Cohomology of group theoretic Dehn fillings

By

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To my dear father and mother,

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION AND MAIN RESULTS

1.1 Introduction

1.1.1. Dehn surgery of 3-manifolds. In 3-dimensional topology, Dehn surgery is an operation of modifying a 3-manifold by cutting off a solid torus and then gluing it back in a different way. The Lickorish-Wallace theorem, which states that every closed orientable connected 3-manifold can be obtained from the 3-dimensional sphere by performing finitely many Dehn surgeries, serves as a motivation of the study of Dehn surgeries.

The second step of the surgery, called *Dehn filling*, can be formalized as follows. Let M be a 3-manifold with toral boundary. Topologically distinct ways of gluing a solid torus to M are parametrized by free homotopy classes of essential simple closed curves of ∂M, called *slopes*. For a slope s, the Dehn filling $M(s)$ is obtained by attaching a solid torus $S^1 \times D^2$ to ∂M such that ∂D^2 is mapped to a curve of the slope s. The following is a particular case of Thurston's hyperbolic Dehn filling theorem.

Theorem 1.1.1 ([33, Theorem [TH1]]). *Let M be a compact orientable 3-manifold with toral boundary such that $M \backslash \partial M$ admits a complete finite-volume hyperbolic structure. Then $M(s)$ is hyperbolic for all but finitely many slopes s.*

1.1.2. Group theoretic Dehn fillings. In group theoretic settings, Dehn filling can be generalized as follows. Let G be a group, let H be a subgroup of G, and let N be a normal subgroup of H. The *group theoretic Dehn filling* associated with the data (G, H, N) is the process of forming the quotient group $G/\langle\langle N\rangle\rangle$, where $\langle\langle N\rangle\rangle$ is the normal closure of N in G.

Under the assumptions of Theorem 1.1.1, let $G = \pi_1(M)$. The natural map $\pi_1(\partial M) \to \pi_1(M)$ is injective. We think of $\pi_1(\partial M)$ as a subgroup of $\pi_1(M)$ and let $H = \pi_1(\partial M)$. Let $N \lhd H$ be the subgroup generated by the slope s. Then $G/\langle\langle N\rangle\rangle = \pi_1(M(s))$ by the Seifert-van Kampen theorem.

Dehn filling is a fundamental tool in group theory. The solution of the virtually Haken conjecture uses Dehn fillings of hyperbolic groups [2]. For a large number of relatively hyperbolic groups, Dehn fillings are used to prove the Farrell-Jones conjecture [5] and solve the isomorphism problem [12]. By considering Dehn fillings of hyperbolically embedded subgroups, [13] constructs purely pseudo-Anosov normal subgroups of mapping class groups. Other applications of Dehn fillings can be found in [3, 16].

In group theoretic settings, Thurston's theorem was first generalized by Osin [27], and independently by Groves-Manning [15] to Dehn fillings of peripheral subgroups of relatively hyperbolic groups. More recently, Dahmani-Guirardel-Osin [13] proved an analog of Thurston's theorem in the more general settings of groups with hyperbolically embedded subgroups (see Theorem 1.1.4 below and the discussion afterwards). We discuss here some examples and refer to Section 2.5 for the definition. We use $H \hookrightarrow_h G$ to indicate that H is a hyperbolically embedded subgroup of G.

Example 1.1.2. If H is a peripheral subgroup of a relatively hyperbolic group G, then $H \hookrightarrow_h G$. For example,

(a) if a group G decomposes as a free product $G = A * B$, then we have $A \hookrightarrow_h G$ and $B \hookrightarrow_h G$;

(b) under the assumptions of Theorem 1.1.1, we have $\pi_1(\partial M) \hookrightarrow_h \pi_1(M)$.

Example 1.1.3. Let G be a group acting acylindrically on a Gromov hyperbolic space and let g be a loxodromic element of G. Then there exists a maximal virtually cyclic subgroup $E(g) \leqslant G$ containing g such that $E(g) \hookrightarrow_h G$. In particular,

(a) if G is a free group and H is a maximal cyclic subgroup of G, then $H \hookrightarrow_h G$;

(b) if G is a hyperbolic group (resp. the mapping class group of a punctured closed orientable surface, outer automorphism group of a finite rank non-abelian free group) and g is a loxodromic (resp. pseudo-Anosov, fully irreducible) element, then $E(g) \hookrightarrow_h G$.

Other examples of hyperbolically embedded subgroups can be found in [13].

Theorem 1.1.4 ([13, Theorem 2.27]). *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then there exists a finite set $\mathcal{F} \subset H\backslash\{1\}$ such that if $N \lhd H$ and $N \cap \mathcal{F} = \emptyset$, then the natural homomorphism $H/N \to G/\langle\langle N \rangle\rangle$ maps H/N injectively onto a hyperbolically embedded subgroup of $G/\langle\langle N \rangle\rangle$.*

Under the assumptions of Theorem 1.1.1, the above theorem, together with some basic facts about relatively hyperbolic groups, implies that $\pi_1(M(s))$ is non-virtually-cyclic and word-hyperbolic for all but finitely many slopes s. Thurston's geometrization conjecture, proved by Perelman, implies that this algebraic statement about $\pi_1(M(s))$ is equivalent to the hyperbolicity of $M(s)$. Thus, the above theorem indeed provides a generalization of Theorem 1.1.1.

1.1.3. Motivation: a question on group cohomology. Note that in the settings of Thurston's theorem, i.e., if $G = \pi_1(M), H = \pi_1(\partial M)$, and $M(s)$ admits a hyperbolic structure, we have

$$H^*(G/\langle\langle N\rangle\rangle; \cdot) \cong H^*(\pi_1(M(s)); \cdot),$$

which can be computed via $M(s)$. Indeed, as $M(s)$ admits a hyperbolic structure, the universal cover of $M(s)$ is $\mathbb{H}^3$, which is contractible, and thus $M(s)$ is a model of $K(G/\langle\langle N\rangle\rangle, 1)$.

However, there are no analogous methods for Dehn fillings of hyperbolically embedded subgroups. The main question motivating our research is the following.

Question 1. For a group G with a subgroup $H \hookrightarrow_h G$ and a normal subgroup $N \lhd H$, what can be said about $H^*(G/\langle\langle N\rangle\rangle; \cdot)$?

In this thesis, we answer this question and discuss some applications. The first task is to understand the structure of $\langle\langle N\rangle\rangle$, which is solved by Chapter 3. In Chapter 4, we combine structural results obtained in Chapter 3 and the Lyndon-Hochschild-Serre spectral sequence to compute $H^*(G/\langle\langle N\rangle\rangle; \cdot)$. In Chapter 5, we estimate the cohomological dimension of $G/\langle\langle N\rangle\rangle$ and discuss some applications to acylindrically hyperbolic groups.

1.2 Main results

1.2.1. Cohen-Lyndon type theorems for $\langle\langle N\rangle\rangle$. In general, $\langle\langle N\rangle\rangle$ does not need to have any particular structure. Nevertheless, it turns out that if N avoids a finite set of bad elements, then $\langle\langle N\rangle\rangle$ enjoys a nice free product structure. In order to state our main results, we introduce the following terminology.

Definition 1.2.1. Let G be a group with a subgroup $H \hookrightarrow_h G$. We say that a property P holds *for all sufficiently deep* normal subgroups $N \lhd H$ if there exists a finite set $\mathcal{F} \subset H \setminus \{1\}$ such that P holds for all normal subgroups $N \lhd H$ with $N \cap \mathcal{F} = \emptyset$.

Definition 1.2.2. Let G be a group with a subgroup H and let $N \lhd H$. We say that the triple (G, H, N) has the *Cohen-Lyndon property* if there exists a left transversal T of $H\langle\langle N\rangle\rangle$ in G such that $\langle\langle N\rangle\rangle$ decomposes as a free product $\langle\langle N\rangle\rangle = \prod^{*}_{t\in T} N^t$, where $N^g = gNg^{-1}$ for $g \in G$.

The latter definition is motivated by the following result [11, Theorem 4.1], which was later generalized by [14, Theorem 1.1] to free products of locally indicable groups.

Theorem 1.2.3 (Cohen-Lyndon). *Let F be a free group and let C be a maximal cyclic subgroup of F. Then for all $f \in C$, the triple $(F, C, \langle f\rangle)$ has the Cohen-Lyndon property.*

By Example 1.1.3, we have $C = E(f) \hookrightarrow_h F$ and thus the above theorem fits in the general framework of group theoretic Dehn fillings. For general hyperbolically embedded subgroups, a weak version of the Cohen-Lyndon property is given in [13, Theorem 2.27].

Theorem 1.2.4 (Dahmani-Guirardel-Osin). *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then for all sufficiently deep $N \lhd H$,*

$$\langle\langle N\rangle\rangle = \prod^{*}_{t\in T} N^t$$

for some subset $T \subset G$.

The main difference between Theorems 1.2.4 and 1.2.3 is that in Theorem 1.2.4, T is just some subset of G, instead of being a left transversal of $H\langle\langle N\rangle\rangle$ in G. Our result improves Theorem 1.2.4.

Theorem 1.2.5. *Suppose that G is a group with a subgroup $H \hookrightarrow_h G$. Then (G, H, N) has the Cohen-Lyndon property for all sufficiently deep $N \lhd H$.*

In the special case where G and H are finitely generated and G is hyperbolic relative to H, Theorem 1.2.5 is proved in [16, Theorem 4.8]. The proofs of [13, Theorem 7.15] and [16, Theorem 4.8] use technicalities such as windmills, very rotating families, and spiderwebs. The proof of Theorem 1.2.5 is easier and only uses surgeries on van Kampen diagrams and geometric properties of geodesic polygons of Cayley graphs.

Remark 1.2.6. In fact, we prove Theorem 1.2.5 in much more general settings of a group G with a family of *weakly hyperbolically embedded subgroups* (see Definition 2.5.4 for the definition). As an application, we also obtain Cohen-Lyndon type theorems for graphs of groups, e.g., amalgamated free products and HNN-extensions (see Corollaries 3.3.8, 3.3.9, and 3.3.10).

Combining Theorem 1.2.5 and Example 1.1.2, we obtain:

Corollary 1.2.7. *Let G be a group acting acylindrically on a Gromov hyperbolic space, and let $g \in G$ be a loxodromic element. Then $(G, E(g), N)$ has the Cohen-Lyndon property for all sufficiently deep $N \lhd E(g)$.*

In the case where $G = F$ and $H = C$, we recover Theorem 1.2.3 for sufficiently deep (but not all) $\langle f \rangle \lhd C$. In the case where G is a free product of locally indicable groups, by considering the action of G on the corresponding Bass-Serre tree, we also recover [14, Theorem 1.1] for sufficiently deep normal subgroups.

1.2.2. Structure of relative relation modules. Let $Rel(G, \langle\langle N \rangle\rangle)$ and $Rel(H, N)$ be the *relative relation modules* of the exact sequences

$$1 \to \langle\langle N \rangle\rangle \to G \to \overline{G} \to 1$$

and

$$1 \to N \to H \to \overline{H} \to 1,$$

respectively, i.e. $Rel(G, \langle\langle N \rangle\rangle)$ (resp. $Rel(H, N)$) is the $\mathbb{Z}\overline{G}$-module (resp. $\mathbb{Z}\overline{H}$-module) whose base set is the abelianization of $\langle\langle N \rangle\rangle$ (resp. N) and the $\overline{G}$-action (resp. $\overline{H}$-action) is induced by conjugation. If G is free, then $Rel(G, \langle\langle N \rangle\rangle)$ is called a *relation module*. For sufficiently deep N, it follows immediately from Theorem 1.1.4 that the natural map identifies $\overline{H}$ with a subgroup of $\overline{G}$. We can then further identify $\mathbb{Z}\overline{H}$ with a subring of $\mathbb{Z}\overline{G}$. Thus, given any $\mathbb{Z}\overline{H}$-module A, it makes sense to talk about the *induced module* of A from $\mathbb{Z}\overline{H}$ to $\mathbb{Z}\overline{G}$, which is denoted by $Ind_{\overline{H}}^{\overline{G}} A = \mathbb{Z}\overline{G} \bigotimes_{\mathbb{Z}\overline{H}} A$.

If $G = F$ and $H = C$, Theorem 1.2.3 directly implies $\mathbb{Z}\overline{G}$-module isomorphisms

$$Rel(F, \langle\langle f \rangle\rangle) \cong \mathbb{Z}[F/C\langle\langle f \rangle\rangle] \cong Ind_{\overline{H}}^{\overline{G}} \mathbb{Z} \cong Ind_{\overline{H}}^{\overline{G}} Rel(C, \langle f \rangle).$$

In general, we have the following corollary of Theorem 1.2.5.

Corollary 1.2.8. *Let G be a group with a subgroup $H \hookrightarrow_h G$. Then for all sufficiently deep $N \lhd H$, there is an isomorphism of $\mathbb{Z}\overline{G}$-modules*

$$Rel(G, \langle\langle N \rangle\rangle) \cong Ind_{\overline{H}}^{\overline{G}} Rel(H, N). \tag{1.1}$$

Remark 1.2.9. Merely knowing that $\langle\langle N\rangle\rangle = \prod^*_{t\in T} N^t$ for some subset $T \subset G$ is not enough to guarantee (1.1). For example, let G be any abelian group and let H be a proper subgroup of G. Then for any subgroup N of H, $\langle\langle N\rangle\rangle = N = \prod^*_{t\in\{1\}} N^t$. But $Rel(G, \langle\langle N\rangle\rangle)$ (resp. $Rel(H, N)$) is a $\mathbb{Z}\overline{G}$-module (resp. $\mathbb{Z}\overline{H}$-module) with the trivial $\overline{G}$-action (resp. $\overline{H}$-action) and thus $Rel(G, \langle\langle N\rangle\rangle) \ncong Ind^{\overline{G}}_{\overline{H}} Rel(H, N)$.

1.2.3. A spectral sequence for Dehn fillings.

Assuming the Cohen-Lyndon property, we obtain a spectral sequence to compute cohomology of Dehn filling quotients. Let G be a group, let H be a subgroup of G, and let N be a normal subgroup of H. For simplicity, let $\overline{G} = G/\langle\langle N\rangle\rangle$ and $\overline{H} = H/N$.

Theorem 1.2.10. *If the triple (G, H, N) has the Cohen-Lyndon property, then for every $\mathbb{Z}\overline{G}$-module A, there exists a spectral sequence of cohomological type.*

$$E^{p,q}_2 = \begin{cases} H^p(\overline{H}; H^q(N; A)) & , \text{ if } q \neq 0 \\ H^p(\overline{G}; A) & , \text{ if } q = 0 \end{cases} \Rightarrow H^{p+q}(G; A). \tag{1.2}$$

Usually, a spectral sequence is used to compute its limit. However, the point of Theorem 1.2.10 is that information about $H^*(G; A)$ and $H^*(\overline{H}; H^q(N; A))$ can be used to deduce properties of $H^*(\overline{G}; A)$ and answer Question 1. To enhance our answer, we also supplement Theorem 1.2.10 by relating the differentials of (1.2) to the differentials of the standard Lyndon-Hochschild-Serre spectral sequence of the extension $1 \to N \to H \to \overline{H} \to 1$ (see Remark 4.0.2). In Chapter 5, we use Theorem 1.2.10 to study certain homological properties of Dehn fillings.

Remark 1.2.11. In fact, we deal with a general version of the Cohen-Lyndon property which is defined for a family of subgroups and normal subgroups. The corresponding generalized version of Theorem 1.2.10 turns out to be useful in Chapter 5 when we construct particular quotients of acylindrically hyperbolic groups.

Remark 1.2.12. Historically, spectral sequences were introduced by Leray [21] in his attempt to compute cohomology of sheafs. In the proof of Theorem 1.2.10, we make use of the Lyndon-Hochschild-Serre spectral sequence, which was discovered by Lyndon [22] and then put into its current form by Hochschild-Serre [17].

Remark 1.2.13. Let G be a group with a subgroup H. Relative cohomology $H^*(G, H; \cdot)$ was introduced by [7], which shows that absolute and relative cohomology groups fit into a long exact sequence.

Proposition 1.2.14 ([7, Proposition 1.1]). *Let G be a group and let H be a subgroup of G. Then for every $\mathbb{Z}G$-module A, there exists a long exact sequence*

$$\cdots \to H^\ell(G,H;A) \to H^\ell(G;A) \to H^\ell(H;A) \to H^{\ell+1}(G,H;A) \to \cdots$$

whose arrows are natural maps of cohomology.

If $H \hookrightarrow_h G$, $N \lhd H$ is sufficiently deep, and some additional assumptions are met, [35, Theorem 1.1] provides a spectral sequence of homological type which computes $H^*(\overline{G},\overline{H};\mathbb{Z}\overline{G})$ from certain combination of homology and cohomology. Clearly, Theorem 1.2.10 (resp. [35, Theorem 1.1]), together with Proposition 1.2.14, can be applied to compute $H^*(\overline{G},\overline{H};\mathbb{Z}\overline{G})$ (resp. $H^*(\overline{G};\mathbb{Z}\overline{G})$). However, (1.2) and the spectral sequence of [35] are essentially different, as there is no homology involved in (1.2).

It is worth noting that if H has finite cohomological dimension, then Theorem 1.2.10 and Proposition 1.2.14 imply $H^\ell(\overline{G},\overline{H};A) \cong H^\ell(G;A)$ for every $\mathbb{Z}\overline{G}$-module A and sufficiently large ℓ (see Remark 1.2.17 below).

1.2.4. Homological properties of Dehn filling quotients. Recall that the *cohomological dimension* of a group G is

$$cd(G) = \sup\{\ell \in \mathbb{N} \mid H^\ell(G,A) \neq \{0\} \text{ for some } \mathbb{Z}G\text{-module } A\}$$

(in this paper, the set $\mathbb{N}$ of natural numbers contains 0, while the set of positive natural numbers is denoted as $\mathbb{N}^+$). A group G is of *type FP_∞* if there is a projective resolution

$$\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

over $\mathbb{Z}G$ such that P_n is finitely generated for each $n \in \mathbb{N}$. A group G is of *type FP* if (a) $cd(G) < \infty$ and (b) G is of type FP_∞.

Theorem 1.2.15. *Let $H \hookrightarrow_h G$ be groups. If $N \lhd H$ is sufficiently deep, then for all $\ell \geqslant cd(H)+2$ and any $\mathbb{Z}\overline{G}$-module A, we have*

$$H^\ell(\overline{G},A) \cong H^\ell(G,A) \bigoplus H^\ell(\overline{H},A). \tag{1.3}$$

In particular,

$$cd(\overline{G}) \leqslant \max\{cd(G), cd(H)+1, cd(\overline{H})\}.$$

Remark 1.2.16. In case G is a free group and $H \leqslant G$ is a maximal cyclic subgroup, the direct sum decomposition (1.3) is proved by [23, Theorem 11.1]. In case $G = G_1 * G_2$ is a free product of locally indicable groups G_1, G_2 and $H \leqslant G$ is the cyclic subgroup generated by an element $g \in G$ such that g is not a proper power and does not conjugate into either G_1 or G_2, (1.3) is proved by [18, Theorem 3]. Note that in these two cases, H is a hyperbolically embedded subgroup of G by Example 1.1.3. Thus, Theorem 1.2.15 recovers the results of [23, 18] for sufficiently deep (but not all) normal subgroups.

Notice that, (1.3) does not hold for $\ell \leqslant cd(H)+1$. For instance, let G be a group freely generated by two elements x and y and let $H = \langle h \rangle \leqslant G$ with $h = xyx^{-1}y^{-1}$. Then $H \hookrightarrow_h G$ by Example 1.1.3 and $cd(H)+1 = 2$. Let $N = \langle h^k \rangle \lhd H$ with k large enough so that N is sufficiently deep. By [23, Theorem 11.1], $H^2(\overline{G};\mathbb{Z}) \cong \mathbb{Z}$, and it is well-known that $H^2(G;\mathbb{Z}) = \{0\}$ and $H^2(\overline{H};\mathbb{Z}) \cong \mathbb{Z}/k\mathbb{Z}$. Thus, $H^2(\overline{G};\mathbb{Z}) \ncong H^2(G;\mathbb{Z}) \bigoplus H^2(\overline{H};\mathbb{Z})$.

Remark 1.2.17. As a by-product of the proof of Theorem 1.2.15, we show that for $\ell \geqslant cd(H)+2$, the natural map $H^\ell(\overline{G};A) \to H^\ell(\overline{H};A)$, induced by the inclusion $\overline{H} \hookrightarrow \overline{G}$, is surjective, and the kernel of this natural map can be identified with $H^\ell(G;A)$. This, together with Proposition 1.2.14, implies $H^\ell(\overline{G},\overline{H};A) \cong H^\ell(G;A)$ for $\ell \geqslant cd(H)+3$, and for $\ell = cd(H)+2$, there is a surjection $H^\ell(\overline{G},\overline{H};A) \twoheadrightarrow H^\ell(G;A)$.

Theorem 1.2.18. *Let $H \hookrightarrow_h G$ be groups. Suppose that $N \lhd H$ is sufficiently deep and $G, \overline{H}$ are of type FP_∞ (resp. FP). If either one of the following conditions holds, then $\overline{G}$ is also of type FP_∞ (resp. FP).*

(a) N is of type FP_∞.

(b) H is of the form $H = K \times F$, where K is a finite group and F is a finite rank free group, and $N \leqslant F$.

Remark 1.2.19. The seemingly unnatural condition (b) of Theorem 1.2.18 will be used in Chapter 5 to deal with acylindrically hyperbolic groups. For acylindrically hyperbolic groups, [13, Theorem 6.14] constructs hyperbolically embedded subgroups of the form described in condition (b). In most of the interesting cases, $N \lhd H$ is a free group of infinite rank and thus is not of type FP_∞, in which case condition (a) does not hold. It is unclear to us though whether the conclusion of Theorem 1.2.18 still holds if conditions (a) and (b) are dropped.

Remark 1.2.20. In Theorem 1.2.18, the condition that $\overline{H}$ is of type FP_∞ is necessary. Indeed, for sufficiently deep Dehn fillings, $\overline{H}$ embeds onto a hyperbolically embedded subgroup of $\overline{G}$. If $\overline{G}$ is of type FP_∞, then [13, Theorem 2.11] implies that $\overline{H}$ is also of type FP_∞.

Remark 1.2.21. In fact, we consider the general case of a family of weakly hyperbolically embedded subgroups. The corresponding general versions of Theorems 1.2.15 and 1.2.18 can be applied to graph of groups (see Remark 1.2.6).

1.2.5. Quotients of acylindrically hyperbolic groups The notion of acylindrically hyperbolic groups was introduced by Osin [28] as a generalization of non-elementary hyperbolic and non-elementary relatively hyperbolic groups. Examples of acylindrically hyperbolic groups can be found in many classes of group that interest group theorists for years, e.g., mapping class groups of surfaces, outer automorphism groups of free groups, small cancellation groups, convergence groups, Cremona groups, tame automorphism groups, etc. We refer to [29] for details and other examples of acylindrically hyperbolic groups.

It is known that acylindrically hyperbolic groups have a lot of quotients. For instance, every acylindrically hyperbolic group G is *SQ-universal* [13], i.e., every countable group can be embedded into a quotient of G. Also, if two finitely generated acylindrically hyperbolic groups G_1 and G_2 are given, one can construct a common acylindrically hyperbolic quotient of G_1 and G_2 [19]. As an application of our main results, we study homological properties of those quotients.

For the following theorems, recall that every acylindrically hyperbolic group G has a maximal finite normal subgroup $K(G)$ [13, Theorem 6.14].

Theorem 1.2.22. *Let G be an acylindrically hyperbolic group, and let C be any countable group. Then C embeds into a quotient $\overline{G}$ of G such that*

(a) $\overline{G}$ is acylindrically hyperbolic;

(b) $cd(\overline{G}) \leqslant \max\{cd(G), cd(C)\}$;

(c) if $K(G) = \{1\}$, then for all $\ell \geqslant 3$ and any $\mathbb{Z}\overline{G}$-module A, we have

$$H^\ell(\overline{G}; A) \cong H^\ell(G; A) \bigoplus H^\ell(C; A);$$

(d) if C is finitely generated, then $C \hookrightarrow_h \overline{G}$;

(e) if G and C are of type FP_∞, then so is $\overline{G}$.

Theorem 1.2.23. *Let G_1 and G_2 be finitely generated acylindrically hyperbolic groups. Then there exists a common quotient G of G_1 and G_2 such that*

(a) G is acylindrically hyperbolic;

(b) $cd(G) \leqslant \max\{cd(G_1), cd(G_2)\}$;

(c) if $K(G_1) = K(G_2) = \{1\}$, then for all $\ell \geqslant 3$ and any $\mathbb{Z}G$-module A, we have

$$H^\ell(G; A) \cong H^\ell(G_1; A) \bigoplus H^\ell(G_2; A);$$

(d) if G_1 and G_2 are of type FP_∞, then so is G.

Remark 1.2.24. Except for the homological conditions, Theorems 1.2.22 and 1.2.23 are proved by [13, Theorem 8.1] and [19, Corollary 7.4], respectively. The benefit of Theorems 1.2.22 and 1.2.23 is that they allow constructions of various acylindrically hyperbolic groups satisfying certain homological conditions.

CHAPTER 2

PRELIMINARIES

We introduce conventions and notations and recall preliminaries in this chapter. In Sections 2.1 and 2.2, we recall the notation of Cayley graphs and van Kampen diagrams. Section 2.3, whose main references are [8, 34], reviews the notions of Gromov hyperbolic spaces and Gromov boundaries. In Sections 2.5 through 2.7, whose main references are [13, 29], we recall the definition and basic information about acylindrically hyperbolic groups and (weakly) hyperbolically embedded subgroups. In Sections 2.6 and 2.7, we review the concepts of isolated components and diagram surgery, which were introduced by Osin [27] and are useful in the proof of Cohen-Lyndon type theorems in Chapter 3.

In Section 2.8, we introduce notations related to direct sums and products of abelian group homomorphisms. Sections 2.9 through 2.13, whose main references are [10, 31], are devoted to a series of concepts related to group cohomology. Section 2.14 defines the Cohen-Lyndon property and introduces related notations. Sections 2.15 and 2.16, whose main references are [31, 36], are devoted to spectral sequences and related concepts, which are used in the Chapter 4 when we study certain spectral sequences with the aid of the Cohen-Lyndon property.

2.1 Words and Cayley graphs

Let X be an alphabet. Given a word w over X, the *length* of w, denoted as $\|w\|$, is the number of letters in w. If X is the generating set of a group G, the *word length* of an element $g \in G$ with respect to X, denoted as $|g|_X$, is the length of a shortest word (geodesic word) w over X such that w represents g in G. If X is understood from the context, we will simply write $|g|$ instead of $|g|_X$.

There are two types of equalities for words over X. Given two words u and v over X, the notation $u \equiv v$ indicates the letter-by-letter equality between u and v and the notation $u =_G v$ indicates that u and v represent the same element of G.

If u is a word over X, then u^{-1} denotes the inverse of u. If, in addition, $g \in G$ and $S \subset H$, then we write $u = g$ to indicate that u represents g in G, and write $u \in S$ to indicate that the element of G represented by u is in S.

The *Cayley graph* $\Gamma(G, X)$ is the labeled directed graph with vertices labeled by elements of G and directed edges labeled by elements of X. In G (resp. $\Gamma(G, X)$), we use 1 to denote the identity (resp. identity vertex). The word metric of $\Gamma(G, X)$ with respect to the alphabet X is denoted as d_X. Let p be an edge path in $\Gamma(G, X)$. Then $\ell_X(p)$ denotes the length of p under d_X. Lab(p) denotes the *label* of p, i.e., $Lab(p)$ is obtained by concatenate labels of edges of p. p^- (resp. p^+) denotes the initial (resp. terminal) vertex of p. If $S \subset \Gamma(G, X)$, then $diam_X(S)$ denotes the diameter of S under d_X. If T is another subset of $\Gamma(G, X)$, then $d_{Hau}(S, T)$ denotes the Hausdorff distance between S and T.

2.2 Van Kampen diagrams

Let G be a group given by the presentation

$$G = \langle \mathcal{A} \mid \mathcal{R} \rangle, \tag{2.1}$$

where $\mathcal{A}$ is a symmetric set of letters and $\mathcal{R}$ is a symmetric set of words in $\mathcal{A}$ (i.e., for every $w \in \mathcal{R}$, every cyclic shift of w or w^{-1} belongs to $\mathcal{R}$).

A *van Kampen diagram* Δ over (2.1) is a finite oriented connected planar 2-complex with labels on its oriented edges such that

(a) Each oriented edge of Δ is labeled by a letter in $\mathcal{A} \cup \{1\}$;

(b) If an oriented edge e of Δ has label $a \in \mathcal{A} \cup \{1\}$, then e^{-1} has label a^{-1}, where e^{-1} (resp. a^{-1}) is the inverse of e (resp. a).

Here, 1 is identified with the empty word over $\mathcal{A}$ and thus $1 = 1^{-1}$. By convention, the empty word of $\mathcal{A}$ represents the identity of G.

Let $p = e_1 \cdots e_k$ be a path in a van Kampen diagram over (2.1). The initial vertex (resp. terminal vertex) of p is denoted as p^- (resp. p^+). The *label* of p, denoted as $Lab(p)$, is obtained by first concatenating the labels of the edges $e_1, ..., e_k$ and then removing all 1's, as 1 is identified with the empty word. Therefore, the label of a path in a van Kampen diagram is a word over $\mathcal{A}$. If w is a word over $\mathcal{A}$, then the notation $Lab(p) \equiv w$ indicates a letter-by-letter equality between $Lab(p)$ and w.

Remark 2.2.1. Suppose that p is a path in a van Kampen diagram over (2.1) with $Lab(p) \equiv w_1 \cdots w_k$. Then we can decompose p in the following way: Let p_{w_1} be the maximal subpath of p such that $p_{w_1}^- = p^-$

and $Lab(p_{w_1}) \equiv w_1$. For $i = 2, ..., k$, let p_{w_i} be the maximal subpath of p such that $p_{w_i}^- = p_{w_{i-1}}^+$ and $Lab(p_{w_i}) \equiv w_i$.

Edges labeled by letters from $\mathcal{A}$ are called *essential edges*, while edges labeled by the letter 1 are called *non-essential edges*. A *face* of Δ is a 2-cell of Δ. Let Π be a face of Δ, the boundary of Π is denoted as $\partial\Pi$. Likewise, the boundary of Δ is denoted by $\partial\Delta$. Note that if we choose a base point for $\partial\Pi$ (resp. $\partial\Delta$), then $\partial\Pi$ (resp. $\partial\Delta$) becomes a path in Δ. For a word w over $\mathcal{A}$, we use the notation $Lab(\partial\Pi) \equiv w$ (resp. $Lab(\partial\Delta) \equiv w$) to indicate that one can pick a base point to turn $\partial\Pi$ (resp. $\partial\Delta$) into a path p so that $Lab(p) \equiv w$.

Remark 2.2.2. Suppose that Δ is a diagram with $Lab(\partial\Delta) \equiv w_1 \cdots w_k$. Then we can decompose $\partial\Delta$ in the following way: Let p_b be vertex of $\partial\Delta$ such that when we use p_b as the base point of $\partial\Delta$, we can turn $\partial\Delta$ into a path p with $Lab(p) \equiv w_1 \cdots w_k$. And then we use Remark 2.2.1 to decompose p and thus decompose $\partial\Delta$.

Consider the following additional assumption on van Kampen diagrams:

(c) For every face Π of a van Kampen diagram Δ over the presentation (2.1), at least one of the following conditions (c1) and (c2) holds.

(c1) $Lab(\partial\Pi)$ is equal (up to a cyclic permutation) to an element of $\mathcal{R}$.

(c2) $\partial\Pi$ either consists entirely of non-essential edges or consists of exact two essential edges with mutually inverse labels (in addition to non-essential edges).

A face satisfying (c_2) is called a *non-essential face*. All other faces are called *essential faces*. The process of adding non-essential faces to a van Kampen diagram is called a *refinement*. Figure 2.1 illustrates a refinement on a van Kampen diagram, where the unlabeled edges are labeled by 1. The interested readers are referred to [25] for a formal discussion. By using refinements, we can ensure

(d) Every face is homeomorphic to a disc, i.e., its boundary has no self-intersection.

Assumption 2.2.3. In the sequel, the above assumptions (c) and (d) will be imposed on van Kampen diagrams.

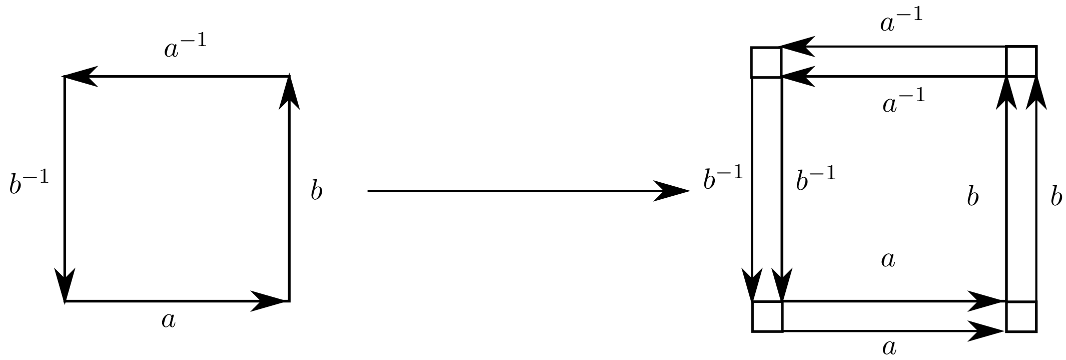


Figure 2.1: A refinement of a van Kampen diagram over the presentation $G = \langle a, b \mid aba^{-1}b^{-1} = 1\rangle$

The well-known van Kampen lemma states that a word w over $\mathcal{A}$ represents 1 in G if and only if there is a van Kampen diagram Δ over (2.1) such that Δ is homeomorphic to a disc (such diagrams are called *disk diagrams*), and that $Lab(\partial\Delta) \equiv w$.

Remark 2.2.4. If a van Kampen diagram Δ is homeomorphic to a disc, and O is a vertex of Δ, then there exists a unique continuous map μ from the 1-skeleton of Δ to the Cayley graph $\Gamma(G, \mathcal{A})$ sending O to the identity vertex, preserving the labels of the essential edges and collapsing non-essential edges to points.

2.3 Gromov hyperbolic spaces and Gromov boundary

Let (S, d) be a geodesic metric space and let Δ be a geodesic triangle consists of three geodesic segments $\gamma_1, \gamma_2, \gamma_3$. For a number $\delta \geqslant 0$, Δ is called δ-*slim* if the distance between every point of γ_i and the union $\gamma_j \cup \gamma_k$ is less than δ, where $i, j, k \in \{1, 2, 3\}, i \neq j, j \neq k, k \neq i$.

Notation 2.3.1. We use (S, d) to denote a space S with metric d. If the metric d is unnecessary or well-understood, we will omit it and write S for a metric space.

S is called a δ-*hyperbolic space* if geodesic triangles in S are all δ-slim. S is called a *Gromov hyperbolic space* if it is δ-hyperbolic for some $\delta \geqslant 0$. Gromov hyperbolic spaces generalize notions such as simplicial trees and complete simply connected Riemannian manifolds with constant negative sectional curvature while preserving most of the interesing properties.

Remark 2.3.2. In literature, properness is often part of the definition of a Gromov hyperbolic space. However, in this thesis, we do not assume that a Gromov hyperbolic space S is proper, i.e. some closed balls of S might not be compact.

Let S be a Gromov hyperbolic space. The *Gromov product* is defined by

$$(x \cdot y)_z = (d(x,z) + d(y,z) - d(x,y))/2.$$

Pick a point $e \in S$, viewing as the base point of the Gromov product. The *Gromov boundary* ∂S_e of S *with respect to* e is defined as follows. A sequence of points $\{s_n\}_{n \geqslant 1} \subset S$ is called a Gromov sequence if $(s_i \cdot s_j)_e \to \infty$ as i and $j \to \infty$. We say that two Gromov sequences $\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1}$ are *equivalent* and write $\{x_n\}_{n\geqslant 1} \sim \{y_n\}_{n\geqslant 1}$ if $(x_n \cdot y_n)_e \to \infty$ as $n \to \infty$. ∂S_e is then defined as the set of all Gromov sequences modulo the equivalence relation $\sim$. Elements of ∂S_e are just equivalence classes of Gromov sequences in S and we say a sequence $\{x_n\}_{n\geqslant 1} \in S$ *tends to* a boundary point $x \in \partial S_e$ and write $x_n \to x$ as $n \to \infty$ if $\{x_n\}_{n\geqslant 1} \in x$.

If e and f are two points of S, then ∂S_e and ∂S_f can be naturally identified [34]. We thus obtain a well-defined notion of the *Gromov boundary* ∂S of S.

2.4 Acylindrically hyperbolic groups

Let (S,d) be a Gromov hyperbolic space and let G be a group acting on S by isometries. The action of G is called *acylindrical* if for every $\epsilon > 0$ there exist $R, N > 0$ such that for every two points $x,\ y$ with $d(x,y) \geqslant R$, there are at most N elements $g \in G$ satisfying both $d(x,gx) \leqslant \epsilon$ and $d(y,gy) \leqslant \epsilon$. The *limit set* $\Lambda(G)$ of G on ∂S is the set of limit points in ∂S of a G-orbit in S, i.e.

$$\Lambda(G) = \{x \in \partial S \mid \text{ there exists a Gromov sequence in } Gs \text{ tending to } x, \text{ for some } s \in S\}.$$

If $\Lambda(G)$ contains more than two points, we say the action of G is *non-elementary*. Acylindrically hyperbolic groups are defined in [28].

Definition 2.4.1. A group G is *acylindrically hyperbolic* if G admits a non-elementary acylindrical action on some Gromov hyperbolic spaces by isometries.

2.5 Hyperbolically embedded subgroups and group theoretic Dehn fillings

Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of G, let X be a subset of G such that G is generated by X together with the union of all $H_\lambda, \lambda \in \Lambda$, and let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$. Consider the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$. Note that $\Gamma(G, X \sqcup \mathcal{H})$ is a metric space under the word metric.

Remark 2.5.1. It is possible that X and $H_\lambda, \lambda \in \Lambda$, as subsets of G, have non-empty intersections with each other. As a consequence, several letters of $X \sqcup \mathcal{H}$ might represent the same element of G. If this is the case, the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ will have multiple edges corresponding to those letters.

Note that for each $\lambda \in \Lambda$, the Cayley graph $\Gamma(H_\lambda, H_\lambda)$ can be identified as the complete subgraph of $\Gamma(G, X \sqcup \mathcal{H})$ whose vertex set is H_λ, and edges are the ones labeled by letters from H_λ.

Definition 2.5.2. Fix $\lambda \in \Lambda$. A (combinatorial) path p in $\Gamma(G, X \sqcup \mathcal{H})$ between vertices of $\Gamma(H_\lambda, H_\lambda)$ is called *H_λ-admissible* if it does not contain any edge of $\Gamma(H_\lambda, H_\lambda)$. Note that a H_λ-admissible path p is allowed to pass through vertices of $\Gamma(H_\lambda, H_\lambda)$. For every pair of elements $h, k \in H_\lambda$, let $\widehat{d}_\lambda(h, k) \in [0, +\infty]$ be the length of a shortest H_λ-admissible path connecting h, k. If no such path exists, set $\widehat{d}_\lambda(h, k) = +\infty$. The laws of summation on $[0, +\infty)$ extend naturally to $[0, +\infty]$ and it is easy to verify that $\widehat{d}_\lambda : H_\lambda \times H_\lambda \to [0, +\infty]$ defines a metric on $\Gamma(H_\lambda, H_\lambda)$ called the *relative metric on* $\Gamma(H_\lambda, H_\lambda)$ *with respect to* X.

Remark 2.5.3. Let p be a path in $\Gamma(G, X \sqcup \mathcal{H})$ with $Lab(p) \equiv h \in H_\lambda$, for some $\lambda \in \Lambda$. For simplicity, we denote $\widehat{d}_\lambda(1, h)$ by $\widehat{\ell}_\lambda(p)$.

Definition 2.5.4. Let G be a group, let $\{H_\lambda\}_{\lambda \in \Lambda}$ be a family of subgroups of G, let X be a subset of G, and let $\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} H_\lambda$. We say that $\{H_\lambda\}_{\lambda \in \Lambda}$ is *weakly hyperbolically embedded into* (G, X) (denoted as $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} (G, X)$) if G is generated by the set X together with union of all $H_\lambda, \lambda \in \Lambda$, and the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ is a Gromov hyperbolic space.

If the collection $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} (G, X)$ and for each $\lambda \in \Lambda$, the metric space $(H_\lambda, \widehat{d}_\lambda)$ is proper, i.e., every ball of finite radius contains only finitely many elements, then $\{H_\lambda\}_{\lambda \in \Lambda}$ is called *hyperbolically embedded into* (G, X) (denoted as $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$).

Further, the collection $\{H_\lambda\}_{\lambda \in \Lambda}$ is called *weakly hyperbolically embedded into* (resp. *hyperbolically embedded into*) G, denoted as $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} G$ (resp. $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h G$), if there exists some subset $X \subset G$ such that $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} (G, X)$ (resp. $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$).

Remark 2.5.5. Note that if the family $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} (G, X)$ for some subset $X \subset G$ and $Y = X \cup X^{-1}$, then we also have $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} (G, Y)$. In the sequel, we always assume that the relative generating set X is symmetric, i.e., $X = X^{-1}$.

Notation 2.5.6. Let G, H be groups and let $X \subset G$. If $\{H\} \hookrightarrow_h (G, X)$, then we drop braces and write $H \hookrightarrow_h (G, X)$ and $H \hookrightarrow_h G$. If H is not a subgroup of G but there is a subgroup $K \hookrightarrow_h G$ such that $H \cong K$, then we will slightly abuse notation and write $H \hookrightarrow_h G$.

Examples of hyperbolically embedded subgroups can be found in acylindrically hyperbolic groups. In particular, we have the following.

Theorem 2.5.7 ([13, Theorem 6.14]). *Let G be an acylindrically hyperbolic group. Then G has a maximal finite normal subgroup $K(G)$. Moreover, for $n \in \mathbb{N}$, there exists a free group F of rank n such that $F \times K(G) \hookrightarrow_h G$.*

Remark 2.5.8. If a group G can be decomposed as a free product $G = G_1 * G_2$, then

$$\{G_1, G_2\} \hookrightarrow_h (G, \emptyset)$$

by [13, Example 4.12]. In this case, the relative metrics

$$\widehat{d_1} : G_1 \times G_1 \to [0, +\infty], \quad \widehat{d_2} : G_2 \times G_2 \to [0, +\infty]$$

with respect to $\emptyset$ satisfy

$$\widehat{d_1}(1,1) = \widehat{d_2}(1,1) = 0, \quad \widehat{d_1}(1, g_1) = \widehat{d_2}(1, g_2) = +\infty$$

for $g_1 \in G_1 \backslash \{1\}, g_2 \in G_2 \backslash \{1\}$.

Note that if $G = G_1 * G_2$, then we also have

$$G_1 \hookrightarrow_h (G, G_2).$$

Proposition 2.5.9 ([13, Proposition 4.35]). *If G, H, K are groups and $X \subset G, Y \subset H$ such that $K \hookrightarrow_h (H, Y)$ and $H \hookrightarrow_h (G, X)$, then $K \hookrightarrow_h (G, X \cup Y)$.*

Theorem 2.5.10 ([13, Theorem 4.24]). *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ and let $X \subset G$. Then the following are equivalent.*

(a) $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_h (G, X)$.

(b) There exists a strongly bounded relative presentation of G with respect to X and $\{H_\lambda\}_{\lambda\in\Lambda}$ with linear relative isoperimetric function.

For the definition of a strongly bounded relative presentation (resp. a linear relative isoperimetric function), the reader is referred to [13, Definition 4.22] (resp. [13, Section 3.3]).

Definition 2.5.11. Suppose that G is a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ for some subset $X \subset G$. For $\lambda \in \Lambda$, let $\widehat{d}_\lambda$ be the relative metric on $\Gamma(H_\lambda, H_\lambda)$ with respect to X. We say that a property P *holds for all sufficiently deep Dehn fillings of* $\{H_\lambda\}_{\lambda\in\Lambda}$ (or *for all sufficiently deep* $N_\lambda \lhd H_\lambda, \lambda \in \Lambda$,) if there exists a number $C > 0$ such that if $N_\lambda \lhd H_\lambda$ and $\widehat{d}_\lambda(1, n) > C$ for all $n \in N_\lambda\backslash\{1\}, \lambda \in \Lambda$, then P holds.

One remarkable property of weakly hyperbolically embedded subgroups is the following group theoretic Dehn filling theorem.

Theorem 2.5.12 ([13, Theorem 7.15]). *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ for some subset $X \subset G$. Then for all sufficiently deep $N_\lambda \lhd H_\lambda, \lambda \in \Lambda$, we have:*

(a) For each $\lambda \in \Lambda$, the natural homomorphism $i_\lambda : H_\lambda/N_\lambda \to G/\langle\langle\mathcal{N}\rangle\rangle$ is injective (i.e., $H_\lambda \cap \langle\langle\mathcal{N}\rangle\rangle = N_\lambda$), where $\mathcal{N} = \bigcup_{\lambda\in\Lambda} N_\lambda$.

(b) $\{i_\lambda(H_\lambda/N_\lambda)\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G/\langle\langle\mathcal{N}\rangle\rangle, \overline{X})$, where $\overline{X}$ is the image of X under the quotient map $G \to G/\langle\langle\mathcal{N}\rangle\rangle$.

(c) There exist subsets $T_\lambda \subset G, \lambda \in \Lambda$, such that $\langle\langle\mathcal{N}\rangle\rangle = \prod^*_{\lambda\in\Lambda, t\in T_\lambda} N^t_\lambda$, where $N^t_\lambda = tN_\lambda t^{-1}$ for $\lambda \in \Lambda$ and $t \in T_\lambda$.

2.6 Isolated components

Let us assume, until the end of Section 2.7, that G is a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} (G, X)$ for some symmetric subset $X \subset G$. For each $\lambda \in \Lambda$, let $\widehat{d}_\lambda$ be the relative metric on $\Gamma(H_\lambda, H_\lambda)$ with respect to X, and let $\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} H_\lambda$. The following terminology goes back to [26].

Definition 2.6.1. Let p be a path in $\Gamma(G, X \sqcup \mathcal{H})$. Fix $\lambda \in \Lambda$. An H_λ*-subpath* q of p is a nontrivial subpath of p labeled by a word over the alphabet H_λ (if p is a cycle, we allow q to be a subpath of some cyclic shift of p). An H_λ-subpath q of p is called an H_λ-*component* if q is not properly contained in any other H_λ-subpath. Two H_λ-components q_1, q_2 of p are called *connected* if there exists a path c in $\Gamma(G, X \sqcup \mathcal{H})$ such that c connects a vertex of q_1 to a vertex of q_2, and that $Lab(c)$ is a letter from H_λ. An H_λ-component q of p is called *isolated* if it is not connected to any other H_λ-component of p.

The key property of isolated components is that, in a *geodesic polygon* (i.e., a polygon in $\Gamma(G, X \sqcup \mathcal{H})$ with geodesic sides) p, the total $\widehat{\ell}$-length of isolated components is uniformly bounded above by a linear function of the number of sides. The following result is proved in [13, Proposition 4.14], which is a straightforward generalization of [27, Proposition 3.2].

Lemma 2.6.2 (Dahmani-Guirardel-Osin). *There exists a positive number D satisfying the following property: Let p be an n-gon in $\Gamma(G, X \sqcup \mathcal{H})$ with geodesic sides $p_1, ..., p_n$ and let I be a subset of the set of sides of p such that every side $p_i \in I$ is an isolated H_{λ_i}-component of p for some $\lambda_i \in \Lambda$. Then*

$$\sum_{p_i \in I} \widehat{\ell}_{\lambda_i}(p_i) \leqslant Dn.$$

Remark 2.6.3. Theorem 2.5.12 asserts the existence of a constant C such that if $\widehat{d_\lambda}(1, n) \geqslant C$ for every $n \in N_\lambda \backslash \{1\}$ and $\lambda \in \Lambda$, then $H_\lambda \cap \langle\langle \mathcal{N} \rangle\rangle = N_\lambda$ for all $\lambda \in \Lambda$. In fact, one can let $C = 4D$, where D is the constant provided by Lemma 2.6.2 (see [13]).

2.7 Diagram surgery

The diagram surgery surveyed in this section was first introduced by Osin [27], where he proved a group theoretic Dehn filling theorem for relatively hyperbolic groups. Later, Dahmani et al. generalized this technique to deal with weakly hyperbolically embedded subgroups [13].

Consider a symmetric set $\mathcal{R}$ of words over the alphabet $X \sqcup \mathcal{H}$ such that G has the presentation

$$G = \langle X \sqcup \mathcal{H} \mid \mathcal{R} \rangle, \tag{2.2}$$

and that for all $\lambda \in \Lambda$, $\mathcal{R}$ contains all words over the alphabet H_λ which represent the identity.

Suppose that N_λ is a normal subgroup of H_λ for each $\lambda \in \Lambda$. Denote the union of $N_\lambda, \lambda \in \Lambda$, by $\mathcal{N}$. The *normal closure* of $\mathcal{N}$ in G, denoted as $\langle\langle \mathcal{N} \rangle\rangle$, is the smallest normal subgroup of G containning $\mathcal{N}$.

Killing $\langle\langle \mathcal{N} \rangle\rangle$ in G is equivalent to adding, to $\mathcal{R}$, all words over H_λ which represent elements of N_λ, for all $\lambda \in \Lambda$, to form a new presentation

$$\overline{G} = G/\langle\langle \mathcal{N} \rangle\rangle = \langle X \sqcup \mathcal{H}, \mathcal{R} \cup \mathcal{S} \rangle, \tag{2.3}$$

where $\mathcal{S} = \bigcup_{\lambda \in \lambda} \mathcal{S}_\lambda$ and $\mathcal{S}_\lambda$ consists of all words over H_λ representing elements of N_λ in G.

In the sequel, let $\mathcal{D}$ be the set of all van Kampen diagrams Δ over (2.3) such that:

(D1) Topologically Δ is a disc with $k \geqslant 0$ holes. The boundary of Δ can be decomposed as $\partial\Delta = \partial_{ext}\Delta \cup \partial_{int}\Delta$, where $\partial_{ext}\Delta$ is the boundary of the disc, and $\partial_{int}\Delta$ consists of disjoint cycles (connected components) $c_1, ..., c_k$ that bound the holes.

