_1 - n_1^{(e)}\Big) \geq \Big(1 - \frac{\gamma}{1-\gamma}\Big)n_1.$$

Remark 4.2. If a simple S-map Δ satisfies the condition $\mathcal{X}(\gamma)$ for some $\gamma < \frac{1}{2}$, then Δ is not spherical.

Main Theorem. *Let Δ be a semisimple S-map with at most 3 contours, whose closure is spherical. Let*

$$k\colon \Delta(2) \to \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \colon \Delta(2) \to [0,1],$$
$$\bar{a} = (k; \lambda_1, \lambda_2, \lambda_3, \lambda_4).$$

Let γ be a real number. Suppose Δ satisfies the condition $\mathcal{A}(\bar{a})$. Suppose

$$2 \cdot \max_{\Delta(2)}\Big(\lambda_1 + (3+2k)\lambda_2 + 3\lambda_3\Big) + \max_{\Delta(2)}\Big((2+k)\lambda_2 + 2\lambda_4\Big) < 1,$$
$$\max_{\Delta(2)}\Big(\lambda_1 + (3+2k)\lambda_2 + 3\lambda_3\Big) \le \gamma.$$

Then the S-map Δ satisfies the condition $\mathcal{X}(\gamma)$, and the condition $\mathcal{Z}(2)$ relative to every proper simple disc submap.

The main idea of the proof is to verify the conditions $\mathcal{X}(\gamma)$ and $\mathcal{Z}(2)$ by simultaneous induction on the number of internal edges of Δ.

Remark 4.3. In the case when all the functions k, λ_1, λ_2, λ_3, λ_4 are constant, the first inequality in the hypotheses is equivalent to

$$2\lambda_1 + (8+5k)\lambda_2 + 6\lambda_3 + 2\lambda_4 < 1.$$

Inductive Lemma 1. *Let Δ be an S-map. Let*

$$k\colon \Delta(2) \to \mathbb{N} \cup \{0\}, \quad \lambda_2, \lambda_4 \colon \Delta(2) \to [0,1].$$

Suppose Δ satisfies the conditions $\mathcal{A}_1(k)$ and $\mathcal{A}_5(\lambda_2, \lambda_4)$. Let γ be a real number such that the following inequality holds point-wise (face-wise):

$$2\gamma + (2+k)\lambda_2 + 2\lambda_4 < 1.$$

Suppose that every proper simple disc S-submap of Δ whose all edges are internal in Δ satisfies the condition $\mathcal{X}(\gamma)$. Then Δ satisfies the condition $\mathcal{Z}(2)$ relative to every proper simple disc submap.

Proof. Suppose that such Δ does not satisfy the condition $\mathcal{Z}(2)$ relative to some proper simple disc submap Δ_0. Let Δ_0 be such a submap with the minimal possible number of internal edges. View Δ_0 as an S-submap. Then in Δ there exists a set of selected paths with at most 2 maximal elements which covers $\bar{\partial}_1\Delta_0$. Let Q be a set consisting either of 1 selected cyclic path q_1 which is a representative of $\bar{\partial}_1\Delta_0$, or of 2 selected paths q_1, q_2 whose product q_1q_2 is a representative of $\partial_1\Delta_0$. Without loss of generality, in the first case assume that $\partial_1\Delta_0 = q_1$ and in the second assume $\partial_1\Delta_0 = q_1q_2$. Let $s = \|Q\|$. Let $c = \partial_1\Delta_0$.

The map Δ_0 is a proper submap of Δ. Indeed, the edges that lie on $\partial_1\Delta_0$ are external in Δ_0 but internal in Δ. Hence, by the assumptions, the S-map Δ_0 satisfies the condition $\mathcal{X}(\gamma)$.

Let f_0 be a face of Δ_0 which has at least $(1-2\gamma)|\partial f_0|$ selected external edges in Δ_0. Such a face exists, otherwise the number S of selected external edges of Δ_0 would satisfy the inequality

$$S < \sum_{f\in\Delta_0(2)} (1-2\gamma)|\partial f|,$$

in contradiction with the condition $\mathcal{X}(\gamma)$.

Let P be a non-overlapping set of arcs of Δ such that every element of P is a selected external arc of Δ_0 (i.e., external and selected in Δ_0) incident to f_0 and lying on a path from Q, and such that P covers the set of all selected external edges of Δ_0 incident to f_0; moreover, let P be such a set with the minimal possible number of elements. For $i = 1, s$, let P_i be the set of those elements of P that lie on q_i. Let $m = \|P\|$, let $m_i = \|P_i\|$, $i = 1, s$. For $i = 1, s$, enumerate all the subpaths of q_i^{-1}

which are oriented arcs corresponding to elements of P_i, according to the order in which they appear on q_i^{-1}: $p_1^{(i)}, \ldots, p_{m_i}^{(i)}$. Check now that for arbitrary i, j, there is no selected subpath of $\bar{\partial} f_0$ with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$.

Suppose on the contrary that for some $i \in \{1, s\}$ and some $j \in \{\, n \in \mathbb{N} \mid 1 \leq n \leq m_i - 1\,\}$, there is a selected subpath of $\bar{\partial} f_0$ with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$. Let $p_j^{(i)} x p_{j+1}^{(i)}$ be the shortest such selected subpath. Then the path x is either selected or trivial; in either case, it is reduced. Let y be the subpath of q_i that starts at the initial vertex of $p_{j+1}^{(i)}$ and ends at the terminal vertex of $p_j^{(i)}$. The path y is reduced, as a subpath of $\bar{\partial}_1 \Delta_0$. Clearly, the cyclic path xy is the contour of some disc submap Φ of the map Δ_0. Since the path $p_{j+1}^{(i)-1} y p_j^{(i)-1}$ is a subpath of q_i, and the path $p_j^{(i)} x p_{j+1}^{(i)}$ is selected, the path x cannot be inverse to the path y, otherwise it would contradict to the minimality of the number of elements of P. Therefore, the map Φ is non-degenerate.

By Proposition 2.1, there exists a maximal simple disc submap of Φ whose contour cycle may be covered by 1 or 2 paths each of which is a nontrivial subpath of either x or y. Let Φ_1 be such a submap of Φ. The map Φ_1 is also a submap of Δ. The contour cycle of Φ_1 is covered by a set of 1 or 2 selected paths. The map Φ_1 clearly has fewer internal edges than Δ_0. This contradicts to the choice of Δ_0—to the minimality of its number of internal edges.

Since for any i and j, no subpath of $\bar{\partial} f_0$ with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$ is selected, the number of maximal selected subpaths of $\bar{\partial} f_0$ is not less than $m_1 - 1 = m - 1$ in the case $s = 1$, and not less than $m_1 + m_2 - 2 = m - 2$ in the case $s = 2$. Hence, in both cases, $m \leq k(f_0) + 2$ by the condition $\mathcal{A}_1(k)$.

Let

$$S_i = \sum_{u \in P_i} |u| = \sum_{j=1}^{m_i} |p_j^{(i)}|, \quad i = 1, s, \qquad S = \sum_{u \in P} |u| = \sum_{i=1}^{s} S_i.$$

Note that S equals the number of selected external edges of f_0 in Δ_0. For each $i = 1, s$, the condition $\mathcal{A}_5(\lambda_2, \lambda_4)$ for Δ implies that

$$S_i \leq \Big(m_i\lambda_2(f_0) + \lambda_4(f_0)\Big)|\partial f_0|.$$

Therefore, on one hand,

$$S \leq \Big(m\lambda_2(f_0) + s\lambda_4(f_0)\Big)|\partial f_0| \leq \Big(\big(2 + k(f_0)\big)\lambda_2(f_0) + 2\lambda_4(f_0)\Big)|\partial f_0|,$$

on the other hand,

$$S \geq (1 - 2\gamma)|\partial f_0|.$$

Hence,

$$\big(2 + k(f_0)\big)\lambda_2(f_0) + 2\lambda_4(f_0) \geq 1 - 2\gamma,$$

and this contradicts to the inequality

$$2\gamma + (2 + k)\lambda_2 + 2\lambda_4 < 1. \quad \square$$

Inductive Lemma 2. *Let Δ be a semisimple S-map with at most 3 contours, whose closure is spherical. Let*

$$k\colon \Delta(2) \to \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3\colon \Delta(2) \to [0, 1].$$

Suppose Δ satisfies the condition $\mathcal{Z}(2)$ relative to every proper simple disc submap. Suppose Δ also satisfies the conditions $\mathcal{A}_1(k)$, $\mathcal{A}_2(\lambda_1)$, $\mathcal{A}_3(\lambda_2)$, $\mathcal{A}_4(\lambda_2, \lambda_3)$. Let γ be a real number such that the following double inequality holds point-wise (face-wise):

$$\lambda_1 + (3 + 2k)\lambda_2 + 3\lambda_3 \leq \gamma < 1.$$

Then Δ satisfies the condition $\mathcal{X}(\gamma)$.

Proof. Let n_1 be the number of edges of Δ, $n_1^{(e)}$ be the number of external edges of Δ, S be the number of selected external edges of Δ. Note that since Δ is simple,

$$\sum_{f\in\Delta(2)} |\partial f| = 2n_1 - n_1^{(e)}.$$

First, estimate the number of all edges that are neither selected external, nor double-selected internal. Denote this number by S'_1. Since S'_1 is less than or equal to the number of non-selected oriented edges of the contours of all the faces of Δ, and Δ satisfies the condition $\mathcal{A}_2(\lambda_1)$,

$$S'_1 \le \sum_{f\in\Delta(2)} \lambda_1(f)|\partial f|.$$

Second, estimate the number of double-selected internal edges. Denote this number by S'_2.

Let U be a minimal by the number of elements non-overlapping set of double-selected internal arcs of Δ covering the set of all double-selected edge of Δ. Let for any $f_1, f_2 \in \Delta(2)$, $B(\{f_1, f_2\})$ be the set of all arcs between f_1 and f_2 that are elements of U (if $f_1 = f_2$, then $B(\{f_1, f_2\}) = B(\{f_1\})$ is the set of all elements of U which are intra-facial arcs of f_1). Let M be the set of all such pairs or singletons X of faces of Δ that $B(X) \ne \emptyset$. Then

$$S'_2 = \sum_{u\in U} |u| = \sum_{X\in M} \sum_{u\in B(X)} |u|.$$

Let N be the set of all pairs of distinct contiguous faces of Δ that have at least one double-selected edge between them.

Let $h_1 \colon U \to \Delta(2)$ and $h_2 \colon N \to \Delta(2)$ be such functions that

(1) each arc $u \in U$ is incident to the face $h_1(u)$;

(2) the full pre-image of each face f of Δ under h_1 consists of at most $3+2k(f)$ arcs;

(3) each pair $P \in N$ contains the face $h_2(P)$;

(4) the full pre-image of each face f of Δ under h_2 consists of at most 3 pairs.

Such functions exist by Estimating Lemmas 1 and 2. (The conditions $\mathcal{A}_1(k)$ and $\mathcal{Z}(2)$ have been used here.)

Let K be the subset of $\Delta(2) \times \Delta(2)$ consisting of all ordered pairs (f_1, f_2) such that $\{f_1, f_2\} \in M$. Define functions $r_1 \colon K \to \mathbb{R}$ and $r_2 \colon K \to \mathbb{R}$ as follows. For any $(f_1, f_2) \in K$, let

$$\begin{aligned} r_1(f_1, f_2) &= \Big\| h_1^{-1}(f_1) \cap B(\{f_1, f_2\}) \Big\| \lambda_2(f_1) |\partial f_1|, \\ r_2(f_1, f_2) &= \Big\| h_2^{-1}(f_1) \cap \{\{f_1, f_2\}\} \Big\| \lambda_3(f_1) |\partial f_1|. \end{aligned}$$

Let $r = r_1 + r_2$.

Take arbitrary distinct faces f_1 and f_2 such that $\{f_1, f_2\} \in M$. For the sake of definiteness, assume that $h_2(\{f_1, f_2\}) = f_1$. Then

$$\begin{aligned} &r(f_1, f_2) + r(f_2, f_1) \\ &\quad = \Big(r_1(f_1, f_2) + r_1(f_2, f_1) \Big) + \Big(r_2(f_1, f_2) + r_2(f_2, f_1) \Big) \\ &\quad = \Big\| h_1^{-1}(f_1) \cap B(\{f_1, f_2\}) \Big\| \lambda_2(f_1) |\partial f_1| + \Big\| h_1^{-1}(f_2) \cap B(\{f_1, f_2\}) \Big\| \lambda_2(f_2) |\partial f_2| \\ &\qquad\quad + \lambda_3(f_1) |\partial f_1| \\ &\quad \geq \Big\| B(\{f_1, f_2\}) \Big\| \min\{\lambda_2(f_1)|\partial f_1|, \lambda_2(f_2)|\partial f_2|\} + \lambda_3(f_1)|\partial f_1|. \end{aligned}$$

By the condition $\mathcal{A}_4(\lambda_2, \lambda_3)$,

$$\begin{aligned}\sum_{u \in B(\{f_1, f_2\})} |u| &\leq \Big\| B(\{f_1, f_2\}) \Big\| \min\{\lambda_2(f_1)|\partial f_1|, \lambda_2(f_2)|\partial f_2|\} + \lambda_3(f_1)|\partial f_1| \\ &\leq r(f_1, f_2) + r(f_2, f_1).\end{aligned}$$

Now, take an arbitrary face f such that $\{f\} \in M$. Then

$$\begin{aligned} r(f, f) &= r_1(f, f) + r_2(f, f) \\ &= \Big\| h_1^{-1}(f) \cap B(\{f\}) \Big\| \lambda_2(f)|\partial f| + 0 \\ &= \Big\| B(\{f\}) \Big\| \lambda_2(f)|\partial f|. \end{aligned}$$

By the condition $\mathcal{A}_3(\lambda_2)$,

$$\sum_{u \in B(\{f\})} |u| \leq \Big\| B(\{f\}) \Big\| \lambda_2(f)|\partial f| = r(f, f).$$

Thus,

$$\begin{aligned} S_2' &= \sum_{u \in U} |u| = \sum_{X \in M} \sum_{u \in B(X)} |u| \\ &\leq \sum_{(f_1, f_2) \in K} r(f_1, f_2) = \sum_{f_1 \in \Delta(2)} \sum_{f_2 : (f_1, f_2) \in K} r(f_1, f_2) \\ &= \sum_{f_1 \in \Delta(2)} \left(\sum_{f_2 : (f_1, f_2) \in K} r_1(f_1, f_2) + \sum_{f_2 : (f_1, f_2) \in K} r_2(f_1, f_2) \right) \\ &= \sum_{f_1 \in \Delta(2)} \Big(\|h_1^{-1}(f_1)\| \lambda_2(f_1)|\partial f_1| + \|h_2^{-1}(f_1)\| \lambda_3(f_1)|\partial f_1| \Big) \\ &\leq \sum_{f_1 \in \Delta(2)} \Big(\big(3 + 2k(f_1)\big) \lambda_2(f_1)|\partial f_1| + 3\lambda_3(f_1)|\partial f_1| \Big) \\ &= \sum_{f \in \Delta(2)} \Big(\big(3 + 2k(f)\big) \lambda_2(f) + 3\lambda_3(f) \Big) |\partial f|. \end{aligned}$$

Eventually,

$$\begin{aligned} S &= n_1 - S_1' - S_2' \\ &\geq n_1 - \sum_{f \in \Delta(2)} \Big(\lambda_1(f) + \big(3 + 2k(f)\big)\lambda_2(f) + 3\lambda_3(f) \Big) |\partial f| \\ &\geq n_1 - \sum_{f \in \Delta(2)} \gamma |\partial f| = n_1 - \gamma\big(2n_1 - n_1^{(e)}\big), \end{aligned}$$

since $\lambda_1 + (3 + 2k)\lambda_2 + 3\lambda_3 \leq \gamma$. □

Proof of the main theorem. Suppose the theorem is not true. Then let Δ be a simple S-map, let

$$k \colon \Delta(2) \to \mathbb{N} \cup \{0\}, \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 \colon \Delta(2) \to [0, 1],$$
$$\bar{a} = (k; \lambda_1, \lambda_2, \lambda_3, \lambda_4),$$

and let γ be a real number such that they all together satisfy the hypotheses of the theorem and do not satisfy the conclusion (i.e., provide a counterexample), and such that the number of internal edges of Δ is the minimal possible under this condition (i.e., the theorem holds whenever the S-map has fewer internal edges). Without loss of generality, assume that

$$\max_{\Delta(2)} \Big(\lambda_1 + (3 + 2k)\lambda_2 + 3\lambda_3 \Big) = \gamma.$$

Then

$$2\gamma + \max_{\Delta(2)} \Big((2 + k)\lambda_2 + 2\lambda_4 \Big) < 1.$$

In particular, $\gamma < 1$.

The hypotheses of Inductive Lemma 1 hold for Δ, $\bar{a}$, and γ. Therefore, by Inductive Lemma 1, Δ satisfies the condition $\mathcal{Z}(2)$ relative to every proper simple disc submap. Therefore, by Inductive Lemma 2, Δ satisfies the condition $\mathcal{X}(\gamma)$. Hence, the theorem holds for Δ, $\bar{a}$, and γ, which gives a contradiction. $\square$

Corollary. *Let Δ be an S-map with at most 3 contours, whose closure is spherical. Let $\lambda_1, \lambda_2 \in [0,1]$. Suppose $2\lambda_1 + 13\lambda_2 < 1$ and Δ satisfies $\mathcal{B}(\lambda_1, \lambda_2)$. Then Δ satisfies $\mathcal{X}(\lambda_1 + 5\lambda_2)$.*

5 Lemma about exposed face

Lemma about exposed face. *Let Δ be a simple disc S-map. Let*

$$k\colon \Delta(2) \to \mathbb{N} \cup \{0\}, \quad \lambda_2, \lambda_4 \colon \Delta(2) \to [0,1], \quad \gamma \in \mathbb{R}.$$

Suppose Δ satisfies the condition $\mathcal{Z}(2)$ relative to every proper simple disc submap. Suppose Δ satisfies the conditions $\mathcal{A}_1(k)$ and $\mathcal{A}_5(\lambda_2, \lambda_4)$. Suppose every simple disc S-submap of Δ satisfies the condition $\mathcal{X}(\gamma)$. Then there exists a face f in Δ satisfying the following property: if P is a non-overlapping set of selected external arcs incident to f, if P covers the set of all selected external edges incident to f, and if the number of elements of P is the minimal possible under these assumptions, then $\|P\| \le k(f)+1$ and

$$\sum_{p\in P} |p| \ge \Big(1 - 2\gamma - \big(2 + k(f)\big)\lambda_2(f) - \lambda_4(f) + \|P\|\lambda_2(f)\Big)|\partial f|.$$

(Such a face f may be called "exposed").

Remark 5.1. It is possible to prove a stronger statement than the one claimed in this lemma. Namely, under the hypotheses of the lemma, either there exists a face f in Δ such that a corresponding set P has at most $k(f)+1$ elements, and the total sum of their lengths is at least $(1-2\gamma)|\partial f|$, or there exist at least two distinct "exposed" faces such as in the conclusion of the lemma.

Remark 5.2. If Δ, $\bar{a} = (k; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$, and γ satisfy the hypotheses of the main theorem, then it follows from the main theorem that Δ, k, λ_2, λ_4, γ satisfy the hypotheses of the lemma about exposed face.

Proof. Case I: there exists a simple disc submap Δ_0 of Δ whose contour is of the form $q_1 q_2$ where q_1 is a selected subpath of the contour cycle of some face f_0, and q_2 is a subpath of $\bar{\partial}_1 \Delta$. Let Δ_0 be a minimal such map (which do not contain any other such map as a submap). View Δ_0 as an S-submap. Let q_1, q_2 and f_0 be as above. It follows from the condition $\mathcal{Z}(2)$ that q_2 is nontrivial.

Pick a face f_1 of Δ_0 which has at least $(1-2\gamma)|\partial f_1|$ selected external edges in Δ_0. Such a face exists because Δ_0 satisfies the condition $\mathcal{X}(\gamma)$.

Let P_1 be a non-overlapping set of double-selected arcs between f_1 and f_0 such that P_1 covers the set of all double-selected edges between f_1 and f_0. Let P_2 be a non-overlapping set of selected external arcs incident to f_1 such that P_2 covers the set of all selected external edge incident to f_1. Moreover, let P_1 and P_2 be such sets with the minimal possible numbers of elements. Note that all the elements of P_1 lie on the path q_1 and all the element of P_2 lie on the path q_2. Let $m_1 = \|P_1\|$, $m_2 = \|P_2\|$. Let $P = P_1 \sqcup P_2$. All the elements of P are selected external arcs of Δ_0 incident to f_1. By the choice of f_1,

$$\sum_{p \in P} |p| \geq (1 - 2\gamma)|\partial f_1|.$$

The goal is to prove that $m_2 \leq k(f_1) + 1$ and

$$\sum_{p \in P_2} |p| \geq \Big(1 - 2\gamma - \big(2 + k(f_1)\big)\lambda_2(f_1) - \lambda_4(f_1) + m_2 \lambda_2(f_1)\Big)|\partial f_1|;$$

this will show that the face f_1 is a desired one.

For $i = 1$, 2, if $P_i \neq \emptyset$, then enumerate all the subpaths of q_i^{-1} which are oriented arcs corresponding to elements of P_i, according to the order in which they appear on

q_i^{-1}: $p_1^{(i)}, \ldots, p_{m_i}^{(i)}$. Check now that for arbitrary i, j, there is no selected subpath of $\bar{\partial} f_1$ with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$.

Suppose that for some $j \in \{\, n \in \mathbb{N} \mid 1 \le n < m_1 \,\}$, there is a selected subpath of $\bar{\partial} f_1$ with initial subpath $p_j^{(1)}$ and terminal subpath $p_{j+1}^{(1)}$. Let $p_j^{(1)} x p_{j+1}^{(1)}$ be the shortest such selected subpath. Then the path x is either selected or trivial; in either case, it is reduced. Let y be the subpath of q_1 that starts at the initial vertex of $p_{j+1}^{(1)}$ and ends at the terminal vertex of $p_j^{(1)}$. The path y is reduced, as a subpath of $\bar{\partial}_1 \Delta_0$. Note that $p_{j+1}^{(1)-1} y p_j^{(1)-1}$ is a subpath of q_1. The cyclic path xy is the contour of some disc submap Φ of Δ. The path x cannot be inverse to y, otherwise it would contradict to the minimality of the number of elements of P_1. Therefore, the map Φ is non-degenerate. The corollary of Proposition 2.1 gives a contradiction with the fact that Δ satisfies the condition $\mathscr{Z}(2)$ relative to every maximal simple disc submap of Φ.

Suppose that for some $j \in \{\, n \in \mathbb{N} \mid 1 \le n < m_2 \,\}$, there is a selected subpath of $\bar{\partial} f_1$ with initial subpath $p_j^{(2)}$ and terminal subpath $p_{j+1}^{(2)}$. Let $p_j^{(2)} x p_{j+1}^{(2)}$ be the shortest such selected subpath. Since the disc map Δ is simple and the number of elements of P_2 is minimal, the path x cannot be trivial. Therefore, the path x is selected and reduced. Let y be the subpath of q_2 that starts at the initial vertex of $p_{j+1}^{(2)}$ and ends at the terminal vertex of $p_j^{(2)}$. The path y is reduced, as a subpath of $\bar{\partial}_1 \Delta_0$. Note that $p_{j+1}^{(2)-1} y p_j^{(2)-1}$ is a subpath of q_2. The cyclic path xy is the contour of some disc submap Φ of Δ. The path x cannot be inverse to y, otherwise it would contradict to the minimality of the number of elements of P_2. Therefore, the map Φ is non-degenerate. By Proposition 2.1, there exists a simple disc submap Δ_1 of Φ whose contour cycle is covered by a set of 1 or 2 paths, each of which is either a subpath of x or a subpath of y. In any case, this leads to a contradiction either with the condition $\mathscr{Z}(2)$, or with the fact that Δ is simple, or with the choice of Δ_0 (its minimality).

