Dynamical Sampling and Systems of Vectors from Iterative Actions of Operators

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Chapter 1

Introduction

The main problem in sampling theory is to reconstruct a function from the values (*samples*) on some discrete subset Ω of its domain (Fig. 1.1). This type of inverse problems are common in many applications such as signal processing, image and audio processing, data analysis as well as in biology, medicine, geology and other fields [10]. For it to be solvable, the function to be reconstructed must be known to belong to a certain class of functions. Moreover, the sampling set Ω must be chosen appropriately.

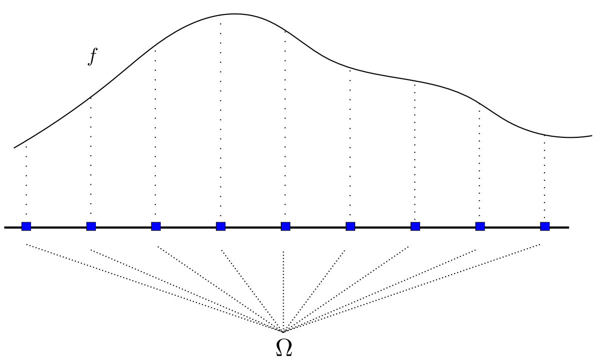

Figure 1.1: Classical sampling

For example, consider a function f from the so-called *Paley – Wiener space* $PW_\sigma(\mathbb{R})$, for some $\sigma > 0$ (called the *bandwidth*), that is, the space of functions whose Fourier transforms vanish outside of the interval $[-\sigma, \sigma]$:

$$PW_\sigma(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subset [-\sigma, \sigma] \right\}.$$

Here, the Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i \xi t} dt.$$

The Paley – Wiener spaces are often used for modeling analog signals in signal processing. The reason is that computers can process only those sound frequencies that lie within a certain range. An important property of the Paley – Wiener spaces is that any function $f \in PW_\sigma(\mathbb{R})$ can be uniquely recovered from its uniform samples on $\frac{1}{2\sigma}\mathbb{Z}$ due to Nyquist – Shannon – Kotelnikov sampling theorem.

Theorem. *Any function $f \in PW_\sigma(\mathbb{R})$ can be uniquely recovered from its uniform samples on $T\mathbb{Z}$, for $T \leq \frac{1}{2\sigma}$, by the formula*

$$f(t) = \sum_{k\in\mathbb{Z}} f(kT) \cdot \operatorname{sinc}\left(\frac{x-kT}{T}\right)$$

for $\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}$, *where the series converge in $L^2(\mathbb{R})$ and uniformly on compact sets.*

Claude Shannon, in his paper titled "Communication in the Presence of Noise" [60] published in 1949, saw a possibility for application of this theorem to signal processing and communications. The theorem itself was independently proved by several authors, including, E. T. Whittaker and V. Kotelnikov [66]. In the mathematical and engineering literature, the theorem can be found under several different names like Nyquist – Shannon theorem, Shannon – Kotelnikov theorem, Whittaker – Shannon theorem, etc.

Sampling and reconstruction theory is important because it bridges the modern digital world and the analog world of continuous functions. Many applications of digital signal processing begin by converting a continuous function to a sequence of real or complex numbers. This process is called *analog-to-digital conversion* or *sampling*. The inverse process, converting a sequence of numbers to a continuous function, is called reconstruction.

Since the publication of Shannon's paper to our times, the field of sampling theory has expanded and become more mathematical. There are many overlaps with wavelets, splines, shift-invariant spaces, frame theory, etc. [14, 27, 37, 71]

In many situations of interest, taking samples on an appropriate sampling set Ω is not always practical or even possible—it can be that measuring devices are too expensive or scarce. For example, for functions from $PW_\sigma(\mathbb{R})$, it is well-known that *undersampling* a signal in $PW_\sigma(\mathbb{R})$ by a rate $T > \frac{1}{2\sigma}$ (i.e. taking samples on the sparser grid $\Omega = T\mathbb{Z}$ instead of $\frac{1}{2\sigma}\mathbb{Z}$) will introduce *aliasing*: the samples $f(\Omega)$ will not give enough data to recover f.

Nevertheless, it often happens that the sparseness of the sampling locations can be compensated by involving dynamics and sampling the evolved versions of the function. For example when f is the initial state of a physical process (say, change of temperature or air pollution), we can sample its values at the same sampling locations as the time progresses, and try to recover f from the combination of these *spatio-temporal samples* (Fig. 1.2).

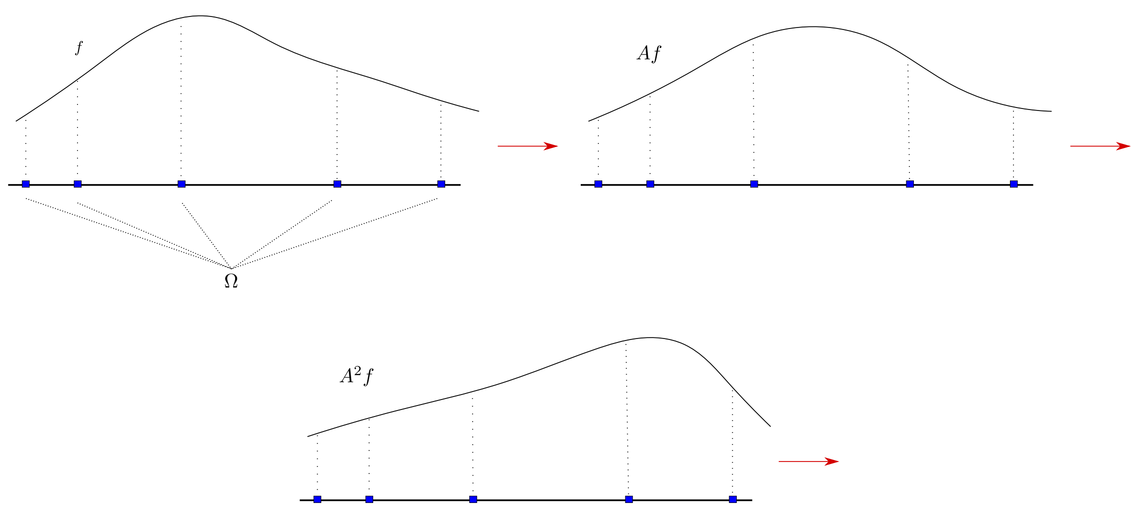

Figure 1.2: The evolution system

This new problem, called *dynamical sampling problem*, was introduced by A. Aldroubi, J. Davis and I. Krishtal in [8], motivated by the work of Y. Lu and M. Vetterli in [43, 55]. They assume the process of evolution is given by a discrete-time dynamical system, $f_n = A^n f$, where A is a linear operator. Lu and Vetterli studied the space-time sampling for the functions in the Paley – Wiener space, in the case when the evolution operator is the heat operator.

The theory developed for the dynamical sampling problem has similarities with the wavelet theory [16, 23, 24, 36, 44, 51, 52, 61]. In dynamical sampling, an operator A is applied iteratively to the function f producing the functions $f_n = A^n f$. The f_n is then, typically, sampled coarsely at each level n. Thus, f cannot be recovered from samples at any single time level. But, similarly to the wavelet transform, the combined data at all time levels is required to reproduce f. However, unlike the wavelet transform, there is a single operator A instead of two complementary operators L (the low-pass operator) and H (the high pass operator). Moreover, A is imposed by the constraints of the problem, rather than designed, as in the case of L and H in wavelet theory. Finally, in dynamical sampling, the spatial sampling grids are not required to be regular.

In inverse problems, given an operator B that represents a physical process, the goal is to recover a function f from the observation Bf. Deconvolution or deblurring are prototypical examples. When B is not bounded below, the problem is considered ill-posed (see e.g., [47]). The dynamical sampling problem can be viewed as an inverse problem when the operator B is the result of applying the operators $S_{X_0}, S_{X_1}A, S_{X_2}A^2, \dots, S_{X_L}A^L$, where S_{X_l} is the sampling operator at time l on the set X_l, i.e., $B_X = [S_{X_0}, S_{X_1}A, S_{X_2}A^2, \dots, S_{X_L}A^L]^T$. However, unlike the typical inverse problem, in dynamical sampling the goal is to find conditions on L, $\{X_i : i = 0, \dots, L\}$, and A such that B_X is injective, well-conditioned, etc.

The dynamical sampling problem has connections and applications to other areas of mathematics, including, C^*-algebras, spectral theory of normal operators, and frame theory [5, 18, 22, 28, 30, 31, 53, 63].

Dynamical sampling has potential applications in plenacoustic sampling, on-chip sensing, data center temperature sensing, neuron-imaging, and satellite remote sensing, and more generally to wireless sensor networks (WSN). In wireless sensor networks, measurement devices are distributed to gather information about a physical quantity to be monitored, such as temperature, pressure, or pollution [38, 42, 55, 43, 57]. The goal is to exploit the evolutionary structure and the placement of sensors to reconstruct an unknown field. When it is not possible to place sampling devices at the desired locations (e.g. when there are not enough devices), then the desired information field can be recovered by placing the sensors elsewhere and taking advantage of the evolution process to recover the signals at the relevant locations. Even when the placement of sensors is not constrained, if the cost of a sensor is expensive relative to the cost of activating the sensor, then the relevant information may be recovered with fewer sensors placed judiciously and activated frequently. Super resolution is another application when an evolutionary process acts on the signal of interest.

In our recent work, we have taken a more operator-theoretic approach to the dynamical sampling problem [12, 6]. We assume that the unknown function f is a vector in some Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ and A is a bounded linear operator on $\mathscr{H}$. The samples are given in the form

$$\langle A^n f, g\rangle \text{ for } 0 \le n < L(g), \ \ g \in \mathscr{G} \tag{1.1}$$

where $\mathscr{G}$ is a countable (finite or infinite) set of vectors in $\mathscr{H}$, and the function $L : \mathscr{G} \to \{1, 2, \ldots, \infty\}$ represents the *"sampling level"* (Fig. 1.3). Then the main problem is to recover the unknown vector $f \in \mathscr{H}$ from the measurements (1.1).

Note that in this abstract setting we don't have a notion of domain on which the vectors are defined as functions so the samples or measurements are modeled as continuous linear functionals on $\mathscr{H}$ and, from Riesz representation theorem, every continuous functional Φ on Hilbert space $\mathscr{H}$ has the form $\Phi(f) = \langle f, g\rangle$ for every $f \in \mathscr{H}$ and some $g \in \mathscr{H}$.

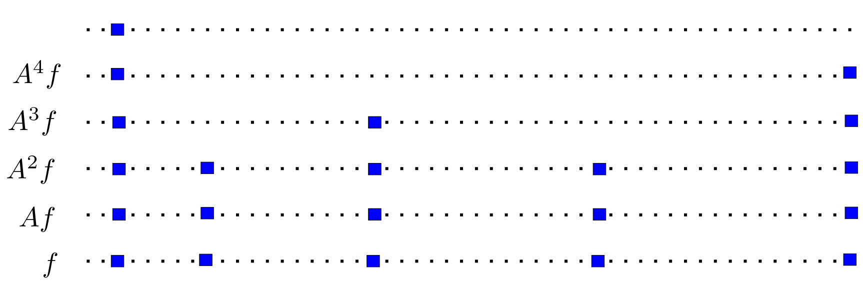

Figure 1.3: Spatio-temporal sampling

It is fundamental in applications that the reconstruction operator $R : \ell^2(X) \to \mathscr{H}$ given by $R(\langle A^n f, g\rangle) = f$ for all $f \in \mathscr{H}$, exists and is well-defined [43]. Moreover, it is very important that the function and data are continuously dependent on each other. This is because very often, the samples $\{\langle A^n f, g\rangle : (g,n) \in X\}$ are corrupted by "noise" $\{\varepsilon_{g,n} : (g,n) \in X\}$, and we require the reconstruction $\tilde{f} = R(\langle A^n f, g\rangle) + R(\varepsilon_{g,n})$ to be close to f when the noise is small. Similarly, the data from two vectors that are "close" should be close too. In other words, we want R to exist and the operators R, R^{-1} to be continuous, which is equivalent to the condition

$$\alpha\|f\|^2 \leq \sum_{g\in\mathscr{G}} \sum_{0\leq n<L(g)} |\langle A^n f, g\rangle|^2 \leq \beta\|f\|^2,$$

for every $f \in \mathscr{H}$, where $\alpha, \beta > 0$ are absolute constants. In this case, it is said that the reconstruction is *stable*.

Using the Hahn-Banach theorem and the relation between an operator and its adjoint, we make the following observations:

(a) Any $f \in \mathscr{H}$ can be recovered from the samples $\{\langle A^n f, g\rangle\}_{g\in\mathscr{G},0\leq n<L(g)}$ if and only if the system $\{(A^*)^n g\}_{g\in\mathscr{G},0\leq n<L(g)}$ is complete in $\mathscr{H}$.

(b) Any $f \in \mathscr{H}$ can be recovered from $\{\langle A^n f, g\rangle\}_{g\in\mathscr{G},0\leq n<L(g)}$ in a stable way if and only if the system $\{(A^*)^n g\}_{g\in\mathscr{G},0\leq n<L(g)}$ is a frame in $\mathscr{H}$.

Because of these equivalences, we drop the $*$ and investigate systems of iterations of the form $\{A^n g\}_{g\in\mathscr{G},0\leq n<L(g)}$, where A is a bounded operator on the Hilbert space $\mathscr{H}$ and $\mathscr{G}$ is a subset of $\mathscr{H}$. The goal is then to find conditions on A, L, and $\mathscr{G}$ such that $\{A^n g\}_{g\in\mathscr{G},0\leq n<L(g)}$ is complete, Bessel, frame, etc.

In Chapter 2, we introduce the main concepts and results used in the dissertation and provide references.

Chapter 3 offers a general introduction to the classical and dynamical sampling problems and reviews some of the existing results.

In Chapter 4, we discuss the iterative systems when the operator is normal. We characterize all countable subsets $\mathscr{G} \subset \mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<\infty}$ is complete in $\mathscr{H}$ when the operator A is a normal reductive operator (Theorem 4.1.1). These results are also extended to the system of vectors $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<L(g)}$, where L is any suitable function from $\mathscr{G}$ to $\mathbb{N}^*$. However, we also show that the system $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<\infty}$ always fails to be a basis for $\mathscr{H}$ when A is a normal operator (Corollary 4.2.2). In fact, if the set $\mathscr{G} \subset \mathscr{H}$ is finite, and A is a reductive normal operator, then $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<L(g)}$ cannot be a basis for $\mathscr{H}$ for any choice of the function L (Corollary 4.2.5). The obstruction to being a basis is the redundancy in the form of non-minimality of the set of vectors $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<L(g)}$. Two of the main theorems in Section 4.3 (Theorems 4.3.1 and 4.3.2) can be reformulated as

Theorem. *If for some set of vectors $\mathscr{G} \subset \mathscr{H}$, $\{A^n g\}_{g\in\mathscr{G},\, n\ge 0}$ is a complete Bessel system in $\mathscr{H}$, then $\|A\| \le 1$ and, for the projection valued spectral measure E_A of A, $E_A|_{S_1}$ is absolutely continuous with respect to the arc length measure (the Lebesgue measure) on the unit circle S_1.*

Conversely, if $A \in B(\mathscr{H})$ is a normal operator, $\|A\| \le 1$ and $E_A|_{S_1}$ is absolutely continuous with respect to the arc length measure on S_1, then there exists a countable set $\mathscr{G} \subset \mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\ge 0}$ is a complete Bessel system.

Theorem. *If A is a normal operator and $\{A^n g\}_{g\in\mathscr{G},\, n\ge 0}$ is a frame in $\mathscr{H}$ for some countable set $\mathscr{G} \subset \mathscr{H}$ then $\|A\| \le 1$ and $E_A|_{S_1} = 0$.*

Conversely, if $\|A\| \le 1$ and $E_A|_{S_1} = 0$ then there exists a countable set $\mathscr{G} \subset \mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\ge 0}$ is a Parseval frame for $\mathscr{H}$.

Thus, the system $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<\infty}$, (or $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<L(g)}$) may be a frame, but cannot be a basis. However, it is difficult for a system of vectors of the form $\{A^n g\}_{g\in\mathscr{G},\, 0\le n<\infty}$ to be a frame as the spectrum of A must be very special. When the set $\mathscr{G}$ is finite such frames do exist, as shown by the constructions in [7]. Surprisingly, the difficulty becomes an obstruction if we normalize the system of iterations to become $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},\, n\ge 0}$ when the operator A is self-adjoint as described in Section 4.5 (Theorem 4.5.2).

In Section 4.6, we apply our results to systems that are generated by the unitary actions of a discrete group Γ on a set of vectors $\mathscr{G} \subset \mathscr{H}$ which is common in many constructions of wavelets and frames.

In Chapter 5, we present our results for the case of general operators. We show that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ being a frame in $\mathscr{H}$ implies that $(A^*)^n f \to 0$ as $n \to \infty$ for every $f \in \mathscr{H}$ (Theorem 5.0.1). Under the additional condition that $\|A\| \leq 1$, we also prove the inverse in the following sense: if for every $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$, then we can choose $\mathscr{G} \subseteq \mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a Parseval frame (Theorem 5.0.3).