(D2) For $i = 1, ..., k$, c_i is labeled by a word from $\mathcal{S}$.

(D3) Each diagram Δ is equipped with a *cut system* that is a collection $T = \{t_1, ..., t_k\}$ of disjoint paths (*cuts*) $t_1, ..., t_k$ in Δ without self-intersections such that, for $i = 1, ..., k$, the two endpoints of t_i belong to $\partial\Delta$, and that after cutting Δ along t_i for all $i = 1, ..., k$, one gets a disc van Kampen diagram $\widetilde{\Delta}$ over (2.2).

See Figure 2.2 for an illustration of a diagram in $\mathcal{D}$.

Lemma 2.7.1. *A word w over $X \sqcup \mathcal{H}$ represents 1 in $\overline{G}$ if and only if there is a diagram $\Delta \in \mathcal{D}$ such that $Lab(\partial_{ext}\Delta) \equiv w$.*

Proof. Let w be a word over $X \sqcup \mathcal{H}$. If there is a diagram $\Delta \in \mathcal{D}$ such that $\partial_{ext}\Delta \equiv w$, by filling the holes of Δ with faces whose boundaries are labeled by words from $\mathcal{S}$, one creates a disc van Kampen diagram over (2.2), whose boundary is labeled by w. Conversely, if w represents 1 in $\overline{G}$, then there exists a disc van Kampen diagram $\overline{\Delta}$ over (2.2) with $Lab(\partial\overline{\Delta}) \equiv w$. By removing all faces of $\overline{\Delta}$ labeled by words from $\mathcal{S}$, we obtain a diagram Δ' satisfying (D1) and (D2). To produce a cut system, choose a vertex O in $\partial_{ext}\Delta'$. Connect O with each component of $\partial_{int}\Delta'$ by a path so that these paths do not cross each other (although they do intersect each other). By passing to a refinement of Δ', one can separate these paths so that they no longer intersect each other and thus creates a diagram Δ satisfying (D1), (D2), and (D3) with $Lab(\partial_{ext}\Delta) \equiv w$. □

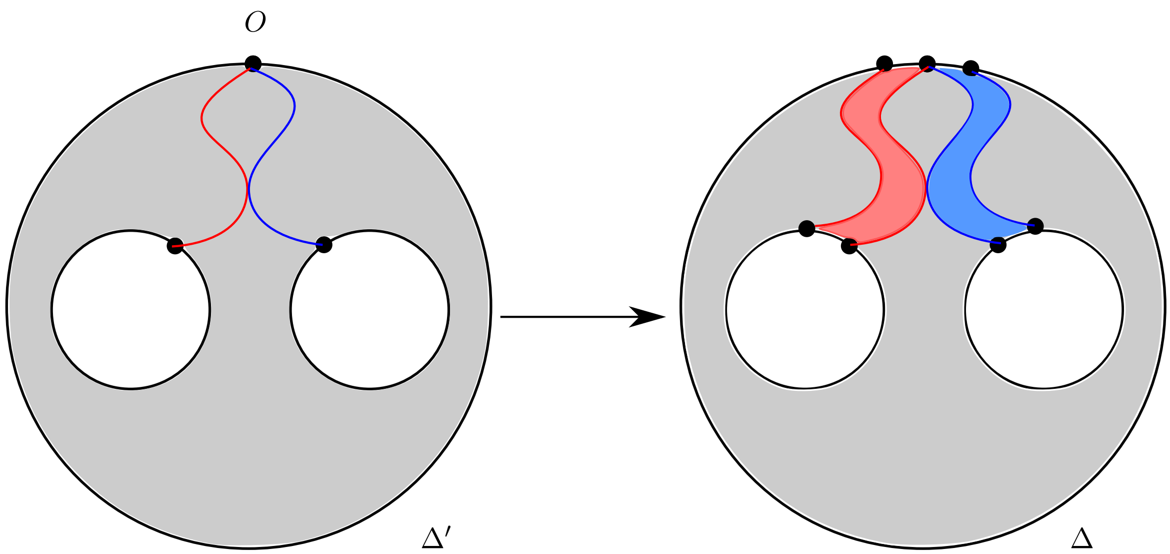


Figure 2.2: How to produce a cut system

Figure 2.2 illustrates the last step of the above proof. The left half shows the diagram Δ' with red and blue paths connect O with two components of $\partial_{int}\Delta'$. By thickening these paths with a refinement, we obtain the right half. The red and blue regions consist of non-essential faces, while the outside red and blue paths form a cut system.

Let Δ be a diagram in $\mathcal{D}$ and let $\widetilde{\Delta}$ be the disc van Kampen diagram resulted from cutting Δ along its set of cuts. Define $\kappa : \widetilde{\Delta} \to \Delta$ to be the map that "sews" the cuts. Fix an arbitrary vertex O in $\widetilde{\Delta}$ and let μ be a map sending the 1-skeleton of Δ to $\Gamma(G, X \sqcup \mathcal{H})$, as described by Remark 2.2.4.

Definition 2.7.2. Let Δ_1 and Δ_2 be two diagrams of $\mathcal{D}$ and let Γ_1 (resp. Γ_2) be the subgraph of the 1-skeleton of Δ_1 (resp. Δ_2) consisting of $\partial\Delta_1$ (resp. $\partial\Delta_2$) and all cuts of Δ_1 (resp. Δ_2). We say that Δ_1 and Δ_2 are *equivalent* if there exists a graph isomorphism $\Gamma_1 \to \Gamma_2$ which preserves labels and orientations of edges, and maps the cuts and boundary of Δ_1 to the cuts and boundary of Δ_2, respectively.

The following Lemmas 2.7.3 and 2.7.8 are results from [13], which are straightforward generalizations of results of [27]. Note that the authors of [13] assume that the presentation (2.2) has a linear relative isoperimetric function, but this assumption is not used in the proofs of those lemmas.

Lemma 2.7.3 ([13, Lemma 7.11] (see also [27, Lemma 4.2])). *Let a, b be two vertices on $\partial\Delta$ and let $\widetilde{a}, \widetilde{b}$ be two vertices on $\partial\widetilde{\Delta}$ such that $\kappa(\widetilde{a}) = a, \kappa(\widetilde{b}) = b$. Then for any path p in $\Gamma(G, X \sqcup \mathcal{H})$ connecting $\mu(\widetilde{a})$*

to $\mu(\widetilde{b})$, there is a diagram $\Delta_1 \in \mathcal{D}$ with the following properties:

(a) Δ and Δ_1 are equivalent.

(b) There is a path q in Δ_1 without self-intersections such that (1) q connects a and b, (2) q has no common vertices with the cuts of Δ_1 except possibly for a, b, and (3) $Lab(q) \equiv Lab(p)$.

Definition 2.7.4. Fix $\lambda \in \Lambda$. An H_λ-*subpath* in $\partial\Delta$ (resp. $\partial\widetilde{\Delta}$) for some $\Delta \in \mathcal{D}$ is a path labeled by a nontrivial word over H_λ. An H_λ-subpath p of $\partial\Delta$ (resp. $\partial\widetilde{\Delta}$) is called an H_λ-*component* if p is not properly contained in any other H_λ-subpath. Two H_λ-components p, q of $\partial\Delta$ are *connected* if there exist H_λ-components a, b in $\partial\widetilde{\Delta}$ such that $\kappa(a)$ (resp. $\kappa(b)$) is a subpath of p (resp. q), and that $\mu(a), \mu(b)$ are connected in $\Gamma(G, X \sqcup \mathcal{H})$ (in the sense of Definition 2.6.1).

Remark 2.7.5. The definitions of H_λ-subpaths, H_λ-components, and connected H_λ-components in $\partial\Delta$ for a van Kampen diagram $\Delta \in \mathcal{D}$ or $\partial\widetilde{\Delta}$ do not depend on the pre-chosen vertex O.

Definition 2.7.6. The *type* of Δ is defined by the formula

$$\tau(\Delta) = (k, \sum_{i=1}^{k} \|Lab(t_i)\|),$$

where k is the number of holes in Δ and $t_1, ..., t_k$ are the cuts. We order the types of diagrams in $\mathcal{D}$ lexicographically: $(k_1, \ell_1) < (k_2, \ell_2)$ if and only if either $k_1 < k_2$ or $k_1 = k_2$ and $\ell_1 < \ell_2$.

Definition 2.7.7. For any word w over $X \sqcup \mathcal{H}$, let $\mathcal{D}(w)$ be the set of diagrams $\Delta \in \mathcal{D}$ such that $Lab(\partial_{ext}\Delta) \equiv w$.

Lemma 2.7.8 ([13, Lemma 7.17] (see also [27, Lemma 5.2])). *Suppose that for every $\lambda \in \Lambda$ and $n \in N_\lambda \backslash \{1\}$, we have $\widehat{d_\lambda}(1, n) > 4D$, where D is the constant given by Lemma 2.6.2. Let w be a geodesic word over $X \sqcup \mathcal{H}$ representing 1 in $\overline{G}$, and let Δ be a diagram in $\mathcal{D}(w)$ of minimal type. Then there exist $\lambda \in \Lambda$ and a connected component c of $\partial_{int}\Delta$ such that c is connected to an H_λ-component of $\partial_{ext}\Delta$.*

2.8 Direct sums and products of abelian group homomorphisms

Let $f_\lambda : X_\lambda \to Y, \lambda \in \Lambda$, be homomorphisms between abelian groups. The *domain sum* of $f_\lambda, \lambda \in \Lambda$, denoted as

$$\bigoplus_{\lambda \in \Lambda}^{Dom} f_\lambda : \bigoplus_{\lambda \in \Lambda} X_\lambda \longrightarrow Y,$$

is defined as follows. For every $\lambda \in \Lambda$, let $i_\lambda : X_\lambda \to \bigoplus_{\lambda\in\Lambda} X_\lambda$ be the natural inclusion. Then $\bigoplus_{\lambda\in\Lambda}^{Dom} f_\lambda$ is the unique abelian group homomorphism such that

$$\bigoplus_{\lambda\in\Lambda}^{Dom} f_\lambda \circ i_\lambda = f_\lambda$$

for $\lambda \in \Lambda$.

If $f_\lambda : X_\lambda \to Y_\lambda, \lambda \in \Lambda$, are abelain group homomorphisms, then we define the *domain-target sum* of f_λ, denoted as

$$\bigoplus_{\lambda\in\Lambda}^{DT} f_\lambda : \bigoplus_{\lambda\in\Lambda} X_\lambda \longrightarrow \bigoplus_{\lambda\in\Lambda} Y_\lambda,$$

by the following rule. For each $\lambda \in \Lambda$, let $p_\lambda : \bigoplus_{\lambda\in\Lambda} Y_\lambda \to Y_\lambda$ be the natural projection. Then $\bigoplus_{\lambda\in\Lambda}^{DT} f_\lambda$ is the unique abelian group homomorphism such that

$$p_\lambda \circ \bigoplus_{\lambda\in\Lambda}^{DT} f_\lambda \circ i_\lambda = f_\lambda$$

for $\lambda \in \Lambda$.

In contrast, if $f_\lambda : X \to Y_\lambda, \lambda \in \Lambda$, are homomorphisms between abelian groups, then the *target product* of $f_\lambda, \lambda \in \Lambda$, denoted as

$$\prod_{\lambda\in\Lambda}^{Tar} f_\lambda : X \longrightarrow \prod_{\lambda\in\Lambda} Y_\lambda,$$

is defined as follows. For each $\lambda \in \Lambda$, let $\pi_\lambda : \prod_{\lambda\in\Lambda} Y_\lambda \to Y_\lambda$ be the coordinate projection. Then $\prod_{\lambda\in\Lambda}^{Tar} f_\lambda$ is the unique abelian group homomorphism such that

$$\pi_\lambda \circ \prod_{\lambda\in\Lambda}^{Tar} f_\lambda = f_\lambda$$

for $\lambda \in \Lambda$.

If $f_\lambda : X_\lambda \to Y_\lambda, \lambda \in \Lambda$, are abelain group homomorphisms, then we define the *domain-target product* of f_λ, denoted as

$$\prod_{\lambda\in\Lambda}^{DT} f_\lambda : \prod_{\lambda\in\Lambda} X_\lambda \longrightarrow \prod_{\lambda\in\Lambda} Y_\lambda,$$

by the following rule. Every element of $\prod_{\lambda\in\Lambda} X_\lambda$ is a tuple $(x_\lambda)_{\lambda\in\Lambda}$. We demand that $\prod_{\lambda\in\Lambda}^{DT} f_\lambda$ sends each

$(x_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} X_\lambda$ to $(f_\lambda(x_\lambda))_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} Y_\lambda$.

2.9 Chain complexes

Let R be a ring. A *graded abelian group* (resp. *R-module*) is an abelian group (resp. R-module) A equipped with a direct sum decomposition $A = \bigoplus_{\ell\in\mathbb{Z}} A_\ell$. By referring to A as a graded abelian group or R-module and writing $A = \bigoplus_{\ell\geqslant k} A_\ell$ for some $k \in \mathbb{Z}$, we assume implicitly that $A_\ell = \{0\}$ for $\ell < k$. A *morphism $f : A \to B$ between graded abelian groups (resp. R-modules) of degree $s \in \mathbb{Z}$* is a group (resp. R-module) homomorphism such that $f(A_\ell) \subset B_{\ell+s}$. For $\ell \in \mathbb{Z}$ and a morphism $f : A \to B$ of degree s between graded abelian groups or R-modules $A = \bigoplus_{\ell\in\mathbb{Z}} A_\ell$ and $B = \bigoplus_{\ell\in\mathbb{Z}} B_\ell$, we write f_ℓ for the *ℓ-component of f*, i.e.,

$$f_\ell : A_\ell \to B_{\ell+s}, \quad f_\ell(a) = f(a)$$

for all $a \in A_\ell$. A *chain complex* (A, d) of abelian groups (resp. R-modules) is a graded abelian group (resp. R-module) A equipped with a morphism $d : A \to A$ of degree -1 such that $d \circ d = 0$. This morphism d is called the *differential* of A. We call A an *exact chain complex* if $\ker(d) = \operatorname{im}(d)$.

In certain cases, we will write a graded abelian group or R-module as $A = \bigoplus_{\ell\in\mathbb{Z}} A^\ell$. If $f : A \to B$ is a morphism between graded abelian groups or R-modules $A = \bigoplus_{\ell\in\mathbb{Z}} A^\ell$ and $B = \bigoplus_{\ell\in\mathbb{Z}} B^\ell$, then we write f^ℓ for the ℓ-component of f. A cochain complex (A, d) of abelian groups (resp. R-modules) is a graded abelian group (resp. R-module) $A = \bigoplus_{\ell\geqslant 0} A^\ell$ equipped with a morphism $d : A \to A$ (the *differential* of A) of degree 1, where superscripts are used instead of subscripts to indicate cochain complexes. A *chain map* $f : (A, d_A) \to (B, d_B)$ between chain or cochain complexes A and B is a graded abelian group morphism of degree 0 such that $f \circ d_A = d_B \circ f$.

Remark 2.9.1. We write (A, d) for a chain or cochain complex. However, if the differential d is understood, we will simply write A instead of (A, d).

2.10 Resolutions and Ext functor

A *projective (resp. free) resolution* of an R-module S over R is an exact chain complex $P = \bigoplus_{\ell\geqslant -1} P_\ell$ of R-modules such that $P_{-1} = S$ and P_ℓ is a projective (resp. free) R-module for $\ell \geqslant 0$. Such a projective resolution is denoted as $P \to S$. In the case where R is the group ring $\mathbb{Z}G$ for some group G, the *standard* *free resolution* $P \to \mathbb{Z}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ is a free resolution (P, ∂) such that P_ℓ is the abelian group freely generated by ordered $(\ell+1)$-tuples of G and the boundary operator ∂ satisfies

$$\partial_\ell(g_0, \cdots, g_\ell) = \sum_{i=0}^{\ell} (-1)^\ell (g_0, \cdots, g_{i-1}, g_{i+1}, \cdots, g_\ell)$$

for all $g_0, \cdots, g_\ell \in G$.

Given a projective resolution $P \to S$ over a ring R and an R-module M, we can apply the functor $Hom_R(\cdot, M)$ to $P \to S$ to form a *deleted cochain complex*

$$Hom_R(P, M) : 0 \longrightarrow Hom_R(P_0, M) \longrightarrow Hom_R(P_1, M) \longrightarrow \cdots$$

whose arrows (except for the left most one) are induced by the differential of P. In contrast, the non-deleted cochain complex is

$$0 \longrightarrow Hom_R(S, M) \longrightarrow Hom_R(P_0, M) \longrightarrow Hom_R(P_1, M) \longrightarrow \cdots$$

By definition, for $\ell \geqslant 0$, the group $Ext_R^\ell(S, M)$ is the cohomology group of the deleted cochain complex $Hom_R(P, M)$ at dimension ℓ. Note that $Ext_R^\ell(S, M), \ell \geqslant 0$, form a graded abelian group

$$Ext_R^*(S, M) = \bigoplus_{\ell \geqslant 0} Ext_R^\ell(S, M).$$

Let R' be a ring and let S', M' be R'-modules. Suppose that a ring homomorphism $R' \to R$ is given. Then S and M can be regarded as R'-modules. The homomorphism $R' \to R$ induces a chain map

$$Hom_R(P, M) \longrightarrow Hom_{R'}(P, M).$$

Suppose further that R'-module homomorphisms $S' \to S$ and $M \to M'$ are given. Let $P' \to S'$ be a projective resolution over R'. Then $S' \to S$ induces a chain map from $P' \to S'$ to $P \to S$, which further induces a chain map

$$Hom_{R'}(P, M) \longrightarrow Hom_{R'}(P', M).$$

Finally, the module homomorphism $M \to M'$ gives rise to a chain map

$$Hom_{R'}(P', M) \longrightarrow Hom_{R'}(P', M').$$

The composition of the above three chain maps gives rise to a chain map

$$Hom_R(P, M) \longrightarrow Hom_{R'}(P', M'),$$

which induces a 0-degree morphism of graded abelian groups

$$NTR : Ext^*_R(S, M) \longrightarrow Ext^*_{R'}(S', M').$$

It is well-known that the definition of NTR does not depend on the choices of resolutions (for example, see [31, Theorem 6.17]). NTR is called the *natural map* induced by $R \to R'$, $S' \to S$, and $M \to M'$.

Remark 2.10.1. If $R = R'$, we will simply say that NTR is induced by $S' \to S$ and $M \to M'$. Moreover, we treat the cases $S = S'$ and $M = M'$ in the same manner.

Suppose that $R = \mathbb{Z}G$ and $R' = \mathbb{Z}H$ for some groups $G \geqslant H$ and the ring homomorphism $R' \to R$ is induced by the inclusion $H \hookrightarrow G$, we will say that NTR is induced by $H \hookrightarrow G$ instead of $\mathbb{Z}H \to \mathbb{Z}G$.

Similarly, an *injective resolution* of the R-module M over R is an exact cochain complex $I = \bigoplus_{\ell \geqslant -1} I^\ell$ of R-modules such that $I^{-1} = M$ and I^ℓ is an injective R-module for $\ell \geqslant 0$. Such an injective resolution is denoted as $M \to I$. Given an injective resolution $M \to I$ over a ring R, we can apply the functor $Hom_R(S, \cdot)$ to $M \to I$ to form a *deleted cochain complex*

$$Hom_R(S, I) : 0 \longrightarrow Hom_R(S, I^0) \longrightarrow Hom_R(S, I^1) \longrightarrow \cdots$$

whose arrows (except for the leftmost one) are induced by the differential of I. In contrast, the *non-deleted cochain complex* is

$$0 \longrightarrow Hom_R(S, M) \longrightarrow Hom_R(S, I^0) \longrightarrow Hom_R(S, I^1) \longrightarrow \cdots$$

One can use injective resolutions to give an alternative definition of $Ext^*_R(S, M)$. For $\ell \geqslant 0$,

$Ext^{\ell}_R(S,M)$ is the cohomology group of the cochain complex $Hom_R(S,I)$ at dimension ℓ. It is well-known that the Ext groups given by the above two definitions can be naturally identified (for example, see [31, Theorem 7.8]).

Furthermore, if $M' \to I'$ is an injective resolution over R', then $M \to M'$ induces a chain map from $M \to I$ to $M' \to I'$, which further induces a chain map

$$Hom_{R'}(S',I) \longrightarrow Hom_{R'}(S',I').$$

The composition of $Hom_R(S,I) \to Hom_{R'}(S,I)$, $Hom_{R'}(S,I) \to Hom_{R'}(S',I)$, and $Hom_{R'}(S',I) \to Hom_{R'}(S',I')$ is a chain map

$$Hom_R(S,I) \longrightarrow Hom_{R'}(S',I'). \tag{2.4}$$

The natural map NTR can also be defined as the cohomology map induced by (2.4) (for example, see [31, Theorem 7.8]).

Remark 2.10.2. $Ext^*_R(S,M)$ is the standard notation for Ext groups. However, in case of computations, we might need to use the resolution $P \to S$ (resp. $M \to I$) and thus write $H^*(Hom_R(P,M))$ (resp. $H^*(Hom_R(S,I)))$ instead of $Ext^*_R(S,M)$.

Remark 2.10.3. We focus on the case $R = \mathbb{Z}G$ for some group G. In this case, we write Hom_G (resp. Ext^*_G) instead of $Hom_{\mathbb{Z}G}$ (resp. $Ext^*_{\mathbb{Z}G}$). If $R = \mathbb{Z}$, then we will simply use Hom in place of $Hom_{\mathbb{Z}}$.

Similarly, if A and B are two R-modules, then we use $A \cong_R B$ to indicate that A is isomorphic to B as R-modules. In the case $R = \mathbb{Z}G$ for some group G, we will simply write $A \cong_G B$ instead of $A \cong_{\mathbb{Z}G} B$

2.11 Group cohomology

Let G be a group and let A be a $\mathbb{Z}G$-module. We use the dot notation $\cdot$ to denote the action of G on A. The *cohomology group* of G *with coefficients in* A is defined as

$$H^*(G;A) = Ext^*_G(\mathbb{Z},A).$$

Suppose that A' is another $\mathbb{Z}G$-module. The set of abelian group homomorphisms $Hom(A,A')$ naturally admits a G-action defined by

$$^{g}f(a) = g \cdot f(g^{-1} \cdot a)$$

for all $g \in G, f \in Hom(A, A')$, and $a \in A$. It is not hard to see that $Hom_G(A, A')$ is a G-invariant subset of $Hom(A, A')$ and thus naturally admits a G-action.

Remark 2.11.1. For clearness, a superscript is used to indicate an action of G, as $g \cdot f(a)$ shall be interpreted as $g \in G$ applied to $f(a) \in A'$ rather than g first applied to f to obtain a function ${}^{g}f$, and then ${}^{g}f$ applied to a.

Let K be a normal subgroup of G with $\overline{G} = G/K$, and let $P \to A$ be a projective resolution over $\mathbb{Z}G$. As $K \leqslant G$, every projective module over $\mathbb{Z}G$ is automatically a projective module over $\mathbb{Z}K$. Thus, $P \to A$ can also be regarded as a projective resolution over $\mathbb{Z}K$. By applying the functor $Hom_K(\cdot, A')$ to $P \to A$ and computing the cohomology of the resulted deleted cochain complex $Hom_K(P, A')$, we obtain

$$Ext_K^*(A, A') = H^*(Hom_K(P, A')).$$

It is easy to check that K acts on $Hom_K(P, A')$ trivially (i.e., K fixes every function of $Hom_K(P, A')$). Therefore, $Hom_K(P, A')$ naturally admits a structure of $\mathbb{Z}\overline{G}$-module. The $\overline{G}$-action on $Hom_K(P, A')$ preserves cocycles and coboundaries of $Hom_K(P, A')$. Hence, $Ext_K^*(A, A')$ also naturally admits a structure of a $\mathbb{Z}\overline{G}$-module. Explicitly, if $\overline{g} \in \overline{G}$ and an element $[f] \in Ext_K^*(A, A')$ is represented by a cocycle $f \in Hom_K(P_\ell, A')$ for some $\ell \geqslant 0$, let $g \in G$ such that g is mapped to $\overline{g}$ under the quotient map $G \to \overline{G}$. Then

$$^{\overline{g}}[f] = [{}^{g}f].$$

A standard fact in group cohomology is that the module structure on $Ext_K(A, A')$ does not depend on particular choices of projective resolutions (for example, see [10, Chapter III.8]). Thus, we obtain a well-defined $\mathbb{Z}\overline{G}$-module structure on $Ext_K^*(A, A')$. In particular, if $A = \mathbb{Z}$, then we obtain a well-defined $\mathbb{Z}\overline{G}$-module structure on $H^*(K; A')$. The iterative cohomology $H^*(\overline{G}; H^*(K; A'))$ is computed with respect to this module structure.

Remark 2.11.2. Let B, B' be $\mathbb{Z}G$-modules with $\mathbb{Z}G$-module homomorphisms $B \to A$, $A' \to B'$. Direct computation shows that the natural map

$$NTR : Ext^*_K(A, A') \longrightarrow Ext^*_K(B, B'),$$

induced by $B' \to A'$ and $A \to B$ is a $\mathbb{Z}\overline{G}$-module homomorphism.

2.12 Coinduced modules

Let G be a group, let H be a subgroup of G, and let A be a module over $\mathbb{Z}H$. The *coinduced module* of A from $\mathbb{Z}H$ to $\mathbb{Z}G$ is

$$CoInd^G_H A = Hom_H(\mathbb{Z}G, A).$$

There is a *standard projection*

$$\pi : CoInd^G_H A \longrightarrow A,\ \pi(f) = f(1)$$

for all $f \in CoInd^G_H A$.

Notation 2.12.1. In the sequel, we consider iterative functions and frequently refer to an element $f \in Hom(A, Hom(B, C))$ for some abelian groups A, B, C. For $a \in A$ and $b \in B$, the notation $f(a, b)$ is used to indicate that we first apply the function f to $a \in A$ and obtain a function $f(a) \in Hom(B, C)$, and then apply $f(a)$ to $b \in B$ and obtain $f(a, b) \in C$.

Suppose that G, H are groups and the $\mathbb{Z}H$-module A is a function module, i.e., A is a $\mathbb{Z}H$-submodule of $Hom(A_1, A_2)$ for some $\mathbb{Z}H$-modules A_1 and A_2. Then for every $f \in CoInd^G_H A$ and $x \in \mathbb{Z}G$, $f(x)$ is a function in $Hom(A_1, A_2)$. For $a \in A_1$, $f(x, a) \in A_2$ is the element obtained by applying $f(x)$ to a.

Recall that $\mathbb{Z}G$ is also a right $\mathbb{Z}G$-module and hence the coinduced module $CoInd^G_H A$ naturally admits a G-action given by

$$g \bullet f(x) = f(x \cdot g)$$

for all $f \in CoInd^G_H A, g \in G$, and $x \in \mathbb{Z}G$, turning $CoInd^G_H A$ into a $\mathbb{Z}G$-module.

2.13 A generalization of Shapiro's lemma

Suppose that G is a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ is a family of subgroups of G, and A_λ is a $\mathbb{Z}H_\lambda$-module for every $\lambda \in \Lambda$. For $\mu \in \Lambda$, the composition of the standard projection $CoInd^G_{H_\mu} A_\mu \twoheadrightarrow A_\mu$ and the coordinate projection $\prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda \twoheadrightarrow CoInd^G_{H_\mu} A_\mu$ is a map

$$p_\mu : \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda \longrightarrow A_\mu.$$

Let A be a $\mathbb{Z}G$-module. Consider the abelian group $Hom_G(A, \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda)$. Every element of this group is a function $\widetilde{f}$ from A to $\prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda$. Define a map

$$Sha_\lambda : Hom_G(A, \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda) \longrightarrow Hom_{H_\lambda}(A, A_\lambda),\ Sha_\lambda(\widetilde{f}) = p_\lambda \circ \widetilde{f}$$

for $\widetilde{f} \in Hom_G(A, \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda)$. Let

$$Sha = \prod_{\lambda\in\Lambda}^{Tar} Sha_\lambda : Hom_G(A, \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda) \longrightarrow \prod_{\lambda\in\Lambda} Hom_{H_\lambda}(A, A_\lambda).$$

Let us construct an inverse of Sha. For $(f_\lambda)_{\lambda\in\Lambda} \in \prod_{\lambda\in\Lambda} Hom_{H_\lambda}(A, A_\lambda)$ and every $\lambda \in \Lambda$, let $\widetilde{f_\lambda} \in Hom_{H_\lambda}(A, CoInd^G_{H_\lambda} A_\lambda)$ such that

$$\widetilde{f_\lambda}(a, x) = f_\lambda(x \cdot a)$$

for all $a \in A$ and $x \in \mathbb{Z}G$, where we employ notations defined in Notation 2.12.1. Let

$$\widetilde{f} = \prod_{\lambda\in\Lambda}^{Tar} \widetilde{f_\lambda} \in Hom(A, \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda).$$

Direct computation shows $\widetilde{f} \in Hom_G(A, \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda)$. Let

$$\rho : \prod_{\lambda\in\Lambda} Hom_{H_\lambda}(A, A_\lambda) \longrightarrow Hom_G(A, \prod_{\lambda\in\Lambda} CoInd^G_{H_\lambda} A_\lambda)$$

be the map sending each $(f_\lambda)_{\lambda\in\Lambda}$ to the corresponding $\widetilde{f}$.

It is easy to check that Sha and ρ are mutual inverses. Thus, Sha is an isomorphism of abelian groups. The map Sha is called *Shapiro's isomorphism*. The following lemma is a generalization of the well-known Shapiro's lemma.

Lemma 2.13.1. *Let G be a group, let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a family of subgroups of G, and let A_λ be a $\mathbb{Z}H_\lambda$-module* *for every $\lambda \in \Lambda$. Then the Shapiro's isomorphism Sha defined above induces an isomorphism*

$$Sha^* : H^*(G; \prod_{\lambda\in\Lambda} CoInd_{H_\lambda}^G A_\lambda) \longrightarrow \prod_{\lambda\in\Lambda} H^*(H_\lambda; A_\lambda).$$

Proof. Let $P \to \mathbb{Z}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. For every $\lambda \in \Lambda$, as $H_\lambda \leqslant G$, $P \to \mathbb{Z}$ can also be regarded as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H_\lambda$.

By applying functors $Hom_G(\cdot, \prod_{\lambda\in\Lambda} CoInd_{H_\lambda}^G A_\lambda)$ and $\prod_{\lambda\in\Lambda} Hom_{H_\lambda}(\cdot, A_\lambda)$ to $P \to \mathbb{Z}$, we obtain cochain complexes $Hom_G(P, \prod_{\lambda\in\Lambda} CoInd_{H_\lambda}^G A_\lambda)$ and $\prod_{\lambda\in\Lambda} Hom_{H_\lambda}(P, A_\lambda)$. The cohomology groups of the these cochain complexes are $H^*(G; \prod_{\lambda\in\Lambda} CoInd_{H_\lambda}^G A_\lambda)$ and $\prod_{\lambda\in\Lambda} H^*(H_\lambda; A_\lambda)$. It is easy to see that the Shapiro's isomorphism Sha is a chain isomorphism and thus induces an isomorphism between cohomology groups. □

2.14 Group triples and Cohen-Lyndon property

Let G be a group and let H be a subgroup of G. Denote by $LT(H, G)$ (resp. $RT(H, G)$) the left (resp. right) transversal of H in G. The notation

$$G = \prod_{\lambda\in\Lambda}^{*} G_\lambda$$

is used to indicate that G is the free product of its subgroups $G_\lambda, \lambda \in \Lambda$.

Definition 2.14.1. Let G be a group with a family $\{H_\lambda\}_{\lambda\in\Lambda}$ of subgroups. For $\lambda \in \Lambda$, let N_λ be a normal subgroup of H_λ. Then the triple $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ is called a *group triple*.

Notation 2.14.2. Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple. Denote $\bigcup_{\lambda\in\Lambda} N_\lambda$ by $\mathcal{N}$ and write $\overline{G}$ for $G/\langle\langle \mathcal{N} \rangle\rangle$. For $\lambda \in \Lambda$, write $\overline{H}_\lambda$ for H_λ/N_λ. Let A be a $\mathbb{Z}\overline{G}$-module. For $\lambda \in \Lambda$, denote by

$$NTR_{H_\lambda} : H^*(G; A) \longrightarrow H^*(H_\lambda; A)$$

the natural map induced by $H_\lambda \hookrightarrow G$. Let

$$NTR_G = \prod_{\lambda\in\Lambda}^{Tar} NTR_{H_\lambda} : H^*(G; A) \longrightarrow \prod_{\lambda\in\Lambda} H^*(H_\lambda; A).$$

For $\lambda \in \Lambda$ and $q \in \mathbb{Z}$, denote by

$$NTR^q_{N_\lambda} : H^q(\langle\langle \mathcal{N} \rangle\rangle; A) \longrightarrow H^q(N_\lambda; A)$$

the natural map corresponding to the inclusion $N_\lambda \hookrightarrow \langle\langle \mathcal{N} \rangle\rangle$, and by

$$NT^q_{\overline{H}_\lambda} : H^q(\overline{G}; A) \longrightarrow H^q(\overline{H}_\lambda; A)$$

the natural map induced by the natural homomorphism $\overline{H}_\lambda \to \overline{G}$. Let

$$NT^q_{\overline{G}} = \prod_{\lambda \in \Lambda} NT^q_{\overline{H}_\lambda} : H^q(\overline{G}; A) \longrightarrow \prod_{\lambda \in \Lambda} H^q(\overline{H}_\lambda; A).$$

For $p, q \in \mathbb{Z}$, let

$$NTR^{p,q}_{\overline{H}_\lambda} : H^p(\overline{G}; H^q(\langle\langle \mathcal{N} \rangle\rangle; A)) \longrightarrow H^p(\overline{H}_\lambda; H^q(N_\lambda; A))$$

be the natural map corresponding to the natural homomorphism $\overline{H}_\lambda \to \overline{G}$ and $NTR^q_{N_\lambda}$. Let

$$NTR^{p,q}_{\overline{G}} = \prod_{\lambda \in \Lambda}^{Tar} NTR^{p,q}_{\overline{H}_\lambda} : H^p(\overline{G}; H^q(\langle\langle \mathcal{N} \rangle\rangle; A)) \longrightarrow \prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A)).$$

Definition 2.14.3. A group triple $(G, \{H_\lambda\}_{\lambda \in \Lambda}, \{N_\lambda\}_{\lambda \in \Lambda})$ has the *Cohen-Lyndon property* if there exists a left transversal $T_\lambda \in LT(H_\lambda \langle\langle \mathcal{N} \rangle\rangle, G)$ for every $\lambda \in \Lambda$ such that

$$\langle\langle \mathcal{N} \rangle\rangle = \prod^{*}_{\lambda \in \Lambda, t \in T_\lambda} N^t_\lambda.$$

2.15 Spectral sequences of cohomological type

Definition 2.15.1. A *bigraded abelian group* $A = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}$ is a direct sum of abelian groups $A^{p,q}, p, q \in \mathbb{Z}$.

Remark 2.15.2. As for graded abelian groups, for $k, \ell \in \mathbb{Z}$, we write $A = \bigoplus_{p \geqslant k, q \geqslant \ell} A^{p,q}$ to indicate that $A^{p,q} = \{0\}$ if either $p < k$ or $q < \ell$.

Definition 2.15.3. Let $A = \bigoplus_{p,q\in\mathbb{Z}} A^{p,q}$ and $B = \bigoplus_{p,q\in\mathbb{Z}} B^{p,q}$ be bigraded abelian groups. A group homomorphism $f : A \to B$ is called a *morphism between bigraded abelian groups of bidegree* (k,ℓ) for some $k,\ell \in \mathbb{Z}$ if $f(A^{p,q}) \subset B^{p+k,q+\ell}$ for all $p,q \in \mathbb{Z}$.

For $p,q \in \mathbb{Z}$, the (p,q)*-component* of f is the map

$$f^{p,q} : A^{p,q} \longrightarrow B^{p+k,q+\ell},\ f^{p,q}(a) = f(a)$$

for all $a \in A^{p,q}$.

Moreover, for $q \in \mathbb{Z}$ (resp. $p \in \mathbb{Z}$), the q*-th row* (resp. p*-th column*) of A is denoted as $A^{*,q}$ (resp. $A^{p,*}$), i.e., $A^{*,q} = \bigoplus_{p\in\mathbb{Z}} A^{p,q}$ (resp. $A^{p,*} = \bigoplus_{p\in\mathbb{Z}} A^{p,q}$). Note that $A^{*,q}$ and $A^{p,*}$ are graded abelian groups. We denote the domain-target sum $\bigoplus^{DT}_{p\in\mathbb{Z}} f^{p,q} : A^{*,q} \to B^{*,q+\ell}$ (resp. $\bigoplus^{DT}_{q\in\mathbb{Z}} f^{p,q} : A^{p,*} \to B^{p+k,*}$) by $f^{*,q}$ (resp. $f^{p,*}$).

Definition 2.15.4. A *(first quadrant) spectral sequence (of cohomological type)* is a sequence of pairs $E = \{(E_r, d_r)\}_{r\geqslant a}$ for some $a \in \mathbb{N}^+$ such that the following properties hold for $r \geqslant a$.

(a) $E_r = \bigoplus_{p,q\geqslant 0} E_r^{p,q}$ is a bigraded abelian group.

(b) $d_r : E_r \to E_r$ is morphism between bigraded abelian groups of bidegree $(r, 1-r)$ such that $d_r \circ d_r = 0$.

(c) for $p,q \in \mathbb{Z}$, $E_{r+1}^{p,q} = \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,q+r-1})$.

The bigraded abelian groups $E_r, r \geqslant a$ are called *pages* of E.

Definition 2.15.5. Let $E = \{(E_r, d_r)\}_{r\geqslant a}$ and $E' = \{(E'_r, d'_r)\}_{r\geqslant a}$ be spectral sequences. A map $MSS : E \to E'$ is called a *morphism between spectral sequences* if for every $r \geqslant a$, MSS restricts to a bigraded abelian group homomorphism $MSS_r : E_r \to E'_r$ of bidegree $(0,0)$ such that

$$MSS_r \circ d_r = d'_r \circ MSS_r$$

and MSS_{r+1} is the cohomology map induced by MSS_r.

If there exists $R \geqslant a$ such that for all $p,q \in \mathbb{Z}$, $MSS_R^{p,q}$ is an isomorphism, then MSS is called an *isomorphism between spectral sequences*.

Definition 2.15.6. Let G be an abelian group. A *filtration* of G is a sequence $(F_kG)_{k\in\mathbb{Z}}$ of abelian groups such that

$$\{0\} \subset \cdots \subset F_{k+1}G \subset F_kG \subset \cdots \subset F_0G = G.$$

If $k < 0$, then $F_kG = G$ by default.

Definition 2.15.7. We say that a spectral sequence $E = \{(E_r, d_r)\}_{r\geqslant a}$ *converges* to a graded abelian group $H = \bigoplus_{\ell\geqslant 0} H^\ell$, denoted as $E_a^{p,q} \Rightarrow H^{p+q}$, if for every $\ell \geqslant 0$, there exist $R > 0$ and a filtration

$$0 = F_{\ell+1}H^\ell \subset \cdots \subset F_0H^\ell = H^\ell$$

of H^ℓ such that $F_kH^\ell/F_{k+1}H^\ell \cong E_r^{l-k,k}$ for $r \geqslant R$.

Remark 2.15.8. In the notation $E_a^{p,q} \Rightarrow H^{p+q}$, the indexes p and q indicate that for sufficiently large r, $E_r^{p,q}$ appears as the quotient of certain terms in a filtration of H^{p+q}. One can use differnet letters for the indexes, say, writing $E_a^{k,\ell} \Rightarrow H^{k+\ell}$ instead of $E_a^{p,q} \Rightarrow H^{p+q}$.

Remark 2.15.9. Note that if $p, q \geqslant 0, p + q = \ell$ and $r \geqslant \max\{a, \ell + 2\}$, then the target of $d_r^{p,q}$ is $E_r^{p+r,q-r+1} = \{0\}$ and the domain of $d_r^{p-r,q-r+1}$ is $E_r^{p-r,q+r-1} = \{0\}$. Thus,

$$E_{r+1}^{p,q} = \ker(d_r^{p,q})/\operatorname{im}(d_r^{p-r,q-r+1}) \cong E_r^{p,q}.$$

Therefore, it suffices to let $R = \max\{a, \ell + 2\}$ in Definition 2.15.7.

Definition 2.15.10. Let $E_1 = \{(E_{1,r}, d_{1,r})\}_{r\geqslant a}$ and $E_2 = \{(E_{2,r}, d_{2,r})\}_{r\geqslant a}$ be two spectral sequences such that

$$E_{1,a}^{p,q} \Rightarrow H_1^{p+q}, \quad E_{2,a}^{p,q} \Rightarrow H_2^{p+q}$$

for some graded abelian groups $H_1 = \bigoplus_{\ell\geqslant 0} H_1^\ell$ and $H_2 = \bigoplus_{\ell\geqslant 0} H_2^\ell$, let $MSS : E_1 \to E_2$ be a morphism between spectral sequences, and let $f : H_1 \to H_2$ be a morphism between graded abelian groups of degree 0. We say that MSS and f are *compatible* if for every $\ell \geqslant 0$, there exist $R > 0$ and filtrations

$$\{0\} = F_{\ell+1}H_1^\ell \subset \cdots \subset F_0H_1^\ell = H_1^\ell, \quad \{0\} = F_{\ell+1}H_2^\ell \subset \cdots \subset F_0H_2^\ell = H_2^\ell$$

such that $f(F_kH_1^\ell) \subset F_kH_2^\ell$ for $k = 0, ..., \ell + 1$, and that for every $r \geqslant R$ and $k = 0, ..., \ell$, there exist isomorphisms

$$\sigma : F_k H_1^\ell / F_{k+1} H_1^\ell \longrightarrow E_{1,r}^{l-k,k}, \quad \tau : F_k H_2^\ell / F_{k+1} H_2^\ell \longrightarrow E_{2,r}^{l-k,k}$$

with $MSS^{l-k,k} \circ \sigma = \tau \circ \overline{f}$, where

$$\overline{f} : F_k H_1^\ell / F_{k+1} H_1^\ell \longrightarrow F_k H_2^\ell / F_{k+1} H_2^\ell$$

is the map induced by f.

Remark 2.15.11. By Remark 2.15.9, it suffices to let $R = \max\{a, \ell + 2\}$ in Definition 2.15.10.

Lemma 2.15.12 ([36, Comparison Theorem 5.2.12]). *Let $E_1 = \{(E_{1,r}, d_{1,r})\}_{r \geqslant a}$ and $E_2 = \{(E_{2,r}, d_{2,r})\}_{r \geqslant a}$ be two spectral sequences such that*

$$E_{1,a}^{p,q} \Rightarrow H_1^{p+q}, \quad E_{2,a}^{p,q} \Rightarrow H_2^{p+q}$$

for some graded abelian groups $H_1 = \bigoplus_{\ell \geqslant 0} H_1^\ell$ and $H_2 = \bigoplus_{\ell \geqslant 0} H_2^\ell$, let $MSS : E_1 \to E_2$ be an isomorphism between spectral sequences, and let $f : H_1 \to H_2$ be a morphism between graded abelian groups. Suppose that MSS and f are compactible, then f is an isomorphism.

Definition 2.15.13. Let $E_\lambda = \{(E_{\lambda,r}, d_{\lambda,r})\}_{r \geqslant a}$, $\lambda \in \Lambda$, be spectral sequences. The *product* of $E_\lambda, \lambda \in \Lambda$, is a sequence $E = \{(E_r, d_r)\}_{r \geqslant a}$ such that for all $p, q \in \mathbb{Z}$ and $r \geqslant a$,

$$E_r^{p,q} = \prod_{\lambda \in \Lambda} E_{\lambda,r}^{p,q}, \quad d_r^{p,q} = \prod_{\lambda \in \Lambda}^{DT} d_{\lambda,r}^{p,q}.$$

Remark 2.15.14. The product of spectral sequences is a spectral sequence as products of exact sequences are exact.

Lemma 2.15.15. *Suppose that $E_\lambda = \{(E_{\lambda,r}, d_{\lambda,r})\}_{r \geqslant a}$, $\lambda \in \Lambda$, are spectral sequences and $H_\lambda, \lambda \in \Lambda$, are graded abelian groups with $E_{\lambda,a}^{p,q} \Rightarrow H_\lambda^{p+q}$ for $\lambda \in \Lambda$. Let $E = \{(E_r, d_r)\}_{r \geqslant a}$ be the product of $E_\lambda, \lambda \in \Lambda$. Then $E_a^{p,q} \Rightarrow \prod_{\lambda \in \Lambda} H_\lambda^{p,q}$.*

Moreover, let $\overline{E} = \{(\overline{E}_r, \overline{d}_r)\}_{r \geqslant a}$ be a spectral sequence and let $\overline{H} = \bigoplus_{\ell \geqslant 0} \overline{H}^\ell$ be a graded abelian group with $\overline{E}_a^{p,q} \Rightarrow \overline{H}^{p+q}$. For $\lambda \in \Lambda$, let $MSS_\lambda : \overline{E} \to E_\lambda$ be a morphism of spectral sequences and let $f_\lambda : \overline{H} \to H_\lambda$ be a degree-0 morphism of graded abelian groups. If for $\lambda \in \Lambda$, MSS_λ is compatible with

f_λ. Then the maps

$$\prod_{\lambda\in\Lambda}^{Tar} MSS_\lambda : \overline{E} \to E, \quad \prod_{\lambda\in\Lambda}^{Tar} f_\lambda : \overline{H} \to \prod_{\lambda\in\Lambda} H_\lambda$$

are also compatible.

Lemma 2.15.15 can be proved by taking products of filtrations. We leave the details to the reader.

Definition 2.15.16. Suppose that $E_i = \{(E_{i,r}, d_{i,r})\}_{r\geqslant a}, i \in I$, form a directed system of spectral sequences. The *direct limit* of $\{E_i\}_{i\in I}$ is a spectral sequence $E = \{(E_r, d_r)\}_{r\geqslant a}$ such that, for $p, q \in \mathbb{Z}$ and $r \geqslant a$, $E_r^{p,q} = \varinjlim E_{i,r}^{p,q}$ and $d_r^{p,q} = \varinjlim d_{i,r}^{p,q}$.

Remark 2.15.17. The direct limit of spectral sequences is a spectral sequence as $\varinjlim$ is an exact functor on the category of abelian groups.

Lemma 2.15.18. *Suppose that $E_i = \{(E_{i,r}, d_{i,r})\}_{r\geqslant a}$ (resp. H_i), $i \in I$, form a directed system of spectral sequences (resp. graded abelian groups). Let E (resp. H) be the direct limit of $\{E_i\}_{i\in I}$ (resp. $\{H_i\}_{i\in I}$). If for $i \in I$, $E_{i,a}^{p,q} \Rightarrow H_i^{p+q}$ and for $i, j \in I$ with $i < j$, the morphisms $E_i \to E_j, H_i \to H_j$ are compatible, then $E_a^{p,q} \Rightarrow H^{p+q}$.*

Lemma 2.15.18 can be proved by taking direct limits of filtrations and then using the fact that $\varinjlim$ is an exact functor. We leave the details to the reader.

Definition 2.15.19. A *double complex* $(C, {}_hd, {}_vd)$ *(of cohomological type)* is a bigraded abelian group C with homomorphisms ${}_hd, {}_vd : C \to C$ between bigraded abelian groups of bidegree $(1, 0)$ and $(0, 1)$, respectively, such that

$${}_hd \circ {}_hd = {}_vd \circ {}_vd = {}_hd \circ {}_vd + {}_vd \circ {}_hd = 0.$$

The map ${}_hd$ (resp. ${}_vd$) is called the *horizontal* (resp. $vertical$) *differential* of C. C is called a *first quadrant double complex* if $C^{p,q} = \{0\}$ whenever either p or q is strictly less than 0.

Notation 2.15.20. When we refer a double complex $(C, {}_hd, {}_vd)$, if the differentials are clear from the context, we will simply write C.

Definition 2.15.21. Let $(C_1, {}_hd_1, {}_vd_1)$ and $(C_2, {}_hd_2, {}_vd_2)$ be double complexes. A *morphism* MDC : $C_1 \to C_2$ *between double complexes* is a morphism between bigraded abelian groups C_1, C_2 of bidegree $(0, 0)$ such that

$$MDC \circ {}_hd_1 = {}_hd_2 \circ MDC, \quad MDC \circ {}_vd_1 = {}_vd_2 \circ MDC.$$

Definition 2.15.22. Let $(C, {}_hd, {}_vd)$ be a first quadrant double complex. The *total complex* $TC = \bigoplus_{\ell \in \mathbb{Z}} TC^\ell$ of C is a cochain complex with $TC^\ell = \bigoplus_{p+q=\ell} C^{p,q}$. The *differential* of TC is $d = {}_hd + {}_vd$.

The *row filtration* of TC is

$$\{0\} \subset \cdots \subset {}_hF_{k+1}TC \subset {}_hF_kTC \subset \cdots {}_hF_0TC = TC,$$

where ${}_hF_kTC = \bigoplus_{q \geqslant k} C^{p,q}$. For $\ell \in \mathbb{N}$, let

$${}_hF_kTC^\ell = {}_hF_kTC \cap TC^\ell.$$

Then

$${}_hF_kTC = \bigoplus_{\ell \geqslant 0} {}_hF_kTC^\ell$$

is a cochain complex under the differential induced by d.

Similarly, the *column filtration* of TC is

$$\{0\} \subset \cdots \subset {}_vF_{k+1}TC \subset {}_vF_kTC \subset \cdots {}_vF_0TC = TC,$$

where ${}_vF_kTC = \bigoplus_{p \leqslant k} C^{p,q}$. For $\ell \in \mathbb{N}$, let

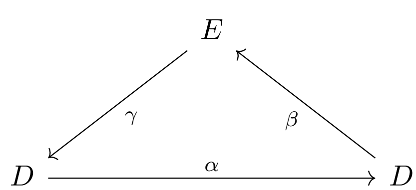
$${}_vF_kTC^\ell = {}_vF_kTC \cap TC^\ell.$$

Then

$${}_vF_kTC = \bigoplus_{\ell \geqslant 0} {}_vF_kTC^\ell$$

is a cochain complex under the differential induced by d.

Definition 2.15.23. A *exact couple* $(D, E, \alpha, \beta, \gamma)$ *(of cohomological type) of degree* $r \in \mathbb{N}$ is a commutative triangle



(2.5)

such that

(a) D and E are bigraded abelian groups;

(b) α, β, and γ are morphisms between bigraded abelian groups of bidegree $(-1, 1), (r-1, 1-r)$, and (1,0), respectively;

(c) exactness holds at each vertex of the triangle diagram (2.5).

