K-THEORY OF UNIFORM ROE ALGEBRAS

By

Ján Špakula

Dissertation

Submitted to the Faculty of the

Graduate School of Vanderbilt University

in partial fulfillment of the requirements

for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

August, 2008

Nashville, Tennessee

Approved:

Professor Guoliang Yu

Professor Dietmar Bisch

Professor Bruce Hughes

Professor Gennadi Kasparov

Professor Thomas Kephart

To my closest family,

Marta and Ján

and

my beloved wife, Iva

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor, Guoliang Yu. Without his support, never-ending encouragement, patience and professional advice this work would not have been possible.

I would also like to thank the faculty of the math department, especially Dietmar Bisch, Bruce Hughes, Gennadi Kasparov and Mark Sapir for many helpful and inspiring conversations, not only on mathematics.

Finally my thanks go to my friends in the graduate program, in particular to Matt Calef, Iva Kozáková, Bogdan Nica and Piotr Nowak, with whom I spent a lot of time discussing mathematical problems.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The theme of this work is the interplay between coarse geometry and K-theory of C*-algebras.

Coarse geometric ideas were introduced to mathematics by the celebrated Mostow's rigidity theorem, and subsequently popularized by Gromov (on the group theory side) and by John Roe (on the analytic side).

Given a metric space (X,d), topology focuses on small–scale properties of the space (for instance, metrics d and $\min(d,1)$ generate the same topology, but the second metric makes the space bounded). Coarse geometry does "the opposite": metrics d and $\max(d,1)$ yield the same "coarse structure". We disregard any local phenomena, and focus on the large–scale structure. This can be visualized as follows: imagine viewing the space from bigger and bigger distance. Any bounded set in the space becomes smaller and smaller, until finally "in the limit" it just becomes a point.

Examples of properties which depend only on the large–scale structure of a space are asymptotic dimension, property A and coarse embeddability into a Hilbert space.

The seminal work of John Roe [Roe88] has provided a connection between the coarse geometic approach and index theory. Index theory is concerned with computing (or at least determining if nonzero) indices of first-order elliptic differential operators on manifolds. This problem was solved for compact manifolds without boundary by the Atiyah–Singer index theorem. In this case, the elliptic operators are actually Fredholm, and the index is an integer. However, this is no longer true for noncompact manifolds. One way to resolve this issue is to consider the index not as a number, but as a K-theory element of a suitable C*-algebra. John Roe [Roe88; Roe96] introduced algebras (subsequently named Roe algebras) whose K-theory can serve a receptacle for indices. They reflect the large–scale structure of the manifold in question, thus linking its coarse geometry to index theory. From this point of view, K-theory of Roe algebras can be viewed as an obstruction group for invertibility of elliptic differential operators, thus having implications for geometric problems such as existence of a metric with positive scalar curvature on a given manifold.

On the other hand, elliptic operators can be organized to form abelian groups — analytic K-homology groups. These groups are a special case of Kasparov's KK-theory [Kas80]. Kasparov developed and pushed much further the original suggestion of Atiyah [Ati70] to use elliptic operators as a basis for the dual homology theory to K-theory.

Putting the pieces together, we obtain an index map from K-homology to K-theory of Roe algebras. Whether or not this index map (also called the coarse assembly map in this case) is an isomorphism is the subject of the coarse Baum–Connes conjecture. This conjecture has implications in geometry — for instance Novikov's conjecture on homotopy invariance of higher signatures.

One of the successes of the coarse geometric approach is a result of Yu [Yu00], which says that if a finitely generated group coarsely embeds into a Hilbert space, then the coarse Baum–Connes conjecture, and consequently Novikov conjecture, is true for this group. The only known examples of groups which do not satisfy the assumption are Gromov's random groups [Gro00].

An important tool in this circle of ideas is the locally finite homology theory, linked to K-homology via a Chern character homomorphism. Schematically:

$$\begin{tikzcd} K\text{-homology} \arrow[r, "\text{index map}"] \arrow[d, "\text{Chern}"] & K\text{-theory of Roe C*-algebras} \\ \text{locally finite homology.} & \end{tikzcd}$$

While Roe C*-algebras appear in the coarse Baum–Connes conjecture, their C*-algebraic properties can be difficult to study directly. Studying uniform Roe C*-algebras (smaller version of Roe algebras) might be more feasible. Results of Guentner, Kaminker and Ozawa; Skandalis, Tu and Yu [GK02; Oza00; STY02] exhibit a connection between a coarse geometric property (property A) of a space and a C*-algebraic property of its uniform Roe C*-algebra (nuclearity). It also follows from their work that if the space in question is actually a Cayley graph of a finitely generated group (or a sufficiently group–like space [BNW07]), nuclearity of its uniform Roe C*-algebra is equivalent to its exactness. Whether it is so for any space with bounded geometry is still an open question.

Following up on these ideas, Ulgen [Ulg05] (building on groupoid results of Tu [Tu99; Tu00]) showed that if a space coarsely embeds into a Hilbert space, its uniform Roe C*-algebra is K-nuclear (the notion is due to Skandalis [Ska88]). For separable C*-algebras (which uniform Roe C*-algebras are unfortunately not, except in trivial cases), K-nuclearity implies K-exactness. K-exactness is a natural K-theoretic generalization of exactness of C*-algebras, defined by Ulgen [Ulg05]. One of the questions addressed in this thesis is the question of K-exactness of uniform Roe C*-algebras. We show that for a large class of expanders, their uniform Roe C*-algebras are not K-exact.

Shifting forus to the index theory side, indices of many natural operators appearing in geometry can be defined in the K-theory of uniform Roe C*-algebras (see [Roe88] and section III.2). These K-theory group tend to be rather large (when nontrivial), see example II.3.4.

K-theory of uniform Roe algebras has similar flavor as the uniformly finite homology theory of Block and Weinberger [BW92]. For instance, in both cases there are analogous criteria for amenability [BW92; Ele97] in terms of vanishing of the fundamental class in the 0-th uniformly finite homology and in K_0 of the uniform Roe C*-algebra respectively.

The second topic of this work (presented first) is to carry out a construction analogous to the usual K-homology/Roe C*-algebras framework, but done in the uniform context. Schematically, we attempt to fill in the box:

$$\begin{tikzcd} \boxed{\text{uniform } K\text{-homology}} \arrow[r, "\text{index map}"] \arrow[d, dotted, "\text{Chern}"] & K\text{-theory of uniform Roe C*-algebras} \\ \text{uniformly finite homology.} & \end{tikzcd}$$

I.1 Results and outline of arguments

In this paper, we construct a uniform version of analytic K-homology theory, prove its basic properties (Mayer–Vietoris sequence, reformulation as K-theory of certain C*-algebras), and construct an index map (or uniform assembly map) into the K-theory of uniform Roe C*-algebras. In an analogy to the coarse Baum–Connes conjecture, this can be viewed as an attempt to provide an algorithm for computing K-theory of uniform Roe algebras. Furthermore, as an application of uniform K-homology, we prove a criterion for amenability.

On the other side of the spectrum, we show that uniform Roe algebras of a large class of expanders are not even K-exact. The argument presented here is a greatly simplified version suggested by the reviewer of our original paper.

The thesis is organized as follows: in chapter II we recall the necessary definitions. Chapter III is devoted to uniform K-homology. Loosely following the exposition [HR00b] of analytic K-homology, we define its uniform version, and prove its properties. The main idea is to quantify "how well approximable by finite rank operators" are various compact operators appearing in the definition of a Fredholm module.

We show that elliptic operators coming from geometry give rise to classes in our theory (section III.2). This is based on estimates present in [Roe88]. Working our way towards a Mayer–Vietoris sequence (section III.4), we prove that the "partial" uniform K-homology groups are isomorphic to K-theory of certain "dual" C*-algebras (section III.3). This is done by showing that uniform K-homology elements can be represented by uniform Fredholm modules with nice properties (analogously to the usual K-homology). The Mayer–Vietoris sequence is then a consequence of the C*-algebra Mayer–Vietoris sequence together with an excision lemma. This lemma is a technical result, which builds on the description of uniform K-homology as a direct limit of K-theories of certain C*-algebras.

The following sections (III.5–III.7) are devoted to the construction of an index map from uniform K-homology to K-theory of uniform Roe algebras. This follows the usual course by first proving that uniform classes can be represented by operators with finite propagation, and then making use of an explicit formula for the index to show that it fits into the K-theory of the uniform Roe algebra. However, some amount of technicalities is necessary, since the index a priori lands in a slightly different C*-algebra. The central issue is that in the definition of uniform Roe C*-algebras we are required to have a fixed basis for the auxiliary Hilbert space; while during the index construction there is no natural basis available. The need to account for all the choices results in slightly complicated notation.

Finally, we prove a criterion for amenability in terms of uniform K-homology. This fits into the picture with the above mentioned criteria [BW92; Ele97].

In chapter IV we turn to the other side: uniform K-homology can be understood as an attempt to compute K-theory of uniform Roe C*-algebras by means of exact sequences (which are present on the K-homology side). However, if there is a lack of exactness in K-theory of some uniform Roe C*-algebras, there is no hope of "computing" that K-theory groups. We show that uniform Roe algebras of expanders coming from some groups with property (τ) are not K-exact. Our original argument based on a construction of Ozawa [Oza03] was greatly simplified thanks to the reviewer of our paper. (It should be pointed out that our original construction worked only in the case of $SL_n(\mathbb{Z})$, $n \geq 3$, with a special choice of subgroups.) We present the simplified version here. The origins of the constructions of this type go back to [Was91], where examples of non-exact C*-algebras were presented. It was an observation of (we believe) Skandalis, who pointed out that sometimes this lack of exactness can be detected at the K-theory level. These construction work for algebras somehow related to the product C*-algebra $\prod_n \mathscr{M}_n(\mathbb{C})$, to which the uniform Roe algebra of a coarse disjoint union certainly is. The result of this construction is a projection, which can be detected in K_0 via a homomorphism similar to the one appearing in Higson's counterexample to the coarse Baum–Connes conjecture [HLS02].

I.2 Further plans

The immediate plan with uniform K-homology is to further explore its connection to known theories and conjectures:

- In an analogy with the Chern character from K-homology into the locally finite homology theory, it should be possible to define a Chern character homomorphism from uniform K-homology into the uniformly finite homology of Block and Weinberger.
- The coarse Baum–Connes conjecture for a Cayley graph of a finitely generated group Γ is equivalent to the Baum–Connes conjecture for this group with coefficients in $\ell^\infty(\Gamma,\mathscr{K})$ [Yu95a]. The argument builds on a Mayer–Vietoris sequence. We propose to prove an analogous statement in the uniform case. Namely, the isomorphism question of the uniform index map for the Cayley graph of Γ is equivalent to the Baum–Connes conjecture for Γ with coefficients in $\ell^\infty(\Gamma)$. With the current status of the Baum–Connes conjecture, this would show that the index map is an isomorphism for, say, finitely generated groups with the Haagerup property [HK97] and for hyperbolic groups [Laf02; MY02]. Alternatively, using the groupoid approach of [STY02], it might be possible to show that the uniform index map is an isomorphism for any space that coarsely embeds into a Hilbert space.
- Use a finite asymptotic dimension argument (following the ideas in [Yu98]) to prove directly that the index map is an isomorphism.

Second, the connections to index theory are to be investigated: in the amenable case, one can use a trace on the uniform Roe algebra (the existence of a trace is in fact equivalent to amenability [Ele97]) to extract numerical information from the index [Roe88]. Since Roe's work, the cyclic cohomology of algebraic uniform Roe algebras was computed [Yu95b]. Furthermore, Lafforgue's construction of smooth subalgebras of groupoid C*-algebras for hyperbolic groupoids [Laf07] can be used to obtain smooth subalgebras of uniform Roe C*-algebras when the space in question is hyperbolic. It is an interesting question if one can fit together these tools to obtain an index theorem.

Turning to K-exactness, a natural question is to try to generalize the argument presented here to expanders which do not come (naturally) from a "mother" group having property (τ). The first candidate is the expander constructed from the sequence of alternating groups [Kas07]. The crucial property that one needs to do so is the following: if we denote $A^{(N)} = \{a_1^{(N)},\dots,a_K^{(N)}\}$ the generating set of $\mathrm{Alt}(N)$

which makes the sequence of Cayley graphs $\mathrm{Cay}(\mathrm{Alt}(N), A^{(N)})$ into an expander, is it true that the sequence $\mathrm{Cay}(\mathrm{Alt}(N) \times \mathrm{Alt}(M), \{(a_i^{(N)}, a_i^{(M)})\})$ ($N > M \to \infty$) constitutes an expander as well?

Another interesting question is the following: Can we use this (explicit) construction of a projection violating K-exactness to produce counterexamples to the coarse Baum–Connes conjecture?

CHAPTER II

PRELIMINARIES

II.1 Conventions

Throughout this paper, metric spaces are assumed to be *proper* (i.e. balls are precompact) and *separable*. Hilbert spaces are assumed to be *separable* as well.

II.2 Coarse geometry

While it is possible to give an abstract definition of a coarse structure (see [Roe03]), for our purposes it is sufficient and more straightforward to assume that our spaces are endowed with a metric. The appropriate notion of maps in the "coarse category" is the following:

Definition II.2.1 (Coarse maps)**.** A (not necessarily continuous) map $g : X \to Z$ between metric spaces (X, d_X) and (Z, d_Z) is said to be *coarse*, if:

- For any $r \geq 0$ there exists $R \geq 0$, such that $d_X(x_1, x_2) \leq r$ implies $d_Z(g(x_1), g(x_2)) \leq R$ for $x_{1,2} \in X$. An equivalent condition is that there exists a non-decreasing function $\rho^+ : \mathbb{R}^+ \to \mathbb{R}^+$, such that $d_Z(g(x_1), g(x_2)) \leq \rho^+(d_X(x_1, x_2))$.
- For any $r \geq 0$ we have $\mathrm{diam}_X(g^{-1}(B(z,r))) < \infty$ for all $z \in Z$, where $B(z,r)$ denotes the ball centered at z with radius r. This condition is referred to as being *cobounded*.

Furthermore, we say that g is called *uniformly cobounded*, if for any $r \geq 0$, we have

$$R_g(r) := \sup_{z \in Z} \mathrm{diam}_X(g^{-1}(B(z,r))) < \infty.$$

Vaguely, the requirements are that "points which are not too far apart do not get mapped too far apart by f", and that f "does not collapse too big chunks of X".

Example II.2.2. A quasi–isometry is an example of a coarse map which is uniformly cobounded. If we denote $A = \{2^n \mid n \in \mathbb{N}\} \subset \mathbb{N}$, $C = \bigcup_{n \in \mathbb{N}} B(2^n, n) \subset \mathbb{N}$ (both with metrics induced from $\mathbb{N}$) and $f : C \to A$ which projects each element of $B(2^n, n)$ onto 2^n, then f is a coarse map, which is not uniformly cobounded.

Definition II.2.3 (Coarse equivalence)**.** Two metric spaces (X, d_X) and (Y, d_Y) are said to be *coarsely equivalent*, if there are coarse maps $f : X \to Y$ and $g : Y \to X$ and a constant C, such that $d_Y((f \circ g)(y), y) < C$ and $d_X((g \circ f)(x), x) < C$ for all $x \in X, y \in Y$.

We say that X *coarsely embeds* to Y, if X is coarsely equivalent to a subset of Y. (A coarse embedding is also called a "uniform embedding" in the literature.)

Example II.2.4. Define $f : \mathbb{R} \to \mathbb{Z}$ by $f(x) = [x]$, and $g : \mathbb{Z} \to \mathbb{R}$ by $g(z) = z$. Then both maps are coarse and they exhibit a coarse equivalence of $\mathbb{R}$ and $\mathbb{Z}$.

When working in the "coarse category", we may try to choose a nice representative in the class of coarsely equivalent spaces:

Definition II.2.5. A metric space Y is said to be *uniformly discrete*, if there is $\delta > 0$, such that $d(x,y) \geq \delta$ whenever $x \neq y \in Y$.

Furthermore, Y is said to have *bounded geometry*, if for any $r \geq 0$ we have

$$\sup_{y \in Y} |B(y,r)| < \infty.$$

While we can always arrange for uniform discreteness, bounded geometry is an intrinsic coarse property of a metric space.

When switching between discrete and "continuous spaces", the following concept proved to be useful:

Definition II.2.6 (Rips complex[1]). Let X be a metric space and let $d \geq 0$. The *Rips complex* $P_d(X)$ is a simplicial polyhedron defined as follows:

- the vertex set of $P_d(X)$ is X,
- any $q+1$ vertices $x_0, x_1, \ldots, x_q$ span a simplex of $P_d(X)$ if and only if

$$d(x_i, x_j) \leq d, \qquad \forall i, j \in \{0, \ldots, q\}.$$

Note that if X has bounded geometry, $P_d(X)$ is locally finite and finite dimensional.

We now turn to various coarse geometric properties, that are of interest in this paper. The first on the list is amenability, which we view through Følner's criterion (see [BW92, section 3]).

Definition II.2.7. Let (Y,d) be a uniformly discrete metric space. For a set $U \subset Y$, we define its *r-boundary* by

$$\partial_r U = \{y \in Y \mid d(y,U) \leq r \text{ and } d(y, Y \setminus U) \leq r\}.$$

Furthermore, we denote by $|U|$ the cardinality of U.
We say that Y is *amenable*, if for any $r, \delta > 0$, there exists a finite set $U \subset Y$, such that

$$\frac{|\partial_r U|}{|U|} < \delta.$$

Remark II.2.8. This definition is equivalent to the usual definition of amenability of groups (existence of an invariant mean) for spaces arising as Cayley graphs of discrete groups. However, we do not require the Følner sets to exhaust the whole space, and so we need to be cautious when applying this to general metric spaces. For instance, taking any uniformly discrete metric space Y, one can make it amenable by attaching an infinite "spaghetti" to it, i.e. an infinite ray. Also note that any "coarse disjoint union finite spaces" is also amenable in this sense, since for a given $r > 0$, we can always select a finite piece U of the space, which is at least r-far from the rest of the space, hence making $\partial_r U = \emptyset$. In particular, this applies to expanders.

[1] Originally defined by Vietoris.

We turn to Yu's property A, which can be viewed as a non-equivariant version of amenability.

Definition II.2.9 ([Yu00; HR00a; STY02])**.** A uniformly discrete metric space (X,d) is said to have *property A*, if for all $R,\varepsilon > 0$, there exists a family of finite non-empty subsets A_x of $X \times \mathbb{N}$ (indexed by $x \in X$), such that

1. for all $x,y \in X$ with $d(x,y) < R$ we have $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} < \varepsilon$,
2. there exists S such that for all $x \in X$ and $(y,n) \in A_x$ we have $d(x,y) \leq S$.

(Δ stands for symmetric difference of sets, and $|.|$ for their cardinality.)

Finally, let us recall one possibility of how to construct a coarse disjoint union of finite spaces. This construction is widely used to view an expanding family of graphs (an expander) as one metric space. Given a sequence $(X_q)_{q\in\mathbb{N}}$ of finite metric spaces, we define their coarse disjoint union $\sqcup_q X_q$ to be the set $\cup_q X_q$ endowed with the metric inherited from individual X_q's together with the condition $d(X_q, X_{q'}) = \max(q,q')$ for $q \neq q'$.

II.3 C*-algebras

We shall define the Roe C*-algebra $C^*(X)$ and the uniform Roe C*-algebra $C^*_u(X)$ for a metric space X. These algebras reflect the large–scale geometry of the space X.

The ultimate goal is to understand these algebras. In particular, it is important to establish connections between properties of the space X and properties of translation algebras. The first instance of this is the proposition saying that if two spaces are coarsely equivalent, their translation algebras are isomorphic (or Morita equivalent in the uniform case). The converse is an open problem. In general, it is difficult to deduce a property of the space from a property of the translation algebra.

While properties of Roe algebras have direct consequences in other areas of mathematics (e.g. Novikov conjecture via the coarse Baum–Connes conjecture), uniform Roe algebras do not. However, uniform Roe algebras, being much smaller, are better suited for direct study. Moreover, some C*-algebraic and K-theoretic properties of the uniform Roe algebras reflect coarse geometric properties of spaces (amenability, property A).

Let Y be a uniformly discrete metric space with bounded geometry. We consider the Hilbert space $\ell^2(Y) \otimes \ell^2(\mathbb{N}) \cong \ell^2(Y \times \mathbb{N})$ (or $\ell^2(Y)$ in the uniform case), and represent bounded operators T on it as matrices $T = (t_{yx})_{x,y\in Y}$ with entries t_{yx} in $\mathscr{B}(\ell^2(\mathbb{N}))$ (or $\mathbb{C}$ respectively).

