Error Patterns in Ordering Fractions among At-Risk Fourth-Grade Students

By

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TABLE OF CONTENTS

Appendix

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Many students persistently misunderstand fractions and have difficulty assessing fraction magnitude (e.g., Brown & Quinn, 2007; U.S. Department of Education, 2008). For example, on the 2013 NAEP assessment, 40% of fourth-grade students could not determine that thirds were bigger than fourths, fifths, and sixths. Similarly, 65% of eighth-grade students could not explain that ½ + 3/8 + 3/8 was greater than one. One reason students struggle with assessing fraction magnitude is that fractions have different properties than whole numbers. One common error students make is overgeneralizing whole number properties to fractions; that is, they incorrectly apply whole number counting properties to fraction concepts (e.g., Ni & Zhou, 2005). For example, students may assume that 1/12 is larger than ½ because the whole number 12 is larger than 2. The following section includes an overview of theories attempting to explain why students tend to operate with whole number bias with fractions. This is followed by an overview of common error patterns with assessing fraction magnitude and goals of the present study.

Theories on Whole Number Bias

There are differing opinions about why students tend to overgeneralize whole number properties to fractions. Evolutionary (e.g., Geary, 2006) and conceptual change approach theories (e.g., Ni & Zhou, 2005; Vamakoussi & Vosniadou, 2010; Vosniadou, Vamvakoussi, & Skopeliti, 2008) both claim that whole numbers are easier to learn than fractions because instruction primarily focuses on the one-to-one counting properties of whole numbers. Whole numbers are successive, discrete, and have predictable calculation properties (e.g., Vosniadou et al., 2008).

Students have difficulty expanding their understanding of number representation when fractions are introduced in the upper elementary grades. Unlike whole numbers, fractions are not successive, they are not discrete, and they do not have predictable calculation properties (e.g., Vosniadou et al., 2008). According to these theories, overgeneralizing whole number properties to fractions is a result of forming a coherent explanatory framework of number as counting number. Students then struggle to reconcile the differing properties between whole numbers and fractions because fractions do not fit into this explanatory framework. As such, whole number knowledge interferes with developing a conceptual understanding of fractions.

Siegler, Thompson, and Schneider (2011) propose a somewhat different theory of numerical development as "a process of progressively broadening the class of numbers that are understood to possess magnitudes and of learning the functions that connect that increasingly broad and varied set of numbers to their magnitudes" (p. 275). According to them, students experience difficulty with fractions because they have a hard time simultaneously considering the numerator and denominator when assessing magnitude. In this way, Siegler et al. hold that whole-number knowledge does not necessarily interfere with fraction learning, but students must be explicitly taught that fractions have magnitude and can be ordered, compared, and placed on the number line. Without explicitly teaching these skills, students rely on the only relevant knowledge they have for assessing fraction magnitude – whole number knowledge.

The distinction between Siegler et al.'s (2011) theory and the earlier ones may seem small, but in contrast to evolutionary and conceptual change approach theories, Siegler et al.'s theory focuses on deficits in instruction as the key issue explaining why students struggle to consolidate properties of whole numbers and fractions. Typical instruction in the United States fails to emphasize how to conceptually assess fraction magnitude. As a result, students rely on whole number properties to determine fraction value and thus struggle to expand their concept of number to include fractions. If instruction does not emphasize magnitude understanding, students will likely continue to operate with whole number bias, which helps explain why whole number comparing patterns are common.

Common Error Patterns with Assessing Fraction Magnitude

It is well documented that individuals have substantial difficulty assessing fraction magnitude (e.g., Bonato, Fabbri, Umilta, & Zorzi, 2007; Schneider & Siegler, 2010; Siegler et al., 2011). I located three studies that investigated the accuracy with which individuals compare fractions and the comparing error patterns on which problem solvers rely. Stafylidou and Vosniadou (2004) investigated the explanatory frameworks for comparing and ordering fractions of 200 average-performing middle- and high- school students. One common misconception was the belief that a fraction is composed of two independent whole numbers, e.g., “the value of a fraction increases when the numbers that comprise it increase” (p. 507). This belief holds that as the whole number values in the numerator or denominator increases, the value of the fraction also increases. Accordingly, many students ordered or compared fractions based on the whole-number value in the numerator or the denominator (e.g., 2/6 > ½, which is incorrect).

Another common misconception was the belief that the value of a fraction increases when the whole numbers that comprise it decrease. This somewhat more advanced misconception reflects some rudimentary understanding that fractions with smaller denominators have bigger parts, but lacks understanding that the numerator and denominator operate synergistically to determine fraction value. Both these misconceptions decreased as students got older (i.e., more fifth-, sixth-, and seventh-grade students committed these errors than did eighth-grade or high school students).

Similar misconceptions about fraction values were found by DeWolf and Vosniadou (2011) among 28 adult undergraduate students enrolled at a prestigious university. The undergraduates were presented with 40 fraction pairs on the computer and asked to determine which fraction was bigger. Twenty of the fraction pairs were consistent with whole number ordering. That is, the larger of the two fractions was also the one with larger whole numbers in the numerator and denominator (e.g., 5/6 > ½). The other 20 fraction pairs were inconsistent with whole number ordering. That is, the larger of the two fractions had smaller whole numbers in the numerator and the denominator (e.g., 4/6 > 5/10). Accuracy and speed of response were superior for items that were consistent with whole number ordering than for items that were inconsistent with whole number ordering, which suggests that estimating fraction magnitude is difficult even for advanced students. This speaks to the difficulty of thinking about the numerator and denominator concurrently to assess a fraction's value when the comparison is inconsistent with whole number knowledge.

Like DeWolf and Vosniadou (2011), Meert, Grégoire, and Noël (2010) found that students were more accurate and quicker to respond when the fraction pairs were consistent with whole number ordering than inconsistent with whole number ordering. Average-achieving fifth- (n = 24) and seventh-grade (n = 44) students identified the larger value within 64 pairs of fractions. Thirty-two of the pairs had the same numerator (i.e., inconsistent with whole number ordering), and 32 pairs had the same denominator (i.e., consistent with whole number ordering). Seventh-grade students were more accurate and quicker to respond for both problem types than were fifth-grade students. However, the same numerator problems were difficult for both fifth- and seventh-grade students, as indicated by slower response times and decreased accuracy.

The Present Study

Across the age range studied, students demonstrated consistent whole number ordering errors when assessing fraction magnitude. Yet, I identified no prior study that investigated error patterns among fourth-grade students, examined the error patterns of students at risk for mathematics difficulties, or used different types of fraction knowledge to predict the probability of committing these errors. Previous intervention research (e.g., Cramer, Post, & delMas, 2002; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., in preparation-a; Fuchs et al., in preparation-b) has shown that teaching students how to assess fraction magnitude significantly improves their conceptual understanding of fractions, but none of these studies investigated how specific types of fraction knowledge predict at-risk students' error patterns. It is clear from NAEP data and the studies reviewed (DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004) that many students struggle with assessing fraction magnitude. Describing the frequency and types of errors among at-risk fourth graders has important implications for designing curriculum to improve students' conceptual understanding of fractions in the era of Common Core State Standards (CCSS, 2012).

Hecht and Vagi (2012) found that fourth- and fifth-grade students identified with mathematics difficulties (operationalized as scoring below the 25th percentile on a standardized mathematics test) had lower than expected procedural and conceptual knowledge about fractions compared to average-achieving students. This is probably true among the present study's at-risk sample. Lower conceptual knowledge likely leads to higher frequencies of errors than the average-achieving samples just reviewed (DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004). In the present study, risk was operationalized as scoring below the 35th percentile on a measure of whole-number knowledge at the start of fourth grade (when fraction knowledge is insufficiently developed to screen on). Because the curriculum primarily focuses on whole-number knowledge in grades K-3, students who continue to demonstrate difficulty with whole numbers in fourth grade will likely have difficulty understanding fraction concepts as they attempt to expand their concept of number to include rational numbers.

The present study had four purposes. The first purpose was to describe fraction ordering error patterns among fourth-grade students identified as at risk for developing mathematics difficulties. As the theories of numerical development suggest (Geary, 2006; Ni & Zhou, 2005; Vamakoussi & Vosniadou, 2010; Vosniadou et al., 2008; Siegler et al., 2011), students demonstrate difficulty with fractions because fraction properties are inconsistent with whole-number properties. Preventing difficulty, when fractions become the focus of instruction at fourth grade is important, especially because fraction knowledge has been found to be a unique predictor of future performance in algebra (Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Brown & Quinn, 2007; Siegler et al., 2012).

