FERMIONS AND BOSONS IN LOOP QUANTUM GRAVITY AND ITS COSMOLOGICAL IMPLICATIONS

by

RUPAM DAS

Dissertation
Submitted to the Faculty of the
Graduate School of Vanderbilt University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in

Physics

December, 2008
Nashville, Tennessee

Approved:
Robert Scherrer
Martin Bojowald
Thomas Kephart
Will Johns
John Ratcliffe

ABSTRACT

FERMIONS AND BOSONS IN LOOP QUANTUM GRAVITY AND ITS COSMOLOGICAL IMPLICATIONS

RUPAM DAS

Dissertation under the direction of Robert J. Scherrer and Martin Bojowald

Canonical gravity in real Ashtekar–Barbero variables is generalized by extending Holst's original vacuum analysis to allow for both fermionic and bosonic matter. For fermions, the effects of emerging torsion on parity and the known canonical (loop) quantization of gravity are discussed on the basis of the classical analysis by eliciting the necessity for adaptations. Then the role of these matter fields in homogeneous models of loop quantum cosmology is explored by discussing their influence on the behavior of the Wheeler-deWitt equation for the wave function of the universe. Next, quantum gravity corrections to equation of state parameters for Maxwell and Dirac fields arising from the discrete geometry of loop quantization are computed to investigate its effect on Big Bang Nucleosynthesis (BBN) to place bounds on these corrections and especially the patch size of discrete quantum gravity states. Finally, an example of the effects of these corrected equation of state on the evolution of dark energy, in particular k-essence and quintessence, is presented.

Approved: ______________________ Date: ______________

This work is dedicated to my beloved spiritual master Gurumayi Swami Chidvilasananda.

ACKNOWLEDGEMENTS

I wish to express my appreciation to those friends and mentors who provided encouragement and support to me throughout this endeavor. I am deeply indebted to Prof. Robert Scherrer for his unyielding support. I am sincerely grateful to Dr. Martin Bojowald for his selfless help with this research at the Institute for Gravitation and the Cosmos. I also thank Prof. Tom Kephart for his support.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER I

INTRODUCTION

Loop Quantum Gravity (LQG) is a non-perturbative, background-independent and canonical quantum theory of gravity, a theoretical endeavor to combine two empirically successful yet apparently incompatible theories of the twientieth century: Quantum Mechanics and General Relativity (GR). The central idea in this approach is to find a gauge theory for GR similar to Yang-Mills theory which respects the crucial lesson of GR that gravity is geometry and then possibly promote this reformulation to a quantum theory by taking advantage of the mathematical machinery available to quantize Yang-Mills theory. An immediate consequence of a successful implementation of this idea is the emergence of a non-perturbative and background-independent classical framework which in turn opens a door for a possible quantization scheme for gravity.

It is obvious that any splitting of the space-time metric, $g_{\mu\nu} = \eta_{\mu\nu} + Gh_{\mu\nu}$ with $\eta_{\mu\nu}$ being a flat space Minkowski metric, $h_{\mu\nu}$ being assigned the role of dynamical variable and Newton's constant G playing the role of the coupling constant, as used in the perturbative approaches developed by particle physicists, disregards the important lesson that gravity is geometry, i.e. the space-time metric plays a dual role: it determines geometry as well as encodes the physical gravitational field. Therefore, a prior background structure or stage on which the dynamics of the gravitational field takes place must be avoided. Hence, a demand of a theory without any metric yet one that captures the dynamics of the very stage ensues. Fortunately, LQG appears to meet this demand with the success of the canonical formulation of general relativity in the real Ashtekar-Barbero variables [1] since this formulation recasts gravity as a gauge theory similar to Yang-Mills theory, which in turn allows one to formulate a quantum theory. A brief review of the basic elements of LQG is presented in this chapter.

1.1 Connection Theory of GR

How can GR be recast as a dynamical theory of connections so that it can be brought closer to gauge theories? Let us recall that the concept of connections enters gauge theories through the local gauge symmetries. The local gauge symmetry, i.e. the invariance of a gauge theory like Yang-Mills theory under local gauge transformations demands an introduction of a gauge connection A_μ in the covariant derivatives of space-time fields such as fermion fields in the following manner

$$D_\mu \equiv \partial_\mu - igA^I_\mu T_I \,, \tag{1.1}$$

where A^I_μ are the gauge connections, T_I are hermitian generators for infinitesimal gauge tranformations and g is the coupling constant of the theory. Here $\mu, \nu, \ldots$ denote space-time indices and $I, J, \ldots$ internal indices for the gauge group. Also, T_I's satisfy $[T_I, T_J] = if^K_{IJ}T_K$ with f^K_{IJ} being the structure constant of the corresponding Lie algebra. Most importantly, the connections A_μ themselves transform under the corresponding gauge group as $A \to g^{-1}Ag + g^{-1}dg$ for $g \in G$ (gauge group) with dg implying the exterior derivative[1]. The connections take values in the Lie algebra of G since they help to describe infinitesimal transformations. Intuitively, a connection can be interpreted as a gauge field or potential which connects geometric objects like vectors or one-forms at two local points on a manifold, e.g. a genralization of the electromagnetic potential. A similar construction may be made for GR such that, in the Hamiltonian framework, all theories share the same kinematics. However, the major difference lies in the dynamics since the dynamics of the connection framework for GR does not require any background while that for the gauge theories for other

[1]Exterior derivatives are defined on differential forms. A differential p-form is a totally anti-symmetric tensor of type $(0,p)$, and the exterior derivative d maps p forms onto $(p+1)$ forms.

interactions does.

Furthermore, it is important to note that all gauge theories are necessarily *constrained* or sometimes called *singular* systems; please see [9, 10] for details. In general, some redundancy in the solutions stemming from the same initial conditions is present in a singular system, and thus the local symmetries relate these apparently different solutions. In other words, some additional restrictions on the initial conditions must be imposed in order to remove the redundancy and these restrictions appear as constraints, not as the form of equations of motion. This always happens as long as the initial data surface is not affected by the local gauge transformations. Thus the constraints are functionals on the phase space of the relevent theory and must be zero for being independent of the time evolution. Therefore, an attempt to find a connection theory for gravity[2] necessarily yields a singular system which is briefly presented below.

The starting point for the connection theory of GR is a classical space-time manifold M which is assumed to be topologically $\Sigma_t \times \mathbb{R}$, equipped with a fixed orientation and Σ_t an oriented, compact or asymptotically flat[3] 3-manifold. Now, some background independent space-time fields must be introduced to capture the gravitational interaction, and it turns out that the tetrad[4] formalism is convenient for this purpose since even the standard model, in particular- fermions, require this formalism, which is explored in detail in Chapter II. Thus, the basic space-time fields consist of a pair, $(e^\mu_I, \omega_\mu{}^{IJ})$, of a tetrad e^μ_I and a Lorentz connection one-form[5] $\omega_\mu{}^{IJ}$ in the convention for the index notations defined above. With these definitions, the

[2] GR must be invariant under space-time diffeomorphisms or covariant under coordinate transformations which are local symmetry transformations.

[3] The complement of a compact set in Σ is diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$

[4] A tetrad or vierbein is a frame field or an orthonormal set of four vector fields, one timetike and three spacelike

[5] The connection one-form $\omega_\mu{}^{IJ}$ takes values in the Lie algebra $so(3,1)$ via map $\omega : V \to so(3,1)$ by definition.

Hilbert-Palatini action, a first order[6] action that encodes GR, is given by [4]

$$\begin{aligned} S_P\left[e,\omega\right] &= \frac{1}{32\pi G}\int_M \epsilon_{IJKL} e^I \wedge e^J \wedge \Omega^{KL} \\ &= \frac{1}{16\pi G}\int_M \mathrm{d}^4x \; e e^\mu_I e^\nu_J F_{\mu\nu}{}^{IJ}(\omega)\,, \end{aligned} \tag{1.2}$$

where the first line is written in terms of differential forms and the second line in components. Here, ϵ_{IJKL} is an alternating tensor on the fixed 4-D vector space V (internal space of tetrads) equipped with a fixed metric η_{IJ} of Lorentzian signature $(-,+,+,+)$ such that its orientation agrees with that fixed on Σ and $\Omega := d\omega + \omega\wedge\omega$ is the curvature of the connection one-form $\omega_\mu{}^{IJ}$. The advantage of using differential forms is that they can be integrated without any background structure such as a metric and thereby render this formalism *background-independent*. In the components, $F^{KL}_{\mu\nu}(\omega) = 2\partial_{[\mu}\omega^{IJ}_{\nu]} + [\omega_\mu, \omega_\nu]^{IJ}$ is the curvature and e is the determinant of co-tetrad e^I_μ needed to make the action integral invariant under general coordinate transformations since the co-tetrads provide an isomorphism between the tangent space T_xM and V at each point $x \in M$ for the chosen topological structure on M. Most importantly, the co-tetrad determines the familiar space-time metric $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$, which implies that the space-time metric is a derived (not fundamental) quantity. This gives rise to a much desired background-independent formalism. This action S_P is invariant under diffeomorphisms on M and local $SO(3,1)$ transformations, i.e.,

$$(e,\omega) \longmapsto (ge, geg^{-1} + gdg^{-1}) \quad \text{for} \quad g \in SO(3,1)\,. \tag{1.3}$$

Now, the Einstein's vacuum equation can be retrieved by extremizing this action with respect to variations in $\omega_\mu{}^{IJ}$ and e^μ_I. While the variations in $\omega_\mu{}^{IJ}$ yield that the connection is completely determined by the tetrad, the variations in e^μ_I reproduce the Einstein's vacuum equation; please see [7] for details.

[6]A first order formalism treats tetrad e^μ_I and the connection $\omega_\mu{}^{IJ}$ independently in the variation while the second order formalism treats $\omega[e]$, i.e. the connection as a function of the tetrads. This distinction is shown to be important in Chapter II.

At this point, one can perform a Legendre transform of this action by carrying out a 3+1 decomposition presented in section 2.2 to obtain the same set of constraints as those derived in the triad version of the ADM formulation, which unfortunately results in losing all reference to connection-dynamics. Most importantly, a passage to a possible quantum theory of gravity in the ADM formulation has been unsuccessful because of the complicated forms of the constraints as they are non-polynomial in the canonical variables. This in turn necessitates the reformulation of canonical gravity in terms of the self-dual[7] (complex) connections known as the Ashtekar variables. The main idea in the Ashtekar formulation is the use of complex variables such as complex tetrads, and Lorentz connections, hence the complex action, yields algebraically simple constraints [7]. Then the real GR can be recovered by imposing appropriate *reality* conditions.

This reformulation of gravity, expressed in Ashtekar variables as a dynamical theory of complex-valued connections, has the advantage of obtaining algebraically simple constraints. However, rather complicated reality conditions have to be imposed on the basic canonical variables in order to recover real, Lorentzian general relativity. Moreover, since holonomies[8] of the complex Ashtekar connections take values in the non-compact gauge group $\mathrm{SL}(2,\mathbb{C})$, this approach prevents one from taking advantage of much of the available mathematical arsenal of gauge theory built upon compact gauge groups. Therefore, real su(2) valued Ashtekar-Barbero connections, that is, $A_a^i = \Gamma_a^i + \gamma K_a^i$ (with the spin connection Γ_a^i, K_a^i being a 1-form derived from extrinsic curvature and the Barbero–Immirzi parameter γ [3, 11] taking any non-zero real value), have mainly been used for the passage to a quantum theory of gravity. The Ashtekar-Barbero connections A_a^i appear naturally in the canonical analysis as projections of real Lorentz connection ω_a^{IJ} on Σ_t, which is shown in details in section 2.2.

[7] A self-dual connection is a connection which satisfies $\omega_a^{IJ} = -\frac{i}{2}\epsilon^{IJ}{}_{KL}\omega_a^{KL}$.
[8] Please see section 1.2.

Real variables were initially introduced by Barbero in a purely canonical formalism [3] which left the relation of the real connection to possible pull-backs[9] that of space-time objects unclear. Holst, motivated by this issue, carried out an analysis in [2] to re-derive Barbero's canonical formulation from an action which generalizes the ordinary Hilbert-Palatini action (1.2) and the Holst action is given by

$$S_H\left[e,\omega\right] = \frac{1}{16\pi G}\int_M \mathrm{d}^4x\; e e^\mu_I e^\nu_J \left(F_{\mu\nu}{}^{IJ}(\omega) - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}F_{\mu\nu}{}^{KL}(\omega)\right) . \tag{1.4}$$

Notice that the Holst action differs from the Hilbert-Palatini action (1.2) by the presence of the second term in (1.4). It turns out that the inclusion of this second term endows the Holst action with a desired property suited to our current purpose. Namely, the addition of the second term does not change the classical equations of motion for GR, yet it seems indispensable for formulating a quantum theory. More precisely, extremizing the Holst action with respect to variations in the real Lorentz connection ω_μ^{IJ} still results in the Levi-Civita spin connection, i.e. $\omega_\mu^{IJ} = e^I_\nu \nabla_\mu e^{\nu J}$, which is completely determined by the tetrads; please see section 2.1. Then upon inserting this connection in the action and using the Bianchi identity, the second term vanishes identically and thus the equations of motion remain unchanged [2]. Therefore, the Holst action differs from the Hilbert Palatini action by at most a canonical transformation, yet it is most suitable for the passage to a quantum theory of GR.

Now, a canonical analysis or the Legendre transform of this action is required for the passage to quantization. An extensive analysis of the Einstein-Cartan action describing fermions coupled to gravity is explored in Chapter II, and it follows that all the necessary classical results for quantization, e.g. canonical variables, symplectic structures, and the first class constraints discussed below, for the vacuum case, as

[9] A pull-back is an induced map between two vector spaces. Let $F : W \to V$ be a linear map and $\phi : V \to \mathbb{R}$ be a one-form, then the pull-back of ϕ by F is defined naturally by $F^\star\phi := \phi \circ F$. Thus, the differential forms are pulled back by a linear map between vector spaces or manifolds. On the other hand, tangent spaces are pushed forward between manifolds.

presented in [2], can easily be recovered in the limit of a vanishing matter field. Therefore, only the main results, without any detail, necessary for loop quantization of GR are presented below.

The Legendre Transform

The basic canonical variables that emerge after the Legendre transformations, as evident from (2.35), are the Ashtekar-Barbero connection $A_a^i = \Gamma_a^i + \gamma K_a^i$ and the densitized[10] triad (of density weight 1) $P_i^a = \sqrt{|\det(\mathcal{E}_i^a)|}\ \mathcal{E}_i^a/\gamma\kappa$ with triads $\mathcal{E}_i^a$ and $\kappa = 8\pi G$. Here, Γ_a^i is the $su(2)$-valued 3-D spin connection compatible with the co-triad e_a^i and γK_a^i the $su(2)$-valued extrinsic curvature with $i, j, \ldots$ denoting the $SO(3)$ internal indices and $a, b, \ldots$ the purely spatial indices. It is interesting to note that the results from the Ashtekar formulation with self-dual (or anti self-dual) connections can be retreived by merely setting $\gamma = \pm i$ respectively in the above definition of A_a^i or in the Holst action (1.4). Thus, the Holst analysis generalizes the Ashtekar's self-dual formalism of gravity.

Next, in terms of these configuration variables, the symplectic structure can be expressed as:

$$\Omega(\delta_1, \delta_2) \;=\; \int_\Sigma d^3x\ (\delta_1 P_i^a \delta_2 A_a^i - \delta_2 P_i^a \delta_1 A_a^i) \tag{1.5}$$

for all tangent vectors δ_1 and δ_2 to the canonical phase space Γ consisting of pairs (A_a^i, P_i^a) of fields on the 3-manifold Σ. Note that the symplectic framework geometrizes the Hamiltonian description of classical systems, thus making it coordinate independent. A symplectic structure or form is a closed and non-degenerate 2-form. The availability of the symplectic form on a symplectic manifold like the phase space Γ enables one to construct vector fields that generate canonical transformations from functions such as the Hamiltonian on that phase space; please see [8] for an extensive

[10] Let us recall that the metric determinant $q = \det(q_{ab})$ is a scalar density of weight +2. Here, $\sqrt{|\det(\mathcal{E}_i^a)|} = \sqrt{q}$ makes P_i^a a densitized triad of weight +1 with a factor $1/\gamma\kappa$.

discussion. Therefore, these phase space functions that generate canonical transformations encode the symmetries of classical systems or, more precisely, constrained systems like GR and thus they are the constraints of the classical system. The smeared constraints for GR in the real Ashtekar-Barbero connection are given by the following relations.

Gauss Constraint:

$$\mathcal{C}_G[\Lambda] \quad := \quad \int_\Sigma \mathrm{d}^3x\ \Lambda^i G_i \quad \text{with} \quad G_i := \mathcal{D}_b P_i^b := \partial_b P_i^b + \epsilon_{ij}{}^k A_b^j P_k^b \tag{1.6}$$
$$\text{and for any smooth field } \Lambda^i \in su(2) \text{ on } \Sigma\ .$$

Diffeomorphism Constraint:

$$\mathcal{C}_{\mathrm{Diff}}[\vec{N}] := \int_\Sigma \mathrm{d}^3x\ N^a C_a \quad = \quad \int_\Sigma \mathrm{d}^3x\ N^a \left(P_j^b F_{ab}^j - A_a^i G_i\right) \tag{1.7}$$

for each smooth vector field N^a on Σ and $F_{ab}^j = 2\partial_{[a}A_{b]}^j + \epsilon^j{}_{kl}A_a^k A_b^l$ is the curvature of the Ashtekar-Barbero connection. Here we have used (1.6) to remove the part that generates internal gauge transformations which is shown below. And finally, *Hamiltonian or scalar Constraint*:

$$C[N] \quad := \quad \frac{\gamma^2\kappa}{2}\int_\Sigma \mathrm{d}^3x\ N\frac{P_i^a P_j^b}{\sqrt{|q|}}\left(\epsilon^{ij}{}_k F_{ab}^k - 2(\gamma^2+1)K_{[a}^i K_{b]}^j\right)\ . \tag{1.8}$$

It turns out that these are first class[11] constraints and, as expected, generate canonical transformations. Hence, let us briefly explore the canonical transformations these constraints generate, which play crucial roles in dynamics of loop quantization.

Now, it follows from the Poisson algebra[12] that the Poisson brackets of the canonical variables (A_a^i, P_i^a) with each of these constraints will reflect the action they will have on these variables. For example, in classical mechanics, the Poisson bracket $\{q, H(q,p)\} = \dot{q}$ reflects the time evolution of the generalized coordinate q generated

[11]By definition, the first class constraints of a classical system weakly Poisson commute with each other; more practically, the Poisson brackets of the first class constraints can be expressed as linear combinations of themselves[9].

[12]A Poisson algebra is a commutative ring with elements such as constraints or phase space functions with the Poisson bracket $\{,\}$ as the binary action.

by the Hamiltonian. Once the transformations of the canonical variables induced by these constraints are known, the behavior of any phase space function can be obtained by computing the Poisson brackets of the function and the constraints.

The effects of Gauss constraint can be seen from the following relations

$$\left\{A^i_a, C_G(\Lambda)\right\} = -(\partial_a \Lambda^i + \epsilon^i{}_{jk} A^j_a \Lambda^k) \quad \text{and} \quad \left\{P^a_i, C_G(\Lambda)\right\} = \epsilon_{ij}{}^k \Lambda^k P^a_i \ , \tag{1.9}$$

Clearly, $C_G(\Lambda)$ generates the internal rotations along Λ^i since the above Poisson bracket imply the infinitesimal gauge transformations of A^i_a and P^a_i, which can easily be derived from (1.3) by using $g = e^{i\Lambda^i}$. Next, the Diffeomosphism constraint implies, as can be deduced from the following Poisson brackets

$$\left\{A^i_a, C_{\text{diff}}(\vec{N})\right\} = \mathcal{L}_{\vec{N}} A^i_a \quad \text{and} \quad \left\{P^a_i, C_{\text{diff}}(\vec{N})\right\} = \mathcal{L}_{\vec{N}} P^a_i \ , \tag{1.10}$$

that it generates diffeomorphisms[13] along $\vec{N}$. An important distinction between the actions of these two constraints is that while the Gauss constraint rotates the fields at a single point locally on Σ, the diffeomorphism constraint moves the fields from one point to another on Σ implying intrinsic non-locality. Finally, the Hamiltonian constraint generates time evolution 'off' Σ as one might expect. With these, the complete Poisson algebra can be computed with some lengthy computations to show that these constraints are indeed first class [4]. This completes the brief review of the connection theory of GR. The next step is to carry out canonical quantization of this reformulation of GR, which is briefly reviewed in the next section.

1.2 Elements Of Loop Quantization

Quantization of a classical system is the search for a corresponding quantum system with the correct classical limit. There is no a priori reason for the existence of a unique quantum system for a given classical system, thus causing the possibility

[13] A diffeomorphism $f : \Sigma \to \Sigma$ is a bijection with smooth f and f^{-1}.

of 'quantization ambiguity.' In general, the simplest quantization of a given classical system seems to be the physically correct one. Therefore, the fundamentals of the currently existing (perhaps the simplest) non-perturbative, background-independent and canonical quantization, known as loop quantization, of gravity is presented; see [4, 5, 6, 21, 25] for details.

A standard quantization program for an unconstrained classical system usually begins with the identification of the phase space, Γ, of the corresponding classical system. The phase space can naturally be endowed with a mathematical (differential) structure of a cotangent bundle[14] over a smooth, orientable manifold $\mathcal{C}$, known as the configuration space of the system. The state of the classical system is completely determined by a point (q, p) on Γ for each $q \in \mathcal{C}$. Next, the dynamical variables are defined by smooth real-valued functions on Γ since these functions are known to induce vector fields with the help of the natural symplectic structure on Γ[8]. The flows or the integral curves generated by these vector fields represent the dynamical trajectories of the system. Thus, a subset $\mathcal{S}$ of the set of smooth real-valued functions on Γ containing the elementary variables, such as configuration and momentum variables, must be chosen judiciously for the quantization purpose since they are to be promoted to self-adjoint quantum operators directly. This subset has to be large enough to generate the full algebra of functions on Γ to encompass the correct number of degrees of freedom in the theory, and at the same time, small enough to be closed under Poisson brackets. Then finding a representation of the elements of $\mathcal{S}$ by operators on a Hilbert space such that the Poisson bracket between any two elementary variables is represented by $i\hbar$ times the commutator of the corresponding operators is known as quantization. Also, real elementary variables are represented by symmetric operators to ensure self-adjointness[7].

So far, the quantization program described above applies to systems without

[14]A cotangent bundle, dual of the tangent bundle, of a smooth manifold may be thought of the collection of all the cotangent vactor spaces at every point in the manifold.

constraints only; it cannot be immediately extended to constrained systems. In the presence of constraints, not all points of Γ are accessible to the physical system. Only those points (configurations and momenta) which satisfy the constraints, i.e. points in the reduced phase space, are allowed, which in turn makes the selection of the elementary variables crucial to a successful quantization. In order to quantize the constraint systems, one usually adopts the two successful quantization strategies: the reduced phase space method and the Dirac quantization procedure. In the reduced phase space method, the constraints are eliminated classically to quantize the resulting constraint-free Hamiltonian system. However, there is no guarantee that the reduced phase space will support a cotangent bundle structure, thereby making the above procedure ineffective for quantization. On the other hand, the elementary variables on the full phase space, hence the contangent bundle, that generate a suitable Poisson algebra are identified first in Dirac's procedure. Then, finding a representation of this algebra on some complex vector space $\mathcal{V}$ offers a quantization of the unconstrained system. In order to obtain the physical states satisfying the constraints, the quantum analogs of the classical constraints are constructed from the elementary operators first and then the quantum constraints are solved to pick out the physical states. Finally, one extracts physics from the physical states by defining an appropriate Hermitian inner product on the space of physical states, $\mathcal{V}_{phys}$. Alternatively, a Hermitian inner product may be defined appropriately on the full complex vector space $\mathcal{V}$ and then $\mathcal{V}_{phys}$ is obtained by solving the constraints. Next, since $\mathcal{V}_{phys}$ is a subspace of $\mathcal{V}$, an inner product on $\mathcal{V}_{phys}$ may be automatically obtained.

Now, it is evident from section 1.1 that the constraints for the connection theory of gravity are complicated and thus difficult to solve classically. Therefore, the Dirac's quantization procedure is adopted for quantizing gravity. Moreover, a Hermitian inner product is defined on the full kinematic vector space of cylindrical functions as shown below. The physical states are obtained by solving the quantum constraints.

Kinematics

First, as discussed above, the classical configuration variables for gravity are SU(2)-connections on a principal fiber bundle[15] over the spatial manifold Σ, represented by smooth su(2)-valued local 1-forms A_a^i; the space $\mathcal{A}$ of all such 1-forms is the classical configuration space. The phase space is coordinatized by the pair (A_a^i, P_i^a), where P_a^i is the conjugate momentum, an su(2)-valued vector density on Σ proportional to the densitized triad. Then the only non-vanishing Poisson bracket is

$$\left\{A_a^i(x), P_j^b(y)\right\} = \delta_j^i \delta_a^b \delta(x, y)\,. \tag{1.11}$$

Now it is tempting to quantize the theory by promoting these basic variables to quantum operators and then turning their Poisson bracket into a commutator. However, such scheme does not work in this case since the commutator between them implies an operator-valued distribution[16] due to the presence of the delta function on the right side as encountered in infinite dimensional field theories, which in turn makes these elementary operators ill-defined at a point. Thus no well-defined quantum analogs for these canonical variables are known in a direct form mainly due to the infinite number of degrees of freedom.

Fortunately, borrowing the techniques from lattice gauge theory, this problem with finding well-defined basic operators can be overcome by smearing the configuration variable A_a^i over some finite number of one dimensional edges and its conjugate momenta P_i^a over two dimensional surfaces. It then becomes possible to build the infinite dimensional quantum theory from these new finite dimensional smeared canonical variables. Thus the elementary classical variables that have well-defined quan-

[15]A principal fiber bundle has a fiber identical to the structure group $G = SU(2)$.

[16]A distribution can be thought of a continuous linear functional on the space of well-behaved test functions.

tum analogs are instead given by (complex valued) matrix elements of holonomies[17] $h_e(A) \in \mathrm{SU}(2)$ along paths e in Σ and fluxes $P(S,f) := \int_S f_i \Sigma^i$, where f are su(2)-valued functions across 2-surfaces S in Σ and $\Sigma^i_{ab} = \epsilon_{abc} P^{ci}$ is the 2-form dual to the densitized triad.

With these elementary quantities, it is natural to construct quantum states known as 'cylindrical functions' on $\mathcal{A}$ through holonomies $h_e(A)$ along edges e of a graph α (a finite set of edges) in Σ. If a graph α has n edges, then, given a C^∞ complex-valued function ψ on $\mathrm{SU}(2)^n$, a cylindrical function Ψ_α on $\mathcal{A}$ can be defined as

$$\Psi_\alpha(A) := \psi(h_{e_1}(A), \ldots, h_{e_n}(A)) \ , \tag{1.12}$$

where the holonomies $h_{e_1}(A), \ldots, h_{e_n}(A)$ associate to every connection $A \in \mathcal{A}$ an n-tuple $(g_1, \ldots, g_n)$ of elements of SU(2). Let Cyl_α denote the space of such functions as Ψ_α with respect to the graph α and let $\mathrm{Cyl} = \cup_\alpha \mathrm{Cyl}_\alpha$ denote the space of all cylindrical functions. A natural inner product on Cyl_α can be introduced by defining a suitable measure $\mathrm{d}\mu_\alpha$ by

$$\langle \Psi_\alpha, \Phi_\alpha \rangle \ = \ \int d\mu_\alpha \overline{\psi}_\alpha \phi_\alpha := \int_{\mathrm{SU}(2)^n} \mathrm{d}\mu^n_{\mathrm{H}} \overline{\psi_\alpha(h_{e_1}, \ldots, h_{e_n})} \phi_\alpha(h_{e_1}, \ldots, h_{e_n}) \tag{1.13}$$

with the Haar measure[18] $\mathrm{d}\mu_{\mathrm{H}}$ on SU(2). The Cauchy completion[19] of Cyl_α with respect to this inner product gives rise to a Hilbert space $\mathcal{H}_\alpha := L^2[\overline{\mathcal{A}}_\alpha, \mathrm{d}\mu_\alpha]$, where $\overline{\mathcal{A}}_\alpha := \mathcal{A}_\alpha / \mathcal{G}^0_\alpha$ is the space of smooth connections restricted to the graph α modulo all local gauge transformations $g_\alpha \in \mathcal{G}^0_\alpha$ which are the identity on the vertices.

[17]The holonomy is the parallel transport of the connection A^i_a along an edge e defined by

$$\frac{d}{dt} h_e(t, t_1;\, A) \ = \ -A_a(e(t))\, \dot{e}^a(t)\, h(t, t_1; A), \quad \text{and} \quad h(t_1, t_1; A) = I \ ,$$

where $\dot{e}^a(t)$ is the tangent to the curve with the parameter t of the curve. The formal solution of this equation is given by $h_e(A) = \mathcal{P}\exp \int_e \mathrm{d}t A^i_a(e) \dot{e}^a(t) \tau_i$, where $\mathcal{P}$ denotes the path ordered integral and τ_i the basis in $su(2)$.

[18]The Haar measure is a way to assign 'invariant volume' under left or right translations to subsets of locally compact topological groups such as $SU(2)$ and subsequently define an integral of functions on these groups.

[19]A space X is Cauchy complete if every Cauchy sequence of elements in X converges to an element in X.

In Cyl_α, cylindrical functions are defined on a finite set of given edges, and thus they do not capture the full gauge invariant information in A. In order to capture the full information, we must consider all possible graphs in Σ, which in turn necessitates the introduction of an inner product on the space of all cylindrical functions in Cyl by extending (1.13). An immediate consequence of this generalization is that the same cylindrical function may be seen as associated with two or more different graphs. For instance, a given edge can be split in two, such that a function cylindrical with respect to the original graph now becomes a cylindrical function with the same connection dependence but is associated with a new graph containing the two separate halves of the split edge instead of the full edge. The above inner product then formally depends on the graphs used to perform the integral, which in turn imposes restrictions on the choice of suitable measures on $\mathrm{SU}(2)^n$. All these so-called cylindrical consistency conditions are satisfied if one uses the Haar measure to define integrations of holonomies [13].

The measure then extends to the full space Cyl and, by completion, defines the full Hilbert space $\mathcal{H} := L^2[\overline{\mathcal{A}}, \mathrm{d}\mu_{\mathrm{AL}}]$ where $\mathrm{d}\mu_{\mathrm{AL}}$ is the Ashtekar–Lewandowski measure constructed in this way and $\overline{\mathcal{A}}$ the space of generalized connections. The latter space represents the quantum configuration space as an enlargement from the classical configuration space $\mathcal{A}$ of connections by distributions. Via the action on graphs embedded in Σ, the Hilbert space carries a unitary action of the spatial diffeomorphism group.

An orthonormal basis on $\mathcal{H}$ is constructed by coloring each edge e of a graph α with a non-trivial irreducible representation j_e of SU(2) and an invariant tensor (a map from the tensor product of all edge representations to the trivial representation) l_v, called an intertwiner, at each vertex v. The intertwiner defines the way to contract in a gauge-invariant way all the matrices obtained by evaluating the edge holonomies for edges incident at v in their assigned representations. Intertwiners for a given vertex with incident edge representations in general allow different independent choices, forming a finite-dimensional vector space whose dimension equals the multiplicity by which the trivial representation occurs in the tensor product of incident representations.[20] Choosing a basis of the intertwiner spaces, the set of all such colored graphs provides an orthonormal basis[21] of $\mathcal{H}$ known as the *spin-network basis* [14]. Let $|S\rangle = |\alpha, j_n, l_m\rangle$ denote a spin-network state for a graph α with n edges and m vertices. Then a corresponding cylindrical function as a functional of connections A is written as

$$\Psi_S[A] \;=\; \langle A|S\rangle \;, \tag{1.14}$$

and thus the connection representation of the Poisson algebra is completed. However, historically, the loop representation, hence the name LQG, of the Poisson algebra was originally constructed from Wilson loops (edges with the same end points); for instance, see [12] for details. It has been shown that these two representations are equivalent and the connection representation generalizes the loop states to spin-network states to allow for edges with different end points.

After these preliminaries, the elementary quantum operators on $\mathcal{H}$ and their actions on the cylindrical functions can be defined by computing the Poisson brackets between the new elementary variables, namely the holonomy $h_e(A)$ and the smeared densitized triad $P(S, f)$. It turns out that the Poisson bracket of the new elementary variables $\{h_e(A), P(S, f)\}$ implies that the bracket vanishes if e and S do not intersect or e lies within the closure of S. However, if they have a simple intersection, then the bracket results in a linear combination of the configuration variables with the coefficients determined by the value of the smearing field f at the intersection point. On the other hand, the Poisson bracket between the momentum observables fails to commute since the vector fields $X_{(S,f)}$ on the configuration space used to define

[20] An example of a normalized interwiner for a vertex in a trivalent graph is the familiar Wigner $3j$ symbol used in the coupling of angular momenta.

[21] It follows from the Peter-Weyl theorem that a basis on the Hilbert space of L^2 functions on $SU(2)$ is given by the matrix elements of the irreducible representations of this group.

$P(S, f) := X_{(S,f)} \cdot P$ do not commute in general. This definition of the momentum variables follows from a similar construction of quantum mechanics on a compact Lie group G[7, 4].

It follows that every smooth function f on G defines a configuration variable and every smooth vector field X^i, a momentum variable $P_X := X^i p_i$ on the cotangent bundle $T^\star(G)$. As on any cotangent bundle, (non-trivial) Poisson brackets between them mirrors the action of vector fields on functions and the Lie bracket between vector fields:

$$\{P_X\ ,\ f\} = -\mathcal{L}_X f; \quad \text{and} \quad \{P_X,\, P_Y\} = -P_{[X,Y]}. \tag{1.15}$$

Most importantly, these observables have unambiguous quantum analogs. In the same spirit, classical configuration variables for the connection theory of gravity are represented by complex-valued, cylindrical functions f on $\bar{\mathcal{A}}$. The corresponding quantum operators $\hat{f}$ are defined to act by multiplication:

$$(\hat{f}\,\Psi)(\bar{A}) \;=\; f(\bar{A})\,\Psi(\bar{A}) \tag{1.16}$$

Next, the momentum operators $\hat{P}_{(S,f)}$, labelled by a 2-surface S and $su(2)$-valued smearing fields f^i on S, are defined by using left (right) invariant vector fields[22] L_i and R_i on $\bar{\mathcal{A}}$ respectively; the subscript i denotes the basis $\tau_i \in su(2)$. The use of left (right) invariant vector fields makes the action of the momentum operators depend only on the Lie derivatives with respect to L_i (R_i), and the corresponding momentum operators are denoted by $\hat{L}_i = \hat{J}_i^{(L)}$ ($\hat{R}_i = \hat{J}_i^{(R)}$). The notation $\hat{J}_i$ makes sense since it implies a generalization of the familiar 'angular momentum' operator which acts on the irreducible representations of $SU(2)$ on the edges and the vertices of spin-network. In fact, a more explicit notation of the operator associated with the edge e and the vertex v connected to the edge e can be denoted by $\hat{J}_i^{(v,e)}$, which can be regarded as the angular momentum operator associated with the edge e. Note that the vertex v

[22] A left (right) invariant vector field is a vector field which is invariant under left (right) translation.

determines if the action of this operator is through the left or right invariant vector field, i.e. $\hat{L}_i = \hat{J}_i$ if the vertex v is the origin of the edge and $\hat{R}_i = \hat{J}_i$ if v is the target of e. Similarly, operators on each vertex of α can also be defined, which can be regarded as the total angular momentum arriving at the corresponding vertex. Finally, following (1.15), the action of $\hat{P}_{(S,f)}$ is given just by the Poisson brackets between the classical momentum and configuration observables: For all $\Psi \in \mathrm{Cyl}$, we have:

$$(\hat{P}_{(S,f)}\Psi)(\bar{A}) \;=\; -i\hbar\mathcal{L}_X\Psi(\bar{A}) = i\hbar\{P(S,f),\,\Psi\}(\bar{A})\;, \tag{1.17}$$

with X being a left or right invariant vector field. More explicitly, if $\Psi \in \mathrm{Cyl}_\alpha$, we have

$$\hat{P}_{(S,f)}\,\Psi = \frac{\hbar}{2}\sum_v f^i(v)\left[\sum_{e \text{ at } v}\kappa(S,e)\hat{J}_i^{(v,e)}\,\Psi\right]\;, \tag{1.18}$$

where $\kappa(S,e) = 0$ if $S \cap e = \emptyset$ or $S \cap e = e$ modulo the end points, +1 if e lies above S, and -1 if e lies below S. The momentum operators (also known as the flux operator since it can be interpreted as the flux of the electric field through S) $\hat{P}_{(S,f)}$ in (1.18) are essentially well-dfined self-adjoint operators on Cyl because only a finite number of terms in the uncountable sum are non-zero when it acts on a cylindrical function. Furthermore, a little reflection reveals that the spin-network states are the eigenstates of this operator and it possesses a discrete spectrum containing *zero*, arising from the first condition $\kappa(S,e) = 0$, i.e. if no intersection exists between the suface S and the edges of the spin-network states. This fact makes the quantization of the Hamiltonian constraint trickier since it contains inverse densitized triads $q^{-\frac{1}{2}}$, which will be discussed later.