Definition II.3.1. We say that $T = (t_{yx}) \in \mathscr{B}(\ell^2(Y \times \mathbb{N}))$ (or $\mathscr{B}(\ell^2(Y))$) has *finite propagation*, if there exists $R \geq 0$, such that $t_{yx} = 0$ whenever $d(x,y) > R$. The smallest such R is called the *propagation* of T and denoted by propagation(T).

We say that $T = (t_{yx}) \in \mathscr{B}(\ell^2(Y \times \mathbb{N}))$ is *locally compact*, if $t_{yx} \in \mathscr{K}(\ell^2(\mathbb{N}))$ for all $x,y \in Y$. (This condition is void in the case $T \in \mathscr{B}(\ell^2(Y))$.)

Definition II.3.2 (Roe algebras)**.** We define the *Roe C*-algebra* C^*Y of a uniformly discrete metric space Y with bounded geometry to be the norm-closure of the algebra of all locally compact finite propagation operators in $\mathscr{B}(\ell^2(Y \times \mathbb{N}))$.

Similarly, the *uniform Roe C*-algebra* C_u^*Y is defined to be the norm-closure of the algebra of all finite propagation operators in $\mathscr{B}(\ell^2(Y))$.

Remark II.3.3. The Roe algebras C^*Y are functorial under coarse maps, while the uniform Roe algebras C_u^*Y are not. The uniform Roe algebras are not functorial even under coarse uniformly cobounded maps, as an example of a one-point and a two-point spaces show. Nevertheless, coarsely equivalent spaces have Morita equivalent uniform Roe C*-algebras, see [BNW07].

Example II.3.4 ($K_*(C_u^*\mathbb{Z})$). One possible way to compute K-theory of the Roe C*-algebra $C^*|\mathbb{Z}|$ (by $|\mathbb{Z}|$ we mean $\mathbb{Z}$ as a metric space) is via the Pimsner–Voiculescu exact sequence (see [Roe96, proposition 4.9]). The same approach can be used to compute the K-theory of $C_u^*\mathbb{Z} \cong \ell^\infty\mathbb{Z} \rtimes \mathbb{Z}$. The action of $\mathbb{Z}$ of $\ell^\infty\mathbb{Z}$ is by translation, and we denote by $\alpha : \ell^\infty\mathbb{Z} \to \ell^\infty\mathbb{Z}$ the translation by 1, i.e. $\alpha(f)(n) = f(n+1)$ for $f \in \ell^\infty\mathbb{Z}$. Then the Pimsner–Voiculescu sequence has the form

$$\begin{tikzcd} K_0(\ell^\infty\mathbb{Z}) \arrow[r, "1-\alpha_*"] & K_0(\ell^\infty\mathbb{Z}) \arrow[r, "\mathrm{incl}_*"] & K_0(C_u^*\mathbb{Z}) \arrow[d] \\ K_1(C_u^*\mathbb{Z}) \arrow[u] & K_1(\ell^\infty\mathbb{Z}) \arrow[l, "\mathrm{incl}_*"'] & K_1(\ell^\infty\mathbb{Z}). \arrow[l, "1-\alpha_*"'] \end{tikzcd}$$

Now $K_0(\ell^\infty\mathbb{Z})$ is isomorphic to the abelian group $B(\mathbb{Z})$ of all bounded sequences from $\mathbb{Z}$ to $\mathbb{Z}$; $K_1(\ell^\infty\mathbb{Z}) = 0$. The kernel of $1-\alpha_* : K_0(\ell^\infty\mathbb{Z}) \to K_0(\ell^\infty\mathbb{Z})$ consists of constant sequences, so $K_1(C_u^*\mathbb{Z}) \cong \mathbb{Z}$. Finally, $K_0(C_u^*\mathbb{Z}) = B(\mathbb{Z})/I(\mathbb{Z})$, where $I(\mathbb{Z}) = \mathrm{image}(1-\alpha_*)$ denotes the group of sequences $(a_i)_{i\in\mathbb{Z}}$ which satisfy $-\infty < \sup_{n\in\mathbb{Z}}(\sum_{i=0}^n a_i) < \infty$. So we see that $K_0(C_u^*\mathbb{Z})$ is in fact uncountable.

Finally, note that the Pimsner–Voiculescu sequence works also for finitely generated free groups, and so a similar computation can be done.

II.4 Property (τ) and expanders

Definition II.4.1. An *expander* is a sequence $(X_n = (V_n, E_n))$ of finite graphs with the properties:

- The maximum number of edges emanating from any vertex is uniformly bounded.
- The number of vertices $|V_n|$ tends to infinity as n increases.
- The first nonzero eigenvalue of the Laplacian, $\lambda_1(X_n)$, is uniformly bounded away from zero, say by $\lambda > 0$.

We understand the sequence as one metric space $\sqcup_n X_n$ via the coarse disjoint union construction.

This concept has found many applications in mathematics and computer science. However, our main interest comes from the fact that expanders are (so far) the only explicit examples of metric spaces with bounded geometry that do not embed coarsely into a Hilbert space (see e.g. [Roe03, proposition 11.29]). Moreover, it is a result of Higson [HLS02], that some expanders give counterexamples to the coarse Baum–Connes conjecture.

The first explicit examples of expanders were constructed by Margulis as $\sqcup_q \Gamma/\Gamma_q$, where Γ is a finitely generated group with property (T) (with a fixed generating set), and $\Gamma_q \leq \Gamma$ is a decreasing sequence of normal subgroups with finite index, such that $\bigcap_q \Gamma_q = \{1\}$. This construction eventually led to Lubotzky's property (τ) [Lub94].

Definition II.4.2. Let Γ be a finitely generated group and $\mathscr{L} = N_i$ a family of finite index normal subgroups of Γ. We say that Γ has *property (τ) with respect to the family $\mathscr{L}$* (written also $\tau(\mathscr{L})$) if the trivial representation is isolated in the set of all unitary representations of Γ, which factor through Γ/N_i. We say that Γ has *property (τ)* if it has this property with respect to the family of all finite index normal subgroups.

This property is so designed that $\sqcup_i \Gamma/N_i$ constitutes an expander.

CHAPTER III

UNIFORM K-HOMOLOGY

III.1 Uniform K-homology groups

Throughout this chapter, X shall stand for a proper separable metric space, and d will denote its metric.

Recall the definition of Fredholm modules:

Definition III.1.1 (Fredholm modules)**.** Let (H,ϕ,S) be a triple consisting of a Hilbert space H, a *-homomorphism $\phi : C_0(X) \to \mathscr{B}(H)$ and an operator $S \in \mathscr{B}(H)$. We say that such a triple is a 0-*Fredholm module* (or even Fredholm module), if for every $f \in C_0(X)$ the following hold:

- (Fredholmness) $(1-S^*S)\phi(f) \in \mathscr{K}(H)$ and $(1-SS^*)\phi(f) \in \mathscr{K}(H)$,
- (pseudolocality) $[S,\phi(f)] \in \mathscr{K}(H)$.

Similarly, we say that a triple (H,ϕ,P) as above is a 1-*Fredholm module* (or odd Fredholm module), if for every $f \in C_0(X)$:

- $(P^2-1)\phi(f) \in \mathscr{K}(H)$ and $(P-P^*)\phi(f) \in \mathscr{K}(H)$,
- (pseudolocality) $[P,\phi(f)] \in \mathscr{K}(H)$.

Remark III.1.2. We can also formulate the Fredholmness condition for even Fredholm modules in another form, which is more convenient for the setting of differential operators: A triple (H,ϕ,T) forms an even Fredholm module, if H is $\mathbb{Z}_2$-graded, $\phi(f)$ is of degree 0 (i.e. even) for all $f \in C_0(X)$ and $T \in \mathscr{B}(H)$ is a pseudolocal operator of degree 1 (odd), satisfying that $(T^2-1)\phi(f) \in \mathscr{K}(H)$ and $(T^*-T)\phi(f) \in \mathscr{K}(H)$ for all $f \in C_0(X)$.

For defining a uniform version of K-homology, we want to "quantify" the compactness of an operator in the following way: given $\varepsilon > 0$ we try to approximate our compact operator within ε by a finite rank operator with the smallest possible rank. In the definition of a Fredholm module, instead of just one compact operator, we really have a collection of compacts, depending on $f \in C_0(X)$, and we require a uniform bound on the ranks of ε–approximants for fixed R — a "scale" in the metric of X. This consideration is sufficient to ensure uniformity on the large scale. However, we want (certain) first order differential operators to give rise to uniform K-homology classes. The approximation properties of compacts arising from the pseudolocality condition really depend not only on the support of f but also on its derivative (just consider the operator $[D,f]$), and so we need to build in also some local control.

Specifically, for a metric space X and $R,L \geq 0$ we denote

$$\begin{aligned} C_R(X) &= \{f \in C_c(X) \mid \operatorname{diam}(\operatorname{supp}(f)) \leq R \text{ and } ||f||_\infty \leq 1\} \\ C_{R,L}(X) &= \{f \in C_R(X) \mid f \text{ is } L\text{-continuous}\}. \end{aligned}$$

We say that $f : X \to Y$ is L-continuous, if there exists a nondecreasing function $\alpha : [0,\infty] \to [0,\infty)$ with $\alpha'(0) \geq \frac{1}{L}$, such that for any $x,y \in X$ we have $d(x,y) \leq \alpha(s) \implies d(f(x),f(y)) \leq s$. Loosely, one could formulate the condition as "locally L-Lipschitz". In particular, if a function is L-Lipschitz, then it is L-continuous (with $\alpha(s) = \frac{1}{L}s$). The converse is true for instance when X is a geodesic space. Hence for practical purposes we can replace this condition with just L-Lipschitz. We use the notion of L-continuity to emphasize the local side of being Lipschitz.

The reason for introducing L-continuity is the following: if X is a manifold and $f \in C_{R,L}(X)$ is differentiable at $x \in X$, then the norm of the derivative df of f at x is at most L. This observation is used in a crucial way in section III.2, when proving that Dirac–type operators produce uniform K-homology classes. If one doesn't require the theory to include such classes, it's possible to just ignore L's and l-'s throughout the paper.

Furthermore $\bigcup_{L\geq 0} C_{R,L}(X)$ is dense in $C_R(X)$. This is completely analogous to saying that (once) differentiable functions are dense in the space of all continous functions. The proof is outlined at the end of this section, in lemma III.1.19.

In the following definition, we introduce the uniformity conditions. We list two versions — one without the "L–dependency" and one featuring L.

Definition III.1.3 (Uniform approximability)**.** Let H be a Hilbert space, X a metric space and $\phi : C_0(X) \to \mathscr{B}(H)$ a *-homomorphism. For $\varepsilon, M > 0$, an operator $T \in \mathscr{B}(H)$ is said to be (ε,M)*–approximable*, if there is a rank-M operator k, such that $\|T - k\| < \varepsilon$.

Let $E(\cdot)$ (or $E(f)$) stand for an expression with operators in $\mathscr{B}(H)$ and terms $\phi(\cdot)$ (or $\phi(f)$). (For instance $E(\cdot) = T\phi(\cdot)$ or $E(f) = [T,\phi(f)]$.)

- For $\varepsilon, M, R > 0$, an expression $E(\cdot)$ is said to be $(\varepsilon,R,M;\phi)$*–approximable*, if for each $f \in C_R(X)$, $E(f)$ is (ε,M)–approximable.
- For $\varepsilon, R, L, M > 0$, an expression $E(\cdot)$ is said to be $(\varepsilon,R,L,M;\phi)$*–approximable*, if for each $f \in C_{R,L}(X)$, $E(f)$ is (ε,M)–approximable.
- An expression $E(\cdot)$ is *uniformly approximable*, if for every $R \geq 0, \varepsilon > 0$ there exists $M > 0$, such that $E(\cdot)$ is $(\varepsilon,R,M;\phi)$–approximable. Furthermore, we write $E_1(\cdot) \sim_{ua} E_2(\cdot)$, if the difference $E_1(\cdot) - E_2(\cdot)$ is uniformly approximable.
- An expression $E(\cdot)$ is *l-uniformly approximable*, if for every $R, L \geq 0, \varepsilon > 0$ there exists $M > 0$, such that $E(\cdot)$ is $(\varepsilon,R,L,M;\phi)$–approximable. Furthermore, we write $E_1(\cdot) \sim_{lua} E_2(\cdot)$, if the difference $E_1(\cdot) - E_2(\cdot)$ is l-uniformly approximable.

We introduce several special cases of uniform approximability:

- We say that an operator $T \in \mathscr{B}(H)$ is *uniform*, if $T\phi(\cdot)$ and $\phi(\cdot)T$ are uniformly approximable (i.e. $T\phi(f) \sim_{ua} 0 \sim_{ua} \phi(f)T$). We also say that T is $(\varepsilon,R,M;\phi)$*–uniform*, if both operators $\phi(f)T$, $T\phi(f)$ are $(\varepsilon,R,M;\phi)$–approximable.

- An operator $T \in \mathscr{B}(H)$ is said to be *uniformly pseudolocal*, if $[T, \phi(\cdot)]$ is uniformly approximable (i.e. $[T, \phi(f)] \sim_{ua} 0$).

- An operator $T \in \mathscr{B}(H)$ is said to be *l-uniformly pseudolocal*, if $[T, \phi(\cdot)]$ is l-uniformly approximable (i.e. $[T, \phi(f)] \sim_{lua} 0$).

Remark III.1.4. The property of being uniformly pseudolocal is obviously stronger than that of being l-uniformly pseudolocal. In the former, we can obtain a bound M on ranks of approximants, which is independent of L (local condition), and depends only on R (support condition) and of course on ε.

Remark III.1.5. The notion of an "l-uniform" operator is in fact equivalent to the notion of a uniform operator given above. More precisely, if $T\phi(\cdot)$ and $\phi(\cdot)T$ are l-uniformly approximable, then they are in fact just uniformly approximable, i.e. we can get a bound on M independent of L. In other words, checking uniformity of operator on "nice" function is sufficient. Indeed, for every $f \in C_R(X)$ we can construct a function $\tilde{f} \in C_{R+1,1}(X)$, such that $f\tilde{f} = f$. Now given $R, \varepsilon > 0$, if M is the constant such that $T\phi(\cdot)$ and $\phi(\cdot)T$ are $(\varepsilon, R+1, 1, M; \phi)$–approximable, then $\phi(f)T = \phi(f)\phi(\tilde{f})T$ and $T\phi(f) = T\phi(\tilde{f})\phi(f)$ are $(\varepsilon, R, M; \phi)$–approximable.

Such a $\tilde{f}$ can be constructed for instance as $\tilde{f}(x) = \max(0, 1 - d(x, \mathrm{supp}(f)))$. One easily checks that this function is 1–Lipschitz, and so $\tilde{f} \in C_{\mathrm{diam}(\mathrm{supp}(f))+1,1}(X)$.

In the view of the previous remark, we can completely disregard the constant L appearing in the definition above, when we work with uniform operators only. This is so for instance in sections III.6–III.7.

Definition III.1.6 (Uniform Fredholm modules)**.** Let (H, ϕ, S) be a 0-Fredholm module. It is said to be *uniform*, if S is l-uniformly pseudolocal and the operators $1 - SS^*$, $1 - S^*S$ are uniform.

Let (H, ϕ, Q) be a 1-Fredholm module. It is said to be *uniform*, if Q is l-uniformly pseudolocal and the operators $1 - Q^2$ and $Q - Q^*$ are uniform.

Remark III.1.7. By using "uniform Fredholm module" (without 0- or 1-) in a statement we shall mean that the statement applies to both 0- and 1- uniform Fredholm modules.

Remark III.1.8. If we are given a Hilbert space H together with a *-homomorphism $\phi : C_0(X) \to \mathscr{B}(H)$ (i.e. an action of $C_0(X)$ on H), we say that (H, ϕ), or just H, is an *X-module*. When no confusion about ϕ can arise, we identify $f \in C_0(X)$ with $\phi(f) \in \mathscr{B}(H)$. Similarly, we omit "$\phi$" from $(\varepsilon, R, M; \phi)$, etc.

Example III.1.9 (Fundamental class). Let Y be a uniformly discrete space. Let S be the unilateral shift operator on $\ell^2\mathbb{N}$ (i.e. a Fredholm operator with index 1). Denote $H = \ell^2 Y \otimes \ell^2\mathbb{N}$, and set $\tilde{S} = \mathrm{diag}(S) \in \mathscr{B}(H)$. Endow H with the multiplication action of $C_0(Y)$. More precisely, define $\phi : C_0(Y) \to \mathscr{B}(H)$ by $\phi(f)(\zeta(y)) = f(y)\zeta(y)$, for $\zeta : Y \to \ell^2\mathbb{N}$, a square summable function, and $y \in Y$, $f \in C_0(Y)$.

It is easy to check that $(H, \phi, \tilde{S})$ is a 0-uniform Fredholm module for Y ($\tilde{S}$ is actually uniformly pseudolocal). This module has pivotal role in our characterization of amenability in section III.8.

The following example is concerned with the K-homology classes coming from elliptic differential operators on manifolds.

Example III.1.10. Let M be an open manifold (without boundary) and S a smooth complex vector bundle over M. Let D be a symmetric elliptic differential operator operating on sections of S. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a chopping function (an odd smooth function, $\chi(t) > 0$ for $t > 0$, $\chi(t) \to \pm 1$ for $t \to \pm\infty$.

Denote $H = L^2(M,S)$ and let $\rho : C_0(M) \to \mathscr{B}(H)$ be the multiplication action. It is proved in [HR00b, section 10.8], that $(H, \rho, \chi(D))$ is a Fredholm module.

Assuming that M (endowed with a Riemannian metric) has bounded geometry and that the operator D is "geometric" (e.g. has finite propagation speed), then this Fredholm module is actually uniform. The proof is outlined in section III.2.

Definition III.1.11 (Homotopy)**.** A collection (H, ϕ_t, S_t), $t \in [0,1]$, of uniform Fredholm modules is a *homotopy*, if:

- $t \mapsto S_t$ is continuous in norm,
- there exists $R > 0$, such that for $f, g \in C_0(X)$ with $d(\mathrm{supp}(f), \mathrm{supp}(g)) \geq R$, we have $\phi_s(f)\phi_t(g) = \phi_t(g)\phi_s(f) = 0$ for all $s,t \in [0,1]$,
- for every $s,t \in [0,1]$ and $R > 0$, there are R' and M, so that every $\phi_t(f)$, $f \in C_R(X)$, is within a rank-M operator from one of the form $\phi_s(g)$, $g \in C_{R'}(X)$.

By an *operator homotopy* we mean a homotopy as above, with the restriction that $\phi_t = \phi_0$ for all $t \in [0,1]$.

Remark III.1.12. Given a Hilbert space H with a homomorphism $\phi : C_0(X) \to \mathscr{B}(H)$, let us recall the definition of *finite propagation* (see II.3.1): $T \in \mathscr{B}(H)$ has finite propagation, if there exists $R \geq 0$, such that for any $f, g \in C_0(X)$ whose supports are at least R apart, we have $\phi(f)T\phi(g) = 0$.

Denote by $\Theta(\phi) \subset \mathscr{B}(H)$ the C*-algebra generated by all the uniform operators having ϕ-finite propagation. The last two conditions in the above definition imply that the algebras $\Theta(\phi_t)$ are the same for all $t \in [0,1]$. More precisely, the next-to-last condition implies that the notion of finite propagation coincides for all ϕ_t's; and the last condition does the same for uniformity.

We now proceed as in [HR00b, section 8.2] in defining a K-homology theory.

Given two uniform Fredholm modules, we can clearly form their direct sum, which becomes again a uniform Fredholm module.

Definition III.1.13 (K_*^u)**.** We define the uniform K-homology group $K_i^u(X)$, $i = 0,1$, to be an abelian group generated by the unitary equivalence classes of uniform i-Fredholm modules (H, ϕ, S) with the following relations:

- if two uniform Fredholm modules $\mathbf{x}, \mathbf{y}$ are homotopic, we declare $[\mathbf{x}] = [\mathbf{y}]$,
- for two uniform Fredholm modules $\mathbf{x}, \mathbf{y}$, we set $[\mathbf{x} \oplus \mathbf{y}] = [\mathbf{x}] + [\mathbf{y}]$.

Recall that a Fredholm module (H, ϕ, S) is called *degenerate*, if the conditions in the definition hold exactly, that is $(1 - S^*S) = (1 - SS^*) = [\phi(f), S] = 0$ for all $f \in C_0(X)$ for the 0- version; and $S - S^* = S^2 - 1 = [\phi(f), S] = 0$ for all $f \in C_0(X)$ for the 1- version. The K_*^u-class of a degenerate Fredholm module is 0: the proof of the analogous result for K-homology [HR00b, 8.2.8]) carries verbatim.