Suppose that $(D, E, \alpha, \beta, \gamma)$ is an exact couple of degree r. Let $d = \beta \circ \gamma$, let

$$E_1 = \bigoplus_{p,q \in \mathbb{Z}} E_1^{p,q}$$

be the bigraded abelian group with

$$E_1^{p,q} = \ker(d^{p,q}) / \operatorname{im}(d^{p-r,q+r-1}),$$

and let

$$D_1 = \bigoplus_{p,q \in \mathbb{Z}} D_1^{p,q}$$

be the bigraded abelian group with

$$D_1^{p,q} = \operatorname{im}(\alpha^{p+1,q-1}).$$

Define morphisms

$$\alpha_1 : D_1 \to D_1, \quad \beta_1 : D_1 \to E_1, \quad \gamma_1 : E_1 \to D_1$$

between bigraded abelian groups by the following rule. Let α_1 be the restriction of α to D_1. Fix integers p, q. For every $y \in D_1^{p,q}$, there exists $x \in D^{p+1,q-1}$ such that

$$\alpha^{p+1,q-1}(x) = y.$$

Let $\beta_1^{p,q}(y)$ be the cohomology class of $E_1^{p+r,q-r}$ represented by $\beta^{p+1,q-1}(x)$. For $[z] \in E_1^{p,q}$, there exists $z \in E^{p,q}$ representing $[z]$. Let $\gamma_1^{p,q}([z]) = \gamma(z)$.

Lemma 2.15.24 ([31, Theorem 11.9]). *The maps $\alpha_1, \beta_1, \gamma_1$ constructed above are well-defined. Moreover, $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ is an exact couple of degree $r + 1$.*

Definition 2.15.25. The exact couple $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ in Lemma 2.15.24 is called the *derived couple* of $(D, E, \alpha, \beta, \gamma)$.

Definition 2.15.26. A *morphism*

$$MEC : (D, E, \alpha, \beta, \gamma) \longrightarrow (D', E', \alpha', \beta', \gamma')$$

between exact couples consists of two maps

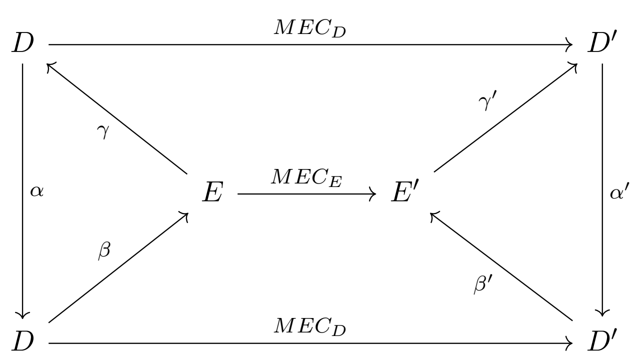
$$MEC_D : D \longrightarrow D', \quad MEC_E : E \longrightarrow E'$$

with the following properties.

(a) $(D, E, \alpha, \beta, \gamma)$ and $(D', E', \alpha', \beta', \gamma')$ are exact couples of the same degree.

(b) MEC_D and MEC_E are maps between bigraded abelian groups of bidegree $(0, 0)$.

(c) The following diagram commutes.



Moreover, we call MEC_D (resp. MEC_E) the D-*component* (resp. E-*component*) of MEC.

Suppose that

$$MEC : (D, E, \alpha, \beta, \gamma) \longrightarrow (D', E', \alpha', \beta', \gamma')$$

is a morphism between degree r exact couples. Let $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ (resp. $(D'_1, E'_1, \alpha'_1, \beta'_1, \gamma'_1)$) be the derived couple of $(D, E, \alpha, \beta, \gamma)$ (resp. $(D', E', \alpha', \beta', \gamma')$). By restricting MEC_D to D_1, we get a map

$$MEC_{1,D_1} : D_1 \longrightarrow D'_1.$$

Recall that E_1 (resp. E'_1) is the cohomology of E (resp. E') with respect to $\beta \circ \gamma$ (resp. $\beta' \circ \gamma'$). Let

$$MEC_{1,E_1} : E_1 \longrightarrow E'_1$$

be the map on cohomology induced by MEC_E. It is easy to check that MEC_{1,D_1} and MEC_{1,E_1} form a morphism

$$MEC_1 : (D_1, E_1, \alpha_1, \beta_1, \gamma_1) \longrightarrow (D'_1, E'_1, \alpha'_1, \beta'_1, \gamma'_1)$$

between degree-$(r+1)$ exact couples. To sum up,

Lemma 2.15.27. *Suppose that*

$$MEC : (D, E, \alpha, \beta, \gamma) \longrightarrow (D', E', \alpha', \beta', \gamma')$$

is a morphism between degree-r exact couples. Then MEC induces a morphism

$$MEC_1 : (D_1, E_1, \alpha_1, \beta_1, \gamma_1) \longrightarrow (D'_1, E'_1, \alpha'_1, \beta'_1, \gamma'_1)$$

between the derived couples.

Lemma 2.15.28 ([31, Theorem 11.10]). *Suppose that $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ is an exact couple of degree 1. For every $r \geqslant 1$, let $(D_{r+1}, E_{r+1}, \alpha_{r+1}, \beta_{r+1}, \gamma_{r+1})$ be the derived couple of $(D_r, E_r, \alpha_r, \beta_r, \gamma_r)$, and let $d_r = \beta_r \circ \gamma_r$. Then the pairs $(E_r, d_r), r \geqslant 1,$ form a spectral sequence.*

Definition 2.15.29. For $r \geqslant 1$, the exact couple $(D_r, E_r, \alpha_r, \beta_r, \gamma_r)$ in Definition 2.15.28 is called the $(r-1)$-*th derived couple* of $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ (the 0-*th* derived couple is just $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$).

The spectral sequence $\{(E_r, d_r)\}_{r \geqslant 1}$ in Lemma 2.15.28 is called the *induced spectral sequence* of the exact couple $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$.

Let

$$MEC_1 : (D_1, E_1, \alpha_1, \beta_1, \gamma_1) \longrightarrow (D'_1, E'_1, \alpha'_1, \beta'_1, \gamma'_1)$$

be a morphism between degree-1 exact couples. By using Lemma 2.15.27 iteratively, we see that MEC_1

induces morphisms

$$MEC_r : (D_r, E_r, \alpha_r, \beta_r, \gamma_r) \longrightarrow (D'_r, E'_r, \alpha'_r, \beta'_r, \gamma'_r), r \geqslant 1$$

between the derived couples. The E_r-components $MEC_{r,E_r}, r \geqslant 1$, form a morphism

$$MSS : \{(E_r, d_r)\}_{r \geqslant 1} \longrightarrow \{(E'_r, d'_r)\}_{r \geqslant 1}$$

between spectral sequences, where $\{(E_r, d_r)\}_{r \geqslant 1}$ (resp. $\{(E'_r, d'_r)\}_{r \geqslant 1}$) is the induced spectral sequence of $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ (resp. $(D'_1, E'_1, \alpha'_1, \beta'_1, \gamma'_1)$).

Lemma 2.15.30. *Let*

$$MEC : (D_1, E_1, \alpha_1, \beta_1, \gamma_1) \longrightarrow (D'_1, E'_1, \alpha'_1, \beta'_1, \gamma'_1)$$

be a map between degree-1 *exact couples. Then* MEC *induces a morphism*

$$MSS : \{(E_r, d_r)\}_{r \geqslant 1} \longrightarrow \{(E'_r, d'_r)\}_{r \geqslant 1}$$

between the induced spectral sequences.

Let C_1 be a first quadrant double complex. Consider the row filtration

$$\{0\} \subset \cdots \subset {}_hF_{k+1}TC_1 \subset {}_hF_kTC_1 \subset \cdots {}_hF_0TC_1 = TC_1$$

of its total complex TC_1. By Definition 2.15.22, ${}_hF_kTC_1$ is a cochain complex for every $k \in \mathbb{Z}$. The short exact sequence

$$0 \longrightarrow {}_hF_{k+1}TC_1 \longrightarrow {}_hF_kTC_1 \longrightarrow {}_hF_kTC_1/{}_hF_{k+1}TC_1 \longrightarrow 0$$

of cochain complexes gives rise to a long exact sequence

$$\cdots \longrightarrow H^\ell({}_hF_{k+1}TC_1) \xrightarrow{\alpha_{1,1}} H^\ell({}_hF_kTC_1) \xrightarrow{\beta_{1,1}} H^\ell({}_hF_kTC_1/{}_hF_{k+1}TC_1) \xrightarrow{\gamma_{1,1}} \cdots$$

of cohomology groups. It follows that $(D_{1,1}, E_{1,1}, \alpha_{1,1}, \beta_{1,1}, \gamma_{1,1})$ is an exact couple of degree 1, where

$$D_{1,1}^{p,q} = H^{p+q}({}_hF_pTC_1), \quad E_{1,1}^{p,q} = H^{p+q}({}_hF_pTC/{}_hF_{p+1}TC_1)$$

for $p, q \in \mathbb{Z}$. $(D_{1,1}, E_{1,1}, \alpha_{1,1}, \beta_{1,1}, \gamma_{1,1})$ is called the *exact couple induced by the row filtration* of TC_1.

Let $E_1 = \{(E_{1,r}, d_{1,r})\}_{r\geqslant 1}$ be the induced spectral sequence of $(D_{1,1}, E_{1,1}, \alpha_{1,1}, \beta_{1,1}, \gamma_{1,1})$. We call E_1 the *spectral sequence induced by the row filtration* of TC_1. Similarly, the column filtration of TC_1 also induces a spectral sequence, which is called the *spectral sequence induced by the column filtration* of TC_1. We summarize the above discussion by the following.

Lemma 2.15.31 ([31, Corollary 11.12]). *If C is a double complex, then the row (resp. column) filtration of TC induces an exact couple and a spectral sequence.*

Lemma 2.15.32. *Let C be a double complex and let E be the spectral sequence induced by the row (resp. column) filtration of TC. Then $E_1^{p,q} \Rightarrow H^{p+q}(TC)$.*

More precisely, let $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ be the exact couple induced by the row (resp. column) filtration of TC and let $(D_r, E_r, \alpha_r, \beta_r, \gamma_r)$ be the $(r-1)$-th derived couple of $(D_1, E_1, \alpha_1, \beta_1, \gamma_1)$ for $r \geqslant 1$. Then for every $k \in \mathbb{N}$,

$$0 = D_{2k+3}^{-k-1,2k+1} \subset D_{2k+2}^{-k-1,2k+1} \subset \cdots \subset D_{k+3}^{-k-1,2k+1} \subset D_{k+2}^{-k-1,2k+1} = H^k(TC) \tag{2.6}$$

is a filtration for $H^k(TC)$ and for $r = k+2, ..., 2k+2$, β_r induces an isomorphism

$$D_r^{-k-1,2k+1}/D_{r+1}^{-k-1,2k+1} \longrightarrow E_r^{r-k-2,2k-r+2}. \tag{2.7}$$

Proof. This is proved in [31, Theorem 11.13] except that the indexes of the D and E terms in (2.6) and (2.7) are not computed there. In order to prove the next Lemma, it is convenient to have those indexes. The reader is encouraged to follow the proof of [31, Theorem 11.13], find the indexes, and check (2.6), (2.7). □

Suppose that another first quadrant double complex C_2 is given. Then the row filtration

$$\{0\} \subset \cdots \subset {}_hF_{k+1}TC_2 \subset {}_hF_kTC_2 \subset \cdots {}_hF_0TC_2 = TC_2$$

of its total complex TC_2 also induces a spectral sequence E_2. Suppose further that there is a morphism $MDC : C_1 \to C_2$ between double complexes. Then MDC induces a map between the cohomology long exact sequences corresponding to the short exact sequences

$$0 \longrightarrow {}_hF_{k+1}TC_1 \longrightarrow {}_hF_kTC_1 \longrightarrow {}_hF_kTC_1/{}_hF_{k+1}TC_1 \longrightarrow 0,$$

$$0 \longrightarrow {}_hF_{k+1}TC_2 \longrightarrow {}_hF_kTC_2 \longrightarrow {}_hF_kTC_2/{}_hF_{k+1}TC_2 \longrightarrow 0$$

for every $k \in \mathbb{Z}$. Therefore, MDC induces a morphism between the induced exact couples and thus induces a morphism between the induced spectral sequences.

Note that MDC also induces a cohomology map

$$MDC^* : H^*(TC_1) \longrightarrow H^*(TC_2).$$

For $k \in \mathbb{N}$ and $r = k+2, ..., 2k+2$, by Lemma 2.15.32, $E_{1,r}^{r-k-2,2k-r+2}$ (resp. $E_{2,r}^{r-k-2,2k-r+2}$) is a subquotient (quotient of a submodule) of $H^*(TC_1)$ (resp. $H^*(TC_2)$). Thus, MDC^* induces a map

$$E_{1,r}^{r-k-2,2k-r+2} \longrightarrow E_{2,r}^{r-k-2,2k-r+2}.$$

Lemma 2.15.33. *Let $MDC : C_1 \to C_2$ be a morphism between first quadrant double complexes C_1, C_2, let $E_1 = \{(E_{1,r}, d_{1,r})\}_{r\geqslant 1}$ and $E_2 = \{(E_{2,r}, d_{2,r})\}_{r\geqslant 1}$ be the spectral sequences induced by the row filtrations of TC_1 and TC_2, respectively, let $MDC^* : H^*(TC_1) \to H^*(TC_2)$ be the cohomological map induced by MDC, and let $MSS : E_1 \to E_2$ be the morphism between spectral sequences induced by MDC. Then MDC^* and MSS are compatible. More precisely, for $k \in \mathbb{N}$ and $r = k+2, ..., 2k+2$, the map*

$$E_{1,r}^{r-k-2,2k-r+2} \longrightarrow E_{2,r}^{r-k-2,2k-r+2}$$

induced by MDC^* can be identified with $MSS_r^{r-k-2,2k-r+2}$.

Moreover, the same conclusion holds with column filtration in place of row filtration.

Proof. We only consider row filtrations. The proof for column filtrations is exactly the same.

Let $(D_{1,1}, E_{1,1}, \alpha_{1,1}, \beta_{1,1}, \gamma_{1,1})$ (resp. $(D_{2,1}, E_{2,1}, \alpha_{2,1}, \beta_{2,1}, \gamma_{2,1})$) be the exact couple induced by the row filtration of TC_1 (resp. TC_2). For $r \geqslant 1$, let $(D_{1,r}, E_{1,r}, \alpha_{1,r}, \beta_{1,r}, \gamma_{1,r})$ (resp. $(D_{2,r}, E_{2,r}, \alpha_{2,r}, \beta_{2,r}, \gamma_{2,r})$) be the $(r-1)$-th derived couple of $(D_{1,1}, E_{1,1}, \alpha_{1,1}, \beta_{1,1}, \gamma_{1,1})$ (resp. $(D_{2,1}, E_{2,1}, \alpha_{2,1}, \beta_{2,1}, \gamma_{2,1})$), let

$$MEC_r : (D_{1,r}, E_{1,r}, \alpha_{1,r}, \beta_{1,r}, \gamma_{1,r}) \longrightarrow (D_{2,r}, E_{2,r}, \alpha_{2,r}, \beta_{2,r}, \gamma_{2,r})$$

be the morphism between exact couples induced by MDC, and let $MEC_{D,r}$ be the $D_{1,r}$-component of MEC_r. By definition, the $E_{1,r}$-component of MEC_r is just MSS_r for $r \geqslant 1$.

Fix $k \in \mathbb{N}$ and let $r \in \{k+2, ..., 2k+3\}$. By definition, the map

$$MEC_{D,r}^{-k-1,2k+1} : D_{1,r}^{-k-1,2k+1} \longrightarrow D_{2,r}^{-k-1,2k+1}$$

is the restriction of

$$MEC_{D,1}^{-k-1,2k+1} : D_{1,1}^{-k-1,2k+1} \longrightarrow D_{2,1}^{-k-1,2k+1}$$

to $D_{1,r}^{-k-1,2k+1}$. Thus,

$$MEC_{D,1}^{-k-1,2k+1}(D_{1,r}^{-k-1,2k+1}) \subset D_{2,r}^{-k-1,2k+1}.$$

The morphism MEC_r gives rise to a commutative digram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & D_{1,r}^{-k,2k} & \xrightarrow{\alpha_{1,r}} & D_{1,r}^{-k-1,2k+1} & \xrightarrow{\beta_{1,r}} & E_{1,r}^{r-k-2,2k-r+2} & \xrightarrow{\gamma_{1,r}} & D_{1,r}^{r-k-1,2k-r+2} & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle MEC_{D,1}^{-k-1,2k+1}} & & \downarrow{\scriptstyle MSS_r} & & & & \\ \cdots & \longrightarrow & D_{2,r}^{-k,2k} & \xrightarrow{\alpha_{2,r}} & D_{2,r}^{-k-1,2k+1} & \xrightarrow{\beta_{2,r}} & E_{2,r}^{r-k-2,2k-r+2} & \xrightarrow{\gamma_{2,r}} & D_{2,r}^{r-k-1,2k-r+2} & \longrightarrow & \cdots \end{array}$$

Note that

$$\begin{aligned} D_{1,r}^{r-k-1,2k-r+2} &= \overbrace{\alpha_{1,1} \circ \cdots \circ \alpha_{1,1}}^{r-1 \text{ times}}(D_{1,r}^{2r-k-2,2k-2r+3}) \\ &= \overbrace{\alpha_{1,1} \circ \cdots \circ \alpha_{1,1}}^{r-1 \text{ times}}(H^{k+1}({}_hF_{2r-k-2}TC_1)) = \{0\}, \end{aligned}$$

$$D_{2,r}^{r-k-1,2k-r+2} = \overbrace{\alpha_{2,1} \circ \cdots \circ \alpha_{2,1}}^{r-1 \text{ times}}(D_{2,1}^{2r-k-2,2k-2r+3})$$

$$= \overbrace{\alpha_{2,1} \circ \cdots \circ \alpha_{2,1}}^{r-1 \text{ times}}(H^{k+1}({}_hF_{2r-k-2}TC_2)) = \{0\},$$

as $2r - k - 2 \geqslant 4k + 4 - k - 2 > k + 1$.

Therefore, $\beta_{1,r}, \beta_{2,r}$ induce isomorphisms

$$\overline{\beta}_{1,r} : D_{1,r}^{-k-1,2k+1}/\alpha_{1,r}(D_{1,r}^{-k,2k}) \longrightarrow E_{1,r}^{r-k-2,2k-r+2},$$

$$\overline{\beta}_{2,r} : D_{2,r}^{-k-1,2k+1}/\alpha_{2,r}(D_{2,r}^{-k,2k}) \longrightarrow E_{2,r}^{r-k-2,2k-r+2},$$

respectively. Note that

$$\alpha_{1,r}(D_{1,r}^{-k,2k}) = D_{1,r+1}^{-k-1,2k+1}, \quad \alpha_{\lambda,r}(D_{2,r}^{-k,2k}) = D_{2,r+1}^{-k-1,2k+1}.$$

As

$$MEC_{2,1}^{-k-1,2k+1}(D_{1,r+1}^{-k-1,2k+1}) \subset D_{2,r+1}^{-k-1,2k+1},$$

the following diagram commutes

$$\begin{array}{ccc}
D_{1,r}^{-k-1,2k+1}/D_{1,r+1}^{-k-1,2k+1} & \xrightarrow{\overline{\beta}_{1,r}} & E_{1,r}^{r-k-2,2k-r+2} \\
\downarrow & & \downarrow MSS_r \\
D_{2,r}^{-k-1,2k+1}/D_{2,r+1}^{-k-1,2k+1} & \xrightarrow{\overline{\beta}_{2,r}} & E_{2,r}^{r-k-2,2k-r+2}
\end{array}$$

where the vertical map on the left is induced by $MEC_{D,1}^{-k-1,2k+1}$.

Thus, $MSS : E_1 \to E_2$ is compatible with $MEC_{D,1}^{-k-1,2k+1}$. By definition,

$$MEC_{D,1}^{-k-1,2k+1} = MDC^*.$$

Thus, the map $MSS : E_1 \to E_2$ is compatible with MDC^*. □

Definition 2.16.1. Let R be a ring and let (C, d) be a cochain complex of R-modules. An *injective Cartan-Eilenberg resolution* (*CE resolution*) of C over R is a double complex $(I, {}_h\delta, {}_v\delta)$ with the following properties.

(a) If $C^p = \{0\}$ for some p, then $I^{p,q} = \{0\}$ for all $q \in \mathbb{Z}$.

(b) $I^{p,q} = \{0\}$ for all $q < 0$.

(c) Note that the 0-th row of I

$$I^{*,0} : \cdots \longrightarrow I^{p,0} \longrightarrow I^{p+1,0} \longrightarrow \cdots$$

is a cochain complex. We demand that there is an injective chain map f (the *augmentation map*) from the cochain complex C to $I^{*,0}$.

(d) For $p \geqslant 0$, let

$${}_hZ^p = \ker(d^p), \quad {}_hB^p = \operatorname{im}(d^{p-1}), \quad {}_hH^p = {}_hZ^p/{}_hB^p$$

be the cocycles, coboundaries, and cohomology of C, respectively. For $p, q \geqslant 0$, let

$${}_hZ^{p,q} = \ker({}_h\delta^{p,q}), \quad {}_hB^{p,q} = \operatorname{im}({}_h\delta^{p-1,q}), \quad {}_hH^{p,q} = {}_hZ^{p,q}/{}_hB^{p,q}.$$

Then the following sequences

$$0 \longrightarrow C^p \xrightarrow{\quad f \quad} I^{p,0} \xrightarrow{\quad {}_v\delta^{p,0} \quad} I^{p,1} \xrightarrow{\quad {}_v\delta^{p,1} \quad} \cdots,$$

$$0 \longrightarrow {}_hZ^p \xrightarrow{\quad f \quad} {}_hZ^{p,0} \xrightarrow{\quad {}_v\delta^{p,0} \quad} {}_hZ^{p,1} \xrightarrow{\quad {}_v\delta^{p,1} \quad} \cdots,$$

$$0 \longrightarrow {}_hB^p \xrightarrow{\quad f \quad} {}_hB^{p,0} \xrightarrow{\quad {}_v\delta^{p,0} \quad} {}_hB^{p,1} \xrightarrow{\quad {}_v\delta^{p,1} \quad} \cdots,$$

$$0 \longrightarrow {}_hH^p \longrightarrow {}_hH^{p,0} \longrightarrow {}_hH^{p,1} \longrightarrow \cdots,$$

are injective resolutions over R, where the unlabeled arrows are the cohomology maps induced by f or ${}_v\delta$. For every $p, q \in \mathbb{Z}$, ${}_hZ^{p,q}$ (resp. ${}_hB^{p,q}$, ${}_hH^{p,q}$) is called the *horizontal cocycle* (resp. *horizontal coboundary, horizontal cohomology*) of I at position (p, q).

Moreover, the notation $(I, {}_h\delta, {}_v\delta) \xrightarrow{f} (C, d)$ (or briefly $I \xrightarrow{f} C, I \to C$, etc.) indicates that I is a CE resolution of C and f is the augmentation.

Definition 2.16.2. Let

$$(I_1, {}_h\delta_1, {}_v\delta_1) \xrightarrow{f_1} (C_1, d_1), \quad (I_2, {}_h\delta_2, {}_v\delta_2) \xrightarrow{f_2} (C_2, d_2)$$

be CE resolutions. A *morphism*

$$MCER : I \longrightarrow J$$

between CE resolutions is a morphism between double complexes I and J.

Let $F : C_1 \to C_2$ be a chain map. We say that $MCER$ and F are *compatible* if

$$MCER \circ f_1 = f_2 \circ F.$$

Lemma 2.16.3 ([36, Lemma 5.7.2]). *Every cochain complex has a CE resolution.*

Lemma 2.16.4 ([36, Exercise 5.7.2]). *Let R be a ring, let C_1 and C_2 be cochain complexes of R-modules,* *and let $I_1 \to C_1, I_2 \to C_2$ be CE resolutions over R. Then for every chain map $f : C \to D$, there exists a morphism $MCER : I_1 \to I_2$ between CE resolutions such that $MCER$ and f are compatible.*

Let $(I, {}_h\delta, {}_v\delta)$ be a CE resolution of some cochain complex over a ring R. As for ordinary resolutions, when we say "apply the functor $Hom_R(\mathbb{Z}, \cdot)$ to I to form a *deleted double complex* $(C, {}_hd, {}_vd)$", we mean that

$$C = \bigoplus_{p,q \geqslant 0} Hom_R(\mathbb{Z}, I^{p,q})$$

and ${}_hd, {}_vd$ are induced by ${}_h\delta, {}_v\delta$, respectively.

CHAPTER 3

COHEN-LYNDON TYPE THEOREMS

The main goal of this chapter is to prove the following generalization of Theorem 1.2.5.

Theorem 3.0.1. *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda} \hookrightarrow_{wh} G$. Then the Cohen-Lyndon property holds for all sufficiently deep Dehn fillings of $\{H_\lambda\}_{\lambda\in\Lambda}$.*

Assuming Theorem 3.0.1, we prove Theorem 1.2.5.

Proof of Theorem 1.2.5. By assumption, $H \hookrightarrow_h (G, X)$ for some subset $X \subset G$. Let $\widehat{d}$ be the relative metric on $\Gamma(H, H)$ with respect to X. Theorem 4.0.1 provides a constant C such that if $N \lhd H$ and $\widehat{d}(n) > C$ for all $n \in N\backslash\{1\}$, then (G, H, N) possesses the Cohen-Lyndon property. As $H \hookrightarrow_h (G, X)$, $\widehat{d}$ is locally finite. In particular,

$$\mathcal{F} = \{h \in H\backslash\{1\} \mid \widehat{d}(h) \leqslant C\}$$

is a finite set. By Theorem 3.0.1, if $N \lhd H$ and $N \cap \mathcal{F} = \emptyset$, then (G, H, N) has the Cohen-Lyndon property, and the desired result follows. $\square$

After the proof of Theorem 3.0.1, we will discuss the application of the Cohen-Lyndon property on relative relation modules.

3.1 Construction of the transversals

Let G be a group with a family of subgroups $\{H_\lambda\}_{wh} \hookrightarrow_{wh} (G, X)$ for some subset $X \subset G$. For $\lambda \in \Lambda$, let $\widehat{d_\lambda}$ be the relative metric with respect to X. The proof of Theorem 3.0.1 relies on constructing a particular left transversal $T_\lambda \in LT(H_\lambda\langle\langle\mathcal{N}\rangle\rangle, G)$ for each $\lambda \in \Lambda$. It is convenient to construct a collection $\{T_\lambda\}_{\lambda\in\Lambda}$ of sets of words over $X \sqcup \mathcal{H}$ satisfying the following properties (P1) through (P3), and think of T_λ as a transversal in $LT(H_\lambda\langle\langle\mathcal{N}\rangle\rangle, G)$ (identifying words over $X \sqcup \mathcal{H}$ and the elements of G represented by those words) for $\lambda \in \Lambda$. Recall that $\|w\|$ is the length of w for a word w over $X \sqcup H$, and that $|g|$ denotes the length of a geodesic word over $X \sqcup \mathcal{H}$ representing an element $g \in G$.

(P1) [$\{T_\lambda\}_{\lambda\in\Lambda}$ is transversal] For each $\lambda \in \Lambda$, $T_\lambda \in LT(H_\lambda\langle\langle\mathcal{N}\rangle\rangle, G)$.

(P2) [$\{T_\lambda\}_{\lambda\in\Lambda}$ is geodesic] If $w \in T_\lambda$ for some $\lambda \in \Lambda$, and $gH_\lambda\langle\langle\mathcal{N}\rangle\rangle = wH_\lambda\langle\langle\mathcal{N}\rangle\rangle$ for some $g \in G$, then $\|w\| \leqslant |g|$. This implies that, for all $\lambda \in \Lambda$, every $w \in T_\lambda$ is a geodesic word over $X \sqcup \mathcal{H}$.

(P3) [$\{T_\lambda\}_{\lambda\in\Lambda}$ is prefix closed] Let $\lambda, \mu \in \Lambda$. If a word $w \in T_\lambda$ can be decomposed as $w \equiv uhv$ with $h \in H_\mu \backslash \{1\}$ (u, v are allowed to be empty words), then $u \in T_\mu$ and $\widehat{d}_\mu(1, h) \leqslant \widehat{d}_\mu(1, h')$ for all $h' \in hN_\mu$.

Lemma 3.1.1. *There exists a collection $\{T_\lambda\}_{\lambda\in\Lambda}$ satisfying (P1), (P2), and (P3).*

Proof. Let $\mathcal{W}$ be the poset of collections $\{W_\lambda\}_{\lambda\in\Lambda}$ of words satisfying (P2) and (P3), while instead of (P1), we only demand that the words of W_λ represent a subset of a transversal in $LT(H_\lambda\langle\langle\mathcal{N}\rangle\rangle, G)$ for every $\lambda \in \Lambda$. We order $\mathcal{W}$ by index-wise inclusion, i.e., $\{U_\lambda\}_{\lambda\in\Lambda}$ is less than $\{V_\lambda\}_{\lambda\in\Lambda}$ if and only if $U_\lambda \subset V_\lambda$ for every $\lambda \in \Lambda$. $\mathcal{W}$ is non-empty because the collection $\{W_\lambda\}_{\lambda\in\Lambda}$ with each W_λ consisting of only the empty word is a member of $\mathcal{W}$. Moreover, the union of any chain of $\mathcal{W}$ is again a member of $\mathcal{W}$. Therefore, Zorn's lemma implies that $\mathcal{W}$ has a maximal member $\{T_\lambda\}_{\lambda\in\Lambda}$. Suppose that $\{T_\lambda\}_{\lambda\in\Lambda}$ does not satisfy (P1), i.e., there exist $\lambda_0 \in \Lambda$ and $g \in G$ such that no element of the coset $gH_{\lambda_0}M$ is represented by a word in T_{λ_0}. Without loss of generality, let us assume that if g' is an element of G such that $|g'| < |g|$, then for each $\lambda \in \Lambda$, $g'H_\lambda\langle\langle\mathcal{N}\rangle\rangle \cap T_\lambda \neq \emptyset$.

Let w be a geodesic word over $X \sqcup \mathcal{H}$ representing g. Consider the collection $\{U_\lambda\}_{\lambda\in\Lambda}$ constructed as follows. For every $\lambda \in \Lambda \backslash \{\lambda_0\}$, let $U_\lambda = T_\lambda$, and construct U_{λ_0} by the following manner: If w contains no letter from $\mathcal{H}$, let $U_{\lambda_0} = T_{\lambda_0} \cup \{w\}$. If w contains at least one letter from $\mathcal{H}$, then w can be decomposed as $w \equiv uhv$ such that $h \in H_\lambda \backslash \{1\}$ for some $\lambda \in \Lambda$ and v contains no letter from $\mathcal{H}$ (u, v are allowed to be empty words). As $\|u\| < \|w\| = |g|$, there exists a word $u' \in T_\lambda$ such that $u' \in uH_\lambda\langle\langle\mathcal{N}\rangle\rangle$. Let h' be an element of H_λ such that $u\langle\langle\mathcal{N}\rangle\rangle = u'h'\langle\langle\mathcal{N}\rangle\rangle$ and let h'' be an element of H_λ such that (a) $h''N_\lambda = h'hN_\lambda$ and (b) if $k \in h''N_\lambda$, then $\widehat{d}_\lambda(1, h'') \leqslant \widehat{d}_\lambda(1, k)$. Set $U_{\lambda_0} = T_{\lambda_0} \cup \{u'h''v\}$.

It is straight-forward to verify that $\{U_\lambda\}_{\lambda\in\Lambda}$ is an element of $\mathcal{W}$. There is a word in U_{λ_0} representing an element in $gH_{\lambda_0}\langle\langle\mathcal{N}\rangle\rangle$, while T_{λ_0} has no such words. It follows that $\{U_\lambda\}_{\lambda\in\Lambda}$ is strictly greater than $\{T_\lambda\}_{\lambda\in\Lambda}$, contradicting the choice of $\{T_\lambda\}_{\lambda\in\Lambda}$. □

3.2 Proof of Theorem 3.0.1

Suppose that the assumptions of Theorem 3.0.1 are met. Recall that Lemma 2.6.2 provides a number $D > 0$ to estimate the total length of isolated components in a geodesic polygon, and that Theorem 2.5.12 and Remark 2.6.3 implies that if $\widehat{d}_\lambda(1, n) \geqslant 4D$ for every $n \in N_\lambda \backslash \{1\}$ and $\lambda \in \Lambda$, then $H_\lambda \cap \langle\langle \mathcal{N} \rangle\rangle = N_\lambda$ for all $\lambda \in \Lambda$. We assume the following condition.

(24D) $\widehat{d}_\lambda(1, n) > 24D$ for all $n \in N_\lambda \backslash \{1\}$ and $\lambda \in \Lambda$.

We prove that (24D) implies the Cohen-Lyndon property of $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$. Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a collection of words over $X \sqcup \mathcal{H}$ satisfying (P1), (P2), and (P3) (by Lemma 3.1.1, such a collection exists) and think of each T_λ as a left transversal in $LT(H_\lambda\langle\langle \mathcal{N} \rangle\rangle, G)$. For every $\lambda \in \Lambda$, we extend T_λ to a set T^{ex}_λ. Roughly speaking, T^{ex}_λ is the set of words obtained from T_λ by replacing letters from H_λ with other letters from the same coset of N_λ in H_λ.

Definition 3.2.1. For every $\lambda \in \Lambda$, let T^{ex}_λ be the set of words with the following property: Every word $w \in T^{ex}_\lambda$ admits a decomposition $w \equiv w_1h_1 \cdots w_kh_kw_{k+1}$ ($w_1, ..., w_{k+1}$ are allowed to be empty words) such that for every $i \in \{1, ..., k\}$, there exists $\lambda_i \in \Lambda$ with the following properties.

(a) For $i = 1, ..., k$, h_i is an element of H_{λ_i} (h_i is allowed to equal 1).

(b) There exists an element $h'_i \in H_{\lambda_i} \backslash \{1\}$ such that $h'_iN_{\lambda_i} = h_iN_{\lambda_i}$ for $i = 1, ..., k$, and that the concatenation $w_1h'_1 \cdots w_kh'_kw_{k+1}$ is a word in T_λ.

Remark 3.2.2. If $k = 0$ in the above definition, conditions (a) and (b) will be satisfied trivially. Thus, T_λ is a subset of T^{ex}_λ for every $\lambda \in \Lambda$.

Definition 3.2.3. Let w be a word over $X \sqcup \mathcal{H}$ and let $\lambda \in \Lambda$. If $w \in T^{ex}_\lambda$, let $rank_\lambda(w)$ be the minimal number k obtained from the decompositions $w \equiv w_1h_1 \cdots w_kh_kw_{k+1}$ satisfying Definition 3.2.1. If $w \notin T_\lambda$, let $rank_\lambda(w) = \infty$.

For every word w over $X \sqcup \mathcal{H}$, the *rank* of w, denoted as $rank(w)$, is the number $\min_{\lambda\in\Lambda}\{rank_\lambda(w)\}$.

Lemma 3.2.4. *Let w be a word in T^{ex}_λ for some $\lambda \in \Lambda$. Suppose that w can be decomposed as $w \equiv uhv$ with $h \in H_\mu \backslash \{1\}$ for some $\mu \in \Lambda$. Let h'' be an element of H_μ such that $h''N_\mu = hN_\mu$. Then $uh''v \in T^{ex}_\lambda$.*

Proof. Let $w \equiv w_1 h_1 \cdots w_k h_k w_{k+1}$ be a decomposition satisfying Definition 3.2.1 and let $h'_1, ..., h'_k$ be as in (b) of Definition 3.2.1.

Without loss of generality, we may assume that $h = h_i$ for some number $i \in \{1, ..., k\}$. Then $uh''v$ can be decomposed as

$$uh''v \equiv w_1 h_1 \cdots w_{i-1} h_{i-1} w_i h'' w_{i+1} h_{i+1} w_{i+2} h_{i+2} \cdots w_k h_k w_{k+1}.$$

By replacing h_j with h'_j for $j \neq i$ and h'' with h'_i, we obtain a word in T_λ and thus $uh''v \in T_\lambda^{ex}$. □

Lemma 3.2.5. *Let w be a word in T_λ^{ex} for some $\lambda \in \Lambda$ with a decomposition $w \equiv w_1 h_1 \cdots w_k h_k w_{k+1}$ satisfying Definition 3.2.1. Then $w_1 \in T_{\lambda_1}$.*

Proof. Let $h'_1, ..., h'_k$ be as in (b) of Definition 3.2.1. Note that the word $w_1 h'_1 \cdots w_k h'_k w_{k+1}$ can be decomposed as

$$w_1 h'_1 ... w_k h'_k w_{k+1} \equiv w_1 h'_1 (w_2 h'_2 \cdots w_k h'_k w_{k+1}).$$

By (P3), $w_1 \in T_{\lambda_1}$. □

It will be shown that $\langle\langle \mathcal{N} \rangle\rangle = \prod^*_{\lambda \in \Lambda, t \in T_\lambda} N_\lambda^t$. For the moment, let

$$K = \langle N_\lambda^t, t \in T_\lambda, \lambda \in \Lambda \rangle \leqslant G.$$

Lemma 3.2.6. *Let w be a word in $\bigcup_{\lambda \in \Lambda} T_\lambda^{ex}$, and let n be an element of N_{λ_0} for some $\lambda_0 \in \Lambda$. Then $wnw^{-1} \in K$.*

Proof. Let μ be an element of Λ with $rank(w) = rank_\mu(w)$. Thus, w admits a decomposition $w \equiv w_1 h_1 \cdots w_k h_k w_{k+1}$ satisfying Definition 3.2.1 with $k = rank(w)$. We perform induction on $rank(w)$. If $rank(w) = 0$, then $w \in T_\mu$ and thus $wnw^{-1} \in K$.

Suppose that, for all $w' \in \bigcup_{\lambda \in \Lambda} T_\lambda^{ex}$ with $rank(w') < rank(w)$ and all $n' \in \bigcup_{\lambda \in \Lambda} N_\lambda$, $w'^{-1} n' w' \in K$. Let $h'_1, ..., h'_k$ be as in (b) of Definition 3.2.1. Thus, there exists $n_1 \in N_{\lambda_1}$ such that $n_1 h'_1 = h_1$ (note that N_{λ_1} is a normal subgroup of H_{λ_1}). Notice that

$$w =_G (w_1 n_1 w_1^{-1})(w_1 h'_1 w_2 h_2 \cdots w_k h_k w_{k+1})$$

and thus

$$wnw^{-1} =_G (w_1n_1w_1^{-1})(w'nw'^{-1})(w_1n_1w_1^{-1})^{-1}, \tag{3.1}$$

where $w' \equiv w_1h_1'w_2h_2\cdots w_kh_kw_{k+1}$.

By replacing h_j with h_j' for $j = 2,...,k$, we can turn w' into a word in T_μ. Thus, $w' \in T_\mu^{ex}$ and $rank(w') \leqslant k-1 < rank(w)$. It follows from the induction hypothesis that $w'n(w')^{-1} \in K$. By Lemma 3.2.5, $w_1 \in T_{\lambda_1}$ and thus $w_1n_1w_1^{-1} \in K$. By (3.1), wnw^{-1} represents a product of elements of K. $\square$

For the next two lemmas, recall that $\|w\|$ denotes the length of a word w over $X\sqcup\mathcal{H}$, and that $|g|$ denotes the length of a geodesic word over $X \sqcup \mathcal{H}$ representing an element $g \in G$.

Lemma 3.2.7. *Let λ be an element of Λ, let u be a word in T_λ^{ex}, let h be a letter of $H_\lambda\backslash\{1\}$, and let v be a word over $X \sqcup \mathcal{H}$ with $\|v\| = \|u\|$. Suppose that every element $m' \in \langle\langle\mathcal{N}\rangle\rangle$ with $|m'| < 2\|u\| + 1$ belongs to K. If the concatenation $uhv \in \langle\langle\mathcal{N}\rangle\rangle$, then $uhv \in K$.*

Proof. If uhv is not a geodesic word, the desired result will follow from the assumptions trivially. So let us assume that uhv is geodesic. Consider a diagram $\Delta \in \mathcal{D}(w)$ of minimal type (see Definition 2.7.6).

We prove Lemma 3.2.7 by an induction on the number of holes in Δ. If Δ has no holes, then it will be a disk van Kampen diagram over (2.2) with boundary labeled by uhv and thus uhv represents $1 \in K$.

Suppose that Δ has $k \geqslant 1$ holes. By Lemma 2.7.8, there exists $\mu \in \Lambda$ and a connected component c of $\partial_{int}\Delta$ such that c is connected to an H_μ-component of $\partial_{ext}\Delta$. Let w be the label of c. Then w is a word over H_μ representing an element $n \in N_\mu$. As $Lab(\partial_{ext}\Delta) \equiv uhv$, we can use Remark 2.2.2 to decompose $\partial_{ext}\Delta$ as the concatenation $p_up_hp_v$ of three paths p_u, p_h, and p_v with $Lab(p_u) \equiv u, Lab(p_h) \equiv h, Lab(p_v) \equiv v$. Depending on where c is connected to, there are three possible cases.

Case 1: c is connected to an H_μ-component of p_u.

In other words, u can be decomposed as $u \equiv u_1h_1u_2$ with $h_1 \in H_\mu\backslash\{1\}$, and p_u can be decomposed as a concatenation $p_{u_1}p_{h_1}p_{u_2}$ of three paths p_{u_1}, p_{h_1}, and p_{u_2} such that $Lab(p_{u_1}) \equiv u_1, Lab(p_{h_1}) \equiv h_1, Lab(p_{u_2}) \equiv u_2$ and c is connected to p_{h_1} (see Remark 2.2.1). By Lemma 2.7.3, passing to an equivalent diagram if necessary, we may assume that there exists a path p_{h_2} in Δ with $Lab(p_{h_2}) \equiv h_2 \in H_\mu$, connecting the common vertex of p_{h_1} and p_{u_1} to a vertex of c. Note that the conjugate $n_1 = h_2nh_2^{-1} \in N_\mu$. Let h_3 be the letter from H_μ such that $h_3 =_G n_1h_1$. Then

$$uhv \equiv u_1h_1u_2hv =_G (u_1n_1^{-1}u_1^{-1})(u_1h_3u_2hv). \tag{3.2}$$

As $h_1 \neq 1$, we have $\|u_1\| \leqslant \|u\| - 1$ and thus $\|u_1 n_1^{-1} u_1^{-1}\| \leqslant 2\|u_1\| - 1 < 2\|u\| + 1$. Note that $u_1 n_1^{-1} u_1^{-1} \in \langle\langle \mathcal{N} \rangle\rangle$. By the induction hypothesis, $u_1 n_1^{-1} u_1^{-1} \in K$.

Let $u_4 \equiv u_1 h_3 u_2$. Note that $\|u_4\| \leqslant \|u\|$. As $uhv, u_1 n_1^{-1} u_1^{-1} \in \langle\langle \mathcal{N} \rangle\rangle$, it follows from (3.2) that $u_4 hv \in \langle\langle \mathcal{N} \rangle\rangle$. If $\|u_4\| < \|u\|$, then $\|u_4 hv\| < 2\|u\| + 1$ and thus $u_4 hv \in K$, by assumption. So let us assume that $\|u_4\| = \|u\|$. By Lemma 3.2.4, $u_4 \in T_\lambda^{ex}$. Let Σ be a disc van Kampen diagram over (2.2) such that

$$Lab(\partial\Sigma) \equiv h_2 w h_2^{-1} h_1 h_3^{-1}.$$

Cut Δ along the path p_{h_2} to produce a diagram $\Delta_1 \in \mathcal{D}$ with

$$Lab(\partial_{ext}\Delta_1) \equiv u_1 h_2 w h_2^{-1} h_1 u_2 hv.$$

Glue Σ to Δ_1 by identifying the paths with label $h_2 w h_2^{-1} h_1$ (perform refinements if the non-essential edges of the two paths do not match) to construct a diagram $\Delta_2 \in \mathcal{D}$ with

$$Lab(\partial_{ext}\Delta_2) \equiv u_4 hv$$

(see Figure 3.1). Note that the number of holes in Δ_2 is strictly less than that of Δ. By the induction hypothesis, $u_4 hv \in K$. By (3.2), uhv is a product of elements of K.

Case 2: c is connected to an H_μ-component of p_v.

This case is symmetric to Case 1 and the proof is left to the reader.

Case 3: c is connected to p_h.

In other words, $\mu = \lambda$ and $h \in H_\lambda \backslash \{1\}$. By Lemma 2.7.3 and passing to an equivalent diagram if necessary, we may assume that there exists a path in Δ, labeled by a letter $h_1 \in H_\lambda$, connecting the common vertex of p_h and p_u to a vertex of c. Note that the conjugate $n_1 = h_1 n h_1^{-1} \in N_\lambda$. Let h_2 be a letter from H_λ such that $h_2 =_G n_1 h$. Consider the equality

$$uhv =_G (un_1^{-1}u^{-1})(uh_2 v). \tag{3.3}$$

As $u \in T_\lambda^{ex}$, Lemma 3.2.6 implies that $un_1^{-1}u^{-1} \in K$. An analysis similar to the one in Case 1 (with uh_2v in place of u_4hv) shows that $uh_2v \in K$. By (3.3), uhv is a product of elements of K. □

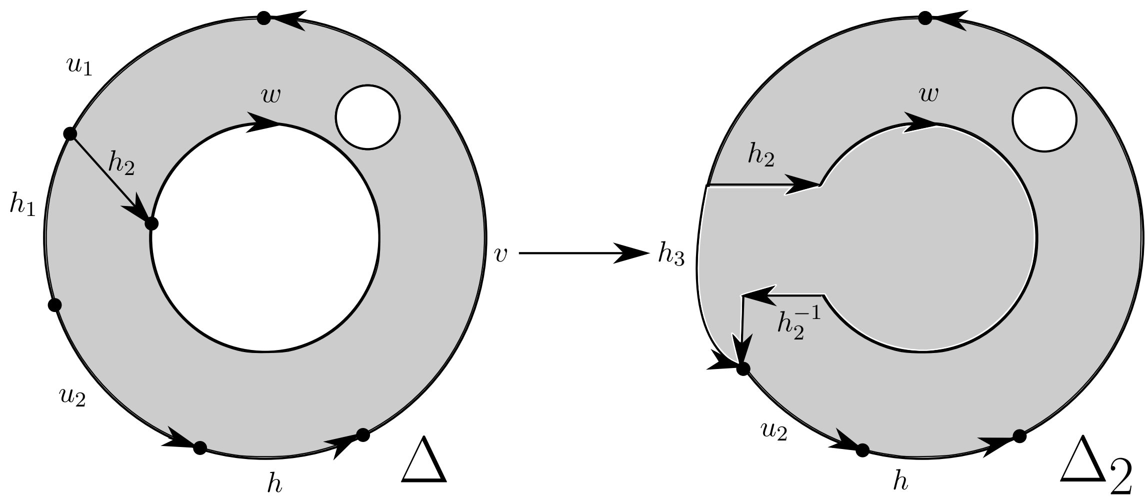


Figure 3.1: An illustration of Case 1 in the proof of Lemma 3.2.7

Definition 3.2.8. Let w be a word representing an element of $\langle\langle \mathcal{N} \rangle\rangle$. Define the number $k(w)$ to be the minimal number of holes of a diagram $\Delta \in \mathcal{D}(w)$. The *type* of w is the pair $\tau(w) = (\|w\|, k(w))$. We order the set of types lexicographically (see Definition 2.7.6).

Remark 3.2.9. If w is a word representing an element of $\langle\langle \mathcal{N} \rangle\rangle$ and Δ is a diagram in $\mathcal{D}(w)$ of minimal type, then Δ necessarily has $k(w)$ holes.

Proposition 3.2.10. $\langle\langle \mathcal{N} \rangle\rangle = K$.

Proof. Clearly, each of the groups $N_\lambda^t, t \in T_\lambda, \lambda \in \Lambda$, is contained in $\langle\langle \mathcal{N} \rangle\rangle$ and thus $K \leqslant \langle\langle \mathcal{N} \rangle\rangle$. Let w be a word over $X \sqcup \mathcal{H}$ such that $w \in \langle\langle \mathcal{N} \rangle\rangle$. Let us show that $w \in K$ by performing induction on the type of w. Note that the base case $\|w\| = k(w) = 0$ is trivial.

Suppose that, for every word w' over $X \sqcup \mathcal{H}$ with $w' \in \langle\langle \mathcal{N} \rangle\rangle$, $\tau(w') < \tau(w)$ implies that $w' \in K$. If w is not a geodesic word, the induction hypothesis will imply $w \in K$. Thus, we may assume that w is geodesic. Consider a diagram $\Delta \in \mathcal{D}(w)$ of minimal type.

By Lemma 2.7.8, there exist $\lambda \in \Lambda$ and a connected component c of $\partial_{int}\Delta$ connected to an H_λ-component of $\partial_{ext}\Delta$. In other words, w can be decomposed as uhv with $h \in H_\lambda \backslash \{1\}$ (u, v are allowed to be empty words), and $\partial_{ext}\Delta$ can be decomposed as a concatenation $p_u p_h p_v$ of three paths p_u, p_h, and p_v such that $Lab(p_u) = u, Lab(p_h) = h, Lab(p_v) = v$ and c is connected to p_h (see Remark 2.2.2). By Lemma 2.7.3 and passing to an equivalent diagram if necessary, we may assume that there exists a path p_{h_1} in Δ with $Lab(p_{h_1}) \equiv h_1 \in H_\lambda$, connecting the common vertex of p_h and p_u to a vertex of c.

Note that, as $h \neq 1$, at least one of $\|u\|$ and $\|v\|$ is at most $(\|w\| - 1)/2$. Without loss of generality, we may assume that $\|v\| \leqslant (\|w\| - 1)/2$. The case $\|u\| \leqslant (\|w\| - 1)/2$ can be analyzed in almost the same way (or just by considering w^{-1} and reversing every edge of Δ if one wishes).

Let $w_1 \equiv Lab(c)$. Thus, $w_1 \in N_\lambda$. Let h_2 be a letter from H_λ such that $h_2 =_G hh_1nh_1^{-1}$. There exists $t \in T_\lambda$ such that t and v^{-1} are in the same left $H_\lambda\langle\langle\mathcal{N}\rangle\rangle$-coset. In other words, there exists $h_3 \in H_\lambda$ such that $th_3v \in \langle\langle\mathcal{N}\rangle\rangle$. Let n_1 be a letter in N_λ such that $n_1 =_G h_3h_1n^{-1}h_1^{-1}h_3^{-1}$.