In this study, I refer to errors in which students order fractions from least to greatest based on the whole number values in the numerator and/or the denominator as *whole number ordering errors*. I hypothesized that these errors were the most common ordering error among at-risk fourth graders; that the next most common error involved students formulating judgments about fraction magnitude based solely on the value of the denominator. That is, students indicate that the fraction with the smallest denominator is the largest fraction and the fraction with the largest denominator is the smallest fraction without concurrently considering the value in the numerator (e.g., 5/8<3/4<½). In this study, I refer to this type of error as a *smallest denominator – biggest fraction ordering error.* This error is more advanced than whole number ordering errors, because it indicates rudimentary knowledge that the smaller denominator means the fraction has bigger parts. This misconception exemplifies the lack of appreciation that the numerator also factors in to the value of the fraction.

The second purpose was to examine performance differences among students from three different academic years over the period in which the school district moved toward implementation of CCSS (2012). This is interesting because CCSS emphasize the measurement interpretation of fractions, in which understanding fraction magnitude (e.g., ordering fractions from least to greatest) is central. As indicated by NAEP (2013) and the studies reviewed above (DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004), many students fail to master this skill. Although the district moved toward implementation of CCSS, I suspected there to be no performance differences among cohorts. This is because the sample included students at risk for mathematics difficulties.

The third purpose was to assess the effect of problem type (i.e., problems with the same numerator versus problems with different numerators and different denominators) on accuracy and whole number ordering errors. (Smallest denominator – biggest fraction ordering errors could not occur on same numerator problems.) I expected greater accuracy on the same numerator problems. In addition, consistent with Stafylidou and Vosniadou's (2004) finding that students believe that fraction values increase as the whole number values that comprise it decrease, whole number ordering errors are likely more common for same numerator problems than problems with different numerators and different denominators. This is because the numbers on the top are the same, and students easily resort to whole number knowledge to order the fractions based on the whole number values in the denominator.

The fourth purpose was to assess the effect of part-whole and measurement understanding of fractions on accuracy, whole number ordering errors, and smallest denominator

– biggest fraction ordering errors. Part-whole and measurement understanding were chosen as predictors of errors because each construct represents a component of conceptual understanding of fractions (e.g., Kilpatrick, Swafford, Findell, 2001) and help to explain individual differences in fraction skills (Hecht, Close, & Santiso, 2003). Each ordering problem could only have one outcome (i.e., each problem was correct or incorrect). If incorrect, only one error type was possible. Therefore, each of these outcomes (i.e., correct, whole number ordering error, smallest denominator – biggest fraction ordering error) was analyzed with a separate statistical model.

In this study, part-whole understanding was indexed by performance on a subset of released fraction items from the NAEP (U.S. Department of Education, 2010). Part-whole understanding emphasizes part to whole relationships. This type of fraction knowledge is often represented as shaded parts of a shape or as pieces of pizza (e.g. Charalambous & Pitta-Pantazi, 2007). The curriculum in the district where this study took place primarily focuses on part-whole understanding. Such understanding is foundational, but not sufficient, for developing an understanding of fraction magnitude (e.g., Charalambous & Pitta-Pantazi, 2007). For example, Smith, Solomon, and Carey (2005) documented that students could accurately judge the relative size of unit fractions (i.e., a fraction with 1 in the numerator) using part-whole explanations but were unable to accurately answer other questions about fraction magnitude properties (e.g., when you multiply a fraction by another fraction, the amount gets smaller). Because part-whole understanding provides an inadequate basis for succeeding with comparisons among non-unit fractions (Smith et al., 2005), I hypothesized that part-whole understanding fails to significantly predict the probability of correctly answering a problem and committing (or not committing) an ordering error when measurement understanding is controlled for in the model.

Measurement understanding was indexed by the accuracy with which students placed fractions on the 0-2 number line and performance on a subset of measurement items from the NAEP (U.S. Department of Education, 2010). Measurement understanding indexes the magnitude value of fractions and knowledge that fractions can be ordered, compared, and placed on the number line (e.g., Siegler et al., 2011; Wu, 2008). I expected superior measurement understanding to significantly predict accuracy and committing a smallest denominator – biggest fraction ordering error. In the first year of a strong focus on fraction learning (i.e., fourth-grade), at-risk students likely do not have advanced measurement understanding. Therefore, an increase in smallest denominator – biggest fraction ordering errors concurrent with an increase in measurement understanding may represent “a transitory phase in the process of fraction learning” and reflect the difficulties students have when they take on new information about ordering fractions (Stafylidou & Vosniadou, 2004, p. 512). By contrast, I expected an inverse relation between measurement understanding and the probability of committing a whole number ordering error. That is, the higher students’ measurement understanding, the less likely they were to commit a whole number ordering error.

To extend the focus further, students in the final cohort also completed an assessment that probed their measurement understanding more deeply, by asking them to provide explanations for how they compared nine pairs of fractions. Understanding how students’ verbal explanations relate to overall accuracy and error patterns is interesting, because sound explanations have been found to significantly predict mathematics achievement (Wong, Lawson, & Keeves, 2002). I expected the relation between student explanations and accuracy and error patterns to parallel the proposed relation between these outcomes and performance on number line and the NAEP measurement items. That is, students’ ability to write sophisticated explanations likely increases the probability of correctly answering a problem and committing a smallest denominator –

biggest fraction ordering error, and decreases the probability of committing a whole number ordering error. This hypothesis is consistent with Wong, Lawson, and Keeves's (2002) finding that sound student explanations significantly predict mathematics achievement (i.e., increasing accuracy and decreasing error).

CHAPTER II

METHOD

Participants included 227 at-risk fourth-grade students. Risk was defined as scoring below the 35th percentile on a measure of whole-number knowledge (Wide Range Achievement Test–4 [WRAT-4]; Wilkinson, 2008). The 227 students were from three cohorts within three larger randomized control trials. In the larger randomized control trials, students had been randomly assigned to intervention or control. For the present study, I relied exclusively on the control group (i.e., those that did not receive intervention), because intervention was designed to disrupt the developmental pattern expected for these students. Cohort 1 included 84 students; cohort 2 included 72 students; and cohort 3 included 71 students. The present study added the analysis of error types to previous reports based on the full sample (Fuchs et al., 2014, in preparation, in preparation), and results presented here do not overlap with those presented with previously reported findings.

Screening Measure

WRAT-4-Arithmetic includes 40 problems of increasing difficulty. At fourth grade, it primarily assesses students' proficiency with whole number computation skill, and includes 37 computation problems (addition, subtraction, multiplication, and division) and three estimation problems. Alpha on the sample for cohorts 1, 2, and 3 were .85, .85, and .74, respectively.

Fraction Measures

Ordering fractions. *Ordering Fractions* is a subtest from the *Fraction Battery-2012-* revised (Schumacher, Namkung, Malone, & Fuchs, 2012), and requires students to order nine sets of three fractions from least to greatest. Three problems have the same numerator and six problems have different numerators and different denominators. All discrepancies were discussed and resolved. Alpha on the sample for cohorts 1, 2, and 3 was .82, .79, and .79, respectively.

Coding of ordering errors. Each item on each student's test was first coded for accuracy. If the student answered a problem incorrectly, their answer was assigned an error type. See Table 1 for the coding sheet and error possibilities by problem. I identified two error types: whole number ordering errors and smallest denominator – biggest fraction ordering errors.

There were three examples of whole number ordering errors, depending on the problem. The first example included incorrectly ordering the fractions from least to greatest based on the whole-number value in the numerator and denominator. That is, students would get the same incorrect answer whether they ordered the three fractions from least to greatest based on the whole number in the numerator or the denominator (e.g., ½<2/6<5/8; problems A, C, and F). The second example included incorrectly ordering the fractions from least to greatest based on the whole-number value in the denominator (i.e., same numerator problems). Students ordered the whole numbers in the denominator from least to greatest (e.g., ½<1/5<1/12; problems B, D, E, G, H, and I). Students did not demonstrate an understanding that when a unit is divided into more parts (i.e., a bigger denominator), each of the parts gets smaller. The third example included incorrectly ordering the fractions from least to greatest based on the whole-number value in the numerator (e.g., ½<2/8<¾; problems D, H, and I).

Smallest denominator – biggest fraction ordering errors occurred when students incorrectly ordered the three fractions from least to greatest considering the size of the parts (i.e., the denominator), but not the number of parts (i.e., the numerator) in each fraction. For example, for 3/10, 7/12, and ½, students would incorrectly order them 7/12<3/10<½. That is, students assumed that the smaller the denominator, the bigger the fraction. This error could occur on problems A, C, D, H, and I.

To ensure that this error type was consistent with how students understood fractions, smallest denominator – biggest fraction ordering errors were only coded as present *if* students also got all of the same numerator problems correct on the test (problems B, E, and G). Accuracy on the same numerator problems indicated that students understood that as the denominator gets smaller, the size of the parts gets smaller. However, they could not transfer this knowledge to a set of fractions that had different numerators and different denominators and failed to account for the number of parts (i.e., the numerator) when ordering the fractions. If students did not get the same numerator problems correct, the coder could not assume that students had an understanding about how the denominator affects the size of the fraction.