At this stage, the surprising *discrete* nature of the fundamental structure of space can be shown to emerge since the spectra of the spatial geometric operators such as area and volume [16, 17, 18] have discrete spectra containing zero. This is shown by building the quantum operators corresponding to the phase space functions representing area and volume from the flux operator $\hat{P}_{(S,f)}$. For example, a well-defined self-adjoint area operator $\hat{\mathbf{A}}_S$ for each surface $S \in \Sigma$ can be constructed from the phase space function for area $A(S) = \int_S \mathrm{d}^2x\,\sqrt{h}$, where h is the determinant of the intrinsic 2-metric h_{ab} on S. Here, S is assumed to be either a closed 2-dimensional sub-manifold of Σ or an open 2-dimensional sub-manifold without boundary.

Now, a suitable 'regularized area function' can be obtained by dividing S into a large number of elementary cells, S_I, with $I = 1, 2, \ldots N$. Next, on each cell, let us introduce an internal triad τ^i to use its elements as test fields f^i such that $P(S_I, \tau^i) := P^i(S_I)$. Then the area function can be approximated by

$$[A_S]_N \;=\; \gamma\kappa \sum_{I=1}^{N} \sqrt{P^i(S_I) P^j(S_I) \eta_{ij}}\;, \tag{1.19}$$

with $\eta_{ij} = -2\mathrm{tr}(\tau_i \tau_j)$ is the Cartan-Killing metric for all $\tau_i, \tau_j \in su(2)$, as the number of cells goes to infinity such that the coordinate size of the cells S_I goes to zero uniformly in I, i.e.

$$\lim_{N\to\infty}\, [A_S]_N \;= A_S\,. \tag{1.20}$$

Now, since each $P^i(S_I)$ gives rise to an unambiguously defined quantum operator, $\hat{P}^i(S_I)\hat{P}^j(S_I)\eta_{ij}$ is a positive definite self-adjoint operator on $\mathcal{H}$ with a well-defined (positive) square-root for each I. Thus the corresponding area operator is given by

$$[\hat{A}_S]_N \;:=\; \gamma\kappa \sum_{I=1}^{N} \sqrt{\hat{P}^i(S_I)\hat{P}^j(S_I)\eta_{ij}}\,. \tag{1.21}$$

The total area operator $\hat{\mathbf{A}}_S$ can be obtained in the limit $N \to \infty$, which corresponds to the operation of removing the regulator. This operator is self-adjoint on $\mathcal{H}$ and also gauge invariant and diffeomorphism covariant.

Next, since (1.18) implies that the flux operators $\hat{P}^i(S_I)$ have well-defined action on spin-networks, the area spectrum can be calculated from the action of $[\hat{A}_S]_N$ on twice differential cylindrical functions on $\bar{\mathcal{A}}$. Again, a non-zero contribution to the sum (1.21) arises only from those S_I which intersect α. For instance, fixing a surface S and considering only those states in Cyl for which the graph has no edge which lies within S and which are gauge invariant at each vertex where S intersects the graph, (in particular if all intersections of S with the graph are at simple bi-valent vertices) the area spectrum is computed to be

$$a_S \;=\; \gamma \ell_{\mathrm{P}}^2 c^{-3} \sum_I \sqrt{j_I(j_I+1)}, \tag{1.22}$$

where $\ell_{\mathrm{P}}^2 = \kappa\hbar$ defines the Planck's length ℓ_{P} and c is the speed of light in vacuum. Then the smallest nonvanishing eigenvalue for $\gamma = 1$ turns out to be $\sim 10^{-66}$ cm^2. A more general spectrum can be calculated by relaxing the above restrictions on the cylindrical functions.

Similarly, a well-defined, self-adjoint and non-negative volume operator $\hat{\mathbf{V}}_R$ corresponding to the phase space function,

$$\begin{aligned} V_R &= (\sqrt{\kappa\gamma})^3 \int_R \mathrm{d}^3x \sqrt{|\det P|} \\ &= (\sqrt{\kappa\gamma})^3 \int_R \mathrm{d}^3x \sqrt{|\frac{1}{3!}\epsilon^{ijk}\,\epsilon_{abc}\,P_i^a P_j^b P_k^c|} \end{aligned} \tag{1.23}$$

for an open subset $R \in \Sigma$, can be constructed. Furthermore, as shown in Chapter III, a well-defined $\hat{\mathbf{V}}_R$ is crucial to the quantization of the Hamiltonian constraint. However, since V_R is a rather complicated non-polynomial function of the densitized triads, the problem with regularization turns out to be considerably more complicated than that for area operators. Hence, the method for constructing $\hat{\mathbf{V}}_R$ and finding the volume spectrum turns out to be technically more subtle; please see [18, 4] for details. Nonetheless, $\hat{\mathbf{V}}_R$ possesses discrete spectrum with contributions only from the nodes of a spin-network,[23] while the area operator receives contributions only from the links that intersect S. Thus, the intertwiners associated with the nodes are quantum

[23]It follows from the presence of ϵ_{abc} in (1.23). The anti-symmeterization forced by ϵ_{abc} on suitably regulated V_R in (1.23) demands that the flux operators $\hat{P}^i(S^a)$'s be independent for non-vanishing contribution and this is satisfied only if each regulated cubic cell contains a node. Intuitively, a node can be imagined as a source so the absence of a source in a cube has vanishing contribution of the total flux across the boundary of the cube.

numbers of the volume and the spins associated with the links are quantum numbers of the area. This enables a spin-network with n nodes to be interpreted as an ensemble of n quanta of volume separated from each other by quanta of area of elementary surfaces. In other words, loop quantization brings about the cellular decomposition of physical space, each cell representing a quantum of volume. Therefore, our physical space appears to be fundamentally discrete in LQG.

So far, the discussion has been focused only on the kinematical Hilbert space $\mathcal{H} = L^2[\overline{\mathcal{A}}, \mathrm{d}\mu_{\mathrm{AL}}]$ spanned by the cylindrical functions in Cyl. This Hilbert space is too large and non-separable because it contains redundancy engendered by the diffeomorphism gauge, i.e. it contains states which are not invariant under diffeomorphism. However, GR demands that the physical states be invariant under internal gauge (encoded in the Gauss constraint) and diffeomorphism. It turns out that the physical Hilbert space $\mathcal{H}_{phys}$ obtained by factoring away both internal and diffeomorphism gauge is indeed separable. In addition, the evolution or the dynamics of the physical states must also satisfy the quantum Hamiltonian constraint. In other words, the physical states must be in the kernel of the quantum constraint equations, which are briefly discussed below.

Dynamics

Since the classical constraints (1.6-1.8) are phase space functions of the canonical variables (A_a^i, P_i^a), the well-defined quantum analogs of these constraints are constructed from the holonomies and the flux operators. However, such a procedure to find well-defined operators on the kinematic Hilbert space $\mathcal{H}$ constructed above is not as straightforward as it may seem, thus leading to one of the major challenges of quantum gravity.

The Gauss Constraint:

The cylindrical functions that are solutions to the quantum Gauss constraint are gauge-invariant under local gauge transformations, i.e. under $SU(2)$. One can solve the quantum Gauss constraint by first promoting $C_G(\Lambda)$ in (1.6) to a well-defined operator $\hat{\mathcal{C}}_G(\Lambda)$ on $\mathcal{H}$ and then finding the states that belong to the kernel $\mathcal{H}^G_{\text{inv}}$ of $\hat{\mathcal{C}}_G(\Lambda)$ for all $\Lambda \in su(2)$. In other words, the set of $\Psi_\alpha \in \text{Cyl}^{(1)}_\alpha$ for any α that satisfies $\hat{\mathcal{C}}_G(\Lambda)\Psi_\alpha = 0$ constitutes the gauge-invariant subspace $\mathcal{H}^G_{\text{inv}}$ of $\mathcal{H}$. Now, the gauge-invariant cylindrical functions can be constructed from holonomies and intertwiners on a given graph α by observing the behavior of holonomies and interwiners under such local gauge transformations. Fortunately, given a local $SU(2)$ rotation $g : \Sigma \to SU(2)$, despite the inhomogeneous transformation behavior of the Ashtekar-Barbero connection A^i_a in (1.3), holonomies transform homogeneously as

$$g \cdot h_e(A) = g(v_i)\, h_e(A)\, (g(v_f))^{-1}, \tag{1.24}$$

for all edges e in Σ with source v_i and target v_f. This follows from the requirement that the definition of holonomy be independent of the choice of bases, meaning the gauge transformations and the parallel translations should commute. Thus, the gauge transformations of holonomies depends only on the end points of edges, i.e. at the vertices. Finally, choosing an invariant intertwiner at each vertex to contract all the incident holonomies makes the spin-network states and therefore the cylindrical functions gauge-invariant by construction.

However, the choice of non-invariant or covariant intertwiners at vertices is also admissible insofar as the gauge-invariant states can be obtained upon imposing necessary conditions on the operators at the vertices. These conditions naturally follow from the quantum Gauss constraint. More explicitly, for any $\Psi_\alpha \in \text{Cyl}^{(1)}_\alpha$, the quantum Gauss constraint[24] becomes [4]

$$\hat{\mathcal{C}}_G(\Lambda)\, \Psi_\alpha = \hbar \sum_v \sum_e \, (\Lambda^i(v) \hat{J}^{(v,e)}_i)\, \Psi_\alpha \tag{1.25}$$

[24]This follows from using the heuristic ansätz $P \to -i\hbar\delta/\delta A$ in the Gauss constraint (1.6).

where the first sum extends over all vertices v of α and the second over all edges e meeting at v. Now, the condition for gauge invariance of cylindrical functions is that these elements of Cyl have zero eigenvalues for every vertex operator, implying $\sum_e \hat{J}_i^{(v,e)} = 0$, i.e. the total angular momentum at each vertex must be zero. Therefore, it follows that either $\mathcal{H}$ (by construction) or $\mathcal{H}^G_{\text{inv}} \subset \mathcal{H}$ are the home for the gauge-invariant states.

The Diffeomorphism Constraint:

The imposition of the diffeomorphism constraint is more complicated due to the following difficulties. First, the shift vector N^a used to obtain smeared constraint in (1.7) is a spatial vector field, so the values of its components depend on spatial coordinates. However, the spatial coordinates have no meaning on the Hilbert space, so what values N^a should take are undetermined, implying N^a cannot be quantized. As a result, the smeared diffeomorphism constraint (1.7) cannot be promoted to a quantum operator and hence finding the home for the diffeomorphism invariant states seems challenging. Second, since diffeomorphisms move graphs, the only element of $\mathcal{H}$ left invariant by the action of all diffeomorphisms is the constant function on $\bar{\mathcal{A}}$. More precisely, the cylindrical functions in Cyl_α becomes orthogonal to those in $\text{Cyl}_{\varphi\cdot\alpha}$ defined on the new graph if the diffeomorphism $\varphi : \Sigma \to \Sigma$ moves α. Thus, the infinitesimal generator of $\hat{\varphi}(\lambda)$ corresponding to the 1-parameter family $\varphi(\lambda)$ fails to exist since this family fails to be weakly continuous in λ. However, these difficulties can be overcome by working directly with finite diffeomorphisms and demand that physical states be invariant under the induced action $\hat{\varphi}$ of appropriate diffeomorphisms φ on Σ.

Now, how can the home for diffeomorphism-invariant states be found? A natural strategy is to use the 'group averaging procedure', generally available for such constraints[19, 20]. Physical states are obtained by averaging elements of Cyl over the group of diffeomorphisms Diff(Σ). Formally, the diffeomorphism-invariant states can be written as

$$\bar{\Phi} := \int_{\text{Diff}} [d\varphi]\, \varphi \star \Phi \ , \tag{1.26}$$

where $d\varphi$ is a measure on Diff(Σ) and $\varphi \star \Phi$ denotes the pull-back of Φ under φ. Now, the problem is that since the group Diff(Σ) is too large, the above integral may not converge to an element in Cyl. Thus $\bar{\Phi}$ does not belong to $\mathcal{H}$ since it is not normalizable.[25] However, it can be shown that a well-defined (linear) action of $\bar{\Phi}$ on a dense subset[26] $\mathcal{H}'$ of $\mathcal{H}$ in the following sense that, for every $\Psi \in \mathcal{H}'$,

$$\bar{\Phi} \cdot \Psi := \int_{\text{Diff}} [d\varphi]\, \langle \varphi \star \Phi | \Psi \rangle \ , \tag{1.27}$$

implies that the home for diffeomorphism-invariant states is the topological dual (if equipped with suitable topology) $\text{Cyl}^\star_{\text{diff}}$ of Cyl. Therefore, the elements of $\text{Cyl}^\star_{\text{diff}}$ are the linear functionals on the elements of Cyl and thus they are genuine distributions on $\bar{\mathcal{A}}$ rather than functions. Also, since the the scalar product on $\mathcal{H}$ is invariant under diffeomorphism, the group averaging procedure naturally endows the solution space with a Hermitian inner product,

$$\langle \bar{\Phi} \,|\, \bar{\Psi} \rangle := \bar{\Phi} \,\cdot\, \Psi \ . \tag{1.28}$$

Finally, the Cauchy completion of $\text{Cyl}^\star$ is the Hilbert space $\mathcal{H}_{\text{diff}}$ for diffeomorphism-invariant states. Thus, the passage to the diffeomorphism-invariant Hilbert space is given by $\text{Cyl} \subset \mathcal{H} \subset \text{Cyl}^\star_{\text{diff}} \subset \mathcal{H}_{\text{diff}}$.

The Scalar Constraint:

While the canonical transformations generated by the Gauss and the diffeomorphism constraints are *kinematical* gauge symmetries of the classical theory since

[25] Similar situation arises even in simple quantum mechanical systems such as a particle in $\mathbb{R}^3$ with a constraint $p_x = 0$. Solutions to the constraint fail to have finite norm in the kinematic Hilbert space $L^2(\mathbb{R}^3)$ and belong to a larger space, e.g. the space of distributions in $\mathbb{R}^3$.

[26] A is dense in X if the only closed subset of X containing A is X itself, i.e. the closure of A is X.

they operate on the 3-D spatial manifold Σ at a 'fixed time', the heart of quantum dynamics lies in the scalar constraint. Implementing it in the quantum theory also by a group averaging procedure is difficult to adopt because the *finite* canonical transformations generated by this constraint are not well-understood even at the classical level. Therefore, the procedure of constructing a quantum operator $\hat{\mathcal{C}}(N)$, similar to that used for the Gauss constraint, corresponding to the classical, smeared scalar constraint function (1.8) and then seeking its kernel is adopted. However, because of the intricate form of this constraint, its implementation is yet to be as clean and complete as that of the other two constraints.

As with area and volume operators, the quantization begins with regularizing the classical expression for the scalar constraint, which is usually achieved by introducing an appropriate partitioning of Σ with a regularization parameter δ. Then the scalar constraint is re-expressed as a Riemann sum under this regularization, involving only those phase space functions which have direct quantum analogs. Finally, the regulator is removed after replacing the classical quantities in the regularized expression by their quantum counterparts. More precisely, if $\mathcal{C}_{R_\delta}(N)$ denotes the regularized scalar constraint, the regulator is removed by taking the limit, $\lim_{\delta\to 0}\mathcal{C}_{R_\delta}(N) = \mathcal{C}(N)$. However, there is considerable freedom in the choice of a permissible classical regulator that satisfies the above limit. Therefore, genuine ambiguities, along with the ambiguities with factor ordering, arise in the regularization procedure and distinct avenues have been pursued to give rise to the *existence* of well-defined strategies. It turns out that the quantum operators corresponding to different choices of the regulators differ from each other and thus define distinct quantum dynamics. Whether any of them is fully viable from a physical perspective is still an *open* issue[22, 23, 24, 25, 26, 27, 28, 29]. A detailed loop quantization of the scalar constraint is presented in Chapter III.

Again, once the quantum operator for the scalar constraint is obtained, the next step is to find the home for its solutions by solving $\hat{\mathcal{C}}(N)|\Psi\rangle = 0$. As one might expect, due to the similar difficulties that arise in the case of diffeomorphism constraints, the true home for the states which are in the kernel of $\hat{\mathcal{C}}(N)$ is the topological dual of Cyl, i.e. $\text{Cyl}^\star$. However, since physical states must be in the kernel of all three constraints, the home for the diffeomorphism-invariant states $\text{Cyl}^\star_{\text{diff}}$ may also be considered as the home for the physical states. Unfortunately, the attempt to obtain the physically viable space for all physical states which have the correct semi-classical limit has been so far unsuccessful. This problem, known as the low energy problem, is currently under serious investigation.

This concludes the review of the fundamentals of LQG. The rest of this paper is devoted to the discussions on various applications, namely the matter like the fermions and bosons in LQG and its cosmological implications. It is important to note that matter fields are put in by hand in the theory. While Chapter II embodies the recount of the canonical formulation of both Dirac's and Maxwell's fields coupled to gravity, Chapter III encompasses the loop quantization of these matter fields. In addition, Chapter IV entails the homogeneous (anisotropic) models of our universe in LQG and the role of parity, and Chapter V delineates the possible observable effects of LQG on Big Bang Nucleosynthesis. Finally, Chapter VI provides an example of an application of the effects from other approaches to quantum gravity including LQG to mainstream cosmology, in particular, dark energy.

CHAPTER II

CANONICAL FORMULATION

When matter is considered coupled to classical or quantum gravity, several important issues arise for fermions. This is, e.g., related to the chirality and possible parity violation of spinors or the fact that they contribute torsion to the space-time geometry. In loop quantum gravity, fermions have been treated occasionally but not yet, as detailed below, in a complete manner. They are therefore revisited here especially with a canonical analysis in mind. In addition, a canonical analysis of Maxwell's field theory is presented to derive some interesting cosmological implications in the latter part of this article.

In this chapter, we further generalize Holst's analysis for pure gravity, as summarized in section 1.1, to allow for fermionic matter. In other words, we present a detailed derivation of the Hamiltonian formulation of the Einstein–Cartan action, which incorporates Holst's action for the gravitational part. This issue has been considered in the literature several times, but the available discussions appear incomplete. In addition to filling this gap in the classical analysis, details of the canonical formulation are crucial for a proper quantization of gravity in the presence of fermions.

In particular, non-zero torsion arising from the coupling of fermionic matter to gravity through the spin connection requires an analysis in terms of more general connections than used in Holst's analysis, which inherit torsion contributions. In our canonical analysis, we derive two sets of constraints: constraints in which variables have been split in their torsion-free terms and explicit torsion contributions, and constraints with torsion implicitly contained in the canonical gravitational variables. These splittings will turn out to be important to understand the canonical structure. To the best of our knowledge, our results for the given Einstein–Cartan action, despite some resemblance to those in [31, 21, 30], are complete and new. Moreover, we generalize the canonical treatment to arbitrary non-minimal coupling of fermions without any inconsistencies as they occur in other approaches.

We present those derivations in a detailed classical part in this paper, which can serve as a guide through some of the tedious but structurally important constructions underlying canonical quantum gravity. These details will show us the crucial changes implied by torsion for the general form of dynamics, and thus also play a role for any quantization based on a formulation in Ashtekar variables. Here our distinction between the two sets of variables, one with explicit torsion terms and one with implicit torsion dependence, will be crucial to show the parity invariance of the non-minimally coupled action. Thus, a discussion of classical parity transformations is also presented.

2.1 Lagrange Formulation

For fermions, one has to use a tetrad e^I_μ rather than a space-time metric $g_{\mu\nu}$, related by $e^I_\mu e^I_\nu = g_{\mu\nu}$, in order to formulate an action with the appropriate covariant derivative of fermions. This naturally leads one to a first-order formalism of gravity in which the basic configuration variables are a connection 1-form and the tetrad. In vacuum the connection would, as a consequence of field equations, be the torsion-free connection compatible with the tetrad. In the presence of matter fields which couple directly to the connection, such as fermions, this is no longer the case and there is torsion [33]. For completeness, we start by demonstrating this well-known origin of torsion in the theory of gravity non-minimally coupled to fermionic matter[1].

[1]Although we closely follow the derivation presented in [34, 35, 31] our presentation adds details to certain key issues.

Einstein–Cartan Action

The basic configuration variables in a Lagrangian formulation of fermionic field theory are the Dirac bi-spinor $\Psi = (\psi \ \ \eta)^T$ and its complex conjugate in $\overline{\Psi} = (\Psi^*)^T \gamma^0$ with γ^α being the Minkowski signature Dirac matrices. We note that ψ and η transform with density weight zero and are spinors according to the fundamental representations of $\mathrm{SL}(2,\mathbb{C})$. Then the non-minimum coupling of gravity to fermions can be expressed by the total action

$$\begin{aligned} S[e,\omega,\Psi] &= S_G[e,\omega] + S_F[e,\omega,\Psi] \\ &= \frac{1}{16\pi G}\int_M \mathrm{d}^4x\ e e^\mu_I e^\nu_J P^{IJ}{}_{KL} F_{\mu\nu}{}^{KL}(\omega) \\ &\quad + \frac{1}{2} i \int_M \mathrm{d}^4x\ e \left[\overline{\Psi}\gamma^I e^\mu_I \left(1 - \frac{i}{\alpha}\gamma_5\right)\nabla_\mu \Psi - \overline{\nabla_\mu \Psi}\left(1 - \frac{i}{\alpha}\gamma_5\right)\gamma^I e^\mu_I \Psi\right], \end{aligned} \tag{2.1}$$

where α is the parameter for non-minimal coupling. Note that we are using the notation of [30] for easier comparison. Also the action is composed of the gravitational contribution S_G and the matter contribution S_F resulting from the fermion field. Here, again $I, J, \ldots = 0,1,2,3$ denote the internal Lorentz indices and $\mu, \nu, \ldots = 0,1,2,3$ the respective space-time indices. For simplicity, we ignore fermionic mass terms or potentials as they do not provide further complications.

The first term in (2.1) is the Holst action [2] of gravity as presented in (1.4), e^μ_I is the tetrad field, e is its determinant, and e^I_μ its inverse. The Lorentz connection in this formulation is again denoted by ω^{IJ}_μ and $F^{KL}_{\mu\nu}(\omega) = 2\partial_{[\mu}\omega^{IJ}_{\nu]} + [\omega_\mu, \omega_\nu]^{IJ}$ is its curvature. In order to write the Holst action in a compact form, we have used the following tensor and its inverse

$$\begin{aligned} P^{IJ}{}_{KL} &= \delta^{[I}_K \delta^{J]}_L - \frac{1}{\gamma}\frac{\epsilon^{IJ}{}_{KL}}{2} \\ P^{-1}{}_{IJ}{}^{KL} &= \frac{\gamma^2}{\gamma^2+1}\left(\delta^{[K}_I \delta^{L]}_J + \frac{1}{\gamma}\frac{\epsilon_{IJ}{}^{KL}}{2}\right) \end{aligned} \tag{2.2}$$

where γ is again the Barbero–Immirzi parameter. Finally, the covariant derivative

∇_μ of Dirac spinors is defined by

$$\nabla_\mu \equiv \partial_\mu + \frac{1}{4}\omega_\mu^{IJ}\gamma_{[I}\gamma_{J]} \quad , \qquad [\nabla_\mu, \nabla_\nu] = \frac{1}{4}F_{\mu\nu}^{IJ}\gamma_{[I}\gamma_{J]} \tag{2.3}$$

in terms of Dirac matrices γ_I (which will always carry an index such that no confusion with the Barbero–Immirzi parameter should arise). Note that we are ignoring the gauge connection required for describing an interaction between charged fermions in the definition of the covariant derivative (2.3). However, this analysis can easily be generalized to incorporate such interactions.

At this point, it is noteworthy that we intend to use the signature $(-+++)$ (instead of $(+---)$ which is common in QFT) for both gravity and fermions since this is the signature most prevalent in the literature for canonical gravity. This demands certain modifications in the representations of the Clifford algebra, where it turns out that changing the signature from $(+---)$ to $(-+++)$ only requires all the Dirac matrices to be multiplied by i (the imaginary unit); see Appendix 0.1 for details.

Now the variation of the first-order action (2.1) with respect to the connection gives rise to the equation of motion for the connection. Using $\delta F_{\mu\nu}^{IJ} = 2\nabla_{[\mu}\delta\omega_{\nu]}^{IJ}$ and the anticommutator $[\gamma_K, \gamma_{[I}\gamma_{J]}]_+ = +2i\epsilon_{IJKL}\gamma^5\gamma^L$ (note the plus sign due to the signature-change) and the commutator $[\gamma_K, \gamma_{[I}\gamma_{J]}]_- = 4\eta_{K[I}\gamma_{J]}$, we obtain

$$\begin{aligned} \frac{\delta S_G}{\delta\omega_\mu^{IJ}} &= \frac{1}{8\pi G}\nabla_\nu(ee_K^{[\mu}e_L^{\nu]})P^{KL}{}_{IJ} && (2.4) \\ \frac{\delta S_F}{\delta\omega_\mu^{IJ}} &= \frac{iee_K^\mu}{8}\overline{\Psi}[\gamma_K, \gamma_{[I}\gamma_{J]}]_+\Psi - \frac{ee_K^\mu}{8\alpha}\overline{\Psi}\gamma_5[\gamma_K, \gamma_{[I}\gamma_{J]}]_-\Psi \\ &= -\frac{1}{4}ee_K^\mu\epsilon_{IJKL}\overline{\Psi}\gamma^5\gamma^L\Psi - \frac{1}{2\alpha}ee_K^\mu\overline{\Psi}\gamma^5\eta_{K[I}\gamma_{J]}\Psi\,. && (2.5) \end{aligned}$$

Note that the first line of the expression (2.5) is also invariant under the above signature-change. In other words, the Einstein-Cartan theory before proceeding to a canonical formulation is independent of the signature. Thus we finally obtain the equation of motion from varying the connection from (2.4) and (2.5)

$$\nabla_\mu(ee_I^{[\mu}e_J^{\nu]}) = 2\pi G e {P^{-1}}_{IJ}{}^{KL}\left(J^\nu_{KL} + \frac{2}{\alpha}e^\nu_K J_L\right), \tag{2.6}$$

where $J^\mu_{KL} := e^\mu_I \epsilon^I{}_{KLJ}J^J$ with the axial fermion current $J^L = -\overline{\Psi}\gamma^5\gamma^L\Psi$ (the minus sign appears due to the signature-change). It is immediate from (2.6) that the presence of a fermion field in the coupled action introduces a torsion component in the connection arising from the fermion current and thus the connection is no longer torsion free, that is, $\nabla_{[\mu}e^I_{\nu]} \neq 0$. In terms of connection variables, this issue has been explored in details in [34, 35, 30]. In order to solve for the connection, let us express it in the form $\omega^{IJ}_\mu = \widetilde{\omega}[e]^{IJ}_\mu + C^{IJ}_\mu$, where $\widetilde{\omega}[e]$ is the torsion free connection determined by the tetrad and C^{IJ}_μ is the tetrad projection of the contorsion tensor, $C^{\rho\sigma}_\mu$, i.e., $C^{IJ}_\mu = C^{\rho\sigma}_\mu e^I_{[\rho}e^J_{\sigma]}$. Then the action of the corresponding covariant derivatives on vectors with internal indices are related as follows:

$$\left(\nabla_\mu - \widetilde{\nabla}_\mu\right)V_I = C_{\mu I}{}^J V_J, \tag{2.7}$$

where $\widetilde{\nabla}_\mu$ is the covariant derivative compatible with the tetrad and hence the corresponding connection is torsion-free and V_J is an internal vector field. Now it follows from (2.7) that the two corresponding curvatures satisfy

$$F^{IJ}_{\mu\nu} = \widetilde{F}^{IJ}_{\mu\nu} + 2\widetilde{\nabla}_{[\mu}C_{\nu]}{}^{IJ} + [C_\mu, C_\nu]^{IJ} \tag{2.8}$$

where $\widetilde{F}$ is the curvature of the torsion-free connection. Inserting (2.8) in (2.1) and using (2.7), we obtain an action composed of the following torsion-free part and an interacting fermion contribution due to torsion:

$$\begin{aligned}
S\left[e,\omega,\Psi\right] &= S_G\left[e,\widetilde{\omega}\right]+S_F\left[e,\widetilde{\omega},\Psi\right]+S_{\text{int}}\left[e,C,\Psi\right] \\
&= \frac{1}{16\pi G}\int_M \mathrm{d}^4x\ ee^\mu_I e^\nu_J P^{IJ}{}_{KL}\widetilde{F}_{\mu\nu}{}^{KL}(\widetilde{\omega}) \\
&\quad +\frac{1}{2}i\int_M \mathrm{d}^4x\ e\left(\overline{\Psi}\gamma^I e^\mu_I\left(1-\frac{i}{\alpha}\gamma_5\right)\widetilde{\nabla}_\mu\Psi-\overline{\widetilde{\nabla}_\mu\Psi}\left(1-\frac{i}{\alpha}\gamma_5\right)\gamma^I e^\mu_I\Psi\right) \\
&\quad +\frac{1}{4}\int_M \mathrm{d}^4x\ e\left(e^\mu_I C_\mu{}^{JK}\epsilon^I{}_{JKL}J^L+\frac{2}{\alpha}C_\mu{}^{JK}e^\mu_J J_K\right) \\
&\quad +\frac{1}{16\pi G}\int_M \mathrm{d}^4x\ ee^\mu_I e^\nu_J P^{IJ}{}_{KL}[C_\mu,C_\nu]^{KL}
\end{aligned} \tag{2.9}$$

Notice that the middle term on the right hand side in (2.8) is ignored since it can be expressed as a total derivative and therefore does not contribute to the variation. The first two terms are just the torsion-free Holst and Dirac action while the last two terms include torsion. With the use of (2.7), the contorsion tensor $C_{\mu I}{}^J$ can be solved by expressing the equation of motion (2.6) as

$$\begin{aligned}
\nabla_\mu(ee^{[\mu}_I e^{\nu]}_J) &= eC_{\mu I}{}^K e^{[\mu}_K e^{\nu]}_J+eC_{\mu J}{}^K e^{[\mu}_I e^{\nu]}_K \\
&= 2\pi Ge\frac{\gamma}{\gamma^2+1}\left(\beta\epsilon_{IJ}{}^{KL}e^\nu_K J_L-2\theta\delta^M_{[I}\delta^N_{J]}e^\nu_M J_N\right),
\end{aligned} \tag{2.10}$$

where $\beta := \gamma+\frac{1}{\alpha}$ and $\theta := 1-\frac{\gamma}{\alpha}$. Now the following equation can be obtained by contracting equation (2.10) with $e^I_\nu e^J_\rho$:

$$C_{\mu J}{}^K e^\mu_K e^J_\rho := S_{\mu J}{}^\mu e^J_\rho = 6\pi G\frac{\gamma\theta}{\gamma^2+1}e^J_\rho J_J. \tag{2.11}$$

Since the tetrads e^I_μ are invertible, this equation implies that $S_{\mu J}{}^\mu = C_{\mu J}{}^K e^\mu_K = 6\pi G\frac{\gamma\theta}{\gamma^2+1}J_J$. Upon inserting this solution in equation (2.10) the equation of motion becomes

$$e^\mu_I S_{\mu J}{}^\nu - e^\mu_J S_{\mu I}{}^\nu = 2S_{[IJ]}{}^\nu \tag{2.12}$$

$$= 4\pi G\frac{\gamma}{\gamma^2+1}\left(\beta\epsilon_{IJ}{}^{KL}e^\nu_K J_L+\theta\delta^M_{[I}\delta^N_{J]}e^\nu_M J_N\right).$$

Again contracting with e^M_ν, we obtain

$$S_{[IJ]}{}^M = 2\pi G\frac{\gamma}{\gamma^2+1}\left(\beta\epsilon_{IJ}{}^{ML}J_L+\theta\eta^M_{[I}J_{J]}\right). \tag{2.13}$$

Notice that $C_{\mu IJ} = C_{\mu[IJ]}$ implies that $S_{IJK} = S_{IJ}{}^{M}\eta_{MK} = e^{\mu}_{I}C_{\mu JK} = S_{I[JK]}$. The following combination of the cyclic permutations of the indices I, J, and K finally yields the expression for $C_{\mu IJ}$:

$$\begin{aligned} e^{\mu}_{I}C_{\mu JK} &= S_{[IJ]K} - S_{[JK]I} + S_{[KI]J} = S_{IJK} \\ &= 2\pi G\frac{\gamma}{\gamma^2+1}\left(\beta\epsilon_{IJKL}J^{L} - 2\theta\eta_{I[J}J_{K]}\right) \end{aligned} \tag{2.14}$$

which depends on the Immirzi parameter γ unless $\alpha = \gamma$.

The following useful identities can be derived after a straightforward but lengthy calculation using the above expression for $C_{\mu IJ}$:

$$\begin{aligned} \frac{1}{16\pi G}e^{\mu}_{I}e^{\nu}_{J}P^{IJ}{}_{KL}[C_{\mu}, C_{\nu}]^{KL} &= \frac{3\pi G}{2}\frac{\gamma^2}{\gamma^2+1}\left(1 - \frac{1}{\alpha^2} + \frac{2}{\gamma\alpha}\right)J_{L}J^{L}, \\ \frac{1}{4}\left(e^{\mu}_{I}C_{\mu}{}^{JK}\epsilon^{I}{}_{JKL}J^{L} + \frac{2}{\alpha}C_{\mu}{}^{JK}e^{\mu}_{J}J_{K}\right) &= -\frac{3\pi G}{2}\frac{\gamma}{\gamma^2+1}2\left(\beta + \frac{\theta}{\alpha}\right)J_{L}J^{L}. \end{aligned} \tag{2.15}$$

In (2.9), we thus obtain a simple interacting term in the total action:

$$\begin{aligned} S\left[e,\omega,\Psi\right] &= S_{G}\left[e,\widetilde{\omega}\right] + S_{F}\left[e,\widetilde{\omega},\Psi\right] + S_{\text{int}}\left[e,C,\Psi\right] \\ &= \frac{1}{2\kappa}\int_{M}\mathrm{d}^4x\; ee^{\mu}_{I}e^{\nu}_{J}\widetilde{F}_{\mu\nu}{}^{IJ}(\widetilde{\omega}) + \frac{1}{2}i\int_{M}\mathrm{d}^4x\; e\left(\overline{\Psi}\gamma^{I}e^{\mu}_{I}\widetilde{\nabla}_{\mu}\Psi - \overline{\widetilde{\nabla}_{\mu}\Psi}\gamma^{I}e^{\mu}_{I}\Psi\right) \\ &\quad + \frac{3\kappa}{16}\frac{\gamma^2}{\gamma^2+1}\left(\frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1\right)\int_{M}\mathrm{d}^4x\; e(\overline{\Psi}\gamma_{5}\gamma_{L}\Psi)(\overline{\Psi}\gamma^{5}\gamma^{L}\Psi), \end{aligned} \tag{2.16}$$

where $\kappa = 8\pi G$ is used from now on. Notice that the second term in the gravitational Holst action containing γ and the term involving non-minimal coupling α in Dirac action are dropped from the above effective action since both these terms can be expressed as boundary terms; please see [2] for details for the second term in Holst action. The non-minimally coupled term in Dirac action can be cast into a boundary term after inserting the solution as follows:

$$
\begin{aligned}
& \frac{1}{2\alpha}\int_M \mathrm{d}^4x\, e\left(\overline{\Psi}\gamma^I e_I^\mu \gamma_5 \widetilde{\nabla}_\mu \Psi - \overline{\widetilde{\nabla}_\mu \Psi}\gamma_5 \gamma^I e_I^\mu \Psi\right) \\
= \ & \frac{1}{2\alpha}\int_M \mathrm{d}^4x\, e e_I^\mu \left(\overline{\Psi}\gamma^I \gamma_5 \partial_\mu \Psi - (\partial_\mu \overline{\Psi})\gamma_5 \gamma^I \Psi - \frac{1}{4}\tilde{\omega}_\mu^{MN}\overline{\Psi}\gamma_5 \left[\gamma^I, \gamma_{[M}\gamma_{N]}\right]_- \Psi\right) \\
= \ & \frac{1}{2\alpha}\int_M \mathrm{d}^4x\, e e_I^\mu \left((\partial_\mu \overline{\Psi}\gamma^I \gamma_5 \Psi) - \frac{1}{4}\tilde{\omega}_\mu^{MN}\overline{\Psi}\gamma_5 \left[\gamma^I, \gamma_{[M}\gamma_{N]}\right]_- \Psi\right) \\
= \ & \frac{1}{2\alpha}\int_M \mathrm{d}^4x\, e\left(e_I^\mu(\partial_\mu J^I) + \tilde{\omega}_\mu^{MN} e_M^\mu J_N\right) \\
= \ & \frac{1}{2\alpha}\int_M \mathrm{d}^4x\, \left(\partial_\mu(e e_I^\mu J^I) - J^N \tilde{D}_\mu(e e_N^\mu)\right) = \frac{1}{2\alpha}\int_M \mathrm{d}^4x\, \left(\partial_\mu(e e_I^\mu J^I)\right). \qquad (2.17)
\end{aligned}
$$

The last term in the simplified action (2.16) describes a four-fermion interaction mediated by a non-propagating torsion. Note that the coupling constant of this interaction depends on the Immirzi parameter in addition to α. The resulting four-fermion interaction term reduces to that for minimal coupling as $\alpha \to \infty$ and is independent of the Immirzi parameter for $\alpha = \gamma$ as shown in [30]. However, unlike in [30], the effective action (2.16) is parity invariant for any value of α, which will be explicitly shown in section 4.33. This discrepency arises from the mismatch between the signature and the Holst action used in [30] and those used in this paper.