In Chapter 6, we consider the dynamical sampling problem when the unknown initial function f is modeled as an element of a shift-invariant space. Necessary and sufficient conditions for stable reconstruction from dynamical samples is found and some special cases are discussed.

Chapter 2

Notation and preliminaries

2.1 Systems of vectors in Hilbert space

Let $\mathscr{H}$ be a complex Hilbert space where the inner product is denoted by $\langle h, g \rangle$ for $h, g \in \mathscr{H}$, and let I be a countable (finite or infinite) set. A system of vectors indexed by a set I is any mapping $i \mapsto h_i$ from I to $\mathscr{H}$ that we denote by $\{h_i\}_{i\in I}$. In the system of vectors, compared to a set of vectors, we allow repetitions. Often in the literature the set I is assumed to have some form of order associated with it because infinite sums can converge conditionally (i.e. convergence depends on the specific order of the elements in the series), but for our investigations the order of summation will not affect the sum of series because frames provide unconditional convergence.

Note that the indexing of the iterative system $\{A^n g\}_{g\in\mathscr{G}, 0\leq n<L(g)}$, which is one of the main objects of our research, is given by the set

$$I = \{(g,n) : \ g \in \mathscr{G}, 0 \leq n < L(g)\}.$$

We denote by $\ell^2(I)$ the set of all $c = \{c_i\}_{i\in I}$, $c_i \in \mathbb{C}$, such that

$$\sum_{i\in I} |c_i|^2 = \|c\|^2_{\ell^2(I)} < \infty.$$

Definition 2.1.1. *Given a system of vectors $\mathscr{E} = \{h_i\}_{i\in I}$ in $\mathscr{H}$, we say that*

- *$\mathscr{E}$ is complete if the closure of the linear span of $\mathscr{E}$ (denoted by $\overline{\operatorname{span}}\,\mathscr{E}$) is equal to $\mathscr{H}$.*

- *$\mathscr{E}$ is minimal if, for every $h_i \in \mathscr{E}$, $h_i \notin \overline{\operatorname{span}}\{h_j\}_{j\in I, j\neq i}$.*

- *$\mathscr{E}$ is a Riesz system if there exists numbers $0 < \alpha \leq \beta$ such that*

$$\alpha \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i h_i \right\|^2 \leq \beta \sum_{i \in I} |c_i|^2,$$

  for every $c = \{c_i\}_{i \in I} \in \ell^2(I)$.

  If a Riesz system is complete, we call it a Riesz basis.

- *$\mathscr{E}$ is a Bessel system if there exists a number $\beta > 0$ such that, for every $f \in \mathscr{H}$,*

$$\sum_{i \in I} |\langle f, h_i \rangle|^2 \leq \beta \|f\|^2.$$

- *$\mathscr{E}$ is a frame in $\mathscr{H}$ if there exists numbers $0 < \alpha \leq \beta$ numbers such that, for every $f \in \mathscr{H}$,*

$$\alpha \|f\|^2 \leq \sum_{i \in I} |\langle f, h_i \rangle|^2 \leq \beta \|f\|^2.$$

  If $\alpha = \beta = 1$ we say that the system is a Parseval frame.

Some of the relations between these different properties of systems of vectors is given below [20, 33, 35]:

(a) A Riesz system is minimal.

(b) A system is a Riesz basis if and only if it is a frame and minimal.

(c) In finite-dimensional space, a finite set of vectors is a frame if and only if it is complete.

The notion of frames was introduced by Duffin and Schaffer [25] in the context of non-harmonic Fourier series as a generalization of the notion of Riesz bases. Later frames found a wide range of applications in mathematics and engineering.

The three main properties of frames that make them useful in applications are

1. Redundancy: many systems of vectors are not minimal, and it is not practical to throw out some of them to achieve minimality. Moreover, the redundancy can be utilized for reducing the reconstruction error in the presence of noise in the measurements.

2. Existence of reconstruction formula: it turns out that if the system of vectors $\{h_i\}_{i\in I}$ is a frame in $\mathscr{H}$, then there exists another system of vectors $\{\eta_i\}_{i\in I}$ called the dual frame of $\{h_i\}_{i\in I}$ such that, for any vector $f \in \mathscr{H}$,

$$f = \sum_{i\in I} \langle f, h_i \rangle \eta_i.$$

The dual frame can be computed using the formula $\eta_i = S^{-1} h_i$ where S is the frame operator defined as

$$S(f) = \sum_{i\in I} \langle f, h_i \rangle h_i.$$

There are iterative methods to compute the operator S^{-1} [35].

For Parseval frames, $\eta_i = h_i$ and we have

$$f = \sum_{i\in I} \langle f, h_i \rangle h_i.$$

which is called Parseval identity, and it makes Parseval frames a natural generalization of orthonormal bases.

3. The stability of the coefficients and the reconstruction in the presence of noise: if f is corrupted by "noise" ε, then, for the coefficients $\{\langle \tilde{f}, h_i \rangle\}_{i\in I}$ of $\tilde{f} = f + \varepsilon$, we get

$$\sum_{i\in I} \left| \langle f, h_i \rangle - \langle \tilde{f}, h_i \rangle \right|^2 \le \beta \|\varepsilon\|^2,$$

where β is the upper frame bound. Thus a small noise added to the function will result in a small change in the coefficients. Similarly, if we have noise in the coefficients $\tilde{c}_i = \langle f, h_i \rangle + \varepsilon_i$ then, for the noisy reconstruction $\tilde{f} = \sum_{i\in I} \tilde{c}_i \eta_i$, from the lower frame bound we get that

$$\|f-\tilde{f}\|^2 \leq \frac{1}{\alpha} \sum_{i \in I} |\varepsilon_i|^2,$$

which means a small error in the coefficients results in a small error in the reconstruction.

2.2 Shift-invariant spaces

As pointed out in the introduction, the Paley – Wiener spaces $PW_\sigma(\mathbb{R})$ are often used for modeling analog signals in signal processing. From the mathematical perspective, we can always assume that $\sigma = \frac{1}{2}$ after scaling. In this case, we use the notation $PW(\mathbb{R})$.

The function

$$\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}$$

is called the *sinc function*.

Theorem 2.2.1. $\{\operatorname{sinc}(t-k)\}_{k \in Z}$ *is an orthonormal basis in* $PW(\mathbb{R})$*, and for any* $f \in PW(\mathbb{R})$*,*

$$f(t) = \sum_{k \in \mathbb{Z}} f(k) \operatorname{sinc}(t-k),$$

where the convergence is understood in the L^2 *sense.*

One important property of the Paley-Wiener space $PW(\mathbb{R})$ is that it is invariant under integer shifts, i.e. if $f(t) \in PW(\mathbb{R})$ then, for every $k \in \mathbb{Z}$, $f(t-k) \in PW(\mathbb{R})$. We call the spaces with this type of property *Shift-Invariant Space*s (SIS). Shift invariant spaces are the typical space of functions considered in sampling theory [10, 13, 67, 62, 64, 46, 72]. Shift invariant spaces also appear in many other fields of analysis such as Wavelet theory (e.g. multiresolution analysis), Splines, Gabor analysis, etc.

Let $\phi \in L^2(\mathbb{R})$ and denote by $V(\phi)$ the span closure of the system $\{\phi(\cdot - k)\}_{k \in \mathbb{Z}}$ of integer translates of ϕ. This kind of spaces are often called principal shift-invariant spaces or singly generated shift-invariant spaces.

In this notation, $PW(\mathbb{R}) = V(\text{sinc})$; moreover the integer shifts of $\text{sinc}(t)$ is an orthonormal basis of $PW(\mathbb{R})$ as we saw above.

The advantage of working with shift-invariant spaces is that, even though $V(\phi)$ is a space of functions defined on $\mathbb{R}$, it has similar properties to $\ell^2(\mathbb{Z})$ and many questions related to sampling can be reduced to $\ell^2(\mathbb{Z})$. A standard assumption is that any function in $V(\phi)$ is uniquely determined by its values on $\mathbb{Z}$. In particular, the condition

$$\sum_{k\in\mathbb{Z}} \hat{\phi}(\xi+k) \neq 0 \ \ a.e.$$

that will be assumed to hold throughout this dissertation, guarantees this property [13, 68].

Proposition ([17, 35]). *Let $\Phi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+k)|^2$, called the periodization of $|\hat{\phi}|^2$. Then*
(a) $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for $V(\phi)$ if and only if there exist $0 < \alpha \leq \beta$ such that

$$\alpha \leq \Phi(\xi) \leq \beta, \ \ a.e. \ \ \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{2.1}$$

(b) $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a frame in $V(\phi)$ if and only if there exist $0 < \alpha \leq \beta$ such that

$$\alpha \cdot \mathbb{1}_{\text{supp}\,(\Phi)}(\xi) \leq \Phi(\xi) \leq \beta \cdot \mathbb{1}_{\text{supp}\,(\Phi)}(\xi), \ \ a.e. \ \ \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

(c) If $\phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a frame if and only if it is a Riesz basis for $V(\phi)$.

To be able to consider the sampling problem, the sampling operator $f \mapsto \{f(k)\}_{k\in Z}$ should be well-defined. For that reason, we need both local and global control for the function ϕ. This can be done by using the Wiener amalgam spaces [10].

Definition 2.2.2. *We say that a measurable function f belongs to the Wiener amalgam* *space $W(L^p)$, $1 \le p < \infty$, if it satisfies*

$$\|f\|_{W(L^p)}^p := \sum_{k\in\mathbb{Z}} \operatorname{ess\,sup}\{|f(t+k)|^p \,:\, t \in [0,1]\} < \infty$$

and, for $p = \infty$,

$$\|f\|_{W(L^\infty)} := \sup_{k\in\mathbb{Z}}\{\operatorname{ess\,sup}\{|f(t+k)| \,:\, t \in [0,1]\}\} < \infty.$$

For $1 \le p \le \infty$, denote $W_0(L^p) = W(L^p) \cap C(\mathbb{R})$.

Proposition 2.2.3 ([10])**.** *(a) $W_0(L^p) \subset L^p$,*

(b) if $a \in L^p$ and $\phi \in W_0(L^1)$, then $a * \phi \in W_0(L^p)$,

(c) if $\phi \in W_0(L^1)$ and $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ is a Riesz basis for $V(\phi)$, then $V(\phi) \subset W_0(L^2)$. Moreover, if $f \in V(\phi)$ and

$$f(t) = \sum_{k\in\mathbb{Z}} c_k \phi(t-k)$$

then $\|f\|_{L^2} \asymp \|c\|_{\ell^2} \asymp \|f\|_{W(L^2)}$.

2.3 Operators in Hilbert space

We denote by $B(\mathscr{H})$ the space of all bounded linear operators mapping the Hilbert space $\mathscr{H}$ to $\mathscr{H}$.

For every operator $A \in B(\mathscr{H})$ there exists another uniquely determined operator $A^* \in B(\mathscr{H})$, called its *adjoint* [22], such that

$$\langle Af, g\rangle = \langle f, A^* g\rangle$$

for every $f, g \in \mathscr{H}$.

Definition 2.3.1. *An operator $A \in B(\mathscr{H})$ is called*

- *normal if*

$$AA^* = A^*A$$

- *self-adjoint if*

$$A = A^*$$

- *unitary if*

$$AA^* = A^*A = Id$$

(where Id is the identity operator) or, equivalently, $\|Af\| = \|f\|$ *for every* $f \in \mathscr{H}$*.*

One of the main tools that we use in this work is the spectral theorem for normal operators described below.

Let μ be a non-negative regular Borel measure on $\mathbb{C}$ with compact support K. Denote by N_μ the operator

$$N_\mu f(z) = zf(z),\ z \in K$$

acting on functions $f \in L^2(\mu)$ (i.e. $f : \mathbb{C} \to \mathbb{C}$, measurable with $\int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty$.)

For a Borel non-negative measure μ, we will denote by $[\mu]$ the class of Borel measures that are mutually absolutely continuous with μ.

Theorem 2.3.2 (Spectral theorem)**.** *For any normal operator A on* $\mathscr{H}$*, there are mutually singular compactly supported non-negative Borel measures* μ_j*,* $1 \le j \le \infty$*, such that A is equivalent to the operator*

$$N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus \cdots$$

i.e. there exists a unitary transformation

$$U : \mathscr{H} \to (L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$$

such that

$$UAU^{-1} = N_{\mu_\infty}^{(\infty)} \oplus N_{\mu_1} \oplus N_{\mu_2}^{(2)} \oplus \cdots . \tag{2.2}$$

Moreover, if M is another normal operator with corresponding measures $\nu_\infty, \nu_1, \nu_2, \ldots$ then M is unitarily equivalent to A if and only if $[\nu_j] = [\mu_j],\ j = 1, \ldots, \infty$.

A proof of the theorem can be found in [22] (Ch. IX, Theorem 10.16) and [21] (Theorem 9.14).

Example 1. *Let A be the 8×8 diagonal matrix*

$$A = \begin{pmatrix} \lambda_1 I_2 & 0 & 0 \\ 0 & \lambda_2 I_3 & 0 \\ 0 & 0 & \lambda_3 I_3 \end{pmatrix}$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and I_j denotes the $j \times j$ identity matrix. For this case, the theorem above gives: $\widetilde{\mathscr{H}} = (L^2(\mu_2))^{(2)} \oplus (L^2(\mu_3))^{(3)},\ \mu_2 = \delta_{\lambda_1},\ \mu_3 = \delta_{\lambda_2} + \delta_{\lambda_3}$, where δ_x is the Dirac measure at x. If $g = (g_1, \ldots, g_8)^T$, then $Ug = \tilde{g} = \left(\tilde{g}_j\right)$. In particular, $\widetilde{g_3}(\lambda_2) = \begin{pmatrix} g_3 \\ g_4 \\ g_5 \end{pmatrix}$, $\widetilde{g_3}(\lambda_3) = \begin{pmatrix} g_6 \\ g_7 \\ g_8 \end{pmatrix}$ and $\widetilde{g_3}(z) = \vec{0}$ for $z \neq \lambda_2, \lambda_3$ (in fact for $z \neq \lambda_2, \lambda_3$, $\widetilde{g_3}(z)$ can take any value since the measure μ_3 is concentrated on $\{\lambda_2, \lambda_3\} \subset \mathbb{C}$). We have

$$\begin{aligned} \langle Uf, Ug \rangle &= \int_{\mathbb{C}} \langle \tilde{f}(z), \tilde{g}(z) \rangle d\mu(z) \\ &= \int_{\mathbb{C}} \langle \tilde{f}_2(z), \tilde{g}_2(z) \rangle d\mu_2(z) + \int_{\mathbb{C}} \langle \tilde{f}_3(z), \tilde{g}_3(z) \rangle d\mu_3(z) \\ &= \langle \tilde{f}_2(\lambda_1), \tilde{g}_2(\lambda_1) \rangle + \langle \tilde{f}_3(\lambda_2), \tilde{g}_3(\lambda_2) \rangle + \langle \tilde{f}_3(\lambda_3), \tilde{g}_3(\lambda_3) \rangle \\ &= \sum_{j=1}^{8} f_j \overline{g}_j = \langle f, g \rangle. \end{aligned}$$

Since the measures μ_j are mutually singular, there are mutually disjoint Borel sets $\{\mathscr{E}_j\}$ such that μ_j is concentrated on $\mathscr{E}_j$ for every $1 \leq j \leq \infty$.

We will define the scalar measure μ, (usually called *the scalar spectral measure)* associated with the normal operator A to be

$$\mu := \sum_{1\leq j\leq\infty} \mu_j. \tag{2.3}$$

The Borel function $J : \mathbb{C} \to \mathbb{N}^* \cup \{0\}$ given by

$$J(z) = \begin{cases} j, & z \in \mathscr{E}_j \\ 0, & otherwise \end{cases} \tag{2.4}$$

is called multiplicity function of the operator A.

From Theorem 2.3.2, every normal operator is uniquely determined, up to a unitary equivalence, by the pair $([\mu], J)$.

For $j \in \mathbb{N}$, define Ω_j to be the set $\{1,...,j\}$ and Ω_∞ to be the set $\mathbb{N}$. Note that $\ell^2(\Omega_j) \cong \mathbb{C}^j$, for $j \in \mathbb{N}$, and $\ell^2(\Omega_\infty) = \ell^2(\mathbb{N})$. For $j = 0$, we define $\ell^2(\Omega_0)$ to be the trivial space $\{0\}$.

Let $\mathscr{W}$ be the Hilbert space

$$\mathscr{W} := (L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$$

associated with the operator A and let $U : \mathscr{H} \to \mathscr{W}$ be the unitary operator given by Theorem 2.3.2. If $g \in \mathscr{H}$, we will denote by $\widetilde{g}$ the image of g under U. Since $\widetilde{g} \in \mathscr{W}$ we have $\widetilde{g} = (\widetilde{g}_j)_{j\in\mathbb{N}^*}$, where $\widetilde{g}_j$ is the restriction of $\widetilde{g}$ to $(L^2(\mu_j))^{(j)}$. Thus, for any $j \in \mathbb{N}^*$, $\widetilde{g}_j$ is a function from $\mathbb{C}$ to $\ell^2(\Omega_j)$ and

$$\sum_{j\in\mathbb{N}^*} \int_{\mathbb{C}} \|\widetilde{g}_j(z)\|^2_{\ell^2(\Omega_j)} d\mu_j(z) < \infty.$$

Let P_j be the projection defined for every $\widetilde{g} \in \mathscr{W}$ by $\widetilde{f} = P_j\widetilde{g}$ where $\widetilde{f}_j = \widetilde{g}_j$ and $\widetilde{f}_k = 0$ for $k \neq j$.