If a student's answer did not fit into any category, their error was coded as a no category error. No category errors typically included students writing the wrong fractions in the blanks, or writing the same fraction twice (therefore, an error type could not be assigned). If students did not attempt the problem, this was also coded. Two independent coders scored all of the tests and entered the data. On the first scoring attempt, coders scored the tests with 98.65% accuracy.

Part-whole understanding. The NAEP part-whole items include eight released fourth- and eighth-grade problems from the National Assessment of Education Progress (U.S. Department of Education, 2010). Seven questions ask students to identify or write fractions using a picture, and one question assesses students' part-whole understanding with a word problem. The maximum score for the part-whole items is 9. Alpha on the sample for cohorts 1, 2, and 3 was .50, .47, and .59, respectively.

Measurement understanding. Three measures indexed students' measurement understanding. *Fraction Number Line* (Hamlett, Schumacher, & Fuchs, 2011, adapted from Siegler et al., 2011) requires students to place proper fractions, improper fractions, and mixed numbers on the 0-2 number line. Students are presented with a number line on a computer screen and instructed to place the fraction where they think it goes. The fractions include: 12/13, 7/9, 5/6, ¼, 2/3, ½, 1/19, 3/8, 7/4, 3/2, 4/3, 7/6, 15/8, 1 1/8, 1 1/5, 1 5/6, 1 2/4, 1 11/12, 5/5, and 1. The fractions are presented in random order. Scores reflect students' percentage of absolute error, which indicates the average percent difference between where the student placed the fraction and where the fraction actually goes. Lower scores indicate greater accuracy. Therefore, students' scores were multiplied by -1 so a positive relation between number line and the outcome reflected superior performance. Test-retest reliability for cohort 2 was .80.

The NAEP measurement items include 11 released fourth- and eighth-grade problems from the National Assessment of Education Progress (U.S. Department of Education, 2010). Four questions require students to write or identify equivalent fractions, four questions require students to order or compare fractions, and three questions require students to write or identify fractions on a number line The maximum score for the measurement items is 15. Alpha on the sample for cohorts 1, 2, and 3 was .67, .77, and .71, respectively.

Explaining Comparing Problems (cohort 3 only) from the *Fraction Battery-2013*-revised (Schumacher, Namkung, Malone, & Fuchs, 2013) requires students to explain why one fraction is greater than or less than another fraction. Students are instructed to place the greater than or less than sign between the two fractions and then write an explanation for and draw a picture to show how to think about the comparison in values and to reveal why their answer makes sense. There are nine items. Three problems have the same numerator, three have the same denominator, and three require students to compare a fraction to ½. Scoring awards credit for four components of sound explanations. Each component is either present or absent, and each component is weighted to account for the sophistication of the explanation.

The first component indexes the accuracy of sign placement between the two fractions being compared (e.g., 4/5 > 4/8). Students receive a “1” if the sign is correct. The second and third components assess whether students demonstrate an understanding of how the numerator and denominator work together to make an amount. Because these components indicate a more advanced understanding of fractions, they are weighted more heavily. Students must demonstrate that the (a) numerator indicates the number of parts in a fraction and (b) denominator indicates the size of the parts in the fraction. These two components are scored independently, and students receive a “2” if the component is present, and a “0” if the component is not present. For example, when comparing 4/5 and 4/8, credit is awarded if the explanation includes information about the size of the parts (2 points) and the number of parts (2 points), e.g., “Both fractions have the same number of parts, but fifths are bigger than eighths so 4/5 is greater than 4/8.”

The last component assesses the tenability of a picture the student draws to support the verbal explanation. A correct picture includes drawing two units the same size (e.g., rectangles that start and end at the same place), correctly dividing the unit into equal parts (the denominator), and correctly shading in the appropriate number of parts (the numerator). This component receives a “1” if present. The maximum score of each of the nine items is 6 points, for a maximum total score of 54. See the appendix for examples and non-examples of sound explanations. Two independent coders scored all of the tests and entered the data. On the first scoring attempt, coders scored the tests with 98.08% accuracy. All discrepancies were discussed and resolved. Alpha for cohort 3 was .90.

Fraction Instruction in the Classroom

The district uses enVision Math, which includes two units on fractions (Scott Foresman-Addison Wesley, 2011). Fraction content includes adding and subtracting, constructing equivalent fractions, word problems, and explaining concepts using words and pictures. Instruction largely relies on part-whole understanding and procedures.

Supplemental survey data about classroom instruction were also collected for cohorts 2 and 3 to determine what teachers relied on for fractions instruction and how they taught comparing fractions. For cohort 2, 2% of teachers exclusively relied on the textbook, 72% used a combination of the textbook and CCSS, and 26% used only CCSS in their classroom. For cohort 3, 0% of teachers exclusively relied on the textbook, 85% used a combination of the textbook and CCSS, 13% used only CCSS, and 2% used another curriculum in their classroom.

For the comparing fractions survey, we asked teachers what percentage of instructional time was spent teaching how to compare fractions using the following: (a) cross multiplying, (b) number lines, (c) benchmark fractions, (d) finding common denominators, (e) drawing a picture of each fraction, (f) reference manipulatives, (g) thinking about the meaning of the numerator and denominator, and (h) other. For cohort 2, these respective percentages were: (a) 12.5%, (b) 14%, (c) 10%, (d) 27%, (e) 19%, (f) 7%, (g) 8.5%, and (h) 3%. For cohort 3, these respective percentages were: (a) 21%, (b) 12%, (c) 13%, (d) 21%, (e) 15%, (f) 7%, (g) 8%, and (h) 2%.

Procedure

Students in cohorts 1, 2, and 3 were screened with the WRAT-4 in the fall of fourth grade (2012, 2013, and 2014, respectively). *Ordering Fractions*, *Fraction Number Line*, the NAEP, and *Explaining Comparing Problems* (cohort 3 only) were administered in the spring of fourth grade, at which time fraction instruction in their classroom had already occurred. We administered *Explaining Comparing Problems* only for cohort 3. Students in cohorts 1 and 2 took posttests in a whole-class setting; students in cohort 3 completed posttests in small-groups of two to six students.

Testers were graduate research assistants employed by the local university. All research assistants received training on testing protocol during two four-hour sessions. Research assistants practiced administering the tests and passed a fidelity check before administering the tests in schools. Research assistants were also trained on scoring. They received a score sheet outlining scoring protocol and the project coordinator was available to answer any questions. Two independent research assistants scored and entered the data for each test. All scoring discrepancies were discussed and resolved.

Data Analysis

Data analysis included three steps. The first was to determine how accurately students ordered fractions and how frequently they committed errors. The second step included running four cross-classified random-effects logistic regression models (CCREM) for each outcome (i.e., accuracy, whole number ordering errors, and smallest denominator – biggest fraction ordering errors). The models were run using the `lme4` command in R (Bates et al., 2013). The data structure included three levels. Problem-by-problem student responses were at Level 1. Because each student answered the same nine problems, responses were cross-classified by student and problem at Level 2. Students were also nested within cohort at Level 3. See Figure 1 for a depiction of the cross-classified hierarchical data structure.

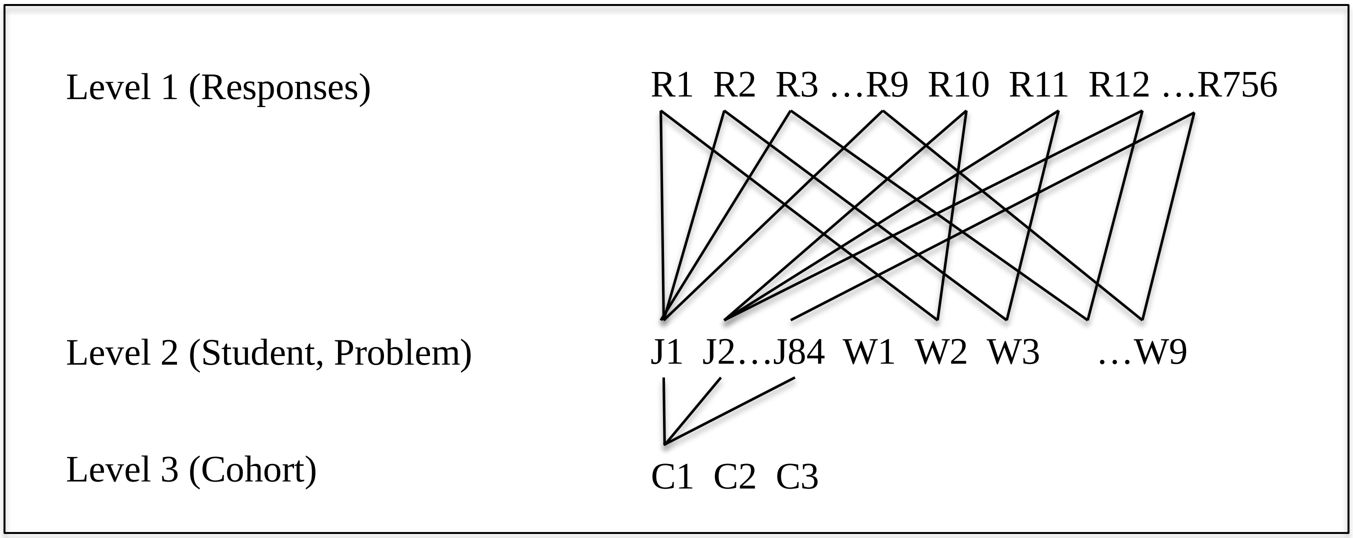


Figure 1. Cross-classified hierarchical data structure (depicted for cohort 1).