From this action (2.16) it is not clear whether it can be consistently formulated canonically. First, equations of motion have been used and solutions were inserted in the original action. As a consequence, its first term is the Holst action for pure gravity (without torsion) and the second term is the Dirac action. This makes the formulation difficult to interpret as a first order theory since variations of the combined action would tell us that the connection cannot be torsion-free although its gravitational part is suggestively written as the free Holst action. Similarly, a canonical formulation of the first two terms together would result in a Gauss constraint that has contributions both from gravity and matter as we will write explicitly in the next section. Also here, fermion contributions through torsion change the expression that one would expect from the torsion-free Holst action. Therefore, we do not use (2.16) as the

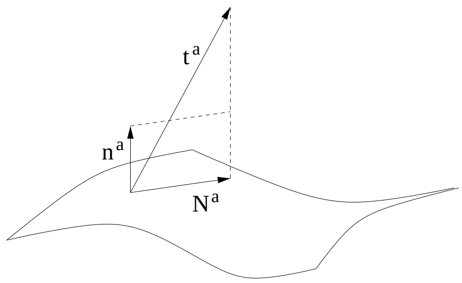

Figure 2.1: Decomposition of the evolution vector field t^a in terms of the normal n^a to spatial slices and a spacelike part N^a.

starting point of a canonical treatment, although at first sight it looks simpler, but the original Einstein–Cartan action (2.1). Avoiding the use of partial solutions to classical equations of motion also has advantages for the quantization procedure.

2.2 Canonical Formulation for Fermions

We present our analysis in a form which differs slightly from Holst's one in the vacuum case. In particular, we put a stronger emphasis on details of the Dirac procedure for constrained systems.

A canonical formalism (Hamiltonian framework) is achieved by performing a Legendre transform of this action S_M, replacing time derivatives of configuration variables by momenta. This, as always, requires one to treat space and time differently and is the reason why the canonical formulation is not manifestly covariant. We introduce a foliation of the space-time (M, g_{ab}) by a family of space-like hypersurfaces Σ_t: $t =$ constant in terms of a time function t on M as illustrated in Fig. 2.1. Canonical variables will depend on which time function one chooses, but the resulting dynamics of observable quantities will remain covariant. Furthermore, let t^a be a timelike vector field whose integral curves intersect each leaf Σ_t of the foliation precisely once and which is normalized such that $t^a\nabla_a t = 1$. This t^a is the 'evolution vector field' along whose orbits different points on all $\Sigma_t \equiv \Sigma$ can be identified. This allows us to write all space-time fields in terms of t-dependent components defined on a spatial manifold Σ. Lie derivatives of space-time fields along t^a are identified with 'time-derivatives' of the spatial fields.

Also, since we are using the Lorentzian signature, the vector field t^μ is required to be future directed. Let us decompose t^μ into normal and tangential parts with respect to Σ_t by defining the lapse function N and the shift vector N^a as $t^\mu = Nn^\mu + N^\mu$ with $N^\mu n_\mu = 0$, where n^μ is the future directed unit normal vector field to the hypersurfaces Σ_t. The space-time metric $g_{\mu\nu}$ induces a spatial metric $q_{\mu\nu}$ by the formula $g_{\mu\nu} = q_{\mu\nu} - n_\mu n_\nu$. Since contractions of $q_{\mu\nu}$ and N^μ with the normal n^μ vanish, they give rise to spatial tensors q_{ab} and N^a. Here, the lower case roman letters, $a, b, c, \ldots$, are used to imply spatial tensorial indices.

Moreover, since we are using a tetrad formulation, in addition to the above foliation of the space-time manifold we need to perform a partial gauge fixing on the internal vector fields of the tetrad to decompose it into an internal unit time-like vector and a triad. Let us fix a constant internal vector field $n_I = -\delta_{I,0}$ with $n^I n_I = -1$. Now we allow only those tetrads which are compatible with the fixed n^I in the sense that $n^a := n^I e^a_I$ must be the unit normal to the given foliation. This implies that $e^a_I = \mathcal{E}^a_I - n^a n_I$ with $\mathcal{E}^a_I n_a = \mathcal{E}^a_I n^I = 0$ so that $\mathcal{E}^a_I$ is a triad [2].

Now using $n^a = N^{-1}(t^a - N^a)$ to project fields normal and tangential to Σ_t, one can decompose the Einstein–Cartan action (2.1) as

$$\begin{aligned} S\,[e,\omega,\Psi] \;=\; & \frac{1}{2\kappa}\int_M \mathrm{d}^4x\; N\sqrt{q}(\mathcal{E}^a_I\mathcal{E}^b_J P^{IJ}{}_{KL}F_{ab}{}^{KL}(\omega) - 2N^{-1}n_I t^a\mathcal{E}^b_J P^{IJ}{}_{KL}F^{KL}_{ab}(\omega) \\ & +2N^{-1}N^a n_I\mathcal{E}^b_J P^{IJ}{}_{KL}F_{ab}{}^{KL}(\omega)) \\ & +\frac{i}{2}\int_M \mathrm{d}^4x\; N\sqrt{q}(\overline{\Psi}\gamma^I\left(\mathcal{E}^a_I - N^{-1}(t^a - N^a)n_I\right)\left(1-\frac{i}{\alpha}\gamma_5\right)\nabla_a\Psi \\ & -\overline{\nabla_a\Psi}\left(1-\frac{i}{\alpha}\gamma_5\right)\gamma^I\left(\mathcal{E}^a_I - N^{-1}(t^a - N^a)n_I)\Psi\right), \end{aligned} \tag{2.18}$$

where we have used $|e| = N\sqrt{q}$ with $|\det(e^i_a)| = \sqrt{q}$ and the decomposed tetrad

[2] While using a partial gauge fixing may be questionable for general gauge transformations, Barros e Sa showed in [48] that such a partial gauge-fixing is harmless in our context (as in Holst's) since only the gauge parameter without any time derivatives appears in the analysis, corresponding to a canonical gauge.

$e^a_I = \mathcal{E}^a_I - n^a n_I$. (We assume positively oriented triads at this stage, thus ignoring sign factors. However, including sign factors yields an overall sign factor before the action, which will reveal nothing new about the parity transformations discussed later)

To proceed with the canonical formulation we need to expand each term in this Holst action into spatial and temporal components of tensors, as well as expand the Dirac spinors of the fermion action in the Weyl representation. The second term in the gravitational part of (2.18) provides the Lie derivative of

$$ {}^{+}\!A^j_b := \omega_b{}^{j0} + \frac{1}{2\gamma}\epsilon^j{}_{kl}\omega_b{}^{kl} \tag{2.19}$$

along t^μ. Only this combination of the Lorentz connection is dynamical and has a non-vanishing momentum. The other independent combination

$$ {}^{-}\!A^j_b := \omega_b{}^{j0} - \frac{1}{2\gamma}\epsilon^j{}_{kl}\omega_b{}^{kl} \tag{2.20}$$

is non-dynamical and will appear in Lagrange multipliers. The momentum of the dynamical connection ${}^{+}\!A^i_a$ is then given by $-\sqrt{q}\mathcal{E}^a_i/\kappa$. It is, however, traditional to perform a minor canonical transformation and use the canonical pair

$$ A^i_a := -\gamma\, {}^{+}\!A^i_a \quad , \quad P^a_i := \frac{1}{\gamma\kappa}\sqrt{q}\mathcal{E}^a_i \tag{2.21}$$

since, as we will see shortly, A^i_a is the Ashtekar–Barbero connection. Moreover, we define

$$ \begin{aligned} \Gamma^i_b &:= -\frac{\gamma}{2}({}^{+}\!A^i_b - {}^{-}\!A^i_b) = -\frac{1}{2}\epsilon^i{}_{kl}\omega^{kl}_b \,, \\ K^i_b &:= -\frac{1}{2}({}^{+}\!A^i_b + {}^{-}\!A^i_b) = -\omega^{i0}_b \end{aligned} \tag{2.22}$$

such that $A^i_a = \Gamma^i_a + \gamma K^i_a$. We will from now on use the variable P^a_i as the momentum to avoid confusion about the different triads used, but will keep referring to ${}^{\pm}\!A^i_a$ as long as it is of interest to use both the dynamical as well as non-dynamical connections.

In addition to the gravitational canonical pair (2.21), from (2.18) we can easily identify $(\psi, -i\sqrt{q}\psi^\dagger)$ and $(\eta, -i\sqrt{q}\eta^\dagger)$ as the dynamical variables and their conjugate momenta for matter fields. On the other hand, the lapse function N, the shift vector N^d as well as ${\omega_t}^{i0}$, ${\omega_t}^{ij}$, and ${}^{-}\!A^i_a$ are non-dynamical variables. (It is interesting to note that all the terms involving ${}^{-}\!A^i_a$ disappear for $\gamma = \pm i$ to reproduce Ashtekar's formulation in the (anti-)self dual connection formulation, respectively.)

Following Dirac's procedure, we thus obtain primary constraints from all the vanishing momenta of non-dynamical variables. They generate secondary constraints which turn out to be of mixed type and are not purely first class. Variations by N and N^a will, as we will see, give the usual diffeomorphism and Hamiltonian constraints, but several secondary constraints resulting from variations by non-dynamical connection components remain:

$$\frac{\delta\mathcal{L}}{\delta{\omega_t}^{m0}} = -\gamma\partial_b P^b_m - \gamma\epsilon^{\ j}_{mi}P^b_j\left(\frac{1-\gamma^2}{2\gamma}\ {}^{+}\!A^i_b + \frac{1+\gamma^2}{2\gamma}\ {}^{-}\!A^i_b\right) - \frac{1}{2\alpha}\sqrt{q}J_m = 0 \qquad (2.23)$$

$$\frac{\delta\mathcal{L}}{\delta{\omega_t}^{mn}} = -\frac{{\epsilon^j}_{mn}}{2}\partial_b P^b_j - \gamma P^b_{[m}\ {}^{+}\!A_{|b|n]} + \frac{\sqrt{q}}{4}{\epsilon^k}_{mn}J_k = 0 \qquad (2.24)$$

and

$$\begin{aligned}\frac{\delta\mathcal{L}}{\delta({}^{-}\!A^l_c)} &= \frac{1+\gamma^2}{2}{\epsilon^j}_{lk}P^c_j{\omega_t}^{k0} + \frac{1+\gamma^2}{2}{\epsilon^j}_{kl}P^{[c}_j N^{a]}({}^{+}\!A^k_a + {}^{-}\!A^k_a) \qquad (2.25)\\ &+\mathrm{sgn}(\mathrm{e^i_a})\frac{1+\gamma^2}{2\gamma\kappa}\epsilon^{\mathrm{acd}}\partial_{\mathrm{a}}(\mathrm{e_{dl}N}) + \frac{\gamma^2(1+\gamma^2)\kappa}{2\sqrt{\mathrm{q}}}\frac{\mathrm{N}}{\kappa}\mathrm{P^a_{[k}P^c_{l]}}({}^{+}\!\mathrm{A^k_a} - {}^{-}\!\mathrm{A^k_a})\\ &+\frac{N^c}{4}\sqrt{q}\left(\gamma+\frac{1}{\alpha}\right)J_l - \frac{\gamma\kappa N}{4}P^c_l\left(\gamma+\frac{1}{\alpha}\right)J^0 - \frac{\gamma\kappa N}{4}\ {\epsilon^j}_{lk}P^c_j\left(1-\frac{\gamma}{\alpha}\right)J^k = 0\,,\end{aligned}$$

where $J^i := \psi^\dagger\sigma^i\psi + \eta^\dagger\sigma^i\eta$, $J^0 := \psi^\dagger\psi - \eta^\dagger\eta$ and $P^a_i P^b_j \epsilon^{ijk} = \mathrm{sgn}(\mathrm{e^i_a})\gamma^2\kappa^2\sqrt{\mathrm{q}}\epsilon^{\mathrm{abc}}\mathrm{e^k_c}$ is used at several stages. This agrees with Holst's equations for the case $J^i = 0 = J^0$. Here and in Appendix 0.2, the sign factors are included appropriately since they are crucial for determining the parity transformations proposed in section 2.3.

As seen shortly, these equations can be solved for the non-dynamical ω^{kl}_t and ${}^{-}\!A^l_c$, which demonstrates that there are second class constraints. After solving these, (2.24) remains as the Gauss constraint in addition to the diffeomorphism and Hamiltonian constraints. These remaining constraints must then be first class because their number and generated symmetries correspond to the known local symmetries of the action. In what follows, we will solve for the non-dynamical variables which are determined by second class constraints and insert the solutions back into the action rather than formally computing Dirac brackets. Varying the resulting Hamiltonian, which turns out to be totally constrained, then results in the dynamical equations to be solved on the surface where second class constraints are satisfied.

Solving (2.23), (2.24) and (74) also provides some information on the appearance of torsion terms. We first express (2.24) as

$$\partial_b P^{bj} = \frac{1}{2}\sqrt{q}J^j - \gamma\epsilon^j{}_{mn}P^{bm}\,{}^{+}\!A^n_b. \tag{2.26}$$

and insert it into (2.23):

$$\epsilon_{ijk}P^{bj}(\,{}^{+}\!A^k_b + {}^{-}\!A^k_b) = \frac{\beta\gamma}{1+\gamma^2}\sqrt{q}J_i, \tag{2.27}$$

where we are using β (and θ below) as defined in the previous section. On the other hand, first solving (2.24) for ${}^{+}\!A^j_b$ and then inserting into (2.23), we obtain

$$\partial_b P^{bj} - \gamma\epsilon^j{}_{mn}P^{bm}\,{}^{-}\!A^n_b = \frac{\gamma}{2(1+\gamma^2)}\left(\frac{1-\gamma^2}{2\gamma} - \frac{1}{\alpha}\right)\sqrt{q}J^j. \tag{2.28}$$

which combined with (2.26) and (2.28) gives

$$2\partial_b P^{bm} - \gamma\epsilon_{ij}{}^{m}P^{bj}(\,{}^{+}\!A^i_b - {}^{-}\!A^i_b) = \frac{\theta}{1+\gamma^2}\sqrt{q}J^m. \tag{2.29}$$

In terms of Γ^i_a and K^i_a, (2.27) and (2.29) take the form

$$\epsilon_{ijk}K^j_b P^{bk} = [K_b, P^b]_i = \frac{\gamma\beta}{2(1+\gamma^2)}\sqrt{q}J_i\,, \tag{2.30}$$

and

$$D_b P^{bm} \equiv \partial_b P^{bm} + \epsilon_{ij}{}^{m}\Gamma^i_b P^{bj} = \frac{\theta}{2(1+\gamma^2)}\sqrt{q}J^m\,. \tag{2.31}$$

Without fermions, K^i_b and Γ^i_b would be identified with extrinsic curvature and the spin connection respectively. The non-vanishing commutator of K^i_b with the densitized triad and the non-vanishing covariant derivative of P^a_i by Γ^i_b emphasize the existence of torsion due to the fermion current as expected from the last section. In particular, $K_{ab} := e^i_b K^i_a$ cannot be symmetric since its antisymmetric part vanishes only if the commutator

$$[K_b, P^b]^k = \epsilon^{ijk} K^i_b P^b_j = K_{ab} \epsilon^{ijk} \mathcal{E}^a_i P^b_j = \gamma\kappa K_{[ab]} \epsilon^{ijk} P^a_i P^b_j$$

vanishes. Moreover, that the definition of Γ^i_a and K^i_a is meaningful can be seen by further manipulating (74) and comparing it with the computation of the spin connection in the presence of torsion (see App. 0.2) which results in

$$\Gamma^k_b = \widetilde{\Gamma}^k_b + \frac{\gamma\kappa}{4(1+\gamma^2)} \left(\theta\ \epsilon_{ij}{}^k e^i_b J^j - \beta e^k_b J^0\right) , \tag{2.32}$$

where $\widetilde{\Gamma}^k_b$ is the torsion-free su(2)-valued spin connection as derived in (70). The second part here is precisely the torsion contribution which can be obtained from the tetrad projection of the contorsion tensor C^{IJ}_μ in (2.14):

$$\begin{aligned} C^j_a &:= \frac{1}{2} q^\nu_a \epsilon^{IJ}{}_{KL} n_I C_\nu{}^{KL} \\ &= \frac{\gamma\kappa}{4(1+\gamma^2)} \left(\theta\ \epsilon^j{}_{kl} e^k_a J^l - \beta e^j_a J^0\right) , \end{aligned} \tag{2.33}$$

where $q^\nu_a := \delta^\nu_a + n_a n^\nu$ is the spatial projection and again $n_I = -\delta^0_I$. As a consistency test, we observe that the expressions for C^j_a derived from the two different methods, through the spin connection in the appendix and through the triad projection of the contorsion tensor, are in complete agreement.

Recalling the definition of Γ^i_a in (2.22), (2.32) can be expressed as

$$\begin{aligned} {}^-\!A^k_b &= {}^+\!A^k_b + \frac{2}{\gamma}\widetilde{\Gamma}^k_b + \frac{\kappa}{2(1+\gamma^2)} \left(\theta\ \epsilon_{ij}{}^k e^i_b J^j - \beta e^k_b J^0\right) \\ &=: {}^+\!A^k_b + \frac{2}{\gamma}(\widetilde{\Gamma}^k_b + C^k_b) =: {}^+\!A^k_b + \frac{2}{\gamma}\Gamma^k_b\,. \end{aligned} \tag{2.34}$$

This provides a solution for the non-dynamical $\bar{A}_b^k$ in terms of dynamical fields. One can directly determine $\omega_t{}^{k0} = \omega_t{}^{k0}(\bar{A}, \Gamma, N, N^a)$ from (74), but we will not require this because it turns out to drop out of the action once second class constraints are solved. Only ω_t^{mn} then remain free as undetermined multipliers of the Gauss constraint. This completes the solution of second class constraints, which we can now use in the action to analyze the remaining first class system.

First class constraints

Let us first look at the Gauss constraint in more detail. After (2.34) is inserted into the second term of (2.18) and integrated by parts, we obtain

$$\begin{aligned}
&-\gamma \int_{\Sigma\times\mathbb{R}} \mathrm{d}x^3 \mathrm{d}t n_I t^a P_J^b P^{IJ}{}_{KL} F_{ab}^{KL}(\omega) \\
=& \int_{\Sigma\times\mathbb{R}} \mathrm{d}x^3 \mathrm{d}t \left(P_j^b \mathcal{L}_t A_b^j - \Lambda^i \mathcal{D}_b P_i^b - \frac{1+\gamma^2}{\gamma} \omega_t{}^{k0} \epsilon_{kl}{}^j C_b{}^l P_j^b \right) . \qquad (2.35)
\end{aligned}$$

We have used (2.32) and (2.33) together with $\epsilon_{klj}\Gamma_b^l P_j^b = \epsilon_{klj}\tilde{\Gamma}_b^l P_j^b + \epsilon_{klj} C_b^l P_j^b$ and defined $\Lambda^l := \frac{1}{2}\epsilon^l{}_{mn}\ \omega_t{}^{mn} - \frac{1}{\gamma}\omega_t{}^{l0}$. The covariant derivative $\mathcal{D}_a$ refers to the Ashtekar connection in (2.21).

This equation again demonstrates that (A_a^i, P_i^a) is the new canonical pair where the Ashtekar–Barbero connection A_a^i inherits torsion contributions from the spin connection term as well as the extrinsic curvature term. The remaining terms provide the gravitational contribution to the Gauss constraint. Similarly, the contribution to the Gauss constraint from fermionic matter can be obtained as

$$\begin{aligned}
&\frac{i}{2}\sqrt{q} t^a \left(\overline{\Psi}\gamma^0 \left(1 - \frac{i}{\alpha}\gamma_5\right) \nabla_a \Psi - c.c. \right) \\
=& -i\sqrt{q}(\theta_L(\psi^\dagger\dot{\psi} - \dot{\eta}^\dagger\eta) - \theta_R(\dot{\psi}^\dagger\psi - \eta^\dagger\dot{\eta})) + \frac{1}{4}\epsilon^i{}_{mn}\omega_t{}^{mn}\sqrt{q}J_i - \frac{\sqrt{q}}{2\alpha}\omega_t{}^{k0} J_k
\end{aligned}$$

where $\theta_L := \frac{1}{2}\left(1 + \frac{i}{\alpha}\right)$ and $\theta_R := \frac{1}{2}\left(1 - \frac{i}{\alpha}\right)$. All terms containing ω_t^{mn} or ω_t^{m0} contribute to the Gauss constraint

$$G[\Lambda^i] = \int \mathrm{d}^3x\Lambda^i \left(\mathcal{D}_b P^b_i - \frac{1}{2}\sqrt{q}J_i\right) \tag{2.36}$$

$$= \int \mathrm{d}^3x\Lambda^i \left(\gamma[K_b, P^b]_i - \frac{\gamma\beta}{2(1+\gamma^2)}\sqrt{q}J_i\right) \tag{2.37}$$

using (2.33) with (2.30) and (2.31).

With (2.30) and (2.31), this takes the form

$$\Lambda^i(\mathcal{D}_b P^b_i - \tfrac{1}{2}\sqrt{q}J_i) = \frac{1}{2}\epsilon^i{}_{mn}\omega_t{}^{mn}(\mathcal{D}_b P^b_i - \tfrac{1}{2}\sqrt{q}J_i) \tag{2.38}$$

such that all terms containing $\omega_t{}^{k0}$ indeed disappear when solutions to the second class constraints are used. (Moreover, after splitting the connection completely in torsion-free components and torsion contributions, one can see that the Gauss constraint just takes the usual torsion-free form $\widetilde{\mathcal{D}}_b P^b_i$ without any torsion contributions. For this decomposition, however, also equations of motion must be partially solved as done later.)

The diffeomorphism constraint is obtained by varying the action by N^a, whose gravitational part is thus given by the last term in (2.18):

$$D_{\rm grav}[N^a] = \int \mathrm{d}^3x N^a P^b_j \left(F^j_{ab} - (\gamma^2+1)\epsilon^j{}_{kl}K^k_a K^l_b\right) , \tag{2.39}$$

where $F^j_{ab} = 2\partial_{[a}A^j_{b]} + \epsilon^j{}_{kl}A^k_a A^l_b$ is the curvature of A^j_a and K^i_a is understood as a functional of A^i_a and the other fields through (2.32). The contribution from the fermion action is

$$D_{\rm Dirac}[N^a] = \int \mathrm{d}^3x N^a \left(-i\sqrt{q}\left(\theta_L(\psi^\dagger\mathcal{D}_a\psi - \overline{\mathcal{D}_a\eta}\eta) - c.c.\right) + \frac{\beta}{2}K^i_a\sqrt{q}J_i\right) , \tag{2.40}$$

where we have employed the covariant derivatives, e.g., $\mathcal{D}_a = \partial_a + A^l_a\tau_l$, related to A^i_a. Combined, the total diffeomorphism constraint is

$$C_a = P^b_j(F^j_{ab} - (\gamma^2+1)\epsilon^j{}_{kl}K^k_a K^l_b) - i\sqrt{q}\left(\theta_L(\psi^\dagger\mathcal{D}_a\psi - \overline{\mathcal{D}_a\eta}\eta) - c.c.\right)$$
$$+\frac{\beta}{2}K^i_a\sqrt{q}J_i \tag{2.41}$$

$$= P^b_j F^j_{ab} - i\sqrt{q}\left(\theta_L(\psi^\dagger\mathcal{D}_a\psi - \overline{\mathcal{D}_a\eta}\eta) - c.c.\right) - \frac{\gamma^2+1}{\gamma}K^j_a G_j \tag{2.42}$$

where the Gauss constraint has been used.

As a secondary constraint from varying N, we have the Hamiltonian constraint with gravitational contribution

$$H_{\rm grav}[N] = \int {\rm d}^3x \frac{\kappa}{2\sqrt{q}} N P^a_i P^b_j \epsilon^{ij}{}_k \left(F^k_{ab} - (1+\gamma^2) R^k_{ab}\right)$$

where F^k_{ab} and R^k_{ab} denote curvatures of A^k_a and Γ^k_a respectively. This can be expressed more dircetly in terms of A^i_a and K^i_a using

$$\begin{aligned} P^{IJ}{}_{KL} F_{ab}{}^{KL}(\omega) &= F_{ab}{}^{ij} - \frac{1}{\gamma}\epsilon^j{}_{kl} F_{ab}{}^{k0} \\ &= -\left(\epsilon^{ij}{}_k F^k_{ab} - 2(\gamma^2+1) K^i_{[a} K^j_{b]}\right) - 2\frac{\gamma^2+1}{\gamma}\, \epsilon^{ij}{}_k D_{[a} K^k_{b]} \end{aligned} \quad (2.43)$$

where we used the covariant derivative D of Γ^i_a. This appears multiplied with $P^a_i P^b_j$, such that the last term can, using the Gauss constraint, be reexpressed as

$$\begin{aligned} -\frac{\gamma^2+1}{\gamma}\epsilon^{ij}{}_k P^a_i P^b_j D_{[a} K^k_{b]} &= \frac{\gamma^2+1}{\gamma}\, \epsilon^{ij}{}_k P^a_i P^b_j \left(\widetilde{D}_{[a} K^k_{b]} + \epsilon^k{}_{mn} C^m_a K^n_b\right) \\ &= \frac{\beta}{2} P^b_j K^j_b \sqrt{q} J^0 + \frac{\beta P^a_i}{2} \widetilde{D}_a(\sqrt{q} J^i) - \frac{\theta}{4}\sqrt{q}\epsilon_{ijk} K^i_b P^b_j J^k \\ &\quad + \frac{1+\gamma^2}{\gamma}\sqrt{q}\widetilde{D}_a\left(\frac{P^a_i G^i}{\sqrt{q}}\right)\,, \end{aligned} \quad (2.44)$$

where we have used (2.33) and (2.37) in the last step. It is clear that there are extra terms compared to the constraint in the absence of torsion. For later comparison we point out that this is the main source of disagreements with other approaches. In the absence of torsion, however, the whole set of extra terms in the last equation vanishes and only contributions from the Gauss constraint are left.

The fermionic Hamiltonian is

$$\begin{aligned} \mathcal{H}_F &= \gamma\kappa N i P^a_i \left(\theta_L(\psi^\dagger \sigma^i \mathcal{D}_a\psi + \overline{\mathcal{D}_a\eta}\sigma^i\eta) - c.c.\right) - \frac{\gamma\kappa\theta}{2}\epsilon^i{}_{mn} P^a_i K^m_a J^n \\ &\quad - \frac{\gamma\kappa\beta}{2} P^a_i K^i_a J^0\,. \end{aligned} \quad (2.45)$$

Combining (2.44) and (2.45) we obtain the total Hamiltonian constraint

$$\begin{aligned}
C &= \frac{\gamma^2\kappa}{2\sqrt{q}}P^a_i P^b_j\left(\epsilon^{ij}{}_k F^k_{ab} - 2(\gamma^2+1)K^i_{[a}K^j_{b]}\right) \\
&\quad +\frac{\gamma\kappa\beta}{2\sqrt{q}}P^a_i\mathcal{D}_a(\sqrt{q}J^i) + (1+\gamma^2)\kappa\widetilde{D}_a\left(\frac{P^a_iG^i}{\sqrt{q}}\right) \\
&\quad +i\gamma\kappa P^a_i\left(\theta_L(\psi^\dagger\sigma^i\mathcal{D}_a\psi + \overline{\mathcal{D}_a\eta}\sigma^i\eta)) - \theta_R(\eta^\dagger\sigma^i\mathcal{D}_a\eta + \overline{\mathcal{D}_a\psi}\sigma^i\psi)\right) \\
&\quad +\frac{\kappa}{4}\left(3-\frac{\gamma}{\alpha}+2\gamma^2\right)\epsilon_{lkn}K^l_aP^a_kJ^n\,. \qquad (2.46)
\end{aligned}$$

Having derived all variations, it is clear that the total Hamiltonian

$$H = \int \mathrm{d}^3x\left((\omega^i\cdot t)G_i + N^aC_a + NC\right) \qquad (2.47)$$

with $\omega^i\cdot t := \frac{1}{2}\epsilon^i{}_{jk}\omega_t{}^{jk}$ is completely constrained. The constraint algebra is lengthy, but it is clear from the number of local symmetries that the remaining constraints must form a first class set. This concludes the canonical analysis.

At this point, we want to emphasize that we have not imposed any restriction on either the non-minimal coupling parameter, α, or the Immirzi parameter, γ. The previously available derivation of the constraints in [30], by contrast, used an intermediate decomposition of the real connection into its (complex-valued) self- and anti-selfdual contributions. The much simpler vacuum constraints for these connections were then combined to yield constraints for the real connection. In order to have the same self- or anti-selfdual connections in the gravitational terms as well as the covariant derivatives in the fermion contributions, this approach requires that $\alpha=\gamma$. The case $\alpha=\gamma$ is indeed of particular interest as it has a special behavior under parity transformations [3], but it is not the most general possibility. For instance, our derivations also include the constraints for minimal coupling as $\alpha\to\infty$.

[3]Note also that the action with minimal coupling was called inconsistent in [30], while our analysis is consistent for any value of α. To clarify potential confusion, one should first note that the viewpoint of [30] is different from our's. While [30] starts from the Einstein–Cartan action with minimal coupling and aims to reproduce its equations of motion from an action involving the Holst term, we only analyze the latter for any type of coupling. What [30] observes is that minimal coupling of the action with the Holst term does not reproduce minimal coupling in the Einstein–Cartan action. From this point of view, one thus has to use non-minimal coupling in the presence of the Holst term to cancel unwanted terms in the coupling for the Einstein–Cartan action. This, however, does not mean that using minimal coupling in the presence of the Holst term would be inconsistent. Since

However, even for $\alpha = \gamma$, the above constraints do not agree with those in [30]. The source of this disagreement, barring calculational mistakes, is not obvious at first sight since the combination of weighted self-dual and anti self-dual gravitational constraints in [30] does indeed give rise to the set of constraints in real Ashtekar-Barbero variables for a torsion-free system. But this happens rather by coincidence, and there is no guarantee that this procedure works because one is combining non-linear expressions out of a linearly decomposed connection. As seen in [30], this combination provides the corrected term in (2.43) quadratic in extrinsic curvature. However, it does not produce the last term in (2.43). In the torsion-free case, the su(2)-commutator of K_a^i and P_j^a vanishes which makes this last term vanish when contracted with P_j^b as it appears in the constraint. This in turn reduces the vacuum Hamiltonian constraint for gravity to the standard expression for a torsion-free system, containing only the first term in (2.44) as it was used in [30]. (The same is true for the diffeomorphism constraint if one wants to arrive at the final expression (2.42) from (2.41).)

In the presence of torsion, $\epsilon_i{}^{jk} K_a^i P_j^a$ does no longer vanish and thus gives rise to extra terms in the Hamiltonian constraint as explicitly displayed in (2.44). In Sec. 2.3, we will see that these terms are necessary for consistency with the Lagrangian derivation of the self-interaction term. (As for the diffeomorphism constraint, the extra term in (2.41) automatically disappears to give (2.42).) The shortcut approach presented in [30] to derive the constraints overlooks this term and is not consistent in the presence of fermions. This explains the disagreements between our derivations and some of the results of [30].

---

non-minimal coupling terms introduce parity violation, one could say that the minimal coupling in the presence of the Holst action is inconsistent with parity preservation. But this again is not an inconsistency of the overall framework, unless one explicitly requires parity preservation.

2.3 Parity transformation of the classical theory

In the presence of fermions, the parity behavior is not fully obvious even in the absence of explicitly parity violating interaction terms. A detailed analysis of transformation properties is then required.

The Torsion Contribution to Extrinsic Curvature

Torsion components play an indirect but important role in the behavior under parity. During the constraint analysis, second class constraints provide the torsion contribution to the connection as seen in (2.32). However, although K_a^i is restricted by the Gauss constraint, constraints do not provide its complete torsion contribution. On the other hand, the transformation properties of the Ashtekar-Barbero connection A_a^i under parity cannot be determined without the knowledge of the torsion contribution to K_a^i, or at least its parity behavior. Thus the splitting of extrinsic curvature into torsion-free and torsion parts is inevitable in order to arrive at a set of consistent parity transformations for gravity with fermions. As in the case of (2.14), we have to solve partially equations of motion for the connection to derive the expression for the torsion part k_a^i of $K_a^i = \tilde{K}_a^i + k_a^i$.

For the canonical pair (A_a^i, P_i^a) the equations of motion are $\mathcal{L}_t A_a^i = \{A_a^i, H\} = \delta H/\delta P_i^a$ and $\mathcal{L}_t P_i^a = \{P_i^a, H\} = -\delta H/\delta A_a^i$ where H is the total Hamiltonian constraint (2.47). While the first equation of motion entails all the dynamics of gravity coupled with matter, the second one yields the expression for the connection. Therefore, our focus will be on solving the second equation of motion. After longer calculations, it takes the form

$$\begin{aligned}
&\mathcal{L}_t P_j^c + (\omega^i \cdot t)\epsilon_{ij}{}^k P_k^c - P_j^c \partial_a N^a - N^a \partial_a P_j^c + P_j^b \partial_b N^c \\
&+N^a \epsilon^i{}_{jk} P_i^c A_a^k + N^c G_j + \mathrm{sgn}(\mathrm{e}_\mathrm{a}^\mathrm{i}) \frac{\epsilon^{\mathrm{abc}}}{\gamma^2 \kappa} \partial_\mathrm{b}(\mathrm{Ne}_\mathrm{aj}) \\
&+\frac{N\sqrt{q}}{\gamma^2\kappa}(e_j^b e_k^c - e_k^b e_j^c) A_b^k \;=\; \frac{1}{2}\kappa N \epsilon^i{}_{jk} P_i^c J^k + \frac{N\kappa}{2\alpha} P_j^c J^0 \,.
\end{aligned} \tag{2.48}$$

In order to solve for k_a^i, we use a technique similar to that used above to solve for C_a^i. First, the use of (2.30) eliminates the term involving G_j. Next, we contract (2.48) with e_c^l and, as an internal tensor with indices l and j, derive its trace and symmetric parts. Combined, this gives

$$\begin{aligned} e_c^l \mathcal{L}_t P_j^c &+ e_c^j \mathcal{L}_t P_l^c - \delta_j^l e_c^k \mathcal{L}_t P_k^c + N^a (P_j^c \partial_a e_c^l + P_l^c \partial_a e_c^j) + e_c^l P_j^b \partial_b N^c + e_c^j P_l^b \partial_b N^c \\ &- \mathrm{sgn}(\mathrm{e}_\mathrm{a}^\mathrm{i}) \left(\delta_j^l \frac{\epsilon^{abc}}{\gamma^2 \kappa} N e_c^l \partial_b (e_{aj}) - N \frac{\epsilon^{abc}}{\gamma^2 \kappa} (e_c^l \partial_b e_{aj} + e_c^j \partial_b e_{al}) \right) \\ &+ \frac{N\sqrt{q}}{\gamma^2 \kappa} (e_j^b A_b^l + e_l^b A_b^j) = -\frac{N\sqrt{q}}{2\alpha} \delta_j^l J^0 \,. \end{aligned} \tag{2.49}$$

The extrinsic curvature contribution is contained in

$$\begin{aligned} \frac{\sqrt{q}}{\gamma \kappa} (e_j^b A_b^l + e_l^b A_b^j) &= (P_j^b \widetilde{\Gamma}_b^l + P_l^b \widetilde{\Gamma}_b^j) + (P_j^b C_b^l + P_l^b C_b^j) \\ &\quad + \gamma (P_j^b \widetilde{K}_b^l + P_l^b \widetilde{K}_b^j) + \gamma (P_j^b k_b^l + P_l^b k_b^j) \,, \end{aligned} \tag{2.50}$$

where we have used the decomposition $A_a^i = \widetilde{A}_a^i + \overline{A}_a^i$ into the torsion-free part $\widetilde{A}_a^i = \widetilde{\Gamma}_a^i + \gamma \widetilde{K}_a^i$ and a torsion contribution $\overline{A}_a^i = C_a^i + \gamma k_a^i$.

To complete the splitting, the torsion-free extrinsic curvature from the usual expression $\tilde{K}_{ab} = \frac{1}{2N}(\dot{q}_{ab} - 2\widetilde{D}_{(a} N_{b)})$ satisfies

$$\begin{aligned} P_j^b \tilde{K}_b^l + P_l^b \tilde{K}_b^j &= -\frac{1}{N} \gamma ((e_c^l \mathcal{L}_t P_j^c + e_c^j \mathcal{L}_t P_l^c - \delta_j^l e_c^k \mathcal{L}_t P_k^c) + (N^a (P_j^c \partial_a e_c^l + P_l^c \partial_a e_c^j) \\ &\quad + e_c^l P_j^b \partial_b N^c + e_c^j P_l^b \partial_b N^c)) \end{aligned} \tag{2.51}$$

for $\tilde{K}_a^i = e_i^b \tilde{K}_{ab}$. Combining this with (77), (2.50), (2.33), we find $e_j^b k_b^l + e^{bl} k_{bj} = \kappa \gamma \theta \delta_j^l J^0 / 2(1+\gamma^2)$. On the other hand, (2.30) as obtained by solving the second class constraints gives $k_b^j e^{bl} - k_b^l e^{bj} = \kappa \gamma \beta \epsilon_i{}^{jl} J^i / 2(1+\gamma^2)$. Thus,

$$k_a^m = \frac{\kappa \gamma}{4(1+\gamma^2)} \left(\beta \epsilon_{ij}{}^m \, e_a^i J^j + \theta e_a^m J^0 \right) \,. \tag{2.52}$$

is the contribution which provides the antisymmetric part of K_{ab}, but also adds to the symmetric term.