Consider the equality

$$w \equiv uhv =_G (uh_2v)(v^{-1}h_3^{-1}t^{-1})(tn_1t^{-1})(th_3v). \tag{3.4}$$

Note that $uh_2v \in \langle\langle\mathcal{N}\rangle\rangle$, as all other brackets in (3.4) represents elements of $\langle\langle\mathcal{N}\rangle\rangle$. As in the proof of Lemma 3.2.7, let Σ be a disc van Kampen diagram over (2.2) with

$$Lab(\partial\Sigma) \equiv hh_1w_1h_1^{-1}h_2^{-1}.$$

Cut Δ along p_{h_1} to produce a diagram $\Delta_1 \in \mathcal{D}$ with $Lab(\partial_{ext}\Delta_1) \equiv uhh_1w_1h_1^{-1}v$. Glue Δ_1 to Σ, identifying the paths labeled by $hh_1w_1h_1^{-1}$ (perform refinements if the non-essential edges of the two paths do not match). Denote the resulting diagram by Δ_2. Clearly, $\Delta_2 \in \mathcal{D}$ and $Lab(\partial_{ext}\Delta_2) \equiv uh_2v$. Note that the number of holes in Δ_2 is strictly less than that of Δ, and that $\|uh_2v\| \leqslant \|u\| + \|v\| + 1 = \|uhv\|$, as uhv is a geodesic word. Thus, $\tau(uh_2v) < \tau(w)$ and the induction hypothesis implies $uh_2v \in K$.

Clearly, $tn_1t^{-1} \in K$. Note also that $th_3v \in K$. Indeed, if either $\|t\| < \|v\|$ or $h_3 = 1$, then $\|th_3v\| < 2\|v\| + 1 = \|w\|$ and the induction hypothesis implies that $th_3v \in K$. If $\|t\| = \|v\|$ and $h_3 \neq 1$, then Lemma 3.2.7 implies $th_3v \in K$.

As $v^{-1}h_3^{-1}t^{-1} \equiv (th_3v)^{-1}$, we also have $v^{-1}h_3^{-1}t^{-1} \in K$. By (3.4), w is a product of elements of K. □

The cutting process in the proof of Lemma 3.2.10 is exactly the same as the one for Lemma 3.2.7. See Figure 3.1 for an illustration.

The goal of the rest of this section is to prove the following.

Proposition 3.2.11. $\langle\langle\mathcal{N}\rangle\rangle = \prod^*_{\lambda\in\Lambda, t\in T_\lambda} N_\lambda^t$.

Proof. Assume, for the contrary, that there exists a word

$$z \equiv \prod_{i=1}^{k} t_i n_i t_i^{-1} \tag{3.5}$$

representing $1 \in G$ such that

(Z1) $k \geqslant 2$;

(Z2) for $i = 1, ..., k$, there exists $\lambda_i \in \Lambda$ such that $n_i \in N_{\lambda_i} \setminus \{1\}$ and $t_i \in T_{\lambda_i}$;

(Z3) $t_i \not\equiv t_{i+1}$ for $i = 1, ..., k$ (subscripts are modulo k, i.e., $n_{k+1} = n_1$, $t_0 = t_k$, etc.).

Without loss of generality, we may also assume

(Z4) z is minimal, i.e., has the minimal k among all other words of the form (3.5) representing 1 in G and satisfying (Z1), (Z2), and (Z3).

The main idea of the proof of Lemma 3.2.11 is to show that the existence of such a word z contradicts Lemma 2.6.2. For this purpose, it is convenient to first cyclically permute z and consider the word

$$w \equiv t_k^{-1} (\prod_{i=1}^{k-1} t_i n_i t_i^{-1}) t_k n_k.$$

In what follows, subscripts are modulo k. Let p_w be the path in $\Gamma(G, X \sqcup \mathcal{H})$ with $Lab(p) \equiv w$ and $p^- = 1$. We use $p_{n_i}, p_{t_i^{\pm 1}}$ to denote subpaths of p_w labeled by $n_i, t_i^{\pm 1}$, respectively. More precisely, p_{n_i} (resp. $p_{t_i}, p_{t_i^{-1}}$) will denote the path in the Cayley graph $\Gamma(G, X \sqcup \mathcal{H})$ with $Lab(p_{n_i}) = n_i$ (resp. $Lab(p_{t_i}) = t_i, Lab(p_{t_i^{-1}}) = t_i^{-1}$) and $p_{n_i}^- = t_k^{-1}(\prod_{j=1}^{i-1} t_j n_j t_j^{-1}) t_i$ (resp. $p_{t_i}^- = t_k^{-1}(\prod_{j=1}^{i-1} t_j n_j t_j^{-1}), p_{t_i^{-1}}^- = t_k^{-1}(\prod_{j=1}^{i-1} t_j n_j t_j^{-1}) t_i n_i$).

Recall that the collection $\{T_\lambda\}_{\lambda \in \Lambda}$ satisfies (P1), (P2), and (P3). Note that, for every $\lambda \in \Lambda$ and every word $t \in T_\lambda$, the word t does not end with a letter from H_λ, by (P2). It follows that p_{n_i} is an H_{λ_i}-component of p_w for $i = 1, ..., k$. Being a cyclic permutation of z, the word w represents 1 in G and thus the terminal vertex of p_w is 1. Hence, p_w is a geodesic $3k$-gon. As $\widehat{\ell}_{\lambda_i}(p_{n_i}) = \widehat{d}_{\lambda_i}(1, n_i)$ for $i = 1, ..., k$, by Lemma 2.6.2 and (24D), there exists some $i \in \{1, ..., k\}$ such that p_{n_i} is not an isolated H_{λ_i}-component of p_w.

The rest of the proof is divided into several lemmas. All of them are stated under the assumptions (and using the notations) of Proposition 3.2.11.

Lemma 3.2.12. *If p_{n_i} is not an isolated H_{λ_i}-component of p_w for some $i \in \{1, ..., k\}$, then there are only three possibilities:*

(a) p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i+1}}$, but not connected to any H_{λ_i}-component of $p_{t_{i-1}^{-1}}$.

(b) p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i-1}^{-1}}$, but not connected to any H_{λ_i}-component of $p_{t_{i+1}}$.

(c) p_{n_i} is connected to both an H_{λ_i}-component of $p_{t_{i+1}}$ and an H_{λ_i}-component of $p_{t_{i-1}^{-1}}$.

Proof. Without loss of generality, let us assume that p_{n_1} is not isolated in p_w. There are six cases to consider (see Figure 3.2 for an illustration).

Case 1: p_{n_1} is connected to an H_{λ_1}-component of either p_{t_1} or $p_{t_1^{-1}}$. In this case, some terminal segment of t_1 represents an element of H_{λ_1}, which contradicts (P2).

Case 2: p_{n_1} is connected to either p_{n_2} or p_{n_k}. If p_{n_1} is connected to p_{n_2}, then $\lambda_1 = \lambda_2$, which in turn implies $t_1, t_2 \in T_{\lambda_1}$. The assumption that p_{n_1} is connected to p_{n_2} also implies $t_1^{-1}t_2 \in H_{\lambda_1}$. By (P1), $t_1 \equiv t_2$, contradicting (Z3). The analysis for the subcase where p_{n_1} is connected to p_{n_k} is similar.

Case 3: p_{n_1} is connected to p_{n_i} for some $i \in \{3, ..., k-1\}$. In other words, there exists $h \in H_{\lambda_1}$ such that the word

$$u \equiv t_1^{-1}(\prod_{j=2}^{i-1} t_j n_j t_j^{-1}) t_i h$$

represents 1 in G. As $\prod_{j=2}^{i-1} t_j n_j t_j^{-1} \in \langle\langle \mathcal{N} \rangle\rangle \lhd G$, we have $t_1^{-1}t_i \in H_{\lambda_1}\langle\langle \mathcal{N} \rangle\rangle$. The assumption that p_{n_1} is connected to p_{n_i} also implies $n_1, n_i \in N_{\lambda_1}$ and thus $t_1, t_i \in T_{\lambda_1}$. By (P1), $t_1 \equiv t_i$. Thus, the word

$$u' \equiv t_1 h t_1^{-1}(\prod_{j=2}^{i-1} t_j n_j t_j^{-1})$$

is a cyclic permutation of u and represents 1 in G. It follows that $t_1 h t_1^{-1} \in \langle\langle \mathcal{N} \rangle\rangle$. By Theorem 2.5.12, Remark 2.6.3, and Condition (24D), we have $h \in N_{\lambda_1}$. Then the word $t_1 h t_1^{-1}(\prod_{j=2}^{i-1} t_j n_j t_j^{-1})$ represents 1 in G, contradicting (Z4).

Case 4: p_{n_1} is connected to an H_{λ_1}-component of p_{t_i} for some $i \in \{3, ..., k\}$. Thus, t_i can be decomposed as $t_i \equiv t_i' h' t_i''$ with $h' \in H_{\lambda_1} \setminus \{1\}$ and there exists $h \in H_{\lambda_1}$ such that the word

$$u \equiv t_1^{-1}(\prod_{j=2}^{i-1} t_j n_j t_j^{-1}) t_i' h$$

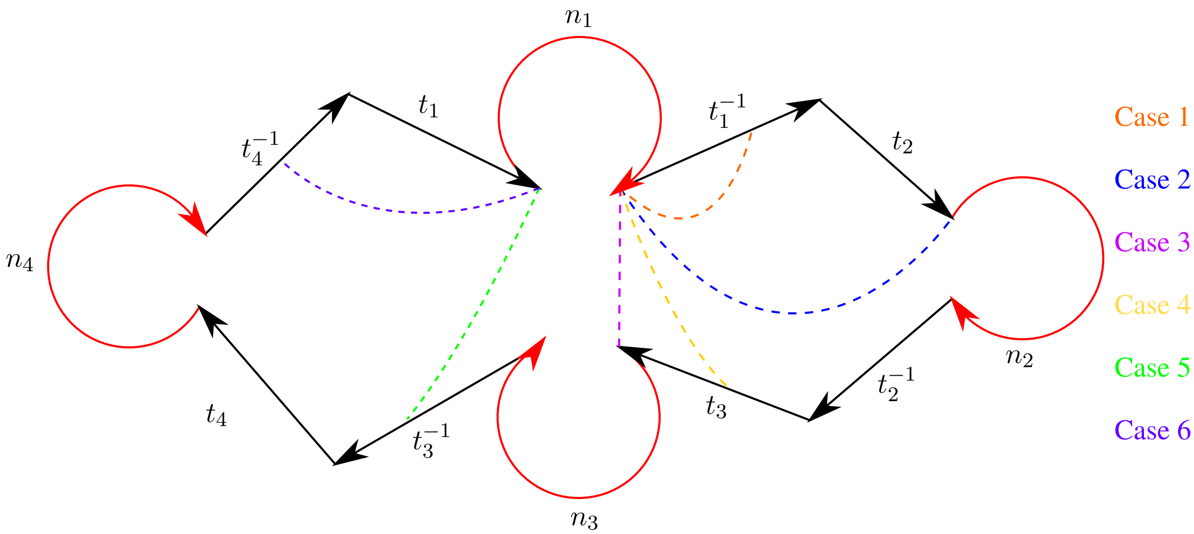


Figure 3.2: Cases 1 through 6 in the proof of Lemma 3.2.12

represents 1 in G. By (P3), t_i' belongs to T_{λ_1}. Arguing as in Case 3, we conclude that the word $t_1ht_1^{-1}(\prod_{j=2}^{i-1} t_jn_jt_j^{-1})$ represents 1 in G, contradicting (Z4).

Case 5: p_{n_1} is connected to an H_{λ_1}-component of $p_{t_i^{-1}}$ for some $i \in \{2, ..., k-1\}$. This case can be reduced to Case 4 by considering w^{-1}.

Thus, the only possibilities left are (a), (b), and (c). □

Lemma 3.2.13. *If p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i+1}}$, then t_{i+1} can be decomposed as $t_{i+1} \equiv uhv$ with $h \in H_{\lambda_i}\backslash\{1\}$ (u, v are allowed to be empty words), $t_i \equiv u$, and $\widehat{d}_{\lambda_i}(1, n_ih) > 12D$.*

Proof. By Definition 2.6.1, t_{i+1} can be decomposed as $t_{i+1} \equiv uhv$ with $h \in H_{\lambda_i}\backslash\{1\}$ such that p_{n_i} is connected to the path p_h in $\Gamma(G, X \sqcup \mathcal{H})$ with $Lab(p_h) \equiv h$ and $p_h^- = t_k^{-1}(\prod_{j=1}^{i} t_jn_jt_j^{-1})u$. By (P3), $u \in T_{\lambda_i}$. The assumption that p_{n_i} is connected to p_h also implies $t_i^{-1}u \in H_{\lambda_i}$ and thus $t_i \equiv u$, by (P1). Another consequence of (P3) is

$$\widehat{d}_{\lambda_i}(1, h) \leqslant \widehat{d}_{\lambda_i}(1, h(h^{-1}n_ih)) = \widehat{d}_{\lambda_i}(1, n_ih).$$

Therefore, the triangle inequality implies

$$\widehat{d}_{\lambda_i}(1, n_i) \leqslant \widehat{d}_{\lambda_i}(1, n_ih) + \widehat{d}_{\lambda_i}(1, h^{-1}) = \widehat{d}_{\lambda_i}(1, n_ih) + \widehat{d}_{\lambda_i}(1, h) \leqslant 2\widehat{d}_{\lambda_i}(1, n_ih)$$

and thus

$$\widehat{d}_{\lambda_i}(1, n_i h) \geqslant \widehat{d}_{\lambda_i}(1, n_i)/2 > 12D,$$

by (24D). $\square$

The next lemma follows from Lemma 3.2.13 by considering w^{-1}.

Lemma 3.2.14. *If p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i-1}^{-1}}$, then t_{i-1} can be decomposed as $t_{i-1} \equiv uhv$ with $h \in H_{\lambda_i} \backslash \{1\}$ (u, v are allowed to be empty words), $t_i \equiv u$, and $\widehat{d}_{\lambda_i}(1, h^{-1} n_i) > 12D$.*

Lemma 3.2.15. *If p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i+1}}$, then $p_{n_{i+1}}$ is not connected to any $H_{\lambda_{i+1}}$-component of $p_{t_i^{-1}}$. If p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i-1}^{-1}}$, then $p_{n_{i-1}}$ is not connected to any $H_{\lambda_{i-1}}$-component of p_{t_i}.*

Proof. If p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i+1}}$, then t_i equals some prefix of t_{i+1}, by Lemma 3.2.13. If, in addition, $p_{n_{i+1}}$ is connected to an $H_{\lambda_{i+1}}$-component of $p_{t_i^{-1}}$, then t_{i+1} equals some prefix of t_i, by Lemma 3.2.14. Thus, $t_i \equiv t_{i+1}$, contradicting (Z3).

The second assertion of the Lemma can be proved by considering w^{-1}. $\square$

Recall that we assume the existence of a word z satisfying (Z1) through (Z4) and construct w, p_w from z. The previous several lemmas reveal some properties of p_w and we are now ready to construct a geodesic polygon p from p_w so that p violates Lemma 2.6.2, and then we can conclude that z does not exist and prove Proposition 3.2.11. The idea is to merge all H_{λ_i}-components connected to p_{n_i} to form an isolated H_{λ_i}-component for $i = 1, ..., k-1$. Of course, one can also merge p_{n_k} with the H_{λ_k}-components connected to it. We do not perform this merging only because it makes the construction more complicated. Pick elements $h_1, ..., h_{k-1} \in \mathcal{H}$ and $g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2}, ..., g_{k-1,1}, g_{k-1,2} \in G$ by the following procedure.

Procedure 3.2.16. *For $i = 1, ..., k-1$, perform the following.*

- *(a) If p_{n_i} is an isolated H_{λ_i}-component in p_w, let $g_{i,1} \in G$ (resp. $g_{i,2} \in G$) be represented by the word $t_k^{-1}(\prod_{j=1}^{i-1} t_j n_j t_j^{-1}) t_i$ (resp. $t_k^{-1}(\prod_{j=1}^{i-1} t_j n_j t_j^{-1}) t_i n_i$), and let $h_i = n_i$.*
- *(b) If, in p_w, p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i+1}}$, but not connected to any H_{λ_i}-component of $p_{t_{i-1}^{-1}}$, then by Lemma 3.2.13, t_{i+1} can be decomposed as $t_{i+1} \equiv u_i h_i' v_i$ with $h_i' \in H_{\lambda_i} \backslash \{1\}$, $t_i \equiv u_i$, and $\widehat{d}_{\lambda_i}(1, n_i h_i') > 12D$. Let h_i be a letter from H_{λ_i} such that $h_i =_G n_i h_i'$, and let $g_{i,1} \in G$ (resp. $g_{i,2} \in G$) be represented by the word $t_k^{-1}(\prod_{j=1}^{i-1} t_j n_j t_j^{-1}) t_i$ (resp. $t_k^{-1}(\prod_{j=1}^{i-1} t_j n_j t_j^{-1}) t_i h_i$).*

(c) If in p_w, p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i-1}^{-1}}$, but not connected to any H_{λ_i}-component of t_{i+1}, then by Lemma 3.2.14, t_{i-1} can be decomposed as $t_{i-1} \equiv u_i h_i' v_i$ with $h_i' \in H_{\lambda_i}\backslash\{1\}$, $t_i \equiv u_i$, and $\widehat{d}_{\lambda_i}(1, h_i'^{-1}n_i) > 12D$. Let h_i be a letter from H_{λ_i} such that $h_i =_G h_i'^{-1}n_i$, and let $g_{i,1} \in G$ (resp. $g_{i,2} \in G$) be represented by the word $t_k^{-1}(\prod_{j=1}^{i-2} t_j n_j t_j^{-1})t_{i-1}n_{i-1}v_i^{-1}$ (resp. $t_k^{-1}(\prod_{j=1}^{i-2} t_j n_j t_j^{-1})t_{i-1}n_{i-1}v_i^{-1}h_i$).

(d) If in p_w, p_{n_i} is connected to both an H_{λ_i}-component of $p_{t_{i+1}}$ and an H_{λ_i}-component of $p_{t_{i-1}^{-1}}$, then by Lemmas 3.2.13 and 3.2.14, t_{i+1} (resp. t_{i-1}) can be decomposed as $t_{i+1} \equiv u_i h_i' v_i$ (resp. $t_{i-1} \equiv u_i' h_i'' v_i'$) with $h_i' \in H_{\lambda_i}\backslash\{1\}$ (resp. $h_i'' \in H_{\lambda_i}\backslash\{1\}$), $t_i \equiv u_i$ (resp. $t_i \equiv u_i'$). Let h_i be a letter from H_{λ_i} such that $h_i =_G h_i''^{-1}n_i h_i'$, and let $g_{i,1} \in G$ (resp. $g_{i,2} \in G$) be represented by the word $t_k^{-1}(\prod_{j=1}^{i-2} t_j n_j t_j^{-1})t_{i-1}n_{i-1}(v_i')^{-1}$ (resp. $t_k^{-1}(\prod_{j=1}^{i-2} t_j n_j t_j^{-1})t_{i-1}n_{i-1}(v_i')^{-1}h_i$).

Lemma 3.2.17. *$g_{i,1}$ and $g_{i,2}$ are vertices on p_w for $i = 1, ..., k-1$. Moreover, the order in which p_w visits these vertices is $g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2}, ..., g_{k-1,1}, g_{k-1,2}$.*

Proof. The first assertion follows directly from the choices of those vertices. Clearly, the path p_w visits $g_{i,1}$ before visiting $g_{i,2}$ for $i = 1, ..., k-1$. Thus, the second assertion will be proved once we show that, for all $i, j \in \{1, ..., k-1\}$ with $i < j$, the path p_w visits $g_{i,2}$ before visiting $g_{j,1}$.

Suppose, for the contrary, that for some $i, j \in \{1, ..., k-1\}$ with $i < j$, the path p_w visits $g_{j,1}$ before visiting $g_{i,2}$. By Lemma 3.2.12, there is only one possibility for this case: $j = i+1$, p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i+1}}$, and $p_{n_{i+1}}$ is connected to an $H_{\lambda_{i+1}}$-component of $p_{t_i^{-1}}$. By Lemma 3.2.15, if p_{n_i} is connected to an H_{λ_i}-component of $p_{t_{i+1}}$, then $p_{n_{i+1}}$ is not connected to any $H_{\lambda_{i+1}}$-component of $p_{t_i^{-1}}$, a contradiction. □

Lemma 3.2.18. *For $i = 1, ..., k-2$, the subpath of p_w from $g_{i,2}$ to $g_{i+1,1}$ consists of at most two geodesic segments.*

Lemma 3.2.18 follows immediately from the choices of the vertices $g_{i,1}$ and $g_{i,2}$, $1 \leqslant i \leqslant k-1$. We are now ready to construct a geodesic polygon p from p_w.

Construction 3.2.19. For $i = 1, ..., k-1$, let p_{h_i} the edge of $\Gamma(G, X \sqcup \mathcal{H})$ with $Lab(p_{h_i}) = h_i$ and $p_{h_i}^- = g_{i,1}$. Let p be the path in $\Gamma(G, X \sqcup \mathcal{H})$ satisfying: p^- is the identity vertex. p first follows the path of p_w (in the direction of p_w) until p visits $g_{1,1}$, and then p travels along p_{h_1} and arrives at $g_{1,2}$. And then p follows the path p_w (in the direction of p_w) until p arrives at $g_{2,1}$ (Lemma 3.2.17 guarantees that p will arrive

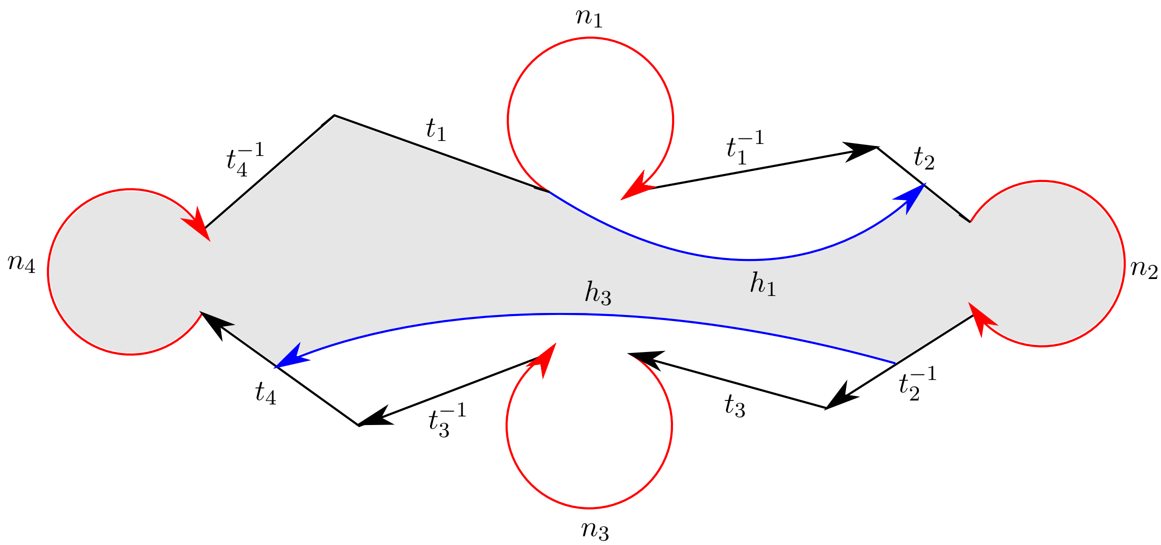


Figure 3.3: The construction of p

at $g_{2,1}$), where p travels along p_{h_2} and then arrives at $g_{2,2}$. The path p continues traveling in this manner until arriving at $g_{k-1,2}$. Finally, p follows the path p_w (in the direction of p_w) and comes back to the identity vertex.

Figure 3.3 illustrates how to construct the geodesic polygon p. In Figure 3.3, the outside boundary with label $t_4^{-1}t_1n_1t_1^{-1}t_2n_2t_2^{-1}t_3n_3t_3^{-1}t_4n_4$ is the geodesic polygon p_w. In the outside boundary, p_{n_2} is an isolated H_{λ_2}-component, p_{n_1} (resp. p_{n_4}) is connected to an H_{λ_1}-component (resp. H_{λ_4}-component) of p_{t_2} (resp. p_{t_1}), and p_{n_3} is connected to both an H_{λ_3}-component of $p_{t_2^{-1}}$ and an H_{λ_3}-component of p_{t_4}. By Lemma 3.2.13, t_1^{-1} cancels with a prefix of t_2. After this cancellation, p_{n_1} merges with an H_{λ_1}-component of p_{t_2} to form p_{h_1}. Similarly, p_{n_3} merges with both an H_{λ_3}-component of $p_{t_2^{-1}}$ and an H_{λ_3}-component of p_{t_4} to form p_{h_3}. The merging process does nothing to n_4, although n_4 is not an isolated H_{λ_4}-component. Finally, p_w becomes p, the boundary of the shaded region.

Remark 3.2.20. It follows easily from the above construction that p_{h_i} is an isolated H_{λ_i}-component of p for $i = 1, ..., k-1$.

Note that the subpath of p_w from 1 to $g_{i,1}$ consists of at most 2 geodesic segments, and the subpath of p_w from $g_{k-1,2}$ to 1 consists of at most 3 geodesic segments. Together with Lemma 3.2.18, these observations imply that p is a polygon in $\Gamma(G, X \sqcup \mathcal{H})$ with at most $3k$ geodesic sides.

Consider the following partition of $\{1, ..., k-1\} = I_1 \sqcup I_2$. A number $1 \leqslant i \leqslant k-1$ belongs to I_1

if in p_w, p_{n_i} is connected to both an H_{λ_i}-component of $p_{t_{i-1}^{-1}}$ and an H_{λ_i}-component of $p_{t_{i+1}}$. Otherwise, i belongs to I_2.

Lemma 3.2.21. $card(I_1) \leqslant (k-1)/2$.

Proof. First suppose $card(I_1) > k/2$. Then there exists a number i such that both i and $i+1$ belong to I_1, contradicting Lemma 3.2.15. Thus, $card(I_1) \leqslant k/2$.

Suppose $card(I_1) = k/2$. Then k is even and $I_1 = \{1, 3, ..., k-3, k-1\}$. For every even number $i \in \{2, 4, ..., k-2, k\}$, Lemma 3.2.15 implies that p_{n_i} is an isolated H_{λ_i}-component of p_w. Note that $\widehat{\ell}_{\lambda_i}(p_{n_i}) = \widehat{d}_{\lambda_i}(1, n_i) > 24D$ for $i = 1, ..., k$, by (24D). Therefore, Lemma 2.6.2, applied to the geodesic $3k$-gon p_w, yields

$$\frac{24Dk}{2} < \widehat{\ell}_{\lambda_2}(p_{n_2}) + \widehat{\ell}_{\lambda_4}(p_{n_4}) + \cdots + \widehat{\ell}_{\lambda_{k-2}}(p_{n_{k-2}}) + \widehat{\ell}_{\lambda_k}(p_{n_k}) < 3kD,$$

a contradiction. □

Thus, $card(I_2) = k - 1 - card(I_1) \geqslant (k-1)/2$. For each $i \in I_2$, p_{h_i} is an isolated H_{λ_i}-component of p with $\widehat{\ell}_{\lambda_i}(p_{h_i}) = \widehat{d}_{\lambda_i}(1, h_i) > 12D$, by Procedure 3.2.16 and Construction 3.2.19. Lemma 2.6.2, applied to the geodesic polygon p, yields

$$6D(k-1) = 12D(k-1)/2 < \sum_{i \in I_2} \widehat{\ell}_{\lambda_i}(p_{h_i}) \leqslant 3kD. \tag{3.6}$$

In other words, $k < 2$, contradicting (Z1). Proposition 3.2.11 is proved. □

Finally, Theorem 4.0.1 follows from Proposition 3.2.10 and Proposition 3.2.11.

Remark 3.2.22. The proof of Theorem 4.0.1 implies that if $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} G$, $N_\lambda \lhd H_\lambda$ for $\lambda \in \Lambda$, and (24D) holds, then for every collection $\{T_\lambda\}_{\lambda \in \Lambda}$ satisfying (P1), (P2), and (P3), we have

$$\langle\langle \mathcal{N} \rangle\rangle = \prod_{\lambda \in \Lambda, t \in T_\lambda}^{*} N^t.$$

Remark 3.2.23. In fact, one can show that if $\{H_\lambda\}_{\lambda \in \Lambda} \hookrightarrow_{wh} G$, $N_\lambda \lhd H_\lambda$ for $\lambda \in \Lambda$, and following condition

(4D) $\widehat{d}_\lambda(1, n) > 4D$ for all $n \in N_\lambda \backslash \{1\}$ and $\lambda \in \Lambda$

holds, then the triple $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ possesses the Cohen-Lyndon property. For the proof, one needs to merge p_{n_k} with the H_{λ_k}-components connected to it in the construction of p, and sharpen the coarse estimate (3.6).

3.3 Relative relation modules

Let H be a group with a normal subgroup N and let $\overline{H} = H/N$. The *relative relation module $Rel(H, N)$ of the exact sequence*

$$1 \to N \to H \to \overline{H} \to 1$$

is the abelianization $\widetilde{N} = N/[N, N]$ equipped with the $\overline{H}$-action by conjugation. More precisely, denote by $\widetilde{n}$ the image of an element $n \in N$ under the quotient map $N \to \widetilde{N}$. Then there is an action of H on $\widetilde{N}$ given by $h\Box\widetilde{n} = \widetilde{hnh^{-1}}$ for all $h \in H, \widetilde{n} \in \widetilde{N}$. Notice that if h belongs to N, then $h\Box\widetilde{n} = \widetilde{hnh^{-1}} = \widetilde{h}\widetilde{n}\widetilde{h}^{-1} = \widetilde{n}$ for all $\widetilde{n} \in \widetilde{N}$, as $\widetilde{h}$ commutes with $\widetilde{n}$. Hence, the action of H gives rises to an action of $\overline{H}$, turning $\widetilde{N}$ into a $\mathbb{Z}\overline{H}$-module. If H is a free group, then $Rel(H, N)$ is called a *relation module*.

The main goal of this section is to prove Proposition 3.3.1, which, together with Theorem 1.2.5, implies Corollary 1.2.8.

Proposition 3.3.1. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple satisfying the Cohen-Lyndon property. Employ the notation defined in Notation 2.14.2. If $N_\lambda \neq \{1\}$ for every $\lambda \in \Lambda$, then*

(a) for every $\lambda \in \Lambda$, the natural map $\overline{H}_\lambda \to \overline{G}$ is injective (i.e., $H_\lambda \cap \langle\langle \mathcal{N} \rangle\rangle = N_\lambda$), identifying $\overline{H}_\lambda$ with a subgroup of $\overline{G}$;

(b) $Rel(G, \langle\langle \mathcal{N} \rangle\rangle) \cong_{\overline{G}} \bigoplus_{\lambda\in\Lambda} Ind_{\overline{H}_\lambda}^{\overline{G}} Rel(H_\lambda, N_\lambda)$.

Remark 3.3.2. If $N_{\lambda_0} = \{1\}$ for some $\lambda_0 \in \Lambda$, then we can consider the subset Λ' such that $N_\lambda \neq \{1\}$ for every $\lambda \in \Lambda'$. It is easy to see that $(G, \{H_\lambda\}_{\lambda\in\Lambda'}, \{N_\lambda\}_{\lambda\in\Lambda'})$ has the Cohen-Lyndon property and thus Proposition 3.3.1 can be applied to $(G, \{H_\lambda\}_{\lambda\in\Lambda'}, \{N_\lambda\}_{\lambda\in\Lambda'})$.

Suppose that the assumptions of Proposition 3.3.1 are satisfied. Let $T_\lambda, \lambda \in \Lambda$, be the transversals provided by Definition 2.14.3. Fix some $\lambda \in \Lambda$ for the moment. Suppose $h \in H_\lambda \cap \langle\langle \mathcal{N} \rangle\rangle$. Then $h \in$

$N_{\langle\langle \mathcal{N} \rangle\rangle}(N_\lambda)$, the normalizer of N_λ in $\langle\langle \mathcal{N} \rangle\rangle$. Note that

$$\langle\langle \mathcal{N} \rangle\rangle = \prod_{\mu \in \Lambda, t \in T_\mu}^{*} N_\mu^t = N_\lambda * (\prod_{t \in T_\lambda \setminus \{1\}}^{*} N_\lambda^t * \prod_{\mu \in \Lambda \setminus \{\lambda\}, t \in T_\mu}^{*} N_\mu^t)$$

and $N_\lambda \neq \{1\}$. Note also the following general fact.

Lemma 3.3.3. *Let $A, B \neq \{1\}$ be groups. Then $N_{A*B}(A) = A$.*

Proof. Suppose that there exists $a \in A \setminus \{1\}$ and $g \in A * B \setminus A$ such that $a^g \in A$. Consider the Bass-Serre tree Tr corresponding to $A * B$. Denote the $A * B$ action on Tr by $\diamondsuit$. The vertex group A fixes a vertex v of Tr and thus a^g fixes v. Clearly, the vertex $g \diamondsuit v$ is also fixed by a^g. As $g \in A * B \setminus A$, $g \diamondsuit v \neq v$ and thus a^g fixes a nontrivial path between v and $g \diamondsuit v$. In particular, a^g fixes an edge of Tr and thus conjugates into the unique edge subgroup $\{1\}$ of $A * B$. It follows that $a^g = 1$, which is in contradiction with $a \neq 1$. □

Therefore, $N_{\langle\langle \mathcal{N} \rangle\rangle}(N_\lambda) = N_\lambda$ and $h \in N_\lambda$. We conclude:

Lemma 3.3.4. *For every $\lambda \in \Lambda$, $H_\lambda \cap \langle\langle \mathcal{N} \rangle\rangle = N_\lambda$.*

Let us consider the relative relation modules $Rel(G, \langle\langle \mathcal{N} \rangle\rangle)$ and $Rel(H_\lambda, N_\lambda), \lambda \in \Lambda$. For every $\lambda \in \Lambda$, let M_λ be the subgroup of G generated by $N_\lambda^t, t \in T_\lambda$. Note that $M_\lambda = \prod_{t \in T_\lambda}^{*} N_\lambda^t$ for every $\lambda \in \Lambda$, as $\langle\langle \mathcal{N} \rangle\rangle = \prod_{\lambda \in \Lambda, t \in T_\lambda}^{*} N_\lambda^t$. Note also that $\langle\langle \mathcal{N} \rangle\rangle = \prod_{\lambda \in \Lambda}^{*} M_\lambda$.

For every $\lambda \in \Lambda$, the composition of natural maps $M_\lambda \hookrightarrow \langle\langle \mathcal{N} \rangle\rangle \to \widetilde{\langle\langle \mathcal{N} \rangle\rangle}$ maps M_λ into the abelian group $\widetilde{\langle\langle \mathcal{N} \rangle\rangle}$ and thus factors through

$$i_\lambda : \widetilde{M_\lambda} \to \widetilde{\langle\langle \mathcal{N} \rangle\rangle}.$$

The homomorphisms $i_\lambda, \lambda \in \Lambda$, extend to an abelian group homomorphism

$$i : \bigoplus_{\lambda \in \Lambda} \widetilde{M_\lambda} \to \widetilde{\langle\langle \mathcal{N} \rangle\rangle}.$$

It is well-known that i is an abelian group isomorphism (for example, see [30, Problem 4 of Exercise 6.2]). Thus, we identify $\widetilde{M_\lambda}$ with its image $i_\lambda(\widetilde{M_\lambda})$ for every $\lambda \in \Lambda$ and write

$$Rel(G, \langle\langle \mathcal{N} \rangle\rangle) = \widetilde{\langle\langle \mathcal{N} \rangle\rangle} = \bigoplus_{\lambda \in \Lambda} \widetilde{M_\lambda}.$$

Fix $\lambda \in \Lambda$ for the moment. By the same argument as the one above, we write

$$\widetilde{M_\lambda} = \bigoplus_{t \in T_\lambda} \widetilde{N_\lambda^t}.$$

Lemma 3.3.5. *$\widetilde{M_\lambda}$ is a $\mathbb{Z}\overline{G}$-submodule of $Rel(G, \langle\langle \mathcal{N} \rangle\rangle) = \bigoplus_{\lambda \in \Lambda} \widetilde{M_\lambda}$. The $\overline{G}$-action on $\widetilde{M_\lambda}$ transitively permutes the summands $\widetilde{N_\lambda^t}, t \in T_\lambda$, and its isotropy group of $\widetilde{N_\lambda}$ is $\overline{H}_\lambda$, i.e., an element $\overline{g} \in \overline{G}$ satisfies $\overline{g} \Box \widetilde{n} \in \widetilde{N_\lambda}$ for all $\widetilde{n} \in \widetilde{N_\lambda}$ if and only if $\overline{g} \in \overline{H}_\lambda$.*

Proof. Fix $t_0 \in T_\lambda$ and $g \in G$. There exists $t_1 \in T_\lambda, h \in H_\lambda$, and $m \in \langle\langle \mathcal{N} \rangle\rangle$ such that

$$gt_0 = t_1 hm. \tag{3.7}$$

Consider the summand $\widetilde{N_\lambda^{t_0}}$. For all $n \in N_\lambda$,

$$g \Box \widetilde{t_0 n t_0^{-1}} = \widetilde{g t_0 n t_0^{-1} g^{-1}} = \widetilde{t_1 hmnm^{-1} h^{-1} t_1^{-1}} = \widetilde{t_1 hnh^{-1} t_1^{-1}} \in \widetilde{N_\lambda^{t_1}},$$

where the fact that the action of $\langle\langle \mathcal{N} \rangle\rangle$ acts trivially on $Rel(G, \langle\langle N \rangle\rangle)$ is used in the second equality. Hence, $g \Box \widetilde{N_\lambda^{t_0}} \subset \widetilde{N_\lambda^{t_1}}$. As $\widetilde{M_\lambda} = \bigoplus_{t \in T_\lambda} \widetilde{N_\lambda^t}$, it follows that $\widetilde{M_\lambda}$ is G-invariant and thus $\widetilde{M_\lambda}$ is also $\overline{G}$-invariant.

The above paragraph shows that g maps $\widetilde{N_\lambda^{t_0}}$ into $\widetilde{N_\lambda^{t_1}}$. Actually, $g \Box \widetilde{N_\lambda^{t_0}} = \widetilde{N_\lambda^{t_1}}$. Indeed, given $n \in N_\lambda$, we find an element x of $N_\lambda^{t_0}$ such that $g \Box \widetilde{x} = \widetilde{n^{t_1}}$. Let $x = n^{t_0 h^{-1}}$. Note that $n^{h^{-1}} \in N_\lambda$, as N_λ is normal in H_λ. Thus, $x \in N_\lambda^{t_0}$. Direct computation shows

$$g \Box \widetilde{x} = \widetilde{gxg^{-1}} = \widetilde{gt_0(h^{-1}nh)t_0^{-1}g^{-1}} = \widetilde{t_1 hm(h^{-1}nh)m^{-1}h^{-1}t_1^{-1}} = \widetilde{t_1 h(h^{-1}nh)h^{-1}t_1^{-1}} = \widetilde{n^{t_1}},$$

where the fact that the action of $\langle\langle \mathcal{N} \rangle\rangle$ on $Rel(G, \langle\langle N \rangle\rangle)$ is trivial is used in the second equality. Hence, $g \Box \widetilde{x} = \widetilde{n^{t_1}}$.

As a consequence, $g \Box \widetilde{N_\lambda^{t_0}} = \widetilde{N_\lambda^{t_1}}$, i.e., the action of G on $\widetilde{M_\lambda}$ permutes the summands $\widetilde{N_\lambda^t}, t \in T_\lambda$. In fact, this permutation is transitive: Let t be any element of T_λ. We wish to find an element of G which maps $\widetilde{N_\lambda^{t_0}}$ to $\widetilde{N_\lambda^t}$. This can be done by tt_0^{-1}:

$$tt_0^{-1} \Box \widetilde{N_\lambda^{t_0}} = \widetilde{N_\lambda^t}.$$

Thus, the action of G on $\widetilde{M_\lambda}$ transitively permutes the summands $\widetilde{N_\lambda^t}, t \in T_\lambda$. The same is thus true for the action of $\overline{G}$ on $\widetilde{M_\lambda}$.

Clearly, for the action of G on $\widetilde{M_\lambda}$, the isotropy group of $\widetilde{N_\lambda}$ contains $H_\lambda\langle\langle\mathcal{N}\rangle\rangle$. Observe that in equation (3.7), if $t_0 = 1$ and $g \notin H_\lambda\langle\langle\mathcal{N}\rangle\rangle$, then $t_1 \neq 1$ as $t_1^{-1}g \in H_\lambda\langle\langle\mathcal{N}\rangle\rangle$. It follows that

$$g\square\widetilde{N_\lambda} = \widetilde{N_\lambda^{t_1}} \neq \widetilde{N_\lambda},$$

i.e., g does not fix $\widetilde{N_\lambda}$ setwise. Therefore, for the action of G on $\widetilde{M_\lambda}$, the isotropy group of $\widetilde{N_\lambda}$ is $H_\lambda\langle\langle\mathcal{N}\rangle\rangle$. As a consequence, for the action of $\overline{G}$ on $\widetilde{M_\lambda}$, the isotropy group of $\widetilde{N_\lambda}$ is $\overline{H}_\lambda$. □

Recall that if $\mathcal{O}$ is a ring, $\mathcal{D}$ is a subring of $\mathcal{O}$, and A is a $\mathcal{D}$-module, the *induced module of A from $\mathcal{D}$ to $\mathcal{O}$*, denoted as $Ind_{\mathcal{D}}^{\mathcal{O}}A$, is the tensor product $\mathcal{O}\bigotimes_{\mathcal{D}} A$. If $\mathcal{O}, \mathcal{D}$ are integral group rings, we simplify notations by dropping $\mathbb{Z}$, e.g., we write Ind_H^G instead of $Ind_{\mathbb{Z}H}^{\mathbb{Z}G}$. For $\lambda \in \Lambda$, Lemma 3.3.5, together with the following Proposition 3.3.6, which is a well-known characterization of induced modules (for example, see [10, Proposition 5.3 of Chapter III]), implies $\widetilde{M_\lambda} \cong_{\overline{G}} Ind_{\overline{H}_\lambda}^{\overline{G}} Rel(H_\lambda, N_\lambda)$.

Proposition 3.3.6. *Let G be a group and let A be a $\mathbb{Z}G$-module. Suppose that the underlying abelian group of A is a direct sum $\bigoplus_{i\in I} A_i$ and that the G-action transitively permutes the summands. If $H \leqslant G$ is the isotropy group of A_j for some $j \in I$. Then A_j is a $\mathbb{Z}H$-module and $A \cong Ind_H^G A_j$ as $\mathbb{Z}G$-modules.*

Proof of Proposition 3.3.1. For every $\lambda \in \Lambda$, $\widetilde{M_\lambda} \cong_{\overline{G}} Ind_{\overline{H}_\lambda}^{\overline{G}} Rel(H_\lambda, N_\lambda)$. Thus,

$$Rel(G, \langle\langle\mathcal{N}\rangle\rangle) = \bigoplus_{\lambda\in\Lambda} \widetilde{M_\lambda} \cong_{\overline{G}} \bigoplus_{\lambda\in\Lambda} Ind_{\overline{H}_\lambda}^{\overline{G}} Rel(H_\lambda, N_\lambda),$$

as desired. □

Example 3.3.7. Let $\mathcal{G}$ be a graph of groups, let $\pi_1(\mathcal{G})$ be the fundamental group of $\mathcal{G}$, let $\{G_v\}_{v\in V\mathcal{G}}$ be the collection of vertex subgroups, and let $\{G_e\}_{e\in E\mathcal{G}}$ be the collection of edge subgroups. By [13, Example 4.12], $\{G_v\}_{v\in V\mathcal{G}} \hookrightarrow_{wh} \pi_1(\mathcal{G})$ with respect to any subset X consisting of stable letters (i.e., generators corresponding to edges of $\mathcal{G}\backslash T\mathcal{G}$, where $T\mathcal{G}$ is a spanning tree of $\mathcal{G}$), and the corresponding relative metric on a vertex group G_v corresponding to a vertex $v \in V\mathcal{G}$ is bi-Lipschitz equivalent to the word metric with respect to the union of the edge subgroups of G_v corresponding to edges incident to v. Thus, we have the following corollary of Theorems 2.5.12, 4.0.1 and Proposition 3.3.1.

Corollary 3.3.8. *Let $\mathcal{G}$ be a graph of groups, let $\pi_1(\mathcal{G})$ be the fundamental group of $\mathcal{G}$, let $\{G_v\}_{v\in V\mathcal{G}}$ be the collection of vertex subgroups, and let $\{G_e\}_{e\in E\mathcal{G}}$ be the collection of edge subgroups. Suppose that, for every $v \in V\mathcal{G}$, N_v is normal subgroup of G_v with*

$$N_v \cap \langle G_e, v \in e\rangle = \emptyset.$$

Then the group triple $(G, \{G_v\}_{v\in V\mathcal{G}}, \{N_v\}_{v\in V\mathcal{G}})$ has the Cohen-Lyndon property, and

$$Rel(G, \langle\langle \mathcal{N} \rangle\rangle) \cong_{\overline{G}} \bigoplus Ind_{\overline{G}_v}^{\overline{G}} Rel(G_v, N_v),$$

where $\mathcal{N} = \bigcup_{v\in V\mathcal{G}} N_v, \overline{G} = G/\langle\langle \mathcal{N} \rangle\rangle$, and $\overline{G}_v = G_v/N_v$ for $v \in V\mathcal{G}$.

In particular,

Corollary 3.3.9. *Let $G = A *_C B$ be an amalgamated free product. If $N \lhd A$ and $N \cap C = \{1\}$, then (G, A, N) has the Cohen-Lyndon property, and*

$$Rel(G, \langle\langle N \rangle\rangle) \cong_{\overline{G}} Ind_{\overline{A}}^{\overline{G}} Rel(A, N),$$

where $\overline{G} = G/\langle\langle N \rangle\rangle$ and $\overline{A} = A/N$.

Corollary 3.3.10. *Let $G = H*_t$ be an HNN-extension with associated subgroups $A, B \leqslant H$. If $N \lhd H$ and $N \cap (A \cup B) = \{1\}$, then (G, H, N) has the Cohen-Lyndon property, and*

$$Rel(G, \langle\langle N \rangle\rangle) \cong_{\overline{G}} Ind_{\overline{H}}^{\overline{G}} Rel(H, N),$$

where $\overline{G} = G/\langle\langle N \rangle\rangle$ and $\overline{H} = H/N$.

Alternatively, Corollary 3.3.9 can be deduced from [20] and both of Corollaries 3.3.9, 3.3.10 can be deduced from the Bass-Serre theory.

CHAPTER 4

COHEN-LYNDON PROPERTY AND SPECTRAL SEQUENCES

The goal of this chapter is the following more general and precise version of Theorem 1.2.10.

Theorem 4.0.1. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple satisfying the Cohen-Lyndon property. Employ the notations defined in Notation 2.14.2, and let A be a $\mathbb{Z}\overline{G}$-module. Then there are spectral sequences*

$$E_{G,2}^{p,q} = H^p(\overline{G}; H^q(\langle\langle \mathcal{N} \rangle\rangle; A)) \Rightarrow H^{p+q}(G; A),$$

$$E_{\mathcal{H},2}^{p,q} = \prod_{\lambda\in\Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A)) \Rightarrow \prod_{\lambda\in\Lambda} H^{p+q}(H_\lambda; A),$$

of cohomological type and there is a morphism

$$MSS : E_G \longrightarrow E_{\mathcal{H}}$$

between spectral sequences such that

(a) MSS and NTR_G are compatible;

(b) $MSS_2^{p,0}$ can be identified with $NT_{\overline{G}}^p$;

(c) for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$, $MSS_2^{p,q}$ is an isomorphism.

Assuming Theorem 4.0.1, we prove Theorem 1.2.10.

Proof of Theorem 1.2.10. Apply Theorem 4.0.1 for the case $|\Lambda| = 1$ and let

$$E_{G,2}^{p,q} = H^p(\overline{G}; H^q(\langle\langle N \rangle\rangle; A)) \Rightarrow H^{p+q}(G; A), \quad E_{H,2}^{p,q} = H^p(\overline{H}; H^q(N; A)) \Rightarrow H^{p+q}(H; A)$$

be the spectral sequences in that theorem. Then there is a morphism $MSS : E_G \to E_H$ such that

$$MSS_2^{p,q} : H^p(\overline{G}; H^q(\langle\langle N \rangle\rangle; A)) \longrightarrow H^p(\overline{H}; H^q(N; A))$$

is an isomorphism for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$. Replace $E_{G,2}^{p,q} = H^p(\overline{G}; H^q(\langle\langle N \rangle\rangle; A))$ with $H^p(\overline{H}; H^q(N; A))$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$. For $p \in \mathbb{Z}$ and $q = 0$, as $MSS_2^{p,0}$ can be identified with $NT_{\overline{G}}^p$, we have $E_{G,2}^{p,0} \cong H^p(\overline{G}; A)$ and thus we can replace $E_{G,2}^{p,0}$ with $H^p(\overline{G}; A)$. After these replacements, we obtain the spectral sequence (1.2). □

Remark 4.0.2. We can describe the differentials of (1.2) as follows. Let $d_r, r \geqslant 2$, be the differential of the spectral sequence (1.2). Then d_r is induced by $d_{H,r}$. More precisely, we think of $MSS : E_G \to E_H$ as a morphism from the spectral sequence (1.2) to E_H and we have a commutative diagram for $r \geqslant 2$:

$$\begin{array}{ccc} E_r & \xrightarrow{MSS_r} & E_{H,r} \\ \downarrow{\scriptstyle d_r} & & \downarrow{\scriptstyle d_{H,r}} \\ E_r & \xrightarrow{MSS_r} & E_{H,r} \end{array}$$

4.1 Idea towards proving Theorem 1.2.10

In this section, we sketch, without assuming Theorem 4.0.1, the proof of Theorem 1.2.10. The proof of Theorem 4.0.1 is a generalization of the following argument.

Sketched proof of Theorem 1.2.10. The Lyndon-Hochschild-Serre spectral sequence for a $\mathbb{Z}\overline{G}$-module A and the group extension

$$1 \to \langle\langle N \rangle\rangle \to G \to \overline{G} \to 1$$

takes the form

$$E_2^{p,q} = H^p(\overline{G}; H^q(\langle\langle N \rangle\rangle; A)) \Rightarrow H^{p+q}(G; A). \tag{4.1}$$

The Cohen-Lyndon property of (G, H, N) gives rise to the following.