The first CCREM assessed the effect of cohort membership (cohort 1 served as the referent group for this analysis) on accuracy, whole number ordering errors, and smallest denominator – biggest fraction ordering errors:

$$\text{logit}(\pi_{i[jk]l}) = \gamma_{000} + \gamma_{001}\text{COHORT1}_l + \gamma_{002}\text{COHORT2}_l + u_{0j} + u_{0k} + e_{i(jk)l}$$

The second CCREM assessed whether problem type (i.e., problems with the same numerator versus problems with different numerators and different denominators) significantly predicted accuracy or committing a whole-number ordering error:

$$\text{logit}(\pi_{i[jk]}) = \gamma_{00} + \gamma_{01}\text{Same N}_k + u_{0j} + u_{0k}$$

Because smallest denominator – biggest fraction ordering errors could only occur on ordering problems with different numerators and different denominators, problem type was not a relevant predictor for this error outcome.

The third CCREM assessed whether part-whole understanding (i.e., NAEP part-whole items) and measurement understanding (i.e., number line and NAEP measurement items) significantly predicted the probability of correctly answering a problem, committing a whole number ordering error, or committing a smallest denominator – biggest fraction ordering error:

$$\text{logit}(\pi_{i[jk]}) = \gamma_{00} + \gamma_{01}\text{NL}_j + \gamma_{02}\text{NAEP\_M}_j + \gamma_{03}\text{NAEP\_PW}_j + u_{0j} + u_{0k}$$

To make the intercept interpretable, all of the student-level predictors were centered at their respective means. Therefore, the intercept could be interpreted as the probability of correctly answering a problem or committing an error at the predictors' respective means.

Analyzing student accuracy included the full sample of 2,043 responses. However, because the purpose of analyzing errors was to compare the probability of committing one error over another error, analysis of these errors excluded correct answers from their respective models. Of the 2,043 responses that could have whole number ordering errors, 379 were correctly answered. Therefore, the total sample for the whole number ordering error outcome included 1,664 responses with errors. That is, students either committed a whole number ordering error or some other type of error (i.e., smallest denominator – biggest fraction ordering error, an error that could not be categorized, or did not attempt the problem). Of the 1,135 responses that could have smallest denominator – biggest fraction ordering errors (this error could only occur on five of the nine ordering problems), 85 were correctly answered. That is, students either committed a smallest denominator – biggest fraction ordering error or some other type of error (i.e., a whole number ordering error, an error that could not be categorized, or did not attempt the problem). Therefore, the total sample for the smallest denominator – biggest fraction ordering error outcome included 1,050 responses with errors.

The fourth CCREM assessed whether students' ability to explain comparing problems (cohort 3 only; $n = 71$ students) significantly predicted the probability of correctly answering a problem (n = 639 responses), committing a whole number ordering error (n = 513 responses), or committing a smallest denominator – biggest fraction ordering error (n = 322 responses):

$$\text{logit}(\pi_{i[jk]}) = \gamma_{00} + \gamma_{01}\text{Explanation}_k + u_{0j} + u_{0k}$$

To contextualize results, the final (third overall) step was to convert all significant logit coefficients to probabilities using the following formula:

$$P = \frac{1}{1 + e^{-\text{logit}(\pi_{i[jk]})}}$$

For all student-level predictors, these probabilities were calculated at the following values: one standard deviation below the mean, at the mean (i.e., the intercept), one standard deviation above the mean, and two standard deviations above the mean. This allowed assessment of how the probability of correctly answering a problem or committing an error changed as a function of an increase or decrease in part-whole and measurement understanding. For problem type, the calculated probability reflects how likely students were to correctly answer a problem or commit a whole number ordering error when the ordering problem had the same numerator versus different numerators and different denominators.

Table 1

Ordering errors coding form

Student/Teacher/Cohort

	Answer		Fraction Understanding Error						
Problem	Correct = 1 Incorrect = 0		WN	WN-D	WN-N	SD-BF	NC-WF	NC	DNA
A.* $\frac{1}{2}$ $\frac{5}{8}$ $\frac{2}{6}$	0	1	1	2	3	4	5	6	7
B. $\frac{1}{2}$ $\frac{1}{5}$ $\frac{1}{12}$	0	1	1	2	3	4	5	6	7
C.* $\frac{3}{10}$ $\frac{7}{12}$ $\frac{1}{2}$	0	1	1	2	3	4	5	6	7
D.* $\frac{3}{4}$ $\frac{1}{2}$ $\frac{2}{8}$	0	1	1	2	3	4	5	6	7
E. $\frac{1}{8}$ $\frac{1}{3}$ $\frac{1}{2}$	0	1	1	2	3	4	5	6	7
F. $\frac{5}{4}$ $\frac{9}{8}$ $\frac{7}{6}$	0	1	1	2	3	4	5	6	7
G. $1\frac{4}{6}$ $1\frac{4}{12}$ $1\frac{4}{10}$	0	1	1	2	3	4	5	6	7
H.* $\frac{7}{12}$ $\frac{10}{6}$ $\frac{3}{2}$	0	1	1	2	3	4	5	6	7
I.* $1\frac{7}{8}$ $1\frac{2}{10}$ $1\frac{1}{2}$	0	1	1	2	3	4	5	6	7

Note. WN = whole number ordering errors; WN.D = whole number (denominator specific) ordering errors; WN-N: whole number (numerator specific) ordering errors; SD-B = smallest denominator – biggest fraction ordering errors; NC-WF = no category errors where students wrote the wrong fractions or the same fraction twice; NC = no category errors (i.e., could not determine an error pattern); DNA = did not attempt. If an error is gray, that means that particular error could not occur on the problem.

*Starred problems were only coded as smallest denominator – biggest fraction ordering errors *if* students also got Problems B, E, and D correct (i.e., the same numerator problems). Answering these problems correct indicates that students understand that as the denominator gets smaller, the parts get bigger. However, students failed to correctly address the value in the numerator for problems that have fractions with different numerators and different denominators. All errors were entered as a “1” for present and a “0” for absent.

CHAPTER III

RESULTS

Of the 2,043 responses, 379 were correctly answered (i.e., 19%). The most common ordering error was whole number ordering errors. Of the 1,664 responses with errors, 1,078 had whole number ordering errors (i.e., 65%). The next most common error was smallest denominator – biggest fraction ordering errors. Of the 1,050 responses with errors (this error could only occur on five of the nine ordering problems), 190 had smallest denominator – biggest fraction ordering errors (i.e., 18%). Students also committed random errors: They wrote the incorrect fraction or the same fraction twice (i.e., 5% of errors) or committed an error that we could not categorize because it did not fit a pattern (i.e., 17% of errors). See Table 2 for frequencies of correct answers and errors by cohort and across the sample.

Effect of Cohort on Accuracy, Whole Number Ordering Errors, and Smallest Denominator – Biggest Fraction Ordering Errors

There were no significant differences among cohorts on accuracy or the frequency of errors. See Table 3 for a summary of the CCREM results. Therefore, cohort membership (i.e., Level 3) was dropped from subsequent analyses.

Effect of Problem Type on Accuracy and Whole Number Ordering Errors

See Table 4 for a summary of the CCREM results. Problem type significantly predicted the probability of correctly answering a problem. The probability of correctly answering a same numerator problem was .155. The probability of correctly answering a problem with different numerators and different denominators was .012. Problem type did not significantly predict the probability of committing a whole number ordering error.

Effect of Part-Whole and Measurement Understanding on Accuracy, Whole Number Ordering Errors, and Smallest Denominator – Biggest Fraction Ordering Errors

See Table 5 for a summary of the CCREM results. Performance on NAEP part-whole items (*M* = 6.33, *SD* = 1.50), number line (*M* = 0.52, *SD* = 0.14), and NAEP measurement items (*M* = 5.09, *SD* = 2.78) significantly predicted the probability of correctly answering a problem. To contextualize results, all significant logit coefficients were converted to probabilities at one standard deviation below the mean, at the mean, one standard deviation above the mean, and two standard deviations above the mean. At these respective values, the probability of correctly answering a problem was .011, .035, .100, and .254.