Let E_A be the projection valued spectral measure for the normal operator A. Then for every μ-measurable set $G \subseteq \mathbb{C}$ and vectors f, g in $\mathscr{H}$ we have the following formula

$$\langle E_A(G)f, g\rangle_{\mathscr{H}} = \int_G \left[\sum_{1 \le j \le \infty} \mathbb{1}_{\mathscr{E}_j}(z) \langle \widetilde{f}_j(z), \widetilde{g}_j(z) \rangle_{\ell^2(\Omega_j)} \right] d\mu(z),$$

which relates the spectral measure of A with the scalar spectral measure of A.

In [40] and [1], the spectral multiplicity of multiplication operator is computed.

As a generalization of self-adjoint operators, we will consider normal reductive operators. Reductive operators were first studied by P. Halmos [32] and J. Wermer [70].

Definition 2.3.3. *A closed subspace $\mathscr{V} \subseteq \mathscr{H}$ is called reducing for the operator A if both $\mathscr{V}$ and its orthogonal complement $\mathscr{V}^{\perp}$ are invariant subspaces of A.*

Notice that, $\mathscr{V} \subseteq \mathscr{H}$ being a reducing subspace for A is equivalent to $\mathscr{V}$ being an invariant subspace both for A and its adjoint A^*, and also equivalent to $AP_{\mathscr{V}} = P_{\mathscr{V}}A$ where $P_{\mathscr{V}}$ is the projection operator onto $\mathscr{V}$.

Definition 2.3.4. *An operator A is called reductive if every invariant subspace of A is reducing.*

It is not known whether every reductive operator is normal. In fact, every reductive operator being normal is equivalent to the veracity of the long-standing *invariant subspace conjecture*, which states that every bounded operator on a separable Hilbert space has a non-trivial closed invariant subspace [26].

Proposition 2.3.5. *[41] A normal operator is reductive if and only if its restriction to every invariant subspace is normal.*

Proposition 2.3.6 ([70])**.** *Let A be a normal operator on the Hilbert space $\mathscr{H}$ and let μ_j be the measures in the representation* (2.2) *of A. Let μ be as in* (2.3)*. Then A is reductive if and only if for any two vectors $f, g \in \mathscr{H}$*

$$\int_{\mathbb{C}} z^n \left[\sum_{1 \le j \le \infty} \mathbb{1}_{\mathscr{E}_j}(z) \langle \widetilde{g}_j(z), \widetilde{f}_j(z) \rangle_{\ell^2(\Omega_j)} \right] d\mu(z) = 0$$

for every $n \geq 0$ *implies* μ_j*-a.e.* $\langle \widetilde{g}_j(z), \widetilde{f}_j(z) \rangle_{\ell^2(\Omega_j)} = 0$ *for every* $j \in \mathbb{N}^*$.

Note that the property in the above proposition is equivalent to the implication

$$\int_{\mathbb{C}} z^n h(z) d\mu(z) = 0 \text{ and } h \in L^1(\mu) \Rightarrow h = 0 \; \mu - \text{a.e.}$$

As proved in [70], being reductive is not entirely a property of the spectrum: it is possible to find two operators with the same spectrum such that one is reductive and the other is not. However, the following sufficient condition holds.

Proposition 2.3.7 ([70])**.** *Let* A *be a normal operator on* $\mathscr{H}$ *whose spectrum* $\sigma(A)$ *has empty interior and* $\mathbb{C} \setminus \sigma(A)$ *is connected. Then* A *is reductive.*

Corollary 2.3.8. *Every self-adjoint operator on a Hilbert space is reductive.*

The fact that self-adjoint operators are reductive is easily derived without the use of Proposition 2.3.7. However, to see how this fact follows from Proposition 2.3.7, simply note that for a self-adjoint operator A, $\sigma(A)$ is a compact subset of $\mathbb{R}$, hence it has empty interior (as a subset of $\mathbb{C}$), and $\mathbb{C} - \sigma(A)$ is connected.

Also the following necessary condition for being reductive holds.

Proposition ([59])**.** *Let* A *be a normal operator. If the interior of* $\sigma(A)$ *is not empty then* A *is not reductive.*

Definition 2.3.9. *1) An operator* $S \in B(\mathscr{H})$ *is called unilateral shift operator if it is an isometry and there exists a subspace* $\mathscr{V} \subset \mathscr{H}$ *such that*

$$\mathscr{H} = \bigoplus_{n \geq 0} S^n(\mathscr{V}).$$

2) An operator $T \in B(\mathscr{H})$ *is called bilateral shift operator if it is unitary and there exists* *a subspace $\mathscr{V} \subset \mathscr{H}$ such that*

$$\mathscr{H} = \bigoplus_{n \in \mathbb{Z}} T^n(\mathscr{V}).$$

Chapter 3

The classical and dynamical sampling problems

3.1 The classical sampling problem

Shift invariant spaces are a common choice in signal processing for modeling analog signals. In this thesis, we work with singly generated shift-invariant spaces but multiply and infinitely generated shift-invariant spaces are often considered in signal processing as well. Spaces with wavelet and Gabor generators are examples of infinitely generated shift-invariant spaces. Wavelets are typically used for modeling images and Gabor frames are used for audio signals, utilizing their symmetries with respect to frequency shifts. [27]

When the underlying domain on which the functions are defined is not discrete, we need to ensure the pointwise evaluation is well-defined. For example, in $L^2(\mathbb{R})$ any two functions that coincide on a set of measure zero are considered to be equal so the sampling in the domain of the function makes no sense here.

Let $\mathscr{H}$ be a Hilbert space of functions defined on the set X.

Definition 3.1.1. *$\mathscr{H}$ is called a Reproducing Kernel Hilbert Space (RKHS) if the evaluation function $\delta_x : g \mapsto g(x)$, $\delta_x : \mathscr{H} \to \mathbb{C}$ is well defined and is continuous for every $x \in X$.*

The point evaluation function is continuous hence, from the Riesz representation theorem, for every $x \in X$ there exist a unique element $K_x \in \mathscr{H}$ such that $f(x) =< f, K_x >$.

In [10], the authors find conditions under which the singly generated shift-invariant spaces become RKHS and thus the sampling problem can be considered.

In recent years compressed sampling has become a popular field of investigation in signal processing and applied mathematics. Just like in dynamical sampling, in compressed sensing the samples are assumed to be scarce resulting in an ill-posed problem, and that makes the recovery of the function impossible unless additional assumptions are made about the function. In compressed sampling, to compensate for the scarceness in samples, it is assumed that many of the coefficients in the representation of the signal are equal to zero. Then the problem becomes to find which of the coefficients are non-zero and obtaining their values using the samples. [29]

3.2 The dynamical sampling problem

The problem of spatio-temporal sampling and reconstruction was first addressed in the engineering literature, in the work of M. Vatterli and Y. Lu [43] when the evolution operator is given by the diffusion process. They consider the one-dimensional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = K\frac{\partial^2 u(x,t)}{\partial x^2} \text{ for } x \in \mathbb{R}, t > 0$$

where K is the diffusion coefficient and can be assumed to be equal to 1 by rescaling the time. They assume the spatio-temporal samples of the function $u(x,t)$ are given at locations $\Omega \subset \mathbb{R}$ and at times

$$\{k\tau\}_{k=0,\dots,L-1}$$

where τ is the uniform sampling length in the time, and the goal is to reconstruct the $u(x,0) = f(x)$. They assume the function f can be well approximated by band-limited functions and restrict their attention to this class. In particular, they point out that for the uniform spatial samples and for the sub-Nyquist rate there is no stable reconstruction (at a sup-Nyquist rate the samples at the 0th level already provide stable reconstruction), and proceed to get results for the case of non-uniform samples.

The work in dynamical sampling for a more general setting was started with the convolution operators on the spaces $\ell^2(\mathbb{Z}_d)$, $\ell^2(\mathbb{Z})$ and $L^2(\mathbb{R})$ with the samples taken on a sparse uniform grid [8, 2, 3, 4, 9].

In [9], the authors assume $\mathscr{H} = \ell^2(\mathbb{Z})$ and A is convolution operator with a kernel $a \in \ell^1(\mathbb{Z})$, i.e. $Af = a * f$. Let $\mathscr{G} = \{e_{mk}\}_{k\in\mathbb{Z}}$ for some $m > 1$ where $\{e_k\}_{k\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. The Fourier transform of a is defined as

$$\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a(k)e^{-2\pi i\xi k},\ \xi \in [0,1].$$

Denote

$$\mathscr{A}_m(\xi) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \hat{a}(\frac{\xi}{m}) & \hat{a}(\frac{\xi+1}{m}) & \dots & \hat{a}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{(L-1)}(\frac{\xi}{m}) & \hat{a}^{(L-1)}(\frac{\xi+1}{m}) & \dots & \hat{a}^{(L-1)}(\frac{\xi+m-1}{m}) \end{pmatrix}.$$

Let $\sigma(\xi)$ denote the smallest singular value of the matrix $\mathscr{A}_m(\xi)$. Let $L(g) = M$ for each $g \in \mathscr{G}$. From [9], the system $\{A^n g\}_{g\in\mathscr{G},0\le n<M}$ is complete in $\ell^2(\mathbb{Z})$ if and only if $\mathscr{A}_m(\xi)$ has a left inverse for a.e. $\xi \in [0,1]$, or equivalently $\sigma(\xi) > 0$ for a.e. $\xi \in [0,1]$, and it forms a frame if and only if $\sigma(\xi) \ge \alpha$ for a.e. $\xi \in [0,1]$ for some $\alpha > 0$. Since $\mathscr{A}_m(\xi)$ is a Vandermonde matrix, iterations $n > m-1$ will not affect the completeness of the system. Thus, we let $M = m$. In that case, $\{A^n g\}_{g\in\mathscr{G},0\le n\le m-1}$ is complete in $\ell^2(\mathbb{Z})$ if and only if $\det\mathscr{A}_m(\xi) \neq 0$ for a.e $\xi \in [0,1]$, and it is a frame if and only if for a.e $\xi \in [0,1]$, $|\det\mathscr{A}_m(\xi)| \ge \alpha\}$ for some $\alpha > 0$.

Although there are infinitely many convolution operators that satisfy this last condition, many natural operators in practice do not. For example, an operator where a is real, even and $\hat{a}$ is strictly decreasing on $[0,\frac{1}{2}]$. For this case, it can be shown that the matrices $\mathscr{A}_m(0)$ and $\mathscr{A}_m(\frac{1}{2})$ are singular, while all the other matrices $\mathscr{A}_m(\xi)$ are invertible. For this case, any set of the form $\mathscr{G} = \{e_{mk}\}_{k\in\mathbb{Z}} \cup \{e_{mlk+1}\}_{k\in\mathbb{Z}}$ where $l \ge 1$, produces a system $\{A^n g\}_{g\in\mathscr{G},0\le n\le m-1}$ which is a frame for $\mathscr{H} = \ell^2(\mathbb{Z})$.

In [9], the results in [3] has been generalized to the multidimensional case when the convolution kernel has what's called strongly quadrantal symmetry.

In [11], the authors allow the operator A to be unknown too. They use a generalization of Prony's method to reconstruct the spectrum of A and eventually the function itself.

In [54] when the locations of the sampling positions are allowed to change, is considered.

In [7], the authors consider the case when $f \in \ell^2(\mathbb{N})$, A is a diagonalizable operator on $\ell^2(\mathbb{N})$, i.e. A is a bounded self-adjoint operator on $\ell^2(\mathbb{N})$ such that there exists a basis of eigenvectors of A. Then there is a bounded invertible operator B such that $A = B^{-1}DB$ and $D = \sum_j \lambda_j P_j$ is an infinite diagonal matrix, where $\sigma(A) = \{\lambda_j\} \subset \mathbb{R}$ is the pure spectrum of A and P_j is the projections onto the eigenspace E_j corresponding to eigenvalue λ_j. Given a set $\Omega \subset \mathbb{N}$, for

$$l_i = \min\left\{n : (A^*)^n e_i \in \operatorname{span}\{e_i, \cdots, (A^*)^{n-1} e_i\}\right\} \ i \in \Omega$$

(l_i is the degree of minimal A-annihilator of basis vector e_i), they prove the following theorem

Theorem. *$\{A^n e_i : i \in \Omega, n = 0, \ldots, l_i - 1\}$ is complete in $\ell^2(\mathbb{N})$ if and only if, for each j, the set $\left\{P_j(Be_i) : i \in \Omega\right\}$ is complete in the corresponding eigenspace E_j.*

They also find a necessary and a sufficient condition for the existence of a single sampling location that allows stable recovery.

Theorem. *There exists an $i_0 \in \mathbb{N}$ such that $\{A^n e_{i_0} : n = 0, 1 \ldots\}$ is a frame for $\ell^2(N)$, if and only if the following are satisfied for the eigenvalues of A*
(i) $|\lambda_j| < 1$ for every j
(ii) $|\lambda_j| \to 1$
(iii) $\{\lambda_j\}$ satisfy Carleson's condition

$$\inf_k \prod_{j \neq k} \frac{|\lambda_j - \lambda_k|}{|1 - \bar{\lambda}_j \lambda_k|} \geq \delta$$

for some $\delta > 0$
(iv) $C_1\sqrt{1 - |\lambda_j|^2} \leq |Be_{i_0}(j)| \leq C_2\sqrt{1 - |\lambda_j|^2}$ where $0 < C_1 \leq C_2$.

An interesting field of application for the dynamical sampling is graph theory and geometric data analysis. The authors in [56] provide one example of such framework where the dynamical sampling can be applied. Let $G=(V,E)$ be a finite, simple, connected graph associated with the weight function w. The operator

$$\mathscr{L} = Id - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$$

is called symmetric Laplacian of the graph, where Id is the identity matrix, D is the degree matrix of G and A is the adjacency matrix of G.

Let $A = D^{-\frac{1}{2}} e^{-\mathscr{L}} D^{\frac{1}{2}}$, and consider the dynamical sampling problem with the spatio-temporal samples

$$f|_{\Omega},\ Af|_{\Omega},\ \cdots,\ A^{L-1}f|_{\Omega} \tag{3.1}$$

where $\Omega \subset V$ and $f \in \ell^2(V)$. Note that $A^n f$ is the discrete time homogeneous heat evolution with the initial state f.

If $f = \delta_x$, where

$$\delta_x(y) = \begin{cases} 1, & y = x \\ 0, & \text{otherwise} \end{cases}$$

then (3.1) is called the spectral signature of the vertex x. Spectral signature is a special case of dynamical samples when the initial state is known to be one of the standard basis vectors

In [56], the authors show that for every finite graph there exists a $\Omega \subset V$ with $|\Omega| < |V|$ such that every vertex has a unique signature. Also, if all the eigenvectors of the Laplacian are different, and at least one of the eigenvectors has all non-zero entries then there exists a single vertex $x \in V$ such that every other vertex has a unique signature for $\Omega = \{x\}$.

They also offer an algorithm, that computes the symmetries of the graph G when the dynamical sampling problem 3.1 has a unique solution. Then conduct numerical experiments and successfully compute the symmetry group of several graphs.

Chapter 4

The case of normal operators

Let $\mathscr{H}$ be an infinite-dimensional separable complex Hilbert space, $A \in B(\mathscr{H})$ be a bounded normal operator and $\mathscr{G}$ a countable (finite or countably infinite) collection of vectors in $\mathscr{H}$. Let L be a function $L : \mathscr{G} \to \mathbb{N}^*$, where $\mathbb{N}^* = \{1, 2, \dots\} \cup \{+\infty\}$. We are interested in the structure of the set of iterations of the operator A when acting on the vectors in $\mathscr{G}$ and are limited by the function L. More precisely, we are interested in the following two questions:

(I) Under what conditions on A, $\mathscr{G}$ and L is the iterated system of vectors

$$\{A^n g : g \in \mathscr{G},\ 0 \le n < L(g)\}$$

complete, Bessel, a basis, or a frame for $\mathscr{H}$?

(II) If $\{A^n g : g \in \mathscr{G},\ 0 \le n < L(g)\}$ is complete, Bessel, a basis, or a frame for $\mathscr{H}$ for some system of vectors $\mathscr{G}$ and a function $L : \mathscr{G} \to \mathbb{N}^*$, what can be deduced about the operator A?

We study these and other related questions and we give answers in many important and general cases. In particular, we show that there is a direct relation between the spectral properties of a normal operator and the properties of the systems of vectors generated by its iterative actions on a set of vectors. We are hoping that the questions above and the approach we use can be interesting for research in both, frame theory and operator theory.