Proposition 4.1.1. *If (G, H, N) has the Cohen-Lyndon property, then for $q \in \mathbb{Z}\backslash\{0\}$,*

$$H^q(\langle\langle N \rangle\rangle; A) \cong_{\overline{G}} CoInd_{\overline{H}}^{\overline{G}} H^q(N; A). \tag{4.2}$$

Thus, Shapiro's lemma implies

Proposition 4.1.2. *If (G, H, N) has the Cohen-Lyndon property, then for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$,*

$$H^p(\overline{G}; H^q(\langle\langle N\rangle\rangle; A)) \cong H^p(\overline{H}; H^q(N; A)). \tag{4.3}$$

Notice that for $q = 0$,

$$E_2^{p,0} = H^p(\overline{G}; H^0(\langle\langle N\rangle\rangle; A)) \cong H^p(\overline{G}; A^{\langle\langle N\rangle\rangle}) \cong H^p(\overline{G}; A), \tag{4.4}$$

where $A^{\langle\langle N\rangle\rangle}$ is the $\langle\langle \mathcal{N}\rangle\rangle$-fixed-points of A. As A is a $\mathbb{Z}\overline{G}$-module, the $\langle\langle N\rangle\rangle$-action on A fixes every point and thus $A^{\langle\langle N\rangle\rangle} = A$.

(1.2) is obtained by substituting terms of (4.1) with the terms on the right-hand side of (4.3) and (4.4).

□

A natural way to prove Propostion 4.1.1 is to decompose $H^q(\langle\langle N\rangle\rangle; A)$ into a direct product $\prod_{t\in T} H^q(N^t; A)$, which can be achieved by starting with a model X of the classifying space of N and taking wedge sum of copies of X to obtain a model of the classifying space of $\langle\langle N\rangle\rangle$. The problem with this approach is that one loses information about the action $\overline{G} \curvearrowright H^q(\langle\langle N\rangle\rangle; A)$ and thus cannot derive Proposition 4.1.1. Therefore, we take another approach and consider $Ext^q_{\langle\langle N\rangle\rangle}(\mathbb{Z}[G/H], A)$. By manipulating different projective resolutions, we prove the following $\mathbb{Z}\overline{G}$-module isomorphisms

$$H^q(\langle\langle N\rangle\rangle; A) \cong_{\overline{G}} Ext^q_{\langle\langle N\rangle\rangle}(\mathbb{Z}[G/H], A) \cong_{\overline{G}} CoInd^{\overline{G}}_{\overline{H}} H^q(N; A)$$

for $q \neq 0$.

4.2 Isomorphism of iterative cohomology groups

The goal of this section is the following generalization of Proposition 4.1.2.

Proposition 4.2.1. *Suppose that $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ is a group triple satisfying the Cohen-Lyndon property. Employ the notations defined in Notation 2.14.2. Then for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$, $NTR^{p,q}_{\overline{G}}$ is an isomorphism.*

Remark 4.2.2. Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple and let $\Lambda' = \{\lambda \in \Lambda \mid N_\lambda \neq \{1\}\}$. It is easy to see that $(G, \{H_\lambda\}_{\lambda\in\Lambda'}, \{N_\lambda\}_{\lambda\in\Lambda'})$ also has the Cohen-Lyndon property and if the conclusion of

Proposition 4.2.1 holds for $(G, \{H_\lambda\}_{\lambda\in\Lambda'}, \{N_\lambda\}_{\lambda\in\Lambda'})$, then it also holds for $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$. Thus we will only prove Proposition 4.2.1 for the case where $N_\lambda \neq \{1\}$ for all $\lambda \in \Lambda$.

Assuming Proposition 4.2.1, we prove Proposition 4.1.2.

Proof of Proposition 4.1.2. The isomorphism (4.3) is the special case $|\Lambda| = 1$ of Proposition 4.2.1. □

The proof of Proposition 4.2.1 is a combination of Lemma 2.13.1 and the following generalization of Proposition 4.1.1.

Proposition 4.2.3. *Suppose that $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ is a group triple satisfying the Cohen-Lyndon property and $N_\lambda \neq \{1\}$ for $\lambda \in \Lambda$. Employ the notations defined in Notation 2.14.2 and think of $\overline{H}_\lambda, \lambda \in \Lambda$, as subgroups of $\overline{G}$. Then there is a $\mathbb{Z}\overline{G}$-module homomorphism*

$$\eta : H^*(\langle\langle \mathcal{N} \rangle\rangle; A) \longrightarrow \prod_{\lambda\in\Lambda} CoInd^{\overline{G}}_{\overline{H}_\lambda} H^*(N_\lambda; A)$$

such that, for $\ell \geqslant 1$, η maps $H^\ell(\langle\langle \mathcal{N} \rangle\rangle; A)$ isomorphically onto $\prod_{\lambda\in\Lambda} CoInd^{\overline{G}}_{\overline{H}_\lambda} H^\ell(N_\lambda; A)$.

Moreover, for every $\mu \in \Lambda$, let

$$Pro_\mu : \prod_{\lambda\in\Lambda} CoInd^{\overline{G}}_{\overline{H}_\lambda} H^*(N_\lambda; A) \longrightarrow CoInd^{\overline{G}}_{\overline{H}_\lambda} H^*(N_\mu; A)$$

be the coordinate projection, and let

$$\pi_\mu : CoInd^{\overline{G}}_{\overline{H}_\lambda} H^*(N_\mu; A) \longrightarrow H^*(N_\mu; A)$$

be the standard projection. Then $NTR_{N_\mu} = \pi_\mu \circ Pro_\mu \circ \eta$.

Assuming Proposition 4.2.3, we prove Proposition 4.1.1.

Proof of Proposition 4.1.1. Without loss of generality, we may assume that $N \neq \{1\}$. In this case, the isomorphism (4.2) is the special case $|\Lambda| = 1$ of Proposition 4.2.3. □

4.2.1. $Ext^*_{\langle\langle \mathcal{N} \rangle\rangle}(\mathbb{Z}[G/H_\lambda], A) \cong_{\overline{G}} CoInd^{\overline{G}}_{\overline{H}_\lambda} H^*(N_\lambda; A)$

In Section 4.2.1and the following Section 4.2.2, let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple. We employ the notations definied in Notation 2.14.2. Suppose

(N1) for all $\lambda \in \Lambda$, $H_\lambda \cap \langle\langle \mathcal{N} \rangle\rangle = N_\lambda$ and thus the natural homomorphism

$$\overline{H}_\lambda = H_\lambda / N_\lambda \longrightarrow \overline{G} = G / \langle\langle \mathcal{N} \rangle\rangle$$

is injective, identifying $\overline{H}_\lambda$ with a subgroup of $\overline{G}$.

For $\lambda \in \Lambda$, we will slightly abuse notations and use $\overline{H}_\lambda$ to denote the subgroup of $\overline{G}$ identified with $\overline{H}_\lambda$.

Let A be a $\mathbb{Z}\overline{G}$-module, and let $P \to \mathbb{Z}$ be the standard free resolution over $\mathbb{Z}G$ with boundary operator ∂. Fix $\lambda \in \Lambda$ for the moment. Note that $P \to \mathbb{Z}$ can also be thought of as a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H_\lambda$, and thus $H^*(N_\lambda; A)$ can be identified with $H^*(Hom_{N_\lambda}(P, A))$. We use the notation $H^*(Hom_{N_\lambda}(P, A))$ to perform calculations (see Remark 2.10.2).

Consider the cochain complex $CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)$ whose differential is given by

$$d\widehat{f}(x, p) = \widehat{f}(x, \partial p)$$

for all $\widehat{f} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A), x \in \mathbb{Z}\overline{G}$, and $p \in P$. Denote the cohomology groups associated with $CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)$ by $H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A))$. Clearly,

$$x \bullet (d\widehat{f}) = d(x \bullet \widehat{f})$$

for all $x \in \mathbb{Z}\overline{G}$ and $\widehat{f} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)$ and thus the cocycles and coboundaries of $CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)$ have natural structures of $\mathbb{Z}\overline{G}$-modules.

It turns out that the order of the operations $CoInd_{\overline{H}_\lambda}^{\overline{G}}$ and H^* can be switched. More precisely, let us consider the map

$$SCH_\lambda : H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)) \longrightarrow CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\lambda}(P, A))$$

constructed as follows. Let

$$[\widehat{f}] \in H^\ell(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A))$$

for some $\ell \geqslant 0$. Then there exists $\widehat{f} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P_\ell, A)$ representing $[\widehat{f}]$. It follows that $d\widehat{f} = 0$, i.e., $\widehat{f}(x)$ is a cocycle in $Hom_{N_\lambda}(P, A)$ for every $x \in \mathbb{Z}\overline{G}$. Denote by Z (resp. B) the set of cocycles (coboundaries) of $Hom_{N_\lambda}(P,A)$ and let Quo be the quotient map sending Z to $H^*(Hom_{N_\lambda}(P,A))$. Then

$$Quo \circ \widehat{f} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\lambda}(P,A)).$$

Let SCH_λ be the function sending every $[\widehat{f}] \in H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P,A))$ to the corresponding $Quo \circ \widehat{f}$. It is easy to check that SCH_λ is well-defined, i.e., independent of the choice of the representative $\widehat{f}$ of the cohomology class $[\widehat{f}]$.

Lemma 4.2.4. *SCH_λ is a $\mathbb{Z}\overline{G}$-module isomorphism.*

Proof. Clearly, SCH_λ is a $\mathbb{Z}\overline{G}$-module homomorphism. Let us show that SCH_λ is injective. Suppose

$$[\widehat{f}] \in H^\ell(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P,A))$$

for some $\ell \geqslant 0$ such that $SCH_\lambda[\widehat{f}] = 0$. Let $\widehat{f} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P_\ell, A)$ be a representative of $[\widehat{f}]$. It follows that $Quo \circ \widehat{f} = 0$, i.e., $\widehat{f}(x) \in B$ for every $x \in \mathbb{Z}\overline{G}$. Let $S \in RT(\overline{H}_\lambda, \overline{G})$. For every $s \in S$, let $\widehat{F}_s \in Hom_{N_\lambda}(P_{\ell-1}, A)$ such that $\widehat{F}_s \circ \partial = \widehat{f}(s)$.

Let $\widehat{F}$ be a function sending every $s \in S$ to $\widehat{F}_s$. As a $\mathbb{Z}\overline{H}_\lambda$-module, $\mathbb{Z}\overline{G}$ is freely generated by $s \in S$ and thus we can $\mathbb{Z}\overline{H}_\lambda$-linearly extend $\widehat{F}$ to a function (still denoted by)

$$\widehat{F} : \mathbb{Z}\overline{G} \to Hom_{N_\lambda}(P_{\ell-1}, A).$$

Clearly, $\widehat{F} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P_{\ell-1}, A)$. Moreover, $\widehat{F} \circ \partial = \widehat{f}$ and thus $[\widehat{f}] = 0$.

Let us show that SCH_λ is also surjective. Given

$$f \in CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\lambda}(P,A)),$$

for every $s \in S$, choose a function $\widetilde{f}_s \in Z$ representing $f(s) \in H^*(Hom_{N_\lambda}(P,A))$. Let $\widetilde{f}$ be a function sending every s to $\widetilde{f}_s$. As a $\mathbb{Z}\overline{H}_\lambda$-module, $\mathbb{Z}\overline{G}$ is freely generated by $s \in S$ and thus we can $\mathbb{Z}\overline{H}_\lambda$-linearly extend $\widetilde{f}$ to a function (still denoted by)

$$\widetilde{f} : \mathbb{Z}\overline{G} \to Hom_{N_\lambda}(P_\ell, A).$$

Clearly, $\widetilde{f} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P_\ell, A)$. As $\widetilde{f}_s \in Z$, we have $\widetilde{f} \circ \partial = 0$ and thus $\widetilde{f}$ represents an element $[\widetilde{f}] \in H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A))$. Moreover, $Quo \circ \widetilde{f} = f$. Thus, $SCH_\lambda[\widetilde{f}] = f$. □

Remark 4.2.5. Let

$$\widetilde{\pi}_\lambda : CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A) \longrightarrow Hom_{N_\lambda}(P, A)$$

be the standard projection. Then $\widetilde{\pi}_\lambda$ induces a map

$$\widetilde{\pi}_\lambda^* : H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)) \longrightarrow H^*(Hom_{N_\lambda}(P, A)).$$

Consider the diagram

$$\begin{array}{ccc} H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)) & \xrightarrow{\widetilde{\pi}_\lambda^*} & H^*(Hom_{N_\lambda}(P, A)) \\ \big\downarrow SCH_\lambda & \nearrow \pi_\lambda & \\ CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\lambda}(P, A)) & & \end{array} \tag{4.5}$$

where

$$\pi_\lambda : CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\lambda}(P, A)) \longrightarrow H^*(Hom_{N_\lambda}(P, A))$$

is the standard projection. We claim that (4.5) commutes. Indeed, given

$$f \in CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\lambda}(P, A)),$$

use the second part of the proof of Lemma 4.2.4 to construct an $\widetilde{f} \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)$ such that $SCH_\lambda[\widetilde{f}] = f$. It is easy to check that $\widetilde{\pi}_\lambda(\widetilde{f}) = f(1)$. As $\widetilde{\pi}_\lambda(\widetilde{f})$ represents $\widetilde{\pi}_\lambda^* \circ SCH_\lambda^{-1}(f)$, we have $\widetilde{\pi}_\lambda^* \circ SCH_\lambda^{-1} = \pi_\lambda$.

Tensoring $P \to \mathbb{Z}$ with $\mathbb{Z}[G/H_\lambda]$ produces a chain complex

$$(P \bigotimes \mathbb{Z}[G/H_\lambda], \epsilon_\lambda) : \cdots \xrightarrow{\epsilon_\lambda} P_1 \bigotimes \mathbb{Z}[G/H_\lambda] \xrightarrow{\epsilon_\lambda} P_2 \bigotimes \mathbb{Z}[G/H_\lambda] \xrightarrow{\epsilon_\lambda} \mathbb{Z}[G/H_\lambda] \longrightarrow 0,$$

where $\epsilon_\lambda = \partial \otimes id_{\mathbb{Z}[G/H_\lambda]}$. G acts on $P_\ell \bigotimes \mathbb{Z}[G/H_\lambda]$ by a diagonal action:

$$g \cdot (p \otimes g' H_\lambda) = g \cdot p \otimes g g' H_\lambda$$

for all $g, g' \in G$ and $p \in P$. Thus, $P \bigotimes \mathbb{Z}[G/H_\lambda]$ is a $\mathbb{Z}G$-module.

Lemma 4.2.6. *Suppose that E is a basis for the $\mathbb{Z}G$-module P and $S \in LT(H_\lambda, G)$. Then the $\mathbb{Z}G$-module $P \bigotimes \mathbb{Z}[G/H_\lambda]$ is freely generated by the set $\widetilde{E} = \{e \otimes sH_\lambda \mid e \in E, s \in S\}$.*

Proof. For $e \in E$, let $\langle e \rangle_P$ be the $\mathbb{Z}G$-submodule of P generated by e. As $P = \bigoplus_{e \in E} \langle e \rangle_P$, we have

$$P \bigotimes \mathbb{Z}[G/H_\lambda] \cong_{\overline{G}} \bigoplus_{e \in E} \big(\langle e \rangle_P \bigotimes \mathbb{Z}[G/H_\lambda] \big).$$

The desired conclusion follows from the fact that, for each $e \in E$, $\langle e \rangle_P \bigotimes \mathbb{Z}[G/H_\lambda]$ is freely generated by elements of the form $e \otimes sH_\lambda, s \in S$. □

Thus, $P \bigotimes \mathbb{Z}[G/H_\lambda] \longrightarrow \mathbb{Z}[G/H_\lambda]$ is a free resolution of $\mathbb{Z}[G/H_\lambda]$ over $\mathbb{Z}G$. By definition, the cohomology group associated with the deleted cochain complex $Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)$ is $Ext^*_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\mathbb{Z}[G/H_\lambda], A)$. We use $Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)$ to perform computations (see Remark 2.10.2).

Lemma 4.2.7. $H^*(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)) \cong_{\overline{G}} H^*(CoInd^{\overline{G}}_{\overline{H}_\lambda} Hom_{N_\lambda}(P, A))$.

Proof. Construct a chain map

$$Iso_\lambda : Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A) \longrightarrow CoInd^{\overline{G}}_{\overline{H}_\lambda} Hom_{N_\lambda}(P, A).$$

By the following procudure. Let

$$\widetilde{f} \in Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P_\ell \bigotimes \mathbb{Z}[G/H_\lambda], A)$$

for some $\ell \geqslant 0$. Recall that $Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P_\ell \bigotimes \mathbb{Z}[G/H_\lambda], A)$ is $\mathbb{Z}\overline{G}$-module and a superscript is used to denote the $\overline{G}$-action (see Remark 2.11.1). As an abelian group, $\mathbb{Z}\overline{G}$ is freely generated by elements of $\overline{G}$ and thus there exists a unique abelian group homomorphism

$$f \in Hom(\mathbb{Z}\overline{G}, Hom_{N_\lambda}(P, A))$$

such that

$$f(\overline{g}, p) = {}^{\overline{g}}\widetilde{f}(p \otimes H_\lambda) = g \cdot \widetilde{f}(g^{-1} \cdot p \otimes g^{-1} H_\lambda)$$

for every $\overline{g} \in \overline{G}, p \in P_\ell$, and $g \in G$ such that g is mapped to $\overline{G}$ under the quotient map $G \to \overline{G}$ (see Notation 2.12.1). Let Iso_λ be the map sending each $\widetilde{f} \in Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P_\ell \bigotimes \mathbb{Z}[G/H_\lambda], A)$ to the corresponding f.

Claim 1. *If* $\widetilde{f} \in Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P_\ell \bigotimes \mathbb{Z}[G/H_\lambda], A)$ *for some* $\ell \geqslant 0$*, then*

$$Iso_\lambda f \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A).$$

Proof of Claim 1. It suffices to prove

$$Iso_\lambda \widetilde{f}(\overline{h}\overline{g}, p) = h \cdot Iso_\lambda \widetilde{f}(\overline{g}, h^{-1} \cdot p)$$

for every $\overline{h} \in \overline{H}_\lambda, \overline{g} \in \overline{G}$, $p \in P_\ell$, and $h \in H_\lambda$ such that h is mapped to $\overline{h}$ under the quotient map $G \to \overline{G}$. Let $g \in G$ such that g is mapped to $\overline{g}$ by the quotient map $G \to \overline{G}$. Direct computation shows

$$\begin{aligned}
(Iso_\lambda \widetilde{f})(\overline{h}\overline{g}, p) =& {}^{\overline{h}\overline{g}}\widetilde{f}(p \otimes H_\lambda) \\
=& hg \cdot \widetilde{f}(g^{-1}h^{-1} \cdot p \otimes g^{-1} H_\lambda) \qquad\qquad \text{as } h \in H_\lambda \\
=& h \cdot Iso_\lambda \widetilde{f}(\overline{g}, h^{-1} \cdot p),
\end{aligned}$$

as desired. □

Claim 2. *Iso_λ is a $\mathbb{Z}\overline{G}$-module homomorphism.*

Proof of Claim 2. Claim 2 follows from the following equality

$$Iso_\lambda({}^{\overline{g}_1}\widetilde{f})(\overline{g}_2, p) = {}^{\overline{g}_2\overline{g}_1}\widetilde{f}(p \otimes H_\lambda) = Iso_\lambda \widetilde{f}(\overline{g}_2\overline{g}_1, p) = (\overline{g}_1 \bullet Iso_\lambda \widetilde{f})(\overline{g}_2, p)$$

for $\ell \geqslant 0, g_1, g_2 \in G, p \in P_\ell$, and $\widetilde{f} \in Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P_\ell \bigotimes \mathbb{Z}[G/H_\lambda], A)$. ☐

Claim 3. *Iso_λ is a chain map.*

Proof of Claim 3. Claim 3 follows from the following equality

$$Iso_\lambda(\widetilde{f} \circ \epsilon_\lambda)(\overline{g}, p) = g \cdot \widetilde{f}((g^{-1} \cdot \partial p) \otimes s^{-1} H_\lambda) = Iso_\lambda \widetilde{f}(\overline{g}, \partial p)$$

for $\ell \geqslant 0, \overline{g} \in \overline{G}, \widetilde{f} \in Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P_\ell \bigotimes \mathbb{Z}[G/H_\lambda], A)$, $p \in P_{\ell+1}$, and $g \in G$ such that g is mapped to $\overline{g}$ by the quotient map $G \to \overline{G}$. ☐

It follows that Iso_λ induces a $\mathbb{Z}\overline{G}$-module homomorphism

$$Iso_\lambda^* : H^*(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)) \longrightarrow H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)).$$

To show that Iso_λ^* is in fact an isomorphism, it suffices to construct an inverse of Iso_λ. Fix $S \in RT(H_\lambda, G)$. Let $f \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P_\ell, A)$ for some $\ell \geqslant 0$. As an abelian group, $\mathbb{Z}[G/H_\lambda]$ is freely generated by $\{s^{-1}H_\lambda \mid s \in S\}$ and thus $P \bigotimes \mathbb{Z}[G/H_\lambda]$, as an abelian group, is freely generated by elements of the form $p \otimes s^{-1}H_\lambda$, where p ranges over all $(\ell+1)$-tuples of G and $s \in S$. It follows that there exists a unique abelian group homomorphism

$$\widetilde{f} \in Hom(P \bigotimes \mathbb{Z}[G/H_\lambda], A)$$

such that

$$\widetilde{f}(p \otimes s^{-1}H_\lambda) = s^{-1} \cdot f(\overline{s}, s \cdot p)$$

for all $p \in P_\ell$ and $s \in S$. Let

$$\tau_\lambda : CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A) \longrightarrow Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)$$

be the map sending each $f \in CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)$ to the corresponding $\widetilde{f}$. Clearly, τ_λ and Iso_λ are mutual inverses and we are done. ☐

4.2.2. Proof of Proposition 4.2.3

Further suppose

(N2) for every $\lambda \in \Lambda$, there exists a left transversal $T_\lambda \in LT(H_\lambda\langle\langle\mathcal{N}\rangle\rangle, G)$ such that

$$\langle\langle\mathcal{N}\rangle\rangle = \prod_{\lambda\in\Lambda, t\in T_\lambda}^{*} N_\lambda^t.$$

Definition 4.2.8. Let $m \in \langle\langle\mathcal{N}\rangle\rangle$. If $m \neq 1$, then m can be uniquely factorized as

$$m = \prod_{i=1}^{k} n_i^{t_i} \tag{4.6}$$

with $t_i \in T_{\lambda_i}, n_i \in N_{\lambda_i}\backslash\{1\}$, and $\lambda_i \in \Lambda$ for $1 \leqslant i \leqslant k$. (4.6) is called the *factorization* of m. *The number of factors of* m, denoted as $\omega(m)$, is the number k in (4.6). If $m = 1$, we let $\omega(m) = 0$.

Apply $Hom_{\langle\langle\mathcal{N}\rangle\rangle}(\cdot, A)$ to the resolution $P \to \mathbb{Z}$ to produce a deleted cochain complex $Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P, A)$, whose cohomology group is $H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P, A)) = H^*(\langle\langle\mathcal{N}\rangle\rangle; A)$. We use $H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P, A))$ for computation (see Remark 2.10.2).

Fix $\lambda \in \Lambda$ for the moment. Consider a $\mathbb{Z}G$-module homomorphism

$$Fg_\lambda : \mathbb{Z}[G/H_\lambda] \longrightarrow \mathbb{Z},\ Fg_\lambda(gH_\lambda) = 1$$

for every left coset gH_λ (Fg_λ "forgets" the coset information). Fg_λ induces a natural $\mathbb{Z}\overline{G}$-module homomorphism (see Remark 2.11.2)

$$Fg_\lambda^* : H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P, A)) \longrightarrow H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)).$$

Extend Fg_λ to a chain map (still denoted by)

$$Fg_\lambda : P \bigotimes \mathbb{Z}[G/H_\lambda] \longrightarrow P,\ Fg_\lambda(p \otimes gH_\lambda) = p$$

for all $p \in P$ and left coset gH_λ. Then Fg_λ^* is the cohomology map induced by the chain map Fg_λ. Let

$$Fg = \bigoplus_{\lambda \in \Lambda}^{Dom} Fg_\lambda : \bigoplus_{\lambda \in \Lambda} (P \bigotimes \mathbb{Z}[G/H_\lambda]) \longrightarrow P.$$

Lemma 4.2.9. *The composition*

$$\pi_\lambda^* \circ Iso_\lambda^* \circ Fg_\lambda^* : H^*(Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P, A)) \longrightarrow H^*(Hom_{N_\lambda}(P, A))$$

is the cohomology map induced by the natural embedding

$$Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P, A) \hookrightarrow Hom_{N_\lambda}(P, A)$$

and thus is the natural map induced by $N_\lambda \hookrightarrow \langle\langle \mathcal{N} \rangle\rangle$.

Proof. In the level of cochains, $\pi_\lambda^* \circ Iso_\lambda^* \circ Fg_\lambda^*$ is induced by

$$\pi_\lambda \circ Iso_\lambda \circ Fg_\lambda : Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P, A) \longrightarrow Hom_{N_\lambda}(P, A).$$

Direct computation shows that $\pi_\lambda \circ Iso_\lambda \circ Fg_\lambda(f) = f$ for all $f \in Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P, A)$. □

Let λ vary in Λ. We construct two auxiliary resolutions

$$R \longrightarrow \bigoplus_{\lambda \in \Lambda} \mathbb{Z}[G/H_\lambda], \quad \widetilde{R} \longrightarrow \mathbb{Z}.$$

For every $\lambda \in \Lambda$, let $Q_\lambda = \bigoplus_{\ell \geqslant -1} Q_{\lambda,\ell}$ be the graded $\mathbb{Z}N_\lambda$-module such that for each $\ell \geqslant -1$, $Q_{\lambda,\ell}$ is the $\mathbb{Z}N_\lambda$-submodule of $P_\ell \bigotimes \mathbb{Z}[G/H_\lambda]$ generated by elements of the form $p \otimes H_\lambda$, where p ranges over all $(\ell+1)$-tuples of elements of N_λ. Clearly, the boundary operator

$$\epsilon_\lambda : P \bigotimes \mathbb{Z}[G/H_\lambda] \longrightarrow P \bigotimes \mathbb{Z}[G/H_\lambda]$$

restricts to a boundary operator (still denoted by) $\epsilon_\lambda : Q_\lambda \to Q_\lambda$, which turns Q_λ into a chain complex. For $\lambda \in \Lambda$, the map Fg_λ sends the chain complex Q_λ isomorphically onto the standard free resolution of $\mathbb{Z}$ over $\mathbb{Z}N_\lambda$. In particular, the chain complex Q_λ is exact.

For every $\lambda \in \Lambda$ and every $t \in T_\lambda$, let $X_{\lambda,t}$ be the set consisting of elements of $\langle\langle \mathcal{N} \rangle\rangle$ whose factorizations do not end with a factor from N_λ^t. Note that

$$X_{\lambda,t} \in LT(N_\lambda^t, \langle\langle \mathcal{N} \rangle\rangle).$$

Let $R_\lambda = \bigoplus_{\ell \geqslant -1} R_{\lambda,\ell}$ be the graded abelian group such that for each $\ell \geqslant -1$, $R_{\lambda,\ell}$ is the subgroup of the abelian group $P_\ell \bigotimes \mathbb{Z}[G/H_\lambda]$ generated by elements of the form

$$xt \cdot p \otimes xtH_\lambda,$$

where $t \in T_\lambda, x \in X_{\lambda,t}$, and p ranges over $(\ell+1)$-tuples of the elements of N_λ. Note that R_λ splits as a direct sum

$$R_\lambda = \bigoplus_{t \in T_\lambda, x \in X_{\lambda,t}} (xt \cdot Q_\lambda)$$

of graded abelian groups. For each summand $xt \cdot Q_\lambda$, the boundary operator ϵ_λ on $P \bigotimes \mathbb{Z}[G/H_\lambda]$ restricts to a boundary operator on $xt \cdot Q_\lambda$, turning $xt \cdot Q_\lambda$ into a chain complex. As a consequence, ϵ_λ induces a boundary operator ϵ'_λ on R_λ. Moreover, the left multiplication of $(xt)^{-1}$ maps the chain complex $xt \cdot Q_\lambda$ isomorphically onto Q_λ and thus $xt \cdot Q_\lambda$ is exact. Thus, R_λ, as a direct sum of exact chain complexes, is an exact chain complex. As $\langle\langle \mathcal{N} \rangle\rangle$ is a normal subgroup of G, it is not hard to show that, for every $\lambda \in \Lambda$ and $t \in T_\lambda$, the $\langle\langle \mathcal{N} \rangle\rangle$-action on $\bigoplus_{x \in X_{\lambda,t}} (xt \cdot Q_\lambda)$ permutes the summands $xt \cdot Q_\lambda$ and thus $\bigoplus_{x \in X_{\lambda,t}} (xt \cdot Q_\lambda)$ is a $\mathbb{Z}\langle\langle \mathcal{N} \rangle\rangle$-module.

In fact, $\bigoplus_{x \in X_{\lambda,t}} (xt \cdot Q_\lambda)$ is a free $\mathbb{Z}\langle\langle \mathcal{N} \rangle\rangle$-module. Indeed, let E be the set consist of tuples of G of the form $(1, g_1, ..., g_\ell), \ell \geqslant 0$. Then E is a basis for the $\mathbb{Z}G$-module P. Let $S \in LT(H_\lambda, G)$. Then the set

$$\widetilde{E} = \{e \otimes sH_\lambda, e \in E, s \in S\}$$

freely generates $P \bigotimes \mathbb{Z}[G/H_\lambda]$ as a $\mathbb{Z}G$-module, by Lemma 4.2.6. Let $U \in LT(\langle\langle \mathcal{N} \rangle\rangle, G)$. Then $P \bigotimes \mathbb{Z}[G/H_\lambda]$, as a $\mathbb{Z}\langle\langle \mathcal{N} \rangle\rangle$-module, is freely generated by the set

$$U \cdot \widetilde{E} = \{u \cdot e \otimes usH_\lambda \mid u \in U, e \in E, s \in S\}.$$

Note that $\bigoplus_{x\in X_{\lambda,t}}(xt\cdot Q_\lambda)$ is generated by a subset of $U\cdot\widetilde{E}$ and thus is a free $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module. It follows that for every $\lambda\in\Lambda$, R_λ is a free $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module as it is a direct sum of free $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-modules.

Lemma 4.2.10. *For every $\lambda\in\Lambda$, $\{xt \mid t\in T_\lambda, x\in X_{\lambda,t}\}\in LT(H_\lambda,G)$.*

Proof. We first prove that $\{xt \mid t\in T_\lambda, x\in X_{\lambda,t}\}$ contains a left transversal of H_λ. Given any $g\in G$ and $\lambda\in\Lambda$, there exists $t\in T_\lambda, m\in M$, and $h\in H_\lambda$ such that $g=tmh$. Let $m'=m^t$. Then $g=m'th$. As $\langle\langle\mathcal{N}\rangle\rangle$ is normal in G, m' belongs to $\langle\langle\mathcal{N}\rangle\rangle$. As $X_{\lambda,t}\in LT(N_\lambda^t,\langle\langle\mathcal{N}\rangle\rangle)$, there exists $x\in X_{\lambda,t}$ and $n\in N_\lambda$ such that $m'=xn^t$. Let $h'=nh\in H_\lambda$. Then

$$g=m'th=xtnt^{-1}th=xtnh=xth'.$$

Next, we verify that any two elements of $\{xt \mid t\in T_\lambda, x\in X_{\lambda,t}\}$ comes from different left cosets of H_λ. Suppose that for some $\lambda\in\Lambda$, there exist $t_1,t_2\in T_\lambda, x_1\in X_{\lambda,t_1}, x_2\in X_{\lambda,t_2}$ with $t_1^{-1}x_1^{-1}x_2t_2\in H_\lambda$. Note that $x_1^{-1}x_2$ is an element of $\langle\langle\mathcal{N}\rangle\rangle$, and thus $m=t_2^{-1}x_1^{-1}x_2t_2$ is an element of $\langle\langle\mathcal{N}\rangle\rangle$. It follows that

$$t_1^{-1}t_2=(t_1^{-1}x_1^{-1}x_2t_2)m^{-1}\in H_\lambda\langle\langle\mathcal{N}\rangle\rangle$$

and thus $t_1=t_2$. Hence,

$$t_1^{-1}x_1^{-1}x_2t_2=t_1^{-1}x_1^{-1}x_2t_1\in\langle\langle\mathcal{N}\rangle\rangle.$$

The assumption $t_1^{-1}x_1^{-1}x_2t_2\in H_\lambda$ then implies

$$t_1^{-1}x_1^{-1}x_2t_2\in\langle\langle\mathcal{N}\rangle\rangle\cap H_\lambda=N_\lambda,$$

where the last equality follows from the assumption at the beginning of Section 4.2.1. In other words, $x_1^{-1}x_2\in N_\lambda^{t_1}$. As neither of the factorizations of x_1 and x_2 ends with a factor from $N_\lambda^{t_1}$, the only possibility for $x_1^{-1}x_2\in N_\lambda^{t_1}$ is $x_1=x_2$. As a consequence, $x_1t_1=x_2t_2$. □

For $\lambda\in\Lambda$, note that $R_{\lambda,-1}$ is generated by $\{xtH_\lambda \mid t\in T_\lambda, x\in X_{\lambda,t}\}$ and thus $R_{\lambda,-1}=\mathbb{Z}[G/H_\lambda]$. It follows that R_λ is a free resolution of $\mathbb{Z}[G/H_\lambda]$ over $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$. Let

$$i_\lambda: R_\lambda\to P\bigotimes\mathbb{Z}[G/H_\lambda]$$

be the embedding of R_λ into $P\bigotimes\mathbb{Z}[G/H_\lambda]$. As $P\bigotimes\mathbb{Z}[G/H_\lambda]\to\mathbb{Z}[G/H_\lambda]$ and $R_\lambda\to\mathbb{Z}[G/H_\lambda]$ are both free resolutions over $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$, we have

Lemma 4.2.11. *For $\lambda\in\Lambda$, i_λ induces a group isomorphism*

$$i_\lambda^*: H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P\bigotimes\mathbb{Z}[G/H_\lambda],A))\longrightarrow H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R_\lambda,A)).$$

Let

$$R=\bigoplus_{\lambda\in\Lambda}R_\lambda,\quad \epsilon'=\bigoplus_{\lambda\in\Lambda}^{DT}\epsilon'_\lambda: R\longrightarrow R,$$

$$i^*=\prod_{\lambda\in\Lambda}^{DT}i_\lambda^*:\prod_{\lambda\in\Lambda}H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P\bigotimes\mathbb{Z}[G/H_\lambda],A))\longrightarrow\prod_{\lambda\in\Lambda}H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R_\lambda,A)).$$

Clearly, $R\to\bigoplus_{\lambda\in\Lambda}\mathbb{Z}[G/H_\lambda]$ is a free resolution under the boundary operator ϵ'. By Lemma 4.2.11, i^* is an isomorphism.

Applying $Hom_{\langle\langle\mathcal{N}\rangle\rangle}(\cdot,A)$ to R and let $Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R,A)$ be the resulted cochain complex. The obvious isomorphism $\prod_{\lambda\in\Lambda}Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R_\lambda,A)\to Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R,A)$ gives rise to an isomorphism

$$j:\prod_{\lambda\in\Lambda}H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R_\lambda,A))\longrightarrow H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R,A)).$$

We construct the second auxiliary resolution. Let $\widetilde{R}=\bigoplus_{\ell\geqslant -1}\widetilde{R}_\ell$ be the graded $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module such that for every $\ell\geqslant 1$, $\widetilde{R}_\ell=R_\ell$, and that $\widetilde{R}_0=\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$, $\widetilde{R}_{-1}=\mathbb{Z}$. Consider the boundary operator $\widetilde{\epsilon}:\widetilde{R}\to\widetilde{R}$ constructed as follows. For all $\ell\geqslant 2$, let $\widetilde{\epsilon}_\ell=\epsilon'_\ell$. For $\ell=1$, note that

$$\widetilde{R}_1=R_1=\bigoplus_{\lambda\in\Lambda}\left(\bigoplus_{t\in T_\lambda,x\in X_{\lambda,t}}(xt\cdot Q_{\lambda,1})\right).$$

If $r\in xt\cdot Q_{\lambda,1}$ for some $\lambda\in\Lambda$, $t\in T_\lambda$, and $x\in X_{\lambda,t}$, let

$$\widetilde{\epsilon}_1(r)=(Fg_\lambda\circ\epsilon'_1(r))\cdot t^{-1}.$$

Here, $Fg_\lambda\circ\epsilon'_1(r)$ is an element of $P_0=\mathbb{Z}G$ and thus we can multiply it by t^{-1} on the right. Finally, let $\widetilde{\epsilon}_0$ be the augmentation of $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$ sending $\widetilde{R}_0=\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$ onto $\mathbb{Z}$.

Lemma 4.2.12. *$\widetilde{\epsilon}$ is a $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module homomorphism.*

Proof. It suffices to prove that $\widetilde{\epsilon}_1$ is a $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module homomorphism. Note that $\widetilde{R}_1$ can be decomposed as a direct sum

$$\widetilde{R}_1 = \bigoplus_{\lambda\in\Lambda} \bigoplus_{t\in T_\lambda, x\in X_{\lambda,t}} (xt \cdot Q_{\lambda,1})$$

of $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-modules. Each direct summand $\bigoplus_{x\in X_{\lambda,t}}(xt \cdot Q_{\lambda,1})$ is a $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module, on which $\widetilde{\epsilon}_1$ is the composition of $Fg_\lambda, \epsilon'_\lambda$, and the right multiplication of t^{-1}. The maps Fg_λ and ϵ'_λ are $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module homomorphisms, and right multiplications are automatically homomorphisms of left modules. Thus, $\widetilde{\epsilon}_1$ is a $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-module homomorphism on each summand $\bigoplus_{t\in T_\lambda, x\in X_{\lambda,t}}(xt \cdot Q_{\lambda,1})$ of $\widetilde{R}_1$. □

Direct computation shows that, under the boundary operator $\widetilde{\epsilon}$, $\widetilde{R}$ becomes a chain complex. Clearly, $\widetilde{R}$ is exact at $\widetilde{R}_\ell$ for every $\ell \geqslant 2$. Note that $\widetilde{\epsilon}(\widetilde{R}_1)$ is a $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$-submodule of $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$ generated by elements of the form $xn^t - x$ with $t \in T_\lambda, x \in X_{\lambda,t}, n \in N_\lambda, \lambda \in \Lambda$, and thus $\widetilde{\epsilon}(\widetilde{R}_1)$ is the augmentation ideal of $\mathbb{Z}\langle\langle\mathcal{N}\rangle\rangle$. Therefore, $\widetilde{R}$ is also exact at $\widetilde{R}_0$.

Lemma 4.2.13. $\ker(\widetilde{\epsilon}_1) = \ker(\epsilon'_1)$.

Proof. For every $\lambda \in \Lambda$, $t \in T_\lambda$, and $x \in X_{\lambda,t}$, denote by $\epsilon'_{1,\lambda,t,x}$ the restriction of ϵ'_1 to $xt \cdot Q_\lambda$. Note that

$$\ker(\epsilon'_1) = \bigoplus_{\lambda\in\Lambda, t\in T_\lambda, x\in X_{\lambda,t}} \ker(\epsilon'_{1,\lambda,t,x}).$$

The restriction of $\widetilde{\epsilon}_1$ on $xt \cdot Q_\lambda$ is $Fg_\lambda \circ \epsilon'_{1,\lambda,t,x}$ composed with the right multiplication of t^{-1}. Thus, for every $\lambda \in \Lambda$, $t \in T_\lambda$, and $x \in X_{\lambda,t}$, $\ker(\epsilon'_{1,\lambda,t,x})$ is contained in $\ker(\widetilde{\epsilon}_1)$. It follows that $\ker(\epsilon'_1) \subset \ker(\widetilde{\epsilon}_1)$.

In order to prove the converse containment, we introduce the following concepts. For $\lambda \in \Lambda$, let E_λ be a set of pairs of elements of N_λ such that

(a) every pair of the form $(n, n), n \in N_\lambda$ belongs to E_λ;

(b) if n_1, n_2 are distinct elements of N_λ, then E_λ contains exactly one of (n_1, n_2) and (n_2, n_1).

Let

$$S = \{(xtn_1, xtn_2) \otimes xtH_\lambda \mid \lambda \in \Lambda, t \in T_\lambda, x \in X_{\lambda,t}, (n_1, n_2) \in E_\lambda\} \subset R_1.$$

For

$$s = (xtn_1, xtn_2) \otimes xtH_\lambda \in S,$$

let

$$\Omega(s) = \max\{\omega(xn_1^t), \omega(xn_2^t)\},$$

where ω is the number of factors of elements of $\langle\langle \mathcal{N} \rangle\rangle$ (see Definition 4.2.8).

For every $\lambda \in \Lambda$, note that E_λ is a basis for the abelian group $Q_{\lambda,1}$. As a consequence, S is a basis for the abelian group R_1 and thus every element $r \in R_1$ can be uniquely written in the form

$$r = \sum_{s \in S} C_{r,s} s$$

where $C_{r,s} \in \mathbb{Z}$ and the above sum makes sense as there are only finitely many non-zero terms.

We call the number $C_{r,s}$ in the above equation the *coefficient of* r with respect to s. Let $rank : R_1 \to \mathbb{N}$ by the function summing the absoute values of the coefficients:

$$rank(r) = \sum_{s \in S} |C_{r,s}|, \quad r \in R_1.$$

Let $r \in \ker(\widetilde{\epsilon}_1)$. We prove $r \in \ker(\epsilon_1')$ by an induction on $rank(r)$. The base case $rank(r) = 0$ implies $r = 0$ and thus $r \in \ker(\epsilon_1')$. So let us suppose that $rank(r) > 0$ and that, for all $r' \in \ker(\widetilde{\epsilon}_1)$ with $rank(r') < rank(r)$, we have $r' \in \ker(\epsilon_1')$.

Let

$$s_0 = (x_0 t_0 n_1, x_0 t_0 n_2) \otimes x_0 t_0 H_{\lambda_0} \in S$$

such that $C_{r,s_0} \neq 0$ and that

(max Ω) if $s \in S$ satisfying $C_{r,s} \neq 0$, then $\Omega(s_0) \geqslant \Omega(s)$.

If $n_1 = n_2$, consider the element $r' \in R_1$ such that $C_{r',s} = C_{r,s}$ for $s \in S \backslash \{s_0\}$ and $C_{r',s_0} = 0$. Direct computation shows

$$rank(r') < rank(r), \quad \widetilde{\epsilon}_1(r - r') = \epsilon_1'(r - r') = 0.$$

Thus, $\widetilde{\epsilon}_1(r') = 0$ and the induction hypothesis implies $\epsilon_1'(r') = 0$. It follows that

$$\epsilon_1'(r) = \epsilon_1'(r - r') + \epsilon_1'(r') = 0.$$

Therefore, $r \in \ker(\epsilon_1')$.

Thus, without loss of generality, let us assume $n_1 \neq n_2$. It follows that at least one of n_1 and n_2 is not the identity of G. Without loss of generality, we may further assume $n_1 \neq 1$ (the case $n_2 \neq 1$ is similar), in which case

$$\Omega(s_0) = \omega(x_0 t_0 n_1 t_0^{-1}).$$

Let us also assume $C_{r,s} > 0$ (otherwise, consider $-r$). Note that

$$\widetilde{\epsilon}_1(r) = \sum_{s \in \mathcal{S}} C_{r,s} \widetilde{\epsilon}_1(s). \tag{4.7}$$

On the right-hand side of (4.7),

$$C_{r,s_0} \widetilde{\epsilon}_1(s_0) = C_{r,s_0}(x_0 t_0 n_2 t_0^{-1} - x_0 t_0 n_1 t_0^{-1}).$$

Thus, s_0 contributes a negative number of $x_0 t_0 n_1 t_0^{-1}$ to $\widetilde{\epsilon}_1(r)$. As $\widetilde{\epsilon}_1(r) = 0$, there exists some $s_1 \in S$ which contributes a positive number of $x_0 t_0 n_1 t_0^{-1}$ to $\widetilde{\epsilon}_1(r)$. In other words, at least one of the following cases happens

(a) $s_1 = (x_1 t_1 n_3, x_1 t_1 n_4) \otimes x_1 t_1 H_{\lambda_1}$ with $C_{r,s_1} < 0$, $n_3 \neq n_4$, and $x_1 t_1 n_3 t_1^{-1} = x_0 t_0 n_1 t_0^{-1}$.

(b) $s_1 = (x_1 t_1 n_3, x_1 t_1 n_4) \otimes x_1 t_1 H_{\lambda_1}$ with $C_{r,s_1} > 0$, $n_3 \neq n_4$, and $x_1 t_1 n_4 t_1^{-1} = x_0 t_0 n_1 t_0^{-1}$.

Let us suppose that Case (a) happens (Case (b) can be treated in the same manner). Note that $n_3 \neq 1$ in this case. Indeed, if $n_3 = 1$, then $n_4 \neq 1$ since $n_4 \neq n_3$. It follows that

$$\begin{aligned} \Omega(s_1) > \omega(x_1 t_1 n_3 t_1^{-1}) & \qquad \text{as } n_3 = 1, n_4 \neq 1, x_1 \in X_{\lambda_1, t_1} \\ = \omega(x_0 t_0 n_1 t_0^{-1}) & \qquad \text{as } x_1 t_1 n_3 t_1^{-1} = x_0 t_0 n_1 t_0^{-1} \\ = \Omega(s_0), & \end{aligned}$$

which contradicts the choice of s_0. Thus, $n_3 \neq 1$, which, together with the assumption $x_1 \in X_{\lambda_1, t_1}$, implies that the factorization of $x_1 t_1 n_3 t_1^{-1}$ ends with $t_1 n_3 t_1^{-1}$.

As $x_0 \in X_{\lambda_0, t_0}$, the factorization of $x_0 t_0 n_1 t_0^{-1}$ ends with $t_0 n_1 t_0^{-1}$. Since $x_1 t_1 n_3 t_1^{-1} = x_0 t_0 n_1 t_0^{-1}$, we have

$$t_0 n_1 t_0^{-1} = t_1 n_3 t_1^{-1} \in t_0 N_{\lambda_0} t_0^{-1} \cap t_1 N_{\lambda_1} t_1^{-1}.$$

As $n_1 \neq 1$, we have

$$t_0 N_{\lambda_0} t_0^{-1} \cap t_1 N_{\lambda_1} t_1^{-1} \neq \{1\}. \tag{4.8}$$

(N2) and (4.8) imply $\lambda_1 = \lambda_0, t_1 = t_0$, which, together with $x_1 t_1 n_3 t_1^{-1} = x_0 t_0 n_1 t_0^{-1}$, implies $n_1 = n_0, x_1 = x_0$ and thus

$$s_1 = (x_0 t_0 n_1, x_0 t_0 n_4) \otimes x_0 t_0 H_{\lambda_0}.$$

Exactly one of (n_2, n_4) and (n_4, n_2) is in E_{λ_0}. Without loss of generality, we assume that $(n_2, n_4) \in E_{\lambda_0}$ (the other case is similar). Let

$$s_2 = (x_0 t_0 n_2, x_0 t_0 n_4) \otimes x_0 t_0 H_{\lambda_0},$$

let $r' \in R_1$ such that $C_{r',s} = C_{r,s}$ for $s \in S \backslash \{s_0, s_1, s_2\}$, and let

$$C_{r',s_0} = C_{r,s_0} - 1, \quad C_{r',s_1} = C_{r,s_1} + 1, \quad C_{r',s_2} = C_{r,s_2} - 1.$$

As $C_{r,s_0} > 0, C_{r,s_1} < 0$, and $\widetilde{\epsilon}_1(r) = 0$, direct computation shows

$$rank(r') < rank(r), \quad \widetilde{\epsilon}_1(r - r') = \epsilon_1'(r - r') = 0.$$

Thus, $\widetilde{\epsilon}_1(r') = 0$ and the induction hypothesis implies $\epsilon_1'(r') = 0$. It follows that

$$\epsilon_1'(r) = \epsilon_1'(r - r') + \epsilon_1'(r') = 0,$$

that is, $r \in \ker(\epsilon_1')$. $\square$

By Lemma 4.2.13, the chain complex $\widetilde{R}$ is also exact at $\widetilde{R}_1$ and thus is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\langle\langle \mathcal{N} \rangle\rangle$. Note that $P \to \mathbb{Z}$ is also a free resolution over $\mathbb{Z}\langle\langle \mathcal{N} \rangle\rangle$. Let

$$\sigma = Fg \circ \bigoplus_{\lambda \in \Lambda}^{DT} i_\lambda : R \longrightarrow P.$$

Then σ gives rise to a chain map

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\widetilde{\epsilon}} & \widetilde{R}_2 & \xrightarrow{\widetilde{\epsilon}} & \widetilde{R}_1 & \xrightarrow{\widetilde{\epsilon}} & \mathbb{Z}\langle\!\langle \mathcal{N} \rangle\!\rangle & \xrightarrow{\widetilde{\epsilon}} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow{\scriptstyle id_{\mathbb{Z}}} & & \\
\cdots & \xrightarrow{\partial} & P_2 & \xrightarrow{\partial} & P_1 & \xrightarrow{\partial} & \mathbb{Z}G & \xrightarrow{\partial} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

Lemma 4.2.14. *σ induces a group isomorphism*

$$\sigma^* : H^*(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(P, A)) \longrightarrow H^*(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\widetilde{R}, A)).$$

Consider the cochain complexes $Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R, A)$ and $Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\widetilde{R}, A)$. The map id_R induces a map id_R^* between these cochain complexes, except at dimension 0:

$$\begin{array}{ccccccccc}
\cdots & \xleftarrow{(\epsilon')^*} & Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R_2, A) & \xleftarrow{(\epsilon')^*} & Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R_1, A) & \xleftarrow{(\epsilon')^*} & Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R_0, A) & \longleftarrow & 0 \\
 & & \downarrow{\scriptstyle id_R^*} & & \downarrow{\scriptstyle id_R^*} & & & & \\
\cdots & \xleftarrow{\widetilde{\epsilon}^*} & Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\widetilde{R}_2, A) & \xleftarrow{\widetilde{\epsilon}^*} & Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\widetilde{R}_1, A) & \xleftarrow{\widetilde{\epsilon}^*} & Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\widetilde{R}_0, A) & \longleftarrow & 0
\end{array}$$

Here, the maps $(\epsilon')^*$ and $\widetilde{\epsilon}^*$ are the duals of ϵ' and $\widetilde{\epsilon}$, respectively. id_R^* induces a group homomorphism (still denoted by)

$$id_R^* : \bigoplus_{\ell \geqslant 1} H^\ell(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R, A)) \longrightarrow \bigoplus_{\ell \geqslant 1} H^\ell(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\widetilde{R}, A)).$$

Clearly, for $\ell \geqslant 2$, id_R^* maps $H^\ell(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R, A))$ isomorphically onto $H^\ell(Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(\widetilde{R}, A))$.