Performance on number line and NAEP measurement items significantly predicted the probability of committing a whole number ordering error. By contrast, performance on NAEP part-whole items did not significantly predict the probability of committing a whole number ordering error (p = .738). The probability of committing a whole number ordering error when students performed one standard deviation below the mean, at the mean, one standard deviation above the mean, and two standard deviations above the mean on all three predictor variables was .986, .727, .092, and .004, respectively.

Performance on NAEP part-whole items, number line, and NAEP measurement items did not significantly predict the probability of committing a smallest denominator – biggest fraction ordering error. Because none of the predictors was statistically significant, probabilities were not calculated.

Effect of Students' Explanations (Cohort 3 Only) on Accuracy, Whole Number Ordering Errors, and Smallest Denominator – Biggest Fraction Ordering Errors

Students' ability to explain comparing problems (*M* = 6.97, *SD* = 3.97) significantly predicted the probability of correctly answering a problem and committing a whole number ordering error. Students' explanations did not significantly predict the probability of committing a smallest denominator **–** biggest fraction ordering error ($p = .065$). See Table 6 for student explanation component scores and Table 7 for a summary of the CCREM results.

Effect of Each Individual Predictor on Probabilities

To determine whether one predictor variable had a greater effect on the probability of correctly answering a problem, committing a whole number ordering error, I also calculated probabilities for each predictor variable individually at one standard deviation below the mean, one standard deviation above the mean, and two standard deviations above the mean while holding the other two predictor variables constant at their respective means. Because part-whole and measurement understanding did not significantly predict the probability of committing a smallest denominator **–** biggest fraction ordering errors, these respective probabilities were not calculated. See Table 8 for these respective probabilities.

Table 2

Frequencies of Correct Answers, Whole Number Ordering Errors, Smallest Denominator – Biggest Fraction Ordering Errors, No Category Errors, and Did Not Attempt Errors

	Correct	%	WN[a]	%	SD-BF[c]	%	NC-WF[a]	%	NC[a]	%	DNA[a]	%
Cohort 1 2011-2012	116	15%	417	65%	52	13%	28	4%	120	19%	23	4%
Cohort 2 2012-2013	137	21%	318	63%	75	22%	27	5%	87	17%	4	1%
Cohort 3 2013-2014	126	20%	343	67%	63	20%	25	5%	73	14%	9	2%
Across cohorts	379 *N* = 2,043	19%	1,078 *n* = 1,664	65%[b]	190 *n* = 1,050	18%	80 *n* = 1,664	5%	280 *n* = 1,664	17%	36 *n* = 1,664	2%

Note: WN = whole number ordering errors; SD-BF = smallest denominator – biggest fraction ordering errors; NC-WF = no category errors where students wrote the wrong fractions or the same fraction twice; NC = no category errors (i.e., could not determine an error pattern); DNA = did not attempt.

[a]Indicates the frequency of errors of the responses that had errors (i.e., subtracting correct answers from the denominator).

[b]46% of WN ordering errors included ordering the three fractions from least to greatest based on the whole number value in the denominator (e.g., ½<1/5<1/12); 33% of WN ordering errors included ordering the three fractions from least to greatest based on the whole number values in the numerator and denominator (e.g., ½<2/6<5/8); 21% of WN ordering errors included ordering the three fractions from least to greatest based on the whole number values in the numerator (e.g., ½<2/8<¾).

[c]Indicates the frequency of smallest denominator – biggest fraction ordering errors of the responses that had errors (i.e., subtracting correct answers from the denominator). This error could only occur on five of the nine ordering problems.

Table 3

Effect of Cohort on Accuracy, Whole Number Ordering Errors, and Smallest Denominator – Biggest Fraction Ordering Errors

Outcome			γ	SE	z	p	Variance
Accuracy	*Fixed Effects*	(intercept)	-3.97	0.64	-6.23	<.001	
		Cohort 2	0.96	0.54	1.78	.075	
		Cohort 3	0.78	0.54	1.43	.152	
	Random effects	Student					7.43
		Problem					1.89
Whole number ordering error	*Fixed Effects*	(intercept)	1.00	0.69	1.45	.148	
		Cohort 2	-0.76	0.96	-0.78	.435	
		Cohort 3	0.54	0.97	0.58	.581	
	Random effects	Student					23.98
		Problem					0.51
Smallest denominator – biggest fraction ordering error	*Fixed effects*	(intercept)	-10.41	1.08	-9.61	<.001	
		Cohort 2	0.74	1.12	0.66	.509	
		Cohort 3	0.47	1.16	0.41	.685	
	Random effects	Student					134.89
		Problem					0.72

Note: Cohort 1 was the referent group.

Table 4

Effect of Problem Type on Accuracy and Whole Number Ordering Errors

Outcome			γ	SE	*z*	*p*	Variance
Accuracy	*Fixed Effects*	(intercept)	-4.38	0.44	-10.06	<.001	
		Same N	2.68	0.51	5.23	<.001	
	Random effects	Student					7.97
		Problem					0.45
Whole number ordering error	*Fixed Effects*	(intercept)	0.74	0.45	1.67	.095	
		Same N	0.41	.52	0.79	.428	
	Random effects	Student					23.55
		Problem					0.47

Table 5

Effect of Measurement (i.e., Number Line and NAEP Measurement Items) and Part-Whole Understanding (i.e., NAEP Part-Whole Items) on Accuracy, Whole Number Ordering Errors, and Smallest Denominator – Biggest Fraction Ordering Errors

Outcome			γ	SE	z	p	Variance
Accuracy	*Fixed Effects*	(intercept)	-3.31	0.54	-6.07	<.001	
		NAEP_PW	0.30	0.14	2.08	.038	
		NL	3.51	1.49	-2.37	.018	
		NAEP_M	0.42	0.08	5.16	<.001	
	Random effects	Student					4.19
		Problem					2.11
Whole number ordering error	*Fixed Effects*	(intercept)	0.98	0.47	2.07	.039	
		NAEP_PW	-0.10	0.30	-0.34	.738	
		NL	-8.70	3.52	-2.47	.013	
		NAEP_M	-0.69	0.19	-3.57	<.001	
	Random effects	Student					23.62
		Problem					0.56
Smallest denominator – biggest fraction ordering error	*Fixed effects*	(intercept)	-9.29	1.04	-8.89	<.001	
		NAEP_PW	0.37	0.44	0.85	.394	
		NL	2.34	4.19	0.56	.576	
		NAEP_M	0.42	0.23	1.84	.066	
	Random effects	Student					94.05
		Problem					0.60

Note: NAEP_PW = NAEP part-whole items; NL = number line; NAEP_M = NAEP measurement items

Table 6

Means and Standard Deviations for Component Scores for Students' Explanations (Cohort 3 Only; n = 71)

	Correct sign placement	Number of parts	Size of parts	Correct Drawing
	M (SD)	*M (SD)*	*M (SD)*	*M (SD)*
Cohort 3 2013-2014	5.58 (2.07)	0.45 (1.18)	0.34 (0.96)	0.61 (1.74)

Table 7

Effect of Students' Explanations (Cohort 3 Only; n = 71) on Accuracy, Whole Number Ordering Errors, and Smallest Denominator – Biggest Fraction Ordering Errors

Outcome			γ	SE	z	p	Variance
Accuracy	*Fixed Effects*	(intercept)	-2.91	0.59	-4.97	<.001	
		Explanation	0.32	0.08	4.00	<.001	
	Random effects	Student					4.39
		Problem					1.53
Whole number ordering error	*Fixed Effects*	(intercept)	4.74	2.09	2.27	.023	
		Explanation	-1.44	0.45	-3.24	.001	
	Random effects	Student					111.23
		Problem					3.38
Smallest denominator – biggest fraction ordering error	*Fixed effects*	(intercept)	-8.13	1.178	-4.56	<.001	
		Explanation	0.66	0.36	1.84	.065	
	Random effects	Student					85.62
		Problem					0.91

Table 8

Effect of Each Individual Predictor on Accuracy and Whole Number Ordering Errors[a]

Outcome	Predictor[b]	-1 SD	+1 SD	+2 SDs
Accuracy	NAEP part-whole	.023	.054	.082
	Number line	.022	.056	.089
	NAEP measurement	.011	.105	.271
	Student explanations	.015	.165	.418
Whole number ordering error	NAEP part-whole	-2.91	0.59	-4.97
	Number line	.899	.443	.192
	NAEP measurement	.948	.282	.055
	Student explanations	>.999	.269	.001

Note: [a]Part-whole and measurement understanding did not significantly predict the probability of committing a smallest denominator – biggest fraction ordering error. Therefore, these effects were not calculated.
[b]These probabilities reflect the effect of each predictor at one standard deviation below the mean, one standard deviation above the mean, and two standard deviations above the mean while holding the other predictor variables constant at their respective means. Student explanations were analyzed in a separate statistical model.