The additive inverse of $[(H,\phi,S)] \in K_0^u(X)$ is $[(H,\phi,-S^*)]$. Similarly, the inverse of $[(H,\phi,P)] \in K_1^u(X)$ is $[(H,\phi,-P)]$. Again, the proof of these facts is just as [HR00b, proof of 8.2.10]. For instance, $\left(\begin{smallmatrix} \cos tS & \sin tI \\ \sin tI & -\cos tS^* \end{smallmatrix}\right)$, $t \in [0,\frac{\pi}{2}]$, is a homotopy showing that $[(H,\phi,S)] + [(H,\phi,-S^*)] = [(H\oplus H,\phi\oplus\phi,\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right))] = 0 \in K_0^u(X)$.

It follows from the facts in the last two paragraphs, that every element of $K_*^u(X)$ can be represented as a class of a single uniform Fredholm module. Furthermore, $[\mathbf{x}] = [\mathbf{y}]$ in $K_*^u(X)$ if and only if there exists a degenerate Fredholm module $\mathbf{z}$, such that $\mathbf{x}\oplus\mathbf{z}$ and $\mathbf{y}\oplus\mathbf{z}$ are unitarily equivalent to a pair of homotopic uniform Fredholm modules. In this case, we say that $\mathbf{x}$ and $\mathbf{y}$ are *stably homotopic*. Therefore, we may reformulate the definition of $K_*^u(X)$ as follows:

Proposition III.1.14. *The group $K_i^u(X)$ is canonically isomorphic to the semigroup of stable homotopy equivalence classes of uniform i-Fredholm modules.*

The uniform K-homology is not functorial under continuous maps in general; we need two extra condition in order to obtain functoriality: one handling the large–scale and one taking care of the local phenomena.

Definition III.1.15 (Uniform coboundedness; see definition II.2.1)**.** A (not necessarily continuous) map $g: X \to Z$ between metric spaces X and Z is said to be *uniformly cobounded*, if for any $r \geq 0$, we have

$$R_g(r) := \sup_{z\in Z} \operatorname{diam}(g^{-1}(B(z,r))) < \infty.$$

Lemma III.1.16. *An L-continuous uniformly cobounded map $g: X \to Z$ descends to a homomorphism on the uniform K-homology groups $g_*: K_*^u(X) \to K_*^u(Z)$.*

In particular, a quasi–isometry induces a map on the level of uniform K-homology groups.

Proof. Take a uniform Fredholm module $(H,\phi: C_0(X) \to \mathscr{B}(H), S)$ of an $K_*^u(X)$-element. We denote by $\tilde{g}: C_0(Z) \to C_0(X)$ the induced *-homomorphism. Then there is a *-homomorphism $\phi\circ\tilde{g}: C_0(Z) \to \mathscr{B}(H)$. By uniform coboundedness, we obtain that if $f \in C_R(Z)$, then $\tilde{g}(f) \in C_{R_g(R)}(X)$. By L-continuity, $f \in C_{R,L'}(Z)$ implies $\tilde{g}(f) \in C_{R_g(R),LL'}(X)$. Hence the uniformity requirements transfer and $(H,\phi\circ\tilde{g},S)$ becomes a uniform Fredholm module representing a $K_*^u(Z)$-element. We define $g_*[(H,\phi,S)] = [(H,\phi\circ\tilde{g},S)]$. □

We now prove a simple lemma analogous to a similar statement in the classical K-homology:

Lemma III.1.17 (Compact perturbations)**.** *If (H,ϕ,T) is a uniform Fredholm module and $K \in \mathscr{B}(H)$ is uniform, then (H,ϕ,T) and $(H,\phi,T+K)$ are operator homotopic.*

Proof. We need to show that $(H,\phi,T+tK)$, $t \in [0,1]$ are uniform Fredholm modules. Fix $\varepsilon, R, L > 0$ and let M be such that all the operators K, $[T,\phi(f)]$, $(1-T^*T)\phi(f)$ and $(1-TT^*)\phi(f)$ (or $(1-T^2)\phi(f)$ and $(T-T^*)\phi(f)$ in the 1-case) are (ε,M)—approximable for $f \in C_{R,L}(X)$.

First, for $f \in C_{R,L}(X)$, we have that $[T+tK,\phi(f)] = [T,\phi(f)] + tK\phi(f) - t\phi(f)K$, which is clearly $(3\varepsilon,3M)$–approximable. Hence the pseudolocality requirement is satisfied.

Let us now deal with the 0-case. Examine the following expression: $1-(T+tK)(T+tK)^* = (1-TT^*) - tKT^* - tTK^* - t^2KK^*$. Taking $f \in C_{R,L}(X)$ and multiplying by $\phi(f)$ the previous formula on the right, each of the elements $(1-TT^*)\phi(f)$, $tTK^*\phi(f)$, $t^2KK^*\phi(f)$ is going to be $(\|T\|\|K\|\varepsilon, M)$–approximable by assumption. We can rewrite the remaining term as follows $tKT^*\phi(f) = tK\phi(f)T^* + tK[T^*, \phi(f)]$, and so it is $(2\|T\|\|K\|\varepsilon, R, 2M)$–approximable. Therefore, $(1-(T+tK)(T+tK)^*)\phi(f)$ is $(5\|T\|\|K\|\varepsilon, 5M)$–approximable. It is clear that similar considerations can be applied to $1-(T+tK)^*(T+tK)$ as well.

Finally, we address the 1-case. It is clear that $((T+tK)-(T^*+tK^*))\phi(f)$ is $(2\varepsilon, 2M)$–approximable. Furthermore for $f \in C_{R,L}(X)$, $(1-(T+tK)^2)\phi(f) = (1-T^2)\phi(f) - tTK\phi(f) - t^2K^2\phi(f) - tK\phi(f)T - tK[T, \phi(f)]$, which is $(5\|T\|\|K\|\varepsilon, 5M)$–approximable. □

As a first application of the previous lemma, we make the following observation:

Remark III.1.18. We may always assume that a K_1^u-element is represented by a uniform 1-Fredholm module (H, ϕ, Q) with Q selfadjoint. It is because if we take any Q, $\frac{1}{2}(Q+Q^*)$ is selfadjoint and $Q - \frac{1}{2}(Q+Q^*) = \frac{1}{2}(Q-Q^*)$ is uniform. Moreover, the procedure of replacing Q by a selfadjoint operator can be applied to whole homotopies.

We finish the section by a lemma promised earlier.

Lemma III.1.19. *Let X be a metric space. Given any compactly supported continuous function $f : X \to \mathbb{R}$ and $\varepsilon > 0$, there exists $L > 0$ and an L-continuous function $g : X \to \mathbb{R}$, such that $\|f-g\|_\infty < \varepsilon$.*

Proof. Without loss of generality we can assume that $f(X) \subset [0,1]$. Take an integer N, such that $\frac{1}{N} < \varepsilon$, and set $U_n = f^{-1}[0, \frac{n}{N}]$, $n = 0, \dots, N$. Then $U_0 \subset U_1 \subset \cdots \subset U_N = X$ are closed sets. By uniform continuity, there exists $\delta > 0$, such that $d(x,y) < \delta$ implies $|f(x) - f(y)| < \frac{1}{N}$. This implies that $N_\delta(U_n) \subset U_{n+1}$.

Define $g : X \to \mathbb{R}$ as follows: for $x \in X$, let $n(x)$ be such that $x \in U_{n(x)}$, but $x \notin U_{n(x)-1}$ (where we set $U_{-1} = \emptyset$). Now set $g(x) = \frac{n-1}{N} + \frac{1}{N} \cdot \min(1, \frac{1}{\delta} d(x, U_{n(x)-1}))$ if $n(x) > 0$, and $g(x) = 0$ if $n(x) = 0$. It is clear from the construction that $\|f-g\|_\infty \leq \frac{1}{N} < \varepsilon$ and it's easy to verify that g is $\frac{1}{N\delta} < \frac{\varepsilon}{\delta}$-continuous. □

III.2 Dirac–type operators

In this section, we outline the proof of the fact that "geometric" operators on Riemannian manifolds with bounded geometry give rise to uniform Fredholm modules. The hard work was already done in [Roe88], where it is shown that such geometric operators have index defined in the algebraic K-theory of the algebra $\mathscr{U}_{-\infty}(M)$ of operators given by smooth uniformly bounded kernels, the precursor of the uniform Roe algebra.

Recall the setting: Let M be a complete Riemannian manifold (without boundary) and S a Clifford bundle over M. More precisely, denote by $\mathrm{Cliff}(M)$ the complexified bundle of Clifford algebras $\mathrm{Cliff}(T_xM)$ (equipped with a natural connection), and let S be a smooth complex vector bundle over M equipped with an action of $\mathrm{Cliff}(M)$ and a compatible connection. The bundle S is graded, if in addition it is equipped with an involution ε anticommuting with the Clifford action of tangent vectors (see remark III.1.2).

A "geometric" operator will be a first-order differential operator D defined by the composition

$$\Gamma(S) \to \Gamma(T^*M \otimes S) \to \Gamma(TM \otimes S) \to \Gamma(S),$$

where the arrows are given by the connection, metric and Clifford multiplication, respectively. In local coordinates, this operator has the form

$$D = \sum e_k \tfrac{\partial}{\partial x_k}.$$

The signature and Dirac operators are of this type. The main properties of these operators is that they are elliptic, and have finite propagation (in a sense that there exists a constant C, such that $\mathrm{supp}(e^{itD}\xi) \subset N_{Ct}(\mathrm{supp}(\xi))$ for all $\xi \in \Gamma_c(S)$)[1].

Denote $H = L^2(S)$ and let $\rho : C_0(M) \to \mathscr{B}(H)$ be the multiplication action. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a chopping function (an odd smooth function, $\chi(t) > 0$ for $t > 0$, $\chi(t) \to \pm 1$ for $t \to \pm\infty$). Then $(H, \rho, \chi(D))$ is a Fredholm module (see [HR00b, sections 10.6 and 10.8]). This is true in more general context, namely for any first-order elliptic differential operator on a complex smooth vector bundle. However to obtain uniformity, bounded geometry assumption and some analysis from [Roe88] is required.

Following [Roe88, section 2], we say that a Riemannian manifold M has *bounded geometry*, if it has positive injectivity radius and the curvature tensor is uniformly bounded, as are all its covariant derivatives. A bundle S has bounded geometry, if its curvature tensor, as well as all its covariant derivatives, are uniformly bounded. By [Roe88, proposition 2.4], bounded geometry can be seen by existence of nice coordinate patches, such that the Christoffel symbols comprise a bounded set in the Fréchet structure on C^∞.

For the record, let us collect all the assumptions and the conclusion into a proposition.

Proposition III.2.1. *Let D be a geometric operator (as described above) on a Clifford bundle S over a complete Riemannian manifold with bounded geometry. For any chopping function χ, the triple $(L^2(S), \rho, \chi(D))$ is a uniform Fredholm module.*

The idea of the proof (which will be made more precise afterwards) is as follows: it is proved in [Roe88, theorem 5.5], that if $\varphi \in C_0(\mathbb{R})$ satisfies $\varphi^{(k)}(t) \leq C_k(1+|t|)^{m-k}$, then $\varphi(D)$ extends to a continuous map between Sobolev spaces $W^r \to W^{r-m}$ for any r. Now a bounded piece of our manifold can be transferred to a torus. The Fourier coefficients of a W^{-k}–function on a torus decay faster than $s \mapsto \frac{1}{s^k}$. Hence the finite rank approximants to the inclusion $W^{r-m} \hookrightarrow W^r$ can be constructed just by truncating the Fourier series — and knowing the rate of decay of the coefficients tells us how big rank do we need for a given $\varepsilon > 0$ — independently on the position of our bounded piece in the manifold. Putting the facts together, $\varphi(D) : W^r \to W^{r-m} \hookrightarrow W^r$ is uniformly approximable.

Before giving the proof of the above proposition, we cite [Roe88, theorem 5.5]. We need to introduce some notation. Define (global) Sobolev spaces $W^k(S)$ as the completion of $\Gamma_c(S)$ in the norm $\|\xi\|_k = (\|s\|^2 + \|Ds\|^2 + \cdots + \|D^k s\|^2)^{1/2}$. If $L \subset M$, then $\|\xi\|_{k,L} = \inf\{\|\zeta\|_k \mid \zeta \in W^k(S), \xi = \zeta \text{ on a nbhd of } L\}$. An operator $A : W^k(S) \to W^l(S)$ is called *quasilocal*, if there exists a function $\mu : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\mu(r) \to 0$ as $r \to \infty$ and for each $K \subset M$ and each $\xi \in W^k(S)$ supported within K one has

$$\|A\xi\|_{l, M \setminus N_r(K)} \leq \mu(r)\|\xi\|_k.$$

We call μ a dominating function for A. Finally, we set $S^m(\mathbb{R})$ to be the set of functions $\varphi \in C^\infty(\mathbb{R})$, which

[1] Recall that $N_\delta(Y)$ denotes the δ–neighborhood of a set Y; and $\Gamma(S)$ denotes the set of smooth sections of S.

satisfy inequalities of the form

$$|\varphi^{(k)}(\lambda)| < C_k(1+|\lambda|)^{m-k}$$

and define the Schwartz space $\mathscr{S}(\mathbb{R}) = \bigcap S^m(\mathbb{R})$.

Theorem III.2.2 ([Roe88, theorem 5.5])**.** *Let D be a geometric operator on a Clifford bundle S over a complete manifold M with bounded geometry. If $\varphi \in S^m(\mathbb{R})$, then $\varphi(D)$ continuously extends to a quasilocal operator $W^r(S) \to W^{r-m}(S)$.*

Proof of proposition III.2.1. Fix now a function $\varphi \in S^m(\mathbb{R})$ ($m \leq -1$). We are going to show that $\varphi(D)$ is a uniform operator[2]. By the above theorem, there is a dominating function μ for $\varphi(D)$, and $\varphi(D)$ extends to a bounded operator $L^2(S) \to W^{-m}(S)$.

Fix now also $\varepsilon > 0$ and $R > 0$. Pick any open subset $U \subset M$ with $\operatorname{diam}(U) \leq R$. Consider now he restriction of $\varphi(D)$ to sections supported on U (denoted $L^2(S|_U)$). This is sufficient to obtain uniformity, since $\varphi(D)$ is selfadjoint. Since $\mu(r) \to 0$, there is $r_0 > 0$, such that $\mu(r_0) < \varepsilon/2$. Now decompose $\varphi(D)|_{L^2(S|_U)} : L^2(S|_U) \to W^{-m}(S|_{N_{r_0}(U)}) \oplus W^{-m}(S|_{M\setminus N_{r_0}(U)})$. By quasilocality, the second component has norm at most $\varepsilon/2$. It remains to prove that the restrictions of $\varphi(D)$ to $L^2(S|_U) \to W^{-m}(S|_{N_{r_0}(U)}) \hookrightarrow L^2(S|_{N_{r_0}(U)})$ are approximable by finite rank operators, such that the ranks depend only on $\varepsilon > 0$ and $R \geq \operatorname{diam}(U)$.

We can now reduce to the case of a torus with a trivial bundle. This just follows from a partition of unity argument and the existence of nice coordinate patches (from bounded geometry). Also note that for a given R, there is a uniform bound on how many of these patches are needed to cover any subset of M with diameter less than $R + 2r_0$.

On the torus T^n with the trivial bundle $E = T^n \times \mathbb{C}^n$, we can use Fourier series. Denote by $P_N : L^2(E) \to L^2(E)$ the orthogonal projection given by replacing the $\vec{q}$–Fourier coefficient ($\vec{q} \in \mathbb{Z}^n$) of a function by 0 if $|\vec{q}| > N$ (in other words, we truncate the Fourier series at N). The absolute values of the Fourier coefficients of a function in $W^{-m}(E)$ decrease at least as fast as $|\vec{q}|^m$. Consequently, the finite–rank maps $W^{-m}(E) \hookrightarrow L^2(E) \xrightarrow{P_N} L^2(E)$ approximate the inclusion $W^{-m}(E) \to L^2(E)$ in norm for $m \leq -1$. Moreover, for a given $\varepsilon > 0$, the rank of an ε–approximant depends only on ε and m. This concludes the proof of the fact that $\varphi(D)$ is uniform if $\varphi \in S^m(\mathbb{R})$ with $m \leq -1$.

The passage from $\varphi \in S^m(\mathbb{R})$, $m \leq -1$, to $\varphi \in C_0(\mathbb{R})$ is by the usual approximation argument (together with the fact that uniform operators from a C*-algebra, see III.3.2). Summarizing, for $\varphi \in C_0(\mathbb{R})$ we have that $\varphi(D)$ is a uniform operator.

Now if χ is any chopping function, then $\chi(D)^2 - 1 = (\chi^2 - 1)(D)$ and $\chi^2 - 1 \in C_0(\mathbb{R})$, hence the Fredholmness condition follows from the previous argument. Furthermore, the difference of two chopping functions is also in $C_0(\mathbb{R})$, and so we are free to choose one particular chopping function (we choose $\chi(t) = \frac{t}{\sqrt{1+t^2}}$) to prove that $\chi(D)$ is l-uniformly pseudolocal. We apply a useful formula from [Kas88, lemma 4.4]:

$$\chi(D) = \frac{2}{\pi}\int_0^\infty \frac{D}{1+\lambda^2+D^2}\,\mathrm{d}\lambda$$

[2]Note that the notion of a uniform operator from [Roe88] is different from ours.

(convergence in the strong topology), so that

$$[\rho(f),\chi(D)] = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2+D^2}\left((1+\lambda^2)[\rho(f),D]+D[\rho(f),D]D\right)\frac{1}{1+\lambda^2+D^2}\,\mathrm{d}\lambda.$$

Fix $\varepsilon > 0$ and $L > 0$. We have estimates

- $\|\frac{D}{1+\lambda^2+D^2}\| \leq \frac{1}{2\lambda}$,
- for a smooth $f \in C_{R,L}(M)$, $[\rho(f),D]$ is the multiplication operator by the derivative of f, and so we have that $\|[\rho(f),D]\| \leq L$.

Consequently, the integral in the last display converges in norm; and there exists $k > 0$ and $\lambda_1,\dots,\lambda_k$, such that the integral can be approximated within $\varepsilon > 0$ by the sum of the integrands with $\lambda = \lambda_1,\dots,\lambda_k$. Now each of the operators $\frac{D}{1+\lambda^2+D^2}$, $\frac{1}{1+\lambda^2+D^2}$ is uniform by the previous considerations: $\frac{t}{1+\lambda^2+t^2}, \frac{1}{1+\lambda^2+t^2} \in S^{-1}(\mathbb{R})$. This finishes the proof. □

We finish the section by an observation, which can be applied to obtain uniform Fredholm modules for non–geometric elliptic operators. We assume that a finitely generated discrete group Γ acts cocompactly on M (this assumption implies that M has bounded geometry), and that D commutes with this action. The vague reason for uniformity is that D "looks the same" on each translate of a fundamental domain (which is bounded), and so the approximation properties of D at any place of M are the same as those over a fixed fundamental domain. In this case, just knowing that $\varphi(D)$ is locally compact for $\varphi \in C_0(R)$ upgrades to:

Claim 1. For any $\varphi \in C_0(\mathbb{R})$, the operator $\varphi(D)$ is uniform.

Proof. For a given $R > 0$, we can find a bounded open set $U \subset M$, such that the collection $\{U\gamma\}_{\gamma\in\Gamma}$ covers M and has Lebesgue number at least R. Construct a continuous function $f : M \to [0,1]$, which is 1 on $\overline{U}$ and 0 outside a small neighborhood of U. Then for any function $g \in C_R(M)$ there is a $\gamma \in \Gamma$, such that $g \cdot f^\gamma = g$ (by f^γ we denote the translate of f by γ). Then $\rho(g)\varphi(D) = \rho(gf^\gamma)\varphi(D) = \rho(g)\rho(f^\gamma)\varphi(D^\gamma) = \rho(g)(\rho(f)\varphi(D))^\gamma$. Hence (ε,N)–approximability of $\rho(g)\varphi(D)$ is not worse than the one of $\rho(f)\varphi(D)$ (which is a compact operator, independent of g). This proves that $\varphi(D)$ is uniform. □

Pseudolocality can be now deduced in the same way as in the geometric case from the claim, provided that $\|[\rho(f),D]\|$ is bounded independently of $f \in C_{R,L}(M)$.

III.3 Dual algebras

In the analytic K-homology, one can use the Voiculescu's theorem and standard normalizing procedure to express K-homology as a K-theory of a certain C*-algebra. In this section, we first work on a fixed X-module (H,ϕ) to obtain a similar isomorphism for the "partial" uniform K-homology groups (proposition III.3.3). To work around the Voiculescu's theorem, we express the uniform K-homology as a direct limit of "partial" uniform K-homology groups (proposition III.3.9).