Thus, for arbitrary i and j, there is no selected path with initial subpath $p_j^{(i)}$ and terminal subpath $p_{j+1}^{(i)}$. Hence, it follows from the condition $\mathcal{A}_1(k)$ that

$$m_1 \leq k(f_1) + 1, \quad m_2 \leq k(f_1) + 1, \quad m_1 + m_2 \leq k(f_1) + 2.$$

By the condition $\mathcal{A}_5(\lambda_2, \lambda_4)$,

$$\begin{aligned}\sum_{p \in P_1} |p| &\leq \Big(m_1\lambda_2(f_1) + \lambda_4(f_1)\Big)|\partial f_1| \\ &\leq \Big(\big(k(f_1) + 2 - m_2\big)\lambda_2(f_1) + \lambda_4(f_1)\Big)|\partial f_1|.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{p \in P_2} |p| &\geq \Big(1 - 2\gamma - \big(k(f_1) + 2 - m_2\big)\lambda_2(f_1) - \lambda_4(f_1)\Big)|\partial f_1| \\ &= \Big(1 - 2\gamma - \big(2 + k(f_1)\big)\lambda_2(f_1) - \lambda_4(f_1) + m_2\lambda_2(f_1)\Big)|\partial f_1|.\end{aligned}$$

Case II: there exists no simple disc submap Δ_0 of Δ whose contour would be of the form q_1q_2 where q_1 would be a selected subpath of the contour cycle of some face, and q_2 would be a subpath of $\bar{\partial}_1\Delta$.

Consider an arbitrary face f of Δ. Let P be a non-overlapping set of selected external arcs incident to f such that P covers the set of all selected external edges incident to f, and the number of elements of P is the minimal possible. Then $\|P\| \leq k(f) + 1$. Indeed, if P is empty, this inequality is obvious. Suppose P is not empty. For every selected subpath q of $\bar{\partial} f$, at most one element of P lies on q (because the number of elements of P is minimal, Δ is a simple disc map, Δ satisfies $\mathcal{Z}(2)$ relative to its simple disc submaps, and it is not Case I). If the set of all selected subpath of $\bar{\partial} f$ has no maximal element, then P consists of one element (because there exists a selected subpath q of $\bar{\partial} f$ such that all the elements of P lie on q), and

$\|P\| = 1 \leq k(f)+1$. If the set of all selected subpath of $\bar{\partial} f$ has at least one maximal element, then every element of P lies on some maximal selected subpath of $\bar{\partial} f$, and $\|P\| \leq k(f) < k(f)+1$ (by $\mathcal{A}_1(k)$).

Now, pick a face f_0 in Δ which has at least $(1-2\gamma)|\partial f_0|$ selected external edges. The face f_0 is a desired one. $\square$

CHAPTER II

BOUNDEDLY GENERATED GROUPS

6 Proof of Theorem 1

Proof. Let F_n be a free group of rank n, $n \in \mathbb{N}$. Let $\mathfrak{B} = \{x_1, \dots, x_n\}$ be a basis of F_n. Let λ be a positive number less than $1/13$. Let S be an infinite subset of F_n such that all the elements of S are cyclically reduced relative to $\mathfrak{B}$ and are not proper powers in F_n. Suppose that the symmetrization of S relative to $\mathfrak{B}$ satisfies the small cancellation condition $C'(\lambda)$ relative to $\mathfrak{B}$. The goal is to prove that there exists an infinite simple 2-generated group P and a homomorphism $\phi\colon F_n \to P$ such that ϕ maps S surjectively onto P.

Let S' be an infinite subset of S which satisfies the following three properties if the elements of F_n are regarded as reduced group words in the alphabet $\mathfrak{B}$:

(1) If $w_1 s v_1$ or $(w_1 s v_1)^{-1}$ belongs to S', and $w_2 s v_2$ or $(w_2 s v_2)^{-1}$ belongs to S', then either $w_1 = w_2$ and $v_1 = v_2$, or $|s| \le \lambda \min\{|w_1 s v_1|, |w_2 s v_2|\}$. (Here, the phrase "$w_1 s v_1$ belongs to S'" means that the concatenation of the group words w_1, s, v_1 is a reduced group word representing an element of S'.)

(2) If u_1 and u_2 are elements of S', then $u_1 \ne u_2^{-1}$.

(3) If u is an element of S', then $|u| \ge 5$.

Such a set S' may be obtained from S by first leaving out all the elements of length less than 5 and then picking one representative out of each of the equivalence classes of the following equivalence relation: call elements u_1 and u_2 equivalent if u_2 is a cyclic shift of u_1 or u_1^{-1}. Note that condition (1) implies that if $u \in S'$, then any common subword of u and u^{-1} is of length at most $\lambda|u|$ (though, it also follows from the condition $C'(\lambda)$ for S).

Let

$$\lambda_1 = \frac{5}{13} - 5\lambda, \quad \lambda_2 = \lambda, \quad \gamma = \frac{5}{13}.$$

Then $\lambda_1, \lambda_2 \in [0, 1]$ and

$$2\lambda_1 + 13\lambda_2 < 1,$$

$$\lambda_1 + 5\lambda_2 = \gamma.$$

Let F_{n+2} be a free group of rank $n+2$ with a basis $\mathfrak{A} = \{x_1, \dots, x_n, a, b\}$, containing F_n as a subgroup. Note that F_{n+2} is the inner free product of the subgroup F_n and the subgroup generated by $\{a, b\}$. In the rest of this proof, all the elements of F_{n+2} shall be regarded as reduced group words in the alphabet $\mathfrak{A}$. The product of two elements of F_{n+2} shall mean the concatenation of the group words; it may be not reduced. The usual group multiplication will not be used.

Split the set S' into the disjoint union of two infinite sets S_1 and S_2. The first step of "constructing" the group P consists in imposing three systems of relations on the group F_{n+2}. The first system of relations uses elements of S_1 and ensures that every element of the obtained quotient group G is represented by some element of S_1 (in particular, the quotient homomorphism maps F_n onto G); the second system uses elements of S_2 and ensures that the quotient group G has no nontrivial finite homomorphic images; the third system ensures that the elements of G represented by a and b generate the whole of G. The main theorem shall be used for proving that the quotient group G is not trivial. The second step of constructing P consists in taking the quotient of G over its maximal proper normal subgroup.

Let $v_{1,1}$, $v_{1,2}$, $v_{1,3}$, ... be a list of all reduced group words in the alphabet $\mathfrak{A}$. Using infiniteness of S_1 and finiteness of $\mathfrak{B}$, it is easy to show that there exists a system

$\{u_{1,i}\}_{i=1}^{+\infty}$ of pairwise distinct elements of S_1 such that $\lambda_1|u_{1,i}| \geq |v_{1,i}|$ for every $i \in \mathbb{N}$. For every $i \in \mathbb{N}$, let

$$r_{1,i} = u_{1,i}v_{1,i}^{-1}.$$

(Here, $r_{1,i}$ is the concatenation of $u_{1,i}$ and $v_{1,i}^{-1}$; it does not need to be reduced.) Let $\mathcal{R}_1 = \{\, r_{1,i} \mid i = 1, 2, \dots \}$.

Let a function from a set X to a group G be called *trivial* if it maps all the elements of X to the neutral element of G. Let M be a set of nontrivial finite groups such that every nontrivial finite group is isomorphic to exactly one group in M. Such a set M exists and is countable. Informally speaking, M is a set of up to isomorphism all nontrivial finite groups. Let T be the set of all ordered pairs (G, ψ) where $G \in M$, and ψ is a nontrivial function from $\mathfrak{A}$ to G. Clearly, T is countable. Let (G_1, ψ_1), (G_2, ψ_2), …be a list of all elements of T (without recurrences). Let $u_{2,1}$, $u_{2,2}$, $u_{2,3}$, …be a list (without recurrences) of elements of S_2 of length at least $1/\lambda_1$. For every $i \in \mathbb{N}$, let $v_{2,i}$ be a group word over $\mathfrak{A}$ of minimal length such that the values of $v_{2,i}$ and $u_{2,i}$ in G_i with respect to ψ_i are not equal. Clearly, the length of every such $v_{2,i}$ is either 0 or 1. For every $i \in \mathbb{N}$, let

$$r_{2,i} = u_{2,i}v_{2,i}^{-1}.$$

Let $\mathcal{R}_2 = \{\, r_{2,i} \mid i = 1, 2, \dots \}$.

Let $S_3 = \{u_{3,1}, \dots, u_{3,n}\}$ be a set consisting of n reduced group words of length at least $1/\lambda_1$ in the alphabet $\{a, b\}$ which satisfies the same three conditions as those required of S' above. (Such a set S_3 exists.) For every $i \in \{1, \dots, n\}$, let $v_{3,i} = x_i$ and

$$r_{3,i} = u_{3,i}v_{3,i}^{-1}.$$

Let $\mathcal{R}_3 = \{\, r_{3,i} \mid i = 1, \dots, n \,\}$.

Let

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3.$$

Let G be the group defined by the presentation $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$. This group may be naturally identified with a quotient group of F_{n+2}. Let $\phi_1 : F_{n+2} \to G$ be the corresponding epimorphism.

Because of the relations $r = 1$, $r \in \mathcal{R}_1$, the homomorphism ϕ_1 maps S_1 onto G.

Because of the relations $r = 1$, $r \in \mathcal{R}_2$, the group G has no nontrivial finite quotients. Indeed, suppose G has a nontrivial finite quotient. Then for some i there exists an epimorphism ϕ_2 from G onto G_i such that $\phi_2 \circ \phi_1$ extends ψ_i. On one hand, the value of the word $r_{2,i}$ in G_i with respect to ψ_i is nontrivial by the construction of $r_{2,i}$. On the other hand, the value of $r_{2,i}$ in G with respect to ϕ_1 is 1 since $r_{2,i} \in \mathcal{R}$; therefore, the value of $r_{2,i}$ in G_i with respect to $\phi_2 \circ \phi_1$ is 1. This gives a contradiction.

Because of the relations $r = 1$, $r \in \mathcal{R}_3$, the group G is generated by $\phi_1(a)$ and $\phi_1(b)$.

The only property of G that still needs to be established, is that G is nontrivial.

Suppose G is trivial. Then, by van Kampen's Lemma, for any group word v in the alphabet $\mathfrak{A}$, there exists a disc diagram Δ over the presentation $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ such that the label of the contour of Δ is v. In particular, there exists a reduced disc diagram over $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ whose contour has length 1. Let Δ by such a diagram. Clearly, it is simple.

Define a selection on Δ as follows. Consider all the faces of Δ one-by-one. Take an arbitrary face Π of Δ on which the selection has not been defined yet. Let i and j be such indices that the label of some representative of $\bar{\partial}\Pi$ is either $r_{i,j}$ or $r_{i,j}^{-1}$. Then there exist paths p and q such that pq is a representative of $\bar{\partial}\Pi$, and either $\ell(p) = u_{i,j}$ and $\ell(q) = v_{i,j}^{-1}$, or $\ell(p) = u_{i,j}^{-1}$ and $\ell(q) = v_{i,j}$. In both cases, such a path p is reduced since the word $u_{i,j}$ is reduced. Let the selection on Π consist of all the nontrivial subpaths of this p. The diagram Δ has become a *diagram with selection* or an *S-diagram*.

Now, check that the S-diagram Δ satisfies the condition $\mathcal{B}(\lambda_1, \lambda_2)$.

The condition $\mathcal{B}_0$ follows directly from the construction of the selection.

In order to check the condition $\mathcal{B}_1(\lambda_1)$, consider an arbitrary face Π and the corresponding subpaths p and q of its contour cycle used above in defining the selection on Π. Note that p is the only maximal selected subpath of $\bar{\partial}\Pi$, and q is the "complement" of p in $\bar{\partial}\Pi$. Take i and j such that p is labelled by $u_{i,j}$ or $u_{i,j}^{-1}$, and q is labelled by $v_{i,j}^{-1}$ or $v_{i,j}$. By the construction of the words $u_{i,j}$ and $v_{i,j}$, the inequality $\lambda_1|u_{i,j}| \geq |v_{i,j}|$ holds. Therefore,

$$|q| \leq \lambda_1|p| \leq \lambda_1|pq| = \lambda_1|\partial\Pi|.$$

The condition $\mathcal{B}_1(\lambda_1)$ follows.

To check the condition $\mathcal{B}_2(\lambda_2)$, means to prove that for all faces Π of Δ, the length of every double-selected oriented arc t which is a subpath of $\bar{\partial}\Pi$ is at most $\lambda_2|\partial\Pi|$.

Consider first an arbitrary intra-facial double-selected oriented arc t. Let Π be the face whose contour cycle has t as a subpath. Let p be the maximal selected subpath of $\bar{\partial}\Pi$, and q be such that pq is a representative of $\bar{\partial}\Pi$ (the same notation as above). Let $u = \ell(p)$. By the choice of the selection, either u or u^{-1} belongs to $S' \cup S_3$. The contour cycle of Π has the form $\langle ts_1t^{-1}s_2\rangle$. Since t and t^{-1} are selected, and there is exactly one maximal selected subpath of $\bar{\partial}\Pi$, either ts_1t^{-1} or $t^{-1}s_2t$ is selected. So, either ts_1t^{-1} or $t^{-1}s_2t$ is a subpath of p. In either case, $\ell(t)$ is a common subword of u and u^{-1}. Therefore,

$$|t| \leq \lambda|p| \leq \lambda|\partial\Pi| = \lambda_2|\partial\Pi|$$

(see property (1) of S' and S_3).

Consider next an arbitrary inter-facial double-selected oriented arc t. Let Π_1 be the face whose contour cycle has t as a subpath, and Π_2 be the face whose contour cycle has t^{-1} as a subpath (t is between Π_1 and Π_2). Let p_1 be the maximal selected subpath of $\bar{\partial}\Pi_1$, and p_2 be the maximal selected subpath of $\bar{\partial}\Pi_2$. By the choice of the selection, $|p_1| \leq |\partial\Pi_1|$ and $|p_2| \leq |\partial\Pi_2|$. Let $u_1 = \ell(p_1)$ and $u_2 = \ell(p_2)$. One of the words u_1, u_1^{-1} and one of the words u_2, u_2^{-1} belong to $S' \cup S_3$. The label of t is a common subword of u_1 and u_2^{-1}. If $|t| > \lambda|p_1|$ or $|t| > \lambda|p_2|$, then the pair of faces $\{\Pi_1, \Pi_2\}$ is cancellable (follows from the properties of S' and S_3, the fact that S' and S_3 use disjoint alphabets, and the construction of $\mathcal{R}$). Since Δ is reduced, this cannot happen. Therefore,

$$|t| \leq \lambda|p_1| \leq \lambda|\partial\Pi_1| = \lambda_2|\partial\Pi_1|.$$

Thus, Δ satisfies the condition $\mathcal{B}_2(\lambda_2)$.

Since Δ satisfies the condition $\mathcal{B}(\lambda_1, \lambda_2)$, by the corollary of the main theorem, the length of its contour is at least $(1-2\gamma)\sum_{f\in\Delta(2)}|\partial\Pi|$. Since

$$\begin{aligned}(1-2\gamma)\sum_{f\in\Delta(2)}|\partial\Pi| &\geq (1-2\gamma)\min\Big\{\,|r|\;\Big|\;r\in\mathcal{R}\,\Big\}\\ &\geq (1-2\gamma)\min\Big\{\,|u|\;\Big|\;u\in S_1\cup S_2\cup S_3\,\Big\} \geq \frac{3}{13}\cdot 5 > 1,\end{aligned}$$

this contradicts to the assumption that the length of the contour of Δ is 1. Hence, the group G could not be trivial.

By Zorn's Lemma, there exists a maximal proper normal subgroup N of the group G since G is finitely generated. Let $P = G/N$. Let $\phi_2 : G \to P$ be the quotient homomorphism. Then P is a desired infinite simple 2-generated group, and $\phi = \phi_2 \circ \phi_1|_{F_n}$ is a desired epimorphism $F_n \to P$. $\square$

Proof of the corollary of Theorem 1 (*see Introduction*). Take a free group F of rank 27 with a basis $\{a_1, \ldots, a_{27}\}$. Let

$$S = \{\, a_1^m \cdots a_{27}^m \mid m \in \mathbb{N}\,\}.$$

The symmetrization of the set S obviously satisfies the condition $C'(\frac{2}{27}+\varepsilon)$ relative to the basis $\{a_1, \ldots, a_{27}\}$ for any $\varepsilon > 0$. Therefore, by Theorem 1, there exists an infinite simple 2-generated group G and a homomorphism $\phi\colon F \to G$ such that ϕ maps S onto G. Let $x_i = \phi(a_i)$, $i = 1, \ldots, 27$. $\square$

7 Proof of Theorem 2

7.a Group construction

Take an arbitrary integer $n \geq 63$. Choose a positive $\lambda_1 < 1$ such that

$$\left(4 + \frac{2n\lambda_1}{1-\lambda_1}\right)\lambda_1 \leq \frac{1}{n}. \tag{1}$$

(It suffices if, for example, $0 < \lambda_1 \leq 1/(5n)$.)

Let $\mathfrak{A} = \{x_1, \ldots, x_n\}$ be an n-letter alphabet. Call a group word w over $\mathfrak{A}$ *regular* if it has the form $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$ where $k_i \in \mathbb{Z}$. (In particular, regular group words are cyclically reduced.) Call a group word w *counter-regular* if w^{-1} is regular. The empty word is both regular and counter-regular. So is every group word that is a *letter power.*

Impose an order on the set of all reduced group words in the alphabet $\mathfrak{A}$ to make it order-isomorphic with the set of natural numbers. For example, use the *deg-lex* order: x_1, x_1^{-1}, x_2, $\ldots$, x_n^{-1}, x_1^2, x_1x_2, $x_1x_2^{-1}$, $\ldots$. Define a sequence of sets $\{\mathcal{R}_i\}_{i=0}^{+\infty}$ inductively.

First, let $\mathcal{R}_0 = \emptyset$.

Second, if $i > 0$ and every group word over $\mathfrak{A}$ equals a regular word modulo the relations $r = 1$, $r \in \mathcal{R}_{i-1}$, then let $\mathcal{R}_i = \mathcal{R}_{i-1}$.

Last, if $i > 0$ and some group word over $\mathfrak{A}$ is not equal to any regular word modulo the relations $r = 1$, $r \in \mathcal{R}_{i-1}$, then let w_i be the least reduced group word (with respect to the chosen order) that does not start with $x_1^{\pm 1}$, does not end with $x_n^{\pm 1}$, and is not equal to any regular word modulo the relations $r = 1$, $r \in \mathcal{R}_{i-1}$ (clearly such w_i exists). Choose a natural number m_i such that

$$i > j \Rightarrow m_i \neq m_j, \tag{2}$$

$$i > j \Rightarrow nm_i + |w_i| \geq nm_j + |w_j|, \tag{3}$$

$$\lambda_1\Big(nm_i + |w_i|\Big) \geq |w_i|. \tag{4}$$

(These conditions may be satisfied by choosing a sufficiently large m_i.) Define

$$r_i = x_1^{m_i} x_2^{m_i} \dots x_n^{m_i} w_i^{-1}, \tag{5}$$

and let $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{r_i\}$. Note that $|r_i| = nm_i + |w_i|$.

Eventually, let

$$\mathcal{R} = \bigcup_{i=1}^{+\infty} \mathcal{R}_i, \tag{6}$$

and let G be the group defined by the presentation $\langle\, \mathfrak{A} \,\|\, r = 1, r \in \mathcal{R} \,\rangle$.

Observe that all elements of $\mathcal{R}$ are cyclically reduced.

Along with $\{\mathcal{R}_i\}_{i=0}^{+\infty}$, the sequences $\{w_i\}_{i=1,\dots}$, $\{m_i\}_{i=1,\dots}$, and $\{r_i\}_{i=1,\dots}$, infinite or finite, have been constructed.

Note that the above construction scheme does not specify $\mathcal{R}$ uniquely and is flexible as to the choice of $\{w_i\}_{i=1,\dots}$ and $\{m_i\}_{i=1,\dots}$.

7.b Properties of the group

In this section some properties of the defined above group G are established. In particular, it is shown that G is an example that provides the negative answer to

Bludov's question.

Adopt the notation of subsection 7.a. In particular, define n and λ_1 the same way. Let $\lambda_2 = 2/n$, $\gamma = \lambda_1 + 5\lambda_2$. Then

$$2\lambda_1 + 13\lambda_2 < 1 \tag{7}$$

and

$$1 - 2\gamma - 2\lambda_1 - \frac{2n\lambda_1^2}{1-\lambda_1} \geq 1 - \frac{21}{n}. \tag{8}$$

In particular $\gamma < 1/2$. It is known that $21 \times 3 = 63$; this explains why 63.

Inequality (4) is equivalent to

$$m_i \geq \frac{1-\lambda_1}{n}\Big(nm_i + |w_i|\Big). \tag{9}$$

Combining (4), (9), and (3), obtain

$$j \leq i \;\Rightarrow\; |w_j| \leq \frac{n\lambda_1}{1-\lambda_1} m_i. \tag{10}$$

Inequality (1) implies that $\lambda_1 < 1/(4n) < 1/(2n+1)$. It follows from this and from (10) that

$$j \leq i \;\Rightarrow\; |w_j| < \frac{m_i}{2}. \tag{11}$$

If w is a group word over $\mathfrak{A}$, let $[w]_G$, or $[w]_{\mathcal{R}}$, or simply $[w]$, denote the element of G represented by w. Let $a_1, \ldots, a_n$ be the elements of G represented by the one-letter group words $x_1, \ldots, x_n$, respectively (in terms of "brackets," $a_i = [x_i]$).

In this section a selection on a diagram Δ is called *special* if the contour cycle of every face Π of Δ has two subpaths s and t such that:

- $\langle st \rangle = \bar{\partial}\Pi$;
- s is the only maximal selected subpath of $\bar{\partial}\Pi$;

- there is $m \in \mathbb{N}$ such that either $\ell(s) = x_1^m x_2^m \dots x_n^m$ or $\ell(s) = x_n^{-m} x_{n-1}^{-m} \dots x_1^{-m}$;
- $|s| > \dfrac{n}{2n-2}|\partial\Pi|$ (note that $|\partial\Pi| = |s| + |t|$).

On every diagram Δ over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$, there exists a unique special selection. (In fact, if Δ is a diagram over an arbitrary group presentation, and a special in the above sense selection on Δ exists, then it is unique.) A diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ together with a special selection shall be called a *special* S-diagram. Note that if s is the maximal selected subpath of the contour cycle of a face Π of a special S-diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$, then $|s| \geq (1 - \lambda_1)|\partial\Pi|$, which is stronger than the inequality in the definition of a special selection.

If Δ is a diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$, Π is a face of Δ, then define the *rank* of the face Π to be such j that the label of some representative of $\bar{\partial}\Pi$ is $r_j^{\pm 1}$. Clearly, the rank of a face is well-defined. Let the rank of a face Π be denoted by $\operatorname{rank}(\Pi)$. Note that if $\operatorname{rank}(\Pi_1) \geq \operatorname{rank}(\Pi_2)$, then $|\partial\Pi_1| \geq |\partial\Pi_2|$. The ranks of two faces in a special S-diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ are equal if and only if the lengths of the maximal selected subpaths of their contour cycles are equal (follows from (2)).

Proposition 7.1. *Every weakly reduced special S-diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ satisfies the condition $\mathcal{B}(\lambda_1, \lambda_2)$.*

Proof. Let Δ be a weakly reduced special S-diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$. Clearly, Δ satisfies $\mathcal{B}_0$.

If Π is a face of Δ, s is the maximal selected subpath of $\bar{\partial}\Pi$, t is the path such that $\langle st \rangle = \bar{\partial}\Pi$, then $\ell(s) = (x_1^{m_j} x_2^{m_j} \dots x_n^{m_j})^{\pm 1}$ and $\ell(t) = w_j^{\mp 1}$ where $j = \operatorname{rank}\Pi$. Since $|w_j| \leq \lambda_1 |r_j|$ (see (4)), have that $|s| \geq (1 - \lambda_1)|\partial\Pi|$. Hence, Δ satisfies $\mathcal{B}_1(\lambda_1)$.