CHAPTER IV

DISCUSSION

There were four purposes to the present study. The first was to describe fraction ordering error patterns among fourth-grade students at risk for developing mathematics difficulties. The second purpose was to determine whether performance differed in terms of accuracy and error patterns among students from three different academic years over the period in which the school district moved toward implementation of CCSS (2012). The third purpose was to determine whether students were more accurate and more likely to commit whole number ordering errors on problems with the same numerator versus problems with different numerators and different denominators. The fourth purpose was to determine whether part-whole and measurement understanding of fractions significantly predicted the probability of correctly answering a problem, committing a whole number ordering error, and committing a smallest denominator – biggest fraction ordering error.

In this section, I discuss findings for each research question by outcome (i.e., accuracy, smallest denominator – biggest fraction ordering errors, and whole number ordering errors). I present findings on accuracy and smallest denominator – biggest fraction ordering errors first to then focus on findings related to the most pervasive ordering error – whole number ordering errors. Therefore, I center the majority of the discussion on how measurement understanding relates to the probability of committing whole number ordering errors. Next, I discuss practical implications and curriculum recommendations as they relate to remediating whole number ordering errors, followed by a discussion of study limitations.

Accuracy

Consistent with previous research (DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004), students in the present study demonstrated substantial difficulty assessing fraction magnitude, with no significant differences among cohorts on accuracy for these low performers as the school district moved toward CCSS implementation. Across years, after their first year of intensive instruction on fractions, fourth-grade students correctly ordered fractions on only 19% of problems.

Contrary to what I expected, part-whole understanding significantly predicted the probability of correctly answering a problem. This is likely because students can accurately assess fraction magnitude using part-whole knowledge when fractions have the same numerator (Smith et al., 2005), and the same numerator problems were slightly easier for students than problems with different numerators and different denominators (examples where part-whole understanding serves students well for accurately assessing fraction magnitude). However, although significant, part-whole understanding had a minimal effect in increasing the probability of correctly answering a problem. The probability of correctly answering a problem at one standard deviation below the mean for NAEP part-whole items was 2%; at two standard deviations above the mean, the probability only increased to 8%. Although part-whole understanding is foundational for developing an understanding of fraction magnitude (e.g., Charalambous, & Pitta-Pantazi, 2007), students cannot accurately use part-whole knowledge for more difficult problems with different numerators and different denominators. This was evidenced by the fact that the probability of correctly ordering fractions with different numerators and different denominators was only 1.2%, as compared to a 16% probability when fractions had the same numerator.

The curriculum in the district where this study took place primarily relied on teaching part-whole understanding of fractions and did not emphasize measurement understanding (i.e., understanding fraction magnitude). However, based on the CCREM results, it appears that performance on the NAEP measurement items and the ability to explain comparing problems (with the latter measure indexed only in cohort 3) had the strongest effect on the probability of correctly answering a problem. When students performed one standard deviation below the mean on the NAEP measurement items (and at the mean on number line and NAEP part-whole items), there was a 1% probability of correctly ordering fractions. However, when students performed two standard deviations above the mean, the probability of correct ordering increased to 27%.

Similarly, when students performed one standard deviation below the mean on explaining comparing problems (cohort 3 only), there was a 2% probability of correctly answering a problem. By contrast, when students performed two standard deviations above the mean, the probability of correctly answering a problem increased to 42%. Considering students only correctly answered 19% of the problems overall, the effect of performance on NAEP measurement items and explaining comparing problems (cohort 3 only) was substantial. This result is not surprising, as research suggests that verbal explanations are positively related to overall mathematics achievement (e.g., Wong et al., 2002). If students are capable of accurately describing how the numerator and denominator work together to create magnitude, they are more likely to accurately order fractions from least to greatest.

Although significant, performance on number line had a less substantial effect than performance on NAEP measurement items and explaining comparing problems (cohort 3 only). The probability of correctly answering a problem when a student performed one standard deviation below the mean on number line was 2%. At two standard deviations above the mean, the probability increased to only 9%. I expected performance on number line to parallel the effect of performance on NAEP measurement items on accuracy.

Of course, it is not surprising that increased measurement understanding significantly predicted students' ability to correctly order fractions from least to greatest, in that ordering also indexes measurement understanding. But it is interesting that performance on NAEP measurement items and explaining comparing problems had a more substantial effect in increasing the probability of correctly answering a problem than did performance on number line. One reason for this is that the NAEP measurement items tested a broader range of measurement understanding than did number line. That is, the NAEP measurement items tested students' ability to order fractions from least to greatest, compare fractions, place fractions on the number line, and construct equivalent fractions.

Performance on number line, on the other hand, was a narrow measure testing whether students could spatially represent fraction magnitude on a 0-2 number line. Although previous research suggests that students' ability to accurately place fractions on the number line significantly predicts students' development of understanding about fractions (e.g., Jordan et al., 2013; Siegler et al., 2011), many students have difficulty placing fractions on the number line (e.g., Opfer & Devries, 2008; Siegler et al., 2011). To combat this difficulty, instruction must focus on developing students' fluency with a range of fraction magnitude concepts, including equivalence, ordering, and comparing (Smith, 2002). This will not only improve students' ability to place fractions on the number line, but also improve students' conceptual understanding. The greater students' accuracy on a range of measurement tasks (i.e., ordering, comparing, number line, equivalence), the more likely they are to accurately order fractions.

Smallest Denominator – Biggest Fraction Ordering Errors

Smallest denominator – biggest fraction ordering errors did not occur as frequently as expected. Of the errors made on the five problems with potential for this error (i.e., 1,050 responses), 18% were smallest denominator – biggest fraction ordering errors. There were no significant differences among cohorts on the frequency of this error. Contrary to what I expected, performance on number line, NAEP measurement items, and NAEP part-whole items did not significantly predict the probability of committing a smallest denominator – biggest fraction ordering error.

The low frequency of smallest denominator – biggest fraction errors could explain why part-whole and measurement understanding failed to predict the probability of this error, as it only occurred 190 times in the sample. Although results do not speak to how part-whole and measurement understanding are related to the probability of committing this error, this type of error may still represent a transitional phase as students attempt to assimilate fractions into their numerical framework (Stafylidou & Vosniadou, 2004). However, because students made very few of these errors, I could not examine the tenability of this hypothesis. Future research should investigate whether this type of error represents a transition from operating with whole number bias to assimilating fractions into a single numerical framework.

Whole Number Ordering Errors

The most interesting finding of this study was that whole number ordering errors were pervasive among the sample of at-risk fourth-grade students. The majority of students ordered fractions from least to greatest based on the whole number values in the numerator and/or denominator, and there were no significant differences among cohorts on the frequency of whole number ordering errors. Of the errors made, 65% were in this error category. Students were just as likely to commit a whole number ordering error when fractions had the same numerator as when fractions had different numerators and different denominators.

Part-whole understanding. As expected, part-whole understanding did not significantly predict the probability of committing whole number ordering errors when measurement understanding was controlled. That is, increased part-whole understanding did not significantly decrease the probability of committing a whole number ordering error. This is interesting because students were just as likely to commit whole number ordering errors on problems with the same numerator (examples where part-whole understanding can lead to accurate answers). Same numerator problems were no easier than problems with different numerators and different denominators for students who tended to operate with whole number bias. As previously discussed, part-whole understanding is foundational but not sufficient for understanding fraction magnitude. Based on these results, it appears that only focusing on teaching part-whole understanding will not decrease the likelihood of operating with whole number bias.

Measurement understanding. Unlike part-whole understanding, performance on number line, NAEP measurement items, and explaining comparing problems (cohort 3 only) all significantly predicted the probability of whole number ordering errors. Superior measurement understanding dramatically decreased whole number ordering errors. When a student performed one standard deviation below the mean on number line, the probability of committing a whole number ordering error was 90%; the probability of committing this error decreased to 19% at two standard deviations above the mean. For NAEP measurement items, when a student performed one standard deviation below the mean, the probability of committing a whole number ordering error was 95%; the probability decreased to 5% at two standard deviations above the mean. For explaining comparing problems (cohort 3 only), the probability of committing a whole number ordering error was 99.99% one standard deviation below the mean; the probability decreased to 0.1% at two standard deviations above the mean.

Parallel to the observed relation with accuracy, performance on NAEP measurement items and explaining comparing problems had the strongest effect on decreasing the probability of committing whole number ordering errors. As previously discussed, the NAEP measurement items tested a broader range of magnitude topics than did number line. Although the number line is a good representational tool for teaching fraction magnitude topics (e.g., Keijzer and Terwel, 2003), many students struggle with the number line (Opfer & Devries, 2008; Siegler et al., 2011). Because of this, students' measurement understanding must also include deep understanding of and fluency with equivalence, ordering, and comparing to reduce the likelihood of committing whole number ordering errors.