Consider the coboundaries of R and $\widetilde{R}$ at dimension 1. Let $f \in Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R_0, A)$, let $\lambda \in \Lambda$, let $t \in T_\lambda$, let $x \in X_{\lambda,t}$, and let $n_1, n_2 \in N_\lambda$. Denote $(n_2 n_1^{-1})^{xt}$ by m. Then

$$\begin{aligned}
&((\epsilon')^* f)((xtn_1, xtn_2) \otimes xtH_\lambda) \\
=& f(xtn_2 \otimes xtH_\lambda) - f(xtn_1 \otimes xtH_\lambda) \\
=& f(m \cdot (xtn_1 \otimes xtH_\lambda)) - f(xtn_1 \otimes xtH_\lambda) && \text{as } n_1, n_2 \in N_\lambda \lhd H_\lambda \\
=& m \cdot f(xtn_1 \otimes xtH_\lambda) - f(xtn_1 \otimes xtH_\lambda) && \text{as } m \in \langle\!\langle \mathcal{N} \rangle\!\rangle, f \in Hom_{\langle\!\langle \mathcal{N} \rangle\!\rangle}(R_0, A) \\
=& f(xtn_1 \otimes xtH_\lambda) - f(xtn_1 \otimes xtH_\lambda) && \text{as the } \langle\!\langle \mathcal{N} \rangle\!\rangle - \text{action on } A \text{ is trivial} \\
=& 0.
\end{aligned}$$

Thus, $\epsilon^* f$ is the 0-function on R_1.

Let $\widetilde{f} \in Hom_{\langle\langle\mathcal{N}\rangle\rangle}(\widetilde{R}_0, A)$. Then

$$\begin{aligned}
&\widetilde{\epsilon}^* \widetilde{f}((xtn_1, xtn_2) \otimes xtH_\lambda) \\
=& f(xtn_2t^{-1}) - f(xtn_1t^{-1}) \\
=& f(m \cdot (xtn_1t^{-1})) - f(xtn_1t^{-1}) \\
=& m \cdot f(xtn_1t^{-1}) - f(xtn_1t^{-1}) && \text{as } m \in \langle\langle\mathcal{N}\rangle\rangle, \widetilde{f} \in Hom_{\langle\langle\mathcal{N}\rangle\rangle}(\widetilde{R}_0, A) \\
=& f(xtn_1t^{-1}) - f(xtn_1t^{-1}) && \text{as the } \langle\langle\mathcal{N}\rangle\rangle - \text{action on } A \text{ is trivial} \\
=& 0.
\end{aligned}$$

Thus, $\widetilde{\epsilon}^* \widetilde{f}$ is also the 0-function on $\widetilde{R}_1$. Therefore,

$$\begin{aligned}
H^1(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R, A)) &= \ker(\epsilon_1^*)/\operatorname{im}(\epsilon_0^*) = \ker(\epsilon_1^*) = \ker(\widetilde{\epsilon}_1^*) \\
&= \ker(\widetilde{\epsilon}_1^*)/\operatorname{im}(\widetilde{\epsilon}_0^*) = H^1(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(\widetilde{R}, A)).
\end{aligned}$$

Lemma 4.2.15. *For $\ell \geqslant 1, id_R^*$ maps $H^\ell(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R, A))$ isomorphically (in the sense of abelian groups) onto $H^\ell(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(\widetilde{R}, A))$.*

Proof of Proposition 4.2.3. Fix $\ell \geqslant 1$. It is easy to check that the following diagram commutes.

$$\begin{array}{ccc}
H^\ell(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P, A)) & \xrightarrow{Fg^*} & \prod_{\lambda\in\Lambda} H^\ell(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)) \\
\big\downarrow \sigma^* & & \big\downarrow j^*\circ i^* \\
H^\ell(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(\widetilde{R}, A)) & \xleftarrow{id_R^*} & H^\ell(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(R, A))
\end{array} \tag{4.9}$$

In (4.9), Fg^* is a $\mathbb{Z}\overline{G}$-module homomorphism. By Lemmas 4.2.11, 4.2.14, and 4.2.15, σ^*, $j \circ i^*$, and id_R^* are group isomorphisms. Thus, Fg^* is also a group isomorphism and thus is a $\mathbb{Z}\overline{G}$-module isomorphism.

Recall that Lemma 4.2.7 constructs a $\mathbb{Z}\overline{G}$-module isomorphism Iso_λ^*. Let

$$Iso^* = \prod_{\lambda\in\Lambda}^{DT} Iso_\lambda^* : \prod_{\lambda\in\Lambda} H^*(Hom_{\langle\langle\mathcal{N}\rangle\rangle}(P \bigotimes \mathbb{Z}[G/H_\lambda], A)) \longrightarrow \prod_{\lambda\in\Lambda} H^*(CoInd_{\overline{H}_\lambda}^{\overline{G}} Hom_{N_\lambda}(P, A)).$$

Denote $SCH \circ Iso^* \circ Fg^*$ by η. Lemmas 4.2.4 and 4.2.7 imply that the map

$$\eta : H^*(Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P, A)) \longrightarrow \prod_{\lambda \in \Lambda} CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\lambda}(P, A))$$

is a $\mathbb{Z}\overline{G}$-module homomorphism and maps $H^\ell(Hom_{\langle\langle \mathcal{N} \rangle\rangle}(P, A))$ isomorphically onto $\prod_{\lambda \in \Lambda} CoInd_{\overline{H}_\lambda}^{\overline{G}} H^\ell(Hom_{N_\lambda}(P, A))$.

Fix $\mu \in \Lambda$. Let

$$\widetilde{\pi}_\mu^* : CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(Hom_{N_\mu}(P, A)) \longrightarrow H^*(Hom_{N_\mu}(P, A))$$

be the standard projection. Then

$$\begin{aligned}
\pi_\mu \circ Pro_\mu \circ \eta =& \pi_\mu \circ Pro_\mu \circ SCH \circ Iso^* \circ Fg^* \\
=& \pi_\mu \circ SCH_\mu \circ Iso_\mu^* \circ Fg_\mu^* \\
=& \widetilde{\pi}_\mu^* \circ SCH_\mu^{-1} \circ SCH_\mu \circ Iso_\mu^* \circ Fg_\mu^* && \text{by Remark 4.2.5} \\
=& \widetilde{\pi}_\mu^* \circ Iso_\mu^* \circ Fg_\mu^* \\
=& NTR_\mu && \text{by Lemma 4.2.9,}
\end{aligned}$$

as desired. □

4.2.3. Proof of Proposition 4.2.1

Suppose that the assumptions of Proposition 4.2.1 hold. By Remark 4.2.2, we may assume that $N_\lambda \neq \{1\}$ for $\lambda \in \Lambda$. Let

$$Sha^* : H^*(\overline{G}; \prod_{\lambda \in \Lambda} CoInd_{\overline{H}_\lambda}^{\overline{G}} H^*(N_\lambda; A)) \longrightarrow \prod_{\lambda \in \Lambda} H^*(\overline{H}_\lambda; H^*(N_\lambda; A))$$

be the isomorphism given by Lemma 2.13.1, and let $NTR_{\overline{G}}$ be the natural map defined in Notation 2.14.2.

Fix $p \in \mathbb{Z}$ and $q \in \mathbb{Z} \backslash \{0\}$. By Proposition 4.2.3 and the definition of Sha^*, there is a commutative diagram

$$\begin{array}{ccc} H^p(\overline{G}; H^q(\langle\langle \mathcal{N} \rangle\rangle; A)) & \xrightarrow{NTR^{p,q}_{\overline{G}}} & \prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A)) \\ \downarrow \eta^* & \nearrow Sha^* & \\ H^p(\overline{G}; \prod_{\lambda \in \Lambda} CoInd^{\overline{G}}_{\overline{H}_\lambda} H^q(N_\lambda; A)) & & \end{array}$$

where η^* is the natural map induced by the map $\eta : H^*(\langle\langle \mathcal{N} \rangle\rangle; A) \to CoInd^{\overline{G}}_{\overline{H}_\lambda} H^*(N_\lambda; A)$.

η maps $H^q(\langle\langle \mathcal{N} \rangle\rangle; A)$ isomorphically onto $CoInd^{\overline{G}}_{\overline{H}_\lambda} H^q(N_\lambda; A)$ and thus η^* maps $H^p(\overline{G}; H^q(\langle\langle \mathcal{N} \rangle\rangle; A))$ isomorphically onto $H^p(\overline{G}, \prod_{\lambda \in \Lambda} CoInd^{\overline{G}}_{\overline{H}_\lambda} H^q(N_\lambda; A))$. As Sha^* is an isomorphism, we deduce that $NTR^{p,q}_{\overline{G}}$ is an isomorphism.

4.3 Morphisms of Lyndon-Hochschild-Serre spectral sequences

4.3.1. Lyndon-Hochschild-Serre spectral sequences

Until the end of Section 4.3.3.let (G, H, N) be a group triple such that the natural map

$$\overline{H} = H/N \longrightarrow \overline{G} = G/\langle\langle N \rangle\rangle$$

is injective. We think of $\overline{H}$ as a subgroup of $\overline{G}$. Let A (resp. B) be a $\mathbb{Z}\overline{G}$-module (resp. $\mathbb{Z}\overline{H}$-module), and let $\mathcal{L} : A \to B$ be a $\mathbb{Z}\overline{H}$-linear map.

The *Lyndon-Hochschild-Serre (LHS) spectral sequence for the triple* $(G, \langle\langle N \rangle\rangle, A)$ is a spectral sequence

$$_hE^{p,q}_{G,2} = H^p(\overline{G}; H^q(\langle\langle N \rangle\rangle; A)) \Rightarrow H^{p+q}(G; A)$$

constructed as follows. Choose an injective resolution $A \to I_A$ over $\mathbb{Z}G$. Apply the functor $Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, \cdot)$ to $A \to I_A$ to obtain a deleted cochain complex $(Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A), \epsilon_G)$. Let

$$(J_G, {}_h\delta_G, {}_v\delta_G) \xrightarrow{f_G} (Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A), \epsilon_G)$$

be a CE resolution over $\mathbb{Z}\overline{G}$. Apply the functor $Hom_{\overline{G}}(\mathbb{Z}, \cdot)$ to J_G to form a deleted double complex

$(C_G, {}_hd_G, {}_vd_G)$. Let (TC_G, d_G) be the total complex of C_G. By Lemma 2.15.31, the row filtration of TC_G induces a spectral sequence

$$ {}_hE_G = \{({}_hE_{G,r}, {}_hd_{G,r})\}_{r\geqslant 1}. $$

The LHS spectral sequence for $(G, \langle\langle N\rangle\rangle, A)$ is the spectral sequence $\{({}_hE_{G,r}, {}_hd_{G,r})\}_{r\geqslant 2}$ resulted from deleting the E_1 page of ${}_hE_G$.

Remark 4.3.1. There is no essential reason for deleting the E_1 page in the construction of LHS spectral sequences. We take this approach only because it simplifies the construction of spectral sequence morphism in the proof of Theorem 4.0.1.

Similarly, there is an LHS spectral sequence

$$ {}_hE_{H,2}^{p,q} = H^p(\overline{H}; H^q(N; B)) \Rightarrow H^{p+q}(H; B) $$

for the tuple (H, N, B) constructed as follows. Pick an injective resolution $B \to I_B$ over $\mathbb{Z}H$. Apply the functor $Hom_N(\mathbb{Z}, \cdot)$ to $B \to I_B$ to obtain a deleted cochain complex $(Hom_N(\mathbb{Z}, I_B), \epsilon_H)$. Let

$$ (J_H, {}_h\delta_H, {}_v\delta_H) \xrightarrow{f_H} (Hom_N(\mathbb{Z}, I_B), \epsilon_H) $$

be a CE resolution over $\mathbb{Z}\overline{H}$. Apply the functor $Hom_{\overline{H}}(\mathbb{Z}, \cdot)$ to J_H to form a deleted double complex $(C_H, {}_hd_H, {}_vd_H)$. Let (TC_H, d_H) be the total complex of C_H. By Lemma 2.15.31, the row filtration of TC_H induces a spectral sequence

$$ {}_hE_H = \{({}_hE_{H,r}, {}_hd_{H,r})\}_{r\geqslant 1}. $$

The LHS spectral sequence for (H, N, B) is the spectral sequence $\{({}_hE_{H,r}, {}_hd_{H,r})\}_{r\geqslant 2}$ resulted from deleting the E_1 page of ${}_hE_H$.

As $H \leqslant G$, every injective $\mathbb{Z}G$-module is automatically an injective $\mathbb{Z}H$-module. Thus, $A \to I_A$ can also be regarded as an injective resolution over $\mathbb{Z}H$. $\mathcal{L}$ gives rise to a chain map $I_A \to I_B$, which induces a chain map

$$ \mathcal{L}^* : Hom_{\langle\langle N\rangle\rangle}(\mathbb{Z}, I_A) \longrightarrow Hom_N(\mathbb{Z}, I_B). $$

As $\overline{H} \leqslant \overline{G}$, every injective $\mathbb{Z}\overline{G}$-module is automatically an injective $\mathbb{Z}\overline{H}$-module. Thus, J_G can be regarded as a CE resolution over $\mathbb{Z}\overline{H}$. By Lemma 2.16.4, $\mathcal{L}^*$ induces a morphism

$$MCER : J_G \longrightarrow J_H$$

between CE resolutions. $MCER$ induces a morphism

$$MDC : C_G \longrightarrow C_H$$

between double complexes, which further induces a morphism

$${}_hMSS : {}_hE_G \longrightarrow {}_hE_H$$

between spectral sequences. For future reference, we note the following lemma.

Lemma 4.3.2. *Under the above assumptions, $\mathcal{L}^*$ and the inclusion $\overline{H} \hookrightarrow \overline{G}$ induces a morphism ${}_hMSS$: ${}_hE_G \to {}_hE_H$ between spectral sequences.*

Note that MDC also induces a cohomology map

$$MDC^* : H^*(TC_G) \longrightarrow H^*(TC_H).$$

Notation 4.3.3. Let

$$NAB_G : H^*(G; A) \longrightarrow H^*(H, B)$$

be the natural map induced by the inclusion $H \hookrightarrow G$.

For $q \in \mathbb{Z}$, let

$$NAB_N^q : H^q(\langle\langle N \rangle\rangle; A) \longrightarrow H^q(N; B)$$

be the natural map induced by $\mathcal{L}$ and the inclusion $N \hookrightarrow \langle\langle N \rangle\rangle$.

For $p, q \in \mathbb{Z}$, let

$$NAB_{\overline{G}}^{p,q} : H^p(\overline{G}; H^q(\langle\langle N \rangle\rangle; A)) \longrightarrow H^p(\overline{H}; H^q(N; B))$$

be the natural map induced by NAB_N^q and the inclusion $\overline{H} \hookrightarrow \overline{G}$.

The goal of the upcoming Sections 4.3.2and 4.3.3is the following.

Proposition 4.3.4. *Under the above assumptions,*

(a) $_hMSS$ *is compatible with* NAB_G*;*

(b) for $p, q \in \mathbb{Z}$*,* $_hMSS_2^{p,q}$ *can be identified with* $NAB_G^{p,q}$*.*

Proposition 4.3.4 should be well-known, but we are unable to find a reference for it, so we provide the proof for the convenience of the reader.

4.3.2. Compatibility of $_hMSS$ and NAB_G

The goal of Section 4.3.2.is to prove part (a) of Proposition 4.3.4. By Lemma 2.15.33, $_hMSS$ and MDC^* are compatible. Thus, it suffices to identify MDC^* with NAB_G.

Recall that $A \to I_A$ is an injective resolution over $\mathbb{Z}G$. Applying $Hom_G(\mathbb{Z}, \cdot)$ to this resolution gives rise to a deleted cochain complex

$$Hom_G(\mathbb{Z}, I_A) : 0 \longrightarrow Hom_G(\mathbb{Z}, I_A^0) \longrightarrow Hom_G(\mathbb{Z}, I_A^1) \longrightarrow \cdots \tag{4.10}$$

Consider the column filtration of TC_G

$$\{0\} \subset \cdots \subset {_vF_{p+1}}TC_G \subset {_vF_p}TC_G \subset \cdots \subset {_vF_0}TC_G = TC_G. \tag{4.11}$$

By Lemma 2.15.31, (4.11) gives rise to a spectral sequence

$$_vE_G = \{({_vE_{G,r}}, {_vd_{G,r}})\}_{r \geqslant 1}.$$

Note that the 0-th row of $_vE_{G,r}$ is a cochain complex

$$_vE_{G,1}^{*,0} : 0 \longrightarrow {_vE^{0,0}} \xrightarrow{{_vd_{G,1}}} {_vE^{1,0}} \xrightarrow{{_vd_{G,1}}} \cdots \tag{4.12}$$

We construct a chain map

$$_vCh_G : Hom_G(\mathbb{Z}, I_A) \longrightarrow {_vE_{G,1}^{*,0}}$$

by the following procedure. Let

$$x \in Hom_G(\mathbb{Z}, I_A^p) \subset Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A^p)$$

for some $p \geqslant 0$. Recall that

$$(J_G, {}_h\delta_G, {}_v\delta_G) \xrightarrow{f_G} (Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A), \epsilon_G)$$

is a CE resolution. Let $y \in Hom(\mathbb{Z}, J_G^{p,0})$ such that $y(k) = kf_G(x)$ for all $k \in \mathbb{Z}$. As f_G is a $\mathbb{Z}\overline{G}$-module homomorphism, y is in fact an element of $C_G^{p,0} = Hom_{\overline{G}}(\mathbb{Z}, J_G^{p,0})$.

Lemma 4.3.5. *$d_G(y) \in {}_vF_{p+1}TC_G$, i.e., ${}_vd_G(y) = 0$.*

Proof. By Definition 2.16.1,

$$0 \longrightarrow Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A^p) \xrightarrow{f_G} J_G^{p,0} \xrightarrow{{}_v\delta_G} J_G^{p,1} \xrightarrow{{}_v\delta_G} \cdots$$

is an injective resolution of $Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A^p)$ over $\mathbb{Z}\overline{G}$. Thus, after applying the functor $Hom_{\overline{G}}(\mathbb{Z}, \cdot)$, the resulted non-deleted cochain complex

$$0 \longrightarrow Hom_{\overline{G}}(\mathbb{Z}, Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A^p)) \xrightarrow{f_G^*} C_G^{p,0} \xrightarrow{{}_vd_G} C_G^{p,1} \xrightarrow{{}_vd_G} \cdots \tag{4.13}$$

is still exact at $C_G^{p,0}$, where f_G^* is the map induced by f_G.

Let

$$y' \in Hom_{\overline{G}}(\mathbb{Z}, Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A^p))$$

such that $y'(k) = kx$ for all $k \in \mathbb{Z}$. Direct computation shows $y = f_G^*(y')$. As (4.13) is exact at $C_G^{p,0}$, we have ${}_vd_G(y) = {}_vd_G \circ f_G^*(y') = 0$. □

Recall that ${}_vE_{G,1}^{p,0} = H^p({}_vF_pTC_G/{}_vF_{p+1}TC_G)$ and the cohomology is computed with respect to the differential induced by $d_G = {}_hd_G + {}_vd_G$ (see Lemma 2.15.31). Thus, every element of ${}_vE_{G,1}^{p,0}$ is represented by an element $z \in {}_vF_pTC_G^p$ such that $d_G(z) \in {}_vF_{p+1}TC_G^{p+1}$. Note that $y \in C_G^{p,0} \subset {}_vF_pTC_G^p$. By Lemma 4.3.5, y represents an element $[y] \in {}_vE_{G,1}^{p,0}$. Let ${}_vCh_G$ be the map such that, for every $p \geqslant 0$ and every $x \in Hom_G(\mathbb{Z}, I^p)$, ${}_vCh_G$ maps x to the corresponding $[y] \in {}_vE_{G,1}^{p,0}$.

We note the following.

Lemma 4.3.6 ([31, Theorem 11.38]). ${}_vCh_G : Hom_G(\mathbb{Z}, I_A) \to {}_vE^{*,0}_{G,1}$ *is a chain isomorphism.*

Remark 4.3.7. In [31], the cochain complexes $Hom_G(\mathbb{Z}, I_A)$ and ${}_vE^{*,0}_{G,1}$ are identified with the chain map being implicit. For the purpose of this paper, we need an explicit description of the chain map. The reader is encouraged to read the proof in [31] and check that the identification is given by ${}_vCh_G$.

Similarly, the column filtration

$$\{0\} \subset \cdots {}_vF_{p+1}TC_H \subset {}_vF_pTC_H \subset \cdots {}_vF_0TC_H = TC_H$$

gives rises to a spectral sequence

$${}_vE_H = \{({}_vE_{H,r}, {}_vd_{H,r})\}_{r \geqslant 1}$$

by Lemma 2.15.31. The 0-th row of ${}_vE_{H,1}$

$${}_vE^{*,0}_{H,1} : 0 \longrightarrow {}_vE^{0,0}_{H,1} \xrightarrow{{}_vd_{H,1}} {}_vE^{1,0}_{H,1} \xrightarrow{{}_vd_{H,1}} \cdots$$

is a cochain complex.

As above, we construct a chain map

$${}_vCh_H : Hom_H(\mathbb{Z}, I_B) \longrightarrow {}_vE^{*,0}_{H,1}$$

by the following procedure. Let

$$x \in Hom_H(\mathbb{Z}, I^p_B) \subset Hom_N(\mathbb{Z}, I^p_B)$$

for some $p \geqslant 0$. Recall that

$$J_H \xrightarrow{f_H} Hom_N(\mathbb{Z}, I_B)$$

is a CE resolution. Let $y \in Hom(\mathbb{Z}, J^{p,0}_H)$ such that $y(k) = kf_H(x)$ for all $k \in \mathbb{Z}$. As f_H is a $\mathbb{Z}\overline{H}_\lambda$-module homomorphism, y is in fact an element of $C^{p,0}_H = Hom_{\overline{H}}(\mathbb{Z}, J^{p,0}_H)$. Moreover, by the same argument as the one above, we see that y represents an element $[y] \in {}_vE^{p,0}_{H,1}$. Let ${}_vCh_H$ be the map such that, for every $p \geqslant 0$ and every $x \in Hom_H(\mathbb{Z}, I_B^p)$, ${}_vCh_H$ maps x to the corresponding $[y] \in {}_vE_{H,1}^{p,0}$. We note the following lemma (see also Remark 4.3.7).

Lemma 4.3.8 ([31, Theorem 11.38]). *${}_vCh_H : Hom_H(\mathbb{Z}, I_B) \to {}_vE_{H,1}^{*,0}$ is a chain isomorphism.*

Recall that $\mathcal{L}^*$ induces morphisms $MCER$ and MDC. MDC further induces a morphism

$$ {}_vMSS : {}_vE_G \longrightarrow {}_vE_H $$

between spectral sequences.

Lemma 4.3.9. *For $p \geqslant 0$, the diagram*

$$ \begin{array}{ccc} Hom_G(\mathbb{Z}, I_A^p) & \xrightarrow{\mathcal{L}^*} & Hom_H(\mathbb{Z}, I_B^p) \\ \downarrow {\scriptstyle {}_vCh_G} & & \downarrow {\scriptstyle {}_vCh_H} \\ {}_vE_{G,1}^{p,0} & \xrightarrow{{}_vMSS_1} & {}_vE_{H,1}^{p,0} \end{array} $$

commutes.

Proof. Given $x \in Hom_G(\mathbb{Z}, I_A^p)$ for some $p \geqslant 0$, let $y \in C_G^{p,0}$ such that $y(k) = kf_G(x)$ and let $[y] \in {}_vE_{G,1}^{p,0}$ be the cohomology class represented by y. By definition, ${}_vCh_G(x) = [y]$. Let $z \in C_H^{p,0}$ such that $z = MDC(y)$. Then

$$ d_H(z) = d_H \circ MDC(y) = MDC \circ d_G(y) \in {}_vF_{p+1}TC_H $$

and thus z represents an element of ${}_vE_{H,1}^{p,0}$. Let $[z] \in {}_vE_{H,1}^{p,0}$ be the cohomology class represented by z. As ${}_vMSS$ is induced by $MCER$, we have

$$ {}_vMSS_1 \circ {}_vCh_G(x) = {}_vMSS_1([y]) = [MCER \circ y] = [z]. $$

Note that $\mathcal{L}^*(x) \in Hom_H(\mathbb{Z}, I_B^p)$. Let $z' \in C_H^{p,0}$ such that $z'(k) = kf_H \circ \mathcal{L}^*(x)$. Then z' represents an element of ${}_vE_{H,1}^{p,0}$. Let $[z'] \in {}_vE_{H,1}^{p,0}$ be the cohomology class represented by z'. Then $[z'] = {}_vCh_H \circ \mathcal{L}^*(x)$, by definition.

Since MDC is induced by $MCER$, we have

$$\begin{aligned}
& z(k) \\
=& MCER \circ y(k) \\
=& kMCER \circ f_G(x) \\
=& kf_H \circ \mathcal{L}^*(x) && \text{as } MCER \text{ is induced by } \mathcal{L}^* \\
=& z'(k).
\end{aligned}$$

Therefore, $z = z'$. □

As a matter of fact, ${}_vE^{p,q}_{G,2} = {}_vE^{p,q}_{H,2} = \{0\}$ for all $q \neq 0$ (for example, see [31, Theorem 11.38]). Thus, Lemma 2.15.32 implies that $H^p(TC_G)$ (resp. $H^p(TC_H)$) can be identified with ${}_vE^{p,0}_{G,2}$ (resp. ${}_vE^{p,0}_{H,2}$). Lemma 2.15.33 implies that the cohomology map MDC^* can be identified with

$${}_vMSS^{*,0}_2 = \bigoplus_{p\in\mathbb{Z}}^{DT} {}_vMSS^{p,0}_{H,2} : {}_vE^{*,0}_{G,2} \longrightarrow {}_vE^{*,0}_{H,2}.$$

By Lemmas 4.3.6, 4.3.8, and 4.3.9, the cochain complex ${}_vE^{*,0}_{G,1}$ (resp. ${}_vE^{*,0}_{\lambda,1}$) can be identified with $Hom_G(\mathbb{Z}, I^p_A)$ (resp. $Hom_H(\mathbb{Z}, I^p_B)$) via the chain map ${}_vCh_G$ (resp. ${}_vCh_H$), while the chain map ${}_vMSS_1$ can be identified with $\mathcal{L}^*$. By Definition 2.15.27, ${}_vMSS^{*,0}_2$ is the cohomology map induced by ${}_vMSS_1$. Note that the cohomology map induced by $\mathcal{L}^*$ is NAB_G. We conclude this subsection by the following.

Lemma 4.3.10. *MDC^* can be identified with NAB_G.*

4.3.3. Identifying ${}_hMSS^{p,q}_2$ with $NAB^{p,q}_{\overline{G}}$

The goal of Section 4.3.3is to finish the proof of Proposition 4.3.4. Recall that the row filtration

$$\{0\} \subset \cdots {}_hF_{p+1}TC_G \subset {}_hF_pTC_G \subset \cdots {}_hF_0TC_G = TC_G.$$

induces the spectral sequence ${}_hE_G$. Note that the 0-th row of ${}_hE_{G,1}$

$${}_hE^{*,0}_{G,1} : 0 \longrightarrow {}_hE^{0,0}_{G,1} \xrightarrow{{}_hd_{G,1}} {}_hE^{1,0}_{G,1} \xrightarrow{{}_hd_{G,1}} \cdots$$

is a cochain complex.

Recall that

$$(J_G, {}_h\delta_G, {}_v\delta_G) \xrightarrow{f_G} (Hom_{\langle\!\langle N\rangle\!\rangle}(\mathbb{Z}, I_A), \epsilon_G)$$

is a CE resolution. For $p, q \in \mathbb{Z}$, let

$${}_hZ_G^{p,q} = \ker({}_h\delta_G^{p,q}), \quad {}_hB_G^{p,q} = \operatorname{im}({}_h\delta_G^{p-1,q}), \quad {}_hH_G^{p,q} = {}_hZ_G^{p,q}/{}_hB_G^{p,q}$$

be the horizontal cocycle, coboudnary, and cohomology of J_G at position (p, q), respectively. Fix $q \geqslant 0$ for the moment. By Definition 2.16.1, the vertical differential ${}_v\delta_G$ induces an injective resolution

$$0 \longrightarrow H^q(\langle\!\langle N\rangle\!\rangle; A) \longrightarrow {}_hH_G^{q,0} \longrightarrow {}_hH_G^{q,1} \longrightarrow \cdots$$

Applying $Hom_{\overline{G}}(\mathbb{Z}, \cdot)$ to this resolution gives rise to a deleted cochain complex

$$Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,*}) : 0 \longrightarrow Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,0}) \longrightarrow Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,1}) \longrightarrow \cdots.$$

We construct a chain map

$${}_hCh_G : Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,*}) \longrightarrow {}_hE_{G,1}^{*,0}$$

by the following procedure. Let $x \in Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,p})$ for some $p \geqslant 0$. Note that every term in the short exact sequence

$$0 \longrightarrow {}_hB_G^{q,p} \longrightarrow {}_hZ_G^{q,p} \longrightarrow {}_hH_G^{q,p} \longrightarrow 0$$

is an injective module (see Definition 2.16.1). Thus, after applying $Hom_{\overline{G}}(\mathbb{Z}, \cdot)$, the resulted sequence

$$0 \longrightarrow Hom_{\overline{G}}(\mathbb{Z}, {}_hB_G^{q,p}) \longrightarrow Hom_{\overline{G}}(\mathbb{Z}, {}_hZ_G^{q,p}) \longrightarrow Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,p}) \longrightarrow 0$$

is still exact. In particular, there exists $y \in Hom_{\overline{G}}(\mathbb{Z}, {}_hZ_G^{q,p})$ such that $y(k) \in {}_hZ_G^{q,p}$ represents $x(k) \in {}_hH_G^{q,p}$ for every $k \in \mathbb{Z}$. As

$${}_hZ_G^{q,p} \subset C_G^{q,p} \subset {}_hF_pTC_G,$$

we may think of y as an element of ${}_hF_pTC_G^{p+q}$. By the same argument as the one in Lemma 4.3.5, $d_G(y) \in {}_hF_{p+1}TC_G$ and thus y represents an element

$$[y] \in {}_hE_{G,1}^{p,q} = H^{p+q}({}_hF_pTC_G/{}_hF_{p+1}TC_G).$$

Let ${}_hCh_G$ be the map such that, for every $p \geqslant 0$, ${}_hCh_G$ maps every $x \in Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,p})$ to the corresponding $[y] \in {}_hE_{G,1}^{p,q}$. It is easy to check that ${}_hCh_G$ is well-defined, i.e., ${}_hCh_G(x)$ does not depend on the choice of $y \in Hom_{\overline{G}}(\mathbb{Z}, {}_hZ_G^{q,p})$ such that $y(k) \in {}_hZ_G^{q,p}$ represents $x(k) \in {}_hH_G^{q,p}$ for $k \in \mathbb{Z}$.

Lemma 4.3.11 ([31, Theorem 11.38]). *${}_hCh_G$ is a chain isomorphism.*

Remark 4.3.12. In [31], the cochain complexes $Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,*})$ and ${}_hE_{G,1}^{*,0}$ are identified with the chain map being implicit. The reader is encouraged to read the proof in [31] and check that the identification is given by ${}_hCh_G$.

Recall that the row filtration

$$\{0\} \subset \cdots {}_hF_{p+1}TC_H \subset {}_hF_pTC_H \subset \cdots {}_hF_0TC_H = TC_H.$$

induces the spectral sequence ${}_hE_H$. The 0-th row of ${}_hE_{H,1}$

$${}_hE_{H,1}^{*,0} : 0 \longrightarrow {}_hE_{H,1}^{0,0} \xrightarrow{{}_hd_{H,1}} {}_hE_{H,1}^{1,0} \xrightarrow{{}_hd_{H,1}} \cdots$$

is a cochain complex.

Recall that

$$(J_H, {}_h\delta_H, {}_v\delta_H) \xrightarrow{f_H} (Hom_{\langle\langle N\rangle\rangle}(\mathbb{Z}, I_B), \epsilon_H)$$

is a CE resolution. For $p, q \in \mathbb{Z}$, let

$${}_hZ_H^{p,q} = \ker({}_h\delta_H^{p,q}), \quad {}_hB_H^{p,q} = \operatorname{im}({}_h\delta_H^{p-1,q}), \quad {}_hH_\lambda^{p,q} = {}_hZ_H^{p,q}/{}_hB_H^{p,q}$$

be the horizontal cocycle, coboudnary, and cohomology of J_H at position (p, q), respectively. Fix $q \geqslant 0$ for the moment. By Definition 2.16.1, the vertical differential ${}_v\delta_H$ induces an injective resolution

$$0 \longrightarrow H^q(N;A) \longrightarrow {}_hH_H^{q,0} \longrightarrow {}_hH_H^{q,1} \longrightarrow \cdots$$

Applying $Hom_{\overline{H}}(\mathbb{Z},\cdot)$ to this resolution gives rise to a deleted cochain complex

$$Hom_{\overline{H}}(\mathbb{Z},{}_hH_H^{q,*}) : 0 \longrightarrow Hom_{\overline{H}}(\mathbb{Z},{}_hH_H^{q,0}) \longrightarrow Hom_{\overline{H}}(\mathbb{Z},{}_hH_H^{q,1}) \longrightarrow \cdots.$$

We construct a map

$${}_hCh_H : Hom_{\overline{H}}(\mathbb{Z},{}_hH_H^{q,*}) \longrightarrow {}_hE_{H,1}^{*,0}$$

by the following procedure. Let $x \in Hom_{\overline{H}}(\mathbb{Z},{}_hH_H^{q,p})$ for some $p \geqslant 0$. By the same argument as the one above, there exists $y \in Hom_{\overline{H}}(\mathbb{Z},{}_hZ_H^{q,p})$ such that $y(k) \in {}_hZ_H^{q,p}$ represents $x(k) \in {}_hH_H^{q,p}$ for every $k \in \mathbb{Z}$. As

$${}_hZ_H^{q,p} \subset C_H^{q,p} \subset {}_hF_pTC_H^{p+q},$$

we may think of y as an element of ${}_hF_pTC_H^{p+q}$. By the same argument as the one in Lemma 4.3.5, $d_H(y) \in {}_hF_{p+1}TC_H$ and thus y represents an element

$$[y] \in {}_hE_{H,1}^{p,q} = H^{p+q}({}_hF_pTC_H/{}_hF_{p+1}TC_H).$$

Let ${}_hCh_H$ be the map such that, for every $p \geqslant 0$, ${}_hCh_H$ maps every $x \in Hom_{\overline{H}}(\mathbb{Z},{}_hH_H^{q,p})$ to the corresponding $[y] \in {}_hE_{H,1}^{p,q}$. It is easy to check that ${}_hCh_H$ is well-defined, i.e., ${}_hCh_H(x)$ does not depend on the choice of $y \in Hom_{\overline{H}}(\mathbb{Z},{}_hZ_H^{q,p})$ such that $y(k) \in {}_hZ_H^{q,p}$ represents $x(k) \in {}_hH_H^{q,p}$ for $k \in \mathbb{Z}$.

Lemma 4.3.13 ([31, Theorem 11.38] (see also Remark 4.3.7)). *${}_hCh_H$ is a chain isomorphism.*

Recall that the chain map $\mathcal{L}^*$ induces morphisms $MCER, MDC$, and ${}_hMSS$. $MCER$ induces a map

$$\overline{MCER} : {}_hH_G^{q,*} \longrightarrow {}_hH_H^{q,*},$$

which further induces a chain map

$$\overline{MCER}^* : Hom_{\overline{G}}(\mathbb{Z},{}_hH_G^{q,*}) \longrightarrow Hom_{\overline{H}}(\mathbb{Z},{}_hH_H^{q,*}).$$

Lemma 4.3.14. *For $p \geqslant 0$, the diagram*

$$\begin{array}{ccc} Hom_{\overline{G}}(\mathbb{Z}, H_G^{q,p}) & \xrightarrow{\overline{MCER}^*} & Hom_{\overline{H}}(\mathbb{Z}, H_H^{q,p}) \\ \Big\downarrow {}_hCh_G & & \Big\downarrow {}_hCh_H \\ {}_hE_{G,1}^{p,q} & \xrightarrow{\quad {}_hMSS_1 \quad} & {}_hE_{H,1}^{p,q} \end{array}$$

commutes.

Proof. Given $x \in Hom_{\overline{G}}(\mathbb{Z}, H_G^{q,p})$ for some $p \geqslant 0$, let $y \in Hom_{\overline{G}}(\mathbb{Z}, {}_hZ_G^{q,p})$ such that $y(k) \in {}_hZ_G^{q,p}$ represents $x(k) \in {}_hH_G^{q,p}$ for all $k \in \mathbb{Z}$, and let $[y] \in {}_hE_{G,1}^{p,q}$ be the cohomology class represented by y. By definition, ${}_hCh_G(x) = [y]$.

Let

$$z = MCER \circ y \in Hom_{\overline{H}}(\mathbb{Z}, J_H^{q,p}).$$

As $MCER$ is a morphism of double complexes, z in fact belongs to $Hom_{\overline{H}}(\mathbb{Z}, {}_hZ_H^{q,p})$ and thus represents an element $[z] \in {}_hE_{H,1}^{p,q}$. By definition, ${}_hMSS_1 \circ {}_hCh_G(x) = [z]$.

Note that $\overline{MCER}^*(x) = \overline{MCER} \circ x$. As $MCER$ is a morphism between double complexes, for every $k \in \mathbb{Z}$, $z(k) \in {}_hZ_H^{q,p}$ represents $\overline{MCER} \circ x(k) \in {}_hH_H^{q,p}$. By definition, ${}_hCh_H \circ \overline{MCER}^*(x) = [z]$. Therefore, ${}_hMSS_1 \circ {}_hCh_G = {}_hCh_H \circ \overline{MCER}^*$. □

As $MCER$ is induced by $\mathcal{L}^*$, the following diagram commutes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom_{\langle\langle N \rangle\rangle}(\mathbb{Z}, I_A^q) & \xrightarrow{f_G} & J_G^{q,1} & \xrightarrow{{}_v\delta_G} & J_G^{q,2} & \xrightarrow{{}_v\delta_G} & \cdots \\ & & \Big\downarrow \mathcal{L}^* & & \Big\downarrow MCER & & \Big\downarrow MCER & & \\ 0 & \longrightarrow & Hom_N(\mathbb{Z}, I_B^q) & \xrightarrow{f_H} & J_H^{q,1} & \xrightarrow{{}_v\delta_H} & J_H^{q,2} & \xrightarrow{{}_v\delta_H} & \cdots \end{array} \tag{4.14}$$

As f_G, f_H, and $\mathcal{L}^*$ are chain maps, (4.14) induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^q(\langle\langle N\rangle\rangle; A) & \xrightarrow{\overline{f_G}} & {}_hH_G^{q,1} & \longrightarrow & {}_hH_G^{q,2} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \overline{\mathcal{L}^*}} & & \downarrow{\scriptstyle \overline{MCER}} & & \downarrow{\scriptstyle \overline{MCER}} & & \\
0 & \longrightarrow & H^q(N; B) & \xrightarrow{\overline{f_H}} & {}_hH_H^{q,1} & \longrightarrow & {}_hH_H^{q,2} & \longrightarrow & \cdots
\end{array} \tag{4.15}$$

where $\overline{f_G}, \overline{f_H}, \overline{\mathcal{L}^*}$ are the maps induced by $f_G, f_H, \mathcal{L}^*$, respectively.

Applying the functors $Hom_{\overline{G}}(\mathbb{Z}, \cdot)$ and $Hom_{\overline{H}}(\mathbb{Z}, \cdot)$ to (4.15) gives rise to

$$\begin{array}{ccccccc}
0 & \longrightarrow & Hom_{\overline{G}}(\mathbb{Z}, H^q(\langle\langle N\rangle\rangle; A)) & \xrightarrow{\overline{f_G}} & Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom_{\overline{H}}(\mathbb{Z}, H^q(N; B)) & \xrightarrow{\overline{f_H}} & Hom_{\overline{H}}(\mathbb{Z}, {}_hH_H^{q,1}) & \longrightarrow & \cdots
\end{array} \tag{4.16}$$

In (4.16), the leftmost vertical map is NAB_N^q and all other vertical maps are $\overline{MCER}^*$. It follows that the cohomology map induced by $\overline{MCER}^*$ is $NAB_{\overline{G}}^{*,q} = \bigoplus_{p\in\mathbb{Z}}^{DT} NAB_{\overline{G}}^{p,q}$.

By Lemmas 4.3.11, 4.3.13, and 4.3.14, the cochain complex ${}_hE_G^{q,*}$ (resp. ${}_hE_H^{q,*}$) can be identified with $Hom_{\overline{G}}(\mathbb{Z}, {}_hH_G^{q,*})$ (resp. $Hom_{\overline{H}}(\mathbb{Z}, {}_hH_H^{q,*})$) via the chain map ${}_hCh_G$ (resp. ${}_hCh_H$), while the chain map ${}_hMSS_1$ can be identified with $\overline{MCER}^*$. By Definition 2.15.27, ${}_hMSS_2$ is the cohomology map induced by ${}_hMSS_1$. Thus,

Lemma 4.3.15. *For $p, q \in \mathbb{Z}$, ${}_hMSS_2^{p,q}$ can be identified with $NAB_{\overline{G}}^{p,q}$.*

Proof of Proposition 4.3.4. Proposition 4.3.4 is a combination of Lemmas 2.15.33, 4.3.10, and 4.3.15. □

4.4 Proof of Theorem 4.0.1

Under the assumptions of Theorem 4.0.1, let $E_G = \{(E_{G,r}, d_{G,r})\}_{r\geqslant 2}$ be the LHS spectral sequence for the triple $(G, \langle\langle\mathcal{N}\rangle\rangle, A)$. For $\lambda \in \Lambda$, let $E_{H_\lambda} = \{(E_{H_\lambda,r}, d_{H_\lambda,r})\}_{r\geqslant 2}$ be the LHS spectral sequence for the triple $(H_\lambda, N_\lambda, A)$. Recall that E_G (resp. E_{H_λ}) results from deleting the E_1 page of $\{(E_{G,r}, d_{G,r})\}_{r\geqslant 1}$ (resp. $\{(E_{H_\lambda,r}, d_{H_\lambda,r})\}_{r\geqslant 1}$).

Employ notations defined in Notation 2.14.2. Let us first construct, for every $\lambda \in \Lambda$, a morphism $MSS_\lambda : E_G \to E_{H_\lambda}$ of spectral sequences.

Let $\Lambda' = \{\lambda \in \Lambda \mid N_\lambda \neq \{1\}\}$. Note that the group triple $(G, \{H_\lambda\}_{\lambda\in\Lambda'}, \{N_\lambda\}_{\lambda\in\Lambda'})$ has the Cohen-Lyndon property. By Proposition 3.3.1, for $\lambda \in \Lambda'$, we may think of $\overline{H}_\lambda$ as a subgroup of $\overline{G}$. By Lemma 4.3.2, the inclusion $\overline{H}_\lambda \hookrightarrow \overline{G}$ induces a morphism

$$\{(E_{G,r}, d_{G,r})\}_{r\geqslant 1} \longrightarrow \{(E_{H_\lambda,r}, d_{H_\lambda,r})\}_{r\geqslant 1}$$

between spectral sequences. By restricting the domain of this morphism to E_G and the target of this morphism to E_{H_λ}, we obtain a morphism

$$MSS_\lambda : E_G \longrightarrow E_{H_\lambda}$$

between LHS spectral sequences.

Let $\lambda \in \Lambda \backslash \Lambda'$. Then for $r \geqslant 2$,

$$E^{p,q}_{H_\lambda,r} = \begin{cases} H^p(H_\lambda; H^0(\{1\}; A)) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

Note that $E^{p,0}_{H_\lambda,r}$ can be naturally identified with $H^p(H_\lambda; A)$.

For $r \geqslant 2$, define a bigraded abelian group homomorphism $MSS_{\lambda,r} : E_{G,r} \to E_{H_\lambda,r}$ by the following.

(1) $MSS^{p,q}_{\lambda,r}$ is the identically zero map for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$.

(2) For $q = 0$, let $R > r$ be sufficiently large such that $E^{p,0}_{G,R}$ naturally embeds into $H^p(G; A)$ (such an R exists as $E^{p,q}_{G,2} \Rightarrow H^{p+q}(G; A)$). By the definition of spectral sequences, there is a natural quotient map $E^{p,0}_{G,r} \to E^{p,0}_{G,R}$. Let $MSS^{p,0}_{\lambda,r}$ be the composisition

$$E^{p,0}_{G,r} \to E^{p,0}_{G,R} \to H^p(G; A) \to H^p(H_\lambda; A) \cong E^{p,0}_{H_\lambda,r}$$

(the definition of $MSS_{\lambda,r}^{p,0}$ does not depend on the choice of R).

It is easy to check that $MSS_{\lambda,r}, r \geqslant 2$, constructed above form a morphism $MSS_\lambda : E_G \to E_{H_\lambda}$ between spectral sequences.

Claim. *For $\lambda \in \Lambda$,*

(a) MSS_λ is compatible with NTR_{H_λ};

(b) for $p, q \in \mathbb{Z}$, $MSS_{\lambda,2}^{p,q}$ can be identified with $NTR_{\overline{H}_\lambda}^{p,q}$.

Proof of the claim. If $\lambda \in \Lambda'$, then (a) and (b) follow from Proposition 4.3.4. If $\lambda \in \Lambda \backslash \Lambda'$, then (a) and (b) follow directly from the definition of MSS_λ. □

Let $E_{\mathcal{H}}$ be the product of $E_{H_\lambda}, \lambda \in \Lambda$, and let

$$MSS = \prod_{\lambda \in \Lambda}^{Tar} MSS_\lambda : E_G \longrightarrow E_{\mathcal{H}}.$$

By Lemma 2.15.15 and the claim above, MSS is compatible with NTR_G. For $p, q \in \mathbb{Z}$, $MSS_2^{p,q}$ can be identified with $NTR_{\overline{G}}^{p,q}$. By Proposition 4.2.1, for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$, $NTR_{\overline{G}}^{p,q}$ is an isomorphism and thus $MSS_2^{p,q}$ is also an isomorphism.

For $q = 0$ and $p \in \mathbb{Z}$, it is well-known that $H^0(\langle\langle \mathcal{N} \rangle\rangle; A)$ can be natrually identified with the $\langle\langle \mathcal{N} \rangle\rangle$-fixed-points of A, and for $\lambda \in \Lambda$, $H^0(N_\lambda; A)$ can be naturally identified with the N_λ-fixed-points of A. As A is a $\mathbb{Z}\overline{G}$-module, the $\langle\langle \mathcal{N} \rangle\rangle$-action on A fixes every point. Thus, we have natural isomorphisms

$$H^p(\overline{G}; H^0(\langle\langle \mathcal{N} \rangle\rangle; A)) \cong H^p(\overline{G}; A),$$

$$\prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; H^0(N_\lambda; A)) \cong \prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; A),$$

and $NTR_{\overline{G}}^{p,0}$ can be natrually identified with $NT_{\overline{G}}^p$. Thus, $MSS_2^{p,0}$ can be identified with $NT_{\overline{G}}^p$.

CHAPTER 5

APPLICATIONS

Theorem 4.0.1 provides a morphism between spectral sequences with special properties. In this chapter, we first perform computations with spectral sequences to extract certain information from such a morphism. And then we use the extracted information to prove Theorems 1.2.15, 1.2.18, 1.2.22, and 1.2.23.

5.1 Computations with spectral sequences

Let $E_1 = \{(E_{1,r}, d_{1,r})\}_{r\geqslant 2}, E_2 = \{(E_{2,r}, d_{2,r})\}_{r\geqslant 2}$ be two spectral sequences and let

$$MSS : E_1 \to E_2$$

be a morphism between spectral sequences. Recall that, for $r \geqslant 2$, the differentials $d_{1,r}, d_{2,r}$ and the map MSS_r are morphisms between bigraded abelian groups, and we use superscripts to denote the components. The following lemma is an immediate consequence of our assumptions.

Lemma 5.1.1. *For $p, q \in \mathbb{Z}$ and $r \geqslant 2$,*

$$MSS_r^{p,q}(\ker(d_{1,r}^{p,q})) \subset \ker(d_{2,r}^{p,q}), \quad MSS_r^{p,q}(\mathrm{im}(d_{1,r}^{p-r,q+r-1})) \subset \mathrm{im}(d_{2,r}^{p-r,q+r-1}).$$

As MSS_{r+1} is the cohomology map induced by MSS_r, it follows that

(a) $MSS_{r+1}^{p,q}$ is surjective if and only if

$$MSS_r^{p,q}(\ker(d_{1,r}^{p,q})) + \mathrm{im}(d_{2,r}^{p-r,q+r-1}) = \ker(d_{2,r}^{p,q});$$

(b) $MSS_{r+1}^{p,q}$ is injective if and only if the preimage of $\mathrm{im}(d_{2,r}^{p-r,q+r-1})$ under $MSS_r^{p,q}$ is $\mathrm{im}(d_{1,r}^{p-r,q+r-1})$.

Suppose that $MSS_2^{p,q}$ is an isomorphism for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$. Ideas of this section are illustrated by the example below.

Example 5.1.2. Suppose

(a) $E_{1,2}^{p,q} \Rightarrow H_1^{p+q}$ and $E_{2,2}^{p,q} \Rightarrow H_2^{p,q}$ for some graded abelian groups $H_1 = \bigoplus_{p \geqslant 0} H_1^p$ and $H_2 = \bigoplus_{p \geqslant 0} H_2^p$;

(b) There is a morphism $f : H_1 \to H_2$ compatible with MSS.

For simplicity, let us further assume

(c) $E_{1,2}^{p,q} = E_{2,2}^{p,q} = \{0\}$ whenever $q \neq 0, 1$.

Under these additional assumptions, we derive properties of E_1, E_2, and MSS. Recall that for $p \in \mathbb{Z}$, f^p denotes the p-component of f.