If u is a double-selected oriented arc, then $\ell(u)$ is a subword of a word of the form $x_l^m x_{l+1}^m$ or $x_{l+1}^{-m} x_l^{-m}$ (since Δ is weakly reduced). Therefore, if such u is a subpath of the contour cycle of a face Π, then $|u| < (2/n)|\partial\Pi| = \lambda_2|\partial\Pi|$. Hence, Δ satisfies $\mathcal{B}_2(\lambda_2)$. $\square$

Corollary 7.1.a. *If Δ is a weakly reduced special S-diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ with at most 3 contours and with spherical closure, and S is the number of selected external edge of Δ, then*

$$S \geq (1-2\gamma) \sum_{\Pi\in\Delta(2)} |\partial\Pi|.$$

Proof. Follows from the proposition and the corollary of the main theorem. $\square$

Corollary 7.1.b. *The group presentation $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ is strongly aspherical in the sense that no spherical diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ is weakly reduced.*

Proof. Every weakly reduced special spherical S-diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ has at least $(1-2\gamma)\cdot 2$ selected external edges and therefore does not exist ($\gamma < 1/2$). $\square$

Proposition 7.2. *Let Δ be a special non-degenerate S-diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$. Let $\mathfrak{B}$ be an arbitrary non-empty subset of $\mathfrak{A}$, let $k = |\mathfrak{B}|$. Let S be the number of selected external edges of Δ whose labels are in $\mathfrak{B}$. Then*

$$S < \frac{k}{n} \sum_{\Pi\in\Delta(2)} |\partial\Pi|.$$

Proof. If Π is a face of Δ, and s is the maximal selected subpath of $\bar{\partial}\Pi$, then $\ell(s) = (x_1^{m_j} x_2^{m_j} \ldots x_n^{m_j})^{\pm 1}$ where $j = \operatorname{rank}\Pi$. Therefore, the number of selected external edges of Δ incident to Π whose labels are in $\mathfrak{B}$ is at most $(k/n)|s| < (k/n)|\partial\Pi|$. Since this is true for every face Π of Δ, the desired inequality follows. $\square$

Definition. Let $\langle\,\mathfrak{B}\,\|\,\mathcal{S}\,\rangle$ be a finite group presentation. A function $f\ \mathbb{N} \to \mathbb{R}$ is called an *isoperimetric function* of $\langle\,\mathfrak{B}\,\|\,\mathcal{S}\,\rangle$ if for every group word w over $\mathfrak{B}$ equal to 1 modulo the relations $r = 1$, $r \in \mathcal{S}$, there exists a disc diagram Δ over $\langle\,\mathfrak{B}\,\|\,\mathcal{S}\,\rangle$ with at most $f(|w|)$ faces such that $w = \ell(\partial_1\Delta)$. The minimal isoperimetric function of $\langle\,\mathfrak{B}\,\|\,\mathcal{S}\,\rangle$ is called the *Dehn function* of $\langle\,\mathfrak{B}\,\|\,\mathcal{S}\,\rangle$.

Proposition 7.3. *If* $\langle\, \mathfrak{B} \,\|\, \mathcal{S} \,\rangle$ *is a finite subpresentation of* $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ *(*$\mathfrak{B}$ *is a subset of* $\mathfrak{A}$*, and* $\mathcal{S}$ *is a subset of* $\mathcal{R}$*), then the function* $f\, \mathbb{N} \to \mathbb{R}$ *given by the formula*

$$f(k) = \frac{k}{1 - 2\gamma}$$

is an isoperimetric function of $\langle\, \mathfrak{B} \,\|\, \mathcal{S} \,\rangle$ *(recall that* $\gamma < 1/2$*).*

Proof. If Δ is a weakly reduced disc diagram over $\langle\, \mathfrak{B} \,\|\, \mathcal{S} \,\rangle$, and $\langle q \rangle = \bar{\partial}_1 \Delta$, then, as follows from Corollary 7.1.a of Proposition 7.1,

$$|q| \geq (1 - 2\gamma) \sum_{\Pi \in \Delta(2)} |\partial \Pi| \geq (1 - 2\gamma) \|\Delta(2)\|.$$

Hence, $\|\Delta(2)\| \leq f(|q|)$. ☐

Corollary 7.3.a. *Every finite subpresentation of* $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ *presents a hyperbolic group.*

Proof. Use the characterization of hyperbolic groups in terms of isoperimetric functions of their finite presentations. According to Theorem 2.5, Theorem 2.12, and Corollary of the latter in [ABC+91], a group is hyperbolic if and only if it has a finite presentation with a linear isoperimetric function. (Note that in [ABC+91] all isoperimetric functions in the sense of the last definition are called "Dehn functions.") ☐

Let *deg-lex* be the order on the set of all group words over $\mathfrak{A}$ described as follows. Two group words are compared first by length, second alphabetically according to the following order on the group letters: $x_1 < x_1^{-1} < x_2 < \cdots < x_n < x_n^{-1}$. For example, $x_n x_n < x_1 x_1 x_1$ and $x_1 x_2 x_3 < x_1 x_3 x_2$.

Proposition 7.4. *If the order on reduced group words used in subsection* 7.a *for choosing* $\{w_i\}_{i=1,\ldots}$ *is deg-lex,* N *is a positive integer, and* $m_i = N|w_i| + i$ *for every* i *for which* r_i *is defined, then the presentation* $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ *is recursive.*

Proof. If the presentation $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ is finite, then it is recursive. (It will be shown in subsection 7.c that it cannot be finite.) Now, assume that the presentation is infinite.

Let C be the set of all 4-tuples $(\mathcal{S}, E, u, v)$ such that $\mathcal{S}$ is a finite set of group words over $\mathfrak{A}$, E is a rational number, u and v are group words over $\mathfrak{A}$, and there exists a disc diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$ with at most E edges such that the label of its contour is uv^{-1}. The set C is recursive.

Let q be a rational number such that $q \geq 1/(1-2\gamma)$. Let D be the set of all 3-tuples $(\mathcal{S}, u, v)$ such that

$$\Big(\mathcal{S}, \frac{1+qL}{2}(|u|+|v|), u, v\Big) \in C$$

if L is the maximum of the lengthes of the elements of $\mathcal{S}$. The set D is recursive. This set describes consequences of a given finite set of relations as explained below.

Let f be the function $\mathbb{N} \to \mathbb{R}, k \mapsto qk$. Consider an arbitrary finite $\mathcal{S}$ such that f is an isoperimetric function of $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$. Let u and v be arbitrary group words over $\mathfrak{A}$. Then the relation $u = v$ is a consequence of the relations of $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$ if and only if $(\mathcal{S}, u, v) \in D$.

Consider the following algorithm:

Input: `a positive integer` j`.`

Step 1: `Produce the set` $\mathcal{S}$ `of outputs of this algorithm on the inputs` 1`, ...,` $j-1$`. (Do not stop if the algorithm does not stop on at leas one of those inputs; if` $j=1$`, then` $\mathcal{S}=\emptyset$`.) Let` L `be the maximum of the lengthes of the elements of` $\mathcal{S}$ `if` $\mathcal{S}$ `is non-empty, and` 0 `otherwise.`

Step 2: `Find the least (with respect to deg-lex) group word` w `over` $\mathfrak{A}$ `such that for every regular group word` u `of` `length at most` $(n+1)|w| + n^4 L$`,` $(\mathcal{S}, u, w) \notin D$`.` `(Do not stop if there is no such` w`.)` `Let` $m = N|w| + j$`.`

Output: $x_1^m x_2^m \dots x_n^m w^{-1}$.

It shall be shown by induction that on every input $j \in \mathbb{N}$, the algorithm stops and gives r_j as the output. It will follow that $\mathcal{R}$ is recursive because the length of r_j is an increasing function of j.

Take an arbitrary $k \in \mathbb{N}$. If $k > 1$, assume that it is already proved that for every positive integer $j < k$, the algorithm gives r_j as the output on the input j. Consider the work of the algorithm on the input k.

By the inductive assumption, the set $\mathcal{S}$ produced on Step 1 coincides with $\mathcal{R}_{k-1}$. In particular, f is an isoperimetric function of $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$. Note that $L = |r_{k-1}|$ if $k > 1$, and $L = 0$ if $k = 1$.

By the choice of w_k, this word is not equal to any regular group word modulo the relations of $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$. Therefore, Step 2 of the algorithm is carried out in finite time, and the obtained group word w is not greater than w_k with respect to the deg-lex order.

To complete the inductive step and the proof of this proposition, it suffices to show that the group word w obtained on Step 2 is w_k. Suppose w is distinct from w_k.

Since $w < w_k$ relative to deg-lex, it follows from the choice of w_k that w equals some regular group word modulo the relations of $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$. Let u be such a regular group word. By the choice of w,

$$|u| > (n+1)|w| + n^4 L.$$

Let Δ be a weakly reduced special disc S-diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$ such that $\ell(\partial_1 \Delta) = uw^{-1}$. Let b and p be the paths such that $\ell(b) = u$, $\ell(p) = w^{-1}$, and $\langle bp \rangle = \bar{\partial}_1 \Delta$. Let b_1, b_2, $\dots$, b_n be the subpaths of b such that $b = b_1 b_2 \dots b_n$, and for every

$i = 1, 2, \ldots, n$, the label of b_i is a power of x_i. For every $i = 1, 2, \ldots, n$, let B_i be the set of all edges that lie on b_i. Let Σ be the sum of the degrees of all the faces of Δ. Let T be the number of selected external edges of Δ that lie on b.

Since $\ell(b)$ is regular, no two oriented edges of b are inverse to each other. Hence,

$$|b| \le \Sigma + |p|.$$

Let Σ_1 be the sum of the degrees of all faces of Δ that are incident with edges from at least 3 distinct sets from among $B_1, \ldots, B_n$. If $1 \le i_1 < i_2 < i_3 \le n$, then there is at most one face that is incident to edges from all three sets B_{i_1}, B_{i_2}, B_{i_3}. Therefore, $\Sigma_1 \le n^3 L$ (the degree of every face of Δ is at most L). Let Σ_2 be the sum of the degrees of all faces of Δ that are incident with edges from no more than 2 distinct sets from among $B_1, \ldots, B_n$.

One one hand,

$$T \le \Sigma_1 + \frac{2}{n}\Sigma_2 \le n^3 L + \frac{2}{n}\Sigma.$$

On the other hand, by Proposition 7.1 and the Main Lemma,

$$T + |p| \ge (1 - 2\gamma)\Sigma.$$

Therefore,

$$\begin{aligned}
(1 - 2\gamma)\Sigma &\le n^3 L + \frac{2}{n}\Sigma + |p|,\\
\left(1 - 2\gamma - \frac{2}{n}\right)\Sigma &\le |p| + n^3 L,\\
\frac{n - 23}{n}\Sigma &\le |p| + n^3 L,\\
\Sigma &\le n|p| + n^4 L,\\
|b| &\le (n+1)|p| + n^4 L.
\end{aligned}$$

In other terms, $|u| \le (n+1)|w| + n^4 L$. This gives a contradiction.

Thus, $w = w_k$, and the inductive step is done. $\square$

Everything is ready now to start proving properties of G.

Property 1. *For every $(n-21)$-element subset I of the set $\{1,\dots,n\}$, the system $\{a_i\}_{i\in I}$ freely generate a free subgroup F of G (of rank $n-21$). Moreover, every two elements in such a free subgroup F are conjugate in G only if they are conjugate in F.*

Proof. Suppose that Property 1 does not hold. Take a set $I\subset\{1,\dots,n\}$ of $n-21$ elements that provides a counterexample. Let $\mathfrak{B}=\{\,x_i \mid i\in I\,\}$.

If the system $\{a_i\}_{i\in I}$ does not freely generate a free subgroup of G, then there is a weakly reduced disc diagram Δ over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ such that the label of its contour is a cyclically reduced non-empty group word over the alphabet $\mathfrak{B}$. Clearly, such a diagram Δ is non-degenerate (has a face).

If the system $\{a_i\}_{i\in I}$ does freely generate a free subgroup F of G, but there are two elements of F that are conjugate in G but not in F, then let v_1 and v_2 be two cyclically reduced group words over $\mathfrak{B}$ that represent two such elements. The group words v_1 and v_2 are not cyclic shifts of each other, and neither of them represents the identity of G. Since $[v_1]$ and $[v_2]$ are nontrivial but are conjugate in G, there exists a weakly reduced annular diagram Δ over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ such that the label of one of its contours is v_1, and the label of its other contour is v_2^{-1} (see Lemma V.5.2 in [LS01]). Clearly, such Δ is non-degenerate.

Thus, to obtain a contradiction and complete the proof, it is enough to show that there is no weakly reduced non-degenerate disc nor annular diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ with fewer than $n-20$ distinct basic letters on its contour(s).

Let Δ be an arbitrary special weakly reduced non-degenerate disc or annular S-diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$. (Recall that a special selection exists on every diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$.) Suppose that Δ has fewer than $n-20$ distinct basic letters of $\mathfrak{A}$ on its contour(s). Let S be the total number of selected external edges of Δ. On one hand, by Proposition 7.2,

$$S < \frac{n-21}{n} \sum_{\Pi \in \Delta(2)} |\partial \Pi|.$$

On the other hand, by Corollary 7.1.a of Proposition 7.1,

$$S \geq (1-2\gamma) \sum_{\Pi \in \Delta(2)} |\partial \Pi|.$$

Since

$$1 - 2\gamma > \frac{n-21}{n},$$

have a contradiction. ☐

The proof of the next property is not so straightforward.

Property 2. *For every element g of G, there exist unique $k_1, \ldots, k_n \in \mathbb{Z}$ such that $g = a_1^{k_1} \ldots a_n^{k_n}$.*

The uniqueness is the "hard part" of Property 2. It shall be proved by contradiction. Main steps of the proof are stated below as Lemmas 7.1–7.16. These lemmas share some common assumptions about an S-diagram Δ.

Assume that Δ is a special disc S-diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ whose contour is of the form $p_1 p_2$ such that $\ell(p_1)$ and $\ell(p_2^{-1})$ are distinct regular group words, and $\ell(p_1 p_2)$ is cyclically reduced. Suppose, moreover, that Δ is such an S-diagram with the minimal possible number of faces. In particular, if Δ' is a nontrivial disc diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$, and $\ell(\partial_1 \Delta')$ is regular or counter-regular, then $\|\Delta'(2)\| \geq \|\Delta(2)\|$.

Lemma 7.1. *The diagram Δ is weakly reduced.*

Proof. This easily follows from the minimality of the number of faces of Δ. ☐

Lemma 7.2. *The diagram Δ is simple.*

Proof. Use the minimality of Δ again. Some maximal simple disc S-subdiagram of Δ satisfies all the assumptions made about Δ (this follows, for example, from Proposition 3.1 of [Mur05]), and hence it must be the whole of Δ. $\square$

Lemma 7.3. *If Δ' is a disc S-subdiagram of Δ, and S is the number of selected external edge of Δ' (external in Δ'), then*

$$S \geq (1 - 2\gamma) \sum_{\Pi \in \Delta'(2)} |\partial \Pi|.$$

Proof. Follows directly from Lemma 7.1 and Corollary 7.1.a of Proposition 7.1. $\square$

Lemma 7.4. *If Δ' is a disc subdiagram of Δ, p_1 and p_2 are paths, $\langle p_1 p_2 \rangle = \bar{\partial}_1 \Delta'$, and the reduced forms of $\ell(p_1)$ and $\ell(p_2^{-1})$ are regular, then Δ' either is degenerate or coincides with Δ.*

Proof. Suppose Δ', p_1, and p_2 are such as in the hypotheses of the lemma.

Case 1: $\ell(p_1)$ and $\ell(p_2^{-1})$ are freely equal. Then the label of every representative of $\bar{\partial}_1 \Delta'$ is freely trivial and therefore Δ' is degenerate. Indeed, if Δ' was non-degenerate, there would exist a disc diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ with the same label of the contour as Δ but with fewer faces (faces of Δ', possibly together with some of the others, could be "eliminated" from Δ), which would contradict the minimality of Δ.

Case 2: $\ell(p_1)$ and $\ell(p_2^{-1})$ are not freely equal. Then take Δ' and repeatedly *fold* labelled external edges and "cut off," whenever necessary, "branches" with freely trivial contour labels until obtain a disc diagram Δ'' such that the labels of the representatives of $\bar{\partial}_1 \Delta''$ are cyclically reduced. Clearly, Δ'' is a non-degenerate diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$, the label of every representative of $\bar{\partial}_1 \Delta''$ is a cyclically reduced form of $\ell(p_1 p_2)$, and $\|\Delta''(2)\| \leq \|\Delta'(2)\|$. By the minimality of Δ, such a diagram Δ'' cannot have fewer faces than Δ. Therefore, $\Delta'(2) = \Delta(2)$ and, since $\partial_1 \Delta$ is cyclically reduced, Δ' coincides with Δ. $\square$

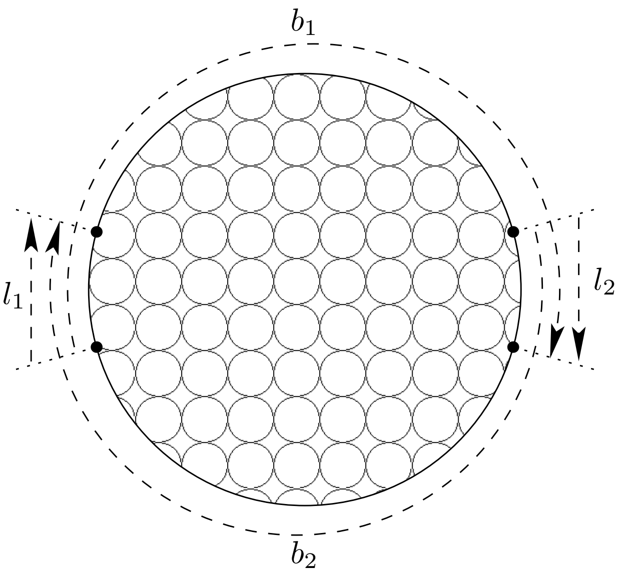


Figure 1: The paths b_1, b_2, l_1, and l_2 in Δ.

Lemma 7.5. *The diagram Δ has more than 1 face.*

Proof. This easily follows from the form of elements of $\mathcal{R}$. □

Lemma 7.6. *There are at least $n - 20$ distinct basic letters on $\partial_1\Delta$.*

Proof. Follows from Property 1. □

Let b_1 be the maximal subpath of $\bar{\partial}_1\Delta$ whose label is regular but not counter-regular (is not a letter power). Let b_2 be the maximal subpath of $\bar{\partial}_1\Delta$ whose label is counter-regular but not regular. Since, according to Lemma 7.6, there are at least 2 distinct basic letters on $\partial_1\Delta$, the paths b_1 and b_2 are well-defined. (If, say, the label of some representative of $\bar{\partial}_1\Delta$ is $x_1x_2x_3\dots x_{42}x_{43}x_{42}x_2x_1$, then $\ell(b_1) = x_1^2x_2\dots x_{43}$, and $\ell(b_2) = x_{43}x_{42}x_2x_1^2$.) The label of each of the paths b_1 and b_2 has at least 2 distinct basic letters. Let l_1 be the initial subpath of b_1 that is a terminal subpath of b_2, and l_2 be the initial subpath of b_2 that is a terminal subpath of b_1. Both l_1 and l_2 are nontrivial (see Fig. 1). The labels of l_1 and l_2 are letter powers.

Lemma 7.7. *Every selected external arc of Δ lies on at least one of the paths b_1 or b_2.*

Proof. Follows from the fact that the label of every selected oriented arc is either regular or counter-regular. □

The following observation is obvious but deserves mentioning because it is used implicitly in the proofs of Lemmas 7.8, 7.10, and 7.13 several times.

Suppose p_1, p_2, and q are paths in a diagram over $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$, and q is nontrivial. Then

- if the products $p_1 q$ and $q^{-1} p_2$ are defined and their labels are regular, then $\ell(q)$ is a letter power, and the reduced form of $\ell(p_1 p_2)$ is regular;
- if the products $p_1 q$ and $q p_2$ are defined and their labels are regular, then $\ell(p_1 q p_2)$ is regular.

Since Lemma 7.7 and many of the remaining lemmas deal with selected external arcs or selected external oriented arcs, it is advisable to review these concepts. An oriented arc of a graph (or a 2-complex) is a simple path whose all vertices, except, possibly, the end-vertices, have degree two in the given graph. A (non-oriented) arc is a pair of mutually inverse oriented arcs. An arc is incident to a face Π if (and only if) at least one of the associated oriented arcs is a subpath of $\bar{\partial}\Pi$. Every external arc of a (semi-)simple map is incident to exactly one face. An oriented arc is selected if (and only if) it is selected as a path (in particular, it must be a subpath of the contour cycle of some face). An external arc is selected if (and only if) it is incident to some face Π and the associated oriented arcs that is a subpath of $\bar{\partial}\Pi$ is selected. Distinct maximal selected external arcs of Δ never overlap.

Lemma 7.8. *In the diagram Δ, every face is incident with at most 1 maximal selected external arc.*

Proof. Consider an arbitrary face Π of Δ. Suppose Π is incident with at least 2 distinct maximal selected external arcs.

Let u_1 and u_2 be distinct maximal selected external oriented arcs of Δ that are subpaths of $\bar{\partial}\Pi$. (They do not overlap.) Let s be the maximal selected subpath of $\bar{\partial}\Pi$. Without loss of generality, assume that u_1 precedes u_2 as a subpath of s. Let v be such a path that $u_1 v u_2$ is a subpath of s. Since Δ is simple (see Lemma 7.2), and u_1 and u_2 are maximal, it follows that v is nontrivial, and the first and last oriented edges of v are internal in Δ.

Suppose v has a simple cyclic subpath p. Then p is the contour of a proper simple disc subdiagram of Δ. Since $\ell(p)$ is regular or counter-regular, this contradicts the minimality of Δ. Hence, the path v is simple and not cyclic (equivalently, every vertex occurs in v at most once).

Let v_1 be the minimal subpath of $\bar{\partial}_1\Delta$ such that u_1^{-1} and u_2^{-1} are respectively its initial and terminal subpaths. Let v_2 be the path such that $\langle v_1 v_2\rangle = \bar{\partial}_1\Delta$. The paths $v_1 v^{-1}$ and $v_2 v$ are cyclically reduced and are the contours of two disc subdiagrams of Δ. Let Δ_1' and Δ_2' be the subdiagrams with the contours $v_1 v^{-1}$ and $v_2 v$, respectively. Note that $\Pi \in \Delta_1'(2)$, and $\Delta(2)$ is the disjoint union of $\Delta_1'(2)$ and $\Delta_2'(2)$. The diagram Δ_1' is simple. The diagram Δ_2' is non-degenerate. Both Δ_1' and Δ_2' are proper subdiagrams of Δ.