The questions above, in their formulation have similarities with problems involving cyclical vectors in operator theory, and our analysis relies on the spectral theorem for normal operators with multiplicity [22]. There have been some attempts to generalize multiplicity theory to non-normal operators [49]. Although it cannot be generalized entirely, some aspects of it have been extended to general operators. In finite dimensions, the spectral theorem for normal operators, represents the underlying space as a sum of invariant subspaces. For general operators, the decomposition into invariant subspaces leads to Jordan's theorem. In the infinite-dimensional case, the extension leads to a decomposition into invariant subspaces, and one of the goals is to give conditions under which these subspaces $\{S_n\}$ form Riesz basis or equivalently unconditional basis, see [49, 65] and the references therein (this notion of Riesz basis is related but different from the one in Definition (2.1.1)). The multiplicity of a spectral value for a normal operators has also been extended. For general operators, a global multiplicity (called multicyclicity) is particularly useful in the context of control theory: using multicyclicity theory for a completely non-unitary contraction A, a formula for $\min|\mathscr{G}|$ such that $\{A^n g : g \in \mathscr{G},\, n \geq 0\}$ is complete in $\mathscr{H}$ was obtained (see [50, 48] and the references therein). For a normal operator A, this number can be deduced from Theorem 4.1.1 below.

Our main goal in this chapter is to find frames or other types of systems through the iterative action of a normal operator, and we use the full power of the spectral theorem for normal operators. We consider both Problem (I) and (II) above, in the general separable Hilbert space setting, and for general normal operators. Problem (I) has already been studied in [7] for the special case when $A \in \mathscr{B}(\mathscr{H})$ is a self-adjoint operator that can be unitarily mapped to an infinite diagonalizable matrix in $\ell^2(\mathbb{N})$. Thus, all the results in [7] are subsumed by the corresponding theorems of this chapter. The present chapter contains new theorems that are not generalizations of those in [7]. In particular, those related to Problem (II) and those that are connected to the action of a group of unitary operators.

4.1 Complete systems with iterations

This section is devoted to the characterization of completeness of the system of vectors $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ where A is a reductive normal operator on a Hilbert space $\mathscr{H}$ and $\mathscr{G}$ is a set of vectors in $\mathscr{H}$. This is done by "diagonalizing" the operator A using multiplicity theory for normal operators, and the properties of reductive operators. We use the notation introduced in Section 2.3.

Theorem 4.1.1. *Let A be a normal operator on a Hilbert space $\mathscr{H}$, and let $\mathscr{G}$ be a countable set of vectors in $\mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $\mathscr{H}$. Let $\mu_\infty, \mu_1, \mu_2, \ldots$ be the measures in the representation (2.2) of the operator A. Then for every $1 \leq j \leq \infty$ and μ_j-a.e. z, the system of vectors $\{\widetilde{g}_j(z)\}_{g\in\mathscr{G}}$ is complete in $\ell^2\{\Omega_j\}$.*

If in addition to being normal, A is also reductive, then $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ being complete in $\mathscr{H}$ is equivalent to $\{\widetilde{g}_j(z)\}_{g\in\mathscr{G}}$ being complete in $\ell^2\{\Omega_j\}$ μ_j-a.e. z for every $1 \leq j \leq \infty$.

Example 2. *Let A be a convolution operator on $\mathscr{H} = L^2(\mathbb{R})$ given by $Af = a * f$, where $a \in L^1(\mathbb{R})$ is a real valued, even function (hence the Fourier transform $\hat{a}$ of a is real valued even function) such that $\hat{a}$ is strictly decreasing on $[0,\infty)$. For example, A can be the discrete-time heat evolution operator given by the convolution with the Gaussian kernel $a(x) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$. Since $a \in L^1(\mathbb{R})$, $\hat{a}$ is continuous, and the spectrum of A is the compact interval $I = [0, \frac{1}{\sqrt{4\pi}}] \subset \mathbb{R}$. Hence as a subset of $\mathbb{C}$, I satisfies the assumption of Proposition 2.3.7 and thus A is reductive. Moreover, the facts that $\hat{a}$ is real valued, even function, strictly decreasing on $[0,\infty)$, imply that $\mu_j = 0$ for $j \neq 1, 2$. In fact, using [1, Theorem 5], we get that $\mu_j = 0$ for $j \neq 2$. Then, using Theorem 4.1.1, for a set of functions $\mathscr{G} \subset L^2(\mathbb{R})$, the system of iterations $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $L^2(\mathbb{R})$ if and only if $\{(\hat{g}(\xi), \hat{g}(-\xi))\}_{g\in\mathscr{G}}$ is complete in $\mathbb{R}^2$ for a.e. $x \in \mathbb{R}$.*

Definition 4.1.2. *For a given set $\mathscr{G}$, let $\mathscr{L}$ be the class of functions $L : \mathscr{G} \to \mathbb{N}^*$ such that*

$$\operatorname{cl}\left(\operatorname{span}\{A^n g\}_{g\in\mathscr{G},\, 0\leq n<L(g)}\right) = \operatorname{cl}\left(\operatorname{span}\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}\right) \tag{4.1}$$

where cl *denotes the taking closure.*

Remark 4.1.3. Note that condition (4.1) is equivalent to

$$A^{L(h)} h \in \operatorname{cl}\left(\operatorname{span}\{A^n g\}_{g\in\mathscr{G},\, 0\leq n<L(g)}\right). \tag{4.2}$$

for every $h \in \mathscr{G}$ such that $L(h) < \infty$.

In particular, $\mathscr{L}$ contains the constant function $L(g) = \infty$ for every $g \in \mathscr{G}$. It also contains the function

$$l(g) = \min\left\{\left\{m \mid A^m g \in \operatorname{span}\{g, Ag, \ldots, A^{m-1}g\}\right\}, \infty\right\} \qquad \text{for every } g \in \mathscr{G}. \tag{4.3}$$

When $l(g)$ is finite, it is called *the degree of the annihilator* of g.

Because of condition (4.1), the reduced system $\{A^n g\}_{g\in\mathscr{G},\,0\le n<L(g)}$ will be complete in $\mathscr{H}$ if and only if $\{A^n g : g \in \mathscr{G}, n \ge 0\}$ is complete in $\mathscr{H}$. Therefore, Theorem 4.1.1 holds if we replace $\{A^n g\}_{g\in\mathscr{G},n\ge 0}$ by $\{A^n g\}_{g\in\mathscr{G},\,0\le n<L(g)}$ as long as $L \in \mathscr{L}$. In particular,

Theorem 4.1.4. *If A is a normal and reductive operator, then $\{A^n g\}_{g\in\mathscr{G},\,0\le n<L(g)}$ is complete in $\mathscr{H}$ if and only if $L \in \mathscr{L}$ and $\{\widetilde{g}_j(z)\}_{g\in\mathscr{G}}$ is complete in $\ell^2\{\Omega_j\}$ μ_j-a.e. z for every $1 \le j \le \infty$.*

Although when $L \in \mathscr{L}$, $\{A^n g\}_{g\in\mathscr{G},\,n\ge 0}$ and $\{A^n g\}_{g\in\mathscr{G},\,0\le n<L(g)}$ are either both complete or both incomplete, the system $\{A^n g\}_{g\in\mathscr{G},\,0\le n<L(g)}$ may form a frame while $\{A^n g\}_{g\in\mathscr{G},\,n\ge 0}$ may not, since the possible extra vectors in $\{A^n g\}_{g\in\mathscr{G},\,n\ge 0}$ may damage the upper frame bound. This difference in behavior between the two systems makes it important to study $\{A^n g\}_{g\in\mathscr{G},\,0\le n<L(g)}$ for $L \in \mathscr{L}$.

Example 3. *Let $\mathscr{H} = \ell^2(\mathbb{Z})$ and A be convolution operator with a kernel $a \in \ell^1(\mathbb{Z})$, i.e. $Af = a * f$. Let $\mathscr{G} = \{e_{mk}\}_{k\in\mathbb{Z}}$ for some $m > 1$ where $\{e_k\}_{k\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. The Fourier transform of a is defined as*

$$\hat{a}(\xi) = \sum_{k\in\mathbb{Z}} a(k)e^{-2\pi i\xi k},\ \xi \in [0,1].$$

Denote

$$\mathscr{A}_m(\xi) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \hat{a}(\frac{\xi}{m}) & \hat{a}(\frac{\xi+1}{m}) & \dots & \hat{a}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{(L-1)}(\frac{\xi}{m}) & \hat{a}^{(L-1)}(\frac{\xi+1}{m}) & \dots & \hat{a}^{(L-1)}(\frac{\xi+m-1}{m}) \end{pmatrix}.$$

Let $\sigma(\xi)$ denote the smallest singular value of the matrix $\mathscr{A}_m(\xi)$. Let $L(g) = M$ for each $g \in \mathscr{G}$. From [9], the system $\{A^n g\}_{g\in\mathscr{G},0\le n<M}$ is complete in $\ell^2(\mathbb{Z})$ if and only if $\mathscr{A}_m(\xi)$ has a left inverse for a.e. $\xi \in [0,1]$, or equivalently $\sigma(\xi) > 0$ for a.e. $\xi \in [0,1]$, and it forms a frame if and only if $\sigma(\xi) \ge \alpha$ for a.e. $\xi \in [0,1]$ for some $\alpha > 0$. Since $\mathscr{A}_m(\xi)$ is a Vandermonde matrix, iterations $n > m-1$ will not affect the completeness of the system. Thus, we set $M = m$. In that case, $\{A^n g\}_{g\in\mathscr{G},0\le n\le m-1}$ is complete in $\ell^2(\mathbb{Z})$ if and only if $\det \mathscr{A}_m(\xi) \neq 0$ for a.e $\xi \in [0,1]$, and it is a frame if and only if for a.e. $\xi \in [0,1]$, $|\det \mathscr{A}_m(\xi)| \ge \alpha$ for some $\alpha > 0$.

Although there are infinitely many convolution operators that satisfy this last condition, many natural operators in practice do not. For example, an operator where a is real, even and $\hat{a}$ is strictly decreasing on $[0,\frac{1}{2}]$. For this case, it can be shown that the matrices $\mathscr{A}_m(0)$ and $\mathscr{A}_m(\frac{1}{2})$ are singular, while all the other matrices $\mathscr{A}_m(\xi)$ are invertible. For this case, any set of the form $\mathscr{G} = \{e_{mk}\}_{k\in\mathbb{Z}} \cup \{e_{mlk+1}\}_{k\in\mathbb{Z}}$ where $l \ge 1$, produces a system $\{A^n g\}_{g\in\mathscr{G},0\le n\le m-1}$ which is a frame for $\mathscr{H} = \ell^2(\mathbb{Z})$.

The proof of Theorem 4.1.1 below, also shows that, for normal reductive operators, completeness in $\mathscr{H}$ is equivalent to the system $\left\{N^n_{\mu_j}\widetilde{g}_j\right\}_{g\in\mathscr{G},\, n\ge 0}$ being complete in the space $(L^2(\mu_j))^{(j)}$ for every $1 \le j \le \infty$, i.e. the completeness of $\{A^n g\}_{g\in\mathscr{G},\, n\ge 0}$ is equivalent to the completeness of its projections onto the mutually orthogonal subspaces $UP_jU^*\mathscr{H}$ of $\mathscr{H}$. This should be contrasted to the fact that, in general, completeness of a set of vectors $\{h_n\} \subset \mathscr{H}$ is not equivalent to the completeness of its projections on subspaces whose orthogonal sum is $\mathscr{H}$. We have

Theorem 4.1.5. *Let A be a normal reductive operator on a Hilbert space $\mathscr{H}$, and let $\mathscr{G}$ be* *a countable system of vectors in $\mathscr{H}$. Then, $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $\mathscr{H}$ if and only if the system $\left\{N^n_{\mu_j}\widetilde{g}_j\right\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $(L^2(\mu_j))^{(j)}$ for every $1\leq j\leq\infty$.*

Proof of Theorem 4.1.1. Since $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $\mathscr{H}$,

$$U\{A^n g : g\in\mathscr{G}, n\geq 0\} = \{(N^n_{\mu_j}\widetilde{g}_j)_{j\in\mathbb{N}^*} : g\in\mathscr{G}, n\geq 0\}$$

is complete in $\mathscr{W} = U\mathscr{H}$. Hence, for every $1\leq j\leq\infty$, the system $\mathscr{S} = \left\{N^n_{\mu_j}\widetilde{g}_j\right\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $(L^2(\mu_j))^{(j)}$.

To finish the proof of the first statement of the theorem we use the following lemma which is an adaptation of [40, Lemma 1].

Lemma 4.1.6. *Let $\mathscr{S}$ be a complete countable set of vectors in $(L^2(\mu_j))^{(j)}$, then for μ_j-almost every z $\{h(z) : h\in\mathscr{S}\}$ is complete in $\ell^2(\Omega_j)$.*

Since S is complete in $(L^2(\mu_j))^{(j)}$, Lemma 4.1.6 implies that $\{z^n\widetilde{g}_j(z)\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $\ell^2(\Omega_j)$ for each $j\in\mathbb{N}^*$. But $\operatorname{span}\{z^n\widetilde{g}_j(z)\}_{g\in\mathscr{G},\, n\geq 0} = \operatorname{span}\{\widetilde{g}_j(z)\}_{g\in\mathscr{G}}$. Thus, we have proved the first part of the theorem.

Now additionally assume that A is also reductive. Let

$$\widetilde{f}\in (L^2(\mu_\infty))^{(\infty)}\oplus L^2(\mu_1)\oplus (L^2(\mu_2))^{(2)}\oplus\cdots$$

and

$$\langle UA^n g, \widetilde{f}\rangle = \sum_{1\leq j\leq\infty}\int_{\mathbb{C}} z^n\langle\widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)}d\mu_j(z) = 0$$

for every $g\in\mathscr{G}$ and every $0\leq n<\infty$. Since the measures μ_j, $1\leq j\leq\infty$, are mutually singular, we get that

$$\begin{aligned}&\sum_{1\leq j\leq\infty}\int_{\mathbb{C}} z^n < \langle\widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)}d\mu_j(z) \\ &\quad = \textstyle\int_{\mathbb{C}} z^n\left[\sum_{1\leq j\leq\infty}\mathbb{1}_{\mathscr{E}_j}\langle\widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)}\right]d\mu(z)\end{aligned}\tag{4.4}$$

for every $g \in \mathscr{G}$ and every $n \geq 0$ with μ as in (2.3).

Using the fact that the operator A is reductive, from Proposition 2.3.6, we conclude that

$$\langle \widetilde{g}_j(z), \widetilde{f}_j(z) \rangle_{\ell^2(\Omega_j)} = 0 \ \ \mu_j\text{-a.e. } z.$$

Since, by assumption $\{\widetilde{g}_j(z)\}_{g \in \mathscr{G}}$ is complete in $\ell^2(\Omega_j)$ for μ_j-a.e z, we obtain

$$\widetilde{f}_j = 0 \ \ \mu_j\text{-a.e. } z \qquad \text{for every } j \in \mathbb{N}^*.$$

Thus $\widetilde{f} = 0$ μ-a.e., and therefore, $\{A^n g\}_{g \in \mathscr{G},\, n \geq 0}$ is complete in $\mathscr{H}$. □

4.2 Minimality property and basis

The goal of this section is to study the conditions on the operator A and the set of vectors $\mathscr{G}$ such that the system $\{A^n g\}_{g \in \mathscr{G},\, 0 \leq n < L(g)}$ is minimal or a basis for $\mathscr{H}$. We start with the following proposition:

Proposition 4.2.1. *If A is a normal operator on $\mathscr{H}$ then, for any set of vectors $\mathscr{G} \subset \mathscr{H}$, the system of iterates $\{A^n g\}_{g \in \mathscr{G}, n \geq 0}$ is not a complete and minimal system in $\mathscr{H}$.*

Note that Proposition 4.2.1 is trivial if the $\dim \mathscr{H} < \infty$ and becomes interesting only when $\dim \mathscr{H} = \infty$. As a corollary of Proposition 4.2.1 we get

Corollary 4.2.2. *If A is a normal operator on $\mathscr{H}$ then, for any set of vectors $\mathscr{G} \subset \mathscr{H}$, the system of iterates $\{A^n g\}_{g \in \mathscr{G}, n \geq 0}$ is not a basis for $\mathscr{H}$.*

If we remove the completeness condition in the statement of Proposition 4.2.1 above, then the operator $Af = zf$ on the unit circle with arc length measure gives an orthogonal system when iterated on the vector $g \equiv 1$, i.e., for this case $\{z^n g\}_{n \geq 0}$ is minimal since it is an orthonormal system. However, if in addition to being normal, we assume that A is reductive then the statement of proposition 4.2.1 remains true without the completeness condition since, by Proposition 2.3.5, the restriction of A onto $\mathrm{cl}\left(\mathrm{span}\{A^n g_{g\in\mathscr{G},n\geq 0}\right)$ will be a normal operator and we will have a minimal complete system contradicting the claim of Proposition 4.2.1. Thus, we have the following corollary

Corollary 4.2.3. *If A is a reductive normal operator on $\mathscr{H}$, then, for any countable system of vectors $\mathscr{G}\subset\mathscr{H}$, the system of iterates $\{A^n g\}_{g\in\mathscr{G},n\geq 0}$ is not a minimal system.*

As another corollary of Proposition 4.2.1, we get

Corollary 4.2.4. *Let A be a reductive normal operator on $\mathscr{H}$, $\mathscr{G}$ a countable system of vectors in $\mathscr{H}$ and let $L\in\mathscr{L}$. If for some $h\in\mathscr{G}$, $L(h)=\infty$, then the system $\{A^n g\}_{g\in\mathscr{G},\,0\leq n<L(g)}$ is not a basis for $\mathscr{H}$.*

Proof. Let $V=\mathrm{cl}\left(\mathrm{span}\{A^n h\}_{n\geq 0}\right)$ where $L(h)=\infty$. V is a closed invariant subspace for A hence, by Proposition 2.3.5, the restriction of A on V is also normal, therefore, from Proposition 4.2.1, $\{A^n h\}_{n\geq 0}$ is not minimal. □

In particular, since $\dim\mathscr{H}=\infty$ (the assumption in this paper), if $|\mathscr{G}|<\infty$, then there exists $g\in\mathscr{G}$ such that $L(g)=\infty$. Thus we have

Corollary 4.2.5. *Let A be a reductive normal operator. If $|\mathscr{G}|<\infty$, then for any $L\in\mathscr{L}$ the system $\{A^n g\}_{g\in\mathscr{G},\,0\leq n<L(g)}$ is never a basis for $\mathscr{H}$.*

Proof of Proposition 4.2.1. We prove that if $\{A^n g\}_{g\in\mathscr{G},n\geq 0}$ is complete in $\mathscr{H}$, then for any $m\geq 0$, $\{A^n g\}_{g\in\mathscr{G},n=0,m,m+1,\ldots}$ is also complete in $\mathscr{H}$, which implies non-minimality.