The expression for k_a^m can independently be verified by computing it from (2.14):

$$\begin{aligned} k_a^m &= -C_a^{m0} = -q_a^\nu n_J C_\nu^{JM} \\ &= \frac{\kappa\gamma}{4(1+\gamma^2)} \left(\beta \epsilon_{ij}{}^m \; e_a^i J^j + \theta e_a^m J^0\right) . \end{aligned} \tag{2.53}$$

With (2.33) and (2.52), the Ashtekar–Barbero connection as split into its torsion and torsion-free parts is

$$A_a^i \;=\; \widetilde{A}_a^i + \overline{A}_a^i \;=\; (\widetilde{\Gamma}_a^i + \gamma \widetilde{K}_a^i) + \frac{\kappa\gamma}{4}\; \epsilon^i{}_{kl} e_a^k J^l - \frac{\kappa\gamma}{4\alpha} e_a^i J^0 \,, \tag{2.54}$$

where the first term is completely torsion-free and only the J-terms represent the torsion contribution.

Parity transformation

In this section, we first define the parity trasnformation for both canonical gravitational variables and fermionic matter fields such that it respects the background independence of a theory of gravity non-minimally coupled with fermions. Parity conservation can then be determined by testing either whether the effective action (2.16) in the Lagrangian formulation, or constraints as well as the symplectic structure of the Hamiltonian formulation are left invariant. As we will see, the torsion contributions to the connection play an important role in this, and we will be led to split all the constraints into their torsion-free and torsion parts to verify the parity behavior.

In a background-independent setting, we cannot refer to spatial coordinates changing their sign under parity reversal. Instead, as usually in formulations on curved manifolds we use the fact that triads change their orientations under parity reversal as one of the primary contributions to the parity transformation: $e_i^a \to -e_i^a$

[4]. For Dirac spinors, we use the conventional field theory definition $\Psi \to \gamma^0\Psi$. These basic definitions imply

$$J^0 = \Psi\gamma^0\gamma^5\Psi \to -J^0 \quad , \quad J^i = \Psi\gamma^i\gamma^5\Psi \to J^i ,$$
$$\Gamma^i_a = \widetilde{\Gamma}^i_a + C^i_a \to \widetilde{\Gamma}^i_a - \frac{\gamma\kappa}{4(1+\gamma^2)}\left(\theta\ \epsilon_{ij}{}^k e^i_b J^j + \beta e^k_b J^0\right),$$
$$K^i_a = \widetilde{K}^i_a + k^i_a \to -\widetilde{K}^i_a - \frac{\gamma\kappa}{4(1+\gamma^2)}\left(\beta\ \epsilon_{ij}{}^k e^i_b J^j - \theta e^k_b J^0\right). \tag{2.55}$$

Here we have used $\widetilde{K}^i_a = \widetilde{K}_{ab}e^{bi} \to -\widetilde{K}^i_a$. It is interesting to note that both Γ^i_a and K^i_a transform like torsion-free $\widetilde{\Gamma}^i_a$ and $\widetilde{K}^i_a$ for $\alpha = \gamma$, a result expected from [30]. Also note that the Ashtekar-Barbero connection A^i_a does not transform like either Γ^i_a or K^i_a due to their different transformation properties. In other words, A^i_a transforms as follows:

$$\begin{aligned} A^i_a &= \left(\widetilde{\Gamma}^i_a - \frac{\gamma\kappa}{4\alpha}\ e^i_a J^0\right) + \gamma\left(\widetilde{K}^i_a + \frac{\kappa}{4}\ \epsilon^i{}_{jk} e^j_a J^k\right) \to \left(\widetilde{\Gamma}^i_a - \frac{\gamma\kappa}{4\alpha}\ e^i_a J^0\right) \\ &\quad -\gamma\left(\widetilde{K}^i_a + \frac{\kappa}{4}\ \epsilon^i{}_{jk} e^j_a J^k\right). \end{aligned} \tag{2.56}$$

It is obvious from (2.56) that A^i_a transforms like the combination of torsion-free $\widetilde{\Gamma}^i_a$ and $\widetilde{K}^i_a$. Next, the symplectic structure transforms as

$$\begin{aligned} \int_\Sigma \mathrm{d}^3x P^a_i \mathcal{L}_t A^i_a &= \int_\Sigma \mathrm{d}^3x P^a_i \mathcal{L}_t \left(\left(\widetilde{\Gamma}^i_a - \frac{\gamma\kappa}{4\alpha}\ e^i_a J^0\right) + \gamma\left(\widetilde{K}^i_a + \frac{\kappa}{4}\ \epsilon^i{}_{jk} e^j_a J^k\right)\right) \\ &\to -\int_\Sigma \mathrm{d}^3x P^a_i \mathcal{L}_t\left(\widetilde{\Gamma}^i_a - \frac{\gamma\kappa}{4\alpha}\ e^i_a J^0\right) \\ &\quad +\gamma\int_\Sigma \mathrm{d}^3x P^a_i \mathcal{L}_t\left(\widetilde{K}^i_a + \frac{\kappa}{4}\ \epsilon^i{}_{jk} e^j_a J^k\right) \\ &= \gamma\int_\Sigma \mathrm{d}^3x P^a_i \mathcal{L}_t\left(\widetilde{K}^i_a + \frac{\kappa}{4}\ \epsilon^i{}_{jk} e^j_a J^k\right) = \int_\Sigma \mathrm{d}^3x P^a_i \mathcal{L}_t A^i_a, \end{aligned} \tag{2.57}$$

where we have used the fact that $\left\{P^a_i, \widetilde{\Gamma}^i_a - \frac{\gamma\kappa}{4\alpha}\ e^i_a J^0\right\}_{PB} = 0$. Therefore, the symplectic structure is invariant under the proposed parity transformations.

---

[4]More generally, the triads can be allowed to transform as $e^a_j \to \Lambda^i_j e^a_i$, where Λ^i_j is an orthogonal transformation matrix with determinant -1. Also, the gamma matrices transform like $\gamma^0 \to \gamma^0$and $\gamma^i \to -\ \Lambda^i_j\gamma^j$. It is easy to check that the torsion-free spin connection and the extrinsic curvature transform as $\widetilde{\Gamma}^i_a \to -\Lambda^i_j\widetilde{\Gamma}^j_a$ and $\widetilde{K}^i_a \to \Lambda^i_j\widetilde{K}^j_a$. Finally, the transformation of A^i_a can be obtained from these two one-forms. Our arguments about parity invariance remain unchanged if this more general transformation is used.

Now, we present the split constraints to further check consistency of the above parity transformations. As seen, the Gauss constraint can already be split in this way and formulated in torsion-free variables without using equations of motion. It is easy to check that the split Gauss constraint, $\widetilde{G}_i = \gamma\epsilon_{ij}{}^k \widetilde{K}^j_a, P^a_k = 0$, is easily satisfied. Splitting the diffeomorphism constraint into torsion and torsion-free components is more involved, and after a long computation we obtain

$$
\begin{aligned}
C_a &= P^b_j(\widetilde{F}^j_{ab} + 2\partial_{[a}\overline{A}^j_{b]} + \epsilon^j{}_{lm}\overline{A}^l_a\overline{A}^m_b + \epsilon^j{}_{lm}\overline{A}^l_a\widetilde{A}^m_b + \epsilon^j{}_{lm}\widetilde{A}^l_a\overline{A}^m_b) - \frac{1+\gamma^2}{\gamma}K^i_a G_i \\
&\quad -\frac{1}{2}i\sqrt{q}(\psi^\dagger\widetilde{D}_a\psi + \eta^\dagger\widetilde{D}_a\eta - c.c.) - \frac{1}{2}C^i_a\sqrt{q}J_i - \frac{\gamma}{2}K^i_a\sqrt{q}J_i \qquad (2.58)\\
&= 2\gamma P^b_j\widetilde{D}_{[a}\widetilde{K}^j_{b]} + \mathrm{sgn}(e^i_a)\frac{\gamma\kappa}{4}\epsilon_{ca}{}^b P^c_l\widetilde{D}_b(\sqrt{q}J^l) - \frac{1}{2}i\sqrt{q}(\psi^\dagger\widetilde{D}_a\psi + \eta^\dagger\widetilde{D}_a\eta - c.c.) \\
&\quad +\mathrm{sgn}(e^i_a)P^f_l\left(\epsilon_{cf}{}^b\Gamma^c_{ba} - \epsilon_{ca}{}^b\Gamma^c_{bf}\right)\sqrt{q}J^l \\
&\quad +\left(\frac{\gamma\kappa}{4}\epsilon^{jkl}J_k e_{al} - \frac{\gamma\kappa}{4\alpha}e^j_a J^0 - \frac{1+\gamma^2-\gamma^3}{\gamma}K^j_a\right)\widetilde{G}_j\,,
\end{aligned}
$$

where Γ^c_{ab} is the torsion-free Christoffel connection which can be expressed in terms of triads and co-triads as in (69) and we have used (2.54) and $\gamma[\widetilde{K}_b, P^b]_i = \widetilde{G}_i$ to arrive at the final expression. Again, it is obvious that the split diffeomorphism constraint (2.58) (modulo Gauss constraint) is invariant under parity transformations.

Finally, the Hamiltonian constraint, after a lengthy calculation, turns out to be

$$
\begin{aligned}
C &= \frac{\kappa\gamma^2}{2\sqrt{q}} P^a_i P^b_j (\epsilon^{ij}{}_k (\widetilde{F}^k_{ab} + 2\partial_{[a}\overline{A}^k_{b]} + \epsilon^k{}_{lm}\overline{A}^l_a\overline{A}^m_b + \epsilon^k{}_{lm}\overline{A}^l_a\widetilde{A}^m_b + \epsilon^k{}_{lm}\widetilde{A}^l_a\overline{A}^m_b) \\
&\quad -2(\gamma^2+1)K^i_{[a}K^j_{b]}) + \frac{1}{2} i\gamma\kappa P^a_i (\psi^\dagger\sigma^i\widetilde{D}_a\psi - \eta^\dagger\sigma^i\widetilde{D}_a\eta - c.c.) + \frac{\gamma^2\kappa P^a_i}{2\sqrt{q}}\widetilde{D}_a(\sqrt{q}J^i) \\
&\quad +\frac{\gamma^2\kappa}{2} P^b_j K^j_b J^0 \\
&\quad +\frac{\gamma\kappa}{2}[K_a, P^a]_j J^j - \frac{3\kappa}{8\sqrt{q}}\frac{\gamma^2}{1+\gamma^2} q(J_0)^2 + (1+\gamma^2)\kappa\widetilde{D}_a\left(\frac{P^a_i G^i}{\sqrt{q}}\right) \qquad (2.59) \\
&= \frac{\kappa\gamma^2}{2\sqrt{q}} P^a_i P^b_j \left(\epsilon^{ij}{}_k \widetilde{R}^k_{ab} - 2\widetilde{K}^i_{[a}\widetilde{K}^j_{b]}\right) + \frac{1}{2} i\gamma\kappa P^a_i (\psi^\dagger\sigma^i\partial_a\psi - \eta^\dagger\sigma^i\partial_a\eta - c.c.) \\
&\quad +\frac{\gamma\kappa\theta}{2} P^b_j \widetilde{\Gamma}^j_b J^0 + \frac{\gamma^3\kappa^2}{4\alpha\sqrt{q}}\epsilon^{ij}{}_k P^a_i e^k_b J^0 \partial_a P^b_j + \frac{3\kappa}{16}\frac{\gamma^2}{1+\gamma^2}\left(\frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1\right)\sqrt{q}(J_0)^2 \\
&\quad -\frac{3\kappa}{16}\frac{\gamma^2}{1+\gamma^2}\left(\frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1\right)\sqrt{q}J_l J^l + \kappa\widetilde{D}_a\left(\frac{P^a_i \widetilde{G}^i}{\sqrt{q}}\right) + \frac{\kappa}{2}(1+\frac{\gamma^2}{2})\widetilde{G}_i J^i \,.
\end{aligned}
$$

It is easy to check that the Hamiltonian constraint (2.59) (modulo Gauss constraint) is also parity invariant. In fact, this is expected since the the usual torsion-free gravitational and Dirac contributions to the effective action (2.16) requires it to be parity invariant. Comparing the interaction term in (2.16) with the following interaction term in (2.59),

$$
\begin{aligned}
H_{\text{int}} &= \frac{3\kappa}{16}\frac{\gamma^2}{1+\gamma^2}\left(\frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1\right)\sqrt{q}(J_0)^2 \\
&\quad -\frac{3\kappa}{16}\frac{\gamma^2}{1+\gamma^2}\left(\frac{1}{\alpha^2} - \frac{2}{\alpha\gamma} - 1\right)\sqrt{q}J_l J^l \,, \qquad (2.60)
\end{aligned}
$$

we conclude that our canonical formulation of gravity with fermions is consistent with the Lagrangian formulation. In other words, splitting the constraints into torsion-free/torsion parts allows us to perform a non-trivial cross-check by comparing our constraints with the interaction Hamiltonian of the effective action.

It is obvious that the above Hamiltonian constraint reduces to that of the vacuum theory if the current and charge density vanish since the interaction term disappears. A comparison of the interacting Hamiltonian constraint (2.60) with (2.16) (for minimal coupling, i.e. $\alpha \to \infty$) reveals that the corresponding interacting terms are in complete agreement. Thus, the derivations in this subsection provide a cross-check of the canonical calculations performed so far. Ignoring the interaction term in (2.60), on the other hand, provides the Hamiltonian constraint of a second-order formalism which can be compared directly with the Appendix of [36] (for $\gamma = 1$). Notice that the derivation sketched in [36] does not work purely in real variables and assumes properties of the projection from complex variables. As the comparison with our results shows, the calculations of [36] leave some extra terms in the constraint which are absent in a complete derivation based only on real variables.

2.4 Canonical Formulation for Maxwell's field

In this section, a relatively simple canonical formulation for Maxwell's field is presented. The basic configuration variable in a Lagrangian formulation of Maxwell's field theory is the vector potential A_a which determines the field strength tensor

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \,, \tag{2.61}$$

where ∇_a is the covariant derivative operator. Notice that ∇_a can be replaced by the partial derivative operator ∂_a even on a curved space-time since the field strength tensor F_{ab} is antisymmetric. The action for the free Maxwell field in an arbitrary background space is given by

$$\begin{aligned} S_M &= -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}F_{ab}F^{ab} \\ &= -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}F_{ab}F_{cd}g^{ac}g^{bd} \end{aligned} \tag{2.62}$$

where g is the determinant of the Lorentzian space-time metric g_{ab}. From the action one obtains Maxwell's equations as the Euler–Lagrange equations extremizing S_M. However, the total action can be obtained from adding this action to the Holst action (1.4) and, as a consequence, background-independent Maxwell's equations are obtained by extremizing the total action.

One crucial difference between the above action (2.62) and the Einstein-Cartan action (2.1) lies in the manner each corresponding field couples to gravity. While Dirac fields couple to gravity through connection as well as metric, Maxwell's fields couple only through metric, which can be seen from the covariant derivatives in (2.62). Therefore, an important ramification of coupling of Maxwell's field to gravity only through metric is the absence of torsion in the theory. Extremizing the Maxwell's action with respect to variations in the Lorentz connection yields

$$\nabla_\mu(ee_I^{[\mu}e_J^{\nu]}) = 0 \,, \tag{2.63}$$

which is the same as the one derived from the Holst action alone, thus implying no torsion.

Maxwell Hamiltonian

Let us again recall, as illustrated in Fig. 2.1, the decomposition of t^a into normal and tangential parts with respect to Σ_t by defining the lapse function N and the shift vector N^a as $t^a = Nn^a + N^a$ with $N^a n_a = 0$, where n^a is the unit normal vector field to the hypersurfaces Σ_t. The space-time metric g_{ab} induces a spatial metric q_{ab} by the formula $g_{ab} = q_{ab} - n_a n_b$. Now using $n^a = N^{-1}(t^a - N^a)$ and $q^{ab} = g^{ab} + n^a n^b$ to project fields normal and tangential to Σ_t, we can decompose the field strength tensor F_{ab} and the action S_M as follows:

$$F_{ab}n^a = \frac{1}{N}\left(F_{ab}t^a - N^a F_{ab}\right)\frac{1}{N}\left(\dot{A}_b - \partial_b\left(A_a t^a\right) - N^a F_{ab}\right) \,, \tag{2.64}$$

$$\begin{aligned} F_{ab}F^{ab} &= F_{ab}F_{cd}g^{ac}g^{bd} \\ &= F_{ab}F_{cd}\left(q^{ac} - n^a n^c\right)\left(q^{bd} - n^b n^d\right) = F_{ab}F_{cd}q^{ac}q^{bd} - 2F_{ab}F_{cd}n^a n^c q^{bd} \\ &= F_{ab}F_{cd}q^{ac}q^{bd} - \frac{2}{N^2}\left(\dot{A}_b - \partial_b\left(A_a t^a\right) - N^a F_{ab}\right) \times \\ &\qquad \left(\dot{A}_d - \partial_d\left(A_a t^a\right) - N^c F_{cd}\right) q^{bd} \,, \end{aligned} \tag{2.65}$$

where $\dot{A}_b = \mathcal{L}_t A_b = t^a \partial_a A_b + A_a \partial_b t^a$, and the action takes the form

$$\begin{aligned}
S_M &= -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}F_{ab}F^{ab} = -\frac{1}{16\pi}\int \mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3xN\sqrt{q}F_{ab}F^{ab} \\
&= -\frac{1}{16\pi}\int \mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3xN\sqrt{q}(-\frac{2}{N^2}\left(\dot{A}_b - \partial_b\left(A_a t^a\right) - N^a F_{ab}\right)\times \\
&\qquad \left(\dot{A}_d - \partial_d\left(A_a t^a\right) - N^c F_{cd}\right)q^{bd} + F_{ab}F_{cd}q^{ac}q^{bd}) \\
&= \int \mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3x(\frac{\sqrt{q}}{8\pi N}\left(\dot{A}_b - \partial_b\left(A_a t^a\right) - N^a F_{ab}\right) \\
&\qquad \left(\dot{A}_d - \partial_d\left(A_a t^a\right) - N^c F_{cd}\right)q^{bd} - \frac{N\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}).
\end{aligned} \tag{2.66}$$

It follows that the conjugate momentum π^a to the configuration variable A_a is given by

$$\pi^e = \frac{\delta S_M}{\delta \dot{A}_e} = \frac{\sqrt{q}}{4\pi N}\left(\dot{A}_d - \partial_d\left(A_a t^a\right) - N^c F_{cd}\right)q^{ed}, \tag{2.67}$$

which is a densitized vector field because of the presence of $\sqrt{q}$. Its physical interpretation is as the electric field measured by an observer with 4-velocity n^a. Now the action can be expressed in terms of the canonical variables A_a and π^a,

$$S_M\left(A_a,\pi^a\right) = \int \mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3x\left(\frac{2\pi N}{\sqrt{q}}\pi^a\pi^b q_{ab} - \frac{N\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\right). \tag{2.68}$$

We can cast the action in equation (2.68) into the desired form
$S_M = \int \mathrm{d}t\left[\int_{\Sigma_t}\mathrm{d}^3x\pi^a\dot{A}_a - H_M\right]$ by writing the integrand in the following manner:

$$\begin{aligned}
S_M\left(A_a,\pi^a\right) &= \int \mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3x\left[\frac{4\pi N}{\sqrt{q}}\pi^a\pi^b q_{ab} - N\left(\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\right)\right] \\
&= \int \mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3x[\pi^a\left(\dot{A}_a - \partial_a\left(A_d t^d\right) - N^c F_{ca}\right) - N(\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} \\
&\qquad +\frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd})] \\
&= \int \mathrm{d}t\int_{\Sigma_t}\mathrm{d}^3x[\pi^a\dot{A}_a + \left(A_d t^d\right)\partial_a\pi^a - N^c\pi^a F_{ca} - N(\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} \\
&\qquad +\frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd})]
\end{aligned} \tag{2.69}$$

having integrated by parts in the second term. This completes the Legendre transform and we can read off the equations of motion from equation (2.69). First, since the momentum conjugate to the time component of A_a is absent, extremization of the action with respect to $A_a t^a$ results in

$$G = \partial_a \pi^a = 0 \tag{2.70}$$

as the usual Gauss constraint. The total Hamiltonian of the Maxwell field then is

$$\begin{aligned} H_M &= \int_{\Sigma_t} d^3x [- \left(A_d t^d\right) \partial_a \pi^a + N^c \pi^a F_{ca} \\ &\qquad + N \left(\frac{2\pi}{\sqrt{q}} \pi^a \pi^b q_{ab} + \frac{\sqrt{q}}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd} \right)] \end{aligned} \tag{2.71}$$

with two contributions

$$D_c = \pi^a F_{ca} \tag{2.72}$$

and

$$\mathcal{H} = \frac{2\pi}{\sqrt{q}} \pi^a \pi^b q_{ab} + \frac{\sqrt{q}}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd} \tag{2.73}$$

which, when added to the gravitational Hamiltonian (1.7) and (1.8), give matter contributions to the diffeomorphism and Hamiltonian constraint, respectively. From (2.73) we obtain the usual expression $\int \mathrm{d}^3 x \mathcal{H}$ for the energy of the electromagnetic field.

2.5 Conclusion

This chapter presents a detailed derivation of the complete canonical formulation of gravity non-minimally coupled to fermions in Ashtekar variables. This includes generalizations of basic results in the recent and some older literature, such as the torsion-mediated four-fermion interaction, and puts them on a firm canonical basis. In particular, we have used this for a demonstration of parity invariance of classical solutions, which required us to derive all contributions to the Ashtekar connection explicitly and to write several new versions of the canonical constraints, with explicit or implicit torsion contributions. The different forms of the constraints are needed to understand the parity behavior, and they also facilitate comparisons with earlier derivations and allow crucial cross-checks of the results. Here, we have noticed that our analysis fills in several gaps of previously available derivations and generalizes them to arbitrary non-minimal coupling.

We have also presented a detailed canonical analysis of the Maxwell theory coupled to gravity through metric only. This canonical formulation of gravity coupled to fermions as well as photons will enable us to proceed with loop quantization , which is discussed in the next chapter.

CHAPTER III

LOOP QUANTIZATION

After extending the canonical formulation of gravity to incorporate matter fields like Dirac's and Maxwell's fields in the previous chapter, the natural next step is to discuss loop quantizations of these matter fields. Therefore, in this chapter, the quantization of fermions non-minimally coupled to gravity, followed by a summary of the quantization of bosons, is presented to explore the possibility of interesting physical effects in the following chapters.

3.1 Quantization of Dirac Fields

Quantizations of fermions in canonical gravity have already been developed in [25, 36] (see also [41, 42, 43] for earlier work). However, these developments were not based on a systematic derivation of the canonical formulation from a Holst-type action and several features related to the torsion-dependence of the Ashtekar–Barbero connection were overlooked or remained implicit. Corresponding adaptations which become necessary in a consistent quantization could thus appear to draw suspicions about the validity of the basic strategy of a loop quantization as used in [25, 36]. (Some concerns have, for instance, been voiced in [37].) In addition to that, we here raise the question of parity invariance of the quantum theory which may be a concern given that the classical verification of parity required us to partially solve equations of motion to see the correct transformation behavior.

Before starting the quantization, the first question concerns the choice of basic variables. We have two sets, given by the canonical variables (A_a^i, P_j^b) in the presence of torsion as well as the torsion-free components $(\tilde{A}_a^i, P_j^b)$ with explicit expressions for torsion in terms of the fermion current in (2.54). However, as we have seen, equations of motion are required to find the torsion contribution to extrinsic curvature in explicit form. The use of classical equations of motion is not suitable for a quantization, and there is thus no choice but to use the canonical variables with implicit torsion terms.

Half-densitized fermions

In addition to torsion terms, there will be a further contribution to the connection once we formulate the fermions in terms of half-densities as required for consistency [36]. For fermions, we have the canonical pair (ψ, π) with $\pi = -i\sqrt{q}\psi^\dagger$. These canonical variables cannot be promoted to operators on a Hilbert space with a suitable inner product in a way incorporating the reality condition $\pi^\dagger = i\sqrt{q}\psi$ by satisfying $\hat{\pi}^\dagger = i\widehat{\sqrt{q}}\hat{\psi}$: First, if $f(A)$ is a non-trivial real valued function of the connection A, then the inconsistent relation

$$0 = 0^\dagger = ([\hat{\pi}, f(A)])^\dagger = i[\widehat{\sqrt{q}}, f(A)]\hat{\psi} \neq 0 \tag{3.1}$$

ensues. Here the first commutator is expected to vanish since the corresponding classical Poisson bracket vanishes. On the contrary, the classical Poisson bracket corresponding to the second commutator is non-zero; hence the inconsistency arises. A second problem can be seen to arise from the symplectic structure obtained from the fermion Liouville form

$$\begin{aligned}\Theta = -i\int_{\Sigma_t} \mathrm{d}^3x\sqrt{q}\left(\theta_L\psi^\dagger\dot{\psi} - \theta_R\dot{\psi}^\dagger\psi\right) &= \int_{\Sigma_t}\mathrm{d}^3x\left(\pi\dot{\psi} - \frac{i}{2}\theta_R\gamma\kappa\psi^\dagger\psi e^i_c\dot{P}^c_i\right) \\ &\quad - \int_{\Sigma_t}\mathrm{d}^3x\theta_R\mathcal{L}_t(\pi\psi)\,. \end{aligned} \tag{3.2}$$

Here, it follows from the second term of the first integral that the connection A^i_a acquires an imaginary correction term $\frac{1}{2}i\theta_R\psi^\dagger\psi e^i_a$, which endows the theory with a complex connection. This, in turn, would require the use of a complexification of SU(2) in holonomies, for which, due to the non-compactness, none of the loop quantization techniques relying on the existence of a normalized Haar measure would be available (see e.g. [13]).

Both problems were solved by Thiemann who observed in [36] that, in order to obtain a well-defined canonical loop quantization with a real Ashtekar–Barbero connection also in the presence of fermions, one should cast fermion fields into Grassmann-valued half-densities. Thus $\xi := \sqrt[4]{q}\psi$ instead of ψ (and $\chi := \sqrt[4]{q}\eta$ instead of η) is considered to be the classical canonical variable, and $\pi_\xi = -i\xi^\dagger$ is the conjugate momentum for ξ. The inconsistencies in (3.1) are naturally removed as the new canonical variables imply the reality condition $\pi_\xi^\dagger = i\xi$ without any appearance of $\sqrt{q}$.

In half-densities, the symplectic structure becomes

$$\begin{aligned}\Theta &= -i\int_{\Sigma_t} \mathrm{d}^3x\ \sqrt{q}\left(\theta_L(\psi^\dagger\dot{\psi} - \dot{\eta}^\dagger\eta) - \theta_R(\dot{\psi}^\dagger\psi - \eta^\dagger\dot{\eta})\right) \\ &= \int_{\Sigma_t} \mathrm{d}^3x\left(\pi_\xi\dot{\xi} + \pi_\chi\dot{\chi}\right) + \int_{\Sigma_t} \mathrm{d}^3x\ \frac{\gamma\kappa}{4\alpha}P_i^c\mathcal{L}_t(e_c^iJ^0)\,, \end{aligned} \tag{3.3}$$

where we have ignored total time derivatives which would drop out of the action for appropriate boundary conditions. The classical anti-Poisson brackets for Grassmann-valued fields are $\{\xi_A(x), \pi_{\xi B}(y)\}_+ = \delta_{AB}\delta(x,y)$. Moreover, as the extra term shows, $\sqrt[4]{q}$ can be absorbed in spinors without changing the symplectic structure of the gravitational variables only when $\alpha \to \infty$, i.e. for minimal coupling. Combining the last term in (3.3) with the gravitational Liouville term $\int \mathrm{d}^3xP_i^c\mathcal{L}_tA_c^i$, a real-valued correction term $\frac{\gamma\kappa}{4\alpha}e_a^iJ^0$ must be added to the Ashtekar-Barbero connection A_a^i. This is a new feature that is present in the non-minimally coupled theory if the fermion fields are expressed in terms of half-densities. Therefore, the new canonical connection can be written as

$$\mathcal{A}_a^i\ :=\ A_a^i + \frac{\gamma\kappa}{4\alpha}e_a^iJ^0\ = \widetilde{\Gamma}_a^i + \mathcal{C}_a^i + \gamma K_a^i\,, \tag{3.4}$$

where

$$\mathcal{C}_a^i\ := \frac{\theta\gamma^2\kappa}{4(1+\gamma^2)}\left(\frac{1}{\gamma}\ \epsilon^j{}_{kl}e_a^kJ^l - e_a^jJ^0\right). \tag{3.5}$$

Absorbing the correction term into the torsion contribution to the spatial spin connection allows one to keep K_a^i unchanged in the course of expressing all the constraints in terms of the corrected connection. Note that the corrected torsion contribution, $\mathcal{C}_a^i$, to the spin connection vanishes for $\alpha = \gamma$. (If one would use the fully split connection (2.54) based on partial solutions of the equations of motion, the new contribution in the presence of half-densities would cancel the J^0-dependence of $\mathcal{A}_a^i$ completely.)

In terms of the corrected connection and half-densities, the total Dirac Hamiltonian constraint (modulo the Gauss constraint) in (2.46) takes the smeared form

$$\begin{aligned} H_{\rm total} &= \int_{\Sigma_t} \mathrm{d}^3x\ N\Big(\frac{\gamma^2\kappa}{2\sqrt{q}}P_i^a P_j^b \left(\epsilon^{ij}{}_k\mathcal{F}_{ab}^k - 2(\gamma^2+1)K^i_{[a}K^j_{b]}\right) \\ &\quad -\frac{\gamma\kappa\beta P_i^a}{\sqrt{q}}\mathcal{D}_a\left(\pi_\xi\tau^i\xi + \pi_\chi\tau^i\chi\right) - i\frac{2\gamma\kappa P_i^a}{\sqrt{q}}\left(\theta_L\pi_\xi\tau^i\mathcal{D}_a\xi - \theta_R\pi_\chi\tau^i\mathcal{D}_a\chi - c.c.\right) \\ &\quad +\frac{\gamma\kappa\beta}{2\sqrt{q}(1+\gamma^2)}\left(3-\frac{\gamma}{\alpha}+2\gamma^2\right)(\pi_\xi\tau_l\xi + \pi_\chi\tau_l\chi)(\pi_\xi\tau^l\xi+\pi_\chi\tau^l\chi) \\ &\quad +i\frac{\gamma^3\kappa^2}{4\alpha q}\epsilon^{ij}{}_kP_i^a e_b^k(\pi_\xi\xi - \pi_\chi\chi)\mathcal{D}_aP_j^b + \frac{3\gamma\kappa\theta}{8\alpha\sqrt{q}}(\pi_\xi\xi-\pi_\chi\chi)(\pi_\xi\xi-\pi_\chi\chi)\Big)\ , \end{aligned} \tag{3.6}$$

where $\mathcal{F}_{ab}^k$ is the curvature and $\mathcal{D}$, now and in the rest of the paper, is the covariant derivative related to the corrected connection $\mathcal{A}$.

Quantum representation

The ordinary kinematical constructions of loop quantum gravity do not refer to torsion or torsion-freedom and thus go through unchanged. We thus present only the bare concepts relevant for the construction of constraint operators.

Fermion fields

The space of all Grassmann-valued half-densitized 2-component spinors $\xi(x)$ and $\chi(x)$ constitutes the classical configuration space $\mathcal{F}$ for fermion fields. The loop quantization [36] then promotes smeared objects

$$\Xi_A(x) := \int_{\Sigma_t}\mathrm{d}^3y\sqrt{\delta(x,y)}\xi_A\ := \lim_{\epsilon\to 0}\int_{\Sigma_t}\mathrm{d}^3y\sqrt{\frac{\chi_\epsilon(x,y)}{\epsilon^3}}\xi_A$$

to operators, where $\chi_\epsilon(x,y)$ is the characteristic function of a box of Lebesgue measure ϵ^3 centered at x. Note that Ξ_A are scalar Grassmann valued functions since the δ distribution is a density of weight one. It is also easy to see that Ξ and their adjoint satisfy anti-Poisson brackets similar to those presented above for ξ. Upon quantization, the anti-Poisson bracket is replaced by the anti-commutator $[\hat{\Xi}_A(x), \hat{\pi}_B(y)]_+ = i\hbar\delta_{AB}\delta_{x,y}$ with $\delta_{x,y}$ being the Kronecker symbol (rather than a δ-distribution thanks to the smearing involved in Ξ_A).

This algebra can be represented on a non-separable Hilbert space $\mathcal{H}_F = L^2(\overline{S}, \mathrm{d}\mu_F) = \bigotimes_{v\in\Sigma} L^2(S_v, \mathrm{d}\mu_v)$ where each copy $\mathcal{H}_v$ for any point v in space is an ordinary Grassmann-valued Hilbert space of multi-linear functions of $\Xi_A(v)$ and $\bar{\Xi}_A(v)$ of two-component spinors in their Grassmann space S_v, with integration measure $\mathrm{d}\mu_v = \mathrm{d}\overline{\Xi}_v \mathrm{d}\Xi_v e^{\overline{\Xi}_v \Xi_v}$. The full space of the fields can then be written as $\overline{S} := \bigotimes_{v\in\Sigma} S_v$ with measure $\mathrm{d}\mu_F(\overline{\Xi}, \Xi) = \prod_{v\in\Sigma} \mathrm{d}\mu_v$. On this space, $\hat{\Xi}_A$ acts as a multiplication operator, and its momentum $\hat{\pi}_B = -i\hbar\partial/\partial\Xi_B$ by a derivative. In addition, we have a second copy of these point-wise Hilbert spaces for χ smeared to X.

A dense subset of functions in this Hilbert space is formed by cylindrical functions which are superpositions only of products of finitely many vertex-wise Grassmann-factors. These functions can be seen to arise if one starts with a cyclic state independent of Ξ and X and uses the $\hat{\Xi}_v$ and $\hat{X}_v$ as "creation" operators. Since all the constraints depend on the fermion only via currents, which are polynomials in Ξ_A and X_A, they can easily be represented on this subspace of cylindrical functions.

Gravitational variables

Classical configuration variables and the corresponding loop quantization for torsion-free gravity are discussed in section 1.2. It follows from section 2.2 that classical configuration variables for gravity with torsion are also SU(2)-connections on a principal fiber bundle over the spatial manifold Σ, represented by smooth su(2)-valued local 1-forms $\mathcal{A}_a^i$ from (3.4); the space $\mathcal{A}$ of all such 1-forms is the classical configuration space. The phase space is coordinatized by the pair $(\mathcal{A}_a^i, P_i^a)$, where P_a^i

is the conjugate momentum, an su(2)-valued vector density on Σ proportional to the densitized triad. Then the only non-vanishing Poisson bracket is

$$\left\{\mathcal{A}_a^i(x), P_j^b(y)\right\} = \delta_j^i \delta_a^b \delta(x,y)\,. \tag{3.7}$$

No well-defined quantum analogs for these canonical variables exist in a direct form without smearing. The elementary classical variables that have well-defined quantum analogs are rather given by (complex valued) matrix elements of holonomies $h_e(\mathcal{A}) = \mathcal{P}\exp(\int_e \mathcal{A}_a^i \tau_i \dot{e}^a \mathrm{d}t) \in \mathrm{SU}(2)$ along paths e in Σ and fluxes $F_S^{(f)}(P) := \int_S f_i n_a P_i^a \mathrm{d}^2 y$, where f are su(2)-valued functions across 2-surfaces S in Σ and n_a is the (metric-independent) co-normal to the surface.

Furthermore, all the fundamental constructions of cylindrical functions, spin-network states and Hilbert space remain unchanged in the presence of torsion. By construction, the Ashtekar-Barbero connection inherits the total torsion contribution and thus the effect of torsion on the system is concealed in holonomies which are used in states and as basic multiplication operators of loop quantum gravity. Consequently, the functions of connections that represent the quantum state of the system and operators containing holonomies are endowed with all contributions from torsion in quantum kinematics. A complete split of torsion-free and torsion components is possible only once equations of motion are partially used. This is not available at the kinematical level, which thus has no choice but to refer to the unsplit torsion connection.