The labels of u_1 and u_2 are both regular or both counter-regular. Therefore, each of the paths u_1^{-1} and u_2^{-1} is a subpath of one of the paths b_1 or b_2 (see Lemma 7.7). Consider the following 4 cases (see Fig. 2; shaded subdiagrams are the "sources of contradiction"):

Case 1: u_1^{-1} and u_2^{-1} are subpaths of b_1. Then either v_1 or $u_2^{-1} v_2 u_1^{-1}$ is a subpath of b_1 as well. Suppose v_1 is a subpath of b_1. Then $\ell(v_1)$ is regular, $\ell(u_1)$ and $\ell(u_2)$ are letter powers, and $\ell(u_1 v u_2)$ is regular. Therefore, $v_1 v^{-1}$ cannot be the contour of a proper non-degenerate disc subdiagram (see Lemma 7.4). This gives a contradiction. Suppose $u_2^{-1} v_2 u_1^{-1}$ is a subpath of b_1. Then $\ell(u_2^{-1} v_2 u_1^{-1})$ is regular, and $\ell(u_1 v u_2)$ is counter-regular. Therefore, $v_2 v$ cannot be the contour of a proper non-degenerate

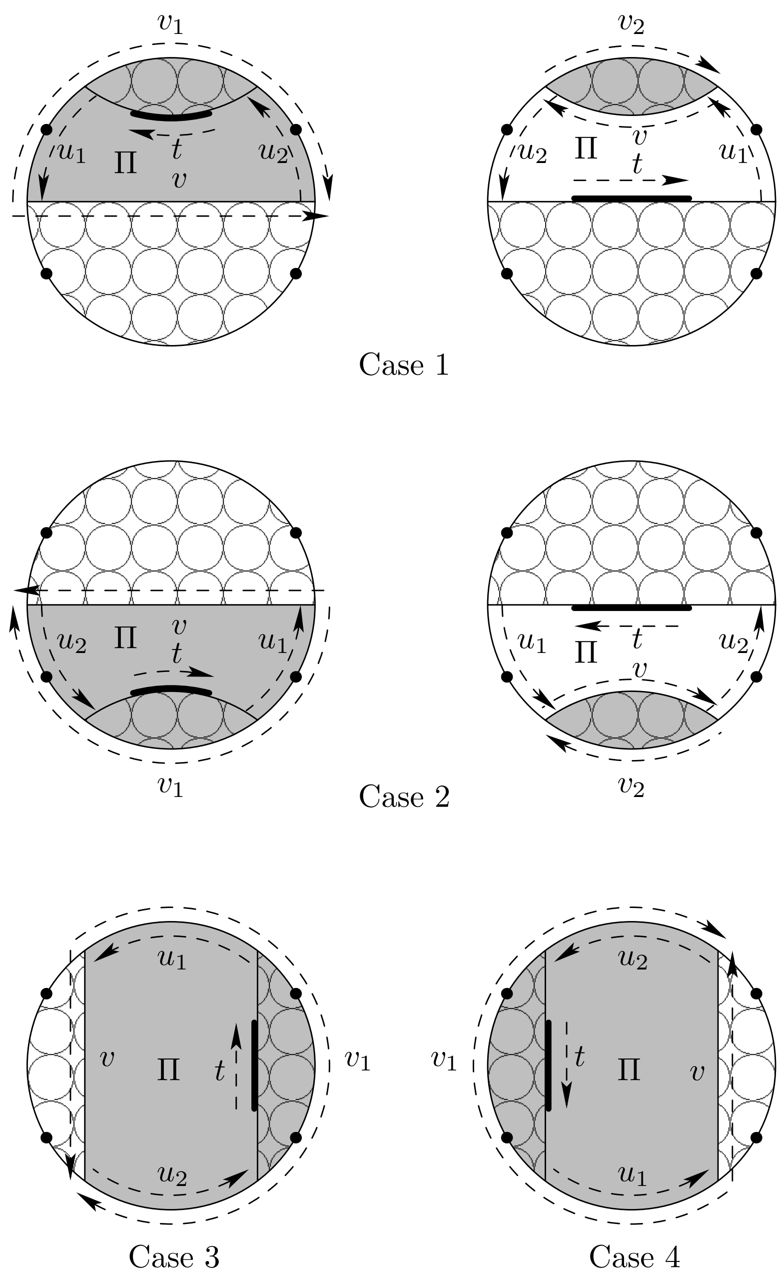


Figure 2: Cases 1–4, Lemma 7.8.

disc subdiagram. This gives a contradiction.

Case 2: u_1^{-1} and u_2^{-1} are subpaths of b_2. When proving impossibility of Case 1, it was essentially shown that no face of Δ can be incident with 2 distinct maximal selected external arcs lying on b_1. Since all the assumptions made about Δ hold for its mirror copy as well, the same statement appropriately reformulated must hold for the mirror copy of Δ. Namely, no face of the mirror copy of Δ can be incident with 2 distinct maximal selected external arcs lying on b_2^{-1} (b_2^{-1} plays the same role for the mirror copy of Δ as b_1 does for Δ). This means that Case 2 is impossible.

Case 3: u_1^{-1} is a subpath of b_1, and u_2^{-1} is a subpath of b_2. Pick a subpath p_1 of b_1 and a subpath p_2 of b_2 such that u_1^{-1} is an initial subpath of p_1, u_2^{-1} is a terminal subpath of p_2, and $p_1p_2 = v_1$. Then $\ell(p_1)$ is regular, $\ell(p_2)$ is counter-regular, and $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular or counter-regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular, then $\ell(v^{-1}p_1)$ is regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is counter-regular, then $\ell(p_2v^{-1})$ is counter-regular. In either case $\langle p_2v^{-1}p_1\rangle = \langle v_1v^{-1}\rangle$ cannot be the contour cycle of a proper non-degenerate disc subdiagram. This gives a contradiction.

Case 4: u_1^{-1} is a subpath of b_2, and u_2^{-1} is a subpath of b_1. Pick a subpath p_1 of b_2 and a subpath p_2 of b_1 such that u_1^{-1} is an initial subpath of p_1, u_2^{-1} is a terminal subpath of p_2, and $p_1p_2 = v_1$. Then $\ell(p_1)$ is counter-regular, $\ell(p_2)$ is regular, and $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular or counter-regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular, then $\ell(p_2v^{-1})$ is regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is counter-regular, then $\ell(v^{-1}p_1)$ is counter-regular. In either case $\langle p_2v^{-1}p_1\rangle = \langle v_1v^{-1}\rangle$ cannot be the contour cycle of a proper non-degenerate disc subdiagram. This gives a contradiction.

(Case 4 is symmetric with Case 3, but this symmetry is not simply the mirror symmetry as between Cases 1 and 2. Say, if Case 4 was indeed the case for Δ, then the mirror copy of Δ would be just another example for the same Case 4, not Case 3.)

A contradiction is obtained in each of the 4 cases, and no other cases exist. $\square$

Lemma 7.9. *There are a face Π and a path s' in Δ such that:*

- *s' is a selected subpath of $\bar{\partial}\Pi$;*
- *s' is a maximal selected external oriented arcs of Δ;*
- *the $n-40$ basic letters x_{21}, x_{22}, …, x_{n-20} all occur in $\ell(s')$.*

Proof. Let for every face Π of Δ, $S(\Pi)$ denote the number of selected external edges of Δ incident to Π. By Lemma 7.3,

$$\sum_{\Pi\in\Delta(2)} S(\Pi) \geq (1-2\gamma) \sum_{\Pi\in\Delta(2)} |\partial\Pi|.$$

Hence, there exists a face Π of Δ such that

$$S(\Pi) \geq (1-2\gamma)|\partial\Pi| > \frac{n-21}{n}|\partial\Pi|$$

(recall that $1-2\gamma > 1-21/n$).

Let Π be such a face as above. Let s be the maximal selected subpath of $\bar{\partial}\Pi$. Let s' be the (only) maximal selected external oriented arc of Δ that is a subpath of s (see Lemma 7.8). Then

$$|s'| = S(\Pi) > \frac{n-21}{n}|\partial\Pi| > \frac{n-21}{n}|s|.$$

Let $j = \mathrm{rank}(\Pi)$. Then either $\ell(s) = x_1^{m_j} x_2^{m_j} \dots x_n^{m_j}$ or $\ell(s) = x_n^{-m_j} x_{n-1}^{-m_j} \dots x_1^{-m_j}$. Therefore, $\ell(s')$ has at least $n-20$ distinct basic letters, and all of the basic letters x_{21}, x_{22}, …, x_{n-20} occur in it. $\square$

Lemmas 7.1–7.9, as well as 7.10–7.16, assert some properties of Δ. Observe that none of these properties in fact can distinguish between Δ and its mirror copy, i.e., each of these properties holds for Δ if and only if it holds for the mirror copy of Δ. Thus, at this point analogues of Lemmas 7.1–7.9 for the mirror copy of Δ shall be assumed proved. Moreover, the initial assumptions about the S-diagram Δ are also true about its mirror copy.

Even though the statement of Lemma 7.7 is about the S-diagram Δ and the paths b_1 and b_2, it still may be viewed as an assertion of a property of Δ because, according to the way they are chosen, b_1 and b_2 are uniquely determined for the given Δ. Therefore, the analog of Lemma 7.7 for the mirror copy of Δ states:

> Every selected external arc of the mirror copy of Δ lies on at least one of the paths b_2^{-1} or b_1^{-1}.

In the proofs of Lemmas 7.10–7.13, it is convenient in some cases to pass to the mirror copy of Δ to reduce the number of cases to consider.

Lemma 7.10. *Let Π_1 be a face of Δ incident to a selected external edge of Δ, and Π_2 be another face of Δ. Let s_1 and s_2 be the maximal selected subpaths of $\bar{\partial}\Pi_1$ and $\bar{\partial}\Pi_2$, respectively. Let s_1' be the maximal selected external oriented arc of Δ that is a subpath of s_1. Let s_{1-}' and s_{1+}' be the paths such that $s_1 = s_{1-}'s_1's_{1+}'$. Let q_1 be the path such that $\langle s_1'^{-1}q_1\rangle = \bar{\partial}_1\Delta$. Suppose there are at least 2 distinct basic letters in $\ell(s_1')$. Suppose the paths s_2 and $(s_{1-}'q_1s_{1+}')^{-1}$ have a common oriented edge. Then they have exactly one maximal common nontrivial subpath.*

Proof. Observe that to prove that there is exactly one maximal common nontrivial subpath of s_2 and $(s_{1-}'q_1s_{1+}')^{-1}$, it suffices to prove the same for s_2^{-1} and $s_{1-}'q_1s_{1+}'$. This justifies passing to the mirror copy of Δ.

Since the paths s_2 and $(s_{1-}'q_1s_{1+}')^{-1}$ have a common oriented edge, they have at least one maximal common nontrivial subpath.

The label of s_1' is regular or counter-regular. Therefore, s_1' is a subpath of b_1^{-1} or b_2^{-1}. If it is a subpath of b_2^{-1} but not of b_1^{-1}, pass from Δ, b_1, b_2, l_1, l_2, s_1, s_2, s_{1-}', s_{1+}', s_1', q_1 to the mirror copy of Δ, b_2^{-1}, b_1^{-1}, l_1^{-1}, l_2^{-1}, s_1^{-1}, s_2^{-1}, $s_{1+}'^{-1}$, $s_{1-}'^{-1}$, $s_1'^{-1}$, q_1^{-1},

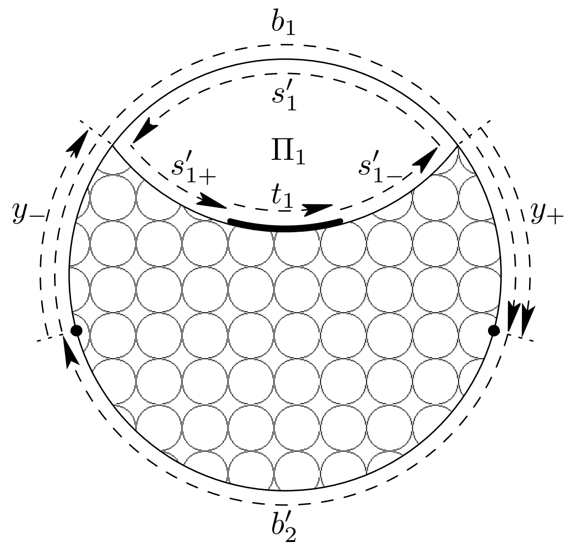


Figure 3: The face Π_1 in Δ, Lemma 7.10.

respectively. Hence, it may and shall be assumed that s'_1 is a subpath of b_1^{-1} (see Fig. 3).

Let t_1 be the path such that $\langle s_1 t_1 \rangle = \bar{\partial}\Pi_1$. Let y_- and y_+ be the paths such that $b_1 = y_- s_1'^{-1} y_+$. Let b'_2 be the subpath of b_2 such that $\langle b_1 b'_2 \rangle = \bar{\partial}_1 \Delta$. Then $b_2 = l_2 b'_2 l_1$ and $q_1 = y_+ b'_2 y_-$.

Since $\ell(b_1^{-1})$ is counter-regular, and $\ell(s'_1)$ is not a letter power (it has at least 2 distinct basic letters), $\ell(s_1)$ is counter-regular as well. Hence, $\ell(s_{1+}'^{-1} s_1'^{-1} y_+)$ and $\ell(y_- s_1'^{-1} s_{1-}'^{-1})$ are regular.

Let $j_1 = \operatorname{rank}(\Pi_1)$. Then $\ell(s_1) = x_n^{-m_{j_1}} x_{n-1}^{-m_{j_1}} \ldots x_1^{-m_{j_1}}$ and $\ell(t_1) = w_{j_1}$.

Since $\ell(s'_1)$ has at least 2 distinct basic letters, $\ell(y_- s'_{1+})$ and $\ell(s'_{1-} y_+)$ have disjoint sets of basic letters. This also implies that l_1 is a proper initial subpath of $y_- s_1'^{-1}$, and l_2 is a proper terminal subpath of $s_1'^{-1} y_+$.

Suppose s_2 and $(s'_{1-} q_1 s'_{1+})^{-1}$ have at least two distinct maximal common nontrivial subpaths. Then let u_1 and u_2 be two paths such that:

- u_1 and u_2 are nontrivial subpaths of distinct maximal common subpaths of s_2 and $(s'_{1-} q_1 s'_{1+})^{-1}$,

- u_1 precedes u_2 as a subpath of s_2, and
- each of the paths u_1 and u_2 is a maximal common subpath of s_2 and one of the paths y_-^{-1}, y_+^{-1}, $s_{1-}'^{-1}$, $s_{1+}'^{-1}$, or $b_2'^{-1}$.

Clearly, such u_1 and u_2 exist and are non-overlapping oriented arcs of Δ. Let v be the path such that u_1vu_2 is a subpath of s_2. Since u_1 and u_2 are subpaths of distinct maximal common subpaths of s_2 and $(s_{1-}'q_1s_{1+}')^{-1}$, the path u_1vu_2 is not a subpath of $(s_{1-}'q_1s_{1+}')^{-1}$.

Consider the following 16 cases (see Fig. 4–8):

(Notice that in a certain sense Cases 6, 8, 10, 12, 14, and 16 are "symmetric" with Cases 5, 7, 9, 11, 13, and 15, respectively.)

Case 1: each of the paths u_1 and u_2 is a subpath of q_1^{-1}. Then u_1 and u_2 are external selected oriented arcs of Δ. Since there is only one maximal selected external arc of Δ incident to Π_2 (see Lemma 7.8), the path u_1vu_2 is a subpath of q_1^{-1}. This gives a contradiction.

Case 2: both u_1 and u_2 are subpaths of one of the paths $s_{1-}'^{-1}$ or $s_{1+}'^{-1}$. Let z be the one of the paths s_{1-}' or s_{1+}' such that both u_1 and u_2 are subpaths of z^{-1}. Then $\ell(z)$ is counter-regular. Suppose u_2 precedes u_1 as a subpath of z^{-1}. Let p be the path such that $u_1^{-1}pu_2^{-1}$ is a subpath of z. Then $\ell(u_1^{-1}pu_2^{-1})$ is counter-regular. Therefore, $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular. The cyclic path $pu_2^{-1}v^{-1}u_1^{-1}$ is the contour of a disc subdiagram of Δ not containing the face Π_1 but containing the face Π_2. This contradicts the minimality of Δ. Therefore, u_1 precedes u_2 as a subpath of z^{-1}. Let p be the path such that $u_2^{-1}pu_1^{-1}$ is a subpath of z. Then $\ell(u_2^{-1}pu_1^{-1})$ is counter-regular. Therefore, $\ell(u_1vu_2)$ is regular. The cyclic path pv is the contour of a disc subdiagram of Δ not containing the faces Π_1 and Π_2. By Lemma 7.4 (by the minimality of Δ), this subdiagram is degenerate. Since z and v are reduced, $v = p^{-1}$. Therefore, u_1vu_2 is a subpath of z^{-1}. This gives a contradiction.

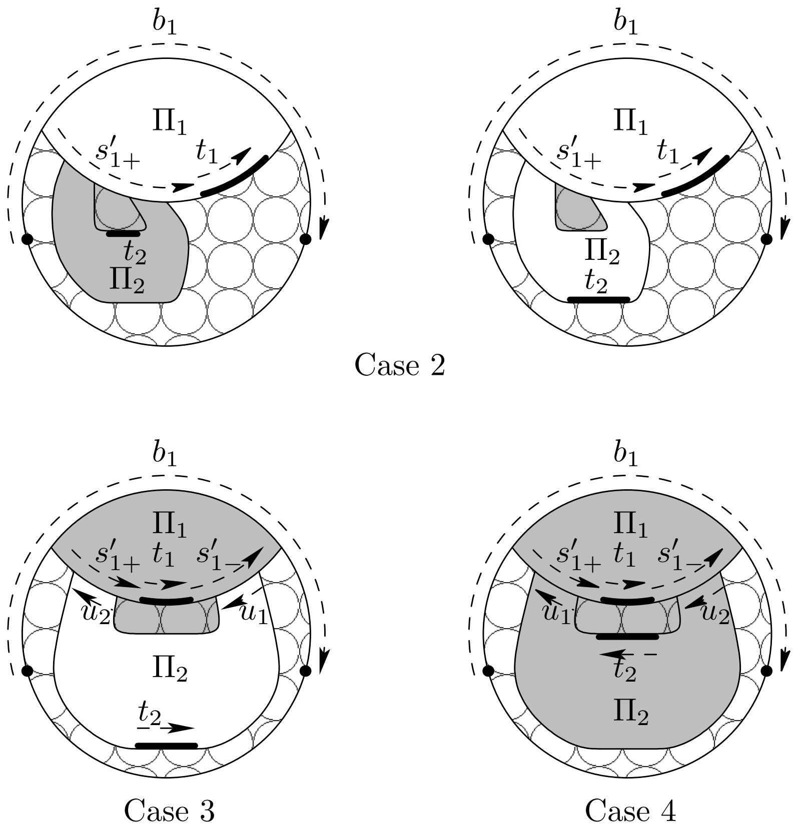
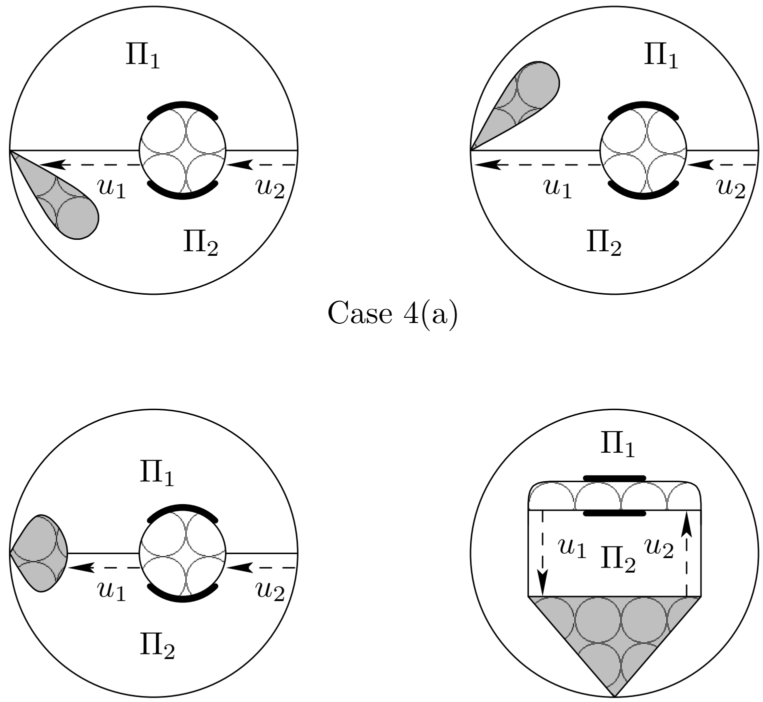


Figure 4: Cases 2, 3, 4, Lemma 7.10.



Case 4(a)

Case 4(b)

Case 4(c)

Figure 5: Special subcases of Case 4, Lemma 7.10.

Case 3: u_1 is a subpath of $s_{1-}'^{-1}$, and u_2 is a subpath of $s_{1+}'^{-1}$. Let p be the path such that $u_1^{-1}p^{-1}u_2^{-1}$ is a subpath of s_1. Then $\ell(u_2pu_1)$ is regular. Therefore, $\ell(u_1vu_2)$ is counter-regular. The cyclic path pu_1vu_2 is the contour of a disc subdiagram of Δ containing the face Π_1 but not containing the face Π_2. This contradicts the minimality of Δ (see Lemma 7.4).

Case 4: u_1 is a subpath of $s_{1+}'^{-1}$, and u_2 is a subpath of $s_{1-}'^{-1}$. Since the labels of s'_{1-} and s'_{1+} have disjoint sets of basic letters, $\ell(s_2)$ is regular. Let p be the path such that $u_2^{-1}p^{-1}u_1^{-1}$ is a subpath of s_1. Then $\ell(u_1vu_2)$ and $\ell(u_1pu_2)$ are regular. If v is trivial, then the cyclic path $pv^{-1} = p$ is cyclically reduced because $\ell(p)$ is regular. If v is nontrivial, then the cyclic path pv^{-1} is cyclically reduced by the maximality of u_1 and u_2, and because the pathes v and p are reduced. Suppose the path pv^{-1} is not simple. Then some subpath of some cyclic shift of vp^{-1} is the contour of a proper simple disc subdiagram of Δ. Let Δ' be such a subdiagram. At least one of the following three subcases takes place:

(a) there is a subpath of v or p^{-1} which represents $\bar{\partial}_1\Delta'$, or

(b) there are paths v' and p' such that $\langle v'p'^{-1}\rangle = \bar{\partial}_1\Delta'$, and the paths v' and p' are either terminal subpaths of v and p, respectively, or initial subpaths of v and p, respectively, or

(c) there are an initial subpath p'_- and a terminal subpath p'_+ of the path p such that $\langle vp'^{-1}_+p'^{-1}_-\rangle = \bar{\partial}_1\Delta'$.

In subcase (c) note that $\ell(p'_-p'_+)$ is regular. Lemma 7.4 easily yields a contradiction in each subcase. Hence, the cyclic path pv^{-1} is simple. Therefore, it is the contour of a simple disc subdiagram of Δ containing the faces Π_1 and Π_2. By Lemma 7.4, $\langle pv^{-1}\rangle = \bar{\partial}_1\Delta$. Therefore, $v = q_1^{-1}$ and u_1vu_2 is a subpath of $(s'_{1-}q_1s'_{1+})^{-1}$. This gives a contradiction.

Case 5: u_1 is a subpath of s'^{-1}_{1-}, and u_2 is a subpath of y_-^{-1}. Let p be the path such that $u_2^{-1}pu_1$ is a subpath of $y_-s'^{-1}_1s'^{-1}_{1-}$. Then $\ell(u_2^{-1}pu_1)$ is regular. Therefore, $\ell(u_1vu_2)$ is counter-regular. The cyclic path pu_1v is the contour of a disc subdiagram of Δ containing the face Π_1 but not containing the face Π_2. This contradicts the minimality of Δ (see Lemma 7.4).

Case 6: u_1 is a subpath of y_+^{-1}, and u_2 is a subpath of s'^{-1}_{1+}. Let p be the path such that $u_2pu_1^{-1}$ is a subpath of $s'^{-1}_{1+}s'^{-1}_1y_+$. Then $\ell(u_2pu_1^{-1})$ is regular. Therefore, $\ell(u_1vu_2)$ is counter-regular. The cyclic path pvu_2 is the contour of a disc subdiagram of Δ containing the face Π_1 but not containing the face Π_2. This contradicts the minimality of Δ.

Case 7: u_1 is a subpath of y_-^{-1}, and u_2 is a subpath of s'^{-1}_{1-}. Let p be the path such that $u_1^{-1}pu_2$ is a subpath of $y_-s'^{-1}_1s'^{-1}_{1-}$. The cyclic path $pv^{-1}u_1^{-1}$ is cyclically reduced. Suppose it is not simple. Then at least one of the following two subcases takes place:

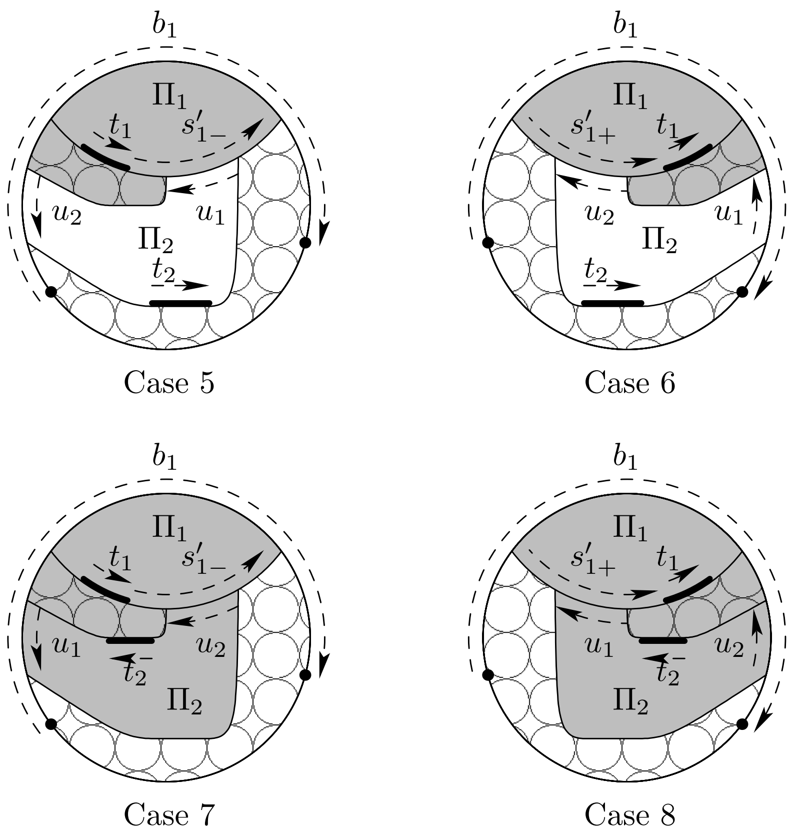


Figure 6: Cases 5–8, Lemma 7.10.