Assume $\{A^n g\}_{g\in\mathscr{G},n\geq 0}$ is complete in $\mathscr{H}$. Let $\delta>0$ and $f\in\mathscr{H}$ be a vector such that $\widetilde{f}(z)=0$ for any $z\in\overline{\mathbb{D}}_\delta$ where $\overline{\mathbb{D}}_\delta$ is the closed unit disc of radius δ centered at 0. Then for a fixed m, $\frac{\widetilde{f}}{z^m}$ is in $U\mathscr{H}$ and hence can be approximated arbitrarily closely by finite linear combinations of the vectors in $\{z^n\widetilde{g}\}_{g\in\mathscr{G},n\geq 0}$. Let $\widetilde{f}^{(1)},\widetilde{f}^{(2)},\ldots$ be a sequence in $U\mathscr{H}$ such that $\widetilde{f}^{(s)}\to\frac{\widetilde{f}}{z^m}$ in $U\mathscr{H}$ and $\widetilde{f}^{(s)}$ is a finite linear combinations of the vectors in $\{z^n\widetilde{g}\}_{g\in\mathscr{G},n\geq 0}$ for each s. Since z^m is bounded on the spectrum of A, it follows that $z^m\widetilde{f}^{(s)}\to\widetilde{f}$. Finally, we note that $z^m\widetilde{f}^{(s)}$ is a finite linear combination of the vectors $\{z^n\widetilde{g}\}_{g\in\mathscr{G},n\geq m}$.

For a general $f \in \mathscr{H}$, we have that

$$\lim_{\delta \to 0} \|\widetilde{f} - \widetilde{f} \mathbb{1}_{\overline{\mathbb{D}}_\delta^c}\|^2_{L^2(\mu)} = \sum_{j \in \mathbb{N}^*} \|\widetilde{f}_j(0)\|^2_{\ell^2(\Omega_j)} \mu(\{0\}) = \mu_{J(0)}(\{0\}) \|\widetilde{f}_{J(0)}(0)\|^2_{\ell^2(\Omega_{J(0)})} \quad (4.5)$$

where $J(0)$ is the value of the multiplicity function defined in (2.4) at point $z = 0$.

From Theorem 4.1.1, for any $\varepsilon > 0$ there exists a finite linear combination $\widetilde{h} \in U\mathscr{H}$ of vectors $\{\widetilde{g}\}_{g \in \mathscr{G}}$ such that $\mu_{J(0)}(\{0\}) \|\widetilde{f}_{J(0)}(0) - \widetilde{h}_{J(0)}(0)\|_{\ell^2(\Omega_{J(0)})} < \frac{\varepsilon}{2}$. Define $\widetilde{w} := \widetilde{f} - \widetilde{h}$. Using (4.5) for w, we can pick δ so small that $\|\widetilde{w} - \widetilde{w} \mathbb{1}_{\overline{\mathbb{D}}_\delta^c}\|^2_{L^2(\mu)} < \frac{\varepsilon}{2}$. Let $\widetilde{u}$ be a finite linear combination of $\{z^n \widetilde{g}\}_{g \in \mathscr{G}, n \geq m}$ such that $\|\widetilde{w} \mathbb{1}_{\overline{\mathbb{D}}_\delta^c} - \widetilde{u}\|^2_{L^2(\mu)} < \frac{\varepsilon}{2}$. Then $\|\widetilde{w} - \widetilde{u}\|^2_{L^2(\mu)} < \varepsilon$, i.e., $\|\widetilde{f} - \widetilde{h} - \widetilde{u}\|^2_{L^2(\mu)} < \varepsilon$. Hence in this case we get that any vector $f \in \mathscr{H}$ is in the closure of the span of $\{\widetilde{g}\}_{g \in \mathscr{G}} \cup \{z^n \widetilde{g}\}_{g \in \mathscr{G}, n \geq m} = \{z^n \widetilde{g}\}_{g \in \mathscr{G}, n = 0, m, m+1, \ldots}$. $\square$

Without the condition that A is normal, the statement of the Corollary 4.2.5 may not be true. Let S be the unilateral shift operator on $\ell^2(\mathbb{N})$ and e_n be the n-th canonical basis vector in $\ell^2(\mathbb{N})$. Then we have $S^n e_1 = e_n$, thus in this case the iterated system is not only a Riesz basis, but an orthonormal basis.

Even though we cannot have bases for $\mathscr{H}$ by iterations of a countable system $\mathscr{G}$ by a normal operator when the system $\{A^n g\}_{g \in \mathscr{G},\, 0 \leq n < L(g)}$ is complete, the non-minimality suggest that we may still have a situation in which the system is a frame leading us to the next section.

4.3 Complete Bessel systems and frames of iterations

It is shown in [7] that it is possible to construct frames from iteration $\{A^n g\}_{n \geq 0}$ of a single vector g for some special cases when the operator A is an infinite matrix acting on $\ell^2(\mathbb{N})$, has point spectrum and g is chosen appropriately [7]. However, it is also shown that generically, $\{A^n g\}_{n \geq 0}$ does not produce a frame for $\ell^2(\mathbb{N})$. Since a frame must be a Bessel system, we study the Bessel properties of $\{A^n g\}_{g \in \mathscr{G},\, n \geq 0}$ when A is normal. In addition, we find conditions that must be satisfied when the system $\{A^n g\}_{g \in \mathscr{G},\, n \geq 0}$ has the lower frame bound property for the case where $\mathscr{G}$ is finite.

Denote by $\mathbb{D}_r$ the open disk in $\mathbb{C}$ of radius r centered at the origin, by $\overline{\mathbb{D}}_r$ its closure, and by S_r its boundary, that is $S_r = \overline{\mathbb{D}}_r \setminus \mathbb{D}_r$. For a set $E \subset \mathbb{C}$, we will use the notation $\mathbb{C} \setminus E$ or E^c for the complement of E. Then we have the following theorem.

Theorem 4.3.1. *Let $A \in \mathscr{B}(\mathscr{H})$ be a normal operator, μ be its scalar spectral measure, and $\mathscr{G}$ a countable system of vectors in $\mathscr{H}$.*

(a) If $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $\mathscr{H}$ and for every $g \in \mathscr{G}$ the system $\{A^n g\}_{n\geq 0}$ is Bessel in $\mathscr{H}$, then $\mu\left(\mathbb{C} \setminus \overline{\mathbb{D}}_1\right) = 0$ and $\mu|_{S_1}$ is absolutely continuous with respect to arc length measure (Lebesgue measure) on S_1.

(b) If $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is frame in $\mathscr{H}$, then $\mu\left(\mathbb{C} \setminus \mathbb{D}_1\right) = 0$.

The converse of Theorem 4.3.1 is true in the following sense.

Theorem 4.3.2. *Let $A \in \mathscr{B}(\mathscr{H})$ be a normal operator, and μ be its scalar spectral measure.*

(a) If $\mu\left(\mathbb{C} \setminus \overline{\mathbb{D}}_1\right) = 0$ and $\mu|_{S_1}$ is absolutely continuous with respect to arc length measure on S_1, then there exists a countable set $\mathscr{G} \subset \mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a complete Bessel system.

(b) If $\mu\left(\mathbb{C} \setminus \mathbb{D}_1\right) = 0$ then there exists a countable set $\mathscr{G} \subset \mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a Parseval frame for $\mathscr{H}$.

Example 4. *Let A be the convolution operator as in Example 2. If there exists a complete Bessel system by iterations of A, then from Theorem 4.3.1 (a), $\hat{a}(0) \leq 1$. Conversely, if $\hat{a}(0) \leq 1$, then the conditions in Theorem 4.3.2 (b) are satisfied and hence there exists a set of vectors $\mathscr{G} \subset L^2(\mathbb{R})$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a Parseval frame in $L^2(\mathbb{R})$. From the proof of the theorem, to construct the set $\mathscr{G}$, we take an orthonormal basis $\mathscr{O}$ in* $\operatorname{cl}\left((1-|\hat{a}|^2)^{\frac{1}{2}} L^2(\mathbb{R})\right) = L^2(\mathbb{R})$*, then $\mathscr{G} = (1-|\hat{a}|^2)\mathscr{O}$. Note that $\mathscr{G}$ is already complete in $L^2(\mathbb{R})$. A natural question will be, what is the smallest $\mathscr{G}$ (in terms of its span closure) such* *that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a frame? We will see from Theorem 4.3.6, that $\mathscr{G}$ can not be finite for such a convolution operator since its spectrum is continuous.*

Using the previous two theorems, we get the following necessary and sufficient conditions for the system $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ to be a complete Bessel system in $\mathscr{H}$.

Corollary 4.3.3. *Let $A \in \mathscr{B}(\mathscr{H})$ be a normal operator, and μ be its scalar spectral measure. Then the following are equivalent.*

1. *There exists a countable set $\mathscr{G} \subset \mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a complete Bessel system.*
2. *$\mu\left(\mathbb{C}\setminus\mathbb{D}_1\right) = 0$ and $\mu|_{S_1}$ is absolutely continuous with respect to arc length measure on S_1.*

For the case of iterates $\{A^n g\}_{g\in\mathscr{G},\, 0\leq n<L(g)}$, where $L \in \mathscr{L}$ as defined in Remark 4.1.2, one has the following theorem.

Theorem 4.3.4. *Let A be a normal operator on a Hilbert space $\mathscr{H}$ and $\mathscr{G}$ a system of vectors in $\mathscr{H}$, and assume $L \in \mathscr{L}$. If $\{A^n g\}_{g\in\mathscr{G},\, 0\leq n<L(g)}$ is a complete Bessel system for $\mathscr{H}$, then for each $g \in \mathscr{G}$ with $L(g) = \infty$, the set $\{x \in \overline{\mathbb{D}}_1^c \,|\, \widetilde{g}(x) \neq 0\}$ has μ-measure 0.*

When the system $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ has the frame bound property and $\mathscr{G}$ is finite, we have the following necessary condition.

Theorem 4.3.5. *Let $A \in \mathscr{B}(\mathscr{H})$ be a normal operator, and μ be its scalar spectral measure. If $|\mathscr{G}| < \infty$ and $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ satisfy the lower frame bound, then, for every $0 < \varepsilon < 1$, $\mu\left(\overline{\mathbb{D}}_{1-\varepsilon}^c\right) > 0$.*

As a corollary of 4.3.1, we get that

Theorem 4.3.6. *Let A be a bounded normal operator in an infinite-dimensional Hilbert space $\mathscr{H}$. If the system of vectors $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a frame for some $\mathscr{G} \subset \mathscr{H}$ with $|\mathscr{G}| < \infty$, then $A = \sum_j \lambda_j P_j$ where P_j are projections such that $\operatorname{rank} P_j \leq |\mathscr{G}|$ (i.e. the global multiplicity of A is less than or equal to $|\mathscr{G}|$).*

Combining Theorem 4.3.6 with the result in [7], where A was assumed to be a diagonal operator on $\ell^2(\mathbb{N})$, we get the following characterization for a general normal operator $A \in \mathscr{B}(\mathscr{H})$ when $|\mathscr{G}| = 1$.

Theorem 4.3.7. *Let A be a bounded normal operator in an infinite-dimensional Hilbert space $\mathscr{H}$. Then $\{A^n g\}_{n \geq 0}$ is a frame for $\mathscr{H}$ if and only if*

i) $A = \sum_j \lambda_j P_j$, where P_j are rank one orthogonal projections

ii) $|\lambda_k| < 1$ for all k

iii) $\{\lambda_k\}$ satisfy Carleson's condition

$$\inf_n \prod_{k \neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta \tag{4.6}$$

for some $\delta > 0$

iv)

$$0 < C_1 \leq \frac{\|P_j g\|}{\sqrt{1 - |\lambda_k|^2}} \leq C_2 < \infty,$$

for some constants C_1, C_2.

Example 5. *Let $\mathscr{H} = \ell^2(\mathbb{N})$, A a semi-infinite diagonal matrix whose entries are given by $a_{jj} = \lambda_j = 1 - 2^{-j}$ for $j \in \mathbb{N}$, and let $g \in \ell^2(\mathbb{N})$ be given by $g(j) = \sqrt{1 - \lambda_j^2}$. Then, the sequence $\lambda_j = 1 - 2^{-j}$ satisfies Carleson's condition (see e.g. [34]), and g satisfies condition (v). Thus, $\{A^n g\}_{n \geq 0}$ are a frame for $\ell^2(\mathbb{N})$.*

For the special case defined by (4.3), we get the following necessary condition on the measure μ.

Theorem 4.3.8. *Suppose A is a normal operator, and $\{A^n g\}_{g \in \mathscr{G},\, n = 0,1,\dots,l(g)}$ (where $l(g)$ is given by (4.3)) is a complete Bessel system for $\mathscr{H}$. Then*

(a) If $l(g) = \infty$ then $\{x \in \overline{\mathbb{D}}_1^c : \widetilde{g}(x) \neq 0\}$ has μ-measure 0.

(b) The restriction of μ on $\overline{\mathbb{D}}_1^c$ is concentrated on at most a countable set, i.e., either $\mu(\overline{\mathbb{D}}_1^c) = 0$, or there exists a countable set $E \subset \overline{\mathbb{D}}_1^c$ such that $\mu|_{\overline{\mathbb{D}}_1^c}\left(E^c \cap \overline{\mathbb{D}}_1^c\right) = 0$.
(c) $\mu|_{S_1}$ is a sum of a discrete and an absolutely continuous measure (with respect to arc length measure) on S_1.

In fact, if for every $g \in \mathscr{G}$, $l(g) < \infty$ then without the condition that the system is Bessel, but with the completeness condition alone, we get that the measure μ is concentrated on a countable subset of $\mathbb{C}$, as stated in the following theorem.

Theorem 4.3.9. *Let A be a normal operator and $\mathscr{G} \subset \mathscr{H}$ be a system of vectors such that, for every $g \in \mathscr{G}$, $l(g) < \infty$ and $\{A^n g\}_{g\in\mathscr{G},\, n=0,1,\dots,l(g)}$ is complete in $\mathscr{H}$. Then there exists a countable set $E \subset \mathbb{C}$ such that $\mu(E^c) = 0$. Moreover, every g is supported, with respect to the measure μ, on a finite set of cardinality not exceeding $l(g)$.*

4.4 Proofs of Theorems in Section 4.3

Proof of Theorem 4.3.1. (a) Suppose $\mu(\overline{\mathbb{D}}_1^c) > 0$, then $\mu_k(\overline{\mathbb{D}}_1^c) > 0$ for some k, $0 \le k \le \infty$. Thus, there exists $\varepsilon > 0$ such that $\mu_k(\overline{\mathbb{D}}_{1+\varepsilon}^c) > 0$. Since the system of vectors $\{A^n g\}_{g\in\mathscr{G},\, n\ge 0}$ is complete in $\mathscr{H}$, it follows from Theorem 4.1.1 that there exists a $g \in \mathscr{G}$ such that $\mu_k(\overline{\mathbb{D}}_{1+\varepsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)) > 0$.