Combined Hilbert space of gravity and fermions

For the combined system, we simply take the tensor product $\mathcal{H} \otimes \mathcal{H}_F$ as the Hilbert space, which acquires the tensor product of the basic representations. All cylindrical states can be written in the form $\psi(h_{e_1}, \cdots, h_{e_m}, \Xi_{v_1}, \cdots, \Xi_{v_m}, X_{w_1}, \cdots, X_{w_l})$ with integer n, m and l. Especially for the gravitational dependence it is useful to use special cylindrical states based on spin networks [14, 15]: graphs together with a labeling j_e of their edges by irreducible SU(2)-representations $\rho^{(j_e)}$, and of vertices with spinor representations σ_v of SU(2) (obtained from tensor products of the fundamental representation given by the basic 2-spinors) as well as contractors C_v in vertices to contract the matrix-represented holonomies of edges incoming and outgoing at v. Such states take the form

$$\prod_{v,e} C_v{}^{\nu_1^v,\ldots,\nu_{n_v}^v,\nu^v}{}_{\mu_1^v,\ldots,\mu_{m_v}^v,\mu^v}\,\rho^{(j_e)}(h_e(\mathcal{A}))^{\mu_e}_{\nu_e}\sigma_v(\Xi_v,X_v)^{\mu^v}_{\nu^v} \tag{3.8}$$

where for all vertex labels ν_i^v are to be contracted with indices ν_e on represented matrices $\rho^{(j_e)}(h_e(\mathcal{A}))^{\mu_e}_{\nu_e}$ of all n_v outgoing edges as well as the spinor index ν^v, and μ_i^v with indices μ_e of all m_v incoming edges as well as the spinor index μ^v.

Constraints

General relativity is a background independent theory and is fully constrained in the canonical formulation. Thus the quantization of the constraints is necessary to obtain physical states. Having identified elementary operators and their quantum representation, this kinematical structure is now used to construct a set of quantum operators corresponding to constraints relevant for the system. Subsequently, these quantum constraints have to be solved to obtain physical states. The existence of torsion may change the form of each of the quantum constraint operators and consequently influence their solutions. Here, we will show that extra terms can be quantized consistently.

Kinematical constraints

We first express the Gauss constraint in terms of half-densities and the new canonical connection $\mathcal{A}_a^i$:

$$G_i := \mathcal{D}_b P_i^b - \frac{1}{2}\sqrt{q}J_i \;=\; \mathcal{D}_b P_i^b + \pi_\xi\tau_i\xi + \pi_\chi\tau_i\chi\,. \tag{3.9}$$

Upon smearing the constraint with an su(2)-valued function Λ^i on Σ, it is easy to see, as shown in section 1.1 for the torsion-free case, that $G[\Lambda^i] = \int_\Sigma \mathrm{d}^3x\ \Lambda^i G_i$ generates internal SU(2) rotations on the phase space of general relativity:

$$\{\mathcal{A}_a^i, G[\Lambda]\} = -\mathcal{D}_a\Lambda^i \quad \text{and} \quad \{P_i^a, G[\Lambda]\} = \epsilon_{ij}{}^k\Lambda^j P_k^a$$

together with a spinor transformation in the fundamental representation of SU(2). Thus, the quantization of the Gauss constraint is carried out in a similar fashion as it is done in the torsion-free case, restricting gauge invariant states to be supported on $\mathcal{A}/\mathcal{G}$. For our configuration variables, we have the transformations $h_e \mapsto g_{e(0)} h_e g_{e(1)}^{-1}$, $\Xi_v \mapsto g_v \Xi_v$ and $X_v \mapsto g_v X_v$ under a gauge transformation $g\colon v \mapsto g_v \in \mathrm{SU}(2)$. A spin network state, when gauge transformed, acquires at each vertex v factors of $\rho^{(j_e)}(g_v^{-1})$ from all incoming edges, $\rho^{(j_e)}(g_v)$ from outgoing edges and $f_v(g_v)$ from spinor factors in the state. For a gauge invariant state, these factors must cancel each other when contracted with the C_v in (3.8), which implies that representation matrices (including the spinor) must be multiplied by contraction with an intertwiner of all relevant representations to the trivial one. The resulting gauge invariant states satisfy the quantum constraint equation $\hat{G}[\Lambda^i]\Psi_\alpha = 0$ for all Λ^i.

Similarly, one can use the action of the spatial diffeomorphism group on the phase space by computing infinitesimal canonical transformations generated by $D[N^a] = \int_\Sigma \mathrm{d}^3x\ N^a C_a$. In terms of half-densities and the corrected connection, the constraint turns out to be

$$\begin{aligned} D[N^a] &= \int_\Sigma \mathrm{d}^3x\ N^a\big(2P_j^b\partial_{[a}\mathcal{A}_{b]}^j - \mathcal{A}_a^i\partial_b P_i^b \\ &\quad +\frac{1}{2}(\pi_\xi\partial_a\xi - (\partial_a\pi_\xi)\xi + \pi_\chi\partial_a\chi - (\partial_a\pi_\chi)\chi)\big) \end{aligned} \tag{3.10}$$

up to contributions from the Gauss constraint. Again, as in the torsion-free gravity shown in section 1.1, this constraint generates transformations

$$\{\mathcal{A}_a^i, D[N^a]\} = N^b\mathcal{F}_{ba}^i + \mathcal{D}_a(N^c\mathcal{A}_c^i) = \mathcal{L}_{\vec{N}}\mathcal{A}_a^i$$

and

$$\{P^a_i, D[N^a]\} = N^b\partial_b P^a_i - P^b_i\partial_b N^a + P^a_i\partial_b N^b = \mathcal{L}_{\vec{N}} P^a_i$$

as well as the correct Lie derivative $\delta\xi = N^a\partial_a\xi + \frac{1}{2}\xi\partial_a N^a$ of half-densitized fermions. Hence, this constraint can be quantized as in the torsion-free case via the finite action of the diffeomorphism group. A finite diffeomorphism φ is represented on cylindrical states by

$$\begin{aligned} & \hat{D}_\varphi\psi(h_{e_1},\ldots,h_{e_m},\Xi_{v_1},\ldots,\Xi_{v_n},X_{w_1},\ldots,X_{w_l}) \\ = \; & \psi(h_{\varphi(e_1)},\ldots,h_{\varphi(e_m)},\Xi_{\varphi(v_1)},\ldots,\Xi_{\varphi(v_n)},X_{\varphi(w_1)},\ldots,X_{\varphi(w_l)}) \end{aligned} \tag{3.11}$$

simply by moving the graph (which presents a unitary transformation with respect to the Ashtekar–Lewandowski measure). Thus, invariant states can be determined by constructing a new, diffeomorphism invariant Hilbert space via group averaging.

Hamiltonian constraint

While the Gauss and diffeomorphism constraints generate the canonical transformations that represent the well-known kinematical gauge symmetries in the classical phase space independently of torsion, the scalar constraint entails the essence of dynamics of the theory. Hence the scalar quantum operator describes quantum dynamics of the physical states which must be in accordance with the presence of torsion. Unfortunately, a complete quantization of this scalar constraint is yet to be satisfactorily realized. Therefore, we present only the necessary adaptations to the existing quantization attempts. In this approach, it is essential to re-express the classical expression of the scalar constraint in terms of those phase space functions which can be promoted to well-defined operators.

Our starting point is expression (3.6) of the Hamiltonian constraint in half-densitized fermions. The fermion terms in the Dirac Hamiltonian coupled with gravity, can be quantized using the strategy developed by Thiemann in [25]. Note that this Dirac Hamiltonian is different from the one presented in [25] (which took a second order viewpoint) in two aspects: the covariant derivative $\mathcal{D}$ now contains the Ashtekar–Barbero connection with torsion and the interaction term is new. Also the gravitational term has torsion contributions which have to be taken into account when applying the standard quantization strategy of [21].

As usual, the expression involving extrinsic curvature K_a^i would vanish for $\gamma = 1$ in Euclidean signature which in turn implies that the first term in the gravitational constraint reduces to the scalar constraint $H^E[N]$ of Euclidean general relativity. Then let us write the scalar constraint for gravity alone as

$$H[N] = \sqrt{\gamma} H^E[N] - 2(1+\gamma^2)\mathcal{T}[N], \tag{3.12}$$

where

$$\mathcal{T}[N] := \sqrt{\frac{\gamma}{4\kappa}} \int_{\Sigma_t} \mathrm{d}^3x \; N \frac{P_i^a P_j^b}{\sqrt{|\det P|}} K^i_{[a} K^j_{b]} \,. \tag{3.13}$$

In order to quantize the scalar constraint for gravity, it is first necessary to express it in terms of classical phase space functions which have well-defined quantum analogs. In this regard, the following classical objects and relationships are crucial as building blocks: The total volume $V = (\gamma\kappa)^{3/2} \int_\Sigma \mathrm{d}^3x \sqrt{|\det P|}$ of Σ, the co-triad

$$e_a^i(x) := \frac{\sqrt{\kappa\gamma}}{2} \mathrm{sgn}\det(e_d^l) \epsilon_{abc} \epsilon^{ijk} \frac{P_j^b P_k^c}{\sqrt{\det\; P}} = \frac{2}{\gamma\kappa} \left\{ \mathcal{A}_a^i(x), V \right\} \,, \tag{3.14}$$

the integrated trace of extrinsic curvature

$$K \;:=\; \gamma\kappa \int_\Sigma \mathrm{d}^3x \; K_a^i P_i^a \,. \tag{3.15}$$

as well as expansions

$$h_e(\mathcal{A}) \;=\; 1 + \delta s^a \tau_i \mathcal{A}_a^i + O(\delta^2) \tag{3.16}$$

$$h_{\alpha_{IJ}}(\mathcal{A}) \;=\; 1 + \delta^2 s_I^a s_J^b \mathcal{F}_{ab}^k \tau_k + O(\delta^3) \tag{3.17}$$

of holonomies along small open edges e in direction s^a of coordinate length δ or small square loops α_{IJ} of coordinate area δ^2 with sides in the directions s_I^a.

The first step in a regularization of a spatial integral is to introduce a triangulation of Σ as the union of tetrahedra with edges of coordinate length δ and edges at a given vertex pointing in directions s_I^a, $I = 1, 2, 3$. To use this for a construction of operators, the positions and directions of tetrahedra are usually adapted to vertices and edges of the graph underlying a state to be acted on. The coordinate volumes of tetrahedra then replace the integration measure: $\epsilon^{abc} \mathrm{d}^3x \to \delta^3 \epsilon^{IJK} s_I^a s_J^b s_K^c$. Moreover, internal tensors can be written in terms of Pauli matrices, such as $\epsilon_{kmn} = -4\mathrm{tr}(\tau_k \tau_m \tau_n)$. The tangents s_I^a, factors of δ and Pauli matrices can then be combined with Poisson brackets to obtain

$$\tau_k \delta s_I^a \{\mathcal{A}_a^k, O\} \to -\frac{1}{i\hbar} h_{s_I} [h_{s_I}^{-1}, \hat{O}] \tag{3.18}$$

in terms of holonomies with their well-defined quantization, where O could be the volume if (4.37) is used, or the integrated trace of extrinsic curvature K. For fine triangulations, $\delta \ll 1$, the error in replacing connection components by holonomies is small, and it goes to zero in the limit where all edge lengths of tetrahedra vanish. Similarly, covariant derivatives can be combined to $\delta s_I^a \mathcal{D}_a$ and then regularized to a difference of values at the endpoints of a small edge in direction s_I^a. If there are always three factors where δ can be absorbed and the quantized contributions vanish only when acting on vertices of a graph, a well-defined operator results even in the limit when the regulator is removed because for finite graphs finitely many terms remain in the triangulation sum.

We first turn to the matter terms which arise in (3.6). Some of them agree with the Dirac Hamiltonian used in [25], and can thus be quantized along the same lines. However, our analysis has provided extra terms which must be ensured to have well-defined quantum expressions, too. The current interaction terms can directly be quantized with fermion operators and using

$$
\begin{aligned}
\frac{\mathrm{sgn}\,\det(e^i_a)}{\sqrt{q}} &= \frac{1}{6q}\epsilon^{abc}\epsilon_{ijk}e^i_a e^j_b e^k_c \\
&= \frac{36\epsilon^{abc}\epsilon_{ijk}}{\gamma^3\kappa^3}\{\mathcal{A}^i_a, V^{1/3}\}\{\mathcal{A}^j_b, V^{1/3}\}\{\mathcal{A}^k_c, V^{1/3}\}
\end{aligned}
$$

for a quantizable expression in terms of commutators of holonomies and the volume operator. Edge tangents of the holonomies for the three Poisson brackets provide the elementary coordinate volumes of the triangulation, while half-densitized fermions in the current products will simply be vertex-wise operators.

Terms of the form $q^{-1/2}P^a_i\mathcal{D}_a O$ where O is an expression of fermions can be reformulated using $\gamma\kappa P^a_i = \frac{\mathrm{sgn}\,\det(e^i_a)}{2}\epsilon^{abc}\epsilon_{ijk}e^j_b e^k_c$ in which we can again absorb the inverse $\sqrt{q}$ after expressing the co-triads as Poisson brackets. Here, we will have two holonomies requiring an edge tangent vector as well as the covariant derivative which will become a directional derivative once the triangulation volumes are expressed via edge vectors: we use the expansion $h_e(\delta)O(e(\delta)) - O(e(0)) \approx \delta\dot{e}^a\mathcal{D}_a O$ where $h_e(\delta)$ is a holonomy along an edge e of coordinate length δ. Also these terms can thus be quantized by standard techniques, which involves a discretization of the derivative.

Finally, we have to turn $q^{-1}\epsilon^{ijk}P^a_i e^k_b\mathcal{D}_a P^b_j$ into an expression which can be quantized. We first rewrite this as

$$
\begin{aligned}
&\frac{\gamma^2\kappa^2}{q}\epsilon^{ijk}P^a_i e^k_b\partial_a P^b_j = -\frac{\gamma^2\kappa^2}{q}\epsilon^{ijk}P^a_i P^b_j\partial_a e^k_b \\
= \; & -\frac{\mathrm{sgn}\,\det(e^i_d)}{\sqrt{q}}\epsilon^{abc}e^k_c\partial_a e^k_b = -\mathrm{sgn}\,\det(e^i_d)\epsilon^{abc}\frac{e^k_c}{q^{1/4}}\partial_a\frac{e^k_b}{q^{1/4}}
\end{aligned}
$$

which provides two factors of co-triads and one partial derivative. Each of them will be combined with a tangent vector to provide either holonomies or a discretized derivative. The inverse powers of $q^{1/4}$ can be absorbed by choosing appropriate positive powers of volume in Poisson brackets expressing the co-triads. (Note that this is the reason why we had to move one $q^{-1/4}$ past the partial derivative, because absorbing a single $q^{-1/2}$ would require the ill-defined logarithm of volume.)

For the gravitational part of the constraint, the curvature components $\mathcal{F}^k_{ab}$ appear in a term which can be expressed as $\int \mathrm{d}^3x \epsilon^{abc} F^k_{ab} \epsilon_{cde} \epsilon^{ijk} P^d_j P^e_k / \sqrt{|\det P|}$. After triangulation, this takes the form $\epsilon^{IJK} s^a_I s^b_J s^c_K \mathrm{tr}(F^k_{ab} \tau_k \tau_l \{A^l_c, V\})$ which can be written in terms of holonomies via $\epsilon^{IJK} \mathrm{tr}(h_{IJ} h_K \{h^{-1}_K, V\})$.

It remains to quantize the extrinsic curvature terms, where our goal is to express K^i_a in terms of Poisson brackets such as $\{\mathcal{A}^i_a, K\}$ and $\{\mathcal{A}^i_a, V\}$ which can be promoted to commutators of well-defined operators. In the torsion-free case the integrated extrinsic curvature is used in the expression $K^i_a = \frac{1}{\kappa\gamma}\{A^i_a, K\}$ for extrinsic curvature components. This relation, proven e.g. in [38], turns out to be one of the main places where torsion changes the quantization procedure of the Hamiltonian constraint. Viewing (3.15) as a functional of the canonical pair $(\mathcal{A}^i_a, P^b_j)$, i.e. expressing K^i_a in terms of $\mathcal{A}^i_a$ and Γ^i_a, yields

$$\begin{aligned}\{\mathcal{A}^j_b(y), K\} &= \kappa(\mathcal{A}^j_b(y) - \Gamma^j_b(y)) - \kappa \int_\Sigma \mathrm{d}^3x\ P^a_i(x) \frac{\delta \Gamma^i_a(x)}{\delta P^b_j(y)} \\ &= \kappa\gamma K^j_b(y) + \frac{\kappa^2\gamma^2\theta}{4(1+\gamma^2)} \left(\frac{1}{\gamma} \epsilon^j{}_{kl} e^k_b(y) J^l(y) + \frac{1}{2} e^j_b(y) J^0(y) \right). \end{aligned} \quad (3.19)$$

Here, we have used $\Gamma^i_a = \widetilde{\Gamma}^i_a + \mathcal{C}^i_a$ (which only requires solutions to second class constraints) in the second step together with (2.33) and the fact that

$$\kappa \int_\Sigma \mathrm{d}^3x\ P^a_i(x) \frac{\delta \widetilde{\Gamma}^i_a(x)}{\delta P^b_j(y)} = 0$$

, which can be proven by a direct calculation or using the fact that

$$\tilde{F} := \kappa\gamma \int_\Sigma \mathrm{d}^3x\ P^a_i(x) \widetilde{\Gamma}^i_a(x)$$

is the generating functional of $\widetilde{\Gamma}$. (Due to the presence of torsion, unless $\theta = 0$ the functional $F := \kappa\gamma \int_\Sigma \mathrm{d}^3x\ P^a_i(x) \Gamma^i_a(x)$ no longer generates a canonical transformation to (K^i_a, P^j_b) since $\{\mathcal{A}^i_a, F\} \neq \Gamma^i_a$. Many of the differences between torsion and torsion-free canonical gravity are reflected in this property of the canonical structure.)

Together with (4.37) it is then straightforward to show that

$$\begin{aligned} K_a^i &= \frac{1}{\gamma\kappa}\left\{\mathcal{A}_a^i, K\right\} - \frac{\theta}{2\gamma(1+\gamma^2)\sqrt{q}}\,\epsilon^i{}_{kl}\left\{\mathcal{A}_a^k, V\right\}\sqrt{q}J^l \\ &\quad -\frac{\theta}{4(1+\gamma^2)\sqrt{q}}\left\{\mathcal{A}_a^i, V\right\}\sqrt{q}J^0\,. \end{aligned} \tag{3.20}$$

With these classical identities, the contributions $H^E[N]$ and $\mathcal{T}[N]$ to the Hamiltonian constraint become

$$H^E[N] = \frac{1}{\kappa^2\gamma^{\frac{3}{2}}}\int_\Sigma \mathrm{d}^3x\ N(x)\epsilon^{abc}\mathcal{F}_{ab}^k(x)\left\{A_c^k(x), V\right\}\mathrm{sgn}\det(e_d^l)\,, \tag{3.21}$$

and

$$\begin{aligned} \mathcal{T}[N] &= \frac{1}{2\kappa^2\gamma}\int_\Sigma \mathrm{d}^3x\ N(x)\epsilon^{abc}\epsilon_{kmn}\left\{\mathcal{A}_a^k(x), V\right\}K_b^m K_c^n \mathrm{sgn}\det(e_d^l) \\ =\ & \frac{1}{2\kappa^4\gamma^3}\int_\Sigma \mathrm{d}^3x\ N(x)\epsilon^{abc}\epsilon_{kmn}\left\{\mathcal{A}_a^k(x), K\right\}\left\{\mathcal{A}_b^m(x), K\right\} \\ &\qquad \left\{\mathcal{A}_c^n(x), V\right\}\mathrm{sgn}\det(e_d^l) \qquad\qquad (3.22) \\ -\ & \frac{2\theta}{\gamma^3\kappa^3(1+\gamma^2)}\int_\Sigma \mathrm{d}^3x\ N(x)\epsilon^{abc}\ \epsilon_{kmn}\ \epsilon^n{}_{ij}\left\{\mathcal{A}_a^k(x), V^{\frac{1}{2}}\right\}\left\{\mathcal{A}_b^m(x), K\right\} \\ &\qquad \left\{\mathcal{A}_c^i(x), V^{\frac{1}{2}}\right\}\sqrt{q}J^j\mathrm{sgn}\det(e_d^l) \\ -\ & \frac{\theta}{\gamma^3\kappa^2(1+\gamma^2)}\int_\Sigma \mathrm{d}^3x\ N(x)\epsilon^{abc}\ \epsilon_{kmn}\left\{\mathcal{A}_a^k(x), V^{\frac{1}{2}}\right\}\left\{\mathcal{A}_b^m(x), K\right\} \\ &\qquad \left\{\mathcal{A}_c^n(x), V^{\frac{1}{2}}\right\}\sqrt{q}J^0\mathrm{sgn}\det(e_d^l) \\ +\ & \frac{27\theta^2}{8\gamma^2\kappa^2(1+\gamma^2)^2}\int_\Sigma \mathrm{d}^3x\ N(x)\ \epsilon^{abc}\ \epsilon_{kmn}\ \epsilon^m{}_{ij}\left\{\mathcal{A}_a^k(x), V^{\frac{1}{3}}\right\}\left\{\mathcal{A}_b^i(x), V^{\frac{1}{3}}\right\} \\ &\qquad \left\{\mathcal{A}_c^n(x), V^{\frac{1}{3}}\right\}\sqrt{q}J^j\sqrt{q}J^0\mathrm{sgn}\det(e_d^l) \\ +\ & \frac{27\theta^2}{32\gamma\kappa^2(1+\gamma^2)^2}\int_\Sigma \mathrm{d}^3x\ N(x)\ \epsilon^{abc}\ \epsilon_{kmn}\left\{\mathcal{A}_a^k(x), V^{\frac{1}{3}}\right\}\left\{\mathcal{A}_b^m(x), V^{\frac{1}{3}}\right\} \\ &\qquad \left\{\mathcal{A}_c^n(x), V^{\frac{1}{3}}\right\}\sqrt{q}J^0\sqrt{q}J^0\mathrm{sgn}\det(e_d^l) \\ +\ & \frac{27\theta^2}{8\gamma^3\kappa^2(1+\gamma^2)^2}\int_\Sigma \mathrm{d}^3x\ N(x)\ \epsilon^{abc}\ \epsilon_{mkn}\left\{\mathcal{A}_a^j(x), V^{\frac{1}{3}}\right\} \\ &\qquad \left\{\mathcal{A}_b^m(x), V^{\frac{1}{3}}\right\}\left\{\mathcal{A}_c^k(x), V^{\frac{1}{3}}\right\}\sqrt{q}J^n\sqrt{q}J_j\mathrm{sgn}\det(e_d^l)\,. \end{aligned}$$

Here, we have already absorbed inverse powers of $\sqrt{q}$ in the Poisson brackets, while keeping one factor of $\sqrt{q}$ with each current component to make the product quadratic in half-densities of fermions without other metric components.

It is thus clear that the presence of torsion introduces non-trivial additional terms in the gravitational Hamiltonian constraint when it is written in a form suitable for quantization.

While no changes to the torsion-free construction of the Hamiltonian constraint are required for expressing $\mathcal{F}^i_{ab}$ and $\mathcal{A}^i_a$ in terms of holonomies, there is a further difference to the treatment of K in [21]. This quantity is not directly related to a basic variable, but can be obtained from a Poisson bracket $\{H^E[1], V\}$ where both ingredients are already written as quantizable functions of basic quantities. With Γ^i_a having contributions from torsion, we obtain, using (71) and the trace of (2.33),

$$\begin{aligned} \left\{H^E[1], V\right\} &= \sqrt{\gamma}\frac{\gamma\kappa}{2}\int_\Sigma \mathrm{d}^3x\ \left(\epsilon^{ij}{}_k P^a_i e^c_j \partial_a e^k_c + 2P^b_n(\Gamma^n_b + \gamma K^n_b)\right) \\ &= \gamma^{3/2}\kappa \int_\Sigma \mathrm{d}^3x\ \left(P^b_n \mathcal{C}^n_b + \gamma P^b_n K^n_b\right) \\ &= \gamma^{\frac{3}{2}} K - \frac{3\theta}{4}\gamma^{\frac{3}{2}}\frac{\gamma}{1+\gamma^2}\int_\Sigma \mathrm{d}^3x\ \sqrt{q}J^0, \end{aligned} \tag{3.23}$$

which implies

$$K = \gamma^{-\frac{3}{2}}\left\{H^E[1], V\right\} - i\frac{6\alpha^2\gamma\kappa\theta}{(1+\gamma^2)(1+\alpha^2)}\int_\Sigma \mathrm{d}^3x\ (\theta_R \pi_\xi \xi - \theta_L \pi_\chi \chi)\,. \tag{3.24}$$

Again, the presence of torsion implies that K can no longer be expressed just as the Poisson bracket of $H^E[1]$ and V; the extra term involving the fermion charge density in (3.24) is necessary if the torsion is included in the connection. This result is consistent since splitting the torsion contribution from K^i_a and taking the trace of (2.52) reduces K to the Poisson bracket $\gamma^{-\frac{3}{2}}\left\{H^E[1], V\right\}$ without any extra terms. The additional term in (3.24), however, does not have much effect since it only depends on the canonical fermion half-densities, and thus drops out of the Poisson bracket with $\mathcal{A}^i_a$ in (3.22) which is the only form in which K appears.

It is interesting to note that, for $\alpha = \gamma$, the equations (3.20), (3.22), (3.23), and (3.24) take the standard forms of the torsion-free case (without any extra terms) since θ vanishes. This results since the torsion contribution to the spatial spin connection,

$\mathcal{C}_a^i$, vanishes for $\alpha = \gamma$ when the fermion fields are expressed in half-densities as shown in (3.5). Therefore, except for the extra terms in (3.6), the strategy for a loop quantization of the gravitational sector of gravity non-minimally coupled to fermions is exactly the same as that in vacuum for $\alpha = \gamma$. Although this is the case which was also addressed in [30], we emphasize that the complete canonical derivation for real variables has to be done to recognize the roles of all possible contributions to the variables and constraints. In particular, there are extra terms in (3.6) whose correct form must be used to quantize the Hamiltonian constraint.

For $\alpha \neq \gamma$, the quantization of the scalar constraint of gravity with fermions demands the quantization of the non-trivial extra terms in (3.22) in addition to the terms appearing in (3.6). This can be carried out using the standard strategy: All extra terms have the structure $\int \mathrm{d}^3x N\epsilon^{abc}\epsilon_{kmn}\{\mathcal{A}_a^k, O_1\}\{\mathcal{A}_b^m, O_2\}\{\mathcal{A}_c^i, O_3\}O_i^n$ where O_1, O_2 and O_3 are either powers of V or K, and O_i^n is $\epsilon_{ij}^n\sqrt{q}J^j$, $\delta_i^n\sqrt{q}J^0$, $\epsilon_{ij}^n qJ^0J^j$, $\delta_i^n q(J^0)^2$ and qJ^nJ_i, respectively, in all the required terms. The operators $\hat{O}_i$ are obtained either as the volume operator or its commutator with the Euclidean part of the Hamiltonian constraint. The current terms also provide vertex operators directly in terms of the smeared fermion operators $\hat{\Xi}_v$ and $\hat{X}_v$. For J_0, this can directly be multiplied with the commutators, while J^i can be inserted into the trace through $\tau_i J^i$. We do not list the long expressions for complete operators here, but it is clear now that well-defined quantizations exist for all the extra terms. This provides quantizations of all terms in (3.22), completing the quantization of the gravitational constraint in the presence of torsion.

Parity

In loop quantum gravity, the parity behavior is not manifest because the Ashtekar connection transforms as $\Gamma_a^i + \gamma K_a^i \mapsto \Gamma_a^i - \gamma K_a^i$ under parity, which does not result in a straightforward transformation of its holonomies. For states in the connection representation, there is thus no simple parity transformation on the Hilbert space for which one could check invariance of the theory. Sometimes the relation between K_a^i and extrinsic curvature is changed in the definition of basic variables, making use of $\mathrm{sgn}\det(e_c^j)K_{ab}e_i^b$ with a sign factor which would make the redefined K_a^i and thus the whole Ashtekar connection invariant under a reversal of the triad orientation. However, the symplectic structure would be invariant under this transformation only if a corresponding sign factor is included in the momentum, now being $\det(e_b^j)P_i^a$ instead of P_i^a. This momentum would also be invariant under orientation reversal. With all the basic gravitational variables being invariant under orientation reversal, one would simply loose any possibility to implement non-trivial parity transformations at all. Thus, the only possibility is to work with a theory whose parity behavior is rather concealed.

While this may appear only as a technical problem in vacuum or with non-fermionic matter, it becomes acute in the presence of fermions and torsion. (Note that a second order formalism, where fermions would not imply torsion contributions to the connection and thus allow a parity behavior as in the vacuum theory, is unnatural for the connection variables of the Ashtekar formulation as it underlies the loop quantization.) As our classical discussion in Sec. 2.3 showed, the precise behavior of the variables and constraints under parity transformations is no longer obvious in the presence of torsion. Even classically, the behavior is fully determined only on-shell, making use not only of the constraints but also of some equations of motion. While the classical solution space turns out to be parity invariant for any α, specific torsion contributions to Γ_a^i and K_a^i acquire a behavior different from the torsion-free parity behavior unless $\alpha = \gamma$. This observation, consistent with [30], indicates that the situation of parity after quantization, where information about solutions of equations of motion cannot be used, may be much more involved.

In fact, now the non-trivial parity behavior is hidden in holonomies used as basic operators. At the kinematical level, there is no way of knowing what unitary transformation could possibly represent a change in parity, given that even classically one would have to make use of constraints and equations of motion to determine that. In the classical case, the behavior of the theory under parity became obvious only after explicitly splitting off the torsion contributions from the basic variables — a procedure which we are denied in the quantum theory. Triads have a much simpler (and obvious) behavior under parity, but this, too, is difficult to implement at the quantum level because no triad representation exists in the full theory [44]. Thus, the triad transformation cannot simply be implemented at the state level.

It is thus quite likely that loop quantum gravity provides for parity violating effects especially once fermions are included, even if the classical fermion interactions used preserve parity. With the hidden nature of torsion contributions and parity in the quantum formulation, the precise form and magnitude of those parity violating effects is not easy to discern. But some implications can be explored either with effective equations (in their canonical form as described in [55, 39]) which would allow one to perform some of the steps required in the classical analysis of parity, or with symmetry reduced models. An advantage of the latter would be that some models exist (such as those introduced in [67, 72, 74, 59, 81]) which do allow a triad representation and thus a more direct implementation of parity transformations.

3.2 Quantization of Maxwell's Fields

Quantizations of Maxwell's fields have already been developed in [25, 4], so the following summary is presented for the sake of completeness and of its relevence to the applications explored in the following chapters. Since Maxwell's field A_a is a $U(1)$ connection, the kinematical constructions here closely mirror that of gravitational sector presented in section 1.2, albeit a bit simpler in this case due to abelian $U(1)$.

Quantum representation

It follows from (2.66) that the canonical variables in the Maxwell's theory are given by (A_a, π^a), where A_a is the Maxwell vector potential and π^a, the Maxwell electric field. As usual, the only non-vanishing Poisson bracket is

$$\left\{A_a(x), \pi^b(y)\right\} = \delta_a^b\,\delta(x,y). \tag{3.25}$$

The basic configuration variables are holonomies $h_{\mathrm{Max}}(e) := \exp(-i\int_e A_a \dot{e}^a \mathrm{d}t)$ and its momenta are smeared along 2-surfaces S in Σ as $P_{\mathrm{Max}}(S,g) := \int_S \mathrm{d}^2y\, g_a \pi^a(y)$ for all test 1-forms g_a on S, e.g. (metric-independent) co-normal n_a to the surface. Note that holonomies are now elements of $U(1)$. The Poisson bracket between these elementary variables is given by:

$$\{h_{\mathrm{Max}}(e),\, P_{\mathrm{Max}}(S,g)\} = -i(\int_e g)\, h_{\mathrm{Max}}(e)\,, \tag{3.26}$$

implying the Poisson algebra of elementary variables is closed as needed.

Convenient orthonormal basis states $F_{\alpha,\vec{n}}$, called *flux networks*, that span the Hilbert space for Maxwell's theory $\mathcal{H}_{\mathrm{Max}} := L^2[\bar{\mathcal{A}}_{u(1)}, \mathrm{d}\mu_{AL}(u(1))]$ can be constructed as follows. Given a graph α, let us assign an orientation to the edges $(e_1,\ldots,e_N)$, label them by integers $(n_1,\ldots,n_N)$ (corresponding to the irreducible representation of $U(1)$) and set

$$F_{\alpha,\vec{n}}(h_{\mathrm{Max}}(e)) = [h_{\mathrm{Max}}(e_1)]^{n_1}\ldots[h_{\mathrm{Max}}(e_n)]^{n_N}\ . \tag{3.27}$$

Note that if the orientation of an edge e_I is reversed, the state is unchanged if n_I is replaced by $-n_I$. Also, notice from (3.27) that since the $U(1)$ elements are complex numbers of unit modulus, these elements incident at each vertex can be multiplied easily with a constant intertwiner to obtain well-defined cylindrical functions. Next, the Poisson bracket relation (3.26) leads to the definition of the smeared electric operator $\hat{P}_{\mathrm{Max}}(S,g)$:

$$\hat{P}_{\mathrm{Max}}(S,g)\Psi = i\hbar\,\{P_{\mathrm{Max}}(S,g), \Psi\}\ , \tag{3.28}$$

capturing the expectation that $\hat{P}_{\text{Max}}(x)$ should be represented by $i\hbar\delta/\delta h_{\text{Max}}(x)$. On the flux network states, the action reduces to:

$$\hat{P}_{\text{Max}}(S,g)F_{\gamma,\vec{n}} \;=\; -\hbar\left(\sum_I n_I \int_{e_I} g\right) F_{\gamma,\vec{n}}. \tag{3.29}$$

If the surface S (support of g) has non-trivial intersection just with a single edge e_I of α, then the flux network $F_{\alpha,\vec{n}}$ is an eigenstate of $\hat{P}_{\text{Max}}(S,g)$ and the eigenvalue just measures n_I, the 'electric flux carried by the oriented edge e_I'. Thus the electric flux is quantized and each edge of the flux network $F_{\alpha,\vec{n}}$ can be interpreted as carrying an integral multiple of the fundamental quantum.

With these constructions, the combined Kinematical Hilbert space of the Einstein-Maxwell theory is given by the tensor product $\mathcal{H}_{\text{EM}} = \mathcal{H}\otimes\mathcal{H}_{\text{Max}}$ corresponding to the group $G = SU(2)\times U(1)$.

Constraints

As expected, the Einstein-Maxwell theory again has a set of three first class constraints, which are obtained by combining the gravitational constraints (1.6-1.7) with the smeared Maxwell's constraints (2.70-2.72). It is straightforward to show that the Maxwell's Gauss constraint,

$$\mathcal{C}_G^{\text{Max}}[\Lambda] \;:=\; \int_\Sigma \mathrm{d}^3x\ \Lambda G^{\text{Max}}$$

with $G^{\text{Max}} := \partial_b \pi^b$ for any smooth test function Λ on Σ, generates the familiar infinitesimal gauge transformation $A_a \to A_a + \partial_a\Lambda$. Hence, the action of the Gauss constraint for the group $SU(2)\times U(1)$ naturally factors on $\mathcal{H}_{\text{EM}} = \mathcal{H}\otimes\mathcal{H}_{\text{Max}}$: $\hat{\mathcal{C}}_G^{\text{EM}} = \hat{\mathcal{C}}_G \otimes \hat{\mathcal{C}}_G^{\text{Max}}$, where $\hat{\mathcal{C}}_G$ is the Gauss constraint operator on the quantum geometry Hilbert space $\mathcal{H}$ and $\hat{\mathcal{C}}_G^{\text{Max}}$ that on the Maxwell Hilbert space. Imposition of this constraint selects the *gauge invariant sub-space* of $\mathcal{H}_{\text{EM}}$. While the gauge invariant subspace of $\mathcal{H}$ has already been obtained in section 1.2, the invariant subspace of the

Maxwell Hilbert space $\mathcal{H}_{\rm Max}$ is obtained by simply restricting the flux network states as follows: at each vertex the sum of the labels n_I assigned to the incoming edges is equal to the sum of the labels assigned to the outgoing edges. This is expected since it reflects the Gauss's law in electromagnetism. Note that the solution space is a *sub-space* on $\mathcal{H}_{\rm EM}$.

Similarly, it can be shown that the Maxwell's diffeomorphism constraint

$$C^{\rm Max}_{\rm diff}(\vec{N}) := \int {\rm d}^3x\, N^a C^{\rm Max}_a \;=\; \int_\Sigma {\rm d}^3x\; N^a \left(\pi^b F_{ab} - A_a G^{\rm Max}\right)$$

generates diffeomorphisms, which can be seen from the following relations

$$\left\{A_a, C^{\rm Max}_{\rm diff}(\vec{N})\right\} = \mathcal{L}_{\vec{N}} A_a \quad \text{and} \quad \left\{\pi^a, C^{\rm Max}_{\rm diff}(\vec{N})\right\} = \mathcal{L}_{\vec{N}} \pi^a \,. \tag{3.30}$$

The total diffeomorphism constraint $\int {\rm d}^3x\, N^a(\mathcal{C}^G_a + \mathcal{C}^{\rm Max}_a)$ can also be imposed using the general procedure described in section 1.2. Again, the solutions lie in the dual $\mathrm{Cyl}^\star_{\rm EM}$ of $\mathrm{Cyl}_{\rm EM} = [\mathrm{Cyl}\otimes \mathrm{Cyl}_{\rm Max}]$ where $\mathrm{Cyl}_{\rm Max}$ is the space of the cylindrical functions of $U(1)$ connections. Note that a typical cylindrical function on graph γ in $\mathrm{Cyl}_{\rm EM}$ is denoted by $[\Psi_\gamma \otimes F_{\gamma,\vec{n}}]$.