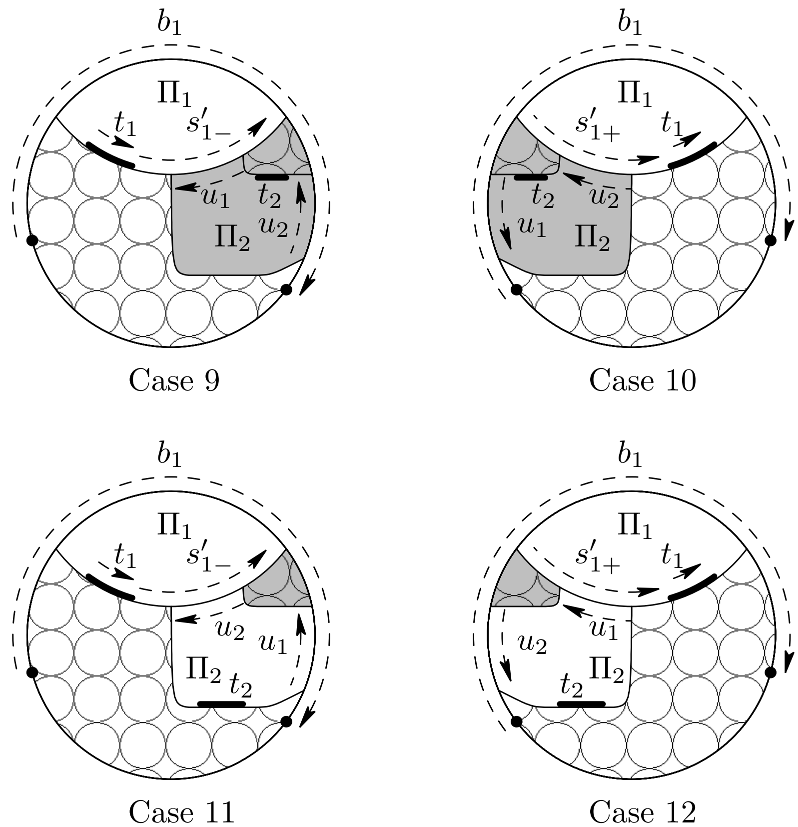


Figure 7: Cases 9–12, Lemma 7.10.

(a) there is a subpath of v or p^{-1} which is the contour of a proper simple disc subdiagram of Δ, or

(b) there are paths v' and p' such that $v'p'^{-1}$ is the contour of a proper simple disc subdiagram of Δ, and v' and p' are terminal subpaths of v and p, respectively.

Lemma 7.4 yields a contradiction in both subcases. Hence, the cyclic path $pv^{-1}u_1^{-1}$ is simple. Therefore, it is the contour of a disc subdiagram of Δ containing the faces Π_1 and Π_2. The label of $u_1^{-1}pu_2$ is regular. Therefore, $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is counter-regular. By Lemma 7.4, $\langle pv^{-1}u_1^{-1}\rangle = \bar{\partial}_1\Delta$. Therefore, u_1vu_2 is a subpath of $(s'_{1-}q_1)^{-1}$. This gives a contradiction.

Case 8: u_1 is a subpath of s'^{-1}_{1+}, and u_2 is a subpath of y_+^{-1}. Arguing as in Case 7, obtain that u_1vu_2 is a subpath of $(q_1s'_{1+})^{-1}$. This gives a contradiction.

Case 9: u_1 is a subpath of s'^{-1}_{1-}, and u_2 is a subpath of y^{-1}_{+}. Let p_1 and p_2 be the paths such that $u_1^{-1}p_1$ is a terminal subpath of s'_{1-}, and $p_2u_2^{-1}$ is an initial subpath of y_+. Then $\ell(u_1^{-1}p_1)$ is counter-regular, $\ell(p_2u_2^{-1})$ is regular, and $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular or counter-regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular, then $\ell(p_2u_2^{-1}v^{-1}u_1^{-1})$ is regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is counter-regular, then $\ell(u_2^{-1}v^{-1}u_1^{-1}p_1)$ is counter-regular. The cyclic path $p_1p_2u_2^{-1}v^{-1}u_1^{-1}$ is the contour of a disc subdiagram of Δ containing the face Π_2 but not containing the face Π_1. This contradicts the minimality of Δ.

Case 10: u_1 is a subpath of y^{-1}_{-}, and u_2 is a subpath of s'^{-1}_{1+}. Contradiction is obtained as in Case 9.

Case 11: u_1 is a subpath of y^{-1}_{+}, and u_2 is a subpath of s'^{-1}_{1-}. Let p_1 and p_2 be the paths such that $u_2^{-1}p_1$ is a terminal subpath of s'_{1-}, and $p_2u_1^{-1}$ is an initial subpath of y_+. Then $\ell(u_2^{-1}p_1)$ is counter-regular, $\ell(p_2u_1^{-1})$ is regular, and $\ell(u_1vu_2)$ is regular or counter-regular. If $\ell(u_1vu_2)$ is regular, then the reduced form of $\ell(p_2v)$ is regular. If $\ell(u_1vu_2)$ is counter-regular, then the reduced form of $\ell(vp_1)$ is counter-regular. The cyclic path p_1p_2v is the contour of a disc subdiagram of Δ not containing the faces Π_1 and Π_2. By Lemma 7.4, p_1p_2v is the contour of a degenerate disc subdiagram. Since $s'_{1-}y_+$ and v are reduced, $v = (p_1p_2)^{-1}$. Therefore, u_1vu_2 is a subpath of $(s'_{1-}y_+)^{-1}$. This gives a contradiction.

Case 12: u_1 is a subpath of s'^{-1}_{1+}, and u_2 is a subpath of y^{-1}_{-}. Arguing as in Case 11, obtain that u_1vu_2 is a subpath of $(y_-s'_{1+})^{-1}$. This gives a contradiction.

Case 13: u_1 is a subpath of s'^{-1}_{1-}, and u_2 is a subpath of b'^{-1}_{2}. Let p_1 and p_2 be the paths such that p_1u_1 is an initial subpath of $y_-s'^{-1}_1s'^{-1}_{1-}$, and $u_2^{-1}p_2$ is a terminal subpath of b'_2. Then $\ell(p_1u_1)$ is regular, $\ell(u_2^{-1}p_2)$ is counter-regular, and $\ell(u_1vu_2)$ is regular or counter-regular. If $\ell(u_1vu_2)$ is regular, then $\ell(p_1u_1v)$ is regular. If $\ell(u_1vu_2)$ is counter-regular, then the reduced form of $\ell(u_1vp_2)$ is counter-regular. The cyclic path $p_2p_1u_1v$ is the contour of a disc subdiagram of Δ containing the face Π_1 but not containing the face Π_2. This contradicts the minimality of Δ.

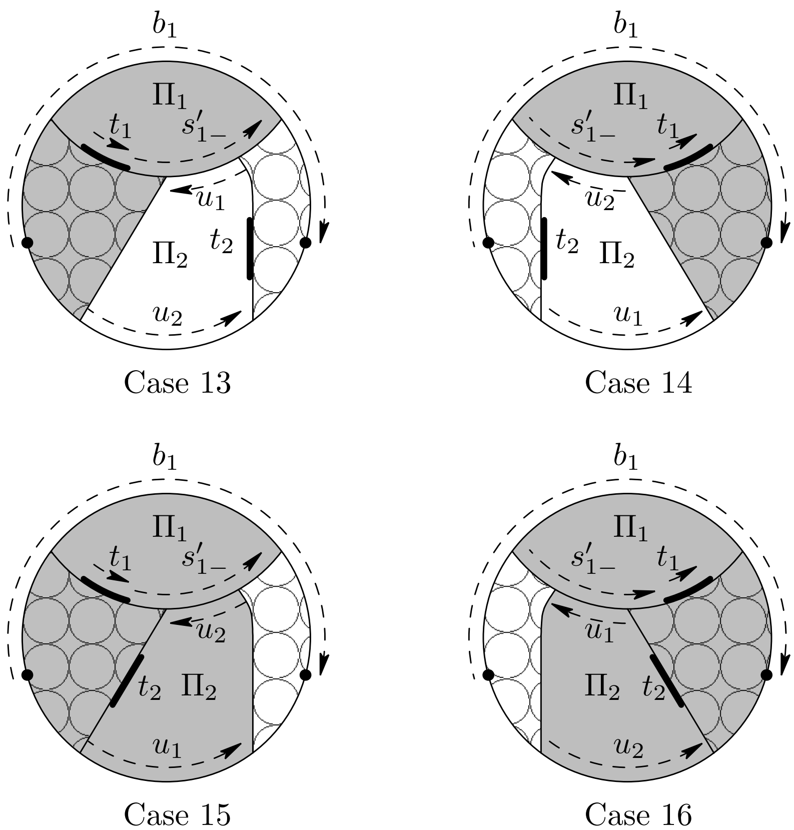


Figure 8: Cases 13–16, Lemma 7.10.

Case 14: u_1 is a subpath of b_2^{-1}, and u_2 is a subpath of $s_{1+}'^{-1}$. Contradiction is obtained as in Case 13.

Case 15: u_1 is a subpath of $b_2'^{-1}$, and u_2 is a subpath of $s_{1-}'^{-1}$. Let p_1 and p_2 be the paths such that $u_1^{-1}p_1$ is a terminal subpath of b_2', and p_2u_2 is an initial subpath of $y_-s_1'^{-1}s_{1-}'^{-1}$. The cyclic path $p_1p_2v^{-1}u_1^{-1}$ is cyclically reduced. Suppose it is not simple. Then at least one of the following two subcases takes place:

(a) there is a subpath of v or p_1^{-1} which is the contour of a proper simple disc subdiagram of Δ, or

(b) there are paths v' and p_2' such that $v'p_2'^{-1}$ is the contour of a proper simple disc subdiagram of Δ, and v' and p_2' are terminal subpaths of v and p_2, respectively.

Lemma 7.4 yields a contradiction in both subcases. Hence, $p_1p_2v^{-1}u_1^{-1}$ is a simple path. Therefore, it is the contour of a disc subdiagram of Δ containing the faces Π_1 and Π_2. The label of $u_1^{-1}p_1$ is counter-regular, the label of p_2u_2 is regular, and the label of $u_2^{-1}v^{-1}u_1^{-1}$ is regular or counter-regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is regular, then the reduced form of $\ell(p_2v^{-1}u_1^{-1})$ is regular. If $\ell(u_2^{-1}v^{-1}u_1^{-1})$ is counter-regular, then $\ell(v^{-1}u_1^{-1}p_1)$ is counter-regular. By Lemma 7.4, $\langle p_1p_2v^{-1}u_1^{-1}\rangle = \bar{\partial}_1\Delta$. Therefore, u_1vu_2 is a subpath of $(s_{1-}'q_1)^{-1}$. This gives a contradiction.

Case 16: u_1 is a subpath of $s_{1+}'^{-1}$, and u_2 is a subpath of $b_2'^{-1}$. Arguing as in Case 15, obtain that u_1vu_2 is a subpath of $(q_1s_{1+}')^{-1}$. This gives a contradiction.

A contradiction is obtained in each of the considered cases, and no other case is possible. Thus, s_2 and $(s_{1-}'q_1s_{1+}')^{-1}$ have exactly one maximal common nontrivial subpath. □

Lemma 7.11. *If Π is a face of Δ, the rank of every other face of Δ is less than the rank of Π, s is the maximal selected subpath of $\bar{\partial}\Pi$, and e_1 and e_n are, respectively, the initial and the terminal subpaths of s of length $|s|/n$, then all oriented edges of e_1 or all oriented edges of e_n are internal in Δ.*

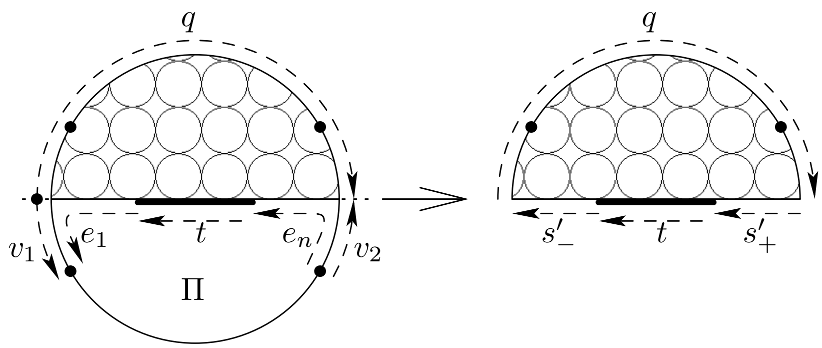


Figure 9: The face Π in Δ and the subdiagram Δ', Lemma 7.11.

Proof. Observe that proving that all oriented edges of e_1 or all oriented edges of e_n are internal in Δ is equivalent to proving that all oriented edges of e_n^{-1} or all oriented edges of e_1^{-1} are internal in the mirror copy of Δ.

Suppose Π, s, e_1, e_n are such as in the hypotheses of the lemma. Assume that $\ell(s)$ is regular (if it is not, pass from Δ, s, e_1, e_n to the mirror copy of Δ, s^{-1}, e_n^{-1}, e_1^{-1}, respectively).

Suppose that some oriented edge of e_1 and some oriented edge of e_n are external in Δ (see Fig. 9).

Let s' be the maximal selected external oriented arc of Δ that is a subpath of s. Let s'_- and s'_+ be such paths that $s = s'_- s' s'_+$. Then s'_- is a proper initial subpath of e_1, and s'_+ is a proper terminal subpath of e_n. Let v_1 and v_2 be the initial and the terminal subpaths of s' such that $s'_- v_1 = e_1$ and $v_2 s'_+ = e_n$. Let t be the path such that $\langle st \rangle = \bar{\partial}\Pi$. Let q be the path such that $\langle q s'^{-1} \rangle = \bar{\partial}_1 \Delta$. Let Δ' be a disc subdiagram of Δ obtained by removing the face Π and all edges that lie on s' together with all intermediate vertices of s'. The contour cycle of Δ' is $\langle s'_+ t s'_- q \rangle$.

Let $j = \operatorname{rank}(\Pi)$. Then $\ell(e_1) = x_1^{m_j}$ and $\ell(e_n) = x_n^{m_j}$. The label of the first oriented edge of s' is x_1; the label of the last oriented edge of s' is x_n. Therefore, s' is a subpath of b_2^{-1}. Note that v_1^{-1} is an initial subpath of l_1, v_2^{-1} is a terminal subpath of l_2, and $v_1^{-1} q v_2^{-1} = b_1$. The reduced form of $\ell(s'_- q s'_+)$ is regular since $\ell(q)$

is regular, $\ell(s'_-)$ is a power of x_1, and $\ell(s'_+)$ is a power of x_n. As it has been assumed, $\operatorname{rank}(\Pi') < j$ for every face Π' of Δ'. Hence, the group word $w_j = \ell(t^{-1})$ equals a regular word (the reduced form of $\ell(s'_- q s'_+)$) modulo the relations $r = 1$, $r \in \mathcal{R}_{j-1}$. This contradicts the choice of w_i. $\square$

Lemma 7.12. *There are two distinct faces* Π_1 *and* Π_2, *and paths* s_1, s'_1, s'_{1-}, s'_{1+}, s_2, s'_2, s'_{2-}, s'_{2+}, z_1, z_2 *in* Δ *such that*:

- s_1 *and* s_2 *are the maximal selected subpaths of* $\bar{\partial}\Pi_1$ *and* $\bar{\partial}\Pi_2$, *respectively*;
- $\ell(s_1)$ *is counter-regular*, $\ell(s_2)$ *is regular*;
- $s_i = s'_{i-} s'_i s'_{i+}$ *for both* $i = 1$ *and* $i = 2$;
- *both* s'_1 *and* s'_2 *are maximal selected external oriented arcs of* Δ;
- $\langle z_1 s'^{-1}_1 z_2 s'^{-1}_2 \rangle = \bar{\partial}_1\Delta$;
- *all basic letters of* $\ell(s'_{2-} z_1 s'_{1+})$ *are in* $\{x_1, \dots, x_{21}\}$, *and all basic letters of* $\ell(s'_{1-} z_2 s'_{2+})$ *are in* $\{x_{n-20}, \dots, x_n\}$.

Proof. Observe that this lemma is equivalent to the analogous lemma about the mirror copy of Δ, i.e., to the statement obtained from this lemma by substituting "the mirror copy of Δ" for "Δ."

Let Π_1 be a face of Δ such as in Lemma 7.9. Let s_1 be the maximal selected subpath of $\bar{\partial}\Pi_1$. If $\ell(s_1)$ is regular, pass from Δ and s_1 to the mirror copy of Δ and s_1^{-1}, respectively. Hence, $\ell(s_1)$ shall be assumed to be counter-regular. Let s'_1 be the maximal selected external oriented arc of Δ that is a subpath of s_1. Let s'_{1-} and s'_{1+} be the paths such that $s_1 = s'_{1-} s'_1 s'_{1+}$. Since all of the basic letters x_{21}, x_{22}, $\dots$, x_{n-20} occur in $\ell(s'_1)$, it follows that all basic letters of $\ell(s'_{1-})$ are in $\{x_{n-20}, \dots, x_n\}$, and all basic letters of $\ell(s'_{1+})$ are in $\{x_1, \dots, x_{21}\}$.

Let t_1 be the path such that $\langle s_1 t_1 \rangle = \bar{\partial}\Pi_1$. Let q_1 be the path such that $\langle s'^{-1}_1 q_1 \rangle = \bar{\partial}_1\Delta$. Let Δ' be a disc S-subdiagram of Δ obtained by removing the face Π_1 and all edges that lie on s_1' together with all intermediate vertices of s_1'. The contour cycle of Δ' is $\langle s_{1+}' t_1 s_{1-}' q_1 \rangle$.

It shall be shown that there is a face Π of Δ' such that Π is incident to no fewer than $(1 - 21/n)|\partial\Pi|$ selected external edges of Δ' that do not lie on t_1. Consider 2 cases:

Case 1: there is a face Π in Δ' such that $\mathrm{rank}(\Pi) \geq \mathrm{rank}(\Pi_1)$. Then let Φ be the maximal simple disc S-subdiagram of Δ' that contains such a face Π. Let Σ be the sum of the degrees of all the faces of Φ. Then $\Sigma \geq |\partial\Pi| \geq |\partial\Pi_1|$. By Lemma 7.3, the number of selected external edges of Φ is at least $(1 - 2\gamma)\Sigma$. Therefore, the number of selected external edges of Φ that do not lie on t_1 is at least

$$(1 - 2\gamma)\Sigma - \lambda_1|\partial\Pi_1| \geq (1 - 2\gamma - \lambda_1)\Sigma > \frac{n - 21}{n}\Sigma.$$

Case 2: the rank of every face of Δ' is less than the rank of Π_1. Let e_{11} and e_{1n} be respectively the initial and terminal subpaths of s_1 of length $|s_1|/n$. Then $\ell(e_{11}) = x_n^{-m_{j_1}}$, and $\ell(e_{1n}) = x_1^{-m_{j_1}}$. Since the rank of Π_1 is greater than the rank of every other face of Δ, all oriented edges of e_{11} or all oriented edges of e_{1n} are internal in Δ by Lemma 7.11. Let e denote e_{11} in the first case or e_{1n} in the second.

If an edge lies on e and is not incident to any face of Δ', then it also lies on t_1. Indeed, both (mutually inverse) oriented edges corresponding to such an edge must occur in $\partial\Pi_1$; one of them is an oriented edge of e and consequently the other cannot be an oriented edge of s_1 (two mutually inverse group letters cannot both occur in $\ell(s_1)$) and has to be an oriented edge of t_1. Since

$$|t_1| = |w_{j_1}| < m_{j_1} = |e|$$

(see (11)), the ratio of the number of edges that lie on t_1 and are incident with faces of Δ' to the number of edges that lie on e but are also incident with faces of Δ' is not greater than

$$\frac{|t_1|}{|e|} = \frac{|w_{j_1}|}{m_{j_1}} \leq \frac{n\lambda_1}{1-\lambda_1}$$

(see (10)). Let Φ be a maximal simple disc S-subdiagram of Δ' such that the ratio of the number of its (external) edges that lie on t_1 to the number of its (external) edges that lie on e is not greater than

$$\frac{n\lambda_1}{1-\lambda_1}.$$

Let Σ be the sum of the degrees of all the faces of Φ. Since all edges that lie on e are labelled with a same basic letter, the number of edges of Φ that lie on e is less than

$$\Big(\lambda_1 + \frac{1-\lambda_1}{n}\Big)\Sigma$$

(because the number of edges with a same label incident to a given face Π of Δ is less than $(\lambda_1 + (1-\lambda_1)/n)|\partial\Pi|$). By Lemma 7.3, the number of selected external edges of Φ is at least $(1-2\gamma)\Sigma$. Therefore, the number of selected external edges of Φ that do not lie on t_1 is greater than

$$\Big(1 - 2\gamma - \frac{n\lambda_1}{1-\lambda_1}\Big(\lambda_1 + \frac{1-\lambda_1}{n}\Big)\Big)\Sigma = \Big(1 - 2\gamma - \lambda_1 - \frac{n\lambda_1^2}{1-\lambda_1}\Big)\Sigma > \frac{n-21}{n}\Sigma$$

(see (8)).

In the both cases, there is a face Π in Φ such that Π is incident to at least $(1-21/n)|\partial\Pi|$ selected external edges of Φ (of Δ') that do not lie on t_1. Let Π_2 be such a face. Let s_2 be the maximal selected subpath of $\bar{\partial}\Pi_2$. Then the paths s_2 and $(s'_{1-}q_1s'_{1+})^{-1}$ have at least $(1-21/n)|\partial\Pi_2|$ common oriented edges. Let $\tilde{s}_2$ be the maximal common nontrivial subpath of s_2 and $(s'_{1-}q_1s'_{1+})^{-1}$ (see Lemma 7.10). Note that $\tilde{s}_2$ is a selected external oriented arc of Δ'. Every selected external edge of Δ'

incident to Π_2 lies either on t_1 or on $\tilde{s}_2$. Therefore,

$$|\tilde{s}_2| \geq \frac{n-21}{n}|\partial\Pi_2| > \frac{n-21}{n}|s_2|.$$

Therefore, $\ell(\tilde{s}_2)$ has at least $n-20$ distinct basic letters, which implies that each of the basic letters x_{21}, …, x_{n-20} occurs in it. The basic letters x_{22}, …, x_{n-21} do not occur on s'_{1-} nor on s'_{1+}, but they occur in $\ell(\tilde{s}_2)$. Hence, the paths $\tilde{s}_2$ and q_1^{-1} have common oriented edges. Therefore, Π_2 is incident to some selected external edges of Δ.

Let s'_2 be the maximal selected external oriented arc of Δ that is a subpath of s_2. Then s'_2 is the maximal common subpath of $\tilde{s}_2$ and q_1^{-1}. All of the basic letters x_{22}, …, x_{n-21} occur in $\ell(s'_2)$ (since they occur in $\ell(\tilde{s}_2)$ but neither in $\ell(s'_{1-})$ nor in $\ell(s'_{1+})$).

Since $\ell(s_1)$ is counter-regular, s'_1 is a subpath of b_1^{-1}. Therefore, s'_2 is a subpath of b_2^{-1}, and $\ell(s_2)$ is regular.

Let s'_{2-} and s'_{2+} be the paths such that $s_2 = s'_{2-}s'_2s'_{2+}$. Let z_1 and z_2 be the paths such that $\langle z_1 s_1'^{-1} z_2 s_2'^{-1}\rangle = \bar{\partial}_1\Delta$. It shall be proved that all basic letters of $\ell(s'_{2-})$ and $\ell(z_1)$ are among x_1, …, x_{21}, and all basic letters of $\ell(s'_{2+})$ and $\ell(z_2)$ are among x_{n-20}, …, x_n.