Let $f \in \mathscr{H}$ be any vector such that $\widetilde{f} = P_k \widetilde{f}$, and $\widetilde{f}(z) = 0$ for $z \in \overline{\mathbb{D}}_{1+\varepsilon}$. Then

$$
\begin{aligned}
&|\langle f, A^n g\rangle| \\
&\quad = \left| \sum_{0\le j\le\infty} \int_{\mathbb{C}} z^n \langle \widetilde{g}_j(z), \widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu_j(z) \right| \\
&\quad = \left| \int_{\overline{\mathbb{D}}_{1+\varepsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} z^n \langle \widetilde{g}_k(z), \widetilde{f}_k(z)\rangle_{\ell^2(\Omega_k)} d\mu_k(z) \right|.
\end{aligned}
$$

For each n, denote by $\lambda_n(f)$ the linear functional on the space $\mathscr{H}_0 := \{f \in \mathscr{H} : \widetilde{f} = P_k \widetilde{f},\ \widetilde{f}(z) = 0 \text{ for } z \in \overline{\mathbb{D}}_{1+\varepsilon}\}$, defined by $\lambda_n(f) = \langle f, A^n g\rangle$. The norm of this functional (on $\mathscr{H}_0$) is

$$\|\lambda_n\|_{op}^2 = \int_{\overline{\mathbb{D}}_{1+\varepsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} |z|^{2n} \|\widetilde{g}_k(z)\|_{\ell^2\{\Omega_k\}}^2 d\mu_k(z)$$
$$\geq (1+\varepsilon)^{2n} \int_{\overline{\mathbb{D}}_{1+\varepsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} \|\widetilde{g}_k(z)\|_{\ell^2\{\Omega_k\}}^2 d\mu_k(z).$$

Since the right side of the last inequality tends to infinity as $n \to \infty$, so does $\|\lambda_n\|_{op}$. Thus, from the uniform boundedness principle there exists an $f \in \mathscr{H}_0$ such that

$$\lim_{n\to\infty} \left| \int_{\overline{\mathbb{D}}_{1+\varepsilon}^c \cap \operatorname{supp}(\widetilde{g}_k)} z^n \langle \widetilde{g}_k(z), \widetilde{f}_k(z) \rangle_{\ell^2(\Omega_k)} d\mu_k(z) \right| = \infty.$$

For such f, $\lambda_n(f) = |\langle f, A^n g \rangle\| \to \infty$ as $n \to \infty$. Thus, we also have that

$$\sum_{n=0}^{\infty} |\langle f, A^n g \rangle|^2 = \infty$$

which is a contradiction to our assumption that $\{A^n g\}_{n\geq 0}$ is a Bessel system in $\mathscr{H}$.

To prove the second part of the statement, let $k \geq 1$ be fixed, and consider the Lebesgue decomposition of $\mu_k|_{S_1}$ given by $\mu_k|_{S_1} = \mu_k^{\mathrm{ac}} + \mu_k^s$ where μ_k^{ac} is absolutely continuous with respect to arc length measure on S_1, μ_k^s is singular and $\mu_k^{\mathrm{ac}} \perp \mu_k^s$. We want to show that $\mu_k^s \equiv 0$.

For a vector $a = (a_1, a_2, \dots, a_k) \in \ell^2(\Omega_k)$, define $Q_r a := a_r$. Fix $1 \leq r \leq k$ and $m \geq 1$. Let $f \in \mathscr{H}$ be the vector such that

i) $Q_r \widetilde{f}_k(e^{2\pi it}) = e^{2\pi imt}$, μ_k^s-a.e.

ii) $Q_r \widetilde{f}_k(e^{2\pi it}) = 0$, μ_k^{ac}-a.e.

iii) $Q_s \widetilde{f}_j(z) = 0$ if $r \neq s$ or $k \neq j$

iv) $\widetilde{f}(z) = 0$ for $z \notin S_1$.

Then for such an f and a fixed $g \in \mathscr{G}$, from the assumption that $\{A^n g\}_{n\geq 0}$ is a Bessel system in $\mathscr{H}$, we have

$$
\begin{aligned}
&\sum_{n\geq 0}\left|\int_{S_1} e^{2\pi int} Q_r\widetilde{g}_k(e^{2\pi it})\overline{e^{2\pi imt}}\, d\mu_k^s(e^{2\pi it})\right|^2 \\
&\quad = \sum_{n\geq 0}\left|\sum_{0\leq j\leq\infty}\int_{\mathbb{C}} z^n\langle\widetilde{g}_j(z),\widetilde{f}_j(z)\rangle_{\ell^2(\Omega_j)}\, d\mu_j(z)\right|^2 \\
&\quad = \sum_{n=0}^{\infty}|\langle A^n g, f\rangle|^2 \leq C\|f\|^2 \leq C\mu(S_1).
\end{aligned}
$$

Thus

$$
\sum_{n\geq 0}\left|\int_{S_1} e^{2\pi i(n-m)t} Q_r\widetilde{g}_k(e^{2\pi it})\, d\mu_k^s(e^{2\pi it})\right|^2 \leq C\mu(S_1).
$$

Since the last inequality holds for every $m\geq 1$, we have

$$
\sum_{n\in\mathbb{Z}}\left|\int_{S_1} e^{2\pi int} Q_r\widetilde{g}_k(e^{2\pi it})\, d\mu_k^s(e^{2\pi it})\right|^2 \leq C\mu(S_1).
$$

This means the Fourier – Stieltjes coefficients of the measure $Q_r\widetilde{g}_k(e^{2\pi it})d\mu_k^s(e^{2\pi it})$ are in $\ell^2(\mathbb{Z})$. Hence, from the uniqueness theorem of the Fourier Stieltjes coefficients ([39], p. 36) and the fact that any element of $\ell^2(\mathbb{Z})$ determines Fourier coefficients of an $L^2(S_1)$ function (with respect to arc length measure), $Q_r\widetilde{g}_k(e^{2\pi it})d\mu_k^s(e^{2\pi it})$ is absolutely continuous with respect to the arc length measure. But the measure μ_k^s is concentrated on a measure zero set as a singular measure, hence $Q_r\widetilde{g}_k(e^{2\pi it})d\mu_k^s(e^{2\pi it})$ is the zero measure. Since the system $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is complete in $\mathscr{H}$, from Theorem 4.1.1 we obtain that $\mu_k^s = 0$ and hence μ_k is absolutely continuous with respect to the arc length measure on S_1. Thus μ is absolutely continuous with respect to the arc length measure on S_1.

(b) Suppose $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a frame with frame bounds α and β. Let $f\in\mathscr{H}$ be any vector such that $\widetilde{f} = 0$ on $\mathbb{C}\setminus S_1$. For such an f, we have that $\|(A^*)^m f\| = \|A^m f\| = \|f\|$

for any $m \in \mathbb{Z}$. Thus, for any $m \in \mathbb{Z}$, we have

$$\begin{aligned} \alpha\|f\|^2 = \alpha\|(A^*)^m f\|^2 &\leq \sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty}|\langle (A^*)^m f, A^n g\rangle|^2 = \sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty}|\langle f, A^{n+m} g\rangle|^2 \quad (4.7)\\ &= \sum_{g\in\mathscr{G}}\sum_{n=m}^{\infty}|\langle f, A^n g\rangle|^2 \leq \beta\|A^m f\|^2 \leq \beta\|f\|^2. \end{aligned}$$

Since (4.7) holds for every m, the right inequality implies $\sum_{n=m}^{\infty}\sum_{g\in\mathscr{G}}|\langle f, A^n g\rangle|^2 \to 0$ as $m \to \infty$. Hence, using the left inequality we conclude that $\|f\| = 0$. Since f is such that $\widetilde{f} = 0$ on $\mathbb{C}\setminus S_1$, but otherwise is arbitrary, it follows that $\mu(S_1) = 0$. But, from Part (a), we already know that $\mu(\mathbb{C}\setminus\overline{\mathbb{D}}_1) = 0$, hence $\mu(\mathbb{C}\setminus\mathbb{D}_1) = 0$. $\square$

Proof of Theorem 4.3.2. Let $\mathscr{H}_1 = \{f \in \mathscr{H} : \widetilde{f}(z) = 0, z \notin S_1\}$ and $\mathscr{H}_2 = \{f \in \mathscr{H} : \widetilde{f}(z) = 0, z \notin \mathbb{D}_1\}$. Then $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. Let $\mathscr{G}_i \subset \mathscr{H}_i$, be complete Bessel systems in $\mathscr{H}_i$, $i = 1, 2$, then, it is not difficult to see that $\mathscr{G}_1 \cup \mathscr{G}_2$ is a complete Bessel system in $\mathscr{H}$. We will proceed by constructing complete Bessel systems for $\mathscr{H}_1$ and $\mathscr{H}_2$. To construct a complete Bessel sequence for $\mathscr{H}_1$, we first consider the operator $N_{\mu_j|_{S_1}}$ on $L^2(\mu_j|_{S_1})$ for a fixed j, with $1 \leq j \leq \infty$, where μ_j is as in the decomposition of Theorem 2.3.2. Since for $f \in \mathscr{H}_1$, $\widetilde{f}(z) = 0$ for $z \notin S_1$, and since $\mu|_{S_1}$ (and hence also $\mu_j|_{S_1}$) is absolutely continuous with respect to the arc lengh measure σ, we have that on the circle S_1, $d\mu_j|_{S_1} = w_j d\sigma$ for some $w_j \in L^1(\sigma)$. Hence on the support E_j of w_j, μ_j and σ are mutually absolutely continuous, i.e., for ν_j defined by $d\nu_j = \mathbb{1}_{E_j} d\sigma$, μ_j and ν_j are mutually absolutely continuous.

Now consider the two functions p_j and q_j such that $p_j(z) = q_j(z) = 0$ for $z \notin S_1$, while on S_1, $p_j(e^{2\pi it}) = \mathbb{1}_{[0,\frac{1}{2}]}(t)$ and $q_j(e^{2\pi it}) = \mathbb{1}_{[\frac{1}{2},1]}(t)$. From the properties of the Fourier series on $L^2(S_1, \sigma)$, the sets $\{z^n p_j(z)\}_{n\geq 0}$ and $\{z^n q_j(z)\}_{n\geq 0}$ are Bessel systems in $L^2(S_1, \sigma)$ with bound 1. Thus, $\{z^n p_j(z)\}_{n\geq 0}$ and $\{z^n q_j(z)\}_{n\geq 0}$ are also Bessel systems in $L^2(S_1, \nu_j)$ with bound 1. Therefore, $\{z^n p_j(z)\}_{n\geq 0} \cup \{z^n q_j(z)\}_{n\geq 0}$ is a Bessel system for $\oplus_{j=1}^{\infty} L^2(S_1, \nu_j)$. By Proposition 2.3.7 and Theorem 2.3.2 the sytem $\{z^n p_j(z)\}_{n\geq 0} \cup \{z^n q_j(z)\}_{n\geq 0}$ is also complete in $\oplus_{j=1}^{\infty} L^2(S_1, \nu_j)$. Thus, $\{z^n p_j(z)\}_{n\geq 0} \cup \{z^n q_j(z)\}_{n\geq 0}$ is a complete Bessel system for $\oplus_{j=1}^{\infty} L^2(S_1, \nu_j)$.

Since μ_j and ν_j are mutually absolutely continuous, the multiplication operator z on $\oplus_{j=1}^{\infty} L^2(S_1, \nu_j)$ is unitarily equivalent to the multiplication operator z on $\oplus_{j=1}^{\infty} L^2(S_1, \mu_j)$ which we denote by V. Hence, $\{z^n V(p_j)(z)\}_{n\geq 0} \cup \{z^n V(q_j)(z)\}_{n\geq 0}$ is a complete Bessel system for $\oplus_{j=1}^{\infty} L^2(S_1, \mu_j)$. Finally, using Theorem 2.3.2, it follows that

$$\{A^n U^{-1} V(p_j(z))\}_{n\geq 0} \cup \{A^n U^{-1} V(q_j(z))\}_{n\geq 0}$$

forms a complete Bessel system for $U\mathscr{H}_1 = \oplus_{j=1}^{\infty} L^2(S_1, \mu_j)$.

The existence of complete Bessel system in $\mathscr{H}_2$ (moreover, a Parseval frame) follows from Part (b) of Theorem 4.3.2 which we prove next.

(b) Let D be the operator $(Id - AA^*)^{\frac{1}{2}}$. Let $\mathscr{O}$ be an orthonormal basis for $\mathrm{cl}(D\mathscr{H})$, and define $\mathscr{G} = \{g = Dh : h \in \mathscr{O}\}$. Then

$$\begin{aligned}
\sum_{n=0}^{m} \sum_{h\in\mathscr{O}} |\langle f, A^n Dh\rangle|^2 &= \sum_{n=0}^{m} \sum_{h\in\mathscr{O}} |\langle D(A^*)^n f, h\rangle|^2 = \sum_{n=0}^{m} \|D(A^*)^n f\|^2 \\
&= \sum_{n=0}^{m} \langle D^2 (A^*)^n f, (A^*)^n f\rangle = \sum_{n=0}^{m} \langle (Id - AA^*)(A^*)^n f, (A^*)^n f\rangle \\
&= \|f\|^2 - \|(A^*)^{m+1} f\|.
\end{aligned}$$

Using Lebesgue's Dominated Convergence Theorem,

$$\|(A^*)^m f\|^2 = \int_{\overline{\mathbb{D}}_1} |z|^{2m} \|\widetilde{f}(z)\|^2 d\mu(z) \to 0$$

as $m \to \infty$ since $|z|^{2m} \to 0$, $\mu - a.e.$ on $\overline{\mathbb{D}}_1$. Hence, from the identity above we get that

$$\sum_{n=0}^{\infty} \sum_{h\in\mathscr{I}} |\langle f, A^n Dh\rangle|^2 = \|f\|^2.$$

Therefore the system of vectors $\mathscr{G} = \{g = Dh : h \in \mathscr{O}\}$ is a tight frame for $\mathscr{H}$. □

The proof of Theorem 4.3.4 is a direct consequence of the proof of (a) in the above theorem.

Proof of Theorem 4.3.5. Suppose $\mu(\overline{\mathbb{D}}^c_{1-\varepsilon}) = 0$ for some $0 < \varepsilon < 1$. Because $|\mathscr{G}| < \infty$ and $\dim(\mathscr{H}) = \infty$, the system $\{A^n g\}_{g\in\mathscr{G},\, n=0,1,\ldots,M}$ is not complete in $\mathscr{H}$ for $M < \infty$. From the Hahn – Banach theorem, there exists a vector $h \in \mathscr{H}$ with $\|h\| = 1$ such that $\langle A^n g, h\rangle = 0$ for every $g \in \mathscr{G}$, and $n = 0, \ldots, M$. Then

$$
\begin{aligned}
&\sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty} |\langle h, A^n g\rangle|^2 \\
&= \sum_{g\in\mathscr{G}}\sum_{n=M+1}^{\infty} \Big| \sum_{0\le j\le\infty} \int_{\mathbb{C}} z^n \langle \widetilde{h}_j(z), \widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu_j(z)\Big|^2 \\
&\le \sum_{g\in\mathscr{G}}\sum_{n=M+1}^{\infty} \Big| \int_{\overline{\mathbb{D}}_{1-\varepsilon}} z^n \sum_{0\le j\le\infty} \mathbb{1}_{\mathscr{E}_j}(z) \langle \widetilde{h}_j(z), \widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)} d\mu(z)\Big|^2 \\
&\le \sum_{g\in\mathscr{G}}\sum_{n=M+1}^{\infty} (1-\varepsilon)^{2n} \Big(\sum_{0\le j\le\infty} \int_{\mathbb{C}} |\langle \widetilde{h}_j(z), \widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)}| d\mu_j(z)\Big)^2.
\end{aligned}
$$

Applying Hölder's inequality several times, we get

$$
\begin{aligned}
&\sum_{0\le j\le\infty} \int_{\mathbb{C}} |\langle \widetilde{h}_j(z), \widetilde{g}_j(z)\rangle_{\ell^2(\Omega_j)}| d\mu_j(z) \\
&\le \sum_{0\le j\le\infty} \int_{\mathbb{C}} \|\widetilde{h}_j(z)\|_{\ell^2(\Omega_j)} \|\widetilde{g}_j(z)\|_{\ell^2(\Omega_j)} d\mu_j(z) \\
&\le \sum_{0\le j\le\infty} \Big(\int_{\mathbb{C}} \|\widetilde{h}_j(z)\|^2_{\ell^2(\Omega_j)} d\mu_j(z)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{C}} \|\widetilde{g}_j(z)\|^2_{\ell^2(\Omega_j)} d\mu_j(z)\Big)^{\frac{1}{2}} \\
&\le \|h\|\|g\|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty} |\langle h, A^n g\rangle|^2 \le \sum_{g\in\mathscr{G}}\sum_{n=M+1}^{\infty} (1-\varepsilon)^{2n} \|h\|^2 \|g\|^2 \\
=& \frac{(1-\varepsilon)^{2(M+1)}}{1-(1-\varepsilon)^2} \|h\|^2 \sum_{g\in\mathscr{G}} \|g\|^2 \ \to 0 \ \text{ as } \ M \to \infty.
\end{aligned}
$$

Therefore the left frame inequality does not hold, and we have a contradiction. $\square$

Proof of Theorem 4.3.6. Define the subspace V_ρ of $\mathscr{H}$ to be $V_\rho = \{f : \operatorname{supp}\tilde{f} \subseteq \overline{\mathbb{D}}_\rho\}$. The restriction of A to V_ρ is a normal operator with its spectrum equal to the part of the spectrum of A inside $\overline{\mathbb{D}}_\rho$. Let $\widetilde{\mathscr{G}} = U\mathscr{G}$ where U is as in Theorem 2.3.2. Let $\widetilde{\mathscr{G}_\rho} = \{\mathbb{1}_{\overline{\mathbb{D}}_\rho}\widetilde{g} : \widetilde{g} \in \widetilde{\mathscr{G}}\}$. Since $\{A^n g\}_{g\in\mathscr{G},n\geq 0}$ is a frame by assumption, $\{z^n\widetilde{w}\}_{\widetilde{w}\in\widetilde{\mathscr{G}_\rho}}$ is a frame for UV_ρ. Thus, since $\rho < 1$, Theorem 4.3.5 implies that V_ρ is finite-dimensional. Hence the restriction of the spectrum of A to $\overline{\mathbb{D}}_\rho$ for any $\rho < 1$ is a finite set of points. We also know from Theorem 4.3.1 (b) that $\mu(\mathbb{D}_1^c) = 0$. Thus, UAU^{-1} has the form $\Lambda = \sum_j \lambda_j P_j$. $\square$

Proof of Theorem 4.3.8. (a) Follows from Theorem 4.3.4.