Finally, the scalar constraint $C_{\rm EM}(N) := \int {\rm d}^3x\, N(\mathcal{C}_G + \mathcal{C}^{\rm Max})$ is imposed by solving the constraint equation, $\hat{C}_{\rm EM}(N)|\Psi\rangle = 0$ for $\Psi \in \mathcal{H}_{\rm EM}$. In order to solve this equation, the classical scalar constraints have to be promoted to well-defined quantum operators. Since the construction of a quantum operator for the gravitational Hamiltonian is discussed in details in section 4.3, the construction of an operator for smeared Maxwell's Hamiltonian expressed in terms of the magnetic field $B^a = \epsilon^{abc}F_{bc}$ (see section 5.3),

$$\mathcal{C}^{\rm Max}(N) = \int_\Sigma {\rm d}^3x\; N\left(\frac{2\pi q_{ab}}{\sqrt{q}}\pi^a\pi^b + \frac{q_{ab}}{8\pi\sqrt{q}}B^aB^b\right)\,, \tag{3.31}$$

is presented below. Notice that the Maxwell's fields couple to gravity through the term $q_{ab}/\sqrt{q}$. Now, it follows from section 4.3 that the quantization of this term can be carried out by expressing it in terms of the Poisson brackets as follows:

$$\frac{q_{ab}(x)}{\sqrt{\det q(x)}} = \frac{16}{\kappa^2\gamma^2}\left\{A_a^i(x), \sqrt{V(x)}\right\}\left\{A_b^i(x), \sqrt{V(x)}\right\} . \tag{3.32}$$

Next, the operator for product of electric fields $\pi^a\pi^b$ at the same point is ill-defined since the electric field becomes an operator-valued distribution. Hence, a well-defined operator for this product can be obtained by point-splitting the product by introducing a two-point smearing function $\chi(x, y) = 1$ if $y \in \mathcal{R}$ (an arbitrary open neighborhood of x), or 0 otherwise. On the other hand, magnetic field $B^a = \epsilon^{abc}F_{bc}$ can be quantized by expressing F_{bc} by $U(1)$ holonomies around small closed loops. With a suitable regularization similar to the one described in section 4.3, the electric fields can be regularized to be elevated to the operator $\hat{P}^{\text{Max}}(S, g)$ furnished by (3.28) and the magnetic field to $h_{\text{Max}}(\alpha)$ for small loops α. Thus, the electric part of Maxwell's Hamiltonian can be promoted to a well-defined self-adjoint operator $\hat{\mathcal{C}}^{\text{Max}}_{\text{elec}}(N)$ and similarly the magnetic part to a well-defined self-adjoint operator $\hat{\mathcal{C}}^{\text{Max}}_{\text{mag}}(N)$; please see [4] for details. Finally, as expected, the solutions to scalar constraint equation lies in the dual $\text{Cyl}^{\star}_{\text{EM}}$ of Cyl_{EM}.

3.3 Conclusion

The main purpose of the paper, however, is to provide a better and more complete foundation for the loop quantization of gravity coupled to fermions than can be found in the existing literature. Also this requires knowledge of the details given in the derivation of the canonical formalism to appreciate which of the established quantization steps of the torsion-free case go through in the presence of torsion, and where adaptations may be necessary. Overall, we find that the quantization of fermion fields and their dynamics given by Thiemann and others goes through in a well-defined manner. In details, however, we have clarified several steps where previously gaps existed, although they were not always realized. For all values of the non-minimal coupling parameter α there are new terms in the constraints due to torsion which are derived here in complete form. We have shown that torsion contributions and terms which arise from using half-densitized spinors cancel in the connection for the case where the non-minimal coupling parameter α equals the Barbero–Immirzi parameter γ. As a consequence, the presence of fermions does not change the quantization procedure much in this case, although there are still additional terms. For $\alpha \neq \gamma$, on the other hand, several additional adaptations to the usual construction steps of the Hamiltonian constraint operator are necessary.

While our results do not challenge the previous claims that all fields necessary for the standard model of particle physics can be quantized by loop techniques, some of the details of a specific quantization have to be corrected. As such Hamiltonians may become relevant for phenomenological considerations, e.g. in cosmology [46, 45], a precise understanding of the quantum states and dynamical operators is not only necessary for a complete quantization but even for potential physical applications. In particular, we have highlighted the fact that current constructions of loop quantum gravity do not suffice to show that it exactly preserves parity.

By contrast, the quantization of the Maxwell theory parallels that of gravity but possesses much technical simplifications due to $U(1)$. This completes the discussion on quantizations of Dirac and Maxwell fields in the full theory of LQG. In the following chapters, concrete examples of such quantizations in the reduced theory are presented to provide some interesting applications in cosmology.

CHAPTER IV

QUANTUM COSMOLOGY

Until now, the domain of our exploration has been the *full* theory of LQG extended to incorporate fermionic and bosonic matter. Because of the mathematical complexity arising from the infinite degrees of freedom, it is difficult to test ideas and constructions illustrated in the full theory. Fortunately, mathematical simplifications can be achieved by imposing physical symmetries such as spatial homogeneity and/or isotropy. The process of symmetry reduction generally freezes all but a finite number of degrees of freedom, e.g. the number degrees of freedom in cosmology is taken to be finite. Therefore, the arena of early cosmolgy provides a suitable test bed for analyzing ideas and constructions introduced in the full theory both at the classical and quantum levels. Cosmological models that exploit the quantum nature of geometry have been explored to make some remarkable discoveries of the nature of the initial singularity and its fate[51]-[94]. The branch of cosmology which specializes loop quantum gravity [6, 4, 5] to cosmological regimes is known as loop quantum cosmology. In this chapter, these cosmological models are extended to allow for matters represented by Dirac's fields followed by Maxwell fields.

4.1 Cosmological models with Fermions

Most cosmological models — classical or quantum — introduce the matter ingredients of the universe as bosonic fields, in particular scalar ones. While this provides a good measure for the implications of matter energy on space-time, some effects of realistic fermionic particles may be overlooked. Especially in homogeneous models of quantum cosmology there is an important difference between bosonic and fermionic models: the exclusion principle forbids large matter energies when symmetry reduction leaves only a few, finitely many fermionic degrees of freedom. A massive universe can then be obtained by only two possibilities: a homogeneous description with many different fermionic species, or inhomogeneity with many local degrees of freedom of a few species (as in [49]).

Both options differ from what is modeled by large values of homogeneous bosonic fields which rather resemble a Bose–Einstein condensate of many identical excitations. In fact, fermion condensates have been suggested for such a purpose, with characteristic effects [50, 46]. This is an effective picture starting from an inhomogeneous perspective in which fermions condense under certain conditions, after which a symmetry reduction can be done. It differs from a fundamental description from fermions in quantum cosmology where constraints due to the exclusion principle cannot as easily be avoided. Potentially fundamental mechanisms which rely on a large amount of bosonic matter, such as bounce scenarios to avoid the big bang singularity, have to be reanalyzed if matter is fermionic. A truly microscopic description will then be achieved. Here, we perform an analysis of the role of fermions in loop quantum cosmology.

Loop quantum cosmology [51] provides a general mechanism for fundamental singularity resolution [64, 65]. Commonly in quantum cosmology, when volume is used as an intrinsic measure of time, evolution must stop at the classical singularity where the volume vanishes. In loop quantum cosmology, by contrast, the timeline is naturally extended, first at the kinematical quantum level, by including orientation into the basic variables: The (densitized) triad knows about the size as well as the orientation of the universe which make it take all real values, not just positive ones. Vanishing size is then no longer a boundary but an interior point of minisuperspace.

What is more, even dynamically the classical singularity is removed because the quantized Hamiltonian constraint equation uniquely extends any wave function defined on minisuperspace across the subset of vanishing sizes. Dynamics is dictated by a difference equation [66, 67] which remains regular where classical relativity and Wheeler–DeWitt quantum cosmology would reach their limits. Rather than being singular, the big bang transition then appears as a place where space flips its orientation — turning its inside out — while it changes from being contracting to being expanding.[1] Still, the region of vanishing volume does remain special in the underlying recurrence scheme. Some coefficients of the difference equation can vanish at labels corresponding to zero volume, which leads to consistency conditions implied by the dynamical law [70, 71]. This is welcome because, at least partially, it frees one from having to pose initial values for a wave function independently of the dynamics. The wave function of the universe is restricted by the theory alone, relaxing the need to pick one solution among many which could correspond to our universe.

While the set of configurations of vanishing volume is not a boundary within the theory, in the presence of ordinary matter one may choose to consider parity transformations as large gauge transformations which complete the gauge group of triad rotations to all orthogonal transformations. Then, one would restrict solutions to only those states which are either even or odd under parity reversal. This would essentially factor out the orientation degree of freedom introduced by the use of triad variables, and again demote the set of vanishing sizes to a boundary rather than an interior regime. This factoring has indeed been assumed in recent constructions of physical Hilbert spaces for specific isotropic models and the corresponding intuitive bounce pictures based on [75].

But if this is used crucially for the constructions, what happens if more realistic matter is included which, as we know from particle physics, cannot be parity invariant? Do properties of the specific solutions based on the assumption of reflec-

[1] Sometimes it is suggested that this small-volume regime is avoided altogether because wave packets may turn around in a bounce at some minimal non-zero volume. This indeed happens for homogeneous models containing sufficiently much kinetic energy of matter [68, 69]. However, this does not appear as a general mechanism which would be valid in this form for generic quantum states or for inhomogeneous situations. Fundamental singularity resolution which deals with the wave function right at vanishing volume is thus required.

tion symmetry depend on the conservation of parity by matter, and if so, how reliable are the conclusions drawn from this assumption? Only the inclusion of parity violating terms, at least as a possibility, can provide a sufficiently general mechanism of singularity resolution.

It may also give rise to new effects related to the role of parity violation in the big bang transition. If this were to happen, an intriguing new link between particle physics and quantum gravity would result. Seeing whether this is indeed the case requires the introduction of fermions, which is available in loop quantum gravity [36, 25, 40] (see also [41, 42, 43]). In general, however, the parity behavior of loop quantum gravity is highly non-trivial due to the fact that the basic variable conjugate to the densitized triad, namely the Ashtekar–Barbero connection, is the sum of a parity-even and a parity-odd term. It does not have a simple parity behavior and, moreover, it appears in quantized expressions only non-linearly through holonomies. Even in vacuum, this makes a direct demonstration of parity invariance of loop quantum gravity — or the lack thereof — very complicated [40].

Here, we introduce a homogeneous model which allows one to analyze the parity behavior in a clear-cut way. At the same time, the model is amenable to the techniques which have been proven useful for explicit constructions of Hamiltonian constraint equations through the difference equations of loop quantum cosmology [66, 67, 72]. The microscopic nature of fermions due to the exclusion principle is explicitly realized. As we will see, quantization of this model does not introduce unexpected parity violations in the absence of classical parity violations. But the inclusion of parity violating matter interactions is possible, which can be used to illustrate the role of parity for singularity removal. Then indeed, wave functions change under triad reflections. The big bang transition through vanishing sizes is a non-trivial event, which represents true local evolution in internal time rather than merely the application of a symmetry transformation.

4.2 Classical Symmetry Reduction

In this section, we provide the formulation of symmetry reduced cosmological models which may have torsion due to the presence of fermions. We follow the symmetry reduction of torsion-free Bianchi class A models [73, 72, 74], combined with the canonical formulation of gravity with fermions [36, 25, 30, 40]. Here, we combine these research lines and explore the symmetry reduction of gravity coupled to fermions in a first-order formalism, implying a theory with torsion. As we will see, there are non-trivial changes in the underlying equations, such that the analysis done here provides a crucial consistency test of the robustness of existing models. At the same time, we clarify the constructions of loop quantum cosmology [51] from the viewpoint of some recent developments.

Diagonalization

Bianchi class A models constitute all homogeneous models with a symmetry group S acting freely on the space manifold $\Sigma \cong S$ and for which standard Hamiltonian formulations exist. The symmetry group is characterized by its structure constants C^I_{JK}, which for class A models satisfy $C^I_{IJ} = 0$ [76] and can be parameterized as $C^K_{IJ} = \epsilon^K{}_{IJ} n^{(K)}$ with three coefficients n^I which either vanish or take values ± 1. Some of these models can be reduced further by imposing rotational symmetry with one axis (where S has isotropy group $F = \mathrm{U}(1)$) or even isotropy ($F = \mathrm{SO}(3)$). Later in this paper we will present a locally rotationally symmetric (LRS [77]) model with torsion in detail.

The action of a symmetry group S on Σ provides invariant 1-forms ω^I_a which are used for the reduction of Ashtekar–Barbero variables. For each $s \in S$, they satisfy $s^*\omega^I_a = \mathrm{Ad}(s)^I_J \omega^J_a$ or, in terms of the Lie-algebra valued 1-form $\Omega_a := \omega^I_a T_I$ with generators T_I of S, $s^*\Omega_a = s^{-1}\Omega_a s$. The left invariant 1-forms then yield the decomposition $A^i_a = V_0^{-1/3} \phi^i_I \omega^I_a$ of an invariant connection with spatially constant coefficients ϕ^i_I (see the Appendix of [51] for more details on invariant connections). Here, we have explicitly included a factor of $V_0 = \int \mathrm{d}^3x |\det(\omega^I_a)|$ of the spatial coordinate volume (or the volume of any finite region used to define the homogeneous variables) as it will be convenient later on. A corresponding decomposition of the densitized triad is given by $E^a_i = V_0^{-2/3} p^I_i X^a_I$ with X^a_I being densitized left invariant vector fields dual to the 1-forms: $\omega^I_a X^a_J = \delta^I_J |\det(\omega^K_b)|$. The symplectic structure of the reduced model is given by

$$\left\{\phi^i_I, p^J_j\right\} = \gamma\kappa\delta^i_j\delta^J_I \tag{4.1}$$

as it follows from $(\gamma\kappa)^{-1}\int \mathrm{d}^3x \dot{A}^i_a E^a_i = (\gamma\kappa)^{-1}\dot{\phi}^i_I p^I_i$.

For the purpose of loop quantization, it is useful to further reduce the number of independent components of the invariant connection and its conjugate momentum. In some cases, this will allow very explicit calculations of matrix elements of the Hamiltonian constraint and the difference equation it implies for physical states [72]. Both the connection and the densitized triad can be cast into diagonal form

$$A^i_a = V_0^{-1/3} c_{(K)} \Lambda^i_K \omega^K_a \quad , \quad E^a_i = V_0^{-2/3} p^{(K)} \Lambda^K_i X^a_K \tag{4.2}$$

with six spatially constant coefficients c_I and p^I which are considered as the only dynamical components while $\Lambda \in \mathrm{SO}(3)$ is fixed up to gauge transformations. Using the same Λ^i_I for A^i_a and E^a_i is consistent with the Gauss constraint for diagonal torsion-free Bianchi class A models which is then solved identically. From the diagonal densitized triad, moreover, we find the co-triad $e^i_a = V_0^{-1/3} a_{(K)} \Lambda^i_K \omega^K_a$ with $|a_1| = \sqrt{|p^2 p^3/p^1|}$ and cyclic. It determines the diagonal anisotropic spatial metric

$$q_{ab} = e^i_a e^i_b = V_0^{-2/3} a^2_{(I)} \delta_{IJ} \omega^I_a \omega^J_b = q_{IJ} \omega^I_a \omega^J_b$$

with three independent scale factors $V_0^{-1/3}|a_I|$.

By construction, c_I, p^I and a_I are independent of coordinates as long as the diagonalized homogeneous form is respected. In particular in a Bianchi I model where $\omega_a^I = \partial_a x^I = \delta_a^I$ in terms of Cartesian coordinates x^I, spatial coordinates can be rescaled arbitrarily without affecting the basic variables. However, the specific values do depend on V_0 and the choice of the integration volume. Obviously, the V_0-dependence is a consequence of the symmetry reduction to homogeneity, since V_0 does not occur at all in an inhomogeneous framework. Thus, the dependence has to be interpreted with care especially after quantization where, fundamentally, the relation to coordinates is lost. As a consequence, the role of V_0 cannot be properly understood if considerations are limited to purely homogeneous models because only the reduction from inhomogeneity shows how V_0 enters; see [78] for a discussion from the point of view of inhomogeneous states.

Note that p^I and a_K are allowed to take negative values to represent different triad orientations while the orientation of $\Lambda \in \mathrm{SO}(3)$ is fixed. A parity transformation then simply implies $p^I \mapsto -p^I$ for the triad components (leaving coordinates unchanged), while the transformation of the c_I is in general more complicated. In fact, we have $A_a^i = \widetilde{\Gamma}_a^i + \gamma K_a^i$ with the parity-even torsion-free spin connection

$$\widetilde{\Gamma}_a^i = \frac{1}{2}\epsilon^{ijk} e_k^b (2\partial_{[b} e_{a]}^j + e_j^c e_a^l \partial_b e_c^l) \tag{4.3}$$

and the odd extrinsic curvature $K_a^i = K_{ab} e_i^b$. In the torsion-free case, it follows from (4.3) that the homogeneous spin connection can be expressed as $\widetilde{\Gamma}_a^i = \widetilde{\Gamma}_{(K)} \Lambda_K^i \omega_a^K$ [74] with

$$\widetilde{\Gamma}_I = \frac{1}{2}\left(\frac{a_J}{a_K} n^J + \frac{a_K}{a_J} n^K - \frac{a_I^2}{a_J a_K} n^I\right) \quad \text{for indices such that} \quad \epsilon_{IJK} = 1 \tag{4.4}$$

and the same Λ_I^i as used for the densitized triad. Similarly $K_a^i = K_{(I)} \Lambda_i^I \omega_a^i$ also with the same Λ_I^i. Then, $c_I = \widetilde{\Gamma}_I + \gamma K_I$ does not have a straightforward parity behavior unless $\widetilde{\Gamma}_I = 0$ (as in the Bianchi I model).

The diagonalization is sufficient to capture the crucial dynamical behavior of Bianchi models, such as the approach to a singularity. For the quantization, it has the advantage that it reduces SU(2) to $\mathrm{U}(1)^3$: holonomies of a homogeneous connection, computed along curves generated by the invariant vector fields X_I^a, take the form $h_I^{(\mu)} = \exp(\mu\phi_I^i\tau_i)$ with a real number μ depending, e.g., on the coordinate length of a curve used to compute the holonomy. For $\phi_I^i = c_{(I)}\Lambda_I^i$, we have

$$h_I^{(\mu_I)} = \exp(\mu_I c_{(I)}\Lambda_I^i\tau_i) = \cos\left(\tfrac{1}{2}\mu_I c_{(I)}\right) + 2\Lambda_I^i\tau_i \sin\left(\tfrac{1}{2}\mu_I c_{(I)}\right) . \qquad (4.5)$$

While any SU(2)-holonomy along X_I^a can be written in this way,[2] the diagonalization implies that Λ_I^i becomes a mere background quantity not subject to dynamics. Thus, it is sufficient to consider only the simple commuting exponentials $\exp(i\mu_I c_{(I)})$ to separate diagonal connections. After a loop quantization, as we will see in detail below, this will have the implication that a triad representation exists, which simplifies the analysis of dynamics considerably. In fact, triad operators will simply be $\hat{p}^I = -i\gamma\ell_{\mathrm{P}}^2\partial/\partial c_I$, with the Planck length $\ell_{\mathrm{P}} = \sqrt{\kappa\hbar}$, which form a complete commuting set. Their eigenstates

$$\langle c_1, c_2, c_3|\mu_1, \mu_2, \mu_3\rangle = \exp\left(\tfrac{1}{2}i(\mu_1c_1 + \mu_2c_2 + \mu_3c_3)\right)$$

(written here in the connection representation) form an orthonormal basis such that the coefficients in

$$|\psi\rangle = \sum_{\mu_1,\mu_2,\mu_3} s_{\mu_1,\mu_2,\mu_3}|\mu_1, \mu_2, \mu_3\rangle$$

form the triad representation of arbitrary states. This explicit representation, which becomes available only after diagonalization [72], has been the basis of all investigations so far in homogeneous loop quantum cosmology. As we will see in this article, arriving at such a representation is less trivial in the presence of torsion.

[2]General curves do not provide this simple form. For instance, along $X_1^a + X_2^a$ holonomies are not of the (almost) periodic form in c_1 or c_2 (but in $\sqrt{c_1^2 + c_2^2}$). If curves are considered which are not even straight with respect to the given symmetry, the behavior is more complicated due to path ordering and do not give rise to almost periodic functions [79]. However, such curves do not play a role in the kinematical symmetry reduction, which uses the given set of X_I^a to introduce particular quantum geometries, just like classical symmetric metrics which are used in adapted coordinates but can look complicated in arbitrary coordinates.

Torsion effects

This scheme of diagonalization of the basic torsion-free gravitational variables relies on the fact that both the connection and its conjugate momentum can be diagonalized with the same Λ_I^i. In other words, the su(2) valued connection and its conjugate momentum are parallel to each other in the tangent space of the internal symmetry group. This can be seen from the torsion-free Gauss constraint which expressed in terms of the diagonalized variables takes the form $p^{(I)}c_{(I)}\epsilon_{ijk}\Lambda_I^j\Lambda_k^I = 0$ and is identically satisfied. However, the presence of torsion via the axial fermion current J_i enters the Gauss constraint (2.36) implying that

$$\phi_I^j p_k^I \epsilon_{ijk} = \frac{1}{2}\sqrt{|\det(p_j^I)|}J_i\,. \tag{4.6}$$

(We only discuss the case where torsion is implied by the coupling to fermions. The implications of torsion on the diagonalizability of basic variables are, however, more general.) For $\phi_I^i = c_{(I)}\Lambda_I^i$ and $p_i^I = p^{(I)}\Lambda_i^I$ as above, this would only allow vanishing spatial components of the fermion current and severely restrict the allowed models. This situation becomes more obvious if we try to express the spin connection including its torsion contribution as $\Gamma_a^i = \Gamma_{(K)}\Lambda_K^i\omega_a^K$ with the same Λ_i^I as used for the triad: One can easily verify that the partial torsion contribution (3.5) to the connection cannot be expressed as $C_a^i = C_{(K)}\Lambda_K^i\omega_a^K$ if $J^i \neq 0$. Then also the Ashtekar–Barbero connection cannot be diagonal in the same basis. Therefore, our first result is that the presence of torsion does not allow us to diagonalize both canonical variables, i.e. the connection and the densitized triad, simultaneously.

Moreover, fermion terms require us to use a connection $\mathcal{A}_a^i$ in (3.4) which carries an extra term compared to the Ashtekar–Barbero connection, depending on the fermion current. We then write the new diagonal variables as

$$\mathcal{A}_a^i = V_0^{-1/3}c_{(K)}\Lambda_K^i\omega_a^K \quad , \quad E_i^a = V_0^{-2/3}p^{(K)}T_i^K X_K^a \tag{4.7}$$

where in general $T_i^I \neq \Lambda_I^i$. Not both Λ_I^i and T_i^I can be fixed because partially they are determined by dynamical fields as, e.g., per the Gauss constraint (4.6). This has an immediate implication for the symplectic structure because c_I and p^I will no longer be canonically conjugate:

$$\int_\Sigma \mathrm{d}^3x E^a_i \mathcal{L}_t A^i_a = p^{(I)} T^I_i \mathcal{L}_t \left(c_{(I)}\Lambda^i_I\right) = p^{(I)} \mathcal{L}_t \left(c_{(I)}\Lambda^i_I T^I_i\right) - c_{(I)} p^{(I)} \Lambda^i_I \mathcal{L}_t T^I_i \,. \qquad (4.8)$$

Thus, it is not c_I which is conjugate to p^I but $c_{(I)}\Lambda^i_I T^{(I)}_i$. This is not a pure connection component but depends on the relative angles between the connection direction Λ^i_I and the triad direction T^I_i in internal space. (It is not possible to fix both Λ^i_I and T^I_i because this would require six parameters while the Gauss constraint allows one to fix only three.) Moreover, some of the angles enter the symplectic structure as independent variables. We can, for instance, (Euler) parameterize T^I_i as the matrix $T(\phi_I) = \exp(\phi_3 T_3)\exp(\phi_2 T_1)\exp(\phi_1 T_3)$ using generators T_I of SO(3). Inserting this in (4.8) shows that the angles ϕ_I acquire canonical momenta given in terms of the angles in Λ^i_I, e.g. ϕ_1 being conjugate to $-\mathrm{tr}((c\cdot\Lambda)(p\cdot T(\phi_1+\pi/2,\phi_2,\phi_3)))$, where c and p here denote the diagonal matrices with components c_I and p^I, respectively. (Taking a derivative of $T(\phi_I)$ amounts to switching sines and cosines, which is the same as shifting an angle by $\pi/2$.)

The corresponding phase space and the constrained system defined on it is rather involved, and so we consider a more special case which still allows the non-trivial implications of torsion to be seen: We are interested in the case where the presence of a fermion current is the sole reason for anisotropy, while the 2-dimensional space transversal to the spatial current is rotationally invariant. We can then assume that there are bases for $\mathcal{A}^i_a$ and E^a_i, respectively, such that

$$\Lambda^j_J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\rho & -\sin\rho \\ 0 & \sin\rho & \cos\rho \end{pmatrix} \quad , \quad T^J_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} , \qquad (4.9)$$

where ρ and ϕ are the only non-vanishing rotation angles. As we will demonstrate below, this allows non-trivial solutions where the fermion current is aligned in the

1-direction. The Liouville term in the action can then be expressed as

$$\begin{aligned}
\frac{1}{\gamma\kappa}\int_\Sigma \mathrm{d}^3x E^a_i \mathcal{L}_t A^i_a &= \frac{1}{\gamma\kappa} p^{(I)}\mathcal{L}_t\left(c_{(I)}\Lambda^i_I T^I_i\right) - c_{(I)}p^{(I)}\Lambda^i_I\mathcal{L}_t T^I_i \\
&= \frac{1}{\gamma\kappa}\big(\dot{c}_1 p^1 + \mathcal{L}_t(c_2\mathrm{cos}(\rho-\phi))p^2 + \mathcal{L}_t(c_3\mathrm{cos}(\rho-\phi))p^3 \\
&\quad -\dot{\phi}(c_2p^2 + c_3p^3)\mathrm{sin}(\rho-\phi)\big) \\
&= \frac{1}{\gamma\kappa}\left(\dot{c}_1p^1 + \dot{\tilde{c}}_2p^2 + \dot{\tilde{c}}_3p^3 + \dot{\phi}p_\phi\right)\,, \qquad (4.10)
\end{aligned}$$

where we introduced

$$\tilde{c}_2 = c_2\mathrm{cos}(\rho-\phi) \quad , \quad \tilde{c}_3 = c_3\mathrm{cos}(\rho-\phi) \quad , \quad p_\phi = -(c_2p^2 + c_3p^3)\mathrm{sin}(\rho-\phi)\,. \quad (4.11)$$

In these components, the symplectic structure is

$$\left\{c_1, p^1\right\} = \gamma\kappa \quad , \quad \left\{\tilde{c}_2, p^2\right\} = \gamma\kappa \quad , \quad \left\{\tilde{c}_3, p^3\right\} = \gamma\kappa \quad , \quad \{\phi, p_\phi\} = \gamma\kappa\,. \quad (4.12)$$

Notice that the presence of torsion at this stage introduces a new kinematical degree of freedom ϕ. It will be removed after solving the Gauss constraint (4.6), which is now non-trivial.

There is a useful interpretation of the canonical variables in the presence of torsion: We can write, e.g.,

$$\tilde{c}_2 = c_2\cos(\rho-\phi) = c_2\Lambda^i_2T^2_i = \phi^i_2T^2_i$$

in terms of the general homogeneous coefficients $\phi^i_I = c_{(I)}\Lambda^i_I$. Since T^I_i gives the direction of E^a_i, we can interpret $\tilde{c}_2$ as a component

$$V_0^{-1/3}\tilde{c}_2 = \mathcal{A}^i_aE^b_i\frac{X^a_2\omega^2_b}{V_0^{-2/3}p^2}$$

of the projection of $\mathcal{A}^i_a$ onto E^a_i. In the absence of torsion, this would be a pure connection component because $\mathcal{A}^i_a$ and E^a_i would be parallel. With torsion, however, $\tilde{c}_2$ is only part of an $\mathcal{A}^i_a$-component: Using the expression (3.4), the projection removes the term $\epsilon^i{}_{kl}e^k_aJ^l$ perpendicular to E^a_k which happens to be the torsion contribution to extrinsic curvature. Moreover, the projection transversal to E^a_i is just (half of) the variable p_ϕ due to the sine, which thus takes a value equal to the torsion contribution. This agrees with the solution of the Gauss constraint (4.14) below. Recall that the identification of the torsion contribution to extrinsic curvature used in (3.4) cannot be completed without partially solving equations of motion. In the projection defining $\tilde{c}_2$ and $\tilde{c}_3$, on the other hand, no equations of motion have been used. Thus, these canonical variables which we are naturally led to at the basic kinematical level present torsion-free contributions without explicitly splitting off torsion. (Something similar happens in inhomogeneous models such as spherical symmetry [80] or Gowdy models [81]. There it is spin connection contributions that are split off by a natural definition of canonical variables which then allows a manageable loop quantization.)

Reduced constraints

In terms of the diagonal variables the Gauss constraint (2.36) becomes

$$G_i = \frac{1}{\gamma\kappa}\epsilon_{ijk}c_{(I)}p^{(I)}\Lambda^j_I T^I_k - \frac{1}{2}\sqrt{|p^1p^2p^3|}T^I_i J_I = -\frac{\epsilon_{i23}}{\gamma\kappa}(c_2p^2+c_3p^3)\sin(\rho-\phi) - \frac{\sqrt{|p^1p^2p^3|}}{2}T^I_i J_I = 0\,. \qquad (4.13)$$

For $i = 2, 3$, it thus implies $J_2 = 0 = J_3$ while the remaining condition

$$\epsilon_{i23}p_\phi = \frac{\gamma\kappa}{2}\sqrt{|p^1p^2p^3|}T^I_i J_I$$

relates J_1 to p_ϕ:

$$p_\phi = \frac{\gamma\kappa}{2}\sqrt{|p^1p^2p^3|}J_1 =: \frac{1}{2}\gamma\kappa\mathcal{J}_1\,, \qquad (4.14)$$

where $\mathcal{J}_i = \xi^\dagger\sigma_i\xi + \chi^\dagger\sigma_i\chi$ denotes the densitized axial fermion current (which is bilinear in half-densitized fermions ξ and χ). The form (4.14) of the Gauss constraint together with the expression for the densitized fermion current in terms of canonical fermion fields makes it clear that the new connection components remain gauge invariant: c_1 and $\tilde{c}_2$ commute with p_ϕ as well as with fermion fields. With the choice

(4.9) of bases the fermion current J_i is aligned along the first (fixed) internal direction: $J_2 = J_3 = 0$. This defines a specific class of models with a non-trivial spatial fermion current, as J_1 may be non-zero.

Similarly, the diffeomorphism constraint (3.10) can be written as

$$\mathcal{D}_a N^a = -c^K_{IJ}\phi^i_K p^J_i N^I = N^1(n^2 c_2 p^2 + n^3 c_3 p^3)\sin(\rho - \phi) = 0\,, \tag{4.15}$$

where $N^a = N^I X^a_I$ with N^I constant and $C^K_{IJ} = \epsilon^K{}_{IJ} n^{(K)}$ to specify different Bianchi class A models are used. We have also imposed that the partial derivatives of spinor fields vanish in a homogeneous model, e.g. $\partial_a \psi = 0$. A conclusion to be drawn from (4.13) and (4.15) is that torsion is strongly restricted in Bianchi Class A models with $n^2 + n^3 \neq 0$ since this implies that $p_\phi = \gamma\kappa\mathcal{J}_1/2 = 0$, and thus all spatial components of the axial vector current vanish.

Finally, the Hamiltonian constraint (3.6) is

$$\begin{aligned}
H_{\rm Bianchi} \;=\;& \frac{\kappa^{-1}}{\sqrt{|p^1p^2p^3|}}\left(n^1c_1p^2p^3 + n^2c_2p^1p^3\cos(\rho-\phi) + n^3c_3p^2p^1\cos(\rho-\phi)\right)\\
&-\frac{\kappa^{-1}\gamma^{-2}}{\sqrt{|p^1p^2p^3|}}\left(c_1p^1c_2p^2\cos(\rho-\phi) + c_1p^1c_3p^3\cos(\rho-\phi) - c_2p^2c_3p^3\right)\\
&+\frac{\kappa^{-1}\gamma^{-2}(1+\gamma^2)}{\sqrt{|p^1p^2p^3|}}\left((c_1-\tilde\Gamma_1)p^1(\tilde\Gamma_2p^2+\tilde\Gamma_3p^3)\cos(\rho-\phi)\right.\\
&\left.-(c_2-\tilde\Gamma_2)p^2\tilde\Gamma_3p^3\right) + \frac{1}{2\sqrt{|p^1p^2p^3|}}\left(\gamma(c_2p^2+c_3p^3)\sin(\rho-\phi)\mathcal{J}_1\right.\\
&\left.+\theta\left(\tilde\Gamma_1p^1 + (\tilde\Gamma_2p^2+\tilde\Gamma_3p^3)\cos(\rho-\phi)\right)\mathcal{J}^0\right)\\
&+\frac{\gamma}{4\alpha}\left(n^1\left|\frac{p^2p^3}{p^1}\right| + n^2\left|\frac{p^1p^3}{p^2}\right| + n^3\left|\frac{p^2p^1}{p^3}\right|\right)\mathcal{J}^0\\
&-\frac{3\gamma\kappa\theta}{16\sqrt{|p^1p^2p^3|}}\left(\frac{2}{\alpha}+\frac{\gamma\theta}{1+\gamma^2}\right)\mathcal{J}_0^2\\
&+\frac{\kappa}{16\sqrt{|p^1p^2p^3|}(1+\gamma^2)}\left(2\gamma\beta\left(3-\frac{\gamma}{\alpha}+2\gamma^2\right)-\theta^2\right)\mathcal{J}_1^2
\end{aligned} \tag{4.16}$$

where α, β and θ are defined in Chapter II. It is important to emphasize that since Γ^i_a is not diagonalized in either Λ^i_I or T^I_i in the presence of torsion, the Hamiltonian constraint in (3.6) expressed in terms of $\tilde\Gamma^i_a$ by splitting torsion from the spin connection is essential to obtain a controlled loop quantization as will be shown below.

The Bianchi I LRS Model with Torsion

If there is an isotropy group $F = \mathrm{U}(1)$ for the action of the symmetry group S, one obtains locally rotationally symmetric (LRS) models. Therefore, two of the diagonal components of the connection as well as of the triad, e.g. the second two for definitiveness, have to equal each other and only two degrees of freedom are left which we choose to be (c_1, p^1) and $(\tilde{c}_2, p^2)$ embedded into the general Bianchi model by

$$(c_1, \tilde{c}_2) \mapsto (c_1, \tilde{c}_2, \tilde{c}_3) = (c_1, \tilde{c}_2, \tilde{c}_2) \quad , \quad (p^1, p^2) \mapsto (p^1, p^2, p^3) = (p^1, p^2, p^2) \,.$$

The symplectic structure can be pulled back by this embedding providing Poisson brackets

$$\left\{c_1, p^1\right\} = \gamma\kappa, \quad \left\{\tilde{c}_2, p^2\right\} = \frac{1}{2}\gamma\kappa, \quad \{\phi, p_\phi\} = \gamma\kappa \tag{4.17}$$

from (4.12), where p_ϕ is now $p_\phi := -2c_2 p^2 \sin(\rho - \phi)$. (Solutions of this symmetry type in the presence of torsion due to spin fluids have been studied in [82, 83].)