The only possibly nontrivial differentials at the second page of E_1 or E_2 are the ones going from the first row to the 0-th row. Two such maps are shown in Figure 5.1, where the unlabeled arrows are $d_{1,2}^{p-2,1}$ and $d_{2,2}^{p-2,1}$, respectively. After finishing the computations at the second page, we obtain the third page, which is shown by Figure 5.2. In Figure 5.2, the line segment connecting $\operatorname{coker}(d_{1,2}^{p-3,0}), \ker(d_{1,2}^{p-2,1})$, and H_1^{p-1} indicates the exact sequence

$$1 \to \operatorname{coker}(d_{1,2}^{p-3,0}) \to H_1^{p-1} \to \ker(d_{1,2}^{p-2,1}) \to 1,$$

which is a consequence of $E_{1,2}^{p,q} \Rightarrow H_1^{p+q}$. Similarly, other line segments in Figure 5.2 indicate different consequences of the limits of E_1 and E_2.

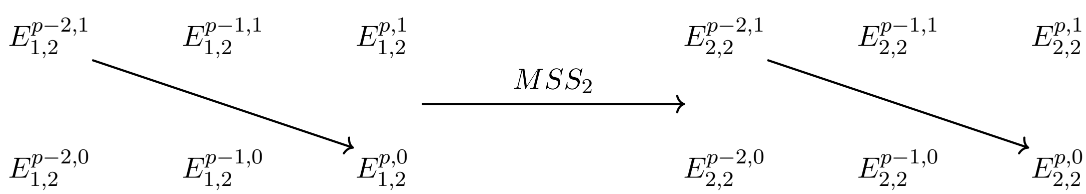


Figure 5.1: The second pages of E_1 and E_2

For $p \in \mathbb{Z}$, the map $MSS_3^{p-2,1}$ results from $MSS_2^{p-2,1}$ by restricting the domain to $\ker(d_{1,2}^{p-2,1})$ and restricting the target to $\ker(d_{2,2}^{p-2,1})$. Thus,

Observation 1. $MSS_3^{p-2,1}$ is injective as $MSS_2^{p-2,1}$ is.

In general, $MSS_3^{p-2,1}$ need not be surjective, although $MSS_2^{p-2,1}$ is surjective. For instance, if

$$\ker(MSS_2^{p,0}) \cap \operatorname{im}(d_{1,2}^{p-2,1}) \neq \{0\}$$

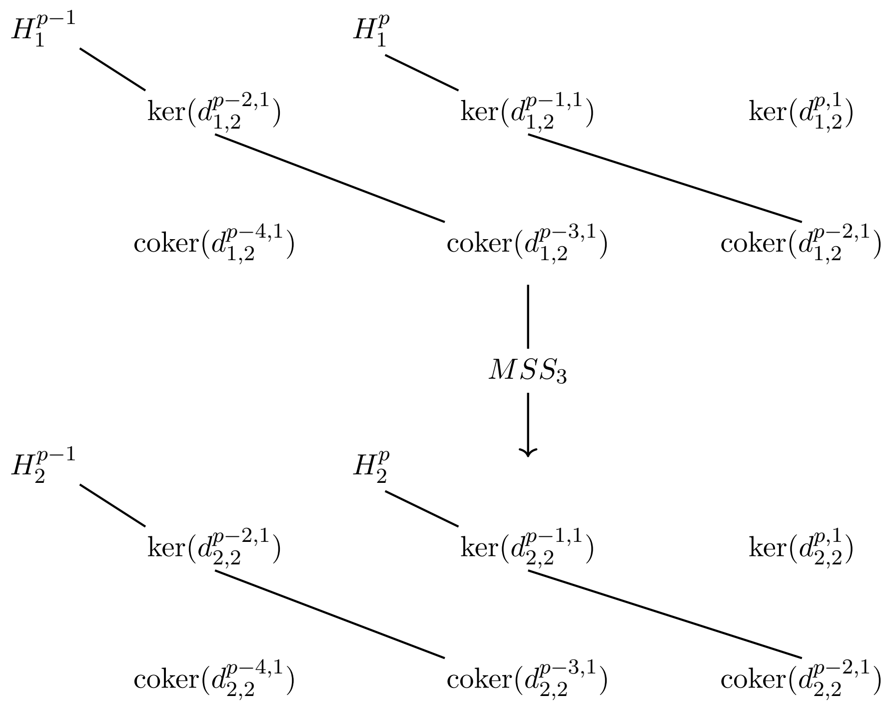


Figure 5.2: The third pages of E_1 and E_2

Then there exists $x \in E_{1,2}^{p-2,1}$ such that

$$d_{1,2}^{p-2,1}(x) \in \ker(MSS_2^{p,0}) \backslash \{0\}.$$

Let $y = MSS_2^{p-2,1}(x)$. Then

$$d_{2,2}^{p-2,1}(y) = d_{2,2}^{p-2,1} \circ MSS_2^{p-2,1}(x) = MSS_2^{p,0} \circ d_{1,2}^{p-2,1}(x) = 0.$$

Thus, $y \in \ker(d_{2,2}^{p-2,1})$. We claim that y has no preimage under $MSS_3^{p-2,1}$. Indeed, $MSS_3^{p-2,1}$ is a restriction of $MSS_2^{p-2,1}$, and $MSS_2^{p-2,1}$ is injective. Therefore, the only candidate for the preimage of y under $MSS_3^{p-2,1}$ is x. But $x \notin \ker(d_{1,2}^{p-2,1})$ and thus x is not in the domain of $MSS_3^{p-2,1}$.

Observation 2. By the above argument, if $MSS_3^{p-2,1}$ is surjective (for example, if f^{p-1} is surjective), then $\ker(MSS_2^{p,0}) \cap \mathrm{im}(d_{1,2}^{p-2,1}) = \{0\}$, that is, $MSS_2^{p,0}$ maps $\mathrm{im}(d_{1,2}^{p-2,1})$ injectively into $E_{2,2}^{p,0}$.

Let us make some other observations. Note that $MSS_2^{p,0}$ maps $\operatorname{im}(d_{1,2}^{p-2,1})$ onto

$$\operatorname{im}(MSS_2^{p,0} \circ d_{1,2}^{p-2,1}) = \operatorname{im}(d_{2,2}^{p-2,1} \circ MSS_2^{p-2,1}).$$

By assumption, $MSS_2^{p-2,1}$ is an isomorphism. In particular, $MSS_2^{p-2,1}$ is surjective. If $d_{2,2}^{p-2,1}$ is also surjective (for example, if $H_2^p = \{0\}$ and thus $\operatorname{coker}(d_{2,2}^{p-2,1}) = \{0\}$), then $d_{2,2}^{p-2,1} \circ MSS_2^{p-2,1}$ will be surjective, which will imply the surjectivity of $MSS_2^{p,0} \circ d_{1,2}^{p-2,1}$. Therefore,

Observation 3. If $H_2^p = \{0\}$, then $MSS_2^{p,0}$ maps $\operatorname{im}(d_{1,2}^{p-2,1})$ surjectively onto $E_{2,2}^{p,0}$.

Now suppose that for some p, f^{p-1} is surjective and $H_2^p = \{0\}$. By Observations 2 and 3, $MSS_2^{p,0}$ maps $\operatorname{im}(d_{1,2}^{p-2,1})$ isomorphically onto $E_{2,2}^{p,0}$. It follows that

(1) $1 \to \ker(MSS_2^{p,0}) \to E_{1,2}^{p,0} \to E_{2,2}^{p,0} \to 1$ is a split exact sequence;

(2) $E_{1,2}^{p,0} = \ker(MSS_2^{p,0}) \bigoplus \operatorname{im}(d_{1,2}^{p-2,1})$ and thus $\operatorname{coker}(d_{1,2}^{p-2,1}) \cong \ker(MSS_2^{p,0})$.

As $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$, another implication of $H_2^p = \{0\}$ is $\ker(d_{2,2}^{p-1,1}) = \{0\}$. By Observation 1, $MSS_3^{p-1,1}$ is injecitve. Thus, a consequence of $\ker(d_{2,2}^{p-1,1}) = \{0\}$ is $\ker(d_{1,2}^{p-1,1}) = \{0\}$, which, together with $E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}$, implies $H_1^p \cong \operatorname{coker}(d_{1,2}^{p-2,1})$. Thus,

Observation 4. If for some p, f^{p-1} is surjective and $H_2^p = \{0\}$, then

$$E_{1,2}^{p,0} = \ker(MSS_2^{p,0}) \bigoplus \operatorname{im}(d_{1,2}^{p-2,1}) \cong H_1^p \bigoplus E_{2,2}^{p,0}.$$

Now drop the assumption $H_2^p = \{0\}$ and instead assume that f^{p-1} and f^p are isomorphisms. As

$$E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}, \quad E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell},$$

we have

$$\ker(d_{1,2}^{p-2,1}) \cong \ker(d_{2,2}^{p-2,1}), \quad \operatorname{coker}(d_{1,2}^{p-2,1}) \cong \operatorname{coker}(d_{2,2}^{p-2,1}).$$

Thus, the five lemma and the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker(d_{1,2}^{p-2,1}) & \longrightarrow & E_{1,2}^{p,0} & \longrightarrow & \operatorname{coker}(d_{1,2}^{p-2,1}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle MSS_2} & & \downarrow{\scriptstyle MSS_2} & & \downarrow{\scriptstyle MSS_2} & & \\ 1 & \longrightarrow & \ker(d_{2,2}^{p-2,1}) & \longrightarrow & E_{2,2}^{p,0} & \longrightarrow & \operatorname{coker}(d_{2,2}^{p-2,1}) & \longrightarrow & 1 \end{array}$$

imply

Observation 5. If for some p, f^{p-1} and f^p are isomorphisms, then $E_{1,2}^{p,0} \cong E_{2,2}^{p,0}$.

The rest of this section aims to prove Observations 4 and 5 in full generality. The following Lemma 5.1.3 is a generalization of Observation 1.

Lemma 5.1.3. *For $r \geqslant 2$,*

(a) $MSS_r^{p,q}$ is injective for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}\backslash\{0\}$;

(b) $MSS_r^{p,q}$ is an isomorphism if $p \in \mathbb{Z}$ and $q \geqslant r-1$.

Proof. We prove these statements by induction on r. The base case $r = 2$ follows from the assumptions.

Suppose that (a) and (b) hold for $r = R \geqslant 2$. Consider the case $r = R+1$. The following Claims 1 and 2 follow directly from the induction hypothesis and Lemma 5.1.1.

Claim 1. *For all $p \in \mathbb{Z}$ and $q \geqslant 1$, $MSS_R^{p,q}$ maps $\ker(d_{1,R}^{p,q})$ injectively into $\ker(d_{2,R}^{p,q})$. If $q \geqslant R$, then $MSS_R^{p,q}$ maps $\ker(d_{1,R}^{p,q})$ isomorphically onto $\ker(d_{2,R}^{p,q})$.*

Claim 2. *For all $q \geqslant R$, $MSS_R^{p+R,q-R+1}$ maps $\operatorname{im}(d_{1,R}^{p,q})$ isomorphically onto $\operatorname{im}(d_{2,R}^{p,q})$.*

(a) and (b) are immediate consequences of Claims 1,2 and Lemma 5.1.1. □

Fix $p \geqslant 2$. Note that for all $r \geqslant 2$, $d_{1,r}^{p,0}$ is a map from $E_{1,r}^{p,0}$ to $E_{1,r}^{p+r,1-r} = \{0\}$. It follows that $\ker(d_{1,r}^{p,0}) = E_{1,r}^{p,0}$ and thus $E_{1,r+1}^{p,0}$ is a quotient of $E_{1,r}^{p,0}$. Similarly, $E_{2,r+1}^{p,0}$ is a quotient of $E_{2,r}^{p,0}$ for all $r \geqslant 2$.

For $r = 2, ..., p+1$, let

$$Q_{1,r} : E_{1,r}^{p,0} \to E_{1,r+1}^{p,0}, \quad Q_{2,r} : E_{2,r}^{p,0} \to E_{2,r+1}^{p,0}$$

be the corresponding quotient maps.

To simplify notations, we also let $Q_{1,1} : E_{1,2}^{p,0} \to E_{1,2}^{p,0}, Q_{2,1} : E_{2,2}^{p,0} \to E_{2,2}^{p,0}$ be the identity maps. For $r = 1, ..., p+1$, let $CQ_{1,r}$ (resp. $CQ_{2,r}$) be the composition of $Q_{1,i}$ (resp. $Q_{2,i}$) for $1 \leqslant i \leqslant r$, i.e.,

$$CQ_{1,r} = Q_{1,r} \circ \cdots Q_{1,1} : E_{1,2}^{p,0} \to E_{1,r+1}^{p,0}, \quad CQ_{2,r} = Q_{2,r} \circ \cdots Q_{2,1} : E_{2,2}^{p,0} \to E_{2,r+1}^{p,0}.$$

Remark 5.1.4. For $r = 2, ..., p+1$, $Q_{1,r}$ (resp. $Q_{2,r}$) is the cohomology map sending every $x \in E_{1,r}^{p,0}$ (resp. $y \in E_{2,r}^{p,0}$) to the cohomology class in $E_{1,r+1}^{p,0}$ (resp. $E_{2,r+1}^{p,0}$) represented by x (resp. y). Thus,

$$\ker(Q_{1,r}) = \operatorname{im}(d_{1,r}^{p-r,r-1}), \qquad \ker(Q_{2,r}) = \operatorname{im}(d_{2,r}^{p-r,r-1}),$$
$$MSS_{r+1}^{p,0} \circ Q_{1,r} = Q_{2,r} \circ MSS_r^{p,0}, \qquad MSS_{r+1}^{p,0} \circ CQ_{1,r} = CQ_{2,r} \circ MSS_2^{p,0}.$$

Lemma 5.1.5.

(a) If $MSS_{r+2}^{p-r-1,r} : E_{1,r+2}^{p-r-1,r} \to E_{2,r+2}^{p-r-1,r}$ is surjective for $r = 1, ..., p-1$, then $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ injectively into $\ker(MSS_{p+2}^{p,0})$.

(b) If $E_{2,p+2}^{p,0} = \{0\}$, then $MSS_2^{p,0}$ maps $\ker(CQ_{1,p+1})$ surjectively onto $E_{2,2}^{p,0}$.

Proof.

(a) By Remark 5.1.4, $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ into $\ker(MSS_{p+2}^{p,0})$. It remains to prove that $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ injectively into $E_{1,p+2}^{p,0}$. Suppose that this is not true. As $CQ_{1,p+1}$ is the composition of $Q_{1,r}$, there exists $1 \leqslant r \leqslant p$ such that $Q_{1,r+1}$ does not map $CQ_{1,r}(\ker(MSS_2^{p,0}))$ injectively into $E_{1,r+2}^{p,0}$. We prove that $MSS_{r+2}^{p-r-1,r}$ is not surjective, which contradicts our assumption. By Lemma 5.1.3, $MSS_{r+1}^{p-2r-2,2r}$ is an isomorphism. It follows that

$$\begin{aligned}
&MSS_{r+1}^{p-r-1,r}(\operatorname{im}(d_{1,r+1}^{p-2r-2,2r})) \\
=\,&\operatorname{im}(MSS_{r+1}^{p-r-1,r} \circ d_{1,r+1}^{p-2r-2,2r}) \\
=\,&\operatorname{im}(d_{2,r+1}^{p-2r-2,2r} \circ MSS_{r+1}^{p-2r-2,2r}) && \text{as } MSS \text{ is a morphism of spectral sequences} \\
=\,&\operatorname{im}(d_{2,r+1}^{p-2r-2,2r}) && \text{as } MSS_{r+1}^{p-2r-2,2r} \text{ is an isomorphism.}
\end{aligned}$$

In view of Lemma 5.1.1, it suffices to show

$$MSS_{r+1}^{p-r-1,r}(\ker(d_{1,r+1}^{p-r-1,r})) \neq \ker(d_{2,r+1}^{p-r-1,r}).$$

By the Remark 5.1.4, we have $\ker(Q_{1,r+1}) = \mathrm{im}(d_{1,r+1}^{p-r-1,r})$. This, together with the assumption that $Q_{1,r+1}$ does not map $CQ_{1,r}(\ker(MSS_2^{p,0}))$ injectively into $E_{1,r+2}^{p,0}$, implies

$$CQ_{1,r}(\ker(MSS_2^{p,0})) \cap \mathrm{im}(d_{1,r+1}^{p-r-1,r}) \neq \{0\}. \tag{5.1}$$

Let W be the preimage of $CQ_{1,r}(\ker(MSS_2^{p,0}))$ under $d_{1,r+1}^{p-r-1,r}$. Note that

$$\begin{aligned}
& d_{2,r+1}^{p-r-1,r} \circ MSS_{r+1}^{p-r-1,r}(W) \\
=& MSS_{r+1}^{p,0} \circ d_{1,r+1}^{p-r-1,r}(W) && \text{as } MSS \text{ is a morphism of spectral sequences} \\
\subset& MSS_{r+1}^{p,0} \circ CQ_{1,r}(\ker(MSS_2^{p,0})) \\
=& CQ_{2,r} \circ MSS_2^{p,0}(\ker(MSS_2^{p,0})) && \text{by Remark 5.1.4} \\
=& \{0\}.
\end{aligned}$$

Thus,

$$MSS_{r+1}^{p-r-1,r}(W) \subset \ker(d_{2,r+1}^{p-r-1,r}).$$

(5.1) implies

$$\ker(d_{1,r+1}^{p-r-1,r}) \subsetneq W.$$

By Lemma 5.1.3, $MSS_{r+1}^{p-r-1,r}$ is injective. Thus,

$$MSS_{r+1}^{p-r-1,r}(\ker(d_{1,r+1}^{p-r-1,r})) \subsetneq MSS_{r+1}^{p-r-1,r}(W) \subset \ker(d_{2,r+1}^{p-r-1,r}).$$

(b) Suppose, for the contrary, that

$$MSS_2^{p,0}(\ker(CQ_{1,p+1})) \neq E_{2,2}^{p,0}.$$

Compare the following two sequences:

$$\{MSS^{p,0}_{r+1} \circ CQ_{1,r}(\ker(CQ_{1,p+2}))\}^{p+1}_{r=1}, \qquad \{E^{p,0}_{2,r+1}\}^{p+1}_{r=1}.$$

Note that

$$MSS^{p,0}_{2} \circ CQ_{1,1}(\ker(CQ_{1,p+1})) = MSS^{p,0}_{2}(\ker(CQ_{1,p+1})) \neq E^{p,0}_{2,2},$$

but

$$MSS^{p,0}_{p+2} \circ CQ_{1,p+1}(\ker(CQ_{1,p+1})) = E^{p,0}_{2,p+2} = \{0\}.$$

Thus, there exists $1 \leqslant r \leqslant p$ such that

$$MSS^{p,0}_{r+1} \circ CQ_{1,r}(\ker(CQ_{1,p+1})) \neq E^{p,0}_{2,r+1}, \tag{5.2}$$

$$MSS^{p,0}_{r+2} \circ CQ_{1,r+1}(\ker(CQ_{1,p+1})) = E^{p,0}_{2,r+2}. \tag{5.3}$$

Let $x \in E^{p,0}_{2,r+1}$. Then $Q_{2,r+1}(x) \in E^{p,0}_{2,r+2}$. By (5.3), there exists $y \in \ker(CQ_{1,p+2})$ such that

$$MSS^{p,0}_{r+2} \circ CQ_{1,r+1}(y) = Q_{2,r+1}(x).$$

Note that

$$\begin{aligned}
0 =& MSS^{p,0}_{r+2} \circ CQ_{1,r+1}(y) - Q_{2,r+1}(x) \\
=& MSS^{p,0}_{r+2} \circ Q_{1,r+1} \circ CQ_{1,r}(y) - Q_{2,r+1}(x) && \text{by the definition of } CQ_{1,r+1} \\
=& Q_{2,r+1} \circ MSS^{p,0}_{r+1} \circ CQ_{1,r}(y) - Q_{2,r+1}(x) && \text{by Remark 5.1.4} \\
=& Q_{2,r+1}(MSS^{p,0}_{r+1} \circ CQ_{1,r}(y) - x).
\end{aligned}$$

In other words,

$$MSS^{p,0}_{r+1} \circ CQ_{1,r}(y) - x \in \ker(Q_{2,r+1}).$$

By Remark 5.1.4, there exists $z \in E_{2,r+1}^{p-r-1,r}$ such that

$$d_{2,r+1}^{p-r-1,r}(z) = MSS_{r+1}^{p,0} \circ CQ_{1,r}(y) - x.$$

By Lemma 5.1.3, $MSS_{r+1}^{p-r-1,r}$ is an isomorphism. Thus, there exists $t \in E_{1,r+1}^{p-r-1,r}$ such that $MSS_{r+1}^{p-r-1,r}(t) = z$. By Remark 5.1.4 again,

$$d_{1,r+1}^{p-r-1,r}(t) \in \ker(Q_{1,r+1}) \subset CQ_{1,r}(\ker(CQ_{1,p+1})).$$

Thus,

$$\begin{aligned} x =& MSS_{r+1}^{p,0} \circ CQ_{1,r}(y) + d_{2,r+1}^{p-r-1,r}(z) \\ =& MSS_{r+1}^{p,0} \circ CQ_{1,r}(y) + d_{2,r+1}^{p-r-1,r} \circ MSS_{r+1}^{p-r-1,r}(t) \\ =& MSS_{r+1}^{p,0} \circ CQ_{1,r}(y) + MSS_{r+1}^{p,0} \circ d_{1,r+1}^{p-r-1,r}(t) \\ =& MSS_{r+1}^{p,0}(CQ_{1,r}(y) + d_{1,r+1}^{p-r-1,r}(t)) \\ \in& MSS_{r+1}^{p,0} \circ CQ_{1,r}(\ker(CQ_{1,p+1})). \end{aligned}$$

As x is arbitrary, we have

$$MSS_{r+1}^{p,0} \circ CQ_{1,r}(\ker(CQ_{1,p+1})) = E_{2,r+1}^{p,0},$$

contradicting (5.2).

□

Lemma 5.1.6. *Let $r \in \{0, ..., p-1\}$ and let $R \geqslant r+2$. If $MSS_{R+1}^{p-r-1,r}$ is surjective, then $MSS_{R}^{p-r-1,r}$ is also surjective.*

Proof. Suppose that $MSS_{R}^{p-r-1,r}$ is not surjective. Note that the target of $d_{1,R}^{p-r-1,r}$ is $E_{1,R}^{p+R-r-1,r-R+1} = \{0\}$. Thus,

$$\ker(d_{1,R}^{p-r-1,r}) = E_{1,R}^{p-r-1,r}.$$

Similarly,

$$\ker(d_{2,R}^{p-r-1,r}) = E_{2,R}^{p-r-1,r}.$$

As $MSS_R^{p-r-1,r}$ is not surjective, we have

$$MSS_R^{p-r-1,r}(\ker(d_{1,R}^{p-r-1,r})) \neq \ker(d_{2,R}^{p-r-1,r}). \tag{5.4}$$

By Lemma 5.1.3, $MSS_R^{p-r-R-1,r+R-1}$ is an isomorphism. It follows that

$$\begin{aligned}
&MSS_R^{p-r-1,r}(\operatorname{im}(d_{1,R}^{p-r-R-1,r+R-1})) \\
=&\operatorname{im}(MSS_R^{p-r-1,r} \circ d_{1,R}^{p-r-R-1,r+R-1}) \\
=&\operatorname{im}(d_{2,R}^{p-r-R-1,r+R-1} \circ MSS_R^{p-r-R-1,r+R-1}) \quad \text{as } MSS \text{ is a morphism of spectral sequences} \\
=&\operatorname{im}(d_{2,R}^{p-r-R-1,r+R-1}) \quad \text{as } MSS_R^{p-r-R-1,r+R-1} \text{ is an isomorphism.}
\end{aligned} \tag{5.5}$$

(5.4), (5.5), and Lemma 5.1.1 imply that $MSS_{R+1}^{p-r-1,r}$ is not surjective, contradicting our assumption. □

Let us further suppose that

$$E_{1,2}^{p,q} \Rightarrow H_1^{p+q}, \quad E_{2,2}^{p,q} \Rightarrow H_2^{p+q}$$

for some graded abelian groups $H_1 = \bigoplus_{\ell \geqslant 0} H_1^\ell, H_2 = \bigoplus_{\ell \geqslant 0} H_2^\ell$ and there is a degree-0 morphism $f : H_1 \to H_2$ compactible with MSS. The following Lemmas 5.1.7 and 5.1.8 are generalizations of Observations 4 and 5, respectively.

Lemma 5.1.7. *If f^{p-1} is surjective and $H_2^p = \{0\}$, then $MSS_2^{p,0}$ is surjective with $\ker(MSS_2^{p,0}) \cong H_1^p$. Moreover,*

$$E_{1,2}^{p,0} \cong E_{2,2}^{p,0} \bigoplus H_1^p.$$

Proof. If $p \leqslant -1$, then $E_{1,2}^{p,0} = E_{2,2}^{p,0} = \{0\}$. If $p = 0$, then $E_{1,2}^{0,0} \cong H_1^0$ as $E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}$, and $E_{2,2}^{0,0} \cong H_2^0 = \{0\}$ as $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$. Thus, the lemma holds in these two cases.

Suppose $p = 1$. By assumption, $H_2^1 = \{0\}$. It follows from Remark 2.15.9 and $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$ that

$$E_{2,3}^{0,1} = E_{2,3}^{1,0} = \{0\}.$$

The same argument as the one in Remark 2.15.9 shows

$$E_{2,2}^{1,0} = E_{2,3}^{1,0} = \{0\}.$$

By Lemma 5.1.3, $MSS_3^{1,0}$ maps $E_{1,3}^{0,1}$ injectively into $E_{2,3}^{0,1}$ and thus $E_{1,3}^{0,1} = \{0\}$. Therefore,

$E_{1,2}^{1,0}$	
$=E_{1,3}^{1,0}$	by the same argument as the one in Remark 2.15.9
$\cong H_1^1$	by $E_{1,3}^{0,1} = \{0\}, E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell},$ and Remark 2.15.9
$\cong E_{2,2}^{1,0} \bigoplus H_1^1$	as $E_{2,2}^{1,0} = \{0\}$.

Let us assume $p \geqslant 2$. As f^{p-1} is surjective and MSS is compatible with f, Remark 2.15.11 implies that for $r = 1, ..., p-1, MSS_{p+1}^{p-r-1,r}$ is surjective. By succesively applying Lemma 5.1.6, we see that $MSS_{r+2}^{p-r-1,r}$ is also surjective. It follows from Lemma 5.1.5 that $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ injectively into $E_{1,p+2}^{p,0}$. Thus,

$$\ker(CQ_{1,p+1}) \cap \ker(MSS_2^{p,0}) = \{0\}. \tag{5.6}$$

By Remark 2.15.9, $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$, and $H_2^p = \{0\}$, we have $E_{2,p+2}^{p,0} = \{0\}$. It follows from Lemma 5.1.5 that $MSS_2^{p,0}$ maps $\ker(CQ_{1,p+1})$ surjectively onto $E_{2,2}^{p,0}$. Together with (5.6), this implies

$$E_{1,2}^{p,0} = \ker(CQ_{1,p+1}) \bigoplus \ker(MSS_2^{p,0}) \tag{5.7}$$

and

$$\ker(CQ_{1,p+1}) \cong E_{2,2}^{p,0}.$$

We have already shown that $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ injectively into $E_{1,p+2}^{p,0}$. Thus, (5.7) implies that $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ isomorphically onto $E_{1,p+2}^{p,0}$.

For $r = 1, ..., p$, as $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$ and $H_2^p = \{0\}$, we have $E_{2,p+2}^{p-r,r} = \{0\}$ by Remark 2.15.9. By Lemma 5.1.3, $MSS_{p+2}^{p-r,r}$ maps $E_{1,p+2}^{p-r,r}$ injectively into $E_{2,p+2}^{p-r,r}$. Thus, $E_{1,p+2}^{p-r,r} = \{0\}$. As $E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}$, Remark 2.15.9 implies

$$H_1^p \cong E_{1,p+2}^{p,0} \cong \ker(MSS_2^{p,0}).$$

Therefore,

$$E_{1,2}^{p,0} \cong \ker(CQ_{1,p+2}) \bigoplus \ker(MSS_2^{p,0}) \cong E_{2,2}^{p,0} \bigoplus E_{1,p+2}^{p,0} \cong E_{2,2}^{p,0} \bigoplus H_1^p.$$

□

Lemma 5.1.8. *If f^{p-1} is surjective and f^p is an isomorphism, then $MSS_2^{p,0}$ is an isomorphism.*

Proof. If $p \leqslant 0$, then Remark 2.15.9, $E_{1,2}^{k,\ell} \Rightarrow H_1^{k+\ell}$, and $E_{2,2}^{k,\ell} \Rightarrow H_2^{k+\ell}$ imply

$$E_{1,2}^{p,0} \cong H_1^p, \quad E_{2,2}^{p,0} \cong H_2^p.$$

As f^p is an isomorphism and MSS is compatible with f^p, Remark 2.15.11 implies that $MSS_2^{p,0}$ is an isomorphism.

Let us suppose $p \geqslant 1$. As MSS is compatible with f and f^{p-1} is surjective, $MSS_{p+1}^{p-r-1,r}$ is surjective for $r = 0, ..., p-1$, by Remark 2.15.11. It follows from Lemma 5.1.6 that $MSS_{r+2}^{p-r-1,r}$ is surjective for $r = 0, ..., p-1$. By Lemma 5.1.5, $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ injectively into $\ker(MSS_{p+2}^{p,0})$.

By Remark 2.15.11 and the assumption that f^p is an isomorphism, $MSS_{p+2}^{p,0}$ is injective. Thus, $\ker(MSS_{p+2}^{p,0}) = \{0\}$. As $CQ_{1,p+1}$ maps $\ker(MSS_2^{p,0})$ injectively into $\ker(MSS_{p+2}^{p,0})$, we have $\ker(MSS_2^{p,0}) = \{0\}$, i.e., $MSS_2^{p,0}$ is injective.

By Remark 2.15.11 and the assumption that f^p is an isomorphism, $MSS_{p+2}^{p,0}$ is surjective. By successively applying Lemma 5.1.6 (with p in place of $p-1$ in part (a)), we see that $MSS_2^{p,0}$ is surjective and thus is an isomorphism. □

5.2 Cohomology of Dehn filling quotients

Theorem 5.2.1. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple satisfying the Cohen-Lyndon property. Employ the notations defined in Notation 2.14.2 and let A be a $\mathbb{Z}\overline{G}$-module. Suppose that for some $p \in \mathbb{N}$, $\prod_{\lambda\in\Lambda} H^p(H_\lambda; A) = \{0\}$ and NTR_G maps $H^{p-1}(G; A)$ surjectively onto $\prod_{\lambda\in\Lambda} H^{p-1}(H_\lambda; A)$. Then $NT_{\overline{G}}^p$ is surjective with $\ker(NT_{\overline{G}}^p) \cong H^p(G; A)$. Moreover,*

$$H^p(\overline{G}; A) \cong \left(\prod_{\lambda\in\Lambda} H^p(\overline{H}_\lambda; A)\right) \bigoplus H^p(G; A). \tag{5.8}$$

Proof. Let MSS be as in Theorem 4.0.1. Note that MSS and NTR_G satisfy the assumptions of Lemma 5.1.7, which yields (5.8) and shows that $MSS_2^{p,0}$ is surjective. By Theorem 4.0.1, $MSS_2^{p,0}$ can be identified with $NT^p_{\overline{G}}$ and thus $NT^p_{\overline{G}}$ is surjective. □

Recall that for a group G, the *cohomological dimension of G* is

$$cd(G) = \sup\{\ell \in \mathbb{N} \mid H^\ell(G, A) \neq \{0\} \text{ for some } \mathbb{Z}G\text{-module } A\}.$$

If $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ is a group triple, let

$$cd(\mathcal{H}) = \sup_{\lambda\in\Lambda}\{cd(H_\lambda)\}, \quad cd(\overline{\mathcal{H}}) = \sup_{\lambda\in\Lambda}\{cd(\overline{H}_\lambda)\}.$$

Also recall the following result of [7] concerning relative cohomology groups.

Proposition 5.2.2 ([7, Proposition 1.1]). *Let G be a group with a family of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$. Then for every $\mathbb{Z}G$-module A, there is a long exact sequence*

$$\cdots \to H^\ell(G, \{H_\lambda\}_{\lambda\in\Lambda}; A) \to H^\ell(G; A) \xrightarrow{NTR_G} \prod_{\lambda\in\Lambda} H^\ell(H_\lambda; A) \to H^{\ell+1}(G, \{H_\lambda\}_{\lambda\in\Lambda}; A) \to \cdots$$

where NTR_G is the natural map defined in Notation 2.14.2.

Corollary 5.2.3. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple satisfying the Cohen-Lyndon property. Then for all $\ell \geqslant cd(\mathcal{H}) + 3$ and every $\mathbb{Z}\overline{G}$-module A, there is an isomorphism*

$$H^\ell(G, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \cong H^\ell(G; A).$$

For $\ell = cd(\mathcal{H}) + 2$, there is a surjection $H^\ell(G, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \twoheadrightarrow H^\ell(G; A)$.

Proof. By Proposition 5.2.2, there is a long exact sequence

$$\cdots \to H^\ell(\overline{G}, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \to H^\ell(\overline{G}; A) \xrightarrow{NT^\ell_{\overline{G}}} \prod_{\lambda\in\Lambda} H^\ell(\overline{H}_\lambda; A) \to H^{\ell+1}(\overline{G}, \{\overline{H}_\lambda\}_{\lambda\in\Lambda}; A) \to \cdots$$

By Theorem 5.2.4, if $\ell \geqslant cd(\mathcal{H}) + 2$, then $NT^\ell_{\overline{G}}$ is surjective and $\ker(NT^\ell_{\overline{G}}) \cong H^\ell(G; A)$, which implies the desired result. □

Corollary 5.2.4. *Let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple satisfying the Cohen-Lyndon property. Then*

$$cd(\overline{G}) \leqslant \max\{cd(G), cd(\mathcal{H})+1, cd(\overline{\mathcal{H}})\}.$$

Proof. If $cd(\overline{G}) \leqslant cd(\mathcal{H})+1$, then the desired conclusion already holds. Thus, let us assume that $cd(G) \geqslant cd(\mathcal{H})+2$. Let $\ell \geqslant cd(\mathcal{H})+2$, and let NTR_G be the natural map defined by Notation 2.14.2, then $\prod_{\lambda\in\Lambda} H^\ell(H_\lambda; A) = \{0\}$ and NTR_G maps $H^{\ell-1}(G; A)$ surjectively onto $\prod_{\lambda\in\Lambda} H^{\ell-1}(H_\lambda; A) = \{0\}$. It follows from Theorem 5.2.1 that

$$H^\ell(\overline{G}; A) \cong \left(\prod_{\lambda\in\Lambda} H^\ell(\overline{H}_\lambda; A)\right) \bigoplus H^\ell(G; A),$$

which implies $cd(\overline{G}) \leqslant \max\{cd(G), cd(\overline{\mathcal{H}})\}$. □

Proof of Theorem 1.2.15. By Theorem 3.0.1, for sufficiently deep $N \lhd H$, the group triple (G, H, N) has the Cohen-Lyndon property. Thus, Theorem 1.2.15 follows from the case $|\Lambda| = 1$ of Theorem 5.2.1 and Corollary 5.2.4. □

Our next result concerns another finiteness property. Recall that a group G is of type FP_∞ if there is a projective resolution $P \to \mathbb{Z}$ over $\mathbb{Z}G$ such that P_ℓ is finitely generated for $\ell \in \mathbb{N}$. Also recall the following characterization of FP_∞.

Theorem 5.2.5 ([10, Chapter VIII Theorem 4.8] (see [9, Theorem 3] for a proof)). *A group G is of type FP_∞ if and only if $H^*(G; \cdot)$ preserves direct limits.*

Lemma 5.2.6. *Let F be a free group of finite rank, let N be a normal subgroup of F, let $\overline{F} = F/N$, let $\{A_i\}_{i\in I}$ be a directed system of $\mathbb{Z}\overline{F}$-modules, and let $A = \varinjlim A_i$. If $\overline{F}$ is of type FP_∞, then for $p, q \in \mathbb{Z}$, the natural maps $A_i \to A, i \in I$, induce an isomorphism*

$$\varinjlim H^p(\overline{F}; H^q(N; A_i)) \cong H^p(\overline{F}; H^q(N; A)). \tag{5.9}$$

Proof. Let

$$E_2^{p,q} = H^p(\overline{F}; H^q(N; A)) \Rightarrow H^{p+q}(F; A)$$

be the LHS spectral sequence for the triple (F, N, A). For $i \in I$, let

$$E_{i,2}^{p,q} = H^p(\overline{F}; H^q(N; A_i)) \Rightarrow H^{p+q}(F; A_i)$$

be the LHS spectral sequence for the triple (F, N, A_i).

Being a subgroup of the free group F, N is also free. By the Stallings-Swan theorem [32, Corollary to Theorem 1], $cd(N) \leqslant 1$. It follows that

(CD1) $E_{i,2}^{p,q} = E_2^{p,q} = \{0\}$ whenever $q \notin \{0, 1\}$.

Thus, if $q \notin \{0, 1\}$, then both sides of (5.9) are $\{0\}$. Therefore, it suffices to prove (5.9) for $q \in \{0, 1\}$.

Note that if $p \leqslant -1$, then both sides of (5.9) are $\{0\}$, and if $q = 0$, then (5.9) follows from Theorem 5.2.5 as there are natural isomorphisms

$$H^0(N; A) \cong A, \quad H^0(N; A_i) \cong A_i, \text{ for } i \in I.$$

Thus, it suffices to prove (5.9) for $p \geqslant 0$ and $q = 1$.

By Proposition 4.3.4, the maps $A_i \to A$, $i \in I$, induce morphisms

$$MSS_i : E_i \to E$$

between spectral sequences. For $i \in I$ and $p \in \mathbb{Z}$, Proposition 4.3.4 implies that the map

$$MSS_{i,2}^{p,1} : E_{i,2}^{p,1} \to E_2^{p,1}$$

can be identified with the natural map

$$H^p(\overline{F}; H^1(N; A_i)) \to H^p(\overline{F}; H^1(N; A))$$

induced by $A_i \to A$. It suffices to show that for $p \geqslant 0$,

$$\varinjlim MSS_{i,2}^{p,1} : \varinjlim H^p(\overline{F}; H^1(N; A_i)) \to H^p(\overline{F}; H^1(N; A)). \tag{5.10}$$

is an isomorphism.

Fix $p \geqslant 0$. We have the following commutative diagram.

$$\begin{array}{ccc} E_{i,2}^{p,1} & \xrightarrow{MSS_i^{p,1}} & E_2^{p,1} \\ \Big\downarrow{\scriptstyle d_{i,2}^{p,1}} & & \Big\downarrow{\scriptstyle d_2^{p,1}} \\ E_{i,2}^{p+2,0} & \xrightarrow{MSS_i^{p+2,0}} & E_2^{p+2,0} \end{array} \tag{5.11}$$

Note that $H^{p+2}(F; A) = \{0\}$. As $E_2^{k,\ell} \Rightarrow H^{k+\ell}(F; A)$, we have $E_r^{p+2,0} = \{0\}$ for sufficiently large r. By (CD1) and the definition of spectral sequences, $E_r^{p+2,0} = E_3^{p+2,0}$ for all $r \geqslant 3$. Thus, $E_3^{p+2,0} = \{0\}$ and, as a consequence, $d_2^{p,1}$ is surjective. Similarly, $d_{i,2}^{p,1}$ is surjective.

If $p \geqslant 1$, then as $H^{p+1}(F; A) = \{0\}$ and $E_2^{k,\ell} \Rightarrow H^{k+\ell}(F; A)$, we have $E_r^{p,1} = \{0\}$ for sufficiently large r. By (CD1), $E_r^{p,1} = E_3^{p,1}$ for all $r \geqslant 3$. Thus, $E_3^{p,1} = \{0\}$. Using (CD1) once again, we see that $d_2^{p,1}$ is injective and thus is an isomorphism. Similarly, $d_{i,2}^{p,1}$ is an isomorphism.

Taking direct limit of (5.11), we obtain

$$\begin{array}{ccc} \varinjlim E_{i,2}^{p,1} & \longrightarrow & E_2^{p,1} \\ \Big\downarrow & & \Big\downarrow \\ \varinjlim E_{i,2}^{p+2,0} & \longrightarrow & E_2^{p+2,0} \end{array} \tag{5.12}$$

By Theorem 5.2.5, the lower horizontal map of (5.12) is an isomorphism. Being direct limits of isomorphisms, the vertical maps of (5.12) are isomorphisms. Thus, the upper horizontal map of (5.12) is also an isomorphism, which proves that $\varinjlim MSS_{i,2}^{p,1}$ is an isomorphism for $p \geqslant 1$.

Suppose $p = 0$. Then $d_{i,2}^{p,1} = d_{i,2}^{0,1}$ and $d_2^{p,1} = d_2^{0,1}$ are not necessarily injective. Let $\ker_i$ (resp. $\ker$) be the kernel of $d_{i,2}^{0,1}$ (resp. $d_2^{0,1}$). By Remark 2.15.9 and $E_2^{k,\ell} \Rightarrow H^{k+\ell}(F; A)$, there is an exact sequence

$$1 \to E_3^{1,0} \to H^1(F; A) \to \ker \to 1. \tag{5.13}$$

By the same argument, we see that there is an exact sequence similar to (5.13) holds for every $i \in I$. As

$E_{i,2}^{-1,1} = E_2^{-1,1}$=$\{0\}$, we have

$$E_{i,3}^{1,0} = E_{i,2}^{1,0}, \quad E_3^{1,0} = E_2^{1,0}.$$

Combining these observations, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & E_{i,2}^{1,0} & \longrightarrow & H^1(F;A_i) & \longrightarrow & \ker_i & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle MSS_{i,2}^{1,0}} & & \downarrow & & \downarrow{\scriptstyle MSS_{i,2}^{0,1}} & & \\ 1 & \longrightarrow & E_2^{1,0} & \longrightarrow & H^1(F;A) & \longrightarrow & \ker & \longrightarrow & 1 \end{array} \tag{5.14}$$

By taking direct limit of (5.14) and using the fact that $\varinjlim$ is an exact functor, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varinjlim E_{i,2}^{1,0} & \longrightarrow & \varinjlim H^1(F;A_i) & \longrightarrow & \varinjlim \ker_i & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & E_2^{1,0} & \longrightarrow & H^1(F;A) & \longrightarrow & \ker & \longrightarrow & 1 \end{array} \tag{5.15}$$

As F has finite rank, F is of type FP_∞. By Theorem 5.2.5, the first and the second vertical maps of (5.15) are isomorphisms. Thus, the five lemma implies that the last vertical map of (5.15) is also an isomorphism.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker_i & \longrightarrow & E_{i,2}^{0,1} & \xrightarrow{d_{i,2}^{0,1}} & E_{i,2}^{2,0} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle MSS_{i,2}^{0,1}} & & \downarrow{\scriptstyle MSS_{i,2}^{0,1}} & & \downarrow{\scriptstyle MSS_{i,2}^{2,0}} & & \\ 1 & \longrightarrow & \ker & \longrightarrow & E_2^{0,1} & \xrightarrow{d_2^{0,1}} & E_2^{2,0} & \longrightarrow & 1 \end{array} \tag{5.16}$$

By taking direct limit of (5.16) and using the fact that $\varinjlim$ is an exact functor, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \varinjlim \ker_i & \longrightarrow & \varinjlim E_{i,2}^{0,1} & \longrightarrow & \varinjlim E_{i,2}^{2,0} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \ker & \longrightarrow & E_2^{0,1} & \longrightarrow & E_2^{2,0} & \longrightarrow & 1 \end{array} \tag{5.17}$$

We have already proved that the first vertical map of (5.17) is an isomorphism. By Theorem 5.2.5 and the assumption that $\overline{F}$ is of type FP_∞, the last vertical map of (5.17) is an isomorphism. Thus, the five lemma implies that the second vertical map of (5.17) is also an isomorphism, which proves that $\varinjlim MSS_{i,2}^{0,1}$ is an isomorphism. □

Lemma 5.2.7. *Let K be a finite group, let F be free group of finite rank, let $H = K \times F$, let N be a normal subgroup of F, let $\overline{H} = H/N$, let $\{A_i\}_{i\in I}$ be a directed system of $\mathbb{Z}\overline{H}$-modules, and let $A = \varinjlim A_i$. If $\overline{H}$ is of type FP_∞, then for $p, q \in \mathbb{Z}$, the natural maps $A_i \to A$ induce an isomorphism*

$$\varinjlim H^p(\overline{H}; H^q(N; A_i)) \cong H^p(\overline{H}; H^q(N; A)).$$

Proof. Note that $\overline{F} = F/N$ has finite index in $\overline{H}$ and thus $\overline{F}$ is of type FP_∞. Fix $p, q \in \mathbb{Z}$. Lemma 5.2.6 asserts that the natural maps $A_i \to A$ induce an isomorphism

$$\varinjlim H^p(\overline{F}; H^q(N; A_i)) \cong H^p(\overline{F}; H^q(N; A)). \tag{5.18}$$

Notice that $\overline{F} \lhd \overline{H}$ and $\overline{H}/\overline{F} \cong K$ is a finite group. In particular, $\overline{H}/\overline{F}$ is of type FP_∞. For $i \in I$, let E_i the LHS spectral sequence for the triple $(\overline{H}, \overline{F}, H^q(N, A_i))$. Let E be the LHS spectral sequence for the triple $(\overline{H}, \overline{F}, H^q(N, A))$, let $\varinjlim E_i$ be the direct limit of $\{E_i\}_{i\in I}$, and let $MSS : \varinjlim E_i \to E$ be the morphism induced by $A_i \to A, i \in I$. Then

$$\begin{aligned} & E_2^{k,\ell} \\ \cong & H^k(\overline{H}/\overline{F}; H^\ell(\overline{F}; H^q(N; A))) \\ \cong & H^k(\overline{H}/\overline{F}; \varinjlim H^\ell(\overline{F}; H^q(N; A_i))) && \text{by (5.18)} \\ \cong & \varinjlim H^k(\overline{H}/\overline{F}; H^\ell(\overline{F}; H^q(N; A_i))) && \text{as } \overline{H}/\overline{F} \text{ is of type } FP_\infty \end{aligned} \tag{5.19}$$

$$\cong \varinjlim E_{2,i}^{k,\ell}.$$

The isomorphisms involved above are natural maps. Thus, $MSS_2 : \varinjlim E_{2,i} \to E_2$ is an isomorphism of bigraded abelian groups. It follows that MSS is an isomorphism between spectral sequences. As $E_{i,2}^{k,\ell} \Rightarrow H^{k+\ell}(\overline{H}; H^q(N; A_i))$ and $E_2^{k,\ell} \Rightarrow H^{k+\ell}(\overline{H}; H^q(N; A))$, (5.19) and Lemmas 2.15.12, 2.15.18 imply the desired result. □

Theorem 5.2.8. *Let Λ be a finite index set and let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple satisfying the Cohen-Lyndon property. Employ the notations defined in Notation 2.14.2. Suppose that $G, \overline{H}_\lambda, \lambda \in \Lambda$, are of type FP_∞. If, for each $\lambda \in \Lambda$, either one of the following conditions holds, then $\overline{G}$ is of type FP_∞.*

(F_1) N_λ is of type FP_∞.

(F_2) H_λ is of the form $K_\lambda \times F_\lambda$, where K_λ is finite and F_λ is a finite rank free group, and $N_\lambda \leqslant F_\lambda$.

Proof. By Theorem 5.2.5, it suffices to prove that the functor $H^*(\overline{G}; \cdot)$ preserves direct limits. Let $\{A_i\}_{i\in I}$ be a directed system of $\mathbb{Z}\overline{G}$-modules and let $A = \varinjlim A_i$. For $i \in I$, let $E_i = \{(E_{i,r}, d_{i,r})\}_{r\geqslant 2}$ be the LHS spectral sequence for the triple $(G, \langle\langle \mathcal{N} \rangle\rangle, A_i)$. Let $E = \{(E_r, d_r)\}_{r\geqslant 2}$ be the direct limit of $\{E_i\}_{i\in I}$. Also let $E_A = \{(E_{A,r}, d_{A,r})\}_{r\geqslant 2}$ be the LHS spectral sequence for the triple $(G, \langle\langle \mathcal{N} \rangle\rangle, A)$.

By Lemma 2.15.18, $E_2^{p,q} \Rightarrow \varinjlim H^{p+q}(G; A_i)$. The maps $A_i \to A, i \in I$, induce

(a) a morphism

$$MSS : E \to E_A$$

between spectral sequences, by Proposition 4.3.4,

(b) a natural map

$$NA_G : \varinjlim H^*(G; A_i) \to H^*(G; A)$$

(c) a natural map

$$NA_{\overline{G}}^{p,q} : \varinjlim H^p(\overline{G}; H^q(\langle\langle \mathcal{N} \rangle\rangle; A_i)) \to H^p(\overline{G}; H^q(\langle\langle \mathcal{N} \rangle\rangle; A))$$

for $p, q \in \mathbb{Z}$.

As there are natural isomorphisms

$$H^0(\langle\!\langle \mathcal{N} \rangle\!\rangle; A) \cong A, \quad H^0(\langle\!\langle \mathcal{N} \rangle\!\rangle; A_i) \cong A_i, \text{ for } i \in I,$$

for $p \in \mathbb{Z}$, $NA^{p,0}_{\overline{G}}$ can be identified with the natural map $\varinjlim H^p(\overline{G}; A_i) \to H^p(\overline{G}; A)$ induced by the maps $A_i \to A, i \in I$. Thus, it suffices to show that $NA^{p,0}_{\overline{G}}$ is an isomorphism, which is done by using Lemma 5.1.8.

For $p \in \mathbb{Z}$ and $q \leqslant -1$, $NA^{p,q}_{\overline{G}}$ is clearly an isomorphism as it is just a map from $\{0\}$ to $\{0\}$.

Fix $p \in \mathbb{Z}$ and $q \geqslant 1$. Let $i, j \in I$ with $i < j$. Consider the following commutative diagram

$$\begin{array}{ccc}
H^p(\overline{G}; H^q(\langle\!\langle \mathcal{N} \rangle\!\rangle; A_i)) & \longrightarrow & H^p(\overline{G}; H^q(\langle\!\langle \mathcal{N} \rangle\!\rangle; A_j)) \\
\downarrow \cong & & \downarrow \cong \\
\prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A_i)) & \longrightarrow & \prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A_j))
\end{array} \tag{5.20}$$

where the horizontal maps are induced by $A_i \to A_j$, and the vertical isomorphisms are given by Proposition 4.2.1. Let i, j vary in I. (5.20) induces a commutative diagram corresponding to direct limits

$$\begin{array}{ccc}
\varinjlim H^p(\overline{G}; H^q(\langle\!\langle \mathcal{N} \rangle\!\rangle; A_i)) & \xrightarrow{NA^{p,q}_{\overline{G}}} & H^p(\overline{G}; H^q(\langle\!\langle \mathcal{N} \rangle\!\rangle; A)) \\
\downarrow \cong & & \downarrow \cong \\
\varinjlim \prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A_i)) & \longrightarrow & \prod_{\lambda \in \Lambda} H^p(\overline{H}_\lambda; H^q(N_\lambda; A))
\end{array} \tag{5.21}$$

whose vertical maps, being direct limits of isomorphisms, are themselves isomorphisms.