(b) If $l(g) < \infty$ then $A^{l(g)}g - \sum_{k=0}^{l(g)-1} c_k A^k g = 0$ for some complex numbers c_k. Call Q the polynomial $Q(z) := z^{l(g)} - \sum_{k=0}^{l(g)-1} c_k z^k$. We have $Q(A)g = 0$ and therefore $0 = U(Q(A)g)(z) = Q(z)\widetilde{g}(z)$ $\mu-$a.e. z.

Let E_g be the set of roots of Q. Hence $\widetilde{g}(z) = 0$ μ a.e. in $(\mathbb{C}\setminus E_g)$. This together with part (a) of the theorem gives us that, for all $g \in \mathscr{G}$,

$$\widetilde{g}(z) = 0 \quad \text{a.e. } \mu \quad \text{in } \bigcap_{g\in\mathscr{G}_F} (\overline{\mathbb{D}}_1^c \setminus E_g) \tag{4.8}$$

where, $\mathscr{G}_F = \{g \in \mathscr{G} : l(g) < \infty\}$.

The set $E := \bigcup_{g\in\mathscr{G}_F} E_g$ is countable and $\bigcap_{g\in\mathscr{G}_F}(\overline{\mathbb{D}}_1^c \setminus E_g) = \overline{\mathbb{D}}_1^c \setminus E$. So (4.8) holds on $\overline{\mathbb{D}}_1^c \setminus E$. It follows that for each $j \in \mathbb{N}^*$, span$\{\widetilde{g}_j(z)\}_{g\in\mathscr{G}}$ is not complete in $\ell^2(\Omega_j)$ μ_j-a.e. $z \in \overline{\mathbb{D}}_1^c \setminus E$ and therefore $\mu_j(\overline{\mathbb{D}}_1^c \setminus E) = 0$. We conclude that $\mu(\overline{\mathbb{D}}_1^c \setminus E) = 0$.

(c) Let $\Delta := \{x \in S_1 : x \in \operatorname{supp} g,\ g \in \mathscr{G},\ l(g) < \infty\}$. From the proof of (b) Δ is countable. Then, since the projection of a Bessel system is Bessel, and the projection of a complete set is complete, following the proof of Theorem 4.3.1(a) we can see that μ is absolutely continuous on $S_1 \setminus \Delta$. $\square$

4.5 Self-adjoint operators

The class of self-adjoint operators is an important subclass of normal reductive operators which has some interesting properties that we study in this section. In particular, we prove that for self-adjoint operators the normalized system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},\, n\geq 0}$ is never a frame. The proof of this fact relies on the following theorem.

Theorem 4.5.1. *Every unit norm frame is a finite union of Riesz basis sequences.*

Theorem 4.5.1 was conjectured by Feichtinger and is equivalent to the Kadison – Singer theorem [19, 15] which was proved recently in [45].

Theorem 4.5.2. *If A is a self-adjoint operator on $\mathscr{H}$ then the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},\, n\geq 0}$ is not a frame for $\mathscr{H}$.*

Remark 4.5.3. *An open problem is whether the theorem remains true for general normal operators. The theorem does not hold if the operator is not normal. For example, the shift operator S on $\ell^2(\mathbb{N})$ defined by $S(x_1,x_2,\dots)=(0,x_1,x_2,\dots)$, is not normal, and $\{S^n e_1\}$ where $e_1=(1,0,\dots)$ is an orthonormal basis for $\ell^2(\mathbb{N})$.*

Remark 4.5.4. It may be that the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},\, n\geq 0}$ is not a frame for $\mathscr{H}$ because it is overly redundant due to the fact that we are iterating $\{A^n g\}_{g\in\mathscr{G}}$ for all $n\geq 0$. We may reduce the redundancy by letting $0\leq n<L(g)$ where $L\in\mathscr{L}$ as defined in Remark 4.1.2. For example, if $\{g\}_{g\in\mathscr{G}}$ is an orthonormal basis for $\mathscr{H}$, then trivially, we can choose $L(g)=1$ and the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},0\leq n<L(g)}$ is an orthonormal basis for $\mathscr{H}$. However, if $\mathscr{G}$ is finite, $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},0\leq n<L(g)}$ cannot be a frame for $\mathscr{H}$ as in the corollary below.

Corollary 4.5.5. *Let $\{g\}_{g\in\mathscr{G}}\subset\mathscr{H}$ and assume that $|\mathscr{G}|<\infty$ and $L\in\mathscr{L}$. Then for a self-adjoint operator A, $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},0\leq n<L(g)}$ is not a frame for $\mathscr{H}$.*

Proof of Theorem 4.5.2. Suppose it is a frame. Using Feichtinger's theorem, we decompose the set $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},\, n\geq 0}$ into a finite union of Riesz sequences. Choose a vector $h\in\mathscr{G}$.

Thus the subsystem $\left\{\frac{A^n h}{\|A^n h\|}\right\}_{n\geq 0}$ can be decomposed into a union of Riesz sequences and therefore a union of minimal sets. Since there are finitely many sequences, the powers of A in one of these sequences must contain infinite number of even numbers $\{2n_k\}$ (in particular, the system $\{A^{2n_k}h\}_{k=1,\dots}$ is a minimal set) such that

$$\sum_{k\geq 1}\frac{1}{n_k}=\infty. \tag{4.9}$$

If we consider the operator A^2, then its spectrum is a subset of $[0,\infty)$. In order to finish the proof of the theorem, we use the following Lemma whose proof is a corollary of the Müntz – Szász theorem [58].

Lemma 4.5.6. *Let μ be a regular Borel measure on $[0,\infty)$ with a compact support and n_k, $k=0,1,\dots,$ be a sequence of natural numbers such that $n_0=0$ and*

$$\sum_{k\geq 1}\frac{1}{n_k}=\infty.$$

For a function $\phi\in L^1(\mu)$, if

$$\int_0^\infty x^{n_k}\phi(x)d\mu(x)=0\, \textit{for every}\ \ k,$$

then $\phi=0$ μ a.e.

Let $V=\mathrm{cl}\,(\mathrm{span}\,\{(A^2)^n h\}_{n\geq 0})$, and let B be the restriction of A^2 on V. Since B is positive definite, its spectrum $\sigma(B)\subset[0,b]$ for some $b\geq 0$. Let μ be the measure defined in (2.3) associated with B. By Theorem 4.1.1, $\mu_j=0$ for all $j\neq 1$ (i.e., $\mu=\mu_1$), and $\widetilde{h}(x)\neq 0$ a.e. μ.

Let $n_k, k\geq 1$ be the sequence of integers chosen above such that $\{A^{2n_k}h\}_{k=1,\dots}$ is a minimal set and (4.9) holds. Set $n_0=0$. Note that both sequences $\{n_k\}_{k\geq 0}$, and $\{n_k\}_{k=0,m,m+1,\dots}$ satisfy the condition of the Lemma 4.5.6, hence $\int_0^b x^{n_k}\widetilde{h}(x)\overline{\widetilde{f}(x)}d\mu(x)=0$ for all $k\geq 0$ implies that $\widetilde{f}=0$ a.e. μ, as well as $\int_0^b x^{n_k}\widetilde{h}(x)\overline{\widetilde{f}(x)}d\mu(x)=0$ for all $k=0,m,m+1,\dots$ implies that $\widetilde{f}=0$ a.e. μ. Thus, $V=\operatorname{cl}(\operatorname{span}\{(A^2)^n h\}_{n\geq 0})=\operatorname{cl}(\operatorname{span}\{(A^2)^{n_k}h\}_{k=0,m,m+1,\dots})=\operatorname{cl}(\operatorname{span}\{(A^2)^{n_k}h\}_{k\geq 0})$ which contradicts the minimality condition. $\square$

Proof of Corollary 4.5.5. Suppose the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G},\,0\leq n<L(g)}$ is a frame for $\mathscr{H}$. Because $\dim\mathscr{H}=\infty$, the set $\mathscr{G}_\infty=\{g\in\mathscr{G}|\ L(g)=\infty\}$ is non-empty. Then the system $\left\{\frac{A^n g}{\|A^n g\|}\right\}_{g\in\mathscr{G}_\infty,0\leq n<L(g)=\infty}$ is a frame for its closure since we get it by removing finite number of vectors from a frame. The closure is an invariant subspace and A restricted to it remains self-adjoint which contradicts Theorem 4.5.2. $\square$

4.6 Applications to groups of unitary operators

In this section, we apply some of our results to discrete groups of unitary operators. These often occur in wavelet, time frequency and frame constructions.

As a corollary of the spectral theorem of normal operators, a normal operator is unitary if and only if its spectrum is a subset of the unit circle. We will need the following Proposition from Wermer [70].

Proposition 4.6.1 ([70]). *For a unitary operator T, the following are equivalent*

1. *T is not reductive*

2. *The arc length measure is absolutely continuous with respect to the spectral measure of T.*

Let π be a unitary representation of a discrete group Γ on Hilbert space $\mathscr{H}$. The order $o(\gamma)$ of an element $\gamma\in\Gamma$ is the smallest natural number m such that $\gamma^m=1$. If no such number exists then we say $o(\gamma)=\infty$. The same way we define the order of an operator $\pi(\gamma)$.

Notice that if $o(\pi(\gamma))<\infty$ then it is reductive and its spectrum is a subset of the set of $o(\pi(\gamma))$-th roots of unity.

Theorem 4.6.2. *Let π be a unitary representation of a discrete group Γ on Hilbert space $\mathscr{H}$ and suppose there exists a set of vectors $\mathscr{G} \subseteq \mathscr{H}$ such that $\{\pi(\gamma)g : \gamma \in \Gamma, g \in \mathscr{G}\}$ is a minimal system. Then, for every $\gamma \in \Gamma$ with $o(\gamma) = \infty$, $\pi(\gamma)$ is non-reductive and hence the arc length measure on S_1 is absolutely continuous with respect to the spectral measure of $\pi(\gamma)$.*

Proof. The minimality condition implies that π is injective and hence $o(\gamma) = o(\pi(\gamma))$. Let $\gamma \in \Gamma$ be such that $o(\gamma) = \infty$, then from the minimality assumption, $\{\pi(\gamma)^n g : \gamma \in \Gamma, g \in \mathscr{G}\}$ is a minimal subsystem. Thus, from Corollary 4.2.3, $\pi(\gamma)$ is non-reductive. The rest follows from Proposition 4.6.1 above. $\square$

Theorem 4.6.3. *Let π be a unitary representation of a discrete group Γ on Hilbert space $\mathscr{H}$ and suppose there exists a set of vectors $\mathscr{G} \subseteq \mathscr{H}$ such that $\Gamma\{\mathscr{G}\} = \{\pi(\gamma)g : \gamma \in \Gamma, g \in \mathscr{G}\}$ is complete in $\mathscr{H}$ and, for every $g \in \mathscr{G}$, $\Gamma\{g\} = \{\pi(\gamma)g : \gamma \in \Gamma\}$ is a Bessel system in $\mathscr{H}$. Then for every $\gamma \in \Gamma$ with $o(\gamma) = \infty$, the measure μ associate with $\pi(\gamma)$ is absolutely continuous with respect to the arc length measure on S_1.*

Proof. Suppose $o(\gamma) = \infty$. The assumption that the system $\{\pi(\gamma)g : \gamma \in \Gamma\}$ is Bessel implies that the kernel of the representation π must be finite, otherwise any vector in the system will be repeated infinitely many times, prohibiting the Bessel property from holding. Thus $o(\gamma) = \infty$ implies $o(\pi(\gamma)) = \infty$.

Pick any vector $\pi(h)g$ where $h \in \Gamma, g \in \mathscr{G}$. Then $\{\pi(\gamma)^n \pi(h)g\}_{n \geq 0}$ is a subsystem of $\Gamma\{g\}$ since $\pi(\gamma)^n \pi(h) \neq \pi(\gamma)^m \pi(h)$ if $n \neq m$. Hence, using the fact that $\{\pi(\gamma)^n \pi(h)g\}_{n \geq 0}$ is a Bessel sequence, from the proof of Theorem 4.3.1(a) we get that, for every $j \in \mathbb{N}^*$, the measure μ_j in the (2.2) representation of $\pi(\gamma)$ is absolutely continuous on $\operatorname{supp}[(\pi(h)g\tilde{)}_j]$. Since $\{\pi(h)g : h \in \Gamma, g \in \mathscr{G}\}$ is complete in $\mathscr{H}$, from Theorem 4.1.1, μ is concentrated on the set $\cup_{0 \leq j \leq \infty} \operatorname{supp}[(\pi(h)g\tilde{)}_j]$ thus we get that the spectrum of $\pi(\gamma)$ is absolutely continuous with respect to arc length measure. $\square$

In fact, it was shown in [69] (lemma 4.19) that the assumptions in the previous theorem hold if and only if π is a subrepresentation of the left regular representation of Γ with some multiplicity. And as a corollary of that, if the conditions of Theorem 4.6.3 hold, it is possible to find another set $\mathscr{G}' \subset \mathscr{H}$ such that $\{\pi(\gamma)g : \gamma \in \Gamma, g \in \mathscr{G}'\}$ is a Parseval frame for $\mathscr{H}$.

Chapter 5

General operators

In this chapter, we consider the case of systems generated by the iterative actions of operators that are not necessarily normal.

Theorem 5.0.1. *If for an operator $A \in B(\mathscr{H})$ there exists a set of vectors $\mathscr{G}$ in $\mathscr{H}$ such that $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a frame in $\mathscr{H}$ then for every $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$.*

Proof. Suppose, for some $\{g\}_{g\in G}$, $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is a frame with frame bounds B_1 and B_2. Let $f \in \mathscr{H}$. Then for any $m \in \mathbb{Z}$ we have

$$\begin{aligned}\sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty}|\langle (A^*)^m f, A^n g\rangle|^2 &= \sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty}|\langle f, A^{n+m} g\rangle|^2 \qquad (5.1)\\ &= \sum_{g\in\mathscr{G}}\sum_{n=m}^{\infty}|\langle f, A^n g\rangle|^2.\end{aligned}$$

Since $\sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty}|\langle f, A^n g\rangle|^2 \leq B_2\|f\|^2$, we conclude that $\sum_{n=m}^{\infty}\sum_{g\in\mathscr{G}}|\langle f, A^n g\rangle|^2 \to 0$ as $m \to \infty$. Thus, from (5.1), we get that $\sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty}|\langle (A^*)^m f, A^n g\rangle|^2 \to 0$ as $m \to \infty$. Using the lower frame inequality, we get

$$B_1\|(A^*)^m f\| \leq \sum_{g\in\mathscr{G}}\sum_{n=0}^{\infty}|\langle (A^*)^m f, A^n g\rangle|^2.$$

Since the right side of the inequality tends to zero as m tends to infinity we get that $(A^*)^m f \to 0$ as $m \to \infty$. □

Corollary 5.0.2. *For any unitary operator $A : \mathscr{H} \to \mathscr{H}$ and any set of vectors $\mathscr{G} \subset \mathscr{H}$, $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ is not a frame in $\mathscr{H}$.*

If for every $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$, then we can get the following existence theorem of frames for $\mathscr{H}$ from iterations.