For the LRS model, the diffeomorphism and the Hamiltonian constraints, (4.15) and (4.16) respectively, further reduce to

$$\mathcal{D}_a N^a = -\frac{1}{2\gamma\kappa} N^1 (n^2 + n^3) p_\phi = 0 \tag{4.18}$$

and

$$
\begin{aligned}
H_{\mathrm{LRS}} &= \frac{\kappa^{-1}}{|p^2|\sqrt{|p^1|}}\Bigg(n^1 c_1 \left(p^2\right)^2 + n^2\tilde{c}_2 p^1 p^2 + n^3 \tilde{c}_2 p^2 p^1 \\
&\quad -\gamma^{-2}\left(2c_1 p^1 \tilde{c}_2 p^2 + \left(\tilde{c}_2 p^2\right)^2 + \frac{1}{4}p_\phi^2\right)\Bigg) \\
&\quad +\frac{\kappa^{-1}\gamma^{-2}(1+\gamma^2)}{|p^2|\sqrt{|p^1|}}\Bigg((c_1-\tilde{\Gamma}_1)p^1(\tilde{\Gamma}_2 p^2 + \tilde{\Gamma}_3 p^3)\frac{2|\tilde{c}_2 p^2|}{\sqrt{p_\phi^2 + 4(\tilde{c}_2 p^2)^2}} \\
&\quad -\left(\frac{\mathrm{sgn}(\tilde{c}_2 p^2)}{2}\sqrt{p_\phi^2 + 4(\tilde{c}_2 p^2)^2} - \tilde{\Gamma}_2 p^2\right)\tilde{\Gamma}_3 p^3\Bigg) \\
&\quad -\frac{1}{2|p^2|\sqrt{|p^1|}}\left(\gamma p_\phi \mathcal{J}_1 - \theta\left(\tilde{\Gamma}_1 p^1 + (\tilde{\Gamma}_2 p^2 + \tilde{\Gamma}_3 p^3)\frac{2|\tilde{c}_2 p^2|}{\sqrt{p_\phi^2 + 4(\tilde{c}_2 p^2)^2}}\right)\mathcal{J}^0\right) \\
&\quad +\frac{\gamma}{4\alpha}\left(n^1\frac{(p^2)^2}{|p^1|} + (n^2+n^3)|p^1|\right)\mathcal{J}^0 - \frac{3\gamma\kappa\theta}{16|p^2|\sqrt{|p^1|}}\left(\frac{2}{\alpha} + \frac{\gamma\theta}{1+\gamma^2}\right)\mathcal{J}_0^2 \\
&\quad +\frac{\kappa}{16|p^2|\sqrt{|p^1|}(1+\gamma^2)}\left(2\gamma\beta\left(3 - \frac{\gamma}{\alpha} + 2\gamma^2\right) - \theta^2\right)\mathcal{J}_1^2\,,
\end{aligned}
\tag{4.19}
$$

where we have used the definitions of $\tilde{c}_2$ and p_ϕ to write

$$
\cos(\rho - \phi) = \frac{2|\tilde{c}_2 p^2|}{\sqrt{p_\phi^2 + 4(\tilde{c}_2 p^2)^2}}\,. \tag{4.20}
$$

To allow a non-vanishing $\mathcal{J}_1$ and to be specific, we work from now on with the Bianchi I model. Here, the diffeomorphism constraint (4.18) vanishes identically and does not impose any restriction on p_ϕ. This has the additional advantage that the resulting Hamiltonian constraint will be free of terms such as $\sqrt{p_\phi^2 + 4(\tilde{c}_2 p^2)^2}$, which lack simple quantizations. (While there are well-defined operators with this classical limit, given that both p_ϕ^2 and $(\tilde{c}_2 p^2)^2$ would be mutually commuting positive operators whose square root can be taken after summing them, not all the operators involved have discrete spectra. Thus, it would not be straightforward to compute explicit matrix elements of the square root operator which would be required for the quantized Hamiltonian. Once the square root is quantized, its inverse in (4.20) could easily be obtained from $2\gamma\kappa p^2\cos(\rho-\phi) = \{\sqrt{p_\phi^2 + 4(\tilde{c}_2 p^2)^2}, p^2\}$.)

For the Bianchi I LRS model, we then have $\widetilde{\Gamma}_I = 0$ and thus the Hamiltonian constraint is finally given by

$$
\begin{aligned}
H_{\mathrm{ILRS}} &= -\frac{\kappa^{-1}\gamma^{-2}}{|p^2|\sqrt{|p^1|}}\left(2c_1p^1\tilde{c}_2p^2+\left(\tilde{c}_2p^2\right)^2+\frac{1}{4}p_\phi^2\right)-\frac{\gamma}{2|p^2|\sqrt{|p^1|}}p_\phi\mathcal{J}_1\\
&\quad -\frac{3\gamma\kappa\theta}{16|p^2|\sqrt{|p^1|}}\left(\frac{2}{\alpha}+\frac{\gamma\theta}{1+\gamma^2}\right)\mathcal{J}_0^2\\
&\quad +\frac{\kappa}{16|p^2|\sqrt{|p^1|}(1+\gamma^2)}\left(2\gamma\beta\left(3-\frac{\gamma}{\alpha}+2\gamma^2\right)-\theta^2\right)\mathcal{J}_1^2\,.
\end{aligned}
\tag{4.21}
$$

This concludes the classical symmetry reduction of canonical gravity non-minimally coupled to fermions.

Parity behavior

Because we are mainly concerned about the role of parity in loop quantum cosmology, we end this section on the classical equations with a discussion of parity invariance. As pointed out in [40], parity invariance in loop quantum gravity is not guaranteed. The Ashtekar connection is a sum of a parity-even and a parity-odd term and thus does not have a straightforward parity behavior. This already occurs in the absence of fermions and torsion, but is aggravated by the parity-mixing terms of torsion contributions due to a fermion current (see (3.5), noting that J_1 is even and J_0 is odd). Classically, one can explicitly split these contributions, which essentially amounts to replacing the Ashtekar connection with extrinsic curvature. However, a complete splitting requires equations of motion to be used, which will not be possible after quantization. It is then not guaranteed that quantum corrections due to the loop quantization will preserve parity even in vacuum or in the absence of parity-violating matter.

The model introduced here provides a clear view on parity in the classical theory as well as after quantization, as we will see below. One key property is that the canonical variables (4.11) we are led to do, in hindsight, perform the splitting into torsion-free and torsion components without using equations of motion. Thus, in the new variables every single term in the Hamiltonian constraint (4.21) has a clear and simple behavior under parity: Among the gravitational variables, only c_1 and p^1 change sign under parity (reversing orientation) while the rest remains unchanged. (Since changing the sign of p^2 in an LRS model implies a reflection of both directions related by the rotational symmetry, it is equivalent to a triad rotation and thus mere gauge.) This is accompanied by the usual parity transformation of the fermions present, which implies that $\mathcal{J}_1$ is parity invariant while $\mathcal{J}_0$ changes sign as these are space and time components of an axial vector. In particular, it is immediately clear from (4.21) that the Hamiltonian constraint is parity invariant for free fermions. Parity violation will only result if suitable interactions are introduced to the model, which can easily be done by adding e.g. $\sqrt{-\det g}\mathcal{V}_\mu\mathcal{J}^\mu$ with the vector current $\mathcal{V}^\mu$ to the action. We will avail ourselves of this possibility in what follows to understand the role of parity in the loop quantized model.

4.3 Quantization of the Bianchi I LRS model

Loop quantum cosmology allows one to complete many of the constructions of full loop quantum gravity in simplified and explicit forms, which then provides indications toward the physical implications of the theory. In this section, we provide a self-contained description of anisotropic models with an emphasis on the effects of fermions, torsion and parity.

Quantum Kinematics

We start with basic variables according to the Poisson structure (4.17). As in any loop quantization, states in the connection representation are constructed by taking exponentials

$$\exp(\mu_1 c_1 \Lambda_1^i \tau_i) \in \mathrm{SU}(2) \quad , \quad \exp(\mu_2 \tilde{c}_2 \Lambda_2^i \tau_i) \in \mathrm{SU}(2) \quad , \quad \exp(ik\phi) \in \mathrm{U}(1)$$
$$\text{for all} \quad \mu_I \in \mathbb{R}, k \in \mathbb{Z}, \Lambda_I^i \in SO(3) \tag{4.22}$$

as they arise in holonomies. Using holonomies in the general setting is important for a background independent basic algebra of variables. This crucial feature is then reflected also in symmetric models based on exponentials of connection components. The parameters μ_I can take any real value, corresponding to evaluating holonomies along straight edges (tangential to X_I^a) of arbitrary length. The variable ϕ, on the other hand, was introduced as a periodic angle in (4.9), such that only strictly periodic functions $\exp(ik\phi)$ with $k \in \mathbb{Z}$ are allowed. This unphysical degree of freedom, which we were led to introduce due to the presence of torsion, will be removed after solving the Gauss constraint.

Matrix elements of the exponentials in (98) form a C^*-algebra of (almost) periodic functions, as seen from (4.5). Any function generated by this set can be written as

$$g(c_1, \tilde{c}_2, \phi) = \sum_{\mu_1,\mu_2,k} \xi_{\mu_1,\mu_2,k} \exp\left(\tfrac{1}{2}i\mu_1 c_1 + \tfrac{1}{2}i\mu_2\tilde{c}_2 + ik\phi\right) \,, \tag{4.23}$$

with coefficients $\xi_{\mu_1,\mu_2,k} \in \mathbb{C}$, where the sum is over finitely many $\mu_1, \mu_2 \in \mathbb{R}$ and $k \in \mathbb{Z}$. Note that while $g(c_1, \tilde{c}_2, \phi)$ is almost periodic in c_1 and $\tilde{c}_2$, it is exactly periodic in ϕ. This provides a complete set of continuous functions on $\overline{\mathbb{R}}_{\mathrm{Bohr}} \times \overline{\mathbb{R}}_{\mathrm{Bohr}} \times S^1$, where $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ is the Bohr compactification of the real line. (By definition, $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ is the compactification of $\mathbb{R}$ such that the set of all continuous functions on it is just the set of almost periodic functions. See e.g. [84] for a recent discussion of further properties.) All spaces in the product are compact Abelian groups and carry a unique normalized Haar measure $\mathrm{d}\mu(c)$ in the case of $\overline{\mathbb{R}}_{\mathrm{Bohr}}$, where

$$\int f(c)\mathrm{d}\mu(c) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(c)\mathrm{d}c \,,$$

and $\mathrm{d}\phi$ for S^1.

By Cauchy completion, we obtain the Hilbert space as a tensor product $\mathcal{H}_{\mathrm{grav}} = \mathcal{H}_{\mathrm{Bohr}}^{\otimes 2} \otimes \mathcal{H}_{S^1}$ with the Hilbert spaces $\mathcal{H}_{\mathrm{Bohr}} = L^2(\overline{\mathbb{R}}_{\mathrm{Bohr}}, \mathrm{d}\mu(c))$ and $\mathcal{H}_{S^1} = L^2(S^1, \mathrm{d}\phi)$ of square integrable functions on the Bohr compactification of the real line and the circle, respectively. Orthonormal bases for these spaces are given by $\langle c|\mu\rangle = \exp(i\mu c/2)$, $\mu \in \mathbb{R}$, and $\langle\phi|k\rangle = \exp(ik\phi)$, $k \in \mathbb{Z}$, respectively, with

$$\langle\mu|\mu'\rangle = \delta_{\mu,\mu'} \quad , \quad \langle k|k'\rangle = \delta_{k,k'} \,. \tag{4.24}$$

The configuration variables act in the obvious manner: For all g_1 and g_2 of the form (4.23), we have

$$\left(\widehat{g_1}g_2\right)(c_1,\tilde{c}_2,\phi) = g_1(c_1,\tilde{c}_2,\phi)g_2(c_1,\tilde{c}_2,\phi) \tag{4.25}$$

and the momentum operators are represented by

$$\hat{p}^1 = -i\gamma\ell_{\mathrm{P}}^2\frac{\partial}{\partial c_1} \quad , \quad \hat{p}^2 = -i\frac{\gamma\ell_{\mathrm{P}}^2}{2}\frac{\partial}{\partial\tilde{c}_2} \quad \text{and} \quad \hat{p}_\phi = -i\gamma\ell_{\mathrm{P}}^2\frac{\partial}{\partial\phi}\,, \tag{4.26}$$

where $\ell_{\mathrm{P}}^2 = \kappa\hbar$. (The densitized triad in general is quantized via fluxes, i.e. 2-dimensional integrations over surfaces. In a homogeneous context, however, this is not required and densitized triad components can directly be promoted to operators. This simple representation exists only due to our use of variables; had we used c_2 instead of $\tilde{c}_2$, the operator $\hat{p}^2$ and thus the volume operator would have been more complicated.) Common eigenstates of all triad operators $\hat{p}^I$ are

$$|\mu_1,\mu_2,k\rangle := |\mu_1\rangle\otimes|\mu_2\rangle\otimes|k\rangle\;, \tag{4.27}$$

with

$$\begin{aligned}\hat{p}^1|\mu_1,\mu_2,k\rangle &= \frac{\gamma\ell_{\mathrm{P}}^2\mu_1}{2}|\mu_1,\mu_2,k\rangle \quad , \quad \hat{p}^2|\mu_1,\mu_2,k\rangle = \frac{\gamma\ell_{\mathrm{P}}^2\mu_2}{4}|\mu_1,\mu_2,k\,\rangle \\ &\text{and} \quad \hat{p}_\phi|\mu_1,\mu_2,k\rangle = \gamma\ell_{\mathrm{P}}^2k|\mu_1,\mu_2,k\rangle\,.\end{aligned} \tag{4.28}$$

From triad operators we construct the volume operator:

$$\hat{V}|\mu_1,\mu_2,k\rangle = |\hat{p}^2|\sqrt{|\hat{p}^1|}\;|\mu_1,\mu_2,k\,\rangle = \frac{\gamma^{\frac{3}{2}}\ell_{\mathrm{P}}^3}{4\sqrt{2}}|\mu_2|\sqrt{|\mu_1|}\;|\mu_1,\mu_2,k\rangle\;. \tag{4.29}$$

The full Hilbert space is a further tensor product of $\mathcal{H}_{\mathrm{grav}}$ with the fermionic Hilbert space $\mathcal{H}_{\mathrm{fermion}}$. We represent the latter as the space of functions $f(\Theta_\alpha)$ of four independent half-densitized Grassmann-valued variables Θ_α, $\alpha = 1, \ldots, 4$, for the four components contained in the fermion fields ξ and χ in this order. The fermionic momenta $\pi_\xi = -i\xi^\dagger$ and $\pi_\chi = -i\chi^\dagger$ then give rise to components $\overline{\Theta}_\alpha$ which are represented as $\hbar\partial/\partial\Theta_\alpha$. In particular, for the axial current components $\mathcal{J}^0 = \xi^\dagger\xi - \chi^\dagger\chi$ and $\mathcal{J}_1 = \xi^\dagger\sigma_1\xi + \chi^\dagger\sigma_1\chi$ we have operators

$$\hat{\mathcal{J}}^0 = \hbar\frac{\partial}{\partial\Theta_1}\Theta_1 + \hbar\frac{\partial}{\partial\Theta_2}\Theta_2 - \hbar\frac{\partial}{\partial\Theta_3}\Theta_3 - \hbar\frac{\partial}{\partial\Theta_4}\Theta_4 \tag{4.30}$$

$$\hat{\mathcal{J}}_1 = \hbar\frac{\partial}{\partial\Theta_2}\Theta_1 + \hbar\frac{\partial}{\partial\Theta_1}\Theta_2 + \hbar\frac{\partial}{\partial\Theta_4}\Theta_3 + \hbar\frac{\partial}{\partial\Theta_3}\Theta_4 \,. \tag{4.31}$$

(The component $\hat{\mathcal{J}}_0$ is subject to ordering ambiguities which we can ignore here.)

The currents are easy to diagonalize: Each 2-spinor copy has eigenstates of $\frac{\partial}{\partial\Theta_2}\Theta_1 + \frac{\partial}{\partial\Theta_1}\Theta_2$ given by $f_0(\Theta) = 1$ and $f^0(\Theta) = \Theta_1\Theta_2$ of eigenvalue zero, $f_\pm(\Theta) = \Theta_1 \pm \Theta_2$ of eigenvalue ± 1. The tensor product of both 2-spinor copies ξ and χ then gives eigenstates of eigenvalues zero, $\pm\hbar$ and $\pm 2\hbar$ for $\hat{\mathcal{J}}_1$. The time component $\hat{\mathcal{J}}^0$ has the same eigenstates.

A general state in $\mathcal{H} = \mathcal{H}_{\text{grav}} \otimes \mathcal{H}_{\text{fermion}}$ can then be written in a form using fermion dependent coefficient functions in the triad eigenbasis (4.27):

$$|s\rangle = \sum_{\mu_1,\mu_2,k} s_{\mu_1,\mu_2,k}(\Theta)|\mu_1,\mu_2,k\rangle \,. \tag{4.32}$$

One can define the coefficients $s_{\mu_1,\mu_2,k}(\Theta)$ for all values of $\mu_1, \mu_2 \in \mathbb{R}$ and $k \in \mathbb{Z}$ in this way. However, gauge invariance implies that the state must be invariant under changing the sign of μ_2 because this corresponds to a triad rotation (without changing orientation). Thus, we require $s_{\mu_1,\mu_2,k}(\Theta) = s_{\mu_1,-\mu_2,k}(\Theta)$.

The remaining sign freedom, $\mathrm{sgn}\mu_1$, is physical and crucial because it determines the relative orientation of the triad. Thus, we have a simple action

$$s_{\mu_1,\mu_2,k}(\Theta_1,\Theta_2,\Theta_3,\Theta_4) \xrightarrow{\widehat{\Pi}} s_{-\mu_1,\mu_2,k}(\Theta_3,\Theta_4,\Theta_1,\Theta_2) \tag{4.33}$$

of the parity operator $\widehat{\Pi}$ on states. For the fermion dependence, we have represented the parity action $\hat{\Pi}\Psi = \gamma^0\Psi$ for Dirac spinors by switching the fermion values Θ_α

corresponding to ξ and χ, respectively. This implies

$$\hat{\Pi}\hat{\mathcal{J}}_0\hat{\Pi}^{-1} = -\hat{\mathcal{J}}_0 \quad , \quad \hat{\Pi}\hat{\mathcal{J}}_1\hat{\Pi}^{-1} = \hat{\mathcal{J}}_1 \,. \tag{4.34}$$

For gravitational operators, a direct calculation shows

$$\hat{\Pi}\hat{p}^1\hat{\Pi}^{-1} = -\hat{p}^1 \quad , \quad \hat{\Pi}\hat{p}^2\hat{\Pi}^{-1} = \hat{p}^2 \tag{4.35}$$

$$\hat{\Pi}\exp\widehat{(i\mu_1 c_1}/2)\hat{\Pi}^{-1} = \exp\widehat{(-i\mu_1 c}_1/2) \quad , \quad \hat{\Pi}\exp\widehat{(i\mu_2\tilde{c}_2}/2)\hat{\Pi}^{-1} = \exp\widehat{(i\mu_2\tilde{c}_2}/2) \tag{4.36}$$

as required.

Finally, we can directly solve the Gauss constraint which requires $\hat{p}_\phi = \frac{1}{2}\gamma\kappa\hat{\mathcal{J}}_1$ and thus allows us to eliminate k as an independent quantity. Using the spectra of the operators already determined, this provides solutions with either $k = 0$ or $k = \pm 1$. In the second case, there is a non-vanishing value of the spatial axial current $\mathcal{J}_1$ of size $\pm 2\hbar$. The values $\pm\hbar$ for the fermion current, which do exist as eigenvalues, are ruled out because they do not correspond to integer k. Both 2-spinors present must thus have the same or opposite $\hat{\mathcal{J}}_1$-eigenvalues, which allows them to be parity eigenstates. The parity behavior of the full state according to (4.33), however, is determined by the μ_1-dependence, which required the dynamics of quantum gravity coupling the triad to fermions.

The allowed values for the current are only microscopic and may not seem of interest to describe a macroscopic universe of large matter content; they all vanish in the classical limit $\hbar \to 0$. Nevertheless, this provides an interesting model where one can study the effects of fermions and parity in loop quantum gravity. Physically, it is also clear why the matter contribution can only be microscopic: As always in homogeneous quantum cosmological models, each field component is reduced to a single degree of freedom for all of space. For the fermion, this allows only one excitation per component due to Pauli's principle. Unlike with scalar matter, one cannot simply make the matter content large by choosing a high "occupation" such as a large momentum of the scalar. Significant fermionic matter can only be included by adding more independent spinor fields, or by introducing inhomogeneity which provides independent field values at different points (represented by fermions at different vertices of a spin network state in loop quantum gravity). Rather than being a limitation, we consider this as an important physical property of quantum cosmology in the presence of realistic fermionic matter.

Quantum Dynamics: The Hamiltonian Constraint

A useful feature of the torsion-free Bianchi I model is that the Lorentzian Hamiltonian constraint is related to the Euclidean part simply by $H = -\gamma^{-2}H^{(E)}$ thanks to $K^i_{[a}K^j_{b]} \propto F^k_{ab}\epsilon_{ijk}$, making use of homogeneity as well as the fact that the spin connection vanishes. This has been used in almost all investigations of loop quantum cosmology so far. If this relation is not used, one can still quantize the Lorentzian constraint following techniques of the full theory [21]. This results in a more complicated constraint operator [67], but without crucial differences.

However, in the presence of torsion, such a simple relationship can be obtained only after splitting the torsion contribution from the spin connection as shown in (3.6), which is now to be quantized: even for the Bianchi I model, Γ^i_a is no longer zero due to torsion. Fortunately, torsion contributions to Γ^i_a, namely C^i_a in (3.5), are completely determined by second class constraints. They can thus be split off and quantized separately together with the matter terms. For the Bianchi I LRS model, one can use a further key simplification which, as pointed out above, allows us to project out torsion contributions without directly computing them. All we need to do is use the new variable $\tilde{c}_2$ instead of c_2. The resulting contribution to the gravitational Hamiltonian constraint is the same as the torsion-free one and thus can be quantized in the same way.

Mimicking the steps done in the full theory [21, 25], one writes curvature components $\mathcal{F}^i_{ab}$ as a product of (point) holonomies $h_I = \cos(\frac{1}{2}\delta_I c_I) + 2\Lambda^i_I\tau_i\sin(\frac{1}{2}\delta_I c_I)$

forming a closed loop, whose "edge lengths" are denoted as δ_1 and δ_2 for the two independent directions. Moreover, using

$$\frac{1}{2}\epsilon_{abc}\epsilon^{ijk}\frac{E^b_j E^c_k}{\sqrt{\det(E^d_l)}} = e^i_a = \frac{2}{\gamma\kappa}\left\{\mathcal{A}^i_a(x), V\right\} \tag{4.37}$$

relevant products of triad components, including their inverse powers, are reduced to a Poisson bracket of the general form $h_I\{h_I^{-1}, V\}$ where V is the spatial volume and h_I again a holonomy. This allows one to write an operator in compact form, which corresponds to a densely defined operator in the full theory:

$$\hat{H}_G = -\frac{4i\mathrm{sgn}(\hat{p}^1\hat{p}^2\hat{p}^3)}{\gamma^3\kappa\ell_{\mathrm{P}}^2\delta_1\delta_2\delta_3}\sum_{IJK}\epsilon^{IJK}\mathrm{tr}\left(h_I h_J h_I^{-1}h_J^{-1}h_K[h_K^{-1},\hat{V}]\right)\,. \tag{4.38}$$

We can now compute the product of holonomies and take the trace explicitly, using the basic properties of Pauli matrices. We do this directly for LRS variables with only two independent holonomies such that $\delta_2 = \delta_3$. Moreover, the sign factor is now solely determined by $\mathrm{sgn}\hat{p}^1$ since $\hat{p}^2\hat{p}^3$ cannot be negative. This results in [72]

$$\begin{aligned}\hat{H}_G \;=\; & -\frac{32i\mathrm{sgn}(\hat{p}^1)}{\gamma^3\kappa\ell_{\mathrm{P}}^2\delta_1\delta_2^2}\Bigg(2\sin(\tfrac{1}{2}\delta_1c_1)\cos(\tfrac{1}{2}\delta_1c_1)\sin(\tfrac{1}{2}\delta_2\tilde{c}_2)\cos(\tfrac{1}{2}\delta_2\tilde{c}_2)\\ & \Big(\sin(\tfrac{1}{2}\delta_2\tilde{c}_2)\hat{V}\cos(\tfrac{1}{2}\delta_2\tilde{c}_2) - \cos(\tfrac{1}{2}\delta_2\tilde{c}_2)\hat{V}\sin(\tfrac{1}{2}\delta_2\tilde{c}_2)\Big)\\ & +\sin^2(\tfrac{1}{2}\delta_2\tilde{c}_2)\cos^2(\tfrac{1}{2}\delta_2\tilde{c}_2)\Big(\sin(\tfrac{1}{2}\delta_1c_1)\hat{V}\cos(\tfrac{1}{2}\delta_1c_1)\\ & -\cos(\tfrac{1}{2}\delta_1c_1)\hat{V}\sin(\tfrac{1}{2}\delta_1c_1)\Big)\Bigg)\,.\end{aligned} \tag{4.39}$$

Because we have implicitly eliminated the torsion contributions from holonomies by our choice of basic variables, we can directly use this expression as it is known from torsion-free models. The torsion contribution will then be added to the constraint operator via the fermion current.

We emphasize that the meaning and form of the parameters δ_1 and δ_2 cannot be fully elucidated purely in homogeneous models. In the absence so far of a derivation from a full, inhomogeneous constraint (which itself is currently subject to changes in its general form depending on ongoing developments) it appears best to refrain from specific, heuristic arguments as to what values they may take. (For instance, there is currently no firm basis for a relation of those parameters to an eigenvalue of the area operator of the full theory, as initially proposed in [85].) We therefore follow a more general route which allows whole classes of these parameters, and confine attention to effects which are insensitive to the specific form. To us, this seems most advisable given that it is not just the numerical values of these parameters but even their possible functional dependence on basic variables which remains open; see Sec. 4.6 for further discussions.

In order to quantize the matter Hamiltonian, we must in particular quantize the inverse volume $1/p^2\sqrt{|p^1|}$. Here, we use the standard procedure [25], first writing

$$\begin{aligned}\frac{1}{\sqrt{|\det(E^a_i)|}} &= \frac{\mathrm{sgn}\det(e^i_a)}{6|\det(E^d_l)}\epsilon^{abc}\epsilon_{ijk}e^i_a e^j_b e^k_c \\ &= \frac{36}{\gamma^3\kappa^3}\mathrm{sgn}\det(e^i_a)\epsilon^{abc}\epsilon_{ijk}\{\mathcal{A}^i_a,V^{1/3}\}\{\mathcal{A}^j_b,V^{1/3}\}\{\mathcal{A}^k_c,V^{1/3}\}\end{aligned} \tag{4.40}$$

based on (4.37), which is then quantized to

$$\begin{aligned}\widehat{\left(\frac{1}{V}\right)} &= \frac{144i\mathrm{sgn}(\hat{p}^1\hat{p}^2\hat{p}^3)}{\gamma^3\ell_{\mathrm{P}}^6\delta_1\delta_2\delta_3}\sum_{IJK}\epsilon^{IJK}\mathrm{tr}\left(h_I[h_I^{-1},\hat{V}^{1/3}]h_J[h_J^{-1},\hat{V}^{1/3}]h_K[h_K^{-1},\hat{V}^{1/3}]\right) \\ &= -\frac{32\cdot 81\mathrm{sgn}(\hat{p}^1)}{\gamma^3\ell_{\mathrm{P}}^6\delta_1\delta_2^2}\left(\sin(\tfrac{1}{2}\delta_1c_1)\hat{V}^{1/3}\cos(\tfrac{1}{2}\delta_1c_1)-\cos(\tfrac{1}{2}\delta_1c_1)\hat{V}^{1/3}\sin(\tfrac{1}{2}\delta_1c_1)\right) \\ &\quad\left(\sin(\tfrac{1}{2}\delta_2\tilde{c}_2)\hat{V}^{1/3}\cos(\tfrac{1}{2}\delta_2\tilde{c}_2)-\cos(\tfrac{1}{2}\delta_2\tilde{c}_2)\hat{V}^{1/3}\sin(\tfrac{1}{2}\delta_2\tilde{c}_2)\right)^2 .\end{aligned} \tag{4.41}$$

The action of this operator as well as the Hamiltonian constraint is easily computed using the action of $\sin(\frac{1}{2}\delta_1c_1)$ and $\cos(\frac{1}{2}\delta_1c_1)$ on the triad eigenstates,

$$\begin{aligned}\cos(\tfrac{1}{2}\delta_1c_1)|\mu_1,\mu_2,k\rangle &= \frac{1}{2}(|\mu_1+\delta_1,\mu_2,k\rangle+|\mu_1-\delta_1,\mu_2,k\rangle) \\ \sin(\tfrac{1}{2}\delta_1c_1)|\mu_1,\mu_2,k\rangle &= -\frac{1}{2}i(|\mu_1+\delta_1,\mu_2,k\rangle-|\mu_1-\delta_1,\mu_2,k\rangle) ,\end{aligned} \tag{4.42}$$

and the volume operator (4.29). From matrix elements of the Hamiltonian constraint one can then write the constraint equation $(\hat{H}_G+\hat{H}_{\mathrm{matter}})|s\rangle = 0$ as a difference equation for coefficients $s_{\mu_1,\mu_2,k}(\Theta)$ of the state in the triad representation. We do this immediately on states solving the Gauss constraint which determines k in terms of the action of $\hat{\mathcal{J}}_1$. Dropping the label k on those states, we have

$$\begin{aligned}
& 2(|\mu_2+3\delta_2|-|\mu_2+\delta_2|)\Big(|\mu_1+2\delta_1|^{1/2}s_{\mu_1+2\delta_1,\mu_2+2\delta_2}(\Theta) \\
& -|\mu_1-2\delta_1|^{1/2}s_{\mu_1-2\delta_1,\mu_2+2\delta_2}(\Theta)\Big)+2(|\mu_2-\delta_2|-|\mu_2-3\delta_2|) \\
& \Big(|\mu_1-2\delta_1|^{1/2}s_{\mu_1-2\delta_1,\mu_2-2\delta_2}(\Theta)-|\mu_1+2\delta_1|^{1/2}s_{\mu_1+2\delta_1,\mu_2-2\delta_2}(\Theta)\Big) \\
& +(|\mu_1+\delta_1|^{1/2}-|\mu_1-\delta_1|^{1/2})\Big(|\mu_2+4\delta_2|s_{\mu_1,\mu_2+4\delta_2}(\Theta) \\
& -2|\mu_2|s_{\mu_1,\mu_2}(\Theta)+|\mu_2-4\delta_2|s_{\mu_1,\mu_2-4\delta_2}(\Theta)\Big) \\
= \; & \frac{81}{16}|\mu_1|^{1/3}|\mu_2|^{1/3}(|\mu_1+\delta_1|^{1/6}-|\mu_1-\delta_1|^{1/6})(|\mu_2+\delta_2|^{1/3}-|\mu_2-\delta_2|^{1/3})^2 \\
& \times\left(\left(1+4\gamma^2-\frac{2\gamma\beta}{1+\gamma^2}\left(3-\frac{\gamma}{\alpha}+2\gamma^2\right)-\frac{\theta^2}{1+\gamma^2}\right)\frac{\hat{\mathcal{J}}_1^2}{\hbar^2}\right. \\
& \left.+3\gamma\theta\left(\frac{2}{\alpha}+\frac{\gamma\theta}{1+\gamma^2}\right)\frac{\hat{\mathcal{J}}_0^2}{\hbar^2}\right)s_{\mu_1,\mu_2}(\Theta)\,.
\end{aligned} \tag{4.43}$$

This equation is based on a non-symmetric constraint operator because in (4.39) we ordered all holonomy factors to the left and kept the commutator terms with the volume operator to the right. It is sometimes useful to have a symmetric ordering, although this is not strictly required for constraints. (But it is required by some methods to derive the physical Hilbert space.) There is only one way to order the constraint symmetrically, namely by introducing $\frac{1}{2}(\hat{H}+\hat{H}^\dagger)$. Other possibilities have been suggested, such as splitting the sines and cosines and writing some to the left, others to the right of the commutator term. They are, for instance, useful to prove self-adjointness [86]. However, this corresponds to splitting the holonomy product $h_Ih_Jh_I^{-1}h_J^{-1}$ into different factors, which cannot be done in a general setting where there would rather be a single holonomy h_α around a closed loop α. The direct symmetrization, on the other hand, is always possible and in our case results in a difference equation

$$
\begin{aligned}
& 2\left((|\mu_2+3\delta_2|-|\mu_2+\delta_2|)|\mu_1+2\delta_1|^{1/2}+(|\mu_2+\delta_2|-|\mu_2-\delta_2|)|\mu_1|^{1/2}\right) \\
& \qquad\qquad\qquad \times \quad s_{\mu_1+2\delta_1,\mu_2+2\delta_2}(\Theta) \\
& -2\left((|\mu_2+3\delta_2|-|\mu_2+\delta_2|)|\mu_1-2\delta_1|^{1/2}+(|\mu_2+\delta_2|-|\mu_2-\delta_2|)|\mu_1|^{1/2}\right) \\
& \qquad\qquad\qquad \times \quad s_{\mu_1-2\delta_1,\mu_2+2\delta_2}(\Theta) \\
& +2\left((|\mu_2-\delta_2|-|\mu_2-3\delta_2|)|\mu_1-2\delta_1|^{1/2}+(|\mu_2+\delta_2|-|\mu_2-\delta_2|)|\mu_1|^{1/2}\right) \\
& \qquad\qquad\qquad \times \quad s_{\mu_1-2\delta_1,\mu_2-2\delta_2}(\Theta) \\
& -2\left((|\mu_2-\delta_2|-|\mu_2-3\delta_2|)|\mu_1+2\delta_1|^{1/2}+(|\mu_2+\delta_2|-|\mu_2-\delta_2|)|\mu_1|^{1/2}\right) \\
& \qquad\qquad\qquad \times \quad s_{\mu_1+2\delta_1,\mu_2-2\delta_2}(\Theta)) \\
& +\left(|\mu_1+\delta_1|^{1/2}-|\mu_1-\delta_1|^{1/2}\right)\Bigg((|\mu_2|+|\mu_2+4\delta_1|)s_{\mu_1,\mu_2+4\delta_2}(\Theta) \\
& -4|\mu_2|s_{\mu_1,\mu_2}(\Theta)+(|\mu_2|+|\mu_2-4\delta_1|)s_{\mu_1,\mu_2-4\delta_2}(\Theta)\Bigg) \\
= \; & \frac{81}{8}|\mu_1|^{1/3}|\mu_2|^{1/3}(|\mu_1+\delta_1|^{1/6}-|\mu_1-\delta_1|^{1/6})(|\mu_2+\delta_2|^{1/3}-|\mu_2-\delta_2|^{1/3})^2 \\
& \times\left(\left(1+4\gamma^2-\frac{2\gamma\beta}{1+\gamma^2}\left(3-\frac{\gamma}{\alpha}+2\gamma^2\right)-\frac{\theta^2}{1+\gamma^2}\right)\frac{\hat{\mathcal{J}}_1^2}{\hbar^2}\right. \\
& \left.+3\gamma\theta\left(\frac{2}{\alpha}+\frac{\gamma\theta}{1+\gamma^2}\right)\frac{\hat{\mathcal{J}}_0^2}{\hbar^2}\right)s_{\mu_1,\mu_2}(\Theta)\;. \qquad (4.44)
\end{aligned}
$$

4.4 Cosmological models with Electromagnetism

A similar investigation of the role Maxwell fields play in loop quantum cosmology can be carried out. However, a thorough investigation is yet to be complete, so, for the sake of completeness, only a brief discussion on electromagnetism in Bianchi models, along with some remarks on the future directions for further interesting research, is presented in this section. First, it is straightforward to derive the equations of motion for the canonical variables A_a and π^a from the Poisson brackets of each of these variables with the matter Hamiltonian H_M. Then

$$\dot{A}_a=\{A_a,H_M\}=\frac{\delta H_M}{\delta\pi^a}=\partial_a\left(A_c t^c\right)+N^cF_{ca}+\frac{4\pi N}{\sqrt{q}}\pi^c q_{ca}, \qquad (4.45)$$

and

$$\dot{\pi}^a = \{\pi^a, H_M\} = -\frac{\delta H_M}{\delta A_a} = \partial_c \left(N^c \pi^a\right) - \partial_d \left(N^a \pi^d\right) - 4\partial_c \left(N\sqrt{q} F_{ef} q^{ec} q^{fa}\right). \quad (4.46)$$

The modified Hamiltonian gives rise to the following new set of equations of motion:

$$\dot{A}_a = \{A_a, H_{\text{eff}}\} = \frac{\delta H_{\text{eff}}}{\delta \pi^a} = \partial_a \left(A_c t^c\right) + N^c F_{ca} + \frac{4\pi N}{\sqrt{q}} \alpha\left(q\right) \pi^c q_{ca}, \quad (4.47)$$

and

$$\begin{aligned} \dot{\pi}^a &= \{\pi^a, H_{\text{eff}}\} = -\frac{\delta H_{\text{eff}}}{\delta A_a} \\ &= \partial_c \left(N^c \pi^a\right) - \partial_d \left(N^a \pi^d\right) - 4\partial_c \left(N\beta\left(q\right)\sqrt{q} F_{ef} q^{ec} q^{fa}\right), \end{aligned} \quad (4.48)$$

where α and β are the correction functions derived in Chapter IV and H_{eff} is the effective Hamiltonian of the Maxwell's field (H_M with α and β inserted).