Definition III.3.1 (Dual algebras)**.** Let H be a Hilbert space and $\phi : C_0(X) \to \mathscr{B}(H)$ be a *-representation. We define $\Psi^0_\phi(X) \subset \mathscr{B}(H)$ to be the set of all l-uniformly pseudolocal operators in $\mathscr{B}(H)$ and $\Psi^{-1}_\phi(X) \subset \mathscr{B}(H)$ to be the set of all uniform operators. Furthermore, we denote $\mathscr{D}^u_\phi(X) = \Psi^0_{\phi\oplus 0}(X) \subset \mathscr{B}(H \oplus H)$.

We begin by easy but crucial observation.

Lemma III.3.2. *Let H be a Hilbert space and $\phi : C_0(X) \to \mathscr{B}(H)$ a *-representation. Then $\Psi^0_\phi(X) \subset \mathscr{B}(H)$ is a C*-algebra. Likewise, $\Psi^{-1}_\phi(X) \subset \Psi^0_\phi(X)$ is a C*-algebra. Furthermore, $\Psi^{-1}_\phi(X)$ is a closed two-sided ideal of $\Psi^0_\phi(X)$.*

Proof. We show that $\Psi^0_\phi(X)$ is norm-closed. Assume that $T \in \mathscr{B}(H)$ is approximable by l-uniformly pseudolocal operators. Take $\varepsilon > 0$ and $R, L \geq 0$. By assumption, there is an l-uniformly pseudolocal operator $S \in \mathscr{B}(H)$, such that $\|T - S\| < \varepsilon/4$. Let M be such that S is $(\varepsilon/2, R, L, M; \phi)$–approximable. Hence for any $f \in C_{R,L}(X)$ there exists $k \in \mathscr{B}(H)$ with $\mathrm{rank}(k) \leq M$ such that $\|[\phi(f), S] - k\| < \varepsilon/2$. Consequently, $\|[\phi(f), T] - k\| \leq \|[\phi(f), (T - S)]\| + \|[\phi(f), S] - k\| < \varepsilon$. In other words, $[\phi(f), T]$ is (ε, M)–approximable. The proof that the norm-limits of uniform operators are again uniform is analogous.

The fact that $\Psi^0_\phi(X)$ is closed under multiplication follows from the identity $[\phi(f), ST] = [\phi(f), S]T + S[\phi(f), T]$. Likewise, using the identity $\phi(f)ST = [\phi(f), S]T + S\phi(f)T$ we obtain that $\Psi^{-1}_\phi(X)$ is an ideal of $\Psi^0_\phi(X)$ (we're using remark III.1.5 here). □

For a fixed X-module (H, ϕ), define a group $K^u_*(X; \phi)$, in the similar manner as $K^u_*(X)$, except we consider only (unitary equivalence classes of) uniform Fredholm modules, whose Hilbert spaces and $C_0(X)$-actions are direct sums (finite or countably infinite) of $(H \oplus H, \phi \oplus 0)$. A glance at the proofs for $K^u_*(X)$ shows that $K^u_*(X; \phi)$ can be characterized also as a group of (unitary equivalence classes of) uniform Fredholm modules over the sums of $(H \oplus H, \phi \oplus 0)$, with homotopies also taken within this category (see III.1.14).

Fix (H, ϕ) for a time being, and let us define a homomorphism

$$\varphi_0 : K_1(\mathscr{D}^u_\phi(X)) \to K^u_0(X; \phi)$$

as follows: If $U \in \mathscr{M}_n(\mathscr{D}^u_\phi(X))$ is a unitary representing a K_1-class, we set $\varphi_0([U]) = [(H^{2n}, (\phi \oplus 0)^n, U)]$. It is immediate that $(H^{2n}, (\phi \oplus 0)^n, U)$ is a uniform Fredholm module. Since homotopies of unitaries translate into operator homotopies of Fredholm modules and the operations on K_1 and K^u_0 are both direct sums, we see that φ_0 is a group homomorphism.

Analogously, we induce a homomorphism

$$\varphi_1 : K_0(\mathscr{D}^u_\phi(X)) \to K^u_1(X; \phi)$$

by assigning to a projection $Q \in \mathscr{M}_n(\mathscr{D}^u_\phi(X))$ the triple $(H^{2n}, (\phi \oplus 0)^n, 2Q - 1)$. It is again easy to check that this triple is actually a uniform 1-Fredholm module. Since operations on K_0 and K^u_1 are both direct sums and homotopies translate to homotopies, we really do get a group homomorphism.

Proposition III.3.3 ("One X-module" picture). *The above defined maps $\varphi_*: K_{1-*}(\mathscr{D}^u_\phi(X)) \to K^u_*(X;\phi)$ are isomorphisms.*

The proof follows the usual route of showing that elements of $K^u_*(X;\phi)$ have nice representatives (cf. [HR00b, sections 8.3 and 8.4]). It is done by following three lemmas.

Lemma III.3.4. *Any element of $K^u_*(X;\phi)$ may be represented by a uniform Fredholm module of the form $(H^{2n},(\phi\oplus 0)^n,S)$, where $\|S\|\leq 1$. Furthermore, the homotopies can be also assumed to have this property.*

Proof. This is a standard cutting argument. We first deal with 0- case. Take any representative $(H^{2n},(\phi\oplus 0)^n,S)$. Consider the matrix $\tilde{S}=\left(\begin{smallmatrix} 0 & S \\ S^* & 0\end{smallmatrix}\right)$. It represents an odd selfadjoint operator in $\mathscr{B}(H^{4n})$, whose square differs from 1 by a uniform operator. Take the cutting function $c:\mathbb{R}\to\mathbb{R}$ given by

$$c(t)=\begin{cases} -1 & \text{if } t<-1 \\ t & \text{if } -1\leq t\leq 1 \\ 1 & \text{if } t>1, \end{cases}$$

By functional calculus, $c(\tilde{S})$ is again an odd selfadjoint operator (since c is odd), but with $\|c(\tilde{S})\|\leq 1$. Denote by T the upper right corner of $c(\tilde{S})$. Then $\|T\|\leq 1$, and $T-S$ is uniform. The last statement can be seen by referring to the theorem on the essential spectrum of selfadjoint operators. The proof is completed by applying Lemma III.1.17.

The 1- case is even more straightforward, since we may take a representative $(H^{2n},(\phi\oplus 0)^n,P)$ with $P=P^*$. Hence we can apply the cutting directly to P and replace it by $c(P)$.

The same procedures can be applied to whole homotopies. □

Lemma III.3.5. *Any element of $K^u_0(X;\phi)$ may be represented by a uniform 0-Fredholm module of the form $(H^{2n},(\phi\oplus 0)^n,S)$, where S is a unitary. Furthermore, the homotopies can also be assumed to have this property.*

Proof. By previous lemma, we may take a representative $(H^{2n},(\phi\oplus 0)^n,S)$, such that $\|S\|\leq 1$. For simplicity, assume $n=1$, so that $S=\left(\begin{smallmatrix} T & S_{12} \\ S_{21} & S_{22}\end{smallmatrix}\right)$, $T,S_{ij}\in\mathscr{B}(H)$. It follows that $\|T\|\leq 1$, so the operator $U=\left(\begin{smallmatrix} T & -\sqrt{1-TT^*} \\ \sqrt{1-T^*T} & T^*\end{smallmatrix}\right)$ is well defined and unitary.

Since S is l-uniformly pseudolocal, T is l-uniformly pseudolocal and for any $\varepsilon>0$, $R,L\geq 0$ there exists $M>0$, such that $\phi(f)S_{12}$ and $S_{21}\phi(f)$ are (ε,M)–approximable for all $f\in C_{R,L}(X)$. Using this and uniformity of $1-SS^*$ and $1-S^*S$, we conclude that $1-T^*T$ and $1-TT^*$ are uniform. Since $\Psi^{-1}_\phi(X)$ is a C*-algebra, so are their square roots. Consequently, $S-U$ is uniform, and another application of lemma III.1.17 finishes the proof.

Again, we can apply this procedure to the whole homotopy. □

Lemma III.3.6. *Any class in $K^u_1(X;\phi)$ can be represented by a uniform 1-Fredholm module of the form $(H^{2n},(\phi\oplus 0)^n,P)$, where $P^2=1$.*

Proof. We proceed similarly as in the previous lemma. Take a representative $(H^{2n},(\phi\oplus 0)^n,P)$, such that $P=P^*$ and $\|P\|\leq 1$. For simplicity, we assume that $n=1$, and so $P=\left(\begin{smallmatrix} Q & P_{12} \\ P_{21} & P_{22}\end{smallmatrix}\right)$, where $Q,P_{ij}\in\mathscr{B}(H)$. It follows that Q is also selfadjoint and contractive. Therefore, the operator $O=\left(\begin{smallmatrix} Q & \sqrt{1-Q^2} \\ \sqrt{1-Q^2} & -Q\end{smallmatrix}\right)$ is selfadjoint with $O^2=1$.

As in the previous proof, we obtain that $1-Q^2$ is uniform and that $P-O$ is uniform as well. This finishes the proof. □

Let us now turn to relationship between $K_*^u(X,\phi)$'s for different ϕ's. We shall need another definition (which is more general than what we need at the moment, but full generality will be required later):

Definition III.3.7. Let X and Z be spaces, let $\varphi: C_0(X)\to C_0(Z)$ be a *-homomorphism, $\phi_X: C_0(X)\to\mathscr{B}(H_X)$ and $\phi_Z: C_0(Z)\to\mathscr{B}(H_Z)$ be *-representations. We say that an isometry $V: H_Z\to H_X$ *uniformly covers* φ, if for every $\varepsilon>0$, $R,L\geq 0$ there exists $M\geq 0$, such that $V^*\phi_X(f)V-\phi_Z(\varphi(f))$ is (ε,M)–approximable for every $f\in C_{R,L}(X)$. In short, $V^*\phi_X(\cdot)V\sim_{lua}\phi_Z(\varphi(\cdot))$.

We specialize the definition as follows: assume that we are given two *-representations ϕ and ϕ' of $C_0(X)$ on Hilbert spaces H and H'. If there exists an isometry $V: H'\to H$ which uniformly covers the identity map $\mathrm{id}: C_0(X)\to C_0(X)$, then we obtain a homomorphism

$$i_V: K_*^u(X,\phi')\to K_*^u(X,\phi)$$

using Proposition III.3.3 and the following claim:

Claim 2. $\mathrm{Ad}(V)$ maps $\Psi^0_{\phi'}(X)$ into $\Psi^0_{\phi}(X)$ (the adjoint map Ad is defined as $\mathrm{Ad}(V)(T)=VTV^*$, and it's a *-homomorphism since V is unitary).

This claim is actually a special case of Claim 3 from the next section (applied to $Z=X$ and $\pi=\mathrm{id}$).

We now introduce a relation $\prec$ on the set $\mathscr{X}$ of (unitary equivalence classes of) *-representations ϕ of $C_0(X)$ on some (separable) Hilbert space, which turns it into a directed system. We define the relation $\prec$ by declaring that $(H,\phi)\prec(E,\rho)$ (or just $\phi\prec\rho$) iff there exists an isometry $V_{\phi,\rho}: H\to E$ which uniformly covers the identity map $\mathrm{id}: C_0(X)\to C_0(X)$. The reflexivity of $\prec$ is obvious and the transitivity becomes clear after a momentary reflection on the definition of uniform covering. Furthermore, for $\phi,\rho\in\mathscr{X}$, we easily see that $\phi\prec\phi\oplus\rho$ and $\rho\prec\phi\oplus\rho$.

The set of $K_*^u(X,\phi)$'s, together with the maps $i_{V_{\phi,\rho}}$ becomes now a directed system indexed by $\mathscr{X}$. The next lemma ensures that we may arbitrarily choose (and fix that choice of) an isometry V_ϕ for each ϕ.

Lemma III.3.8. *We adopt the notation from the definition III.3.7. If two isometries $V_1,V_2: H_Z\to H_X$ uniformly cover φ, then the induced maps on K-theory are the same:*

$$(\mathrm{Ad}(V_1))_*=(\mathrm{Ad}(V_2))_*: K_*(\Psi^0_{\phi_Z}(Z))\to K_*(\Psi^0_{\phi_X}(X)).$$

(Note that by the proof of Claim 3, $\mathrm{Ad}(V_i)$*'s really map* $\Psi^0_{\phi_Z}(Z)$ *into* $\Psi^0_{\phi_X}(X)$*.)*

This lemma is analogous to the second part of [HR00b, Lemma 5.2.4], and the proof carries over verbatim. This lemma also implies that $\prec$ becomes antisymmetric when it descends to $K_*^u(X,\phi)$'s.

For each ϕ there is an obvious homomorphism $j_\phi : K_*^u(X;\phi) \to K_*^u(X)$. It is also clear that j_ϕ's commute with $i_{V_{\phi,\rho}}$'s, which allows us to state the final proposition of this section:

Proposition III.3.9 (Direct limit version)**.** *With the notation above,*

$$K_*^u(X) = \lim_{\phi \in \mathscr{X}} j_\phi(K_*^u(X,\phi)).$$

III.4 Mayer–Vietoris sequence

The goal of this section is to prove the Mayer–Vietoris sequence for uniform K-homology groups:

Proposition III.4.1 (Mayer–Vietoris sequence)**.** *Let $A,B \subset X$ be closed subsets of X, such that $A \cup B = X$,* $\mathrm{int}(A \cap B) \neq \emptyset$ *and $d(A \setminus B, B \setminus A) > 0$.*[3] *Then there is a 6-term exact sequence*

$$\begin{tikzcd}
K_0^u(A\cap B) \arrow[r] & K_0^u(A) \oplus K_0^u(B) \arrow[r] & K_0^u(X) \arrow[d] \\
K_1^u(X) \arrow[u] & K_1^u(A) \oplus K_1^u(B) \arrow[l] & K_1^u(X). \arrow[l]
\end{tikzcd}$$

Before outlining the proof, we need a definition:

Definition III.4.2. Given a Hilbert space H and a *-representation $\phi : C_0(X) \to \mathscr{B}(H)$, we let $\Psi_\phi^0(X,Z) \subset \Psi_\phi^0(X)$ to be the set of all operators $T \in \Psi_\phi^0(X)$ which are uniform on $X \setminus Z$, that is, such that for every $\varepsilon > 0, R \geq 0$, there exists $M > 0$, such that for every $f \in C_R(X)$ with $f|_Z = 0$ we have that $\phi(f)T$ and $T\phi(f)$ are (ε, M)–approximable. Also, we set $\mathscr{D}_\phi^u(X,Z) = \Psi_{\phi\oplus 0}^0(X,Z) \subset \mathscr{B}(H \oplus H)$.

Note that a proof similar to the proof of Lemma III.3.2 yields that $\Psi_\phi^0(X,Z)$ is a closed two-sided ideal of $\Psi_\phi^0(X)$.

Proof of III.4.1. The strategy is to first use the C*-algebra Mayer–Vietoris sequence (with ϕ fixed), and then apply Propositions III.3.3, III.3.9 and Excision Lemma III.4.3 to obtain the result.

Keeping the notation from III.4.1, we have that $\mathscr{D}_\phi^u(X,A) \cap \mathscr{D}_\phi^u(X,B) = \mathscr{D}_\phi^u(X,A\cap B)$ directly from the definitions, and $\mathscr{D}_\phi^u(X,A) + \mathscr{D}_\phi^u(X,B) = \mathscr{D}_\phi^u(X)$ (by a partition of unity argument[4]). Subsequently, from the C*-algebra Mayer–Vietoris sequence, we get that

$$\begin{tikzcd}
K_0(\mathscr{D}_\phi^u(X,A\cap B)) \arrow[r] & K_0(\mathscr{D}_\phi^u(X,A)) \oplus K_0(\mathscr{D}_\phi^u(X,B)) \arrow[r] & K_0(\mathscr{D}_\phi^u(X)) \arrow[d] \\
K_1(\mathscr{D}_\phi^u(X)) \arrow[u] & K_1(\mathscr{D}_\phi^u(X,A)) \oplus K_1(\mathscr{D}_\phi^u(X,B)) \arrow[l] & K_1(\mathscr{D}_\phi^u(X,A\cap B)) \arrow[l]
\end{tikzcd}$$

[3]This last condition just expresses the requirement that "the overlap of A and B does not get arbitrarily thin". It is used only in the next footnote.

[4]Take $f,g \in C_b(X)$ with $f+g=1$, $f|_{X\setminus A} = 0$ and $g|_{X\setminus B} = 0$, f,g are L-continuous for some L (this is possible since $d(A\setminus B, B\setminus A) > 0$). Write $T = T\phi(f) + T\phi(g)$. Now if $h|_A = 0$, then $T\phi(f)\phi(h) = 0$ and $\phi(h)T\phi(f) = [\phi(h),T]\phi(f) + T\phi(h)\phi(f)$.

is exact.

The general Mayer–Vietoris sequence now follows by "taking the direct limit", i.e. using naturality of our constructions, Proposition III.3.9 and Excision Lemma III.4.3. $\square$

It remains to deal with the excision lemma. For the rest of this section, we shall denote by X a proper metric space, and by $Z \subseteq X$ a closed subset of X.

Lemma III.4.3 (Excision lemma)**.** *There is a natural isomorphism*

$$\lim_{\phi} K_*(\mathscr{D}^u_\phi(X,Z)) \cong \lim_{\phi_Z} K_*(\mathscr{D}^u_{\phi_Z}(Z)).$$

By virtue of III.3.9, we may say that the "relative uniform K-homology" $K^u_*(X,Z)$ is isomorphic to $K^u_*(Z)$.

Proof. The strategy is obtain a commutative diagram (notation will be introduced in the course of the proof)

$$\begin{tikzcd}
& K_*(\Psi^0_{\phi_X}(X,Z)) \arrow[rr, "Ad(SW)"] \arrow[dr, "\mathrm{Ad}(W)"] & & K_*(\Psi^0_{\phi'_X}(X,Z)) \arrow[r] \arrow[dr] & {} \\
K_*(\Psi^0_{\phi_Z}(Z)) \arrow[ur, "\mathrm{Ad}(V)"] \arrow[rr, "\mathrm{Ad}(WV)"] & & K_*(\Psi^0_{\phi'_Z}(Z)) \arrow[ur, "\mathrm{Ad}(S)"] \arrow[rr] & & {}
\end{tikzcd} \tag{III.1}$$

starting with the following data: a representation $\phi_X : C_0(X) \to \mathscr{B}(H_X)$, a representation $\phi_Z : C_0(Z) \to \mathscr{B}(H_Z)$ and an isometry $V : H_Z \to H_X$, which uniformly covers π (this gives the first $\nearrow$ in (III.1)). In the diagram, the horizontal arrows shall uniformly cover the identity (on the level of K-theory), and the diagonals heading up will uniformly cover π. This would establish the lemma.

Let us now explain how can we arrange the situation from the previous paragraph. If we start with a $*$-representation $\phi_X : C_0(X) \to \mathscr{B}(H_X)$, it induces a Borel measure on X, and extends to a $*$-representation (also denoted by ϕ_X) of $\ell^\infty(X)$. In particular, we may restrict ϕ_X to a representation $\phi_Z : C_0(Z) \to \mathscr{B}(H_X)$ and let $H_Z = \chi_Z H_X$. Then the inclusion $V : H_Z \hookrightarrow H_X$ actually exactly covers π, i.e. $V^*\phi_X(f)V = \phi_Z(\pi(f)) = \phi(\chi_Z f)$ for all $f \in C_0(X)$.

Conversely, starting with a $*$-representation $\phi_Z : C_0(Z) \to \mathscr{B}(H_Z)$, we obtain a $*$-representation $\phi_X = \phi_Z \circ \pi$ of $C_0(X)$, so that we can put $H_X = H_Z$ and $V = \mathrm{id}$.

The rest of the proof is devoted to obtaining a diagram (III.1) from given ϕ_X, ϕ_Z and V uniformly covering π. We accomplish our goal by a series of claims, very much like [HR00b, proof of 3.5.7].

Claim 3. $\mathrm{Ad}(V)(\Psi^0_{\phi_Z}(Z)) \subset \Psi^0_{\phi_X}(X,Z)$.