First, consider s'_{2-} and z_1.

Case 1: $\tilde{s}_2$ has no common oriented edges with $s_{1+}'^{-1}$. Then s'_2 is an initial subpath of $\tilde{s}_2$. Therefore, the basic letter x_{21}, along with x_{22}, …, x_{n-21}, occurs in $\ell(s'_2)$. Hence, all basic letters of $\ell(s'_{2-})$ are in $\{x_1, \ldots, x_{21}\}$. The label of every edge that lies both on z_1 and b_1 is in $\{x_1, \ldots, x_{21}\}$ since the basic letter x_{21} occurs in $\ell(s'_1)$, and the only maximal common subpath of $z_1 s_1'^{-1}$ and b_1 is an initial subpath of b_1. The label of every edge that lies both on z_1 and b_2 is in $\{x_1, \ldots, x_{21}\}$ since the basic letter x_{21} occurs in $\ell(s'_2)$, and the only maximal common subpath of $s_2'^{-1} z_1$ and b_2 is a terminal subpath of b_2.

Case 2: $\tilde{s}_2$ has at least one oriented edge in common with $s_{1+}'^{-1}$. Then some nontrivial terminal subpath of $s_{1+}'^{-1}$ is an initial subpath of $\tilde{s}_2$, and z_1 is trivial. The terminal subpath of $s_{1+}'^{-1}$ that is an initial subpath of $\tilde{s}_2$ is also a terminal subpath of s_{2-}'. Since $s_{1+}'^{-1}$ and s_{2-}' have a common nontrivial terminal subpath, the sets of basic letters of their labels coincide. Therefore, all basic letters of $\ell(s_{2-}')$ are in $\{x_1, \dots, x_{21}\}$. The label of z_1 is empty.

Second, consider s_{2+}' and z_2 in the same manner.

Case 1: $\tilde{s}_2$ has no common oriented edges with $s_{1-}'^{-1}$. Then s_2' is a terminal subpath of $\tilde{s}_2$. Therefore, the basic letter x_{n-20}, along with x_{22}, $\dots$, x_{n-21}, occurs in $\ell(s_2')$. Hence, all basic letters of $\ell(s_{2+}')$ are in $\{x_{n-20}, \dots, x_n\}$. The label of every edge that lies both on z_2 and b_1 is in $\{x_{n-20}, \dots, x_n\}$ since the basic letter x_{n-20} occurs in $\ell(s_1')$, and the only maximal common subpath of $s_1'^{-1} z_2$ and b_1 is a terminal subpath of b_1. The label of every edge that lies both on z_2 and b_2 is in $\{x_{n-20}, \dots, x_n\}$ since the basic letter x_{n-20} occurs in $\ell(s_2')$, and the only maximal common subpath of $z_2 s_2'^{-1}$ and b_2 is an initial subpath of b_2.

Case 2: $\tilde{s}_2$ has at least one oriented edge in common with $s_{1-}'^{-1}$. Then some nontrivial initial subpath of $s_{1-}'^{-1}$ is a terminal subpath of $\tilde{s}_2$, and z_2 is trivial. The initial subpath of $s_{1-}'^{-1}$ that is a terminal subpath of $\tilde{s}_2$ is also an initial subpath of s_{2+}'. Since $s_{1-}'^{-1}$ and s_{2+}' have a common nontrivial initial subpath, the sets of basic letters of their labels coincide. Therefore, all basic letters of $\ell(s_{2+}')$ are in $\{x_{n-20}, \dots, x_n\}$. The label of z_2 is empty.

Clearly, the faces Π_1, Π_2 and the paths s_1, s_1', s_{1-}', s_{1+}', s_2, s_2', s_{2-}', s_{2+}', z_1, z_2 are desired ones. $\square$

Lemma 7.13. *The diagram Δ has more than 2 faces.*

Proof. Note that Δ obviously has the same number of faces as its mirror copy.

Suppose the statement is not true, i.e., Δ has no more than 2 faces. (Then, by Lemma 7.5, it has exactly 2 faces.)

Let Π_1 and Π_2 be such faces of Δ as in Lemma 7.12. Then the diagram Δ consists of the faces Π_1 and Π_2 attached to each other along a common arc. If $\operatorname{rank}(\Pi_2) > \operatorname{rank}(\Pi_1)$, pass to the mirror copy of Δ and interchange the roles of Π_1 and Π_2. Hence, it shall be assumed that $\operatorname{rank}(\Pi_1) \geq \operatorname{rank}(\Pi_2)$.

Let s_1, s'_1, s'_{1-}, s'_{1+}, s_2, s'_2, s'_{2-}, s'_{2+} be the subpaths of $\bar{\partial}\Pi_1$ and $\bar{\partial}\Pi_2$ such as in Lemma 7.12. Then all basic letters of the labels of s'_{1-}, s'_{1+}, s'_{2-}, s'_{2+} are in $\{x_1, \ldots, x_{21}\} \cup \{x_{n-20}, \ldots, x_n\}$, and s'_i is a subpath of b_i^{-1} for $i = 1$ and $i = 2$.

There are exactly two (mutually inverse) maximal internal oriented arcs in Δ. Denote the one that is a subpath of $\bar{\partial}\Pi_1$ by u.

Let t_1 and t_2 be the path such that $\langle s_1 t_1 \rangle = \bar{\partial}\Pi_1$ and $\langle s_2 t_2 \rangle = \bar{\partial}\Pi_2$. Each of the faces Π_1 and Π_2 is incident to exactly one maximal selected external arc of Δ. Therefore, each of the paths t_1 and t_2 has a common vertex with u.

Suppose t_1 and t_2 do not have common vertices. Then either some (proper) nontrivial initial subpath of s_1 is inverse to a (proper) initial subpath of s_2, or some (proper) nontrivial terminal subpath of s_1 is inverse to a (proper) terminal subpath of s_2. In the both cases, have a contradiction since no proper nontrivial prefix of $\ell(s_2)$ can be a suffix of $\ell(s_1^{-1})$, and no proper nontrivial suffix of $\ell(s_2)$ can be a prefix of $\ell(s_1^{-1})$. (A contradiction may also be obtained by applying Lemma 7.10.) Hence, the paths t_1 and t_2 have at least one common vertex.

Let $j_1 = \operatorname{rank}(\Pi_1)$ and $j_2 = \operatorname{rank}(\Pi_2)$. Let e_{i1} and e_{in} be respectively the initial and the terminal subpaths of s_i of length $m_{j_i} = |s_i|/n$ for $i = 1$ and $i = 2$. Then $\ell(e_{11}) = x_n^{-m_{j_1}}$, $\ell(e_{1n}) = x_1^{-m_{j_1}}$, $\ell(e_{21}) = x_1^{m_{j_2}}$, and $\ell(e_{2n}) = x_n^{m_{j_2}}$. At most one of the two group letters $x_1^{\pm 1}$ and at most one of $x_n^{\pm 1}$ can occur as labels of oriented edges of $\partial_1\Delta$. Therefore, at least one of the paths e_{1n} and e_{21} and at least one of the paths e_{11} and e_{2n} do not have external oriented edges in Δ.

Suppose $\operatorname{rank}(\Pi_1) = \operatorname{rank}(\Pi_2)$. Then

$$|e_{11}| = |e_{1n}| = |e_{21}| = |e_{2n}| = m_{j_1} \geq |w_{j_1}| = |t_1| = |t_2|.$$

Recall that w_{j_1} does not start with $x_1^{\pm 1}$, does not end with $x_n^{\pm 1}$, and is not a letter power. Therefore, every subpath of either $\bar{\partial}\Pi_1$ or $\bar{\partial}\Pi_2$ labelled with a letter power has the length of at most m_{j_1}. Moreover, for every i, there exist exactly one subpath of $\bar{\partial}\Pi_1$ and exactly one subpath of $\bar{\partial}\Pi_2$ of length m_{j_1} labelled with powers of x_i (i.e., with $x_i^{\pm m_{j_1}}$). If e_{1n} has no external oriented edges in Δ, then it is a subpath of u. If e_{21} has no external oriented edges in Δ, then it is a subpath of u^{-1}. In either case u has a subpath labelled with $x_1^{-m_{j_1}}$, and the pair $\{\Pi_1, \Pi_2\}$ is immediately cancellable. This contradicts Lemma 7.1. Hence, $\operatorname{rank}(\Pi_1) > \operatorname{rank}(\Pi_2)$.

Let Δ' be a disc S-subdiagram of Δ obtained by removing the face Π_1 and all edges and intermediate vertices of s'_1. The only face of Δ' is Π_2.

By Lemma 7.11, all oriented edges of e_{11} or all oriented edges of e_{1n} are internal in Δ. Consider the following 5 cases (see Fig. 10):

(Notice that Case 3 is "symmetric" with Case 2, and Case 5—with Case 4.)

Case 1: each of the paths e_{11}, e_{1n}, e_{21}, and e_{2n} has an internal oriented edge. Then the paths s'_{1-} and s'^{-1}_{2+} have a common nontrivial terminal subpath, the paths s'_{1+} and s'^{-1}_{2-} have a common nontrivial initial subpath, and $\bar{\partial}_1\Delta = \langle s'^{-1}_1 s'^{-1}_2 \rangle$. Let v_1 be a common nontrivial terminal subpath of s'_{1-} and s'^{-1}_{2+}, and v_2 be a common nontrivial initial subpath of s'_{1+} and s'^{-1}_{2-}. The contour cycle of Δ' is $\langle s'_{1+} t_1 s'_{1-} s'^{-1}_2 \rangle$. The label of $s'_{1-} s'^{-1}_2 s'_{1+}$ is counter-regular because the labels of s'_{1-}, $v_1 s'^{-1}_2 v_2$, and s'_{1+} are such. Hence, the group word $w_{j_1} = \ell(t_1)$ equals the regular word $\ell(s'_{1-} s'^{-1}_2 s'_{1+})^{-1}$ modulo the relation $r_{j_2} = 1$. This contradicts the choice of w_{j_1}.

Case 2: all oriented edges of e_{1n} are external (consequently, all oriented edges of e_{11} and e_{21} are internal), and some oriented edge of e_{2n} is internal. Then the paths s'_{1-}

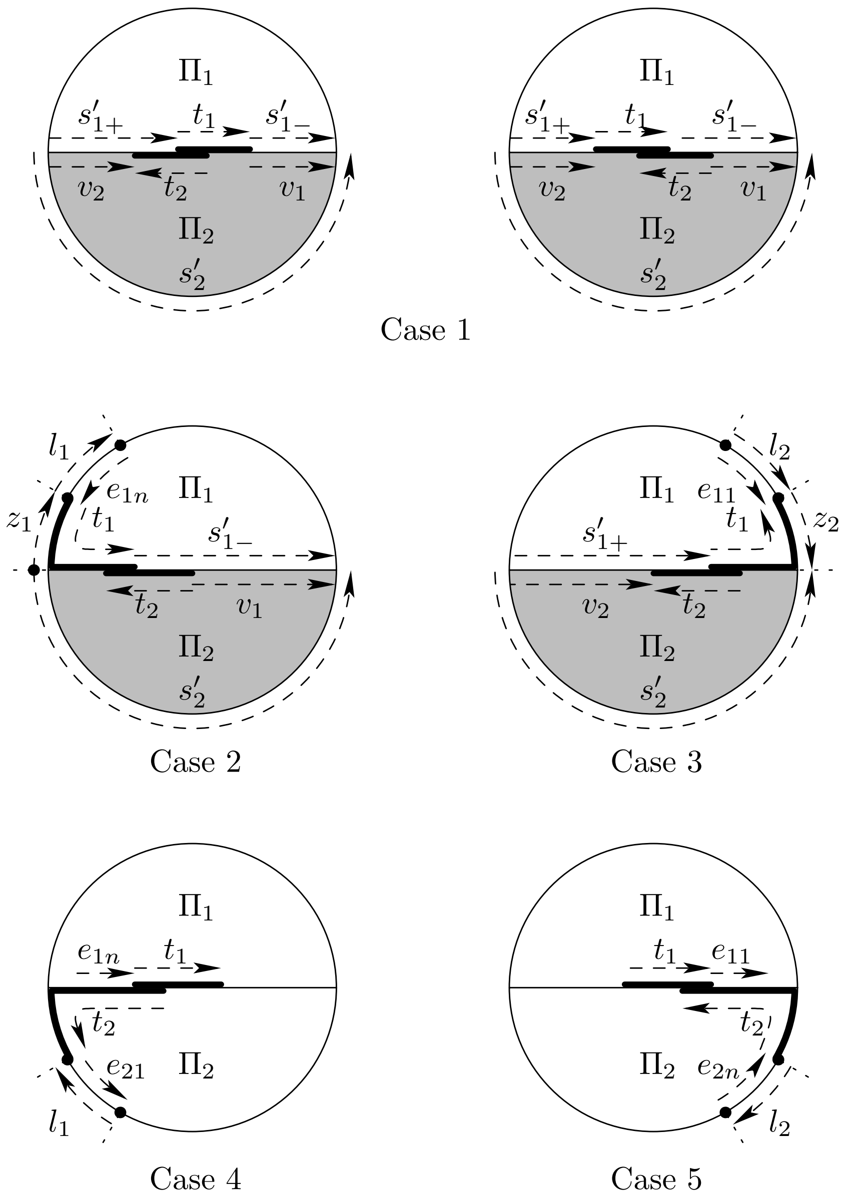


Figure 10: Cases 1–5, Lemma 7.13.

and $s_{2+}'^{-1}$ have a common nontrivial terminal subpath. Let v_1 be a common nontrivial terminal subpath of s_{1-}' and $s_{2+}'^{-1}$. Let z_1 be the path such that $\langle z_1 s_1'^{-1} s_2'^{-1} \rangle = \bar{\partial}_1 \Delta$. Then z_1^{-1} is an initial subpath of t_1. Since $\ell(e_{1n})$ is a power of x_1, the path e_{1n}^{-1} is a (terminal) subpath of b_2. Therefore, $s_2'^{-1} z_1 e_{1n}^{-1}$ is a subpath of b_2. The contour cycle of Δ' is $\langle z_1 t_1 s_{1-}' s_2'^{-1} \rangle$. The label of $s_{1-}' s_2'^{-1} z_1$ is counter-regular because the labels of s_{1-}', $v_1 s_2'^{-1}$, and $s_2'^{-1} z_1$ are such. Hence, the group word $w_{j_1} = \ell(t_1)$ equals the regular word $\ell(s_{1-}' s_2'^{-1} z_1)^{-1}$ modulo the relation $r_{j_2} = 1$. This contradicts the choice of w_{j_1}.

Case 3: all oriented edges of e_{11} are external (consequently, all oriented edges of e_{1n} and e_{2n} are internal), and some oriented edge of e_{21} is internal. Then the paths s_{1+}' and $s_{2-}'^{-1}$ have a common nontrivial initial subpath. Let v_2 be a common nontrivial initial subpath of s_{1+}' and $s_{2-}'^{-1}$. Let z_2 be the path such that $\langle s_1'^{-1} z_2 s_2'^{-1} \rangle = \bar{\partial}_1 \Delta$. Then z_2^{-1} is a terminal subpath of t_1. Since $\ell(e_{11})$ is a power of x_n, the path e_{11}^{-1} is an (initial) subpath of b_2. Therefore, $e_{11}^{-1} z_2 s_2'^{-1}$ is a subpath of b_2. The contour cycle of Δ' is $\langle s_{1+}' t_1 z_2 s_2'^{-1} \rangle$. The label of $z_2 s_2'^{-1} s_{1+}'$ is counter-regular because the labels of $z_2 s_2'^{-1}$, $s_2'^{-1} v_2$, and s_{1+}' are such. Hence, the group word $w_{j_1} = \ell(t_1)$ equals the regular word $\ell(z_2 s_2'^{-1} s_{1+}')^{-1}$ modulo the relation $r_{j_2} = 1$. This contradicts the choice of w_{j_1}.

Case 4: all oriented edges of e_{21} are external. Consequently, all oriented edges of e_{1n} are internal. Then e_{1n} is a terminal subpath of s_{1+}'. The path $s_{1+}'^{-1}$ is a subpath of t_2 (because t_2 has common vertices with both e_{21} and t_1). Therefore, $|t_2| \geq |s_{1+}'| \geq |e_{1n}| = m_{j_1} > |w_{j_2}| = |t_2|$ (see (11)). This gives a contradiction.

Case 5: all oriented edges of e_{2n} are external. Consequently, all oriented edges of e_{11} are internal. Then e_{11} is an initial subpath of s_{1-}'. The path $s_{1-}'^{-1}$ is a subpath of t_2 (because t_2 has common vertices with both e_{2n} and t_1). Therefore, $|t_2| \geq |s_{1-}'| \geq |e_{11}| = m_{j_1} > |w_{j_2}| = |t_2|$ (see (11)). This gives a contradiction.

No other case is possible. Thus, Δ has more than 2 faces. □

Lemma 7.14. *Let* Π_1 *and* Π_2 *be distinct faces of* Δ *such as in Lemma* 7.12. *Then the ranks of* Π_1 *and* Π_2 *are distinct.*

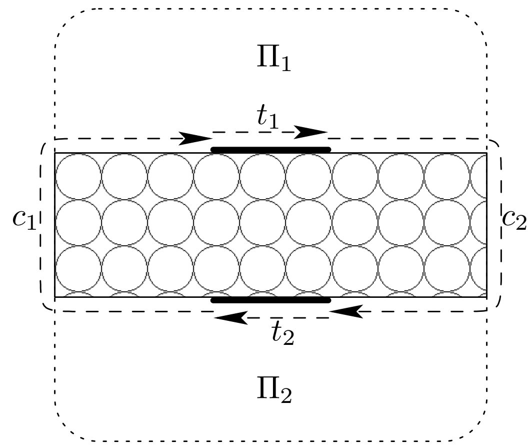


Figure 11: The subdiagram Δ', Lemma 7.14.

Proof. Let Π_1, Π_2, s_1, s'_1, s'_{1-}, s'_{1+}, s_2, s'_2, s'_{2-}, s'_{2+}, z_1, z_2 be such as in Lemma 7.12. Let $c_1 = s'_{2-}z_1s'_{1+}$ and $c_2 = s'_{1-}z_2s'_{2+}$. Let t_1 and t_2 be the paths such that $\langle s_1t_1\rangle = \bar{\partial}\Pi_1$ and $\langle s_2t_2\rangle = \bar{\partial}\Pi_2$. Let Δ' be a disc S-subdiagram of Δ obtained by removing the faces Π_1 and Π_2 and all edges and intermediate vertices of the paths s'_1 and s'_2 (see Fig. 11).

The contour cycle of Δ' is $\langle c_1t_1c_2t_2\rangle$. By Lemma 7.13, the diagram Δ' is non-degenerate. Therefore, by Lemma 7.4, the label of $c_1t_1c_2t_2$ is not freely trivial.

Suppose $\operatorname{rank}(\Pi_1) = \operatorname{rank}(\Pi_2)$. Let $j = \operatorname{rank}(\Pi_1) = \operatorname{rank}(\Pi_2)$. Recall that $\ell(t_1) = w_j$, $\ell(t_2) = w_j^{-1}$, all basic letters of $\ell(c_1)$ are in $\{x_1, \dots, x_{21}\}$, and all basic letters of $\ell(c_2)$ are in $\{x_{n-20}, \dots, x_n\}$. Let $h_1 = \ell(c_1^{-1})$ and $h_2 = \ell(c_2)$. Then $[h_1] = [w_j][h_2][w_j]^{-1}$. Let F be the subgroup of G generated by $\{a_1, \dots, a_{21}\} \cup \{a_{n-20}, \dots, a_n\}$. By Property 1, the group F is free of rank 42 (since $42 \le n - 21$), and elements of F are conjugate in G if and only if they are conjugate in F. The elements $[h_1]$ and $[h_2]$ are in F. Moreover, $[h_1]$ belongs to the subgroup generated by $\{a_1, \dots, a_{21}\}$, and $[h_2]$ belongs to the subgroup generated by $\{a_{n-20}, \dots, a_n\}$. Since $[h_1]$ and $[h_2]$ are conjugate in G, they are conjugate in F and therefore $[h_1] = [h_2] = 1$. Hence, $\ell(c_1t_1c_2t_2)$ is freely trivial. This gives a contradiction. $\square$

Lemma 7.15. *Let Π_1 and Π_2 be distinct faces of Δ both incident to selected external* *edges of Δ. Suppose the rank of Π_1 is greater than the rank of every other face of Δ. Let s_1 and s_2 be the maximal selected subpaths of $\bar{\partial}\Pi_1$ and $\bar{\partial}\Pi_2$, respectively. Suppose the label of one of the paths s_1 or s_2 is regular, and the label of the other is counter-regular. Let s'_1 and s'_2 be the maximal selected external oriented arcs of Δ that are subpaths of s_1 and s_2, respectively. Suppose there are at least 2 distinct basic letters in the label of each of the paths s'_1 and s'_2. Let e_{11} and e_{1n} be respectively the initial and the terminal subpaths of s_1 of length $|s_1|/n$. Then at least one of the paths e_{11} or e_{1n} has the property that every edge that lies on it is internal in Δ and does not lie on s_2.*

Proof. Observe that it suffices to prove that at least one of the paths e_{1n}^{-1} or e_{11}^{-1} has the property that every edge that lies on it is internal in the mirror copy of Δ and does not lie on s_2^{-1}.

Assume that $\ell(s_1)$ is counter-regular (if it is not, pass from Δ, b_1, b_2, l_1, l_2, s_1, s_2, s'_1, s'_2, e_{11}, e_{1n} to the mirror copy of Δ, b_2^{-1}, b_1^{-1}, l_1^{-1}, l_2^{-1}, s_1^{-1}, s_2^{-1}, $s_1'^{-1}$, $s_2'^{-1}$, e_{1n}^{-1}, e_{11}^{-1}, respectively). Then s'_1 is a subpath of b_1^{-1}, and s'_2 is a subpath of b_2^{-1}.

Let s'_{1-} and s'_{1+} be the paths such that $s_1 = s'_{1-}s'_1s'_{1+}$. Let t_1 and t_2 be the paths such that $\langle s_1t_1\rangle = \bar{\partial}\Pi_1$ and $\langle s_2t_2\rangle = \bar{\partial}\Pi_2$. Let q_1 be the path such that $\langle s_1'^{-1}q_1\rangle = \bar{\partial}_1\Delta$. Let Δ' be a disc subdiagram of Δ obtained by removing the face Π_1 and all edges that lie on s'_1 together with all intermediate vertices of s'_1. The contour cycle of Δ' is $\langle s'_{1+}t_1s'_{1-}q_1\rangle$. Let e_{21} and e_{2n} be respectively the initial and terminal subpaths of s_2 of length $|s_2|/n$.

Denote the ranks of Π_1 and Π_2 by j_1 and j_2, respectively. Then

$$\ell(s_1) = x_n^{-m_{j_1}}x_{n-1}^{-m_{j_1}}\dots x_1^{-m_{j_1}}, \qquad \ell(t_1) = w_{j_1},$$

$$\ell(s_2) = x_1^{m_{j_2}}x_2^{m_{j_2}}\dots x_n^{m_{j_2}}, \qquad \ell(t_2) = w_{j_2}^{-1},$$

$$\ell(e_{11}) = x_n^{-m_{j_1}}, \qquad \ell(e_{1n}) = x_1^{-m_{j_1}}, \qquad \ell(e_{21}) = x_1^{m_{j_2}}, \qquad \ell(e_{2n}) = x_n^{m_{j_2}}.$$

At most one of the two group letters $x_1^{\pm 1}$ and at most one of $x_n^{\pm 1}$ can occur as labels of oriented edges of $\partial_1\Delta$ ($\ell(\partial_1\Delta)$ is cyclically reduced). Therefore, at least one of the paths e_{1n} or e_{21} and at least one of the paths e_{11} or e_{2n} do not have external oriented edges in Δ. By Lemma 7.11, all oriented edges of e_{11} or all oriented edges of e_{1n} are internal in Δ.