Fix $\lambda \in \Lambda$. Consider the natural map

$$\varinjlim H^p(\overline{H}_\lambda; H^q(N_\lambda; A_i)) \to H^p(\overline{H}_\lambda; H^q(N_\lambda; A)) \tag{5.22}$$

induced by the maps $A_i \to A, i \in I$.

If (F_1) holds for λ, then Theorem 5.2.5 implies that (5.22) is an isomorphism.

If (F_2) holds for λ, then Lemma 5.2.7 implies that (5.22) is an isomorphism.

Let λ vary in Λ. By taking direct product of (5.22), we obtain an isomorphism

$$\prod_{\lambda\in\Lambda} \varinjlim H^p(\overline{H}_\lambda, H^q(N_\lambda, A_i)) \cong \prod_{\lambda\in\Lambda} H^p(\overline{H}_\lambda, H^q(N_\lambda, A)). \tag{5.23}$$

As $|\Lambda| < \infty$, the operations $\prod_{\lambda\in\Lambda}$ and $\varinjlim$ commute with each other and thus isomorphism (5.23) implies that the lower horizontal map of (5.21) is an isomorphism. By Proposition 4.3.4, MSS and NA_G are compatible and for $p, q \in \mathbb{Z}$, $MSS_2^{p,q}$ can be identified with $NA_{\overline{G}}^{p,q}$. As G is of type FP_∞, Theorem 5.2.5 implies that NA_G is an isomorphism. Thus, Lemma 5.1.8 implies that $NA_{\overline{G}}^{p,0}$ is an isomorphism for all $p \in \mathbb{Z}$. □

Recall that a group G is of type FP if (a) $cd(G) < \infty$ and (b) G is of type FP_∞. The following corollary follows from Corollary 5.2.4 and Theorem 5.2.8.

Corollary 5.2.9. *Let Λ be a finite index set and let $(G, \{H_\lambda\}_{\lambda\in\Lambda}, \{N_\lambda\}_{\lambda\in\Lambda})$ be a group triple satisfying the Cohen-Lyndon property. Suppose that $G, \overline{H}_\lambda, \lambda \in \Lambda$, are of type FP. If, for each $\lambda \in \Lambda$, either one of the following conditions holds, then $\overline{G}$ also is of type FP.*

(F_1) N_λ is of type FP_∞.

(F_2) H_λ is of the form $K_\lambda \times F_\lambda$, where K_λ is finite and F_λ is a finite rank free group, and $N_\lambda \leqslant F_\lambda$.

Proof of Theorem 1.2.18. By Theorem 3.0.1, for sufficiently deep $N \lhd H$, the group triple (G, H, N) has the Cohen-Lyndon property. Thus, Theorem 1.2.18 follows from the case $|\Lambda| = 1$ of Theorem 5.2.8 and Corollary 5.2.9. □

5.3 Cohomology and embedding theorems

We prove Theorem 1.2.22 in this section. Given any acylindrically hyperbolic group G, G has a maximal finite normal subgroup $K(G)$ by Theorem 2.5.7. $G_0 = G/K(G)$ is again acylindrically hyperbolic [19, Lemma 5.10] and $K(G_0) = \{1\}$. By Theorem 2.5.7, there is a non-abelian free group $F \hookrightarrow_h G_0$. It is well-known that F is SQ-universal and thus given any countable group C, there is a normal subgroup $N \lhd F$ such that $C \hookrightarrow F/N$. The main idea of the proof of Theorem 1.2.22 is to choose a particular N so that all statements of Theorem 1.2.22 hold for $\overline{G} = G_0/\langle\langle N\rangle\rangle$, where $\langle\langle N\rangle\rangle$ is the normal closure of N in G_0.

Lemma 5.3.1. *Let F_3 be a free group of rank 3, let $\mathcal{F} \subset F_3$ be a finite set, and let C be a countable group with $cd(C) \geqslant 2$. Then C embeds into a quotient R of F_3 such that*

(1) $card(R) = \infty$;

(2) the natural homomorphism $F_3 \to R$ is injective on $\mathcal{F}$;

(3) $cd(R) \leqslant cd(C)$;

(4) for all $\ell \geqslant 3$ and any $\mathbb{Z}R$-module A, we have $H^\ell(R; A) \cong H^\ell(C; A)$;

(5) if C is finitely generated, then R is hyperbolic relative to C (for the definition of relative hyperbolicity, see [13, Definition 3.6]);

(6) if C is of type FP_∞, then so is R.

Remark 5.3.2. Except for assertions (3), (4), and (6), Lemma 5.3.1 is proved in [13, Lemma 8.4]. We refine the method of [13] so that we can impose homological conditions.

Proof. Let $\{x, y, t\}$ be a free basis of F_3, let $\{c_i\}_{i \in I}$ be a generating set of C, and let $w_i, v_i, i \in I$, be freely reduced words over the alphabet $\{x, y\}$ such that

(a) the words $c_i w_i, i \in I$, satisfy the $C'(1/2)$ small cancellation condition over the free product $\langle x\rangle * \langle y\rangle * C$;

(b) the words $v_i, i \in I$, satisfy the $C'(1/2)$ small cancellation condition over the alphabet $\{x, y\}$;

(c) the words $tc_i w_i t^{-1} v_i, i \in I$, satisfy the $C'(1/6)$ small cancellation condition over the free product $\langle x\rangle * \langle y\rangle * \langle t\rangle * C$.

Let N be the normal subgroup of $F_3 * C$ generated by $tc_i w_i t^{-1} v_i, i \in I$, and let

$$R = (F_3 * C)/N.$$

For $i \in I$, let $\bar{t}$ (resp. $\bar{c}_i, \overline{w}_i, \overline{v}_i$) be the image of t (resp. c_i, w_i, v_i) under the quotient map $F_3 * C \to R$. Note that $\bar{t}\bar{c}_i\overline{w}_i\bar{t}^{-1}\overline{v}_i = 1$ and we can rewrite this equation as $\bar{c}_i = \bar{t}^{-1}\overline{v}_i^{-1}\bar{t}\overline{w}_i^{-1}$. Thus, R is generated by $\bar{t}, \overline{w}_i, \overline{v}_i, i \in I$, and hence is a quotient of F_3. Let

$$Q : F_3 \to R$$

be the corresponding quotient map. We can also think of Q as the restriction of the quotient map $F_3 * C \to R$ to F_3. It follows from the Greendlinger's lemma for free products [24, Chapter V Theorem 9.3] that if $\|w_i\|, \|v_i\|, i \in I$, are sufficiently large, then Q is injective on $\mathcal{F}$ and thus (2) is guaranteed.

Let $L = \langle x \rangle * \langle y \rangle * C$, let $W \leqslant L$ be the subgroup generated by the elements $c_i w_i, i \in I$, and let $V \leqslant L$ be the subgroup generated by the elements $v_i, i \in I$.

Claim. *W is freely generated by $c_i w_i, i \in I$.*

Proof of the claim. Let

$$u \equiv \prod_{k=1}^{n} (c_{i_k} w_{i_k})^{\epsilon_i}$$

be a nontrivial freely reduced word over the alphabet $\{c_i w_i\}_{i \in I}$, where $i_k \in I$ and $\epsilon_k = \pm 1$ for $k = 1, ..., n$. Think of u as a word over the alphabet $\langle x \rangle \cup \langle y \rangle \cup C$ and then reduce u to its normal form $\overline{u}$ corresponding to the free product $\langle x \rangle * \langle y \rangle * C$ (see [24, Chapter IV] for the definition of normal forms). By (a), for each factor $(c_{i_k} w_{i_k})^{\epsilon_i}$ of u, a non-empty subword of $(c_{i_k} w_{i_k})^{\epsilon_i}$ survives in $\overline{u}$. In particular, $\overline{u}$ is a non-empty word and thus u represents a nontrivial element of L. $\square$

Similarly, V is freely generated by $v_i, i \in I$. In particular, W and V are free groups of the same rank $card(I)$.

Note that the relations $\bar{t}\bar{c}_i \overline{w}_i \bar{t}^{-1} \overline{v}_i = 1, i \in I$, can be rewritten as $\bar{t}\bar{c}_i \overline{w}_i \bar{t}^{-1} = \overline{v}_i^{-1}, i \in I$. Thus, R is the HNN-extension of L with associated subgroups W and V. In particular, L embeds into R. As $card(L) = \infty$, we have $card(R) = \infty$, that is, (1) holds.

Since C embeds into L, C embeds into R. By [6, Theorem 3.1], there is a long exact sequence

$$\cdots \to H^{p-1}(W; A) \to H^p(R; A) \to H^p(L; A) \to H^p(W; A) \to \cdots \tag{5.24}$$

for any $\mathbb{Z}R$-module A.

As W is free, for $p \geqslant 3$, (5.24) implies

$$H^p(R; A) \cong H^p(L; A) \cong H^p(C; A),$$

which implies (4). Combining (4) with $cd(C) \geqslant 2$, we see that $cd(R) \leqslant cd(C)$. Hence, (3) holds.

If C is finitely generated, then we can construct R using a finite generating set of C. Then R is the quotient of $F_3 * C$ by adding finitely many relations $tc_i w_i t^{-1} v_i, i \in I$ and thus has a finite relative presentation over C. The Greendlinger's lemma for free products implies that the relative isoperimetric function of R with respect to C is linear. Thus, R is hyperbolic relative to C, which is (5).

If C is of type FP_∞, then C is finitely generated and we can construct R using a finite generating set of C, that is, $card(I) < \infty$. Note that the rank of the free group W is $card(I)$. Thus, W is of type FP_∞. Note also that L is the free product of a finite rank free group F_3 with C and thus is of type FP_∞. By Theorem 5.2.5, $H^*(W;\cdot)$ and $H^*(L;\cdot)$ preserve direct limits. By the five lemma and (5.24), $H^*(R;\cdot)$ also preserves direct limit. It follows from Theorem 5.2.5 that R is of type FP_∞. Thus, (6) also holds. ☐

Proof of Theorem 1.2.22. Recall that by Theorem 2.5.7, G has a maximal finite normal subgroup $K(G)$. Let $G_0 = G/K(G)$. By [19, Lemma 5.10], G_0 is acylindrically hyperbolic.

If $cd(C) = 0$, then $C = \{1\}$. Let $\overline{G} = G_0$. By Theorem 2.5.7, $C \hookrightarrow_h \overline{G}$. Conclusions (a), (b), (c), and (d) hold trivially. As $\overline{G}$ and G are quasi-isometric, [4, Corollary 9] implies (e).

If $cd(C) = 1$, then by the Stallings-Swan theorem [32, Corollary to Theorem 1], C is free. By Theorem 2.5.7, there exists a finitely generated non-cyclic free group F such that $F \hookrightarrow_h G_0$. Let $\overline{G} = G_0$. It is well-known that the free group C embeds into F. Thus, C also embeds into $\overline{G}$. Once again, conclusions (a), (b), (c), and (e) hold trivially. If, in addition, C is finitely generated, then C is a finite rank free group and we can let $F = C$. Thus, (d) also holds.

Let us assume $cd(C) \geqslant 2$. By Theorem 2.5.7, there exists a rank-3 free subgroup $F_3 \hookrightarrow_h G_0$. By Theorems 2.5.12 and 3.0.1, there exists a finite set $\mathcal{F} \subset F_3 \backslash \{1\}$ such that if $N \lhd F_3$ satisfies $N \cap \mathcal{F} = \emptyset$, then

(HE) $F_3/N \hookrightarrow_h G_0/\langle\langle N \rangle\rangle$, where $\langle\langle N \rangle\rangle$ is the normal closure of N in G_0;

(CL) the group triple (G_0, F_3, N) has the Cohen-Lyndon property.

By Lemma 5.3.1, C embeds into an infinite quotient R of F_3 such that $cd(R) \leqslant cd(C)$ and the quotient map $F_3 \to R$ is injective on $\mathcal{F}$. Let N be the kernel of the quotient map $F_3 \to R$. Then $N \cap \mathcal{F} = \emptyset$ and thus (HE) and (CL) hold. Let $\overline{G} = G/\langle\langle N \rangle\rangle$.

As $R = F_3/N$ is infinite, (HE) implies that $\overline{G}$ is acylindrically hyperbolic, that is, statement (a) holds. As C embeds into R, C also embeds into $\overline{G}$.

Consider statement (b). Corollary 5.2.4 implies

$$cd(\overline{G}) \leqslant \max\{cd(G_0), cd(F_3) + 1, cd(R)\}.$$

If $K(G) \neq \{1\}$, then G has torsion and thus $cd(G) = \infty$ by [10, Chapter VIII Corollary 2.5], in which case (b) is a void statement. Thus, let us assume $K(G) = \{1\}$ and thus $G_0 = G$. As $cd(R) \leqslant cd(C)$ and $cd(C) \geqslant 2$, we have

$$cd(\overline{G}) \leqslant \max\{cd(G), cd(F_3) + 1, cd(R)\} \leqslant \max\{cd(G), 2, cd(C)\} = \max\{cd(G), cd(C)\}.$$

Thus, (b) holds. Moreover, (c) follows from Theorem 5.2.1 and statement (4) of Lemma 5.3.1.

If C is finitely generated, then Lemma 5.3.1 implies that R is hyperbolic relative to C. By [13, Proposition 4.28], $C \hookrightarrow_h R$. As $R \hookrightarrow_h \overline{G}$, we have $C \hookrightarrow_h \overline{G}$ by Proposition 2.5.9. Thus, statement (d) holds.

If C is of type FP_∞, then Lemma 5.3.1 implies that R is of type FP_∞. We have already seen that G_0 is of type FP_∞. As F_3 has finite rank, Theorem 5.2.8 implies that $\overline{G}$ is also of type FP_∞. Thus, statement (e) also holds. □

5.4 Common quotients of acylindrically hyperbolic groups

Let G_1 and G_2 be finitely generated acylindrically hyperbolic groups. In this section, we aim to construct a common quotient G of G_1 and G_2 satisfying the conclusions of Theorem 1.2.23.

By Theorem 2.5.7, G_1 (resp. G_2) has a maximal finite normal subgroup $K(G_1)$ (resp. $K(G_2)$). Let $G_{10} = G_1/K(G_1), G_{20} = G_2/K(G_2)$, and $\widetilde{G} = G_{10} * G_{20}$. As G_1 and G_2 are infinite, G_{10} and G_{20} are also infinite and thus there exists $k \in \mathbb{N}$ such that

(AB$_1$) there exists a finite generating set $A = \{a_1, ..., a_k\}$ (resp. $B = \{b_1, ..., b_k\}$) of G_{10} (resp. G_{20});

(AB$_2$) if w is a word over A (resp. B) of length 1 or 2, then $w \neq 1$.

Below, we fix a number k and sets A, B such that they satisfy (AB$_1$) and (AB$_2$) above.

Lemma 5.4.1. *There exists a rank-$(k + 2)$ free subgroup $H_1 \hookrightarrow_h G_{10}$ (resp. $H_2 \hookrightarrow_h G_{20}$) such that if $g \in G_{10}$ (resp. $g \in G_{20}$) satisfying $1 \leqslant |g|_A \leqslant 2$ (resp. $1 \leqslant |g|_B \leqslant 2$), then $g \notin H_1$ (resp. $g \notin H_2$).*

Proof. By [19, Lemma 5.10], G_{10} is acylindrically hyperbolic and $K(G_{10}) = \{1\}$. Thus, by Theorem 2.5.7, there is a rank-$(k+2)((2k+1)^2+1)$ free subgroup $F \hookrightarrow_h G_{10}$. We can decompose F into a free product

$$F = \prod_{1 \leqslant i \leqslant (2k+1)^2+1}^{*} F_i,$$

where each F_i is a free group of rank $k+2$. Note that $F_i \cap F_j = \{1\}$ for $1 \leqslant i < j \leqslant (2k+1)^2+1$.

There are less than $(2k+1)^2$ elemenets $g \in G_{10}$ such that $1 \leqslant |g|_A \leqslant 2$. Therefore, at least one of the $F_i's$, say F_1, does not contain any of such elements. Let $H_1 = F_1$. As H_1 is a free factor of F, we have $H_1 \hookrightarrow_h F$ by Remark 2.5.8. As $F \hookrightarrow_h G_{10}$, Proposition 2.5.9 implies $H_1 \hookrightarrow_h G_{10}$.

The proof for G_{20} is identical and is left to the reader. □

Let $H_1 < G_{10}$ and $H_2 < G_{20}$ be the subgroups provided by Lemma 5.4.1. There exists $X_1 \subset G_{10}$ and $X_2 \subset G_{20}$ such that

$$H_1 \hookrightarrow_h (G_{10}, X_1), \quad H_2 \hookrightarrow_h (G_{20}, X_2).$$

By [13, Corollary 4.27], we may assume that X_1 (resp. X_2) contains all words over A (resp. B) of length at most 2. By Theorem 2.5.10, there exists a strongly bounded relative presentation of G_{10} (resp. G_{20}) with respect to X_1 and H_1 (resp. X_2 and H_2) with linear relative isoperimetric function. By combining the above strongly bounded relative presentations, we obtain a strongly bounded relative presentation of $\widetilde{G} = G_{10} * G_{20}$ with respect to $X_1 \cup X_2$ and $\{H_1, H_2\}$ with linear relative isoperimetric function. By Theorem 2.5.10,

$$\{H_1, H_2\} \hookrightarrow_h (\widetilde{G}, X_1 \cup X_2) \tag{5.25}$$

Let $C = \{c_1, ..., c_{k+2}\}$ (resp. $D = \{d_1, ..., d_{k+2}\}$) be a basis for the free group H_1 (resp. H_2). The Cayley graphs $\Gamma(H_1, C)$ and $\Gamma(H_2, D)$ are Gromov hyperbolic spaces. Let

$$X = X_1 \cup X_2 \cup C \cup D.$$

By (5.25) and [1, Theorem 5.15], we have

(HQ) the Cayley graph

$$S = \Gamma(\widetilde{G}, X)$$

under the word metric d_X is a Gromov hyperbolic space and the natural embeddedings

$$\Gamma(H_1, C) \hookrightarrow S, \quad \Gamma(H_2, D) \hookrightarrow S$$

are (λ, μ)-quasi-isometric embeddedings for some $\lambda, \mu \geqslant 2$.

We note the following structure of $\widetilde{G}$ and X, which helps us estimate length of paths in S.

(FPS) $\widetilde{G} = G_{10} * G_{20}, \quad X_1 \cup C \subset G_{10}, \quad X_2 \cup D \subset G_{20}.$

Let $\widetilde{H_1}, \widetilde{H_2}$ be the subgroups of $\widetilde{G}$ generated, respectively, by

$$\widetilde{C} = \{b_1c_1, ..., b_kc_k, c_{k+1}, c_{k+2}\}, \quad \widetilde{D} = \{a_1d_1, ..., a_kd_k, d_{k+1}, d_{k+2}\}.$$

We are going to prove

$$\{\widetilde{H_1}, \widetilde{H_2}\} \hookrightarrow_h \widetilde{G}.$$

By [13, Theorem 4.42] (see also [13, Remark 4.41]), it suffices to show the following conditions hold for the action of $\widetilde{G}$ on S.

(C_1) For $i = 1, 2$, $\widetilde{H_i}$ acts on S properly.

(C_2) The orbits $\widetilde{H_1}$ and $\widetilde{H_2}$ are quasi-convex in S.

(C_3) For every $\epsilon > 0$, there exists $R > 0$ such that if $g \in \widetilde{G}$ and $i, j \in \{1, 2\}$ satisfy

$$diam_X(g\widetilde{H_i}, (\widetilde{H_j})^{+\epsilon}) \geqslant R,$$

then $i = j$ and $g \in \widetilde{H_i}$, where $(\widetilde{H_j})^{+\epsilon}$ denotes the ϵ-neighborhood of $\widetilde{H_j}$.

Note that there is a natural embedding

$$Emb_1 : \Gamma(\widetilde{H_1}, \widetilde{C}) \hookrightarrow S$$

defined as follows. Emb_1 maps every vertex of $\Gamma(\widetilde{H_1}, \widetilde{C})$ to the vertex of S with the same label. For every edge $e \subset \Gamma(\widetilde{H_1}, \widetilde{C})$ connecting two vertices $v_1, v_2 \in \Gamma(\widetilde{H_1}, \widetilde{C})$. Think of $Lab(e)$ as a word over X and let $Emb_1(e)$ be the path p of S connecting $Emb_1(v_1)$ and $Emb_1(v_2)$ such that $Lab(p) \equiv Lab(e)$.

Similarly, there is a natural embedding

$$Emb_2 : \Gamma(\widetilde{H_2}, \widetilde{D}) \hookrightarrow S.$$

Lemma 5.4.2. *The natural embeddings Emb_1 and Emb_2 are $(2\lambda\mu, \mu)$-quasi-isometric embeddings.*

Proof. We only consider Emb_1. The proof for Emb_2 is similar. Clearly, Emb_1 can increase distance by at most twice. Thus, it suffices to show that the following inequality holds for all $h \in \widetilde{H_1}$.

$$|h|_{\widetilde{C}} \geqslant \frac{|h|_X}{2\lambda\mu} - \mu. \tag{5.26}$$

Fix $h \in \widetilde{H_1}$. Let u be a shortest word over $\widetilde{C}$ such that u represents h in $\widetilde{H_1}$. Note that u can also be regarded as a word over X, i.e., for $i = 1, ..., k$, instead of viewing b_ic_i as a single letter in $\widetilde{C}$, we regard b_ic_i as the concatenation of $b_i, c_i \in X$. Under this point of view, we see that there are two types of subwords w of u:

(T_1) w is a word over B and there is no subword w' of u such that (1) w' is a word over B and (2) w is properly contained in w';

(T_2) w is a word over C and there is no subword w' of u such that (1) w' is a word over C and (2) w is properly contained in w'.

We note the following.

(NT_1) Every subword w of type (T_1) is a word over B of length 1 or 2. Thus, $w \neq 1$ by (AB_2).

(NT_2) Every subword type (T_2) does not represent 1 in $\widetilde{G}$, as C is a basis of the free group H_1.

We construct a new word v from u by replacing every subword w of type (T_1) by a letter $x \in X_2$ such that $x =_{G_{20}} w$ (such an x is called a *subword of v of the first type*) and replacing every subword w of type (T_2) by a geodesic word w' over $X_1 \cup C$ such that $w' =_{G_{10}} w$ (such a w' is called a *subword of v of the second type*). Clearly, $v =_{\widetilde{G}} u$.

(NT_1), (NT_2), and (FPS) imply that v is a geodesic word over X. Let n be the total number of type (T_1) and (T_2) subwords of u. Note that $\|v\| \geqslant n$. We can then estimate $\|u\|$ by distinguishing the cases $n > \|u\|/(2\lambda\mu)$ and $n \leqslant \|u\|/(2\lambda\mu)$:

If $n > \|u\|/(2\lambda\mu)$, then we already have $\|v\| \geqslant n > \|u\|/(2\lambda\mu)$.

If $n \leqslant \|u\|/(2\lambda\mu)$, then the subwords of u of type (T_1) divide u into $\ell \leqslant n+1$ parts, each of which is a subword of type (T_2). Let $w_1, ..., w_\ell$ be these type (T_2) subwords. Note that each type (T_1) subword has length at most 2. Thus, the total length of type (T_1) subwords is at most $2n$. As a consequence, $\sum_{i=1}^{\ell} \|w_i\| \geqslant \|u\| - 2n$. For $1 \leqslant i \leqslant \ell$, let w_i' be the second-type subword of v corresponding to w_i. Then $\|w_i'\| \geqslant \|w_i\|/\lambda - \mu$ by (HQ). Thus, the total length of second type subword of v satisfies

$$\sum_{i=1}^{\ell} \|w_i'\| \geqslant \sum_{i=1}^{\ell} \|w_i\|/\lambda - n\mu \geqslant \frac{\|u\| - 2n}{\lambda} - n\mu.$$

By (NT_1), each first-type subword of v has length at least 1 and thus the total length of first type subwords is at least n. Therefore,

$$\|v\| \geqslant \sum_{i=1}^{\ell} \|w_i'\| + n \geqslant \frac{\|u\| - 2n}{\lambda} - n\mu + n \geqslant \frac{\|u\|}{2\lambda},$$

as $\lambda \geqslant 2$ and $n \leqslant \|u\|/(2\lambda\mu)$. □

Lemma 5.4.2 clearly implies (C_1) and (C_2). Indeed, the action $\widetilde{H_1} \curvearrowright \Gamma(\widetilde{H_1}, \widetilde{C})$ (resp. $\widetilde{H_2} \curvearrowright \Gamma(\widetilde{H_2}, \widetilde{D})$) is proper and the embeddedings Emb_1, Emb_2 are $\widetilde{H_1}, \widetilde{H_2}$-equivariant, respectively. Thus, (C_1) holds. Moreover, Emb_1 (resp. Emb_2) sends the set of vertices of $\Gamma(\widetilde{H_1}, \widetilde{C})$ (resp. $\Gamma(\widetilde{H_2}, \widetilde{D})$) to the orbit $\widetilde{H_1} \subset S$ (resp. $\widetilde{H_2} \subset S$). As S is a Gromov hyperbolic space and Emb_1, Emb_2 are quasi-isometric embeddings, $\widetilde{H_1}$ and $\widetilde{H_2}$ are quasi-convex in S, that is, (C_2) holds.

It remains to prove (C_3). By Remark 2.5.8, $\{G_{10}, G_{20}\} \hookrightarrow_h (\widetilde{G}, \emptyset)$. Consider the Cayley graph

$$\Gamma = \Gamma(\widetilde{G}, G_{10} \sqcup G_{20}).$$

We apply a result of [13] about isolated components. For the convenience of the reader, we adapt Definition 2.6.1 to our situation.

Definition 5.4.3. Let p be a path in Γ. A G_{10}-*subpath* q of p is a nontrivial subpath q of p labeled by a word over the alphabet G_{10} (if p is a cycle, we allow q to be a subpath of some cyclic shift of p). A G_{10}-subpath q of p is called a G_{10}-*component* if q is not properly contained in any other G_{10}-subpath. Two G_{10}-components q_1, q_2 of p are called *connected* if there exists a path c in Γ such that c connects a vertex of

q_1 to a vertex of q_2, and that $Lab(c)$ is a letter of G_{10}.

The notions of G_{20}-subpaths, G_{20}-components, and connected G_{20}-components are defined in the same manner. Moreover, a *component* of a path p is a G_{10} or G_{20}-component of p.

Lemma 5.4.4 ([13, Lemma 4.21] (see also Remark 2.5.8)). *Let $\mathcal{W}$ be the set of words over the alphabet $G_{10} \sqcup G_{20}$ such that $\mathcal{W}$ contains no subwords of type xy, where $x, y \in G_{10}$ or $x, y \in G_{20}$. Then the following hold:*

For every $\epsilon > 0$, there exists $R = R(\epsilon) > 0$ satisfying the following condition. Let p, q be two paths in Γ such that $Lab(p), Lab(q) \in \mathcal{W}$, $\ell_{G_{10} \sqcup G_{20}}(p) \geqslant R$, and p, q are oriented ϵ-close, i.e.,

$$\max\{d_{G_{10} \sqcup G_{20}}(p^-, q^-), d_{G_{10} \sqcup G_{20}}(p^+, q^+)\} \leqslant \epsilon.$$

Then there exist four consecutive components of p which are respectively connected to four consecutive components of q.

Remark 5.4.5. Let p be a path in S. We think of p^-, p^+ as elements of $\widetilde{G}$ and thus p^-, p^+ label vertices of Γ. In Γ, there is a unique geodesic $\overline{p}$ traveling from p^- to p^+. We thus obtain a map $p \mapsto \overline{p}$ from paths in S to geodesics in Γ.

Lemma 5.4.6. *(C_3) holds.*

Proof. Fix $\epsilon > 0$. As S is a Gromov hyperbolic space, there exists $R_1 > 0$ such that if p and q are $(2\lambda\mu, 2\epsilon + \mu)$-quasi-geodesics in S with the same endpoints, then $d_{Hau}(p, q) \leqslant R_1$, where d_{Hau} denotes the Hausdorff distance corresponding to the word metric d_X. There exists $R_2 > 0$ such that if p and q are $(2\lambda\mu, 2R_1 + \mu)$-quasi-geodesics in S with the same endpoints, then $d_{Hau}(p, q) \leqslant R_2$. By Lemma 5.4.4, there exists $R_3 > 0$ such that if p and q are two ϵ-close paths in Γ with

$$Lab(p), Lab(q) \in \mathcal{W}, \quad \ell_{G_1 \sqcup G_2}(p) \geqslant R_3,$$

then there exist four consecutive components of p which are respectively connected to four consecutive components of q.

Let H_3 (resp. H_1') be the subgroup of H_1 generated by c_{k+1}, c_{k+2} (resp. $c_1, ..., c_k$). By Remark 2.5.8, $H_3 \hookrightarrow_h (H_1, H_1')$. Together with $H_1 \hookrightarrow_h (G_{10}, X_1), G_{10} \hookrightarrow_h (\widetilde{G}, G_{20})$, and Proposition 2.5.9, this observation implies $H_3 \hookrightarrow_h (\widetilde{G}, G_{20} \cup X_1 \cup H_1')$. Thus, the relative metric

$$\widehat{d}: H_3 \times H_3 \to [0, +\infty]$$

with respect to $G_{20} \cup X_1 \cup H_1'$ is proper. There exists $R_4 > 0$ such that if $h \in H_3$ and $|h|_X \geqslant R_4$, then $\widehat{d}(1, h) \geqslant 2R_2 + 2$. Also let H_4 be the subgroup of H_2 generated by d_{k+1}, d_{k+2}.

Let

$$R = (R_3 + 1)(\lambda((R_3 + 1)(\lambda(2R_2 + R_4 + \mu) + 4) + 2R_1 + \mu) + 4).$$

Suppose that there exists $g \in \widetilde{G}$ and $i, j \in \{1, 2\}$ such that

$$diam_X(g\widetilde{H_i}, (\widetilde{H_j})^{+\epsilon}) \geqslant R.$$

Without loss of generality, we may assume $i = 1$. There are two cases to consider.

Case 1. $j = 2$.

Then there exist oriented ϵ-close edge paths $p \subset g\widetilde{H_1}$ and $q \subset \widetilde{H_2}$ such that $u = Lab(p)$ is a geodesic word over $\widetilde{C}$, $v = Lab(q)$ is a geodesic word over $\widetilde{D}$, and

$$d_X(p^-, p^+), d_X(q^-, q^+) \geqslant R.$$

Consider a path $r \subset S$ labeled by a word over $\widetilde{C}$. There are two possible reasons for $d_X(r^-, r^+)$ to be large:

(a) $Lab(r)$ contains many subwords of type (T_1), in which case $\ell_{G_{10} \sqcup G_{20}}(\overline{r})$ is large, where $\overline{r}$ is the image of r under the map in Remark 5.4.5.

(b) $Lab(r)$ contains a long subword of type (T_2), in which case $Lab(r)$ contains a long subword over the alphabet $\{c_{k+1}, c_{k+2}\}$.

We observe the following estimate of the length of the longest subword of $Lab(r)$ over $\{c_{k+1}, c_{k+2}\}$.

Claim. *Let $r \subset S$ be a path labeled by a word over $\widetilde{C}$ and let m be the length of the longest subword of $Lab(r)$ over $\{c_{k+1}, c_{k+2}\}$, then*

$$m \geqslant \frac{\|Lab(r)\|}{\ell_{G_{10} \sqcup G_{20}}(\overline{r}) + 1} - 4.$$

Proof of the claim. Let n be the number of type (T_1) subwords of $Lab(r)$. The (T_1) subwords of $Lab(r)$ divide $Lab(r)$ into at most $n+1$ parts, each of which is a (T_2) subword. Note that the total length of type (T_1) subwords is at most $2n$. Thus, there is at least one (T_2) subword with length

$$\frac{\|Lab(r)\| - 2n}{n+1} > \frac{\|Lab(r)\|}{n+1} - 2.$$

By the structure of $\widetilde{C}$, for each type (T_2) subword w of $Lab(r)$, w contains a subword over $\{c_{k+1}, c_{k+2}\}$ of length at least $\|w\| - 2$. Note also that $\ell_{G_{10} \sqcup G_{20}}(\overline{r}) \geqslant n$. Thus, $Lab(r)$ contains a subword over $\{c_{k+1}, c_{k+2}\}$ of length at least $\|Lab(r)\|/(\ell_{G_{10} \sqcup G_{20}}(\overline{r}) + 1) - 4$. □

Consider the images $\overline{p}$ and $\overline{q}$ of p and q under the map in Remark 5.4.5. We distinguish two subcases.

Case 1.1. $\max\{\ell_{G_{10} \sqcup G_{20}}(\overline{p}), \ell_{G_{10} \sqcup G_{20}}(\overline{q})\} \geqslant R_3$.

Without loss of generality, we may assume $\ell_{G_{10} \sqcup G_{20}}(\overline{p}) \geqslant R_3$. By Lemma 5.4.4, there are four consecutive components of $\overline{p}$ connected, respectively, to four consecutive components of $\overline{q}$. It is easy to see that there are three consecutive components x, y, z of $\overline{p}$ such that x, z are G_{20}-components, y is a G_{10}-component, and x, y, z are connected to three consecutive components x', y', z' of $\overline{q}$. Note that x, x' and z, z' are connected by paths labeled by a word over G_{20}, while y, y' are connected by paths labeled by a word over G_{10}. As $\widetilde{G} = G_{10} * G_{20}$, the only possibility is that x, x' and z, z' are connected by the trivial path. Thus, $y^{-1}y'$ is a loop in Γ. However, $Lab(y) \in H_1$ and $Lab(y')$ is a word over A of length 1 or 2. By the construction of H_1 (see Lemma 5.4.1), $Lab(y') \notin H_1$ and thus $(Lab(y))^{-1}Lab(y') \neq 1$, a contradiction. Therefore, Case 1.1 is in fact impossible.

Case 1.2. $\max\{\ell_{G_{10} \sqcup G_{20}}(\overline{p}), \ell_{G_{10} \sqcup G_{20}}(\overline{q})\} \leqslant R_3$.

By the claim and $\|Lab(p)\| \geqslant d_X(p^-, p^+) \geqslant R$, there exists a subpath $p_1 \subset p$ such that $Lab(p_1) \in H_3$ and

$$\|Lab(p_1)\| \geqslant \lambda((R_3+1)(\lambda(2R_2 + R_4 + \mu) + 4) + 2R_1 + \mu).$$

Notice that $Lab(p_1)$ labels a geodesic in $\Gamma(H_1, C)$. Thus, (HQ) implies

$$d_X(p_1^-, p_1^+) \geqslant \frac{\|Lab(p_1)\|}{\lambda} - \mu \geqslant (R_3+1)(\lambda(2R_2 + R_4 + \mu) + 4) + 2R_1.$$

As p and q are oriented ϵ-close, there exist paths $t_1, t_2 \subset S$ such that

$$t_1^- = q^-, \quad t_2^- = p^+, \quad t_1^+ = p^-, \quad t_2^+ = q^+, \quad \ell_X(t_1), \ell_X(t_2) \leqslant \epsilon.$$

By Lemma 5.4.2, q and the conjunction $t_1 p t_2$ are $(2\lambda\mu, 2\epsilon + \mu)$-quasi-geodesics. By our choice of R_1, we have $d_{Hau}(t_1 p t_2, q) \leqslant R_1$ and in particular, p is in the R_1-neighborhood of q. Consequently, there exists a subpath $q_1 \subset q$ such that p_1 and q_1 are oriented R_1-close. Note that

$$\|Lab(q_1)\| \geqslant d_X(q_1^-, q_1^+) \geqslant d_X(p_1^-, p_1^+) - 2R_1 \geqslant (R_3 + 1)(\lambda(2R_2 + R_4 + \mu) + 4)$$

by the triangle inequality. As $\ell_{G_{10} \sqcup G_{20}}(\overline{q}_1) \leqslant \ell_{G_{10} \sqcup G_{20}}(\overline{q}) \leqslant R_3$, by the same argument as the one for the existence of p_1, we see that there exists a subpath $q_2 \subset q_1$ such that $Lab(q_2) \in H_4$ and

$$\|Lab(q_2)\| \geqslant \lambda(2R_2 + R_4 + \mu).$$

Notice that $Lab(q_2)$ labels a geodesic in $\Gamma(H_2, D)$. Thus, (HQ) implies

$$d_X(q_2^-, q_2^+) \geqslant \frac{\|Lab(q_2)\|}{\lambda} - \mu \geqslant 2R_2 + R_4. \tag{5.27}$$

By the same argument as the one for the existence of q_1, we see that there exists a subpath $p_2 \subset p_1$ such that p_2 and q_2 are oriented R_2-close. In other words, there exist words w_1 and w_2 over X such that $\|w_1\|, \|w_2\| \leqslant R_2$ and

$$w_1 Lab(p_2) w_2 (Lab(q_2))^{-1} =_{\widetilde{G}} 1 \tag{5.28}$$

(w_1 and w_2 label short paths between the endpoints of p_2 and q_2). Note that

$$d_X(p_2^-, p_2^+) \geqslant d_X(q_2^-, q_2^+) - 2R_2 \geqslant R_4 > 0. \tag{5.29}$$

Let $g' \in \widetilde{G}$ with

$$g' = Lab(p_2) w_2 (Lab(q_2))^{-1}.$$

By (5.27) and (5.29), we have

$$d_X(p_2^-, p_2^+) + d_X(q_2^-, q_2^+) > 2R_2.$$

By (FPS), $Lab(p_2) \in H_3 < G_{10}$, and $Lab(q_2) \in H_4 < G_{20}$, we have

$$|g'|_X \geqslant d_X(p_2^-, p_2^+) + d_X(q_2^-, q_2^+) - \|w_2\| > R_2.$$

But (5.28) implies $w_1^{-1} = g'$ and thus $|g'|_X \leqslant \|w_1\| \leqslant R_2$, a contradiction. Thus, Case 1.2 is in fact impossible.

As a consequence, Case 1 is impossible.

Case 2. $j = 1$.

Then there exist oriented ϵ-close edge paths $p \subset g\widetilde{H_1}$ and $q \subset \widetilde{H_1}$ such that $u = Lab(p), v = Lab(q)$ are geodesic words over $\widetilde{C}$, and

$$d_X(p^-, p^+), d_X(q^-, q^+) \geqslant R.$$

As for Case 1, we distinguish two subcases.

Case 2.1. $\max\{\ell_{G_{10} \sqcup G_{20}}(\overline{p}), \ell_{G_{10} \sqcup G_{20}}(\overline{q})\} \geqslant R_3$.

Without loss of generality, we may assume $\ell_{G_{10} \sqcup G_{20}}(\overline{p}) \geqslant R_3$. Arguing as in Case 1.1, we see that there is a G_{10}-component y of p and a G_{10}-component y' of q such that y and y' share the same endpoints. By the structure of $\widetilde{C}$, we have $Lab(y), Lab(y') \in H_1$. As H_1 is a free group, we have $Lab(y) \equiv Lab(y')$.

Think of $Lab(p)$ as a word over X and decompose it as

$$Lab(p) \equiv w_1 Lab(y) w_2.$$

Similarly, think of $Lab(q)$ as a word over X and decompose it as

$$Lab(q) \equiv w_3 Lab(y') w_4.$$

As y and y' share the same endpoints, the word $w_3 w_1^{-1}$ labels a path in S from $q^- \in \widetilde{H_1}$ to $p^- \in g\widetilde{H_1}$. Thus, there exists $h_1, h_2 \in \widetilde{H_1}$ with $g = h_1 w_3 w_1^{-1} h_2$. If $w_1, w_3 \in \widetilde{H_1}$, then $g \in \widetilde{H_1}$ and we are done.

Suppose $w_1 \notin \widetilde{H_1}$ (the case $w_3 \notin \widetilde{H_1}$ is similar). By the structure of $\widetilde{C}$, there exists $1 \leqslant i \leqslant k$ such that the first letter of $Lab(y)$ is c_i and the concatenation $w_1 c_i \in \widetilde{H_1}$. As $Lab(y') \equiv Lab(y)$, the first letter of $Lab(y')$ is also c_i. As $Lab(q)$ is a word over $\widetilde{C}$, we have $w_3 c_i \in \widetilde{H_1}$ and thus $g = h_1(w_3 c_i)(c_i^{-1} w_1^{-1}) h_2 \in \widetilde{H_1}$.

Case 2.2. $\max\{\ell_{G_{10}\sqcup G_{20}}(\overline{p}), \ell_{G_{10}\sqcup G_{20}}(\overline{q})\} \leqslant R_3$.

Arguing as in Case 1.2, we see that there are subpaths $p_1 \subset p$ and $q_1 \subset q$ such that

(1) $Lab(p_1), Lab(q_1) \in H_3$;

(2) $d_X(q_1^-, q_1^+) \geqslant 2R_2 + R_4$;

(3) p_1 and q_1 are oriented R_2-close.

By (3), there exist words w_1 and w_2 over X such that

$$w_1 Lab(p_1) w_2 (Lab(q_1))^{-1} =_{\widetilde{G}} 1, \quad \|w_1\|, \|w_2\| \leqslant R_2$$

(w_1 and w_2 label short paths between the endpoints of p_1 and q_1). Decompose $Lab(p)$ and $Lab(q)$ as

$$Lab(p) \equiv u_1 Lab(p_1) u_2, \quad Lab(q) = u_3 Lab(q_1) u_4.$$

By the structure of $\widetilde{C}$, we have $u_1, u_3 \in \widetilde{H_1}$.

Note that the word $u_3 w_1 u_1^{-1}$ labels a path in S from $q^- \in \widetilde{H_1}$ to $p^- \in g\widetilde{H_1}$. Thus, there exist $h_1, h_2 \in \widetilde{H_1}$ with $g = h_1 u_3 w_1 u_1^{-1} h_2$. If $w_1 \in H_3 < \widetilde{H_1}$, then as $h_1, h_2, u_1, u_3 \in \widetilde{H_1}$, we get that $g \in \widetilde{H_1}$, which concludes the proof.

Suppose $w_1 \notin H_3$. Let v_1 (resp. v_2) be the maximal initial (resp. terminal) subword of w_1 (resp. w_2) such that $v_1 \in H_3$ (resp. $v_2 \in H_3$), let v_1' (resp. v_2') be the word resulted from deleting v_1 (resp. v_2) from w_1 (resp. w_2), and let $h, h' \in H_3$ with

$$h = v_1^{-1} Lab(q_1) v_2^{-1}, \quad h' = Lab(p_1).$$

Note that the word $v_1' h' v_2'$ labels an H_3-admissible path in $\Gamma(\widetilde{G}, G_{20} \cup X_1 \cup H_1' \sqcup H_3)$ connecting the vertices labeled by 1 and h, and

$$\|v_1' h' v_2'\| \leqslant \|v_1'\| + \|v_2'\| + 1 \leqslant \|w_1\| + \|w_2\| + 1 \leqslant 2R_2 + 1.$$

Thus,

$$\widehat{d}(1, h) \leqslant 2R_2 + 1.$$

On the other hand,

$$|h|_X \geqslant d_X(q_1^-, q_1^+) - \|v_1\| - \|v_2\| \geqslant d_X(q_1^-, q_1^+) - \|w_1\| - \|w_2\| \geqslant R_4,$$

which contradicts our choice of R_4. Thus, Case 2.2 is in fact impossible. □

We conclude with

Proposition 5.4.7. *$\widetilde{H_1}, \widetilde{H_2}$ satisfy (C_1), (C_2), and (C_3) and thus $\{\widetilde{H_1}, \widetilde{H_2}\} \hookrightarrow_h \widetilde{G}$.*

Proof of Theorem 1.2.23. As $|G_{10}| = |G_{20}| = \infty$, we have $cd(G_{10}), cd(G_{20}) \geqslant 1$. Suppose $cd(G_{10}) = cd(G_{20}) = 1$. Then G_{10} and G_{20} are free by the Stallings-Swan theorem [32, Corollary to Theorem 1]. Without loss of generality, we may assume that the rank of G_{10} is greater than or equal to the rank of G_{20}. It follows that G_{20} is a quotient of G_{10}. Let $G = G_{20}$. Statements (a), (b), and (c) follow trivially. Statement (d) also holds because if G_2 is of type FP_∞, then G_{20} is also of type FP_∞ by [4, Corollary 9].

Thus, let us assume $\max\{cd(G_{10}), cd(G_{20})\} \geqslant 2$. By Theorems 2.5.12, 3.0.1, and Proposition 5.4.7, there exists finite sets $\mathcal{F}_1 \subset \widetilde{H_1}\backslash\{1\}, \mathcal{F}_2 \subset \widetilde{H_2}\backslash\{1\}$ such that if

$$N_1 \lhd \widetilde{H_1}, \quad N_2 \lhd \widetilde{H_2}, \quad N_1 \cap \mathcal{F}_1 = N_2 \cap \mathcal{F}_2 = \emptyset,$$

then

$$\{\widetilde{H_1}/N_1, \widetilde{H_2}/N_2\} \hookrightarrow_h \widetilde{G}/\langle\langle N_1 \cup N_2 \rangle\rangle \tag{5.30}$$

and $(\widetilde{G}, \{\widetilde{H_1}, \widetilde{H_2}\}, \{N_1, N_2\})$ has the Cohen-Lyndon property.

Let $u_i, 1 \leqslant i \leqslant k$, (resp. $v_i, 1 \leqslant i \leqslant k$,) be freely reduced words over $\{c_{k+1}, c_{k+2}\}$ (resp. $\{d_{k+1}, d_{k+2}\}$) satisfying the $C'(1/6)$ small cancellation condition, and let N_1 (resp. N_2) be the normal subgroup of $\widetilde{H_1}$ (resp. $\widetilde{H_2}$) generated by $\{b_1c_1u_1, ..., b_kc_ku_k\}$ (resp. $\{a_1d_1v_1, ..., a_kd_kv_k\}$). By ($AB_2$), $\widetilde{H_1}$ and $\widetilde{H_2}$ are freely generated by $\widetilde{C}$ and $\widetilde{D}$, respectively. Thus, $\widetilde{H_1}/N_1$ and $\widetilde{H_2}/N_2$ can be presented as

$$\widetilde{H_1}/N_1 = \langle b_1c_1, ..., b_kc_k, c_{k+1}, c_{k+2} \mid b_1c_1u_1, ..., b_kc_ku_k\rangle = \langle c_{k+1}, c_{k+2}\rangle, \tag{5.31}$$

$$\widetilde{H_2}/N_2 = \langle a_1d_1, ..., a_kd_k, d_{k+1}, d_{k+2} \mid a_1d_1v_1, ..., a_kd_kv_k\rangle = \langle d_{k+1}, d_{k+2}\rangle, \tag{5.32}$$

where the last equality of (5.31) (resp. (5.32)) follows from eliminating $b_1c_1, ..., b_kc_k$ (resp. $a_1d_1, ..., a_kd_k$)

by Tietze transformations [24, Chapter II].

Thus, $\widetilde{H_1}$ and $\widetilde{H_2}$ are free groups of rank 2. In particular,

$$card(\widetilde{H_1}/N_1) = card(\widetilde{H_2}/N_2) = \infty \tag{5.33}$$

By the Greendlinger's lemma for free groups [24, Chapter V Theorem 4.5], if $\|u_i\|, \|v_i\|, 1 \leqslant i \leqslant k$, are sufficiently large, then

$$N_1 \cap \mathcal{F}_1 = N_2 \cap \mathcal{F}_2 = \emptyset.$$

Let

$$G = \widetilde{G}/\langle\langle N_1 \cup N_2 \rangle\rangle.$$

By (5.30) and (5.33), G is acylindrically hyperbolic, that is, (a) holds.

Let us consider statements (b) and (c). If either $K(G_1)$ or $K(G_2)$ is not $\{1\}$, then (b) holds trivially and (c) is a void statement. Thus, we may assume $K(G_1) = K(G_2) = \{1\}$ and thus $G_{10} = G_1, G_{20} = G_2$. As $(\widetilde{G}, \{\widetilde{H_1}, \widetilde{H_2}\}, \{N_1, N_2\})$ has the Cohen-Lyndon property, Theorem 5.2.4 implies

$$\begin{aligned} cd(G) \leqslant \max\{cd(\widetilde{G}), 2\} &= \max\{cd(G_{10}), cd(G_{20}), 2\} \\ &= \max\{cd(G_{10}), cd(G_{20})\} && \text{as } cd(G_{10}), cd(G_{20}) \geqslant 2 \\ &= \max\{cd(G_1), cd(G_2)\} && \text{as } G_{10} = G_1, G_{20} = G_2. \end{aligned}$$

Therefore, (b) holds. Another cosequence of Theorem 5.2.4 is that, for all $\ell \geqslant 3$ and any $\mathbb{Z}G$-module A, we have

$$\begin{aligned} & H^\ell(G; A) \\ \cong & H^\ell(\widetilde{G}; A) \bigoplus H^\ell(\widetilde{H_1}/N_1; A) \bigoplus H^\ell(\widetilde{H_2}/N_2; A) \\ \cong & H^\ell(\widetilde{G}; A) && \text{as } \widetilde{H_1}/N_1 \text{ and } \widetilde{H_2}/N_2 \text{ are free groups} \\ \cong & H^\ell(G_{10}; A) \bigoplus H^\ell(G_{20}; A) && \text{as } \widetilde{G} = G_{10} * G_{20} \\ \cong & H^\ell(G_1; A) \bigoplus H^\ell(G_2; A) && \text{as } G_{10} = G_1, G_{20} = G_2, \end{aligned}$$

which is (c).

Consider statement (d). Suppose G_1 and G_2 are of type FP_∞. By [4, Corollary 9], G_{10} and G_{20} are also of type FP_∞. As $(\widetilde{G}, \{\widetilde{H_1}, \widetilde{H_2}\}, \{N_1, N_2\})$ has the Cohen-Lyndon property and $\widetilde{H_1}, \widetilde{H_2}$ are free groups of finite rank, Theorem 5.2.8 implies that G is also of type FP_∞. Thus, (d) holds. □

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