Proof. We first show that $VV^* \in \Psi^0_{\phi_X}(X,Z)$. Decompose $H_X = VV^*H_X \oplus (1-VV^*)H_X$. With respect to this decomposition $VV^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and we denote $\phi_X = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}$. The fact that VV^* is ϕ_X–uniformly pseudolocal is equivalent to

$$\phi_{12}(\cdot) \text{ and } \phi_{21}(\cdot) \text{ are l-uniformly approximable.}$$

Using the covering assumption,

$$\phi_{11}(f^*f) = VV^*\phi_X(f^*f)VV^* \sim_{lua} V\phi_Z(\pi(f^*f))V^* = V\phi_Z(\pi(f))^*\phi_Z(\pi(f))V^* \sim_{lua}$$
$$\sim_{lua} VV^*\phi_X(f)^*VV^*\phi_X(f)VV^* = \phi_{11}(f)^*\phi_{11}(f).$$

Since ϕ_X is a *-homomorphism, we have

$$\phi_{21}(f)^*\phi_{21}(f) = \phi_{11}(f^*f) - \phi_{11}(f)^*\phi_{11}(f) \tag{III.2}$$

for each $f \in C_0(X)$. In other words, $\phi_{21}(\cdot)^*\phi_{21}(\cdot)$ is l-uniformly approximable. Using the spectral theorem for compact selfadjoint operators[5], also $\sqrt{\phi_{21}(\cdot)^*\phi_{21}(\cdot)} = |\phi_{21}(f)|$ is l-uniformly approximable. Let $\phi_{21}(f) = u(f)|\phi_{21}(f)|$ denote the polar decomposition. From this formula, it follows that $\phi_{21}(f)$ is l-uniformly approximable as well.

To show that VV^* is uniform on $X \setminus Z$, it suffices to observe that in addition to $\phi_{12}(\cdot)$ and $\phi_{21}(\cdot)$ being l-uniformly approximable, we also have $\phi_{11}(f) = VV^*\phi_X(f)VV^* \sim_{lua} V\phi_Z(\pi(f))V^* = 0$ for $f \in C_0(X \setminus Z)$.

We have shown that $VV^* \in \Psi^0_{\phi_X}(X,Z)$. From this, we easily get that $\mathrm{Ad}(V)$ maps $\Psi^0_{\phi_Z}(Z)$ into $\Psi^*_{\phi_X}(X,Z)$. □

Let $\sigma : C_0(Z) \to C_0(X)$ be a completely positive lift of π that satisfies

- if $f \in C_R(X)$ then $\mathrm{supp}(\sigma(\pi(f))) \subset \{x \in X \mid d(x, \mathrm{supp}(f)) \leq 1\}$,
- there exists L', such that if f is L-continuous then $\sigma(f)$ is $L + L'$-continuous.

In particular, if $g \in C_R(Z)$ then $\sigma(g) \in C_{R+2}(X)$. Such a lift exists.[6] Now $\phi_X\sigma : C_0(Z) \to \mathscr{B}(H_X)$ is a completely positive map, so by the Stinespring's theorem, there exist a Hilbert space H and maps $\rho_{12}, \rho_{21}, \rho_{22}$ such that

$$\phi'_Z = \begin{pmatrix} \phi_X\sigma & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} : C_0(Z) \to \mathscr{B}(H_X \oplus H)$$

is a *-homomorphism. Denote by $W : H_X \to H_X \oplus H$ the obvious inclusion.

Claim 4. The *-homomorphism $\mathrm{Ad}(W)$ maps $\Psi^0_{\phi_X}(X,Z)$ into $\Psi^0_{\phi'_Z}(Z)$. Furthermore WV uniformly covers $\mathrm{id} : C_0(Z) \to C_0(Z)$. In other words, $W^*V^*\phi'_Z(\cdot)VW - \phi_Z(\cdot)$ is l-uniformly approximable on $C_0(Z)$.

Proof. Decomposing into matrices shows that $\mathrm{Ad}(W)(T)$ belongs to $\Psi^0_{\phi'_Z}(Z)$ if and only if $T\rho_{12}(\cdot)$ and $\rho_{21}(\cdot)T$ are l-uniformly approximable. Since ϕ'_Z is a *-homomorphism, we have $\rho_{21}(f)^*\rho_{21}(f) \in \phi_X(C_0(X \setminus Z))$ for all $f \in C_0(Z)$, cf. (III.2). It follows that $\rho^*_{21}(f)\rho_{21}(f)T$ is l-uniformly approximable. Consequently, $T^*\rho^*_{21}(f)\rho_{21}(f)T = (\rho_{21}(f)T)^*(\rho_{21}(f)T)$ is l-uniformly approximable as well, and it follows by the argument in the proof of claim 3 that $\rho_{21}(f)T$ itself is as well. This finishes the first part.

[5] If $k \in \mathscr{K}$ is selfadjoint, then for $\varepsilon > 0$ we can approximate k by a rank-M operator, where M is the sum of dimensions of eigenspaces corresponding to all eigenvalues λ with $|\lambda| > \varepsilon$.

[6] Note that a positive map between commutative C*-algebras is automatically completely positive, and a nice positive linear lift can be constructed using a linear basis and the Urysohn lemma–type construction. The L-continuity can be arranged.

To see that WV uniformly covers id on $C_0(Z)$, just observe that for $f \in C_0(Z)$, we have

$$V^*W^*\phi_Z'(f)WV = V^*\phi_X(\sigma(f))V \sim_{lua} \phi_Z(\pi(\sigma(f))) = \phi_Z(f)$$

by the assumption of V. □

The next step is to consider the Hilbert space $H_X' = H_X \oplus (H_X \oplus H)$ with the *-representation $\phi_X' = \phi_X \oplus \phi_Z'\pi$ of $C_0(X)$. Denote by $S : H_X \oplus H \to H_X'$ the inclusion (H_X is included as the second H_X summand).

Claim 5. *S uniformly covers π. $\mathrm{Ad}(SW)$ is homotopic to a *-homomorphism which uniformly covers $\mathrm{id} : C_0(X) \to C_0(X)$. Hence we are in the position to iterate the construction we've done so far to obtain a commutative diagram (III.1).*

Proof. In fact, S actually exactly covers π, since $S^*\phi_X'S = \phi_Z'\pi$. Continuing with the second part of the claim, note that SW includes H_X into $H_X \oplus H_X \oplus H$ as the second copy of H_X. If we denote by $Y : H_X \to H_X \oplus H_X \oplus H$ the inclusion as the first summand, then $\mathrm{Ad}(Y)$ exactly covers $\mathrm{id} : C_0(X) \to C_0(X)$. Furthermore, $\mathrm{Ad}(SW)$ and $\mathrm{Ad}(Y)$ are homotopic via the homotopy of *-homomorphisms

$$A_t : T \mapsto \begin{pmatrix} \sin^2(\frac{\pi}{2}t)T & \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)T & 0 \\ \sin(\frac{\pi}{2}t)\cos(\frac{\pi}{2}t)T & \cos^2(\frac{\pi}{2}t)T & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t \in [0,1].$$

It remains to verify that A_t maps $\Psi^0_{\phi_X}(X,Z)$ into $\Psi^0_{\phi_X'}(X,Z)$. To this end, it is enough to observe that if $T \in \Psi^0_{\phi_X}(X,Z)$, then $\tilde{T} = \left(\begin{smallmatrix} T & T & 0 \\ T & T & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathscr{B}(H_X \oplus H_X \oplus H)$ is ϕ_X'–l-uniformly pseudolocal and uniform on $C_0(X \setminus Z)$. For $f \in C_0(X)$, we compute

$$[\tilde{T}, \phi_X'(f)] = \begin{pmatrix} T\phi_X(f) - \phi_X(f)T & T\phi_X\sigma\pi(f) - \phi_X(f)T & 0 \\ T\phi_X(f) - \phi_X\sigma\pi(f)T & T\phi_X\sigma\pi(f) - \phi_X\sigma\pi(f)T & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that for showing l-pseudolocality of $\tilde{T}$ it suffices to see that $T\phi_X(f) - \phi_X\sigma\pi(f)T = [T, \phi_X(f)] + (\phi_X(f - \sigma\pi(f)))T$ is l-uniformly approximable. But $f - \sigma\pi(f) \in C_0(X \setminus Z)$, hence the assertion follows from the assumptions on T and the lift σ.

Similarly $\tilde{T}\phi_X'(f) = \left(\begin{smallmatrix} T\phi_X(f) & T\phi_X\sigma\pi(f) & 0 \\ T\phi_X(f) & T\phi_X\sigma\pi(f) & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and the uniformness of $\tilde{T}$ on $C_0(X \setminus Z)$ follows from the observation that $\pi(f) = 0$ for $f \in C_0(X \setminus Z)$. □

This finishes the proof of Lemma III.4.3. □

III.5 Finite propagation representatives

In this section, we prove that any class in a uniform K-homology group to have can be represented by a uniform Fredholm module with the operator having finite propagation. The proof follows the outline of the proof of analogous result in analytic K-homology.

Definition III.5.1. An open cover of X is said to

- have *finite multiplicity*, if for any $R \geq 0$ there is $K \geq 0$, such that any ball with radius R intersects at most K elements of the cover;
- be *uniformly bounded*, if there is a common upper bound for all the diameters of members of the cover.

Remark III.5.2. Any space X with bounded geometry admits uniformly bounded covers with finite multiplicity. However, bounded geometry alone produces such covers with possibly large bound on the diameters of the cover members. Consequently, a priori the propagation might not be made arbitrarily small (see the proof the next proposition). In order to achieve small propagation, we need some small scale (topological) assumption; for instance finite covering dimension would suffice.

Proposition III.5.3 (Uniform K-homology elements have representatives with finite propagation)**.** *Each uniform K-homology element over a space X with bounded geometry can be represented by a uniform Fredholm module (H,ϕ,S), where S is a finite propagation operator. The propagation of S can be made arbitrarily small.*

Furthermore, we may assume that homotopies go through finite propagation operators as well.

Proof. Let (H,ϕ,T) be a uniform Fredholm module. Let $(U_i)_{i\in I}$ be a uniformly bounded open cover with finite multiplicity, and let $(\varphi_i^2)_{i\in I}$ be a continuous partition of unity subordinate to $(U_i)_{i\in I}$. By replacing the sets U_i by $N_\delta(U_i)$, the δ-neighborhoods for a fixed $\delta > 0$ and obtaining a partition of unity for the cover $(N_\delta(U_i))_i$, we can assume that all φ_i's are L_0-continuous for some $L_0 \geq 0$.

Denote $S = \sum_{i\in I} \varphi_i T \varphi_i$. This operator has finite propagation (which is bounded from above by $\sup_i \operatorname{diam}(U_i)$). We prove that (H,ϕ,S) is a uniform Fredholm module which represents the same uniform K-homology element as (H,ϕ,T).

Fix $\varepsilon > 0$ and $R,L > 0$. Let M be such that $[T,\phi(\cdot)]$ is $(\varepsilon,R,2\max(L_0,L),M;\phi)$–approximable and that $T\phi(\cdot)$ and $\phi(\cdot)T$ are $(\varepsilon,R,M;\phi)$–approximable. Denote $S' = S - T = \sum_{i\in I} \varphi_i[T,\varphi_i]$. By finite multiplicity assumption, there is M_1, such that any ball with radius R intersects at most M_1 sets U_i. Take $f \in C_R(X)$ and consider $fS' = \sum_i f\varphi_i[T,\varphi_i]$. This sum has at most M_1 nonzero terms, and each of them is (ε,M)–approximable, hence fS' itself is $(M_1\varepsilon, MM_1)$–approximable. Similarly for $f \in C_{R,L}(X)$,

$$\begin{aligned} S'f &= \textstyle\sum_i \varphi_i[T,\varphi_i]f = \sum_i \varphi_i T \varphi_i f - \varphi_i^2 T f = \sum_i(\varphi_i T \varphi_i f - \varphi_i^2 f T) + \sum_i \varphi_i^2 [f,T] = \\ &= \textstyle\sum_i \varphi_i [T, f\varphi_i] + [f,T]. \end{aligned}$$

The last term is (ε,M)–approximable by assumption, and again only at most M_1 terms in the sum are nonzero, and all of them are (ε,M)–approximable. Consequently, $S'f$ is $(M_1+1)\varepsilon, MM_1+1)$–approximable. Therefore we have proved that S' is uniform. Applying Lemma III.1.17 finishes the first part of the proof.

For the part on homotopies, we just need to observe that the formula $\sum_{i\in I} \varphi_i T \varphi_i$ produces a continuous family if we vary T continuously, thanks to finite multiplicity of the chosen cover. □

III.6 Another picture of uniform Roe C*-algebras

Definition III.6.1. We denote by $C_k^*(Y)$ the norm-closure of the algebra of all locally compact finite propagation operators $T=(t_{yx})$ with uniformly bounded coefficients in $\mathscr{B}(\ell^2(Y\times\mathbb{N}))$, which satisfy the additional condition that the set $\{t_{yx}\mid x,y\in Y\}\subset\mathscr{K}(\ell^2(\mathbb{N}))$ is compact in the norm topology on $\mathscr{K}(\ell^2(\mathbb{N}))$.

Remark III.6.2. The additional condition in the previous definition merely says that up to ε, we have only finitely many entries t_{yx}.

Another way of stating this condition is that for each $\varepsilon>0$ there exists $M\geq 0$, such that each t_{xy}, $x,y\in Y$, is at distance at most ε from a rank-M operator.

We now cite a proposition, which provides an estimate on the norm of an operator in terms of its entries:

Proposition III.6.3 (see [Roe03])**.** *Let Y be a uniformly discrete space with bounded geometry, and let $t=(t_{yz})_{y,z\in Y}$ be a matrix with entries $t_{yz}\in\mathscr{K}(H)$ [or $t_{yz}\in\mathbb{C}$]. For every $P>0$ there is $C>0$, such that if t has propagation at most P, we have $\|t\|\leq C\sup_{y,z}\|t_{yz}\|$, with the operator norm in $\mathscr{B}(\ell^2Y\otimes H)$ [or $\mathscr{B}(\ell^2Y)$ respectively].*

We show that as far as K-theory of uniform Roe algebras is concerned, we may work with $C_k^*(Y)$.

Lemma III.6.4. *If Y be a uniformly discrete metric space with bounded geometry, then $C_k^*(Y)\cong C_u^*Y\otimes\mathscr{K}$.*

Proof. We show that $C_u^*Y\otimes\mathscr{K}(\ell^2(\mathbb{N}))$ is dense in $C_k^*(Y)$ (with the obvious inclusion). Pick $T=(t_{yx})\in C_k^*(Y)$ and $\varepsilon>0$. Denote the propagation of T by p. By proposition III.6.3, there is a constant $C>0$, such that if $S=(s_{yx})$ is a matrix of compacts with propagation at most p, then $\|S\|\leq C\sup_{x,y\in X}\|s_{yx}\|$. Since $\{t_{yx}\mid x,y\in Y\}$ is compact, there is an ε/C-net $t_1,\dots,t_m$ in it. Then clearly T is ε-far from an operator of the form $T_1\otimes t_1+\cdots+T_m\otimes t_m$, where each $T_i\in C_u^*Y$. This shows the density, which implies that $C_k^*(Y)$ and $C_u^*Y\otimes\mathscr{K}(\ell^2(\mathbb{N}))$ are actually isomorphic. □

The definition of $C_k^*(Y)$ as given above uses the standard basis of the auxiliary Hilbert space $\ell^2\mathbb{N}$. In what follows, we develop a usable picture of $C_k^*(Y)$ starting with a general X-module (H,ϕ), instead of the concrete one $(\ell^2Y\otimes\ell^2\mathbb{N},\text{multiplication action})$. Furthermore, this model allows us to translate from "continuous" spaces X (which are needed in order to observe more than just 0–dimensional phenomena in (uniform) K-homology) to their discrete models $Y\subset X$ (which is supposed to be the target if the index/assembly map).

Let us fix a metric space X for the rest of this section.

Definition III.6.5 (Quasi-lattices, partitions)**.** We say that $Y\subset X$ is a *quasi–lattice*, if Y with induced metric is uniformly discrete space with bounded geometry, which is coarsely equivalent to X.

We say that a collection $(V_y)_{y\in Y}$ of subsets of X is a *quasi–latticing partition*, if each V_y is open, $V_x\cap V_y=\emptyset$ if $x\neq y$, $X=\bigcup_{y\in Y}\overline{V_y}$, $\sup_{y\in Y}\operatorname{diam}(V_y)<\infty$ and for every $\varepsilon>0$, $\sup_{y\in Y}\#\{z\in Y\mid V_z\cap\mathrm{Nbhd}_\varepsilon(V_y)\neq\emptyset\}<\infty$.

Remark III.6.6. Not all spaces X have a quasi–lattice, but those with "bounded geometry" in any reasonable sense do. Furthermore, once there is a quasi–lattice, it's easy to produce quasi–latticing partitions (for instance by means of "pick the closest point in Y" map).

Example III.6.7. A useful example to have in mind is the one of a graph X (with edges attached), with Y being its 0-skeleton. More generally, 0-skeleton of a uniformly locally finite simplicial polyhedron (endowed with a geodesic metric) is a quasi–lattice.

Recall that any *-homomorphism $\phi : C_0(X) \to \mathscr{B}(H)$ induces a Borel measure on X, and extends to a representation (also denoted by ϕ) of $\ell^\infty(X)$. We shall use this fact without mentioning explicitly throughout this section.

Definition III.6.8 (Bases choice)**.** Given a metric space X, we define the *bases choice* $\mathscr{A}$ for X to be a 5-tuple $(Y,(V_y)_{y\in Y},H,\phi,\{\mathscr{S}_y\}_{y\in Y})$, where

- $Y \subset X$ is a quasi–lattice of X
- $(V_y)_{y\in Y}$ is a quasi–latticing partition of X
- H is a Hilbert space, $\phi : C_0(X) \to \mathscr{B}(H)$ a non-degenerate $*$-representation,
- $\mathscr{S}_y = (e_i^y)_{i=1}^{N_y}$ is a basis of $H_y = \phi(\chi_{V_y})H$ (where we allow $N_y \in \mathbb{N}\cup\{\infty\}$ and we put by convention that $\mathscr{S}_y = \emptyset$ if $H_y = \{0\}$).

Such a bases choice determines a (possibly non-surjective) isometry $u_{\mathscr{A}} : H = \oplus_y H_y \to \ell^2(Y\times\mathbb{N})$.

Definition III.6.9 (Realizations of $\mathscr{M}_n(C_u^*Y\otimes\mathscr{K})$)**.** Let X be a metric space, $Y \subset X$ a quasi–lattice, $(V_y)_{y\in Y}$ a quasi–latticing partition, and let $\mathscr{A}_i = (Y,(V_y)_{y\in Y},H_i,\phi_i,\{\mathscr{S}_y^i\}_{y\in Y})$, $i = 1,\dots,k$ be bases choices. Define the C*-algebra $C_k^*(X,\mathscr{A}_1,\dots,\mathscr{A}_k) \subset \mathscr{B}(\oplus_{i=1}^k H_i)$ as the closure of the algebra of the operators $T \in \mathscr{B}(\oplus_{i=1}^k H_i)$ satisfying the following conditions:

- T has finite propagation,
- there exists $M \geq 0$, such that each "entry" $T_{j,i;y,x} : \phi_i(\chi_{V_x})H_i : \phi_j(\chi_{V_y})H_j$ only uses the first M basis vectors from bases $\mathscr{S}_x^i$, $\mathscr{S}_y^j$.

There is an injective *-homomorphism

$$\mathrm{Ad}(u_{\mathscr{A}_1}\oplus\cdots\oplus u_{\mathscr{A}_k}) : C_k^*(X,\mathscr{A}_1,\dots,\mathscr{A}_k) \to \mathscr{M}_k(C_k^*(Y)).$$

We call the C*-algebra $C_k^*(X,\mathscr{A}) \subset \mathscr{B}(H)$ the *$\mathscr{A}$-realization of $C_k^*(Y)$*.

Remark III.6.10. Note that $C_k^*(X,\mathscr{A})$ is isomorphic only to a subalgebra of $C_k^*(Y)$ in general, but if each $\mathscr{S}_y$ is infinite, then $C_k^*(Y)$ and $C_k^*(X,\mathscr{A})$ are isomorphic.

Define $\mathrm{supp}(\mathscr{A}) = \{y \in Y \mid \mathscr{S}_y \neq \emptyset\}$. If $\mathrm{supp}(\mathscr{A})$ is coarsely equivalent to Y, we have that $K_*(C_k^*(Y)) \cong K_*(C_k^*(X,\mathscr{A}))$. More precisely, $C_k^*(Y)$ and $C_k^*(X,\mathscr{A})$ are Morita equivalent. Indeed, $\mathscr{M}_\infty(C_k^*(X,\mathscr{A})) \cong \mathscr{M}_\infty(C_u^*(\mathrm{supp}(\mathscr{A})))$, for Morita equivalence of $C_u^*(\mathrm{supp}(\mathscr{A}))$ and C_u^*Y we refer to [BNW07].