Results from this study suggest that accurately explaining how to assess fraction magnitude may be a key component in increasing students' conceptual understanding of fractions. One of the CCSS (2012) fraction standards requires students to "extend core understanding of fraction equivalence and ordering" (www.corestandards.org). This includes accurately comparing fractions with different numerators and different denominators by creating common denominators or comparing to a benchmark fraction such as ½. However, results from the present study suggest that at-risk fourth grade students had not achieved this benchmark, as indicated by the low scores on explaining comparing problems. Of a possible 54 points on the test, the average explanation score was only 7 points.

Moreover, although students are expected to justify their answer with a verbal explanation and a visual fraction model, results from the explaining comparing problems measure (cohort 3 only) suggest these at-risk students have substantial difficulty verbally explaining how to compare fractions. This corroborates previous research that suggests many American students (not just low performing students) struggle to construct verbal explanations. For example, in a descriptive study comparing American and Chinese students' learning of place-value, Yang and Cobb (1995) found that Chinese students constructed far superior verbal explanations about mathematics concepts than their U.S. counterparts. In a similar descriptive study comparing mathematics classroom discourse, Stigler and Perry (1988) showed that Japanese teachers devoted the majority of class time on modeling how to explain the problem-solving process. Teachers stressed the importance of thinking through the correct answer. By contrast, American teachers spent little time on explanations. There is a long history of achievement discrepancies between the United States and Asian countries in mathematics (e.g., Provasnik et al., 2012). The ability to explain concepts and connect ideas in mathematics may be a key factor explaining this achievement gap (e.g., Stigler & Perry, 1988).

Not only did students fail to provide satisfactory verbal explanations, many of the explanations provided were based on whole number knowledge. In fact, 38% of the explanations provided on the explaining comparing problems measure included incorrect whole-number explanations. These problematic whole-number explanations included examples such as stating that "3/6 is bigger than ¾ because 6 is bigger than 4" or cross-multiplying and comparing whole number values to compare the fractions. The prevalence of these types of whole-number explanations indicates students have deep misconceptions about fraction properties and how to assess fraction magnitude.

Instructional implications. The focus of instruction in the district where the study took place primarily emphasized part-whole understanding. However, results suggest that part-whole understanding is insufficient for reducing the likelihood of operating with whole number bias.

Without emphasizing that fractions have magnitude and can be ordered, compared, and placed on the number line, students resort to the only knowledge they have: whole number knowledge. Operating with whole number bias can have a substantial negative effect on students' ability to succeed in higher-level mathematics courses such as algebra (Bailey et al., 2012; Booth & Newton, 2012; Brown & Quinn, 2007; Siegler et al., 2011).

Results from this study corroborate previous research highlighting the importance of explicitly teaching how to conceptually assess fraction magnitude (e.g., Cramer et al., 2003; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., in preparation-a; Fuchs et al., in preparation-b; Siegler et al., 2011; Smith, 2002). However, many American teachers may struggle to do this because they themselves have insufficient fraction understanding. For example, in a descriptive study comparing American and Japanese fourth-grade teachers' understanding of rational numbers, Moseley, Okamoto, and Ishida (2007) found that American teachers relied almost exclusively on part-whole relationships to describe fraction concepts. But they often inaccurately described fraction concepts and struggled to rationalize how to apply part-whole explanations to more difficult concepts like proportional reasoning. By contrast, Japanese teachers' explanations of fraction concepts focused more on quantity relationships. They could apply these explanations to more difficult mathematics concepts such as proportional reasoning.

Like teachers in the Moseley et al. (2007) study, teachers in this study's district may also struggle to rationalize how to apply part-whole explanations to more difficult fraction topics such as ordering fractions with different numerators and different denominators. This explains why many teachers reported that they focused on procedural methods for assessing magnitude (e.g., cross-multiplying). However, teaching procedures without concepts likely leaves students confused and unable to judge the accuracy of their answers (Kilpatrick et al., 2001; Rittle-

Johnson & Siegler, 1998), as both conceptual and procedural knowledge are important for developing conceptual understanding (e.g., Rittle-Johnson, Siegler, & Alibali, 2001).

Results suggest that one potential strategy for reducing whole number bias is revamping instruction to focus on magnitude understanding (i.e., measurement interpretation). Previous research supports this finding. For example, in four years of intervention research, Fuchs et al. (2013), Fuchs et al. (2014), Fuchs et al. (in preparation-a), and Fuchs et al. (in preparation-b) found that at-risk fourth-grade students who received intervention emphasizing the measurement interpretation of fractions significantly outperformed control students on comparing fractions, ordering fractions, number line, and NAEP. In fact, the treatment students from these cited studies made very few ordering errors. The scope and sequence of all four studies included explicitly teaching students conceptual strategies for comparing fractions with the same numerator, the same denominator, and fractions with different numerators and different denominators using ½ as a benchmark. These strategies were extended to ordering and placing fractions on the number line. Part-whole understanding was a component of instruction, but was not the focus of the curriculum.

Revamping instruction to focus on magnitude understanding is not a new idea. The CCSS (2012) and the National Mathematics Advisory Panel (U.S. Department of Education, 2008) both emphasize the importance of teaching fraction magnitude to boost students' conceptual understanding of fractions. However, given there were no differences among cohorts on accuracy or error patterns, as the district moved toward CCSS implementation, it appears that the CCSS are not being effectively implemented – at least for students with risk for mathematics difficulty. This sample of at-risk students, who received their fraction instruction within general education classrooms, had limited measurement understanding. Their scores on the measurement items were much lower than expected compared to average-achieving classmates. For example, Fuchs et al. (2013, 2014) reported that the achievement gap between the at-risk control students (similar to those included in the present study's sample) and a sample of low-risk students (i.e., average-achieving classmates) was approximately one standard deviation on NAEP measurement items. (Average-achieving student data were not collected for number line performance.) However, for students who instead received the intervention program, this achievement gap was nearly eliminated.

Although the pattern of errors is unknown for a sample of average-achieving students (as these data were not collected), previous research (e.g., DeWolf & Vosniadou, 2011; Meert et al., 2010; Stafylidou & Vosniadou, 2004) suggests that even average-achieving students struggle to assess fraction magnitude. This speaks to the importance of improving fraction instruction to benefit all students (Siegler et al., 2011). It is possible to reduce students' tendency to operate with whole number bias (i.e., Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., in preparation-a; Fuchs et al., in preparation-b). However, more work must be done to achieve the standards goals of the CCSS (2012) and improve students' conceptual understanding of fractions.

Limitations

Of course, this study must be viewed in light of limitations. First, this study examined only the differential predictability of two types of fraction understanding – part-whole and measurement. Although part-whole and measurement understanding represent important components of conceptual understanding (e.g., Kilpatrick et al., 2001) and help to explain individual differences in fraction skills (Hecht et al., 2003), there are also other interpretations of fractions. These include defining a fraction as a ratio (i.e., a fraction represents a relationship between two quantities), a quotient (i.e., a/b represents the decimal value of a divided by b), or an operator (i.e., a fraction represents a quantity that can be reached by multiplying smaller units). The present study did not analyze these constructs because they are not typically assessed or taught in fourth grade. This is not to say that these constructs are not important. But laying a foundation for students to understand fraction magnitude (i.e., measurement understanding) in the first year of fraction learning will likely prepare them to consolidate the many interpretations of fractions in the later grades.

Second, the reliability of the NAEP part-whole items was lower than expected (i.e., < .70). It could be that there were fewer NAEP part-whole items than NAEP measurement items, which affects the overall reliability of the test (Cronbach, 1951). However, because the items were released NAEP items actually given to students, I felt this subset of questions was a realistic set of part-whole questions that students may encounter on a standardized test to assess a range of part-whole understanding.

Third, I reported how teachers taught students fractions using a self-report survey. I did not conduct live observations in the classroom to determine whether what they reported was in line with how they were actually teaching fraction concepts. Therefore, these data must be viewed as an approximation of how teachers teach fractions in their classroom.

Despite these limitations, results support previous intervention research (e.g., Cramer et al., 2002; Fuchs et al., 2013; Fuchs et al., 2014; Fuchs et al., in preparation-a; Fuchs et al., in preparation-b) that suggests one strategy for reducing students' tendency to operate with whole number bias is to increase their measurement understanding and allow students to explain how they got their answer. In these studies, instruction explicitly taught students how to compare fractions, order fractions, and place fractions on the number line and give students the opportunity to construct conceptual verbal explanations for how to complete these tasks. In the present study, performance on a wide range of measurement tasks (i.e., NAEP measurement items) and the ability to explain comparison problems had a dramatic effect of reducing whole number ordering errors among fourth-grade students at risk for developing mathematics difficulties.

REFERENCES

Bailey, D. H., Hoard, M. K., Nugent, L., & Geary, D. C. (2012). Competence with fractions predicts gains in mathematics achievement. *Journal of Experimental Child Psychology, 113*, 447-455.

Bates, D., Chambers, J., Dalgaard, D. et al. (2013). R: A language and environment for statistical computing. *R Foundation for Statistical Computing.* Vienna: Austria.