Theorem 5.0.3. *If A is a contraction (i.e., $\|A\| \leq 1$), and for every $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$, then we can choose $\mathscr{G} \subseteq \mathscr{H}$ such that $\{A^n g\}_{g \in \mathscr{G},\, n \geq 0}$ is a tight frame.*

Remark 5.0.4. *The system we find in this case is not very useful since the initial system $\mathscr{G}$ is "too large" (it is complete in $\mathscr{H}$ in some cases). Moreover, the condition $\|A\| \leq 1$ is not necessary for the existence of a frame with iterations. For example, we can take nilpotent operators with large operator norm for which there are frames with iterations.*

Proof. Suppose for any $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$ and $\|A\| \leq 1$. Let $D = (Id - AA^*)^{\frac{1}{2}}$ and $\mathscr{V} = \mathrm{cl}(D\mathscr{H})$. Let $\mathscr{I}$ be an orthonormal basis for $\mathscr{V}$. Then

$$\begin{aligned}
\sum_{n=0}^{m} \sum_{h \in \mathscr{I}} |\langle f, A^n Dh \rangle|^2 &= \sum_{n=0}^{m} \sum_{h \in \mathscr{I}} |\langle D(A^*)^n f, h \rangle|^2 \\
&= \sum_{n=0}^{m} \|D(A^*)^n f\|^2 \\
&= \sum_{n=0}^{m} \langle D^2 (A^*)^n f, (A^*)^n f \rangle \\
&= \sum_{n=0}^{m} \langle (Id - AA^*)(A^*)^n f, (A^*)^n f \rangle \\
&= \|f\|^2 - \|(A^*)^{m+1} f\|.
\end{aligned}$$

Taking limits as $m \to \infty$ and using the fact that $(A^*)^m f \to 0$ we get from the identity above that

$$\sum_{n=0}^{\infty} \sum_{h \in \mathscr{I}} |\langle f, A^n Dh \rangle|^2 = \|f\|^2.$$

Therefore the system of vectors $\mathscr{G} = \{g = Dh : h \in \mathscr{I}\}$ is a tight frame for $\mathscr{H}$. □

Theorem 5.0.5. *If $\dim \mathscr{H} = \infty$, $|\mathscr{G}| < \infty$, and $\{A^n g\}_{g \in \mathscr{G},\, n \geq 0}$ satisfy the lower frame bound, then $\|A\| \geq 1$.*

Proof. Suppose $\|A\| < 1$. Since $\{g\}_{g \in \mathscr{G}}$ is finite and $\dim(\mathscr{H}) = \infty$, for any fixed N there exists a vector $f \in \mathscr{H}$ with $\|f\| = 1$ such that $\langle A^n g, f \rangle = 0$, for every $g \in \mathscr{G}$ and $0 \leq n \leq N$.

Then

$$\sum_{g\in\mathscr{G}}\sum_{n\geq 0}|\langle A^n g, f\rangle|^2 = \sum_{g\in\mathscr{G}}\sum_{n=N}^{\infty}|\langle A^n g, f\rangle|^2 \leq \sum_{g\in\mathscr{G}}\|g\|\sum_{n=N}^{\infty}\|A\|^{2n} \to 0$$

as $N \to \infty$ hence the lower frame bound cannot hold. □

Corollary 5.0.6. *Suppose* $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ *with* $|\mathscr{G}| < \infty$ *satisfy the lower frame bound. Then for any coinvariant subspace* $\mathscr{V} \subset \mathscr{H}$ *of* A *with* $\|P_{\mathscr{V}} A P_{\mathscr{V}}\| < 1$ *we have that* $\dim(\mathscr{V}) < \infty$.

Proof. $\mathscr{V}$ is coinvariant for A that is equivalent to

$$P_{\mathscr{V}} A = P_{\mathscr{V}} A P_{\mathscr{V}}.$$

It follows that $P_{\mathscr{V}} A^n = P_{\mathscr{V}} A^n P_{\mathscr{V}}$. Hence, if $\{A^n g\}_{g\in\mathscr{G},\, n\geq 0}$ satisfy the lower frame inequality in $\mathscr{H}$, then $\{(P_{\mathscr{V}} A P_{\mathscr{V}})^n g\}_{g\in\mathscr{G},\, n\geq 0}$ also satisfy the lower frame inequality for $\mathscr{V}$ and hence from the previous theorem if $\dim(\mathscr{V}) = \infty$, then $\|P_{\mathscr{V}} A P_{\mathscr{V}}\| \geq 1$. □

Chapter 6

Dynamical sampling in shift-invariant spaces

In this chapter, we formulate the problem of dynamical sampling in shift-invariant spaces and study its connection to the dynamical sampling in $\ell^2(\mathbb{Z})$. We show that, in some cases, the problem of dynamical sampling in a shift-invariant space can be reduced to the problem of dynamical sampling in $\ell^2(\mathbb{Z})$. In other cases when this reduction is not possible, we provide specific reconstruction results.

6.1 Formulation of the problem

Let $V = V(\phi)$ be the shift-invariant space generated by ϕ. We assume the atom ϕ is in the space $W_0(L^1) := W(L^1) \cap C(\mathbb{R}^d)$ and satisfies the condition (2.1):

$$\alpha \leq \Phi(\xi) \leq \beta, \text{ a.e. } \xi \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

where $\Phi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+k)|^2$. Then any function f in V has a representation

$$f(t) = \sum_{k\in\mathbb{Z}} c_k \phi(t-k) \tag{6.1}$$

for some $c = (c_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$, also f is continuous and can be sampled at any $x \in \mathbb{R}^d$.

We assume the evolution operator A is given as a convolution with some kernel $a \in L^1(\mathbb{R})$ and the measurements are given by samples with uniform rate m: $\Omega_0 = \cdots = \Omega_{L-1} = m\mathbb{Z}$. Thus, the dynamical sampling problem becomes to recover, preferably in the stable way, the function $f \in V(\phi)$, given the samples

$$y_0 = f|_{m\mathbb{Z}},\ y_1 = (a*f)|_{m\mathbb{Z}},\ \ldots\ y_n = (a^{L-1}*f)|_{m\mathbb{Z}}. \tag{6.2}$$

For $a \in L^1(\mathbb{R}^d)$, we have $a * \phi \in W_0(L^1)$ whenever $\phi \in W_0(L^1)$ from Proposition 2.2.3. This means that A maps $V(\phi)$ onto another SIS $V(a*\phi)$; under these restrictions $a^n f$ lies in a shift-invariant space generated by another function $a*\phi$.

Lemma 6.1.1. *For a vector $g \in \ell^2(\mathbb{Z})$,*

$$\widehat{(S_m g)}\,(\xi) = \frac{1}{m}\sum_{s=0}^{m-1} \hat{f}\left(\frac{\xi+s}{m}\right)$$

where S_m denotes the downsampling operator with rate m given by $S_m(k)g = g(mk)$, $\forall k \in \mathbb{Z}$.

Proof. From the definition

$$\begin{aligned}\sum_{l=0}^{m-1} \hat{f}\left(\frac{\xi+l}{m}\right) &= \sum_{l=0}^{m-1}\sum_{s\in\mathbb{Z}} f(s)e^{\frac{-2\pi i(\xi+l)s}{m}} = \sum_{s\in\mathbb{Z}}\sum_{l=0}^{m-1} f(s)e^{\frac{-2\pi i(\xi+l)s}{m}} \\ &= \sum_{s\in\mathbb{Z}} f(s)e^{\frac{-2\pi i\xi s}{m}}\sum_{l=0}^{m-1} e^{\frac{-2\pi ls}{m}} = m\sum_{s\in\mathbb{Z}} f(s'm)e^{-2\pi i\xi s'}\end{aligned}$$

since

$$\sum_{l=0}^{m-1} e^{\frac{-2\pi ls}{m}} = \begin{cases} m, \text{ when } s \in m\mathbb{Z} \\ 0, \text{ when otherwise} \end{cases}.$$

□

Lemma 6.1.2. *Let Φ_n be the restriction of $A^n\phi \in W_0(L^1)$ to the set of integers: $\Phi_n = A^n\phi|_{\mathbb{Z}}$. Then $\Phi_n \in \ell^1(\mathbb{Z})$ and the corresponding Fourier series of sub-sampled states are*

$$\hat{y}_n(\xi) = \frac{1}{m}\sum_{l\in\mathbb{Z}_m} \hat{c}\left(\frac{\xi+l}{m}\right)\hat{\Phi}_n\left(\frac{\xi+l}{m}\right). \tag{6.3}$$

Proof. The proof of the fact $\Phi_n \in \ell^1(\mathbb{Z})$ can be found in [10]. Notice that, $y_n = S_m(c*\Phi_n)$, where S_m is the downsampling operator. Then using the Lemma 6.1.1 we conclude the proof. □

Remark 6.1.3. *Note that, if the Poisson Summation formula holds for $A^n\phi$, then $\hat{\Phi}_n(\xi)$ is just the periodization of $(A^n\phi)\hat{}(\xi) = \hat{a}^n(\xi)\hat{\phi}(\xi)$:*

$$\hat{\Phi}_n(\xi) = \sum_{k\in\mathbb{Z}} \hat{a}^n(\xi+k)\hat{\phi}(\xi+k).$$

Using the following matrix notation

$$\Phi_L(\xi) = \begin{pmatrix} \hat{\Phi}_0(\frac{\xi}{m}) & \hat{\Phi}_0(\frac{\xi+1}{m}) & \dots & \hat{\Phi}_0(\frac{\xi+m-1}{m}) \\ \hat{\Phi}_1(\frac{\xi}{m}) & \hat{\Phi}_1(\frac{\xi+1}{m}) & \dots & \hat{\Phi}_1(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\Phi}_{L-1}(\frac{\xi}{m}) & \hat{\Phi}_{L-1}(\frac{\xi+1}{m}) & \dots & \hat{\Phi}_{L-1}(\frac{\xi+m-1}{m}) \end{pmatrix},$$

$$\bar{\mathbf{C}}(\xi) = \begin{pmatrix} \hat{c}\left(\frac{\xi}{m}\right) \\ \hat{c}\left(\frac{\xi+1}{m}\right) \\ \vdots \\ \hat{c}\left(\frac{\xi+m-1}{m}\right) \end{pmatrix} \text{ and } \bar{Y}(\xi) = \begin{pmatrix} \hat{y}_0(\xi) \\ \hat{y}_1(\xi) \\ \vdots \\ \hat{y}_{L-1}(\xi) \end{pmatrix}$$

The equations (6.3), for each $n = 1, \dots, L-1$, can be combined together and written in matrix form:

Proposition 6.1.4. *For each $n = 1, \dots, L-1$,*

$$\Phi_L(\xi)C(\xi) = Y(\xi),\ \forall\xi \in [0,1]. \tag{6.4}$$

Corollary 6.1.5. *Let $f \in V = V(\phi)$. We can recover the coefficients sequence $\{c_\lambda\}_{\lambda\in\mathbb{Z}^d}$ in the expansion (6.1) of f from the collection of samples*

$$y_n = (A^n f(n))_{n\in m\mathbb{Z}} \quad n = 0, 1, \dots, m-1,$$

if and only $\det(\Phi_m(\xi)) \neq 0$ *for a.e. $\xi \in [0,1]$. And we can recover in stable way if and only* *if* $|\det(\Phi_m(\xi))| > \alpha$ *a.e. for some* $\alpha > 0$.

Theorem 6.1.6. *If* $\Phi_L(\xi)$ *is singular only when* $\xi \in \{\xi_i\}_{i\in I}$*, where* $|I| < \infty$*, and* J *is a positive integer such that* $|\xi_i - \xi_j| \neq \frac{k}{J}$ *for any* $i, j \in I$ *and* $k \in \{1,\dots,m-1\}$*, then the extra samples* $\{S_{mJ}T_c f\}$ *combined with*

$$y_n(k) = (A^n f(mk))_{k\in\mathbb{Z}}, \;\; n = 0,1,2,...,L-1$$

allow a stable recovery of $f \in V(\phi)$.

6.2 Reduction to $\ell^2(\mathbb{Z})$ case

Under the appropriate conditions on ϕ, the dynamical sampling in SIS reduces to the discrete case $\ell^2(\mathbb{Z})$. To establish this connection we use the following theorem

Theorem 6.2.1. *Let* $V(\phi)$ *be the SIS generated by the function* ϕ*. For a kernel* $a(t)$ *such that* $\hat{a} \in L^\infty(\mathbb{R})$*, the following are equivalent*

1. $a * \phi \in V(\phi)$
2. $a * V(\phi) \subseteq V(\phi)$
3. *there exists a convolutor* b *with* $\hat{b} \in L^\infty$ *such that for any* $c \in \ell^2(\mathbb{Z})$

$$a * (c *_{sd} \phi) = (b *_d c) *_{sd} \phi \tag{6.5}$$

4. *for every* $k \in \mathbb{Z}$ *and a.e.* $\xi \in [0,1]$

$$\hat{a}(\xi + k)\hat{\phi}(\xi + k) = \hat{b}(\xi)\hat{\phi}(\xi + k) \tag{6.6}$$

for some function $\hat{b}(\xi) \in L^2[0,1]$.

Proof. $(1) \Rightarrow (4)$ If $a * \phi \in V(\phi)$, then there exists $(b_k)_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$a * \phi(x) = \sum_{k\in\mathbb{Z}} b_k \phi(x-k). \tag{6.7}$$

Taking the Fourier transform of both sides of the (6.7), for

$$\hat{b}(\xi) = \sum_{k\in\mathbb{Z}} b_k e^{-2\pi i k\xi}$$

we get $\hat{a}(\xi)\hat{\phi}(\xi) = \hat{b}(\xi)\hat{\phi}(\xi)$ which is the same as (6.6), since $\hat{b}$ is 1-periodic.

$(4) \Rightarrow (3)$ From (6.6), we get

$$\sum_k |\hat{a}(\xi+k)|^2 |\hat{\phi}(\xi+k)|^2 = |\hat{b}(\xi)|^2 \sum_k |\hat{\phi}(\xi+k)|^2.$$

Since $\hat{a} \in L^\infty$, using (2.1) we get $|\hat{b}(\xi)|^2 \leq \|\hat{a}\|_\infty^2$, so that $\hat{b} \in L^\infty[0,1]$. The conclusion follows by multiplying both sides of (6.6) with $\hat{c}(\xi)$, where

$$\hat{c}(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{-2\pi i k\xi},$$

and taking the inverse Fourier transform.

$(3) \Rightarrow (2)$ Noting that $\hat{b}\hat{c} \in L^2[0,1]$ implies $b * c \in \ell^2$, we see clearly that the right hand side of (6.5) is in $V(\phi)$.

The implication $(2) \Rightarrow (1)$ is straight forward. $\square$

Note that we can reduce the dynamical sampling problem in $V(\phi)$ to the one in $\ell^2(\mathbb{Z})$ using the theorem above. Specifically, if $\phi \in W_0(L^1)$ and the condition that $\widehat{\Phi}_0(\xi) = \sum_k \hat{\phi}(\xi+k) \neq 0$, then for each $f = c *_{sd} \phi$ we associate $x \in \ell^2$ by $x = f(\mathbb{Z})$. The map $f \mapsto x$ from $V(\phi)$ to ℓ^2 is well-defined, since $\phi \in W_0(L^1)$. Note that the convolution operator $a * f$ corresponds to the discrete convolution $b *_d x$ where b is obtained from a as in Theorem 6.2.1. Hence $S_m(a^n f) = S_m(b^n x)$. By solving the dynamical system on ℓ^2 to obtain x, we can recover f by finding $\hat{c} = \hat{x}/\widehat{\Phi}_0$. Since $\widehat{\Phi}_0$ is continuous and nonzero, $\hat{c} \in L^2[0,1]$.

As a particular case of Theorem 6.2.1, if the sets $E_k = \{\operatorname{supp} \hat{\phi}(\xi + k), \xi \in [-1/2, 1/2]\}$ are disjoint, it is a sufficient conditions for the (6.6) to hold, as we can take

$$\hat{b}(\xi) = \sum_{k \in \mathbb{Z}} \hat{a}(\xi + k)\chi_{E_k} \quad \text{for } \xi \in [-1/2, 1/2].$$

For $\phi = \operatorname{sinc}$, as discussed in the introduction, we get the following corollary.

Corollary 6.2.2. *The dynamical sampling problem in $PW(\mathbb{R}) = V(\operatorname{sinc})$ can be reduced to that of the dynamical sampling in $\ell^2(\mathbb{Z})$ with*

$$\hat{b}(\xi) = \hat{a}(\xi)\chi_{[-1/2,1/2]}(\xi)$$

for $\xi \in [-1/2, 1/2]$.

The condition under which the dynamical sampling problem in SIS can be reduced to that in $\ell^2(\mathbb{Z})$ can be further elucidated by the following theorem which can be proved by solving (6.6).

Theorem 6.2.3. *Let $\phi \in L^2$ be such that $\{\phi(\cdot - k)\ k \in \mathbb{Z}\}$ is a Riesz basis for its closed span $V(\phi)$ with $E = \operatorname{supp} \hat{\phi}$. For a convolutor a such that $\hat{a} \in L^\infty$, and any of the equivalent conditions (1)-(4) of Theorem 6.2.1 is satisfied, then there exists $g \in L^\infty$ such that*

$$\hat{a} = \hat{b}\chi_E + g\chi_{E^c}. \tag{6.8}$$

Conversely, if (6.8) holds, for a 1-periodic $\hat{b} \in L^\infty$, some $g \in L^\infty$ and a measurable set E such that $\sum_j \chi_E(\xi + j) \geq 1$ a.e. ξ then clearly $\hat{a} \in L^\infty$. In addition, for any ϕ with $E = \operatorname{supp} \hat{\phi}$ satisfying (2.1) (i.e., $\{\phi(\cdot - k)\ k \in \mathbb{Z}\}$ is a Riesz basis for $V(\phi)$), the four equivalent conditions of Theorem 6.2.1 are satisfied.

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