We continue by defining a relation between tuples of bases choices, in order to be able to get an inductive limit of realizations of $C_k^*(Y)$. We begin by a notion similar to an inclusion between a pair of bases choices.

Definition III.6.11. Fix a quasi–lattice $Y \subset X$. Let $\mathscr{A}_i = \big(Y, (V^i_y)_{y\in Y}, H_i, \phi_i, \{\mathscr{S}^i_y\}\big)$, $i = 1,2$, be bases choices. We shall write $\mathscr{A}_1 \subseteq \mathscr{A}_2$, if the following conditions are satisfied:

- For each $y \in Y$, $\phi(\chi_{V^1_y})H_1$ is isometric to a subspace of $\phi(\chi_{V^2_y})H_2$ via an isometry v_y.
- each v_y maps n-th vector in the basis $\mathscr{S}^1_y$ to the n-vector in the basis $\mathscr{S}^2_y$.

A weaker version of $\subseteq$, denoted now by $\mathscr{A}_1 \sqsubseteq \mathscr{A}_2$, is defined in the same manner, except the last condition is replaced by

- for all $k \in \mathbb{N}$ there is $l \in \mathbb{N}$, such that for all $y \in Y$ the v_y-images of the first k vectors of $\mathscr{S}^1_y$ are among the linear span of the first l vectors of $\mathscr{S}^2_y$.

We now extend this inclusion to lists. Given two lists of bases choices $(\mathscr{A}_1,\dots,\mathscr{A}_k)$ and $(\mathscr{A}'_1,\dots,\mathscr{A}'_l)$ for X with respect to Y, we shall write $(\mathscr{A}'_1,\dots,\mathscr{A}'_l) \prec (\mathscr{A}_1,\dots,\mathscr{A}_k)$, if there is an injective function $\sigma : \{1,\dots,l\} \to \{1,\dots,k\}$, such that $\mathscr{A}'_i \subseteq \mathscr{A}_{\sigma(i)}$ for all $i = 1,\dots,l$. If this happens, then there is a natural embedding $i : C^*_k(X,\mathscr{A}'_1,\dots,\mathscr{A}'_l) \to C^*_k(X,\mathscr{A}_1,\dots,\mathscr{A}_k)$ (implemented by the *-homomorphism $\mathrm{Ad}(V)$, where $V = \oplus_y v_y$ is the isometric embedding of appropriate Hilbert spaces). This embedding commutes with maps between matrix algebras over $C^*_k(Y)$ as follows:

$$\begin{tikzcd} C^*_k(X,\mathscr{A}'_1,\dots,\mathscr{A}'_l) \arrow[r] \arrow[d, "\mathrm{Ad}(u_{\mathscr{A}'_1}\oplus\cdots\oplus u_{\mathscr{A}'_l})"'] & C^*_k(X,\mathscr{A}_1,\dots,\mathscr{A}_k) \arrow[d, "\mathrm{Ad}(u_{\mathscr{A}_1}\oplus\cdots\oplus u_{\mathscr{A}_k})"] \\ \mathscr{M}_l(C^*_k(Y)) \arrow[r, "h_\sigma\otimes\mathrm{id}"] & \mathscr{M}_k(C^*_k(Y)). \end{tikzcd} \tag{III.3}$$

By $h_\sigma : \mathscr{M}_l(\mathbb{C}) \to \mathscr{M}_k(\mathbb{C})$ we denote the embedding of matrix algebras determined by σ. More precisely, h_σ is the linear extension of the following assignment of matrix units $\mathscr{M}_l(\mathbb{C}) \ni e_{ij} \mapsto e_{\sigma(i)\sigma(j)} \in \mathscr{M}_k(\mathbb{C})$.

Furthermore, if we assume that $\mathscr{A}'_i = \mathscr{A}_{\sigma(i)}$ for $i = 1,\dots,l$, and and if $\mathrm{supp}(\mathscr{A}_j)$ is coarsely equivalent to Y for each $j = 1,\dots,k$, then the top horizontal map induces an isomorphism on K-theory. This is a straightforward generalization of remark III.6.10.

Note that for any bases choice $\mathscr{A} = (Y,(V_y)_{y\in Y},H,\phi,\{\mathscr{S}_y\}_{y\in Y})$, there is another one $\mathscr{A}'$ with $\mathscr{A} \subseteq \mathscr{A}'$, such that $\mathrm{supp}(\mathscr{A}') = Y$. This can be arranged by choosing the Hilbert space of $\mathscr{A}'$ to be $H' = H \oplus \ell^2(Y\times\mathbb{N})$, the direct sum action of $C_0(X)$ and a suitable choice of bases $\mathscr{S}'_y$.

The previous discussion, together with lemma III.6.4, culminates in the following lemma:

Lemma III.6.12 (A picture for $K_*(C^*_uY)$)**.** *Let X be metric space and let $Y \subset X$ be a quasi–lattice. The collection $\mathscr{X}$ of all finite lists $(\mathscr{A}_1,\dots,\mathscr{A}_k)$ of bases choices for X with Y fixed forms a directed system. We have that there is an isomorphism η*

$$\eta : \lim_{\mathscr{X}} K_*(C^*_k(X,\mathscr{A}_1,\dots,\mathscr{A}_k)) \xrightarrow{\cong} K_*(C^*_uY).$$

The following lemma shows that given a finite propagation uniform operator T on an X-module H, we can always find a bases choice $\mathscr{A}$, such that $T \in C^*_k(X,\mathscr{A})$.

Lemma III.6.13. *Let X be metric space, let $Y \subset X$ be a quasi–lattice and let $(V_y)_{y \in Y}$ be a quasi–latticing partition of X. Let H be a Hilbert space and let $\phi : C_0(X) \to \mathscr{B}(H)$ be a *-homomorphism. Given a finite collection $T_1, \dots, T_k \in \mathscr{B}(H)$ of uniform operators with finite propagation, there exists a bases choice $\mathscr{A}$, such that $T_i \in C_k^*(X, \mathscr{A})$ for all $i = 1, \dots, k$.*

Proof. For simplicity, assume that we are given just one $T \in \mathscr{B}(H)$ to deal with (it will be clear that we can follow the procedure outlined below simultaneously for finitely many operators).

Denote $H_y = \phi(\chi_{V_y} H)$ and $T_{yz} = \phi(\chi_{V_y}) T \phi(\chi_{V_z}) \in \mathscr{B}(H_z, H_y)$. Since T has finite propagation and Y is uniformly discrete, there is a K, such that there are at most K nonzero entries in each column and row of the matrix $(T_{xz})_{x,z \in Y}$.

Fix $\varepsilon_1 = 1$ and take $R > \sup_{y \in Y} \operatorname{diam}(V_y)$. It follows from the assumption that there exists M, such that each T_{yz} is (ε_1, M)–approximable. Therefore, for each $y \in Y$, there are $2M$ orthonormal vectors $e_1^y, \dots, e_{2M}^y \in H_y$, for which there are $2M \times 2M$-matrices which in these (partial) bases represent operators $s_y \in \mathscr{B}(H_y)$ with $\|T_{yy} - s_y\| < \varepsilon_1$.

Fix $y \in Y$ for a while and consider the "column" $(T_{yz})_{z \in Y}$. Each of them is (ε_1, M)–approximable, but not necessarily by a matrix in the so far chosen partial basis $e_1^y, \dots, e_{2M}^y$. By adding at most M vectors to the chosen partial bases for H_y and H_z respectively, we can ensure that T_{yz} will be (ε_1, M)–approximable in the partial bases of H_y and H_z. We can do this for each nonzero T_{yz}, $z \in Y$, resulting in having chosen partial basis for H_y having at most $(2+K)M$ elements, and partial bases for H_z's having at most $3M$ elements. Doing this process for all $y \in Y$ results in choosing partial bases for each H_y having at most $(2+2K)M$ elements, now with the property that each T_{yz} is (ε_1, M)–approximable with matrices in the chosen partial bases. We make the partial bases to have exactly $(2+2K)M$ elements by adding arbitrary unit vectors, which are orthogonal to all previously added.

To finish the construction, we choose a sequence of $\varepsilon_n > 0$ converging to 0, and do the above described process for each n, always just adding the newly chosen partial bases to the previous ones. Hence, we have constructed $\mathscr{A} = \mathscr{A}(Y)$. The fact that $T \in C_k^*(X, \mathscr{A})$ follows easily from the construction and the following estimate III.6.3. $\square$

In fact, we can improve the previous lemma to finite collections of uniform operators which are do not necessarily have finite propagation, but are only approximable by finite propagation ones. To carry out the argument, we are going to use the relation $\sqsubseteq$ on bases choices (see definition III.6.11). Note that if $\mathscr{A}_1 \sqsubseteq \mathscr{A}_2$, then $C_k^*(X, \mathscr{A}_1) \subset C_k^*(X, \mathscr{A}_2)$: Let $w \in C_k^*(X, \mathscr{A}_1)$ be finite propagation operator, such that a bound M on the number of basis vectors from $\mathscr{S}_y^1$ which are used in each entry w_{yz} of w. By the last condition in the definition of $\sqsubseteq$, there is a number M', such that for each $y \in Y$, the first M vectors of $\mathscr{S}_y^1$ are in the linear span of the first M' vectors of $\mathscr{S}_y^2$. Consequently, entries w_{yz} use only the first M' vectors of bases $\mathscr{S}_y^2$, and so $w \in C_k^*(X, \mathscr{A}_2)$.

Lemma III.6.14. *Let X be metric space, let $Y \subset X$ be a quasi–lattice and let $(V_y)_{y \in Y}$ be a quasi–latticing partition of X. Let H be a Hilbert space and let $\phi : C_0(X) \to \mathscr{B}(H)$ be a *-homomorphism. Given a finite collection $T_1, \dots, T_k \in \Theta(\phi)$, the C*-algebra generated by uniform operators with finite propagation, there exists a bases choice $\mathscr{A}$, such that $T_i \in C_k^*(X, \mathscr{A})$ for all $i = 1, \dots, k$.*

We isolate a part of the proof of the above lemma as another lemma, as it is useful by itself.

Lemma III.6.15. *Let X be metric space, let $Y \subset X$ be a quasi–lattice and let $(V_y)_{y\in Y}$ be a quasi–latticing partition of X. Let H be a Hilbert space and let $\phi : C_0(X) \to \mathscr{B}(H)$ be a *-homomorphism. Assume that we are given a countable collection $\mathscr{A}_1, \ldots, \mathscr{A}_n, \ldots$ of bases choices. Then there exists a bases choice $\mathscr{A}$, such that $\mathscr{A}_i \sqsubseteq \mathscr{A}$, $i \geq 1$.*

Proof. Denote $\mathscr{A}_n = (Y,(V_y)_{y\in Y}, H, \phi, \{\mathscr{S}_y^n\}_{y\in Y})$. We now define bases $\mathscr{S}_y$ out of $\mathscr{S}_y^n$ (and put $\mathscr{A} = (Y,(V_y)_{y\in Y}, H, \phi, \mathscr{S}_y))$. Assume that we're given a sequence $(\,(e_i)_{i\geq 1}\,)_{n\geq 1}$ of orthonormal bases of $\ell^2\mathbb{N}$. We make one basis out of this sequence as follows: we fix a bijection $\alpha : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ (for instance $\alpha(n,i) = \frac{1}{2}(n+i-1)(n+i-2)+i$; say we think of $\mathbb{N}\times\mathbb{N}$ to be the lattice points in the first quadrant of the plane, and we enumerate the points along the diagonals going from "top–left" to "right–bottom"). Let $(\beta_1,\beta_2) : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ be its inverse. Now take the sequence of vectors $k \mapsto e_{\beta_2(k)}^{\beta_1(k)}$, and apply the Gramm–Schmidt orthogonalization process to it. We obtain a new basis, which obviously has the following property: for each $n \geq 1$ and $i \geq 1$, the vectors $e_1^n, \ldots, e_1^n$ are in the linear span of the first $\alpha(n,i)$ basis vectors of the new basis. Applying this procedure to each sequence $(\mathscr{S}_y^n)_{n\geq 1}$ yields the bases $\mathscr{S}_y$ we need.

A quick glance at the definition of the relation $\sqsubseteq$ for bases choices that $\mathscr{A}$ is as required. $\square$

Proof of lemma III.6.14. For simplicity, we concentrate on the case that $k = 1$, i.e. when we are given one operator $T \in \Theta(\phi)$. Note that T is uniform by the argument of lemma III.3.2. By assumption, T is approximable by a sequence T_n of uniform operators with finite propagation. For each T_n, there is a bases choice $\mathscr{A}_n = (Y,(V_y)_{y\in Y}, H, \phi, \{\mathscr{S}_y^n\}_{y\in Y})$, such that $T_n \in C_k^*(X,\mathscr{A}_n)$. Applying the previous lemma yields a bases choice $\mathscr{A}$, such that $\mathscr{A}_n \sqsubseteq \mathscr{A}$ for each $n \geq 1$. Since $C_k^*(X,\mathscr{A}_n) \subset C_k^*(X,\mathscr{A})$, T_n is a sequence of operators in $C_k^*(X,\mathscr{A})$ which converges to T. This finishes the proof. $\square$

III.7 The uniform index map

Let us now turn to the definition and properties of the index map. In the usual analytic K-homology, there is the index map (often also called the coarse assembly map) from a K-homology $K_*(X)$ of a space X to the K-theory of its Roe algebra $K_*(C^*X)$. But since the Roe algebras of coarsely equivalent spaces are isomorphic, the target group of the index map can be understood as the K-theory $K_*(C^*Y)$ of the Roe algebra of any quasi–lattice $Y \subset X$.

The quickest way to define this map in the usual case is to use the reformulation of the K-homology as a K-theory of a dual algebra (see [HR00b, theorem 8.4.3] and section III.3 for an analogous result in the uniform case) and then the 6-term exact sequence in K-theory, whose boundary maps become the assembly maps. For the details of this construction, see for instance [HR00b, section 12.3].

The goal of this section is to construct a similar index/assembly map in the uniform setting. More precisely, we define a homomorphism $\partial : K_*^u(X) \to K_*(C_u^*Y)$ for a quasi–lattice $Y \subset X$ in a metric space X. However, instead of the C*-algebra route, we take a longer, more hands-on approach.

In this paragraph, we recall the formula for the usual assembly map. If (H,ϕ,S) is a 0-Fredholm module, we can define its index as follows: denote

$$W = \left(\begin{smallmatrix} 1 & S \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -S^* & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & S \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathscr{M}_2(\mathscr{B}(H)).$$

This is an invertible in $\mathscr{M}_2(\mathscr{B}(H))$. Then put $\operatorname{ind}(S) = W \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) W^{-1} \in \mathscr{M}_2(\mathscr{B}(H))$. Concretely,

$$\operatorname{ind}(S) = \begin{pmatrix} SS^* + (1-SS^*)SS^* & S(1-S^*S) + (1-SS^*)S(1-S^*S) \\ S^*(1-SS^*) & (1-S^*S)^2 \end{pmatrix}.$$

A simple computation shows that $\operatorname{ind}(S)$ is actually an idempotent in $\mathscr{M}_2(\mathscr{B}(H))$. Furthermore, $\partial(H,\phi,S) = [\operatorname{ind}(S)] - [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)]$ is a K_0-class in the K-theory group of appropriate algebra, modulo which is S invertible. For example, starting with a finite propagation S, one gets $\partial(H,\phi,S)$ in $K_0(C^*X)$, the K-theory of the Roe C*-algebra.

Starting with a 1-Fredholm module (H,ϕ,Q), its index can be constructed using the formula $\operatorname{ind}(Q) = \exp(-2\pi i \frac{Q+1}{2}) \in \mathscr{B}(H)$. The operator $\operatorname{ind}(Q)$ is invertible[7], but even if we start with a finite propagation Q, $\operatorname{ind}(Q)$ might not have finite propagation. However, it is approximable by finite propagation invertibles in this case, hence still gives a class $[\operatorname{ind}(Q)] \in K_1(C^*X)$.

Let us now turn to the uniform case. Fix a quasi–lattice $Y \subset X$. We define $\partial : K_*^u(X) \to K_*(C_u^*Y)$ in the following lemma:

Proposition III.7.1 (Uniform index map, even case)**.** *Let (H,ϕ,S) be a 0-uniform Fredholm module with S having finite propagation. For any quasi–lattice $Y \subset X$, there exists a bases choice $\mathscr{A} = \mathscr{A}(Y) = ((V_y)_{y\in Y}, H, \{(e_i^y)_{i\in\mathbb{N}}\}_{y\in Y})$, such that* $\operatorname{ind}(S) \in \mathscr{M}_2(\mathscr{B}(H))$ *is an idempotent that belongs to $C_k^*(X,\mathscr{A},\mathscr{A})$. Furthermore, we can define a group homomorphism $\partial : K_0^u(X) \to K_0(C_u^*Y)$ by*

$$\partial[(H,\phi,S)] = \eta_*\left([\operatorname{ind}(S)] - [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)]\right) \in K_0(C_u^*Y),$$

i.e. the right-hand side does not depend on the choices made. Recall that η is described in lemma III.6.12.

Proposition III.7.2 (Uniform index map, odd case)**.** *Let (H,ϕ,Q) be a 1-uniform Fredholm module with Q having finite propagation. For any quasi–lattice $Y \subset X$ there exists a bases choice $\mathscr{A} = \mathscr{A}(Y) = ((V_y)_{y\in Y}, H, \{(e_i^y)_{i\in\mathbb{N}}\}_{y\in Y})$, such that* $\operatorname{ind}(Q) \in \mathscr{B}(H)$ *is an invertible that actually belongs to $C_k^*(X,\mathscr{A})^+$. Furthermore, the map $\partial : K_1^u(X) \to K_1(C_u^*Y)$ defined by*

$$\partial[H,\phi,Q] = \eta_*[\operatorname{ind}(Q)] \in K_1(C_u^*Y)$$

is a group homomorphism.

Proof of the 0*-case.* Picking any quasi–latticing partition $(V_y)_{y\in Y}$, the existence of suitable $\mathscr{A}$ follows from the lemma III.6.13, applied to the four entries of $\operatorname{ind}(S)$, which are uniform and have finite propagation.

It is clear that our construction of the index preserves direct sums. Also, the index of a degenerate element gives zero in the K-theory. Indeed, if (H,ϕ,S) is a degenerate 0-Fredholm module, then

[7] When we talk about invertibles in a non-unital C*-algebra, we mean that they are invertible in the unitization.

$\phi(f)\,\mathrm{ind}(S) = \begin{pmatrix} \phi(f) & 0 \\ 0 & 0 \end{pmatrix}$ for any $f \in C_0(X)$, so by using a partition of unity we obtain that $\mathrm{ind}(S) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Thus, to finish the proof, we need to show the independence of the index on the choice of $\mathscr{A}$, and under homotopies of uniform Fredholm modules. Our proof for homotopies includes the argument for choices of $\mathscr{A}$, since we can just take a constant homotopy, and choose different bases choices at the endpoints. We shall now outline the proof for homotopies.

Assume that we are given a homotopy (H, ϕ_t, S_t) of uniform Fredholm modules. We assume that all S_t have finite propagation (see proposition III.5.3), so that the index as we have defined it can be constructed. Note that the requirements on ϕ_t imply that $B = \Theta(\phi_t)$, the C*-algebra generated by all ϕ-uniform operators with ϕ-finite propagation, does not depend on t.

By applying the index formula to S_t, we obtain a norm-continuous path of projections in $\mathscr{M}_2(B) \subset \mathscr{M}_2(\mathscr{B}(H))$. For the sake of simplicity, let us assume that we have a norm-continuous path of projections (T_t) in B itself.

Choose $\mathscr{A}_0$ and $\mathscr{A}_1$ to be bases choices corresponding to (H, ϕ_0) and (H, ϕ_1) respectively, such that $T_i \in C_k^*(X, \mathscr{A}_i)$, $i = 0, 1$. Now we are in the position to apply lemma III.7.3, which finishes the proof for the even case. □

Proof of the 1*-case.* The operator $\mathrm{ind}(Q) = \exp(-2\pi i \frac{Q+1}{2}) - 1 \in \mathscr{B}(H)$ is uniform ($P = \frac{Q+1}{2}$ satisfies $P^2 \sim_{ua} P$ and so $\exp(-2\pi i P) - 1 \sim_{ua} P(\exp(-2\pi i) - 1) = 0$), but might not have finite propagation. However, from the formula for $\mathrm{ind}(Q)$ and finite propagation of Q it follows that $\mathrm{ind}(Q) - 1 \in \Theta(\phi)$, and so the existence of suitable $\mathscr{A}$ follows from lemma III.6.14 (after we've fixed some quasi–latticing partition $(V_y)_{y \in Y}$).