Suppose the conclusion of the lemma does not hold. Then some oriented edge of e_{11} is either external in Δ or inverse to an oriented edge of e_{2n}, and some oriented edge of e_{1n} is either external in Δ or inverse to an oriented edge of e_{21}. Therefore, as follows from Lemma 7.11, at least one of the following 3 cases takes place:

- some oriented edge of e_{11} is inverse to an oriented edge of e_{2n}, and some oriented edge of e_{1n} is inverse to an oriented edge of e_{21}, or
- some oriented edge of e_{11} is external in Δ, and some oriented edge of e_{1n} is inverse to an oriented edge of e_{21}, or
- some oriented edge of e_{11} is inverse to an oriented edge of e_{2n}, and some oriented edge of e_{1n} is external in Δ.

In every case the path s_2 has a common oriented edge with $s_{1-}'^{-1}$ or $s_{1+}'^{-1}$. Let $\tilde{s}_2$ be the maximal common nontrivial subpath of s_2 and $(s'_{1-}q_1s'_{1+})^{-1}$ (see Lemma 7.10). Now, consider each of the 3 cases separately (see Fig. 12).

(Notice that Case 3 is "symmetric" with Case 2.)

Case 1: some oriented edge of e_{11} is inverse to an oriented edge of e_{2n}, and some oriented edge of e_{1n} is inverse to an oriented edge of e_{21}. Then some nontrivial initial subpath of $\tilde{s}_2^{-1}$ is a terminal subpath of s'_{1-}, and some nontrivial terminal subpath of $\tilde{s}_2^{-1}$ is an initial subpath of s'_{1+} (because $\tilde{s}_2^{-1}$ is a subpath of $s'_{1-}q_1s'_{1+}$). Let v_1 and v_2 be the paths such that $v_1q_1v_2 = \tilde{s}_2^{-1}$. Then v_1 is a nontrivial terminal subpath of s'_{1-}, and v_2 is a nontrivial initial subpath of s'_{1+}. The label of $s'_{1-}q_1s'_{1+}$ is counter-regular since the labels of s'_{1-}, $v_1q_1v_2$, and s'_{1+} are such. As it is assumed, $\operatorname{rank}(\Pi) < j_1$ for every

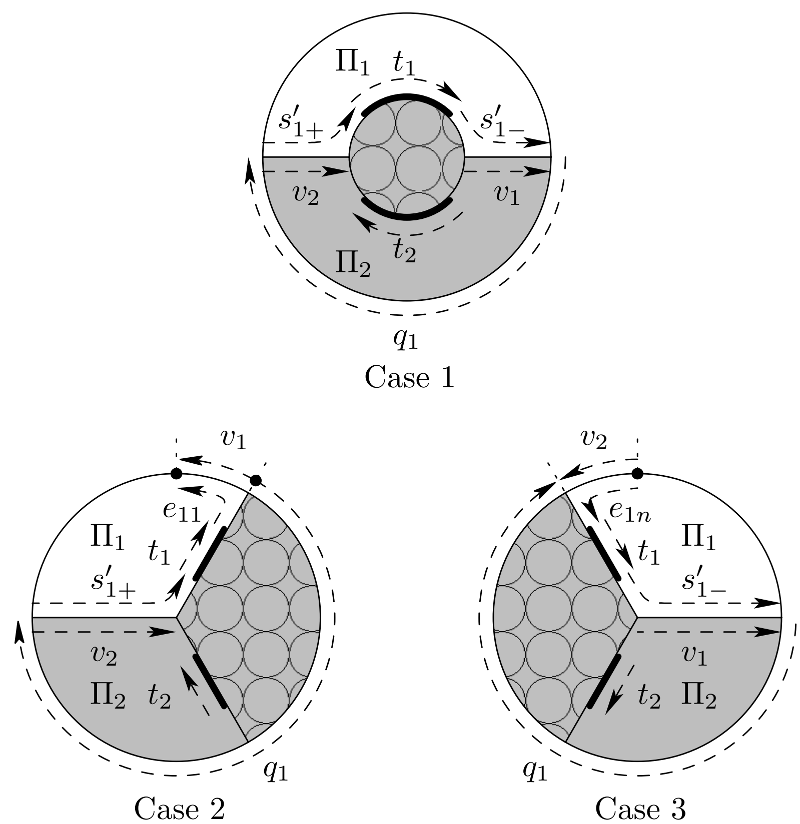


Figure 12: Cases 1–3, Lemma 7.15.

face Π of Δ'. Recall that $\langle s'_{1+}t_1s'_{1-}q_1\rangle = \bar{\partial}_1\Delta'$. Hence, the group word $w_{j_1} = \ell(t_1)$ equals the regular word $\ell(s'_{1-}q_1s'_{1+})^{-1}$ modulo the relations $r = 1$, $r \in \mathcal{R}_{j_1-1}$. This contradicts the choice of w_{j_1}.

Case 2: some oriented edge of e_{11} is external in Δ, and some oriented edge of e_{1n} is inverse to an oriented edge of e_{21}. Then s'_{1-} is a proper initial subpath of e_{11}. Let v_1 be the initial subpath of s'_1 such that $s'_{1-}v_1 = e_{11}$. Then v_1^{-1} is a subpaths of l_2 since $\ell(v_1^{-1})$ is a nontrivial power of x_n. Some nontrivial terminal subpath of $\tilde{s}_2^{-1}$ is an initial subpath of s'_{1+}. Therefore, $s_2'^{-1}$ is a terminal subpath of q_1, and $v_1^{-1}q_1$ is a subpath of b_2 (because $s_2'^{-1}$ is a common subpath of q_1 and b_2). Let v_2 be the path such that $s_2'^{-1}v_2$ is a terminal subpath of $\tilde{s}_2^{-1}$. Then v_2 is a nontrivial initial subpath of s'_{1+}. The label of $q_1s'_{1+}$ is counter-regular since the labels of q_1, $s_2'^{-1}v_2$, and s'_{1+} are such. The reduced form of $\ell(s'_{1-}q_1s'_{1+})$ is counter-regular since $\ell(s'_{1-})$ is a power of x_n, and $\ell(q_1s'_{1+})$ is counter-regular. As it is assumed, $\operatorname{rank}(\Pi) < j_1$ for every face Π of Δ'. Hence, the group word $w_{j_1} = \ell(t_1)$ equals a regular word (the reduced form of $\ell(s'_{1-}q_1s'_{1+})^{-1}$) modulo the relations $r = 1$, $r \in \mathcal{R}_{j_1-1}$. This contradicts the choice of w_{j_1}.

Case 3: some oriented edge of e_{11} is inverse to an oriented edge of e_{2n}, and some oriented edge of e_{1n} is external in Δ. Contradiction is obtained as in Case 2.

No other case is possible. Thus, at least one of the paths e_{11} or e_{1n} has no common oriented edges with s_2^{-1} and no external oriented edges. $\square$

Lemma 7.16. *Let* Π_1, Π_2, s_1, s'_1, s'_{1-}, s'_{1+}, s_2, s'_2, s'_{2-}, s'_{2+}, z_1, z_2 *be such as in Lemma* 7.12. *Let* $c_1 = s'_{2-}z_1s'_{1+}$ *and* $c_2 = s'_{1-}z_2s'_{2+}$. *Then* Δ *has a simple disc S-subdiagram* Φ *such that* Π_1 *and* Π_2 *are not in* Φ, *and the total number of selected external edges of* Φ *that lie on* c_1 *or* c_2 *is greater than*

$$\frac{n-21}{n}\sum_{\Pi\in\Phi(2)}|\partial\Pi|.$$

Proof. Observe that to prove this lemma, it suffices to prove the analogous lemma for the mirror copy of Δ.

By Lemma 7.14, $\operatorname{rank}(\Pi_1) \neq \operatorname{rank}(\Pi_2)$. If $\operatorname{rank}(\Pi_1) < \operatorname{rank}(\Pi_2)$, pass from Δ, Π_1, Π_2, s_1, s'_{1-}, s'_1, s'_{1+}, s_2, s'_{2-}, s'_2, s'_{2+}, z_1, z_2, c_1, c_2 to the mirror copy of Δ, Π_2, Π_1, s_2^{-1}, s'^{-1}_{2+}, s'^{-1}_2, s'^{-1}_{2-}, s_1^{-1}, s'^{-1}_{1+}, s'^{-1}_1, s'^{-1}_{1-}, z_1^{-1}, z_2^{-1}, c_1^{-1}, c_2^{-1}, respectively. Hence, assume that $j_1 = \operatorname{rank}(\Pi_1) > \operatorname{rank}(\Pi_2) = j_2$.

Let t_1 and t_2 be the paths such that $\langle s_1 t_1 \rangle = \bar{\partial}\Pi_1$ and $\langle s_2 t_2 \rangle = \bar{\partial}\Pi_2$. Let Δ' be a disc S-subdiagram of Δ obtained by removing the faces Π_1 and Π_2 and all edges that lie on s'_1 or s'_2 together with all intermediate vertices of s'_1 and s'_2. The contour cycle of Δ' is $\langle c_1 t_1 c_2 t_2 \rangle$. By Lemma 7.13, the diagram Δ' is non-degenerate.

Consider 2 cases:

Case 1: there is a face Π in Δ' such that $\operatorname{rank}(\Pi) \geq \operatorname{rank}(\Pi_1)$. Then let Φ be the maximal simple disc S-subdiagram of Δ' that contains such a face Π. Let Σ be the sum of the degrees of all the faces of Φ. Then $\Sigma \geq |\partial\Pi| \geq |\partial\Pi_1| > |\partial\Pi_2|$. By Lemma 7.3, the number of selected external edges of Φ is at least $(1-2\gamma)\Sigma$. Therefore, the number of selected external edges of Φ that do not lie on t_1 nor on t_2 is at least

$$(1-2\gamma)\Sigma - \lambda_1|\partial\Pi_1| - \lambda_1|\partial\Pi_2| > (1-2\gamma-2\lambda_1)\Sigma > \frac{n-21}{n}\Sigma.$$

Case 2: the rank of every face of Δ' is less than the rank of Π_1. Let e_{11} and e_{1n} be respectively the initial and the terminal subpaths of s_1 of length $m_{j_1} = |s_1|/n$. By Lemma 7.15, at least one of the paths e_{11} or e_{1n} has the property that every edge that lies on it either is incident to a face of Δ' or lies on t_1 or t_2. Let e be one of the paths e_{11} or e_{1n} with this property.

Since

$$|t_1| + |t_2| = |w_{j_1}| + |w_{j_2}| < \frac{1}{2}m_{j_1} + \frac{1}{2}m_{j_1} = m_{j_1} = |e|$$

(see (11)), the ratio of the number of edges that lie on t_1 or t_2 and are incident with faces of Δ' (consequently, do not lie on e) to the number of edges that lie on e and are incident with faces of Δ' (equivalently, do not lie on t_1 nor on t_2) is at most

$$\frac{|t_1|+|t_2|}{|e|} = \frac{|w_{j_1}|+|w_{j_2}|}{m_{j_1}} \leq \frac{2n\lambda_1}{1-\lambda_1}$$

(see (10)). Let Φ be a maximal simple disc S-subdiagram of Δ' such that the ratio of the number of its (external) edges that lie on t_1 or t_2 to the number of its (external) edges that lie on e is at most

$$\frac{2n\lambda_1}{1-\lambda_1}.$$

Let Σ be the sum of the degrees of all the faces of Φ. Since all the edges that lie on e are labelled with a same basic letter, the number of edges of Φ that lie on e is less than

$$\Big(\lambda_1 + \frac{1-\lambda_1}{n}\Big)\Sigma.$$

By Lemma 7.3, the number of selected external edges of Φ is at least $(1-2\gamma)\Sigma$. Therefore, the number of selected external edges of Φ that do not lie on t_1 nor on t_2 is greater than

$$\Big(1-2\gamma-\frac{2n\lambda_1}{1-\lambda_1}\Big(\lambda_1+\frac{1-\lambda_1}{n}\Big)\Big)\Sigma = \Big(1-2\gamma-2\lambda_1-\frac{2n\lambda_1^2}{1-\lambda_1}\Big)\Sigma \geq \frac{n-21}{n}\Sigma$$

(see (8)).

Every external edge of Φ that does not lie on t_1 nor on t_2 lies on c_1 or c_2. Thus, the desired inequality holds in the both cases. □

Proof of Property 2. The existence is obvious from the choice of relators.

Suppose that presentation of some element of G in the form $a_1^{k_1}\dots a_n^{k_n}$ is not unique. Then there exist two distinct regular group words over $\mathfrak{A}$ whose values in G coincide. Consider such a pair of group words with the minimal sum of their lengths. Consider a minimal deduction diagram for the equality of these group words over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$. More precisely, let Δ be a special disc S-diagram over $\langle \mathfrak{A} \parallel \mathcal{R} \rangle$ whose contour has the form p_1p_2 such that $\ell(p_1)$ and $\ell(p_2^{-1})$ are distinct regular group words, and $\ell(p_1p_2)$ is cyclically reduced; moreover, let Δ be such an S-diagram with the minimal possible number of faces.

Choose faces Π_1 and Π_2 and paths s_1, s'_1, s'_{1-}, s'_{1+}, s_2, s'_2, s'_{2-}, s'_{2+}, z_1, z_2 as in Lemma 7.12. Denote the paths $s'_{2-}z_1s'_{1+}$ and $s'_{1-}z_2s'_{2+}$ by c_1 and c_2, respectively (as in the hypotheses of Lemma 7.16). Apply Lemma 7.16: let Φ be a simple disc S-subdiagram of Δ such that the number of selected external edges of Φ that lie on c_1 or c_2 is greater than

$$\frac{n-21}{n}\sum_{\Pi\in\Phi(2)}|\partial\Pi|.$$

By the choice of Π_1 and Π_2, all basic letters of $\ell(c_1)$ are in $\{x_1,\dots,x_{21}\}$, and all basic letters of $\ell(c_2)$ are in $\{x_{n-20},\dots,x_n\}$. Therefore, by Proposition 7.2, the number of selected external edges of Φ that lie on c_1 or c_2 is less than

$$\frac{42}{n}\sum_{\Pi\in\Phi(2)}|\partial\Pi|.$$

Since $n-21\geq 42$, have a contradiction. The uniqueness is proved. $\square$

Remark 7.1. The "symmetry" mentioned in the proofs of Lemmas 7.8, 7.10, 7.13, and 7.15 is the symmetry between the S-diagram Δ and the S-diagram obtained from the mirror copy of Δ by re-labelling according to the following rule: if the label of an oriented edge edge e is x_i^σ, then re-label e with $x_{n+1-i}^{-\sigma}$ (this gives a diagram not over the same presentation).

The proof of the next property uses homological argument.

Property 3. *The group G is torsion-free.*

Proof. The group presentation $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ is strongly aspherical by Corollary 7.1.b of Proposition 7.1; therefore, it is aspherical in the sense of [Ol'91]. Since no element of $\mathcal{R}$ represents a proper power in the free group on $\mathfrak{A}$, the relation module M of $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ is a free G-module by Corollary 32.1 in [Ol'91]. Therefore, there exists a finite-length free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$:

$$0 \to M \to \bigoplus_{x\in\mathfrak{A}} \mathbb{Z}G \to \mathbb{Z}G \to \mathbb{Z} \to 0, \tag{12}$$

where $\mathbb{Z}G$ and $\bigoplus_{x\in\mathfrak{A}} \mathbb{Z}G$ are identified with the G-modules of, respectively, 0- and 1-dimensional cellular chains of the Cayley complex of $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ (see [Bro94]).

Suppose now that G has torsion. Let H be a nontrivial finite cyclic subgroup of G. Every free G-module may be naturally regarded as a free H-module. Hence, the resolution (12) may be viewed a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. This contradicts the fact that all odd-dimensional homology groups of H are nontrivial. $\square$

For more results on group presentations with various forms of asphericity, and torsion in such groups, see [CCH81], [Hue79], [Hue80], and Theorems 32.1, 32.2 in [Ol'91].

Properties 1, 2, 3 of G demonstrate that the answer to Bludov's question is negative.

Remark 7.2. If n was chosen to be less than 63 but greater than 26 ($26 < n < 63$), then Properties 1 and 3 still would hold, but the proof of Property 2 would not work.

For every $i \in \mathbb{N} \cup \{0\}$, let G_i be the group defined by the presentation $\langle \mathfrak{A} \,\|\, \mathcal{R}_i \rangle$ (see subsection 7.a). Also for every $i \in \mathbb{N} \cup \{0\}$, let ϕ_i be the natural epimorphism $G_i \to G_{i+1}$.

Property 4. *All groups G_0, G_1, G_2, … are hyperbolic, and G is isomorphic to the direct limit of $G_0 \overset{\phi_0}{\to} G_1 \overset{\phi_1}{\to} G_2 \overset{\phi_2}{\to} \dots$.*

Proof. Obviously, G is isomorphic to the direct limit of $G_0 \xrightarrow{\phi_0} G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} \dots$. The groups G_0, G_1, G_2, ... are hyperbolic by Corollary 7.3.a of Proposition 7.3. □

It is easy to see that every recursively presented group with a finite regular file basis has solvable word problem. (Note that if a finitely generated group has a recursive presentation with a countably infinite alphabet, then it also has a recursive presentation with a finite alphabet.) In particular, if the group G is recursively presented, then it has solvable word problem. Generally, a recursively presented finitely generated group H has solvable word problem if an only if there exists a recursive set of group words X in a finite alphabet $\mathfrak{B}$ such that relative to some mapping of $\mathfrak{B}$ to H, every element of H has a unique representative in X. In the case of G, the set of all regular group words over $\mathfrak{A}$ may be taken as such a recursive set of representatives.

Property 5 (conditional)**.** *If G is recursively presented, then it has solvable conjugacy problem.*

Proof. Assume G is recursively presented. Then, as noted above, G has solvable word problem.

Let q be an integer such that $q \geq 1/(1-2\gamma)$.

Here is a description of an algorithm that solves the conjugacy problem in G:

Input: `group words` u `and` v `(in the alphabet` $\mathfrak{A}$`).`

Step 1: `If` $[u] \neq 1$ `and` $[v] \neq 1$`, go to the next step; otherwise, test whether` $[u] = [v]$`. If yes, output ''Yes'', if no, output ''No''. Stop.`

Step 2: `Produce the (finite) set` $\mathcal{S}$ `of all group words` s `over` $\mathfrak{A}$ `such that` $[s] = 1$ `and` $|s| \leq q(|v| + |u|)$`.`

Step 3: `Produce a (finite) set` D `of up to isomorphism all annular diagrams over` $\langle\, \mathfrak{A} \,\|\, \mathcal{S} \,\rangle$ `with up to` $q(|v| + |u|)$ `edges.`

Step 4: `If there is` $\Delta \in D$ `such that` u `is the label of one of the contours of` Δ `and` v^{-1} `is the label of the other, then output ‘‘Yes’’; otherwise, output ‘‘No’’. Stop.`

On an input (u, v), this algorithm gives "Yes" as the output if and only if $[u]$ and $[v]$ are conjugate in G; otherwise, it gives "No." Below is an outline of a proof.

It is clear that the algorithm gives "Yes" only if $[u]$ and $[v]$ are conjugate in G. It is also clear that on every input it gives either "Yes" or "No," and terminates.

Now, let u and v be arbitrary group words representing conjugate elements of G, and consider (u, v) as an input to the algorithm.

Case 1: $[u] = 1$ or $[v] = 1$. Then $[u] = 1 = [v]$ and therefore the algorithm gives "Yes" on Step 1.

Case 2: $[u] \neq 1$ and $[v] \neq 1$. Let Δ be a weakly reduced annular disc diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ such that u and v^{-1} are the labels of its two contours (such Δ exists by Lemma V.5.2 in [LS01]). Let $\mathcal{S}$ be the set of group words produced by the algorithm on Step 2. Every element of $\mathcal{R}$ of length at most $q(|v| + |u|)$ is in $\mathcal{S}$. Let D be the set of annular diagrams produced by the algorithm on Step 3. By Proposition 7.1 and the corollary of the Main Theorem,

$$|u| + |v| \geq (1 - 2\gamma) \sum_{\Pi \in \Delta(2)} |\partial\Pi| \geq \frac{1}{q} \sum_{\Pi \in \Delta(2)} |\partial\Pi|$$

(recall that a special selection exists on every diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$). Therefore, the length of the contour of every face of Δ is at most $q(|u| + |v|)$, and the total number of edges of Δ is

$$\begin{aligned} \|\Delta(1)\| = \frac{1}{2}\Big(\sum_{\Pi \in \Delta(2)} |\partial\Pi| + |u| + |v| \Big) &\leq \frac{1}{2}\Big(q(|u| + |v|) + |u| + |v| \Big) \\ &= \frac{q+1}{2}(|u| + |v|) \leq q(|u| + |v|). \end{aligned}$$

Hence, Δ is isomorphic to some element of D, and the algorithm returns "Yes" on Step 4. $\square$

Proof of the theorem (*see Introduction*). Let N be an integer such that $\lambda_1 n N \geq 1$. Let the construction in subsection 7.a be carried out under the following two additional conditions:

(1) the order imposed on the set of all reduced group words is deg-lex;

(2) for every i for which r_i is defined, $m_i = N|w_i| + i$.

Evidently, these conditions are consistent with the rest.

By Proposition 7.4, the group G is recursively presented. Properties 1–5 show that up to isomorphism G is a desired group. (G is *isomorphic* to a direct limit of a sequence of hyperbolic groups with respect to a family of surjective homomorphisms, but a desired group must *be* such a limit.) $\square$

7.c Comments

The group G and its presentation $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ mentioned in this section are defined in subsection 7.a.

Proposition 7.5. *No hyperbolic group can be used as an example to demonstrate the negative answer to Bludov's question.*

Proof. Suppose a hyperbolic group H is an example demonstrating the negative answer. Then H is *boundedly generated* (is the product of a finite sequence of its cyclic subgroups) and not virtually polycyclic. In particular, H is non-elementary. Corollary 4.3 in [Min04] states that every boundedly generated hyperbolic group is elementary. (This fact also follows from Corollary 2 in [Ol'93].) This gives a contradiction. $\square$

Corollary 7.5.a. *The group G is not finitely presented.*

Proof. Suppose G is finitely presented. Then the presentation $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ is finite. By Corollary 7.3.a of Proposition 7.3, G is hyperbolic. This contradicts Proposition 7.5. $\square$

Thus, the set $\mathcal{R}$ and the sequences $\{w_i\}_{i=1,\dots}$, $\{m_i\}_{i=1,\dots}$, $\{r_i\}_{i=1,\dots}$ constructed in subsection 7.a are infinite.

CHAPTER III

BOUNDEDLY SIMPLE GROUPS

8 Proof of Theorem 3

Proof. In this proof, let

$$k = 28, \quad \lambda_1 = 10^{-2}, \quad \lambda_2 = 10^{-4}, \quad \lambda_4 = \frac{1}{28}, \quad \lambda_3 = 4\lambda_4 = \frac{1}{7}.$$

Let $\gamma = 0.45$. These numbers satisfy the inequalities

$$2\lambda_1 + (8+5k)\lambda_2 + 6\lambda_3 + 2\lambda_4 < 1,$$

$$\lambda_1 + (3+2k)\lambda_2 + 3\lambda_3 \leq \gamma.$$

Let $\bar{a} = (k; \lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

Let $\mathfrak{A} = \{x, y\}$ be a 2-letter alphabet. Let (a_1, b_1), (a_2, b_2), $\ldots$ be all the ordered pairs of reduced group words in the alphabet $\mathfrak{A}$ listed without recurrences. Let, moreover, the function $i \mapsto (a_i, b_i)$ be computable.

Pick two non-empty reduced words v_1, v_2 and a system of non-empty reduced words $\{u_{i,j}\}_{\substack{i=1,2,\ldots \\ j=1,\ldots,14}}$ so that the following conditions are satisfied:

(1) If $i \in \mathbb{N}$, then $|u_{i,1}| = |u_{i,2}| = \cdots = |u_{i,14}|$.

(2) If $i \in \mathbb{N}$, then $28|u_{i+1,1}| + 14|a_{i+1}| + |b_{i+1}| > 28|u_{i,1}| + 14|a_i| + |b_i|$.

(3) If $i \in \mathbb{N}$, then $28|u_{i,1}| \geq 99(14|a_i| + |b_i|)$.