Bonato, M., Fabbri, S., Umilt., C., & Zorzi, M. (2007). The mental representation of numerical fractions: Real or integer? *Journal of Experimental Psychology: Human Perception and Performance, 33,* 1410–1419.

Booth, J. L., & Newton, K. L. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology, 37*, 247-253.

Brown, G., & Quinn, R. J. (2007). Investigating the relationship between fraction proficiency and success in algebra. *Australian Mathematics Teacher, 4*, 4-15.

Charalambous, C. Y., & Pitta-Pantazi, D. (2007). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics, 64*, 293-316.

Cronbach, L. J. (1951). Coefficient alpha and the internal structure of tests. *Psychometrika*, *16*, 297-334.

DeWolf, M., & Vosniadou, S. (2011). The whole number bias in fraction magnitude comparisons with adults. In L. Carlson, C. Hoelscher, & T. F. Shipley (Eds.), Proceedings of the 33rd annual conference of the cognitive science society (pp. 1751-1756). Austin, TX: Cognitive Science Society.

Fuchs, L. S., Fuchs, D., Powell, S. R., Seethaler, P. M., Cirino, P. T., Fletcher, J. M. (2008). Intensive intervention for students with mathematics disabilities: Seven principles for effective practice. *Learning Disability Quarterly, 31*, 79-92.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., Jordan, N. C., Siegler, R. S., Gersten, R., & Changas, P. (2013). Improving At-Risk Learners' Understanding of Fractions. *Journal of Educational Psychology*, *105*, 683-700.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Malone, A. S., Wang, A., Hamlett, C. L., Jordan, N. C., Gersten, R., Siegler, R. S., & Changas, P. (in preparation-a). *Effects of intervention to improve at-risk fourth graders' understanding, calculations, and word problems with fractions*. Unpublished manuscript.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Malone, A. S., Wang, A., Hamlett, C. L., Jordan, N. C., Gersten, R., Siegler, R. S., & Changas, P. (in preparation-b).

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A. S., Hamlett, C. L., Jordan, N. C., Gersten, R., Siegler, R. S., & Changas, P. (2014). Does working memory moderate the effects of fraction intervention? An aptitude-treatment interaction. *Journal of Educational Psychology, 106*, 499-514.

Geary, D. C. (2006). Development of mathematical understanding. In W. Damon, et al. (Eds.), *Handbook of child psychology: Cognition, perception, and language* (pp. 777–810). Hoboken, NH: Wiley.

Hamlett, C. L., Schumacher, R. F., & Fuchs, L. S. (2011). *2011 Number Line* adapted from Siegler et al. (2011). Available from L.S. Fuchs, 228 Peabody, Vanderbilt University, Nashville, TN 37203.

Hecht, S. A., Close, L., & Santiso, M. (2003). Sources of individual differences in fraction skills. *Journal of Experimental Child Psychology, 86*, 277–302.

Hecht, S. A., Vagi, K. J., Torgesen, J.K. (2007). Fraction skills and proportional reasoning. In D. B. Berch, M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 121–132). Brookes Publishing: New York.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Psychology, 116*, 45-58.

Keijzer, R., & Terwel, J. (2003). Learning for mathematical insight: a longitudinal comparative study on modeling. *Learning and Instruction 13*, 285–304.

Kilpatrick, J., Swafford, J., & Findell, B., (Eds.) (2001). Adding it up: Helping children learn mathematics. Washington, DC: National Academy Press (by the Mathematics Learning Study Committee/National Research Council).

Meert, G., Grégoire, J., & Noël, M. P. (2010) Comparing the magnitude of two fractions with common components: which representations are used by 10- and 12-year-olds? *Journal of Experimental Child Psychology, 107*, 244-259.

Moseley, B. J., Okamoto, Y., & Ishida, J. (2007). Comparing US and Japanese elementary school teachers' facility for linking rational number representations. *International Journal of Science and Mathematics Education, 5*, 165-185.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2013). *Common Core State Standards (Mathematics Standards).* Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.

Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: the origins and implications of whole number bias. *Educational Psychologist, 40,* 27-52.

Opfer, J. E., DeVries, J. M. (2008). Representational change and magnitude estimation: Why young children can make more accurate salary comparisons than adults. *Cognition, 108,* 843-849.

Provasnik, S., Kastberg, D., Ferraro, D., Lemanski, N., Roey, S., & Jenkins, F. (2012). *Highlights From TIMSS 2011: Mathematics and Science Achievement of U.S. Fourth- and Eighth-Grade Students in an International Context* (NCES 2013-009). Washington, D.C.: National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education.

Rittle-Johnson, B., & Siegler, R. S. (1998). The relations between conceptual and procedural knowledge in learning mathematics: A review. In C. Donlan (Ed.), *The development of mathematical skill* (pp. 75-110). Hove, England: Psychology Press.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*, 346-362.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitudes of fractions. *Journal of Experimental Psychology: Human Perception and Performance, 36*, 1227–1238.

Schumacher, R. F., Namkung, J. M., Malone, A., & Fuchs, L. S. (2012). 2012 Fraction Battery. Available from L.S. Fuchs, 228 Peabody, Vanderbilt University, Nashville, TN 37203.

Schumacher, R. F., Namkung, J. M., Malone, A., & Fuchs, L. S. (2013). 2013 Fraction Battery. Available from L.S. Fuchs, 228 Peabody, Vanderbilt University, Nashville, TN 37203.

Scott Foresman Addison-Wesley (2011). enVision Math Grade 4. San Antonio: Pearson Publishing.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. I., Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23,* 691-697.

Siegler, R. S., Thompson, C. A., & Schneider. M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*, 273-296.

Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology, 51*, 101-140.

Smith, J.P. (2002). The development of students' knowledge of fractions and ratios. In B. Litwiller and G. Bright (eds.), *Making Sense of Fractions, Ratios, and Proportions* (pp. 3–17), Reston, Virginia: National Council of Teachers of Mathematics.

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction, 14,* 503-518.

Stigler, J. W., & Perry, M. (1988). Mathematics learning in Japanese, Chinese, and American Classrooms. In G. B. Saxe & M. Gearheat (Eds.), *Children's mathematics* (pp. 27-54). San Francisco, CA: Jossey-Bass.

U.S. Department of Education (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel.* Washington, D.C.: Author.

U.S. Department of Education. (2010). National Assessment of Educational Progress (NAEP): NAEP questions tool. Retrieved from http://nces.ed.gov/nationsreportcard/itmrlsx/search.aspx?subject_mathematics

Vamakoussi, X. & Vosniadou. S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28,* 181-209.

Vosniadou, S., Vamvakoussi, X., & Skopeliti, I. (2008). The framework theory approach to conceptual change. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (pp. 3–34). Mahwah, NJ: Erlbaum.

Wilkinson, G. S. (2008). Wide Range Achievement Test (4^{th} Edition). Wilmington, DE: Wide Range.

Wong, R. M. F., Lawson, M. J., & Keeves, J. (2002). The effects of self-explanation training on students' problem solving in high-school mathematics. *Learning and Instruction, 12,* 233-262.

Wu, H. (2008). *Fractions, decimals, and rational numbers.* Berkeley, CA: Author.

Yang, M. T., & Cobb, P. (1995). A Cross-Cultural Investigation into the Development of Place-Value Concepts of Children in Taiwan and the United States. *Educational Studies in Mathematics, 28,* 1-33.

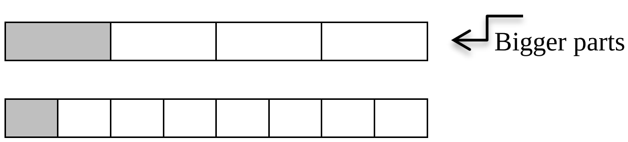
APPENDIX

Examples and Non-Examples of Sound Explanations

Explanation Component	Example
1) Correct sign 1 point	1/4 > 1/8
2) Number of parts (numerator) 2 points	The fractions have the same/different number of parts Draws an arrow on picture to show they have the same number of parts
3) Size of parts (denominator) 2 points	The fractions have the same/different size parts Draws an arrow to the parts and indicates which is bigger/smaller, or that both units have the same size parts
4) Accurate picture 2 points	Units are the same size, divided into equal parts, parts shaded

Example sound explanation: 6 points

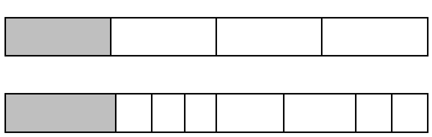
1) Correct sign: ¼ > 1/8 (1 point)
2) Number of parts (2 points)
3) Size of parts (2 points)
4) Accurate picture (1 point)



Same number of parts, but fourths are bigger than eighths

Non-example sound explanation:

Inaccurate picture – ¼ is not larger than 1/8 (0 points)



3/6 is bigger than ¾ because 6 is bigger than 4.