We reduce the independence of the index on homotopies to independence on bases choices. Taking a homotopy (H, ϕ_t, Q_t) of 1-uniform Fredholm modules, we assume that all Q_t have finite propagation. It follows that $U_t = \mathrm{ind}(Q_t)$, $t \in [0, 1]$ is a homotopy of invertibles in $B^+ = \Theta(\phi_0)^+$. Since the set of invertibles is open, by a standard compactness argument we can assume that the homotopy is piecewise-linear. Hence, it is sufficient to assume that we have just one linear path of invertibles from (say) U_0 to U_1 in B^+, and that we are given two bases choices $\mathscr{A}_0$ and $\mathscr{A}_1$, such that $U_i \in C_k^*(X, \mathscr{A}_i)^+$, $i = 1, 2$. Applying lemma III.6.15 gives a bases choice $\mathscr{A}$, such that $\mathscr{A}_i \sqsubseteq \mathscr{A}$ for each $i = 1, 2$. Hence U_0, U_1 and the whole (linear) homotopy between them is actually in $C_k^*(X, \mathscr{A})^+$. So $[U_0] = [U_1] \in K_1(C_k^*(X, \mathscr{A}))$, and the assertion will follow from the independence of the index on the choice of a bases choice.

We find ourselves in the following situation: we are given an invertible $U = 1 + K$, $K \in B = \Theta(\phi)$, and two bases choices $\mathscr{A}_0$, $\mathscr{A}_1$, such that $K \in C_k^*(X, \mathscr{A}_i)$, $i = 0, 1$.

We will think of $\mathscr{M}_k = \mathscr{M}_k(\mathbb{C})$ as $\mathscr{B}(\mathrm{span}(e_1, \ldots, e_k))$ in $\mathscr{K}(\ell^2\mathbb{N})$, where $e_1, e_2, \ldots$ is the standard basis of $\ell^2\mathbb{N}$. Let $A \subset \mathscr{B}(\ell^2(Y) \otimes \ell^2\mathbb{N})$ be the algebra of all finite propagation matrices $(t_{yz})_{y,z \in Y}$ for which there exists $k \in \mathbb{N}$ with $t_{yz} \in \mathscr{M}_k$ for all $y, z \in Y$. Then $C_k^*(Y)$ is the norm closure of A. Denote $u_0 = u_{\mathscr{A}_0}$ and $u_1 = u_{\mathscr{A}_1}$.

We will prove that $\left(\begin{smallmatrix} U & 0 \\ 0 & 1 \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & U \end{smallmatrix}\right) \in C_k^*(X, \mathscr{A}_0, \mathscr{A}_1)^+$. The standard rotation homotopy between these two matrices has the form $1 + \begin{pmatrix} \sin^2(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) \\ \cos(\frac{\pi}{2}t)\sin(\frac{\pi}{2}t) & \cos^2(\frac{\pi}{2}t) \end{pmatrix} K$, and so it is sufficient to prove that actually $\left(\begin{smallmatrix} 0 & K \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ K & 0 \end{smallmatrix}\right) \in C_k^*(X, \mathscr{A}_0, \mathscr{A}_1)$. Equivalently, that $u_0 K u_1^*$ and $u_1 K u_0^* \in A$.

Pick $\varepsilon > 0$. Since $K \in C_k^*(X, \mathscr{A}_i)$, $i = 0, 1$, there exist $\hat{s}_0, \hat{s}_1 \in \mathscr{B}(H)$ with finite propagation, such that

$s_i := u_i \hat{s}_i u_i^* \in A$ and $\|\hat{s}_i - K\| < \varepsilon$ for $i = 0,1$. Since $K \in B$, there exist an operator $\hat{K} \in B$ with finite propagation, such that $\|K - \hat{K}\| < \varepsilon$, $i = 0,1$. Consequently, $\|\hat{s}_i - \hat{K}\| < 2\varepsilon$ for $i = 0,1$.

At this moment, we can apply the proof of claim 6 (with $v = 1$ and $T_0 = K$, otherwise verbatim), to obtain $p \in A$, such that $\|p - u_1 K u_0^*\| < 8\varepsilon$. Letting $\varepsilon \to 0$, we obtain that $u_1 K u_0^* \in A$. Analogous proof shows also $u_0 K u_1^* \in A$. We are done. $\square$

Lemma III.7.3. *Let H be a Hilbert space, $\phi : C_0(X) \to \mathscr{B}(H)$ a *-representation. Denote $B = \Theta(\phi) \subset \mathscr{B}(H)$, the C*-algebra generated by ϕ-uniform operators with ϕ-finite propagation. Assume that T_t, $t \in [0,1]$ is a homotopy of projections in B, and that $\mathscr{A}_0$ and $\mathscr{A}_1$ are two bases choices, such that $T_i \in C_k^*(X, \mathscr{A}_i)$*[8]. *Then $[T_0] = [T_1] \in K_0(C_k^*(X, \mathscr{A}_0, \mathscr{A}_1))$.*

Proof. Since T_t is a homotopy of projections T_t in a C*-algebra B, there exists an invertible element $v_0 \in B$ with $\|v_0\| = 1$, such that $T_1 = v_0^{-1} T_0 v_0$ (see eg. [Bla98, Proposition 4.3.2]). Note that v_0 might not have finite propagation, so we will need to make some approximations further on.

The images of T_0 and T_1 under the inclusions of $C_k^*(X, \mathscr{A}_i)$, $i = 0,1$, into $C_k^*(X, \mathscr{A}_0, \mathscr{A}_1) \subset \mathscr{M}_2(\mathscr{B}(H))$ are the operators $\left(\begin{smallmatrix} T_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & T_1 \end{smallmatrix}\right)$. These two projections are Murray–von Neumann equivalent by the elements $x = \left(\begin{smallmatrix} 0 & T_0 v_0 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \begin{pmatrix} 0 & 0 \\ v_0^{-1} T_0 & 0 \end{pmatrix}$. To finish the argument, we must show that $x, y \in C_k^*(X, \mathscr{A}_0, \mathscr{A}_1)$.

For the rest of the proof, we will think of $\mathscr{M}_k = \mathscr{M}_k(\mathbb{C})$ as $\mathscr{B}(\mathrm{span}(e_1, \ldots, e_k))$ in $\mathscr{K}(\ell^2\mathbb{N})$, where $e_1, e_2, \ldots$ is the standard basis of $\ell^2\mathbb{N}$. Let $A \subset \mathscr{B}(\ell^2(Y) \otimes \ell^2\mathbb{N})$ be the algebra of all finite propagation matrices $(t_{yz})_{y,z \in Y}$ for which there exists $k \in \mathbb{N}$ with $t_{yz} \in \mathscr{M}_k$ for all $y, z \in Y$. Then $C_k^*(Y)$ is the norm closure of A.

We shall give a proof that $y \in C_k^*(X, \mathscr{A}_0, \mathscr{A}_1)$; a proof for x is analogous. Denote $u_0 = u_{\mathscr{A}_0}$ and $u_1 = u_{\mathscr{A}_1}$.

We need to show that $y \in \mathrm{Ad}(u_0 \oplus u_1)(\mathscr{M}_2(C_k^*(Y)))$. This will follow from the following statement: For any $\varepsilon > 0$, there exists $p \in A$, such that $\|p - u_1 v_0^{-1} T_0 u_0^*\| < \varepsilon$. By the choice of $\mathscr{A}_0$ and $\mathscr{A}_1$, we know that there are $\hat{s}_0, \hat{s}_1 \in \mathscr{B}(H)$, such that $u_0 \hat{s}_0 u_0^*, u_1 \hat{s}_1 u_1^* \in A$, $\|\hat{s}_0 - T_0\| < \varepsilon$ and $\|\hat{s}_1 - v_0^{-1} T_0 v_0\| < \varepsilon$. Note that $\hat{s}_0$ and $\hat{s}_1$ have finite propagation. Furthermore, there exists an invertible element $v \in B$ with finite propagation, norm 1, and $\|v - v_0\| < \varepsilon$ and $\|v^{-1} - v_0^{-1}\| < \varepsilon$. It follows that $\|v \hat{s}_1 v^{-1} - T_0\| < 3\varepsilon$.

At this moment, the setting is as follows: we have a finite propagation operators $v, \hat{s}_0, \hat{s}_1$ and T_0, such that $\|\hat{s}_0 - T_0\| < \varepsilon$, $\|v \hat{s}_1 v^{-1} - T_0\| < 3\varepsilon$.

Claim 6. There exists $p \in A$, such that $\|p - u_1 \hat{s}_1 v^{-1} u_0^*\| < 4\varepsilon$.

Proof of claim. Combining the two inequalities with T_0 gives

$$4\varepsilon > \|\hat{s}_0 - v \hat{s}_1 v^{-1}\| = \|v^{-1} \hat{s}_0 - \hat{s}_1 v^{-1}\| \geq \|u_1 v^{-1} u_0^* u_0 \hat{s}_0 u_0^* - u_1 \hat{s}_1 u_1^* u_1 v^{-1} u_0^*\|$$

Denoting $w = u_1 v^{-1} u_0^* \in \mathscr{B}(\ell^2(Y) \otimes \ell^2(\mathbb{N}))$, $s_0 = u_0 \hat{s}_0 u_0^* \in A$, $s_1 = u_1 \hat{s}_1 u_1^* \in A$, we obtain $\|w s_0 - s_1 w\| < 4\varepsilon$. Note that w has finite propagation. Let k be such that all entries s_0 and s_1 belong to $\mathscr{M}_k$. We split the standard basis of $\ell^2(Y) \otimes \ell^2\mathbb{N}$ into two sets $\mathscr{B}_1$ (first k vectors from each $\{y\} \otimes \ell^2\mathbb{N}$) and $\mathscr{B}_2$ (the other basis vectors).

[8] For any bases choice $\mathscr{A}$, $C_k^*(X, \mathscr{A}) \subset B$. The uniformity of $T \in C_k^*(X, \mathscr{A})$ follows the formula $f = \sum_y f \chi_{V_y}$. Note that for fixed $R \geq 0$ and $f \in C_R(X)$, there is a uniform bound on the number of nonzero terms in the sum by bounded geometry.

With respect to this decomposition, we can write $s_0 = \left(\begin{smallmatrix} \star & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $s_1 = \left(\begin{smallmatrix} s_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $w = \left(\begin{smallmatrix} w_{11} & w_{12} \\ \star & \star \end{smallmatrix}\right)$. Consequently,

$$4\varepsilon > \|ws_0 - s_1 w\| = \left\| \left(\begin{smallmatrix} \star & 0 \\ \star & 0 \end{smallmatrix}\right) - \left(\begin{smallmatrix} s_{11}w_{11} & s_{11}w_{12} \\ 0 & 0 \end{smallmatrix}\right) \right\| = \left\| \left(\begin{smallmatrix} \star & s_{11}w_{12} \\ \star & 0 \end{smallmatrix}\right) \right\|.$$

Hence $\|s_{11}w_{12}\| < 4\varepsilon$. If we denote $p = \left(\begin{smallmatrix} s_{11}w_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we immediately see that $p \in A$ and $\|s_1 w - p\| = \left\| \left(\begin{smallmatrix} 0 & s_{11}w_{12} \\ 0 & 0 \end{smallmatrix}\right) \right\| < 4\varepsilon$. ☐

Returning to the proof of the lemma, we conclude

$$\|p - u_1 v_0^{-1} T_0 u_0^*\| \leq \|p - u_1 v^{-1} T_0 u_0^*\| + \varepsilon\|T_0\| \leq$$
$$\leq \|p - u_1 \hat{s}_1 v^{-1} u_0^*\| + \varepsilon\|T_0\| + 3\varepsilon < 4C\varepsilon + \varepsilon\|T_0\| + 3\varepsilon.$$

This finishes the proof. ☐

III.8 Amenability

Let X be a graph (with the edges attached) and let Y be its vertex set. Recall the definition of the fundamental class $\mathbf{S} \in K_0^u(X)$ (see example III.1.9). Let $H = \ell^2 Y \otimes \ell^2\mathbb{N}$, and endow H with the multiplication action of $C_0(X)$. Let $S \in \mathscr{B}(\ell^2\mathbb{N})$ be the unilateral shift. Let $\tilde{S} = \operatorname{diag}(S) \in \mathscr{B}(H)$ and finally denote $\mathbf{S} = [(H, \phi, \tilde{S})]$. It is easy to see that $\mathbf{S} \in K_0^u(X)$, and that $\operatorname{ind}(\tilde{S}) = 1 \otimes p_0 \in \mathscr{B}(\ell^2 Y \otimes \ell^2\mathbb{N})$, where p_0 is a rank one projection (onto $\mathbb{C}e_1 \in \ell^2\mathbb{N}$). We also denote by $0 \in K_0^u(Y)$ the trivial element.

In the view of the characterizations of amenability appearing in [BW92] and [Ele97], the following proposition is not surprising:

Theorem III.8.1. *Let X be a connected graph with the vertex set Y. Then Y is amenable if and only if $\mathbf{S} \neq 0$ in $K_0^u(X)$.*
More generally, if X is not connected, then there exists $C \geq 0$, such that Y is amenable if and only if $\mathbf{S} \neq 0$ in $K_0^u(P_C(Y))$ (recall that $P_C(Y)$ denotes the Rips complex of Y, see definition II.2.6).

Remark III.8.2. Note that the technical assumption that Y is a graph is not too restrictive, since every metric space with bounded geometry is coarsely equivalent to a graph.

Proof. If Y is amenable, then $\partial(\mathbf{S}) = [1] \neq [0] = \partial 0 \in K_0(C_u^* Y)$ by [Ele97], and so $\mathbf{S} \neq 0$. For the convenience of the reader, let us sketch this part of Elek's proof. The idea is that if Y is amenable, then using Følner sets B_n, one can construct a trace on $C_u^* Y$ as an ultralimit of functions $f_n(T) = \frac{1}{|B_n|}\sum_{x \in B_n} t_{xx}$. The trace then distinguishes [1] from [0] in $K_0(C_u^* Y)$.

Let us turn to the reverse implication. Assume that Y is not amenable. Let X be Y with all edges attached, i.e. a space constructed from Y by attaching an interval of length 1 to each pair of vertices with distance 1.

We will proceed to constructing a homotopy connecting $\mathbf{S}$ and 0 in $K_0^u(X)$.

First, we describe a "building block". Denote $I = [0,1]$. Denote $T_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & S \end{smallmatrix}\right)$ and $T_1 = \left(\begin{smallmatrix} S & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathscr{B}(\ell^2 W \otimes \ell^2\mathbb{N})$. Let the action ψ of $C(I)$ on $H_I = \ell^2\mathbb{N} \oplus \ell^2\mathbb{N}$ be $\psi(f)(\eta \oplus \xi) = f(0)\eta \oplus f(1)\xi$. Let us show a homotopy (H_I, ψ_t, T_t) between (H_I, ψ, T_0) and (H_I, ψ, T_1). Define

$$\psi_t(f)(\eta \oplus \xi) = \begin{cases} f(0)\eta \oplus f(1-3t)\xi & 0 \le t \le \frac{1}{3} \\ f(0)\eta \oplus f(0)\xi & \frac{1}{3} \le t \le \frac{2}{3} \\ f(0)\eta \oplus f(3t-2)\xi & \frac{2}{3} \le 1 \end{cases}$$

and

$$T_t = \begin{cases} T_0 & 0 \le t \le \frac{1}{3} \\ \alpha_t \left(\begin{smallmatrix} 1 & 0 \\ 0 & S \end{smallmatrix}\right) \alpha_t^* & \frac{1}{3} \le t \le \frac{2}{3} \\ T_1 & \frac{2}{3} \le 1, \end{cases}$$

where $\alpha_t = \left(\begin{smallmatrix} \cos(\frac{\pi}{2}(3t-1)) & \sin(\frac{\pi}{2}(3t-1)) \\ -\sin(\frac{\pi}{2}(3t-1)) & \cos(\frac{\pi}{2}(3t-1)) \end{smallmatrix}\right)$ is the rotation homotopy. It is clear that operators $\left(\begin{smallmatrix} S^k & 0 \\ 0 & S^l \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} S^{k-1} & 0 \\ 0 & S^{l+1} \end{smallmatrix}\right)$ (on the same Hilbert space with the same action of $C(I)$) are homotopic as well.

Now we turn to $Y \subset X$. Assuming non-amenability of Y and applying [BW92, Theorem 3.1 and Lemma 2.4], for each $y \in Y$ there exists a "tail", i.e. a sequence $(z_i^y)_{i \ge 0} \subset Y$, such that $z_0 = y$, $C = \sup_{y,i}(d(z_i^y, z_{i+1}^y)) < \infty$, satisfying the condition that in every ball fixed radius, the number of tails passing through is uniformly bounded.

In the case when X is connected, we can reduce the general C to the case $C = 1$, i.e. to the situation when the tails actually follow the edges of X. We do this just by refining the tails. We may achieve this without violating the condition on uniform bound on tails passing through balls, since Y has bounded geometry. If we do not assume connectedness, we may get by working with the Rips complex $P_C(Y)$ instead of $X = P_1(Y)$, since any two points with distance $\le C$ are connected by an edge in $P_C(Y)$.

By the above condition on the tails, it is possible to partition the collection of edges contained in all tails $((z_i^y, z_{i+1}^y))_{y \in Y, i \in \mathbb{N}}$ (we allow for multiplicities) into finitely many parts $A_1, \ldots, A_k$, such that no two edges from the same part share a common vertex.

The idea of the rest of the construction is to "send off" the $\tilde{S}$ along the tails off to infinity, and thus connecting $\tilde{S}$ with 1. This is done in k steps. In step j, we simultaneously apply the building block construction to each of the edges in A_j (this is possible by the choice of A_j), thus "transferring" one S along each of those edges. After each step, we obtain a diagonal matrix in $\mathscr{B}(H)$ with various powers of S on the diagonal. The whole homotopy begins with $\tilde{S}$, and ends with 1, since after all k steps the S from each $y \in Y$ was shifted away from y along the tail. □

CHAPTER IV

NON-K-EXACT UNIFORM ROE C*-ALGEBRAS

IV.1 K-exactness and K-nuclearity

The notion of K-exactness was introduced in [Ulg05] as a K-theoretic analogue of exactness of C*-algebras. Recall that a C*-algebra A is exact, if $\cdot \otimes_{\min} A$ is an exact functor, i.e. if we min–tensor every term in a short exact sequence with A, the sequence stays exact. If it does, then we obtain a 6-term exact sequence in K-theory (as below). It may happen that the tensored short exact sequence is not exact for a non–exact C*-algebra, but the induced 6-term sequence still is. This leads to the following definition.

Definition IV.1.1. A C*-algebra A is said to be *K-exact*, if for any exact sequence of C*-algebras $0 \to I \to B \to B/I \to 0$, the map $I \otimes_{\min} A \to \ker(B \otimes_{\min} A \to (B/I) \otimes_{\min} A)$ induces an isomorphism on K-theory.

Proposition IV.1.2 ([Ulg05, 2.3.2])**.** *For a C*-algebra A, the following are equivalent:*

- *A is K-exact,*
- *for any exact sequence of C*-algebras $0 \to I \to B \to B/I \to 0$, there is a cyclic 6-term exact sequence in K-theory:*

$$\begin{array}{ccccc} K_0(I \otimes_{\min} A) & \longrightarrow & K_0(B \otimes_{\min} A) & \longrightarrow & K_0(B/I \otimes_{\min} A) \\ \uparrow & & & & \downarrow \\ K_1(B/I \otimes_{\min} A) & \longleftarrow & K_1(B \otimes_{\min} A) & \longleftarrow & K_1(I \otimes_{\min} A). \end{array}$$

  (Note that if A is not K-exact, there might be no such 6-term K-theory sequence at all.),
- *for any exact sequence of C*-algebras $0 \to I \to B \to B/I \to 0$, the sequences*

$$K_i(I \otimes_{\min} A) \to K_i(B \otimes_{\min} A) \to K_i(B/I \otimes_{\min} A),$$

  is exact in the middle for both $i = 0, 1$.

For separable C*-algebras, a sufficient condition for K-exactness is K-nuclearity. The argument [Ulg05, proposition 3.4.2] can be summarized as follows: the max–tensor product always preserves exact sequences, and if a C*-algebra is K-nuclear, then min–tensor products and max–tensor products with it are KK–equivalent. However, this relies on the key properties of KK-theory (existence and associativity of Kasparov product), which have been proved only for separable C*-algebras in general.

For completeness, we state the definition of K-nuclearity.

Definition IV.1.3. ([Ska88]) A C*-algebra A is called *K-nuclear*, if there is a representative $[(E, \phi, F)]$ of $\mathbf{1} \in KK(A, A)$, such that the map $\phi : A \to \mathscr{B}(E)$ is strictly nuclear, i.e. approximable in the strict topology on $\mathscr{B}(E)$ by completely positive maps that factor through a finite–dimensional C*-algebras.