(4) If $i_1, i_2 \in \mathbb{N}$, $j_1, j_2 \in \{1, 2, 3, \dots, 14\}$, $\sigma_1, \sigma_2 \in \{\pm 1\}$, $u_{i_1,j_1}^{\sigma_1} = w_1 s z_1$ and $u_{i_2,j_2}^{\sigma_2} = w_2 s z_2$, then either $w_1 = w_2$, $z_1 = z_2$, $i_1 = i_2$, $j_1 = j_2$, $\sigma_1 = \sigma_2$, or $|s| \leq 28 \cdot 10^{-4} \min\{|u_{i_1,j_1}|, |u_{i_2,j_2}|\}$.

(5) The products (i.e., concatenations) $v_1 v_1$, $v_2 v_2$, $v_1 v_2$, $v_1^{-1} v_2$, $v_1 v_2^{-1}$, $v_1^{-1} v_2^{-1}$ are reduced. For example, v_1 starts and ends with the letter x, and v_2 starts and ends with the letter y.

(6) If $v \in \{v_1^{\pm 1}, v_2^{\pm 1}\}$ and $u \in \{\, u_{i,j}^{\pm 1} \mid i \in \mathbb{N}, j = 1, \dots, 14 \,\}$, then $|u| \geq 40|v|$ and v is not a subword of u.

Besides that, assume that the function $(i, j) \mapsto u_{i,j}$ is computable. Such words v_1, v_2 and a system $\{u_{i,j}\}_{\substack{i=1,2,\dots \\ j=1,\dots,14}}$ exist.

For every $i = 1, 2, 3, \dots$, let

$$r_i = a_i^{u_{i,1}} a_i^{u_{i,2}} \cdots a_i^{u_{i,14}} b_i^{-1}$$

(recall that $a_i^{u_{i,j}} = u_{i,j} a_i u_{i,j}^{-1}$). Note that by condition (2),

$$|r_1| < |r_2| < |r_3| < \cdots,$$

by condition (3), for every i,

$$14|a_i| + |b_i| \leq \lambda_1 |r_i|,$$

by condition (1), for every i,

$$|u_i| \leq \lambda_4 |r_i|.$$

Define a sequence of sets $\{\mathcal{R}_i\}_{i=0}^{+\infty}$ inductively. Let $\mathcal{R}_0 = \emptyset$. Take $i > 0$. If the relation $a_i = 1$ is a consequence of the relations $r = 1$, $r \in \mathcal{R}_{i-1}$ (for example, if a_i is the empty word), then let $\mathcal{R}_i = \mathcal{R}_{i-1}$. Otherwise, let $\mathcal{R}_i = \mathcal{R}_{i-1} \cup \{r_i\}$. Let

$$\mathcal{R} = \bigcup_{i=1}^{+\infty} \mathcal{R}_i.$$

Let G be the group defined by the presentation $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$. Let H be the subgroup of G generated by $[v_1]$ and $[v_2]$. It easily follows from the imposed relations that G is 14-boundedly simple. Moreover, for any nontrivial element $g \in G$ and for any element $h \in G$, the element h is the product of 14 conjugates of g. The next goal is to prove that the word problem in G is solvable and that H is freely generated by $[v_1]$ and $[v_2]$. It will be done using the fact that every reduced simple disc diagram Δ over $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ satisfies the condition $\mathcal{A}(\bar{a})$ (where the components of $\bar{a}$ are viewed as constant functions on $\Delta(2)$).

In the rest of this proof, a selection on a diagram Δ over $\langle \mathfrak{A} \,\|\, \mathcal{R} \rangle$ will be called *special* if for every face Π of Δ, there is a natural number i, a representative c of $\bar{\partial}\Pi$, and paths $q_0, \ldots, q_{14}, p_1, \ldots, p_{14}, p'_1, \ldots, p'_{14}$ such that the following conditions hold:

- either $\ell(c) = r_i$ or $\ell(c) = r_i^{-1}$;
- if $\ell(c) = r_i$, then

$$c = p'_1 q_1 p_1^{-1} p'_2 q_2 p_2^{-1} \cdots p'_{14} q_{14} p_{14}^{-1} q_0^{-1},$$

  and the selection on Π consists of all nontrivial subpaths of $p_1^{-1}, \ldots, p_{14}^{-1}$ and $p'_1, \ldots, p'_{14}$;
- if $\ell(c) = r_i^{-1}$, then

$$c = q_0 p_{14} q_{14}^{-1} p_{14}'^{-1} p_{13} q_{13}^{-1} p_{13}'^{-1} \cdots p_1 q_1^{-1} p_1'^{-1},$$

  and the selection on Π consists of all nontrivial subpaths of $p_1'^{-1}, \ldots, p_{14}'^{-1}, p_1, \ldots, p_{14}$.

- $\ell(q_0) = b_i$, and for every $j = 1, \ldots, 14$, $\ell(p_j) = \ell(p'_j) = u_{i,j}$ and $\ell(q_j) = a_i$.

Clearly, on every diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ there exists a special selection. A diagram over $\langle\, \mathfrak{A} \,\|\, \mathcal{R} \,\rangle$ together with a special selection will be called a *special S-diagram*. Note that if a simple disc S-map Δ' is cut out from a special S-diagram, and the S-map Δ' is naturally labelled via some pasting morphism, then the S-diagram obtained from Δ' is special.

Consider an arbitrary special S-diagram Δ. Clearly, Δ satisfies the condition $\mathcal{A}_1(k)$. It easily follows from condition (3) that Δ also satisfies $\mathcal{A}_2(\lambda_1)$. It is to be shown that in fact every reduced special disc S-diagram satisfies $\mathcal{A}(\bar{a})$.

Use induction on the number of internal edges of a reduced special disc S-diagram. If a reduced special S-diagram has no internal edges, then it clearly satisfies the condition $\mathcal{A}(\bar{a})$. Suppose that Δ is a reduced special disc S-diagram, and every reduced special disc S-diagram having fewer internal edges than Δ satisfies the condition $\mathcal{A}(\bar{a})$. To complete the inductive proof, it is enough to prove now that Δ satisfies $\mathcal{A}(\bar{a})$. Since it is enough to prove that every maximal simple disc S-subdiagram of Δ satisfies $\mathcal{A}(\bar{a})$, assume that Δ is simple.

The length of every selected subpath of the contour cycle of every face Π of Δ is at most $\lambda_4|\partial\Pi|$. Consider arbitrary (not necessarily distinct) faces Π_1 and Π_2 of Δ. Suppose there exists a double-selected arc t between Π_1 and Π_2 such that $|t| > \lambda_2|\partial\Pi_1|$ (call such arcs "long"). Then, by condition (4), the labels of $\bar{\partial}\Pi_1$ and $\bar{\partial}\Pi_2$ are either equal or mutually inverse. In particular, $|\partial\Pi_1| = |\partial\Pi_2|$. Since Δ is reduced, the labels of $\bar{\partial}\Pi_1$ and $\bar{\partial}\Pi_2$ are equal ($\ell(\partial\Pi_1)$ is a cyclic shift of $\ell(\partial\Pi_2)$). Therefore, if p is a selected subpath of $\bar{\partial}\Pi_2$ on which a given "long" arc lies, then the sum of the lengths of all maximal double-selected arcs between Π_1 and Π_2 that lie on p is at most $\lambda_4|\partial\Pi_1|$. Suppose, moreover, that the faces Π_1 and Π_2 are distinct. Then it follows from the choice of the selection and from the properties of the system $\{u_{i,j}\}_{\substack{i=1,2,\ldots\\ j=1,\ldots,14}}$ that all such "long" double-selected arcs between Π_1 and Π_2 cannot be situated on more than 4 maximal selected subpaths of $\bar{\partial}\Pi_1$ (as well as of Π_2). Therefore, the sum of the lengths of all such "long" maximal double-selected arcs between Π_1 and Π_2 is not greater than $4\lambda_4|\partial\Pi_1| = \lambda_3|\partial\Pi_1|$. Thus, Δ satisfies the conditions $\mathcal{A}_5(\lambda_2, \lambda_4)$ and $\mathcal{A}_4(\lambda_2, \lambda_3)$.

Verification of the condition $\mathcal{A}_3(\lambda_2)$ is more complicated. Suppose that $\mathcal{A}_3(\lambda_2)$ does not hold in Δ. Let Π_0 be a face of Δ and t be a double-selected inward intra-facial oriented arc such that t is a subpath of $\bar{\partial}\Pi_0$ and $|t| > \lambda_2|\partial\Pi_0|$ (such Π_0 and t exist since $\mathcal{A}_3(\lambda_2)$ does not hold). Let s_1 and s_2 be such subpaths of $\bar{\partial}\Pi_0$ that $ts_1t^{-1}s_2$ is a representative of $\bar{\partial}\Pi_0$. Since every reduced special disc S-diagram having fewer internal edges than Δ satisfies the condition $\mathcal{A}_3(\lambda_2)$, the choice of Π_0 and t is in fact unique and $\langle s_2^{-1}\rangle = \bar{\partial}_1\Delta$ (otherwise the cycle represented by s_2^{-1} would cut out an S-diagram having fewer internal edges and still not satisfying $\mathcal{A}_3(\lambda_2)$).

Let i be the number such that $|\partial\Pi_0| = |r_i|$. Then the label of some representative of $\bar{\partial}\Pi_0$ is either r_i or r_i^{-1}. By the construction of r_i (see condition (4)) and by the definition of a special selection, there is exactly one $m \in \{1, 2, \ldots, 14\}$ and one $\sigma \in \{\pm 1\}$ such that $\ell(t)$ is a subword of $u_{i,m}^{\sigma}$. Then either $\ell(s_1)$ or $\ell(t^{-1}s_2t)$ has the form

$$\left(wa_iw^{-1}\right)^{\pm 1},$$

where w is a suffix of $u_{i,m}$.

Case 1: $\ell(s_1)$ has the form

$$\left(wa_iw^{-1}\right)^{\pm 1},$$

w is a suffix of $u_{i,m}$. Let Δ_0 be the disc subdiagram of Δ whose contour is s_1. Note that $\Delta(2) = \Delta_0(2) \cup \{f_0\}$. By the construction of $\{\mathcal{R}_i\}_{i=0}^{+\infty}$, $\ell(s_1)$ is not trivial modulo the set of relation

$$\{\, r = 1 \mid r \in \mathcal{R}_{i-1} \,\} = \{\, r = 1 \mid r \in \mathcal{R},\ |r| < |\partial\Pi_0| \,\}.$$

Pick a face Π_1 in Δ_0 such that $|\partial\Pi_1| \geq |\partial\Pi_0|$. Let Δ_1 be the maximal simple disc S-subdiagram of Δ containing the face Π_1 and not containing the face Π_0. All the external edges of Δ_1 lie on the path s_1. In particular $|\partial_1\Delta_1| \leq |s_1|$. The S-map Δ_1 satisfies $\mathcal{A}_3(\lambda_2)$ since it has fewer internal edges than Δ. Hence, by the main theorem, $|\partial_1\Delta_1| \geq (1-2\gamma)|\partial\Pi_1|$. Therefore, on one hand,

$$|s_1| \geq (1-2\gamma)|\partial\Pi_1| \geq (1-2\gamma)|\partial\Pi_0| = \frac{1}{10}|\partial\Pi_0|.$$

On the other hand,

$$|s_1| < |a_i| + 2|u_{i,m}| \leq \frac{1}{14}|\partial\Pi_0|.$$

Contradiction.

Case 2: $\ell(t^{-1}s_2t)$ has the form

$$\left(wa_iw^{-1}\right)^{\pm 1},$$

w is a suffix of $u_{i,m}$. Let Δ' be an S-diagram cut out of Δ by the cycle represented by $t^{-1}s_2^{-1}t$. (Informally speaking, the S-diagram Δ' may be thought of as obtained from Δ by cutting Δ along t from the boundary inside.) The S-diagram Δ' is a simple reduced special disc S-diagram, it has fewer internal edges than Δ. Therefore, it satisfies $\mathcal{A}_3(\lambda_2)$. Hence, by the main theorem, $|\partial_1\Delta'| \geq (1-2\gamma)|\partial\Pi_0'|$ where Π_0' is the face of Δ' corresponding to Π_0 (note that $|\partial\Pi_0'| = |\partial\Pi_0|$). Therefore, on one hand,

$$|t^{-1}s_2t| \geq (1-2\gamma)|\partial\Pi_0| = \frac{1}{10}|\partial\Pi_0|.$$

On the other hand,

$$|t^{-1}s_2t| \leq |a_i| + 2|u_{i,m}| \leq \frac{1}{14}|\partial\Pi_0|.$$

Contradiction.

Thus, every reduced special disc S-diagram Δ satisfies $\mathcal{A}(\bar{a})$. In particular, Δ, $\bar{a}$, and γ satisfy the assumptions of the main theorem. Hence, the sum of the lengths of the contours of all faces of Δ is at most $\frac{1}{1-2\gamma}|\partial_1\Delta|$, and the number of edges of Δ is at most $\frac{1-\gamma}{1-2\gamma}|\partial_1\Delta|$.

Now, if Δ is an arbitrary reduced disc diagram over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$, the length of the contour of Δ is greater than or equal to the sum of the lengths of the contours of all its maximal simple disc subdiagrams. Hence, again the sum of the lengths of the contours of all faces of Δ is at most $\frac{1}{1-2\gamma}|\partial_1\Delta|$, and the number of edges of Δ is at most $\frac{1-\gamma}{1-2\gamma}|\partial_1\Delta|$.

For every $i = 0, 1, 2, \ldots$, let G_i be the group defined by the presentation $\langle\,\mathfrak{A}\,\|\,\mathcal{R}_i\,\rangle$.

Consider an arbitrary group word w whose value in G is 1. Let Δ be a reduced deduction diagram for w over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$. Let $n = |w|$. Consider an arbitrary $i > 10n$. Then $|r_i| > i > 10n = (1-2\gamma)^{-1}n$. So, no face in Δ has the contour labelled with a cyclic shift of $r_i^{\pm 1}$. Hence, the value of w in G_{10n} is 1 as well.

Recall that an algorithm solves the word problem in $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$ if for every group word w in the alphabet $\mathfrak{A}$, it determines whether or not w is trivial in G, i.e., whether the relation $w = 1$ is a consequence of the relations $r = 1$, $r \in \mathcal{R}$. Here is an informal description of an algorithm that solves the word problem in $\langle\,\mathfrak{A}\,\|\,\mathcal{R}\,\rangle$.

Consider group words only in the alphabet $\mathfrak{A}$. Let `Alg_1` be an algorithm that for every finite set $\mathcal{S}$ of group words and for every group word w decides whether there exists a deduction diagram Δ for w over $\langle\,\mathfrak{A}\,\|\,\mathcal{S}\,\rangle$ with at most $\frac{1-\gamma}{1-2\gamma}|w|$ edges. Clearly, such an algorithm exists. Let `Alg_2` be an algorithm that for every natural i constructs the set $\mathcal{R}_i$. It starts with $\mathcal{R}_0 = \emptyset$, and then for every $l = 1, \ldots, i$ constructs $\mathcal{R}_l$ by using `Alg_1` to check whether $\mathcal{R}_l = \mathcal{R}_{l-1}$ or $\mathcal{R}_l = \mathcal{R}_{l-1} \cup \{r_l\}$. Let `Alg_3` be an algorithm that for every group word w first uses `Alg_2` to construct $\mathcal{R}_{10n}$ where $n = |w|$, and then uses `Alg_1` to determine whether there exists a deduction diagram Δ for w over $\langle\,\mathfrak{A}\,\|\,\mathcal{R}_{10n}\,\rangle$ with at most $\frac{1-\gamma}{1-2\gamma}n$ edges. Such a diagram exists if and only if w is trivial in G_{10n}. Hence, the algorithm `Alg_3` solves the word problem in $\langle\, \mathfrak{A} \,\|\, \mathcal{R}\, \rangle$.

It is left to prove that H is freely generated by $[v_1]$ and $[v_2]$. Suppose it is not. Then there exist $m \geq 1$, $i_1, \dots, i_m \in \{1, 2\}$, $\sigma_1, \dots, \sigma_m \in \{\pm 1\}$ such that

$$\Big(\forall j \in \{1, \dots, m-1\}\Big)\Big(i_j \neq i_{j+1} \text{ or } \sigma_j = \sigma_{j+1}\Big) \text{ and } \Big(i_m \neq i_1 \text{ or } \sigma_m = \sigma_1\Big),$$

and the group word $w = v_{i_1}^{\sigma_1} \cdots v_{i_m}^{\sigma_m}$ is trivial in G. By the choice of v_1 and v_2, the group word w is cyclically reduced. Consider an arbitrary reduced deduction diagram Δ for w over $\langle\, \mathfrak{A} \,\|\, \mathcal{R}\, \rangle$. The contour of Δ is cyclically reduced since w is cyclically reduced. Let Δ_0 be a maximal simple disc submap of Δ such that $\partial_1\Delta_0$ is a subpath of $\partial_1\Delta$. Let $q_0 = \partial_1\Delta_0$. Define a selection on Δ_0 the same way as above (special selection). As it has been shown, the obtained S-diagram satisfies $\mathcal{A}(\bar{a})$. Apply the main theorem and the lemma about exposed face to Δ_0. Let Π be a face of Δ_0 and P be a non-overlapping set of selected external arcs of Δ_0 incident to Π such that $\|P\| \leq k + 1$ and

$$\sum_{p \in P} |p| \geq \Big(1 - 2\gamma - (2 + k)\lambda_2 - \lambda_4 + \|P\|\lambda_2\Big)|\partial\Pi|.$$

Note that elements of P are arcs in Δ_0 but do not need to be arcs in Δ since some intermediate vertex of some element of P may have degree greater than 2 in Δ. Let P_0 be the set of all maximal elements of the set of all arcs that lie on q_0 and are subarcs of elements of P. Then P_0 is a non-overlapping set of arcs, $\|P_0\| \leq \|P\| + 1$, and $\sum_{p\in P_0}|p| = \sum_{p\in P}|p|$. Let p be a longest arc in P_0. Then

$$\begin{aligned}|p| &\geq \frac{1-2\gamma-(2+k)\lambda_2-\lambda_4+\|P\|\lambda_2}{\|P_0\|}|\partial\Pi| \\ &\geq \frac{1-2\gamma-(2+k)\lambda_2-\lambda_4+(\|P_0\|-1)\lambda_2}{\|P_0\|}|\partial\Pi| \\ &= \frac{1-2\gamma-(3+k)\lambda_2-\lambda_4}{\|P_0\|}|\partial\Pi| + \lambda_2|\partial\Pi|.\end{aligned}$$

The picked values of k, λ_2, λ_4 and γ satisfy the inequality

$$1-2\gamma-(3+k)\lambda_2-\lambda_4 \geq 0.$$

Therefore,

$$\begin{aligned}|p| &\geq \frac{1-2\gamma-(3+k)\lambda_2-\lambda_4}{k+2}|\partial\Pi| + \lambda_2|\partial\Pi| \\ &= \frac{1-2\gamma-\lambda_2-\lambda_4}{k+2}|\partial\Pi|.\end{aligned}$$

Therefore, $|p| > \frac{1}{500}|\partial\Pi|$. Let p' be the subpath of $\bar{\partial}\Pi$ corresponding to the arc p (so, p' is a selected oriented arc of Δ). Let i be the number such that $|\partial\Pi| = |r_i|$. Take $j \in \{1, 2, \ldots, 14\}$ and $\sigma \in \{\pm 1\}$ such that $\ell(p')$ is a subword of $u_{i,j}^{\sigma}$. Since $28|u_{i,j}| < |\partial\Pi|$, it follows that

$$|p'| > \frac{1}{20}|u_{i,j}| \geq 2\max\{|v_1|, |v_2|\}.$$

Since p'^{-1} is a subpath of $\bar{\partial}_1\Delta$, and $|p'| \geq 2\max\{|v_1|, |v_2|\}$, at least one of the words $v_1^{\pm 1}$, $v_2^{\pm 1}$ is a subword of $\ell(p')$, and therefore, it is a subword of $u_{i,j}^{\sigma}$. Contradiction with condition (5). Thus, H is freely generated by $[v_1]$ and $[v_2]$. The proof is complete. $\square$

BIBLIOGRAPHY

[ABC$^+$91] Juan M. Alonso, Thomas Brady, Daryl Cooper, Vincent Ferlini, Martin Lustig, Michael L. Mihalik, Michael Shapiro, and Hamish B. Short, *Notes on word hyperbolic groups*, Group theory from a geometrical viewpoint, World Sci. Publ., River Edge, NJ, 1991, Proceedings of the workshop held in Trieste, March 26 – April 6, 1990, pp. 3–63.

[Ale56] Pavel S. Alexandrov, *Combinatorial topology*, Graylock Press, Baltimore, MD, 1956, translated from Russian.

[Blu95] Vasiliy Vasilievich Bludov, *File bases in groups*, Algebra and Logic **34** (1995), no. 3, 131–139, translated from Russian.

[Bou98] Nicolas Bourbaki, *Algebra i. Chapters 1–3*, Springer-Verlag, Berlin, 1998, translated from French, reprint of the 1989 English translation.

[Bro94] Kenneth S. Brown, *Cohomology of groups*, Springer-Verlag, New York, 1994, corrected reprint of the 1982 original.

[CCH81] Ian M. Chiswell, Donald J. Collins, and Johannes Huebschmann, *Aspherical group presentations*, Math Z. **178** (1981), no. 1, 1–36.

[CK83] David Carter and Gordon Keller, *Bounded elementary generation of $SL_n(\mathcal{O})$*, Amer. J. Math. **105** (1983), no. 3, 673–687.

[Hal35] Philip Hall, *On representatives of subsets*, J. London Math. Soc. **10** (1935), 26–30.

[HNN49] Graham Higman, Bernhard Hermann Neumann, and Hanna Neumann, *Embeddind theorems for groups*, J. London Math. Soc. **24** (1949), 247–254.

[Hue79] Johannes Huebschmann, *Cohomology theory of aspherical groups and of small cancellation groups*, J. Pure Appl. Algebra **14** (1979), no. 2, 137–143.

[Hue80] ______, *The homotopy type of a combinatorially aspherical presentation*, Math Z. **173** (1980), no. 2, 163–169.

[Itô55] Noboru Itô, *Über das Produkt von zwei abelschen Gruppen*, Math. Z. **62** (1955), 400–401, in German.

[LR80] John C. Lennox and James E. Roseblade, *Soluble products of polycyclic groups*, Math. Z. **170** (1980), no. 2, 153–154.

[LS01] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Springer-Verlag, Berlin, 2001, Reprint of the 1977 edition.

[Min04] Ashot Minasyan, *On products of quasiconvex subgroups in hyperbolic groups*, Internat. J. Algebra Comput. **14** (2004), no. 2, 173–195.

[MK95] V. D. Mazurov and E. I. Khukhro (eds.), *The Kourovka Notebook: unsolved problems in group theory*, 13th augmented edition ed., Russian Acad. of Sci. Siber. Div., Inst. Math., Novosibirsk, 1995, translated from Russian.

[Mur05] Alexey Yu. Muranov, *Diagrams with selection and method for constructing boundedly generated and boundedly simple groups*, Comm. Algebra **33** (2005), no. 4, 1217–1258, arXiv.org preprint: math.GR/0404472.

[Ol'89] Alexander Yu. Ol'shanskii, *Geometrija opredeljajushchikh sootnoshenij v gruppakh [Geometry of defining relations in groups]*, Nauka, Moscow, 1989, in Russian.

[Ol'91] ———, *Geometry of defining relations in groups*, Kluwer Academic Publishers, Dordrecht, Boston, 1991, translated from Russian.

[Ol'93] ———, *On residualing homomorphisms and G-subgroups of hyperbolic groups*, Internat. J. Algebra Comput. **3** (1993), no. 4, 365–409.

[Osi04] Denis V. Osin, *Small cancellations over relatively hyperbolic groups and embedding theorems*, preprint, 24 pages, URL: http://arXiv.org/abs/math.GR/0411039.

[PR93] Vladimir P. Platonov and Andrei S. Rapinchuk, *Abstract properties of S-arithmetic groups and the congruence problem*, Russian Acad. Sci. Izv. Math. **40** (1993), no. 3, 455–476, translated from Russian.

[San97] Alexey V. Sanin, *O shirine vlozhenija polugrupp v gruppy [On width of embedding of semigroups into groups]*, Fundement. Priklad. Matem. [Fundamental and Applied Math.] **3** (1997), no. 3, 925–936, in Russian.

[Sha99] Yehuda Shalom, *Bounded generation and Kazhdan's property (T)*, IHES Sci. Publ. Math. **90** (1999), 145–168.