Moving towards the context of coarse geometry, it is a result of Ulgen [Ulg05] that uniformly discrete spaces with bounded geometry which coarsely embed into a Hilbert space have K-nuclear uniform Roe C*-algebras. The argument heavily uses the work of Tu [Tu99; Tu00], where he proves that coarse embeddability is equivalent to Haagerup property of the coarse groupoid (see [STY02; Roe03] for the precise definition of the groupoid associated to a coarse space) and that groupoids with Haagerup property are K-amenable. The argument is finished by noting that the uniform Roe C*-algebra is a crossed product of a nuclear C*-algebra $\ell^\infty(X)$ with the coarse groupoid, and that crossed products with K-amenable groupoids yield K-nuclear C*-algebras.

These considerations inevitably lead to the question of what can we say about uniform Roe C*-algebras of spaces that do not uniformly embed into a Hilbert space. The only explicit examples of such spaces (with bounded geometry) are expanders. In what follows, we prove for a large class of expanders that their uniform Roe C*-algebras are not K-exact.

From now on, we write just $\otimes$ instead of $\otimes_{\min}$.

IV.2 Result

Let Γ be a finitely generated discrete group with property $\tau(\mathscr{L})$. For our construction to work, we require two conditions:

($\star$) If $M,N \in \mathscr{L}$ then there exists $L \in \mathscr{L}$, such that $L \subset M \cap N$.

($\star\star$) For each $N \in \mathscr{L}$ there exists an irreducible representation $\pi_N : G_N \to \mathscr{B}(H_N)$, such that $\dim(H_N) \to \infty$ as $N \to \infty$.

Remark IV.2.1. The question whether one can always arrange for ($\star$) to hold seems to be open, cf. [LZ07], question 1.14.

Remark IV.2.2. Since $\mathscr{L}$ is countable, we can construct a metric space $\sqcup_{N\in\mathscr{L}}\Gamma/N$, a coarse disjoint union.

Example IV.2.3. Set $\Gamma = SL_n(\mathbb{Z})$ and $\mathscr{L} = \{SL_n(\mathbb{Z}/p\mathbb{Z})\}$, $n \geq 2$. These choices certainly satisfy the above requirements: $SL_n(\mathbb{Z})$ for $n \geq 3$ has Kazhdan's property (T), and the fact that $SL_2(\mathbb{Z})$ has $\tau(\mathscr{L})$ is Selberg's theorem; ($\star$) is obvious and ($\star\star$) can be seen from the representation theory.

Proposition IV.2.4. *Let Γ be a group having $\tau(\mathscr{L})$ and let $X = \sqcup_{N\in\mathscr{L}}\Gamma/N$. If the conditions ($\star$) and ($\star\star$) hold, then C^*_uX is not K-exact.*

IV.3 Construction

PROOF OF PROPOSITION IV.2.4. For each $N \in \mathscr{L}$, we denote $G_N = \Gamma/N$, $q_N : \Gamma \to G_N$ the quotient map and $\lambda_N : G_N \to \mathscr{B}(\ell^2 G_N)$ the left regular representation of G_N. Denote also $\tilde{\lambda}_N = \lambda_N \circ q_N : \Gamma \to \mathscr{B}(\ell^2 G_N)$, $X = \sqcup_{N\in\mathscr{L}} G_N$ and $\Lambda = \oplus_{N\in\mathscr{L}}\tilde{\lambda}_N : \Gamma \to \mathscr{B}(\ell^2 X)$.

By ($\star\star$), for each $N \in \mathscr{L}$ we have an irreducible representation $\pi_N : G_N \to \mathscr{B}(H_N)$. Denote $\tilde{\pi}_N = \pi_N \circ q_N$, $H = \oplus_{N\in\mathscr{L}} H_N$ and $\pi = \oplus_{N\in\mathscr{L}} \pi_N \circ q_N : \Gamma \to \mathscr{B}(H)$. Let us summarize the notation in the following diagram:

$$\begin{tikzcd} \Gamma \arrow[r, "q_N"] \arrow[dr, "\tilde{\lambda}_N"'] & G_N \arrow[d, "\lambda_N"] \\ & \mathscr{B}(\ell^2 G_N) \end{tikzcd} \qquad \begin{tikzcd} \Gamma \arrow[r, "q_N"] \arrow[dr, "\tilde{\pi}_N"'] & G_N \arrow[d, "\pi_N"] \\ & \mathscr{B}(H_N) \end{tikzcd}$$

Finally, we let $B = \prod_{N\in\mathscr{L}} \mathscr{B}(H_N)$ and $J = \oplus_{N\in\mathscr{L}} \mathscr{B}(H_N)$. We obtain an exact sequence of C*-algebras

$$0 \to J \to B \to B/J \to 0.$$

We shall use this sequence to show that C_u^*X is not K-exact. We construct a projection $e \in C_u^*X \otimes B$, whose K_0-class will violate the exactness of the K-theory sequence

$$K_0(C_u^*X \otimes J) \to K_0(C_u^*X \otimes B) \to K_0(C_u^*X \otimes (B/J)). \tag{IV.1}$$

To construct such e, we let

$$T = \frac{1}{|S|} \sum_{g\in S} (\Lambda \otimes \pi)(g) \in C_u^*X \otimes B \subset \mathscr{B}(\ell^2 X \otimes H) = \mathscr{B}(\oplus_{M,N\in\mathscr{L}} \ell^2 G_N \otimes H_M).$$

Remark IV.3.1. If we denote $s = \frac{1}{|S|}\sum_{g\in S} g \in \mathbb{C}\Gamma$, then $T = (\Lambda \otimes \pi)(s)$. One of the formulations of property (T) is that Γ has property (T) iff 1 is an isolated point of the spectrum of s. In this case, the sought projection e would be just the image of the Kazhdan projection $p \in C^*\Gamma$ (the spectral projection of s corresponding to the eigenvalue 1) under $\Lambda \otimes \pi : C^*\Gamma \to C_u^*X \otimes B$.

Lemma IV.3.2. *The spectrum of T is contained in $[-1,1]$ and contains 1 as an isolated point.*

Assuming this lemma, we set $e \in C_u^*X \otimes B$ to be the spectral projection of T corresponding to the eigenvalue 1. Lemmas IV.3.3 and IV.3.4 then show that e witnesses that the sequence (IV.1) is not exact, thus finish the proof of the proposition.

Note that each $\ell^2 G_N \otimes H_M$ is Γ-invariant, hence the operator T is actually "diagonal", i.e. in the decomposition $\ell^2 X \otimes H = \oplus_{M,N} \ell^2 G_N \otimes H_M$, the only nonzero components of T are $T_{NM} \in \mathscr{B}(\ell^2 G_N \otimes H_M)$. Hence the projection e is diagonal as well. If we denote the components as $e_{NM} \in \mathscr{B}(\ell^2 G_N \otimes H_M)$, then each e_{NM} is actually the projection onto the Γ-invariant vectors in $\ell^2 G_N \otimes H_M$.

Proof of lemma IV.3.2. Taking $N \in \mathscr{L}$, the G_N-action on $\ell^2 G_N \otimes H_N$ is via $\lambda_N \otimes \pi_N$. This representation contains the trivial representation (since λ_N contains the conjugate of π_N, as it does any irreducible representation of G_N), so there are nonzero G_N-invariant vectors in $\ell^2 G_N \otimes H_N$. Therefore, $1 \in \operatorname{spec}(T_{NN})$.

To show that 1 is actually isolated in each $\operatorname{spec}(T_{NM})$ with the uniform bound on the size of the gap, we shall use the property $\tau(\mathscr{L})$ together with the condition ($\star$). Property $\tau(\mathscr{L})$ says that we have such a uniform bound on the size of the spectral gap for the family of representations of Γ which factor through some of G_L, $L \in \mathscr{L}$. Using that $\tilde{\pi}_M$ is contained in $\tilde{\lambda}_M$, we obtain

$$\ker(\tilde{\lambda}_N \otimes \tilde{\pi}_M) = \ker(\tilde{\lambda}_N) \cap \ker(\tilde{\pi}_M) \supseteq \ker(\tilde{\lambda}_N) \cap \ker(\tilde{\lambda}_M) = N \cap M \supseteq L,$$

for some $L \in \mathscr{L}$ by ($\star$). This shows that $\tilde{\lambda}_N \otimes \tilde{\pi}_M$ factors through G_L and the proof is finished. □

Lemma IV.3.3. *The image of e under the map $C_u^*X \otimes B \to C_u^*X \otimes (B/J)$ is* 0.

Proof. Denote $A = \prod_{N \in \mathscr{L}} \mathscr{B}(\ell^2 G_N)$, a product of matrix algebras. It is clear that $T \in (C_u^*X \otimes B) \cap (A \otimes B)$, and so also $e \in A \otimes B \subset \mathscr{B}(\ell^2 X \otimes H)$.

For $N \in \mathscr{L}$, let us examine the $\mathscr{B}(\ell^2 G_N) \otimes \mathscr{B}(H)$–component of e. Denote by $P_N \in \mathscr{B}(\ell^2 X)$ the projection onto $\ell^2 G_N$. It suffices to show that $e_N = (P_N \otimes 1_B) e (P_N \otimes 1_B) \in \mathscr{B}(\ell^2 G_N \otimes H)$ actually belongs to $\mathscr{B}(\ell^2 G_N) \otimes J$, as the following commutative diagram explains:

$$\begin{tikzcd}
A \otimes B \arrow[r, hook, "\prod_N (P_N \otimes 1)\cdot(P_N \otimes 1)"] \arrow[d] & \prod_N (\mathscr{B}(\ell^2 G_N) \otimes B) \arrow[d] \\
A \otimes (B/J) \arrow[r, hook] & \prod_N (\mathscr{B}(\ell^2 G_N) \otimes (B/J)).
\end{tikzcd}$$

Further decompose e_N into $e_{NM} \in \mathscr{B}(\ell^2 G_N \otimes H_M)$. Recall that $e_{NM} \neq 0$ iff $\ell^2 G_N \otimes H_M$ has nonzero invariant vectors. Since the representation $\tilde{\pi}_M$ is irreducible, this is further equivalent to $\tilde{\pi}_M$ being conjugate to a subrepresentation of $\tilde{\lambda}_N$. But by ($\star\star$), this can only happen for finitely many M's, since $\tilde{\lambda}_N$ is fixed and $\dim(H_M) \to \infty$. □

Lemma IV.3.4. $[e] \in K_0(C_u^*X \otimes B)$ *does not come from* $K_0(C_u^*X \otimes J)$.

Proof. For $N \in \mathscr{L}$, denote $C_N = \mathscr{B}(\ell^2 G_N \otimes H_N)$. We construct a *-homomorphism $f : C_u^*X \otimes B \to \prod_N C_N / \oplus_N C_N$, such that $f_*([e]) \neq 0 \in K_0(\prod_N C_N / \oplus_N C_N)$, but $f_*([x]) = 0$ for any $[x] \in K_0(C_u^*X \otimes J)$.

We first embed C_u^*X into a direct limit of C*-algebras A_k, defined below. We enumerate $\mathscr{L} = \{N_k \mid k \in \mathbb{N}\}$ and put $A^q = \mathscr{B}(\ell^2 G_{N_q})$, $A^{0k} = \mathscr{B}(\ell^2(\sqcup_{q \leq k} G_{N_q}))$ and finally $A_k = A^{0k} \oplus \prod_{q>k} A^q$, $k \geq 1$. There are obvious inclusion maps $A_k \hookrightarrow A_l$ for $k < l$, so we can form a direct limit $A_0 = \lim_k A_k$. It follows from the condition on distances $d(G_{N_q}, G_{N_p})$ that each finite propagation operator on $\ell^2 X$ is a member of some $A_k \subset \mathscr{B}(\ell^2 X)$, hence $C_u^*X \hookrightarrow A_0$.

For $k \geq 1$, denote $B^k = \mathscr{B}(H_{N_k})$ (so that $B = \prod_{k \in \mathbb{N}} B^k$) and define f_k as the following composition:

$$A_k \otimes B = \Big(A^{0k} \oplus \prod_{q>k} A^q\Big) \otimes B \hookrightarrow \Big(A^{0k} \otimes B\Big) \oplus \prod_{q>k} \left([A^q \otimes B^q] \oplus \left[A^q \otimes \Big(\prod_{p \neq q} B^p\Big)\right]\right) \xrightarrow{\text{proj}}$$
$$\xrightarrow{\text{proj}} \prod_{q>k} A^q \otimes B^q = \prod_{q>k} C_{N_k} \xrightarrow{\text{quot}} \prod_q C_{N_q} \Big/ \sum_q C_{N_q}.$$

It is easy to see that f_k's commute with inclusions $A_k \otimes B \hookrightarrow A_l \otimes B$, $k < l$. Consequently, we obtain a *-homomorphism $f : C_u^*X \otimes (B \rtimes \Gamma) \to \prod_N C_N / \oplus_N C_N$.

It is known that $K_0(\prod_N C_N \big/ \oplus_N C_N)$ embeds into $\prod_N \mathbb{Z} \big/ \oplus_N \mathbb{Z}$. Examining the construction of f, we see that $f_*([e])$ is the class of the sequence $k \mapsto \operatorname{rank}(e_{N_k N_k})$ in $\prod_{N_k} \mathbb{Z} \big/ \oplus_{N_k} \mathbb{Z}$. As already noted in the proof of the lemma IV.3.2, every term of this sequence is nonzero.

On the other hand, any projection $p \in C_u^* X \otimes J$ has only finitely many nonzero C_N-components, and so $f_*([p]) = 0$ in $\prod_N \mathbb{Z} \big/ \oplus_N \mathbb{Z}$. This obviously extends to the whole of $K_0(C_u^* X \otimes J)$. $\square$

[Ati70] M. F. Atiyah. "Global theory of elliptic operators". In: *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*. Univ. of Tokyo Press, Tokyo, 1970. Pp. 21–30.

[Bla98] B. Blackadar. *K-theory for operator algebras*. 2nd ed. Vol. 5. Mathematical Sciences Research Institute Publications. Cambridge: Cambridge University Press, 1998. Pp. xx+300. ISBN: 0-521-63532-2.

[BW92] J. Block and S. Weinberger. "Aperiodic Tilings, Positive Scalar Curvature, and Amenability of Spaces". In: *Journal of the American Mathematical Society* 5.4 (1992). Pp. 907–918.

[BNW07] J. Brodzki, G. A. Niblo, and N. J. Wright. "Property A, partial translation structures, and uniform embeddings in groups". In: *J. Lond. Math. Soc. (2)* 76.2 (2007). Pp. 479–497. ISSN: 0024-6107.

[Ele97] G. Elek. "The K-Theory of Gromov's Translation Algebras and the Amenability of Discrete Groups". In: *Proceedings of the American Mathematical Society* 125.9 (1997). Pp. 2551–2553.

[Gro00] M. Gromov. "Spaces and questions". In: *Geom. Funct. Anal.* Special Volume, Part I (2000). GAFA 2000 (Tel Aviv, 1999). Pp. 118–161. ISSN: 1016-443X.

[GK02] E. Guentner and J. Kaminker. "Exactness and the Novikov conjecture". In: *Topology* 41.2 (2002). Pp. 411–418. ISSN: 0040-9383.

[HK97] N. Higson and G. G. Kasparov. "Operator K-theory for groups which act properly and isometrically on Hilbert space". In: *Electron. Res. Announc. Amer. Math. Soc.* 3 (1997). 131–142 (electronic). ISSN: 1079-6762.

[HLS02] N. Higson, V. Lafforgue, and G. Skandalis. "Counterexamples to the Baum-Connes conjecture". In: *Geom. Funct. Anal.* 12.2 (2002). Pp. 330–354. ISSN: 1016-443X.

[HR00a] N. Higson and J. Roe. "Amenable group actions and the Novikov conjecture". In: *J. Reine Angew. Math.* 519 (2000). Pp. 143–153. ISSN: 0075-4102.

[HR00b] N. Higson and J. Roe. *Analytic K-homology*. Oxford Mathematical Monographs. Oxford Science Publications. Oxford: Oxford University Press, 2000. Pp. xviii+405. ISBN: 0-19-851176-0.

[Kas80] G. G. Kasparov. "The operator K-functor and extensions of C^*-algebras". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 44.3 (1980). Pp. 571–636, 719. ISSN: 0373-2436.

[Kas88] G. G. Kasparov. "Equivariant KK-theory and the Novikov conjecture". In: *Invent. Math.* 91.1 (1988). Pp. 147–201. ISSN: 0020-9910.

[Kas07] M. Kassabov. "Symmetric groups and expander graphs". In: *Invent. Math.* 170.2 (2007). Pp. 327–354. ISSN: 0020-9910.

[Laf02] V. Lafforgue. "K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes". In: *Invent. Math.* 149.1 (2002). Pp. 1–95. ISSN: 0020-9910.

[Laf07] V. Lafforgue. “*K*-théorie bivariante pour les algèbres de Banach, groupoïdes et conjecture de Baum-Connes. Avec un appendice d’Hervé Oyono-Oyono”. In: *J. Inst. Math. Jussieu* 6.3 (2007). Pp. 415–451. ISSN: 1474-7480.

[Lub94] A. Lubotzky. *Discrete groups, expanding graphs and invariant measures*. Vol. 125. Progress in Mathematics. With an appendix by Jonathan D. Rogawski. Basel: Birkhäuser Verlag, 1994. Pp. xii+195. ISBN: 3-7643-5075-X.

[LZ07] A. Lubotzky and A. Żuk. *On property (τ)*. in preparation, 2007.

[MY02] Igor Mineyev and Guoliang Yu. “The Baum-Connes conjecture for hyperbolic groups”. In: *Invent. Math.* 149.1 (2002). Pp. 97–122. ISSN: 0020-9910.

[Oza00] N. Ozawa. “Amenable actions and exactness for discrete groups”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 330.8 (2000). Pp. 691–695. ISSN: 0764-4442.

[Oza03] N. Ozawa. “An application of expanders to $\mathbb{B}(l_2) \otimes \mathbb{B}(l_2)$”. In: *J. Funct. Anal.* 198.2 (2003). Pp. 499–510. ISSN: 0022-1236.

[Roe88] J. Roe. “An index theorem on open manifolds. I, II”. In: *J. Differential Geom.* 27.1 (1988). Pp. 87–113, 115–136. ISSN: 0022-040X.

[Roe96] J. Roe. *Index theory, coarse geometry, and topology of manifolds*. Vol. 90. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. Pp. x+100. ISBN: 0-8218-0413-8.

[Roe03] J. Roe. *Lectures on coarse geometry*. Vol. 31. University Lecture Series. Providence, RI: American Mathematical Society, 2003. Pp. viii+175. ISBN: 0-8218-3332-4.

[Ska88] G. Skandalis. “Une notion de nucléarité en K-théorie (d’après J. Cuntz)”. In: *K-theory* 1.6 (1988). Pp. 549–573.

[STY02] G. Skandalis, J. L. Tu, and G. Yu. “The coarse Baum-Connes conjecture and groupoids”. In: *Topology* 41.4 (2002). Pp. 807–834. ISSN: 0040-9383.

[Tu99] J. L. Tu. “La conjecture de Baum-Connes pour les feuilletages moyennables”. In: *K-Theory* 17.3 (1999). Pp. 215–264. ISSN: 0920-3036.

[Tu00] J. L. Tu. “The Baum-Connes conjecture for groupoids”. In: *C^*-algebras (Münster, 1999)*. Berlin: Springer, 2000. Pp. 227–242.

[Ulg05] S. Ulgen. “*K*-exact group C^*-algebras”. PhD thesis. Purdue University, 2005.

[Was91] S. Wassermann. “C^*-algebras associated with groups with Kazhdan’s property *T*”. In: *Ann. of Math. (2)* 134.2 (1991). Pp. 423–431. ISSN: 0003-486X.

[Yu95a] G. Yu. “Baum-Connes conjecture and coarse geometry”. In: *K-Theory* 9.3 (1995). Pp. 223–231. ISSN: 0920-3036.

[Yu95b] G. Yu. “Cyclic cohomology and higher indices for noncompact complete manifolds”. In: *J. Funct. Anal.* 133.2 (1995). Pp. 442–473. ISSN: 0022-1236.

[Yu98] G. Yu. “The Novikov conjecture for groups with finite asymptotic dimension”. In: *Ann. of Math. (2)* 147.2 (1998). Pp. 325–355. ISSN: 0003-486X.

[Yu00] G. Yu. “The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space”. In: *Inventiones Mathematicae* 193 (2000). Pp. 201–240.