Reduced Constraints

Now, with the choice of reduced gravitational variables in $SU(2)$ presented in section 4.2, the canonical variabes for Maxwell fields in $U(1)$ can be written as

$$A_a = \phi_I^{\text{EM}} \omega_a^I, \qquad \pi^a = p_{\text{EM}}^I X_I^a\ . \quad (4.49)$$

Then the reduced constraints take the following forms

$$G_{\text{EM}} = -p_{\text{EM}}^J C_{JL}^L \sqrt{g_0} = 0, \quad (4.50)$$

$$D_c N^c = -\phi_M^{\text{EM}} p_{\text{EM}}^I C_{IL}^M N^I = -n^{(M)} \phi_M^{\text{EM}} p_{\text{EM}}^L \epsilon^M{}_{IL} N^I, \quad (4.51)$$

$$\mathcal{H}_M = \frac{V_0^{-2/3}}{\sqrt{q}} a_{(I)}^2 \left(2\pi\alpha(q)\delta_{IJ} p_{\text{EM}}^I p_{\text{EM}}^J + \frac{n^{(I)} n^{(J)}}{2\pi}\, g_0 \beta(q) \delta^{IJ} \phi_I^{\text{EM}} \phi_J^{\text{EM}}\right), \quad (4.52)$$

where $g_0 = \det(\omega_a^I)^2$. The equations of motion become

$$\dot{\phi}_K^{\text{EM}} = \frac{2\pi V_0^{-2/3}}{\sqrt{q}} \left(a_{(K)}^2 + a_{(I)}^2\right) \alpha(q) \delta_{IK} p_{\text{EM}}^I\ , \quad (4.53)$$

$$\dot{p}_{\text{EM}}^K = -\frac{V_0^{-2/3}}{2\pi\sqrt{q}} n^{(I)} n^{(K)} \left(a_{(K)}^2 + a_{(I)}^2\right) \beta(q) \delta^{IK} \phi_I^{\text{EM}}\ . \quad (4.54)$$

It follows immediately from (4.50) that torsion is absent in the presence of photons. For Bianchi I models ($n^{(I)} = 0$), the diffeomorphism constraint (4.51) vanishes identically and so is the magnetic part of the Hamiltonian constraint (4.52). Also, the relation (4.54) implies that the electric fields are constant in time. Therefore, Bianchi I models with electromagnetism do not seem to be interesting cosmological models.

Now, solutions to the equations of motion will depend on the background evolution of the a_I, which is complicated in a Bianchi IX model. However, by taking the ratio of the two equations the a_I-dependence drops out. This provides a closed differential equation for $\phi_K^{\rm EM}(p^K_{\rm EM})$, which can be solved easily to give

$$(\phi_K^{\rm EM})^2 \propto (p^K_{\rm EM})^2 + c_K \tag{4.55}$$

with constants of integration c_K. The constants determine the type of the wave: if they vanish, the electric and magnetic field (whose components are $\phi_K^{\rm EM}$) point in the same direction, but choosing c_K non-zero allows arbitrary relative directions between the fields. The system can thus be used as a model for the local behavior of electromagnetic waves and the role of anisotropy. At least for long wave lengths, the homogeneity assumption should still be valid. (Putting similar corrections directly into the inhomogeneous equations would first have to ensure anomaly-freedom, which is possible at a perturbative level but tedious. The homogeneous approximation would provide indications for possible effects quicker.)

Quantum corrections presented in Chapter V can then be seen to have a potential influence: For inverse volume corrections, for instance, the ratio of $\dot{\phi}_K^{\rm EM}$ and $\dot{p}^K_{\rm EM}$ then depends on the a_I if the two correction functions α and β depend differently on the geometry. This might have consequences for anisotropies in the radiation spectrum. Solutions will be difficult to find analytically, but numerical investigations might be of interest.

Another application would use the resulting matter terms in the difference equation presented below. Since we now have the typical matter ingredients included (fermions, radiation), one could do a general survey of the behavior in anisotropic models and discuss stability, lattice refinement, possible constraints on the Barbero–Immirzi parameter, which we do not pursue here.

4.5 Difference equation for Maxwell fields in Bianchi IX models

For Bianchi IX models, the difference equation for vacuum gravity is derived in [114]. Following the same formalism both for gravity and Maxwell fields, we provide a derivation of the bosonic (photonic) matter terms in the difference equation in this section. Since the quantization of the Maxwell's theory in anisotropic models can be carried out in the same manner as that of the Einstein-Dirac theory presented in section 4.3, we collect the basic elements of loop quantization of the Maxwell's theory in Appendix 0.5.

Now, Since Bianchi IX models with non-vanishing intrinsic curvature must have non-zero spin connection Γ_a^i which, thanks to homogeneity, cannot be made to vanish. While the presence of non-vanishing spin connection in models with fermions makes quantization difficult, it is relatively much easier for photons with the help of the formalism presented in [114]. Therefore, without giving any details for the quantization of the gravitational Hamiltonian for Bianchi IX models, we only offer the qunatization of the matter Hamiltonian below.

The Maxwell Hamiltonian constraint operator turns out to be

$$\begin{aligned} \widehat{H}_{\rm EM} &= 16(\gamma\ell_p^2)^{-2}\sum_{IJ}\mathrm{tr}\left[\left\{h_I(A)\left[h_I^{-1}(A),\hat{V}^{\frac{1}{2}}\right]h_J(A)\left[h_J^{-1}(A),\hat{V}^{\frac{1}{2}}\right]\right\}\right. \\ &\qquad \left.\times\left(4\pi\hat{p}^I_{\rm EM}\hat{p}^J_{\rm EM}+\pi^{-1}n^{(I)}n^{(J)}\hat{\phi}^I_{\rm EM}\hat{\phi}^J_{\rm EM}\right)\right], \end{aligned} \tag{4.56}$$

where

$$\hat{\phi}^I_{\rm EM} := \hat{\phi}^{\rm EM}_M\delta^{MI} \tag{4.57}$$

It is important to note that (4.57) is redefined, without loss of generality, for notational convenience and represents the point holonomy for the $U(1)$ connection. Moreover, it is easy to show that the Maxwell Hamiltonian operator (4.56) reproduces the reduced classical Maxwell Hamiltonian (4.52) in the appropriate classical limit. The first term in the braces of (4.56) comes from $q_{ab}/\sqrt{q}$ in classical Maxwell Hamiltonian, which can be reexpressed as

$$\frac{q_{ab}}{\sqrt{q}} = \eta_{ij}\frac{e^i_a e^j_b}{\sqrt{q}} = \frac{16}{\gamma^2\kappa^2}\ \eta_{ij}\left\{A^i_a, V^{\frac{1}{2}}\right\}\left\{A^j_b, V^{\frac{1}{2}}\right\}\ , \tag{4.58}$$

with the Cartan-Killing metric $\eta_{ij} = -2\mathrm{tr}(\tau_i\tau_j)$. Further, we propose that the following approximation for the $U(1)$ holonomy,

$$\hat{\phi}^{\mathrm{EM}}_M \approx \frac{\sin\left(\phi^{\mathrm{EM}}_M\delta_M(p^I)\right)}{\delta_M(p^I)}\ , \tag{4.59}$$

reproduces the expected classical limit.

Now, in order to obtain the difference equation, let us first make an observation that the operator $\hat{h}_K(A)[\hat{h}^{-1}_K(A), \hat{V}^{\frac{1}{2}}]$ is diagonal in the triad basis, i.e. using (106), (111) and (112), we obtain

$$\begin{aligned}\hat{h}_K(A)\left[\hat{h}^{-1}_K(A), \hat{V}^{\frac{1}{2}}\right] &= \left\{\hat{V}^{\frac{1}{2}} - \cos(\frac{1}{2}c_K)\hat{V}^{\frac{1}{2}}\cos(\frac{1}{2}c_K) - \sin(\frac{1}{2}c_K)\hat{V}^{\frac{1}{2}}\sin(\frac{1}{2}c_K)\right\} \\ &\quad -2\Lambda_K\left\{\sin(\frac{1}{2}c_K)\hat{V}^{\frac{1}{2}}\cos(\frac{1}{2}c_K) - \cos(\frac{1}{2}c_K)\hat{V}^{\frac{1}{2}}\sin(\frac{1}{2}c_K)\right\}\end{aligned} \tag{4.60}$$

implies

$$\begin{aligned}&\left(\hat{V}^{\frac{1}{2}} - \cos(\tfrac{1}{2}c_3)\hat{V}^{\frac{1}{2}}\cos(\tfrac{1}{2}c_3) - \sin(\tfrac{1}{2}c_3)\hat{V}^{\frac{1}{2}}\sin(\tfrac{1}{2}c_3)\right)|\mu_1,\mu_2,\mu_3\rangle \;= \\ &\left(V^{\frac{1}{2}}(\mu_1,\mu_2,\mu_3) - \frac{1}{2}V^{\frac{1}{2}}(\mu_1,\mu_2,\mu_3+1) - \frac{1}{2}V^{\frac{1}{2}}(\mu_1,\mu_2,\mu_3-1)\right)|\mu_1,\mu_2,\mu_3\rangle\end{aligned} \tag{4.61}$$

$$\begin{aligned}&\left(\sin(\tfrac{1}{2}c_3)\hat{V}^{\frac{1}{2}}\cos(\tfrac{1}{2}c_3) - \cos(\tfrac{1}{2}c_3)\hat{V}^{\frac{1}{2}}\sin(\tfrac{1}{2}c_3)\right)|\mu_1,\mu_2,\mu_3\rangle \;= \\ &\quad\frac{i}{2}\left(V^{\frac{1}{2}}(\mu_1,\mu_2,\mu_3+1) - V^{\frac{1}{2}}(\mu_1,\mu_2,\mu_3-1)\right)|\mu_1,\mu_2,\mu_3\rangle\ .\end{aligned} \tag{4.62}$$

Hence, let us first express the proposed Maxwell's Hamiltonian operator (4.56) as follows

$$\begin{aligned}\widehat{H}_{\mathrm{EM}} &= 32(\gamma\ell_p^2)^{-2}\sum_{IJ}\hat{\mathbf{h}}_I\hat{\mathbf{h}}_J\Bigg(4\pi\hat{p}^I_{\mathrm{EM}}\hat{p}^J_{\mathrm{EM}} \\ &\qquad +\pi^{-1}\frac{\sin\left(\phi^{\mathrm{EM}}_I\delta_I(\hat{p}^K)\right)}{\delta_I(\hat{p}^K)}\frac{\sin\left(\phi^{\mathrm{EM}}_J\delta_J(\hat{p}^K)\right)}{\delta_J(\hat{p}^K)}\Bigg)\ ,\end{aligned} \tag{4.63}$$

where

$$\hat{\mathbf{h}}_I := \hat{h}_K(A)\left[\hat{h}_K^{-1}(A), \hat{V}^{\frac{1}{2}}\right] \ . \tag{4.64}$$

Next, inserting (4.61) and (4.62) in (4.63), we obtain the matter part of the difference equation as follows

$$\begin{aligned}
&\left(\widehat{H}_{EM}s\right)_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3} \\
= \quad & 2(\gamma\ell_p^2)^{-2}\Bigg(16\pi\big(\mathcal{V}_1^2 {p_{\mathrm{EM}}^1}^2 + \mathcal{V}_2^2 {p_{\mathrm{EM}}^2}^2 + \mathcal{V}_3^2 {p_{\mathrm{EM}}^3}^2 + 2\mathcal{V}_{12}p_{\mathrm{EM}}^1 p_{\mathrm{EM}}^2 + 2\mathcal{V}_{23}p_{\mathrm{EM}}^2 p_{\mathrm{EM}}^3 \\
& \qquad\qquad +2\mathcal{V}_{31}p_{\mathrm{EM}}^3 p_{\mathrm{EM}}^1\big)s_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3} \\
& -\pi^{-1}\Bigg(\frac{\mathcal{V}_1^2}{\delta_1^2}\left(s_{\mu_1,\mu_2,\mu_3;n_1+2\delta_1,n_2,n_3} + s_{\mu_1,\mu_2,\mu_3;n_1-2\delta_1,n_2,n_3}\right) \\
& \qquad +\frac{\mathcal{V}_2^2}{\delta_2^2}\left(s_{\mu_1,\mu_2,\mu_3;n_1,n_2+2\delta_2,n_3} + s_{\mu_1,\mu_2,\mu_3;n_1,n_2-2\delta_2,n_3}\right) \\
& \qquad +\frac{\mathcal{V}_3^2}{\delta_3^2}\left(s_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3+2\delta_3} + s_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3-2\delta_3}\right) \\
& \qquad +2\frac{\mathcal{V}_{12}}{\delta_1\delta_2}\big(s_{\mu_1,\mu_2,\mu_3;n_1+\delta_1,n_2+\delta_2,n_3} - s_{\mu_1,\mu_2,\mu_3;n_1+\delta_1,n_2-\delta_2,n_3} \\
& \qquad -s_{\mu_1,\mu_2,\mu_3;n_1-\delta_1,n_2+\delta_2,n_3} + s_{\mu_1,\mu_2,\mu_3;n_1-\delta_1,n_2-\delta_2,n_3}\big) \\
& \qquad +2\frac{\mathcal{V}_{23}}{\delta_1\delta_2}\big(s_{\mu_1,\mu_2,\mu_3;n_1,n_2+\delta_2,n_3+\delta_3} - s_{\mu_1,\mu_2,\mu_3;n_1,n_2+\delta_2,n_3-\delta_3} \\
& \qquad -s_{\mu_1,\mu_2,\mu_3;n_1,n_2+\delta_2,n_3-\delta_3} + s_{\mu_1,\mu_2,\mu_3;n_1,n_2-\delta_2,n_3-\delta_3}\big) \\
& \qquad +2\frac{\mathcal{V}_{31}}{\delta_1\delta_2}\big(s_{\mu_1,\mu_2,\mu_3;n_1+\delta_1,n_2,n_3+\delta_3} - s_{\mu_1,\mu_2,\mu_3;n_1+\delta_1,n_2,n_3-\delta_3} \\
& \qquad -s_{\mu_1,\mu_2,\mu_3;n_1-\delta_1,n_2,n_3+\delta_3} + s_{\mu_1,\mu_2,\mu_3;n_1-\delta_1,n_2,n_3-\delta_3}\big)\Bigg)\Bigg),
\end{aligned} \tag{4.65}$$

where, e.g.

$$\begin{aligned}
\mathcal{V}_1^2 \ := \ & 4V(\mu_1,\mu_2,\mu_3) + \left(V^{\frac{1}{2}}(\mu_1+\delta_1,\mu_2,\mu_3) + V^{\frac{1}{2}}(\mu_1-\delta_1,\mu_2,\mu_3)\right)^{\frac{1}{2}} \\
& + \left(V^{\frac{1}{2}}(\mu_1+\delta_1,\mu_2,\mu_3) + V^{\frac{1}{2}}(\mu_1-\delta_1,\mu_2,\mu_3)\right) \\
& \left(V^{\frac{1}{2}}(\mu_1+\delta_1,\mu_2,\mu_3) + V^{\frac{1}{2}}(\mu_1-\delta_1,\mu_2,\mu_3) - 4V^{\frac{1}{2}}(\mu_1,\mu_2,\mu_3)\right) \ ,
\end{aligned} \tag{4.66}$$

$$
\begin{aligned}
\mathcal{V}_{12} \;:=\;& 4V(\mu_1,\mu_2,\mu_3)+\left(V^{\frac{1}{2}}(\mu_1+\delta_1,\mu_2,\mu_3)+V^{\frac{1}{2}}(\mu_1-\delta_1,\mu_2,\mu_3)\right)\\
&\left(V^{\frac{1}{2}}(\mu_1,\mu_2+\delta_2,\mu_3)+V^{\frac{1}{2}}(\mu_1,\mu_2-\delta_2,\mu_3)\right)-2V^{\frac{1}{2}}(\mu_1-\delta_1,\mu_2,\mu_3)\\
&\times\Bigg(V^{\frac{1}{2}}(\mu_1+\delta_1,\mu_2,\mu_3)+V^{\frac{1}{2}}(\mu_1-\delta_1,\mu_2,\mu_3)\\
&+V^{\frac{1}{2}}(\mu_1,\mu_2+\delta_2,\mu_3)+V^{\frac{1}{2}}(\mu_1,\mu_2-\delta_2,\mu_3)\Bigg)\,. \qquad (4.67)
\end{aligned}
$$

4.6 Lattice refinement

So far, we have left the increments δ_1 and δ_2 unspecified. It is clear that as constants they would not influence the recurrence behavior of the difference equation, although specific solutions certainly depend on their values. However, in general δ_1 and δ_2 may not be constant but be functions of μ_1 and μ_2; this captures the way in which the discrete structure of a state underlying spatial expansion and contraction in loop quantum gravity is being refined dynamically [78, 87, 118]: at larger μ_I, an increment of the total size by a Planck-scale amount has a weaker relative influence on the geometry. As a consequence, δ_I decrease with increasing spatial extensions. This can also be seen from more direct considerations of holonomies in inhomogeneous states and how they appear in Hamiltonian constraint operators. Since this involves the dynamical relation between models and a full non-symmetric theory, the precise behavior of lattice refinement has not been completely determined. (Since the parameters δ_I are related to edge lengths of spin network states, a derivation would require one to construct specific inhomogeneous states which correspond to an anisotropic geometry. Then, the action of a full Hamiltonian constraint operator would have to be projected to the homogeneous states; see [87] for further details.) However, consequences of different behaviors can be explored in several models. Sometimes, this is already quite restrictive even though it is impossible to derive a unique form of lattice refinement based solely on homogeneous models.

Non-trivial functions, such as power laws, have a much stronger influence than constants because they make the difference equation non-equidistant. Solutions are then more difficult to analyze and find, even numerically (but see [89, 90]). Only in the special cases where $\delta_1 \propto \mu_1^{x_1}$ and $\delta_2 \propto \mu_2^{x_2}$ can the equation be mapped to an equidistant one by a redefinition of independent variables. However, such cases have been ruled out [91] because they do not provide the correct semiclassical behavior near a horizon of Schwarzschild black holes, whose interior is treated as a homogeneous Kantowski–Sachs model. (The analysis in [91] uses corrections to classical equations due to the use of holonomies in the loop quantization, but ignores other effects such as quantum back-reaction [55, 56]. This type of phenomenological equations may not capture correctly the behavior of strong quantum regimes such as the black hole singularity. However, if these equations do not provide the correct semiclassical behavior in classical regimes, this cannot be corrected by the inclusion of quantum back-reaction. The fact that some refinement schemes are ruled out is thus a reliable feature.) In general, one has to expect functions of the form $\delta_1(\mu_1, \mu_2)$ and $\delta_2(\mu_1, \mu_2)$ with a non-trivial dependence on both arguments (which may not be of power-law form).

As we will see below, a discussion of fundamental singularity resolution only involves the recurrence near $\mu_1 = 0$. This is, fortunately, insensitive to the particular refinement scheme and thus presents a result of much wider generality than anything which applies at larger volume where the specific refinement can be crucial. The constructions and results of this article are thus valid for any functional behavior of the μ_I on phase space variables, even though finding explicit or numerical solutions to the difference equation would be more complicated in general.

4.7 Cosmological Implications

It follows immediately from the difference equation (4.43) or (4.44) that it is parity invariant since all its terms change sign under (4.33). Thus, if $s_{\mu_1,\mu_2}(\Theta_1, \Theta_2, \Theta_3, \Theta_4)$

is a solution, so is $s_{-\mu_1,\mu_2}(\Theta_3,\Theta_4,\Theta_1,\Theta_2)$. In particular, any solution can be written as a combination of even and odd solutions $s_{\mu_1,\mu_2}(\Theta_1,\Theta_2,\Theta_3,\Theta_4)\pm s_{-\mu_1,\mu_2}(\Theta_3,\Theta_4,\Theta_1,\Theta_2)$. This is no longer the case if we had matter interactions violating parity, such as a term proportional to $\mathcal{V}_0\mathcal{J}_0$. In this case, no parity-even or odd solutions would exist. Wave functions for $\mu_1>0$ generically differ from their form for $\mu_1<0$, even though those values are deterministically related via the difference equation. At this stage, the precise form of parity violations in the matter system is crucial to determine the behavior of the wave function near the classical singularity at $\mu_1=0$.

To complete the construction, one would solve the difference equation and determine a physical inner product on the solution space. Ideally, one could then compute the behavior of observables of the system and derive detailed cosmological scenarios including the role of quantum effects. Unfortunately, such complete descriptions at an exact level are possible only in rare, specific models. While such models are instructive mathematically, conclusions drawn are difficult to interpret because one could not be certain about the robustness of results: If specific results are available only in a few special models where exact mathematical solutions in the physical Hilbert space can be found, there is no guarantee that they are not just the very result only of demanding this high mathematical control.

In this context, an aspect of particular interest is the fact that most models of loop quantum cosmology where physical Hilbert spaces have been constructed explicitly [75, 92, 93] specifically assume parity invariance in some form and make use of the corresponding restriction of states when parity is considered as a large gauge transformation. As we have seen here, physical states of quantum cosmology are neither even nor odd in triad reflections if parity violating matter is present. It may thus be misleading to treat parity as a large gauge transformation even in cases where matter preserves parity. Results based on this assumption may be spurious, and one has to re-analyze the constructions of physical Hilbert spaces without the assumption of parity invariant states. Fortunately, the intuitive pictures of bounces which have sometimes been derived from physical observables are insensitive to the specific construction of the physical Hilbert space: They can be derived analytically in a representation independent formalism based on effective equations [69, 94]. Then, the assumption of parity as a large gauge transformation is not necessary, and it can be dropped without affecting the bounce result.

At a fundamental level, singularity resolution is also insensitive to the physical Hilbert space construction and can directly be determined using the difference equation (4.43) or (4.44). (Here, it is important that *all* solutions are non-singular, which then also includes physical ones.) In general, coefficients of a difference equation of the type obtained in loop quantum cosmology may vanish and prevent certain values of s_{μ_1,μ_2} from being determined in a recurrence starting from initial values. This happens for the non-symmetric equation (4.43) where none of the values ψ_{0,μ_2} — right at the classical singularity — is determined by initial values because their coefficients in the difference equation vanish. (The corresponding states $|0,\mu_2\rangle$ are mantic [65].) However, for the difference equations realized such undetermined values, if they arise, drop out completely of the recurrence. In particular, even though values for $\mu_1 = 0$ remain undetermined by initial values in the non-symmetrized version of the equation, coefficients at $\mu_1 < 0$ follow deterministically from coefficients at $\mu_1 > 0$.

In parity preserving models the wave function s_{μ_1,μ_2} for $\mu_1 < 0$ could simply be the mirror image of its cousin at $\mu_1 > 0$, and it had to be symmetric if parity is considered a large gauge transformation. However, if there is parity violation, the transition through $\mu_1 = 0$ constitutes true evolution since values at $\mu_1 < 0$ must now differ from the mirror image at $\mu_1 > 0$. The wave function at $\mu_1 < 0$ cannot be determined simply by reflection, but it has to be derived by local evolution through all intermediate values of μ_1. In this case, the region of $\mu_1 < 0$ can by no means be removed from considerations but must be considered as a physical domain on equal footing with that at $\mu_1 > 0$. In particular, the orientation-reversing big bang transition thus becomes physical and cannot be argued away as a large gauge transformation.

For both forms of difference equations derived here, there are consistency conditions arising due to vanishing coefficients around $\mu_1 = 0$, analogous to dynamical initial conditions [70, 71]. If we evaluate any of the difference equations at $\mu_1 = 0$, matter terms drop out and we obtain the universal relation

$$\begin{aligned}&(|\mu_2 + 3\delta_2| - |\mu_2 + \delta_2|)\, s_{2\delta_1,\mu_2+2\delta_2} - (|\mu_2 - \delta_2| - |\mu_2 - 3\delta_2|)\, s_{2\delta_1,\mu_2-2\delta_2} \\ &= (|\mu_2 + 3\delta_2| - |\mu_2 + \delta_2|)\, s_{-2\delta_1,\mu_2+2\delta_2} - (|\mu_2 - \delta_2| - |\mu_2 - 3\delta_2|)\, s_{-2\delta_1,\mu_2-2\delta_2} \quad (4.68)\end{aligned}$$

valid for all μ_2. In particular, at $\mu_2 = 2\delta_2$ we have $s_{2\delta_1,4\delta_2} = s_{-2\delta_1,4\delta_2}$. At odd integer multiples of $\mu_2 = 2\delta_2$, we obtain a recurrence relation which requires $s_{2\delta_1,2(2n+1)\delta_2} = s_{-2\delta_1,2(2n+1)\delta_2}$ for all integer n.

There are thus reflection symmetry conditions which directly follow from the dynamical law even in the presence of parity-violating terms. (This symmetry has been observed first in the vacuum case [95].) However, evolution away from $\mu_1 = \pm 1$ depends on whether μ_1 is positive or negative if parity is not preserved: Unlike Eq. (4.68), the matter Hamiltonian then enters the recurrence and for parity violating matter the coefficients of the difference equation at negative and positive μ_1 differ. Thus, the wave function is not mirror symmetric even though the dynamical initial condition closely ties the values $s_{\pm 2\delta_1,\mu_2}$ to each other.

4.8 Conclusion

We have introduced fermions into the framework of loop quantum cosmology which gave rise to several non-trivial changes due to the presence of torsion and potential parity non-invariance. We have observed several key features which have a bearing on cosmological scenarios and which do not arise for bosonic matter such as scalar fields as they are used commonly in cosmological models. First, the amount of matter is limited for each fermionic degree of freedom due to the exclusion principle. Thus, large matter contents as they are sometimes used to bring a quantum cosmological model into a semiclassical regime where it may bounce more easily cannot straightforwardly be achieved. The only possibilities are to allow many copies of independent fermions or inhomogeneity where fermionic components at different points will be independent. Physically, both possibilities are quite different from having a single bosonic field of high occupation. The methods used here may also be of interest for a supersymmetric version of loop quantum cosmology along, e.g., the lines of [96] (see also [97]). Fermions in quantum cosmology also play a role for decoherence [98].

This shows that it is crucial to consider the small-volume regime of a quantum cosmological model which cannot be avoided in the absence of much matter energy. Here, the recurrence scheme of an underlying difference equation of loop quantum cosmology becomes essential to determine whether the model is singular or not. As we showed, the singularity resolution mechanism of loop quantum cosmology [65] remains unchanged under the inclusion of fermionic matter even if it violates parity. At the same time, the model we used allows us to show that in its realm parity violations can only arise due to matter interactions, not due to pure gravity. In other models or the full theory, this situation may be different because the basic objects quantized, in particular holonomies, do not transform straightforwardly under parity. The model introduced here thus also serves the purpose of providing one example where parity invariance of pure gravity can be demonstrated after a loop quantization.

If one introduces parity-violating interactions, wave functions cannot be mirror symmetric. Then, the branches at the two opposite orientations of triads are independent of each other, and joined through degenerate geometries by the dynamics of loop quantum cosmology. The big bang transition now becomes a non-trivial event where space turned its inside out in a quantum process which in general cannot be described by an intuitive geometrical picture such as a simple bounce.

CHAPTER V

BIG BANG NUCLEOSYNTHESIS AND LQG

After the discussion of the role of matter fields in loop quantum cosmology in the previous chapter, an application to mainstream cosmology is much desired at this stage. The effects discrete geometry and different matter ingredients on Big Bang Nucleosynthesis (BBN) may be considered one such application, which is presented in this chapter.

In theoretical cosmology, many insights can already be gained from spatially isotropic Friedmann–Robertson–Walker models

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + a(\tau)^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2) \right) \tag{5.1}$$

with $k = 0$ or ± 1. The matter content in such a highly symmetric space-time can only be of the form of a perfect fluid with stress-energy tensor $T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b)$ where ρ is the energy density of the fluid, P its pressure and u^a the 4-velocity vector field of isotropic co-moving observers. Once an equation of state $P = P(\rho)$ is specified to characterize the matter ingredients, the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$ with the Hubble parameter $H = \dot{a}/a$ allows one to determine the behavior of $\rho(a)$ in which energy density changes during the expansion or contraction of the universe. This function, in turn, enters the Friedmann equation $H^2 + k/a^2 = 8\pi G\rho/3$ and allows one to derive solutions for $a(\tau)$.

In general, one would expect the equation of state $P = P(\rho)$ to be non-linear which would make an explicit solution of the continuity and Friedmann equations difficult. It is thus quite fortunate that in many cases linear equations of state $P = w\rho$ with w constant are sufficient to describe the main matter contributions encountered in cosmology at least phenomenologically. The influence of compact objects on cosmological scales is, for instance, described well by the simple dust equation of state

$P(\rho) = 0$. Relativistic matter, mainly electromagnetic radiation, satisfies the linear equation of state $P = \frac{1}{3}\rho$. The latter example is an exact equation describing the Maxwell field, rather than an approximation for large scale cosmology. It is thus, at first sight, rather surprising that the dynamics of electromagnetic waves in a universe can be summarized in such a simple equation of state irrespective of details of the field configuration. The result follows in the standard way from the trace-freedom of the electromagnetic stress-energy tensor and is thus related to the conformal symmetry of Maxwell's equations. That the availability of such a simple equation of state is very special for a matter field can be seen by taking the example of a scalar field ϕ with potential $V(\phi)$. In this case, we have an energy density $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and pressure $P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. Unless the scalar is free and massless, $V(\phi) = 0$ for which we have a stiff fluid $P = \rho$, there is no simple relation between pressure and energy density independently of a specific solution.

Often, one can assume the equation of state parameter w to be constant during successive phases of the universe evolution, with sharp jumps between different phases such as $w = -1$ during inflation, followed by $w = \frac{1}{3}$ during radiation domination and $w = 1$ during matter domination. Observationally relevant details can depend on the precise values of w at a given stage, in particular if one uses an effective value describing a mixture of different matter components. For instance, during big bang nucleosynthesis one is in a radiation dominated phase mainly described by photons and relativistic fermions. Unlike photons, the general equation of state for fermions is more complicated and non-linear, but can in relativistic regimes be approximately given by the same value $w = \frac{1}{3}$ as for photons. In contrast to the case of Maxwell theory, however, there is no strict symmetry such as conformal invariance which would prevent w to take a different value. It is one of the main objectives of the present chapter to discuss possible corrections to this value.

For big bang nucleosynthesis, it turns out, the balance between fermions and photons is quite sensitive. In fact, different values for the equation of state parameters might even be preferred phenomenologically [99]. One possible reason for different equations of state could be different coupling constants of bosons and fermions to gravity, for which currently no underlying mechanism is known. In this chapter, we will explore the possibility whether quantum gravitational corrections to the equations of state can produce sufficiently different values for the equation of state parameters. In fact, since the fields are governed by different actions, one generally expects different, though small, correction terms which can be of significance in a delicate balance.

Note that we are not discussing ordinary quantum corrections of quantum fields on a classical background. Those are expected to be similar for fermions and radiation in relativistic regimes. We rather deal with quantum gravity corrections in the coupling of the fields to the space-time metric, about which much less is known a priori. Thus, different proposals of quantum gravity may differ at this stage, providing possible tests.

An approach where quantum gravitational corrections can be computed is loop quantum cosmology [51]. In such a canonical quantization of gravity, equations of state must be computed from matter Hamiltonians rather than covariant stress-energy tensors. Quantum corrections to the underlying Hamiltonian then imply corrections in the equation of state. This program has been carried out here both for the Maxwell Hamiltonian and Dirac fermions [40, 100]. There are several differences between the treatment of fermions and other fields, which from the gravitational point of view are mainly related to the fact that fermions, in a first order formulation, also couple to torsion and not just the curvature of space-time. After describing the classical derivation of equations of state as well as steps of a loop quantization and its correction terms, we use big bang nucleosynthesis constraints to see how sensitively we can bound quantum gravity parameters. We will start with general remarks on the physics underlying the problem.

5.1 The physical setting

Big bang nucleosynthesis happens at energy scales $E_{\mathrm{BBN}} \sim \mathrm{MeV}$ which are large, but still tiny compared to the Planck energy M_{P}. Also the universe has already grown large compared to the Planck length ℓ_{P} at this stage, and space-time curvature is small. One may thus question why quantum gravity should play any role. There is certainly a fine balance required for successful big bang nucleosynthesis, but the expected quantum gravity terms of the order E/M_{P}, obtained based on dimensional arguments, would have no effect.

However, dimensional arguments do not always work, in particular if more than two parameters L_I of the same dimension, or any large dimensionless numbers are involved. Then, precise calculations have to be done to determine which geometric means $\prod_I L_I^{x_I}$ with $\sum_I x_I = 0$ may appear as coefficients, or which powers of dimensionless numbers occur as factors of correction terms. In loop quantum gravity, we are in such a situation: there is the macroscopic length scale L, which in our case we can take as the typical wave length of fields during nucleosynthesis, and also the Planck length $\ell_{\mathrm{P}} = \sqrt{G\hbar}$ which arises due to the presence of Newton's constant G and Planck's constant $\hbar$. In addition, there is a third and in general independent scale ℓ given by the microscopic size of elementary spatial patches in a quantum gravity state. This is a new feature of the fundamentally discrete theory, for which the precise state of quantum gravity plays an important role. Although ℓ must be proportional to the Planck length, its specific value for a given state can differ numerically. Then, a detailed calculation must show how L, ℓ and ℓ_{P} appear in quantum gravity corrections and which numerical values may arise.

Alternatively, one can work with only two length scales, L and ℓ_{P}, but one has to deal with a large dimensionless parameter $\mathcal{N}$ given by the number of discrete patches of the underlying state in the volume considered, for instance a volume of the size L^3 such that $\mathcal{N} = L^3/\ell^3$. Examples of cosmological phenomena are known where this does play a role for quantum gravity corrections [115, 116], and here we analyze which features arise in the presence of fermions and especially for big bang nucleosynthesis.

There are precedents where such considerations have played important roles. Best known is the evidence for the atomic nature of matter derived by Einstein from the phenomenon of Brownian motion. Also here, there are several orders of magnitude between the expected size of molecules and the resolution of microscopes at that time. However, there is also a large number of molecules which by their sheer number can and do leave sizeable effects on much larger suspended particles. There is, of course, never a guarantee that something analogous has to happen elsewhere. But this is to be checked by calculations and cannot always be ruled out based only on dimensional arguments.

The corrections to the equation of state for radiation and fermions must be derived to check any effects on BBN. Even in relativistic regimes, the coupling of fermions to gravity differs from other fields, e.g. by torsion contributions which arise already from the kinetic term of the Dirac action. One could thus expect that quantum corrections for fermions differ from those to radiation and thus, by throwing off the balance during nucleosynthesis, possibly enhance the effect of quantum gravity corrections. Whether or not this happens cannot be decided without detailed calculations as they are reported and applied here.

5.2 Equation of state

Let us first derive the equation of state for radiation followed by that for fermions. The derivations for both species are exactly the same in the canonical analysis, except with different Hamiltonian.

Radiation

One can determine energy and pressure from our canonical expressions (see also [54]) in order to formulate the equation of state. The matter Hamiltonian is directly related to energy density[1] by

$$\rho = \frac{1}{\sqrt{q}} \frac{\delta H_M}{\delta N}, \tag{5.2}$$

and thus, from equation (3.31), it is

$$\rho = \frac{2\pi}{q} \pi^a \pi^b q_{ab} + \frac{1}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd}. \tag{5.3}$$

The canonical formula for pressure is given by

$$P = -\frac{2}{3N\sqrt{q}} q_{ab} \frac{\delta H_M}{\delta q_{ab}} = \frac{2}{3N\sqrt{q}} q^{ab} \frac{\delta H_M}{\delta q^{ab}} \tag{5.4}$$

as shown in Appendix 0.3. This gives

$$\begin{aligned} P &= \frac{2}{3N\sqrt{q}} q^{ef} \left(\frac{\pi N}{\sqrt{q}} \pi^a \pi^b \left(q_{ab} q_{ef} - 2 q_{ae} q_{bf} \right) + \frac{\sqrt{q} N}{8\pi} q^{ac} F_{ae} F_{cf} - \frac{\sqrt{q} N}{32\pi} F_{ab} F^{ab} q_{ef} \right) \\ &= \frac{2}{3N\sqrt{q}} \left(\frac{\pi N}{\sqrt{q}} \pi^a \pi^b q_{ab} + \frac{\sqrt{q} N}{32\pi} F_{ab} F^{ab} \right) = \frac{1}{3} \left[\frac{2\pi}{q} \pi^a \pi^b q_{ab} + \frac{1}{16\pi} F_{ab} F^{ab} \right]. \end{aligned} \tag{5.5}$$

Finally, the equation of state can easily be obtained from (5.3) and (5.13):

$$w = \frac{P}{\rho} = \frac{1}{3} \tag{5.6}$$

which is the standard result.

Fermions

Let us first write the fermion dependent terms, i.e. the Dirac Hamiltonian, resulting from the gravitational action non-minimally coupled to fermions. It follows from (3.6) that the Dirac Hamiltonian is given by derivative terms and self-interaction

[1] This is the usual term for energy per volume, and does not mean that ρ is a geometrical density.

terms:

$$
\begin{aligned}
H_{\mathrm{Dirac}} \;=\; & \int_{\Sigma_t} \mathrm{d}^3x N \Bigg(-\frac{\beta E_i^a}{\sqrt{q}} \mathcal{D}_a \left(\pi_\xi^T \tau^i \xi + \pi_\chi^T \tau^i \chi \right) \\
& -i \frac{2E_a^i}{\sqrt{q}} \left(\theta_L \pi_\xi^T \tau^i \mathcal{D}_a \xi - \theta_R \pi_\chi^T \tau^i \mathcal{D}_a \chi - c.c. \right) \\
& + \frac{\gamma\kappa\beta}{2\sqrt{q}(1+\gamma^2)} \left(3 - \frac{\gamma}{\alpha} + 2\gamma^2 \right) (\pi_\xi^T \tau_l \xi + \pi_\chi^T \tau_l \chi)(\pi_\xi^T \tau^l \xi + \pi_\chi^T \tau^l \chi) \\
& + \frac{3\gamma\kappa}{8\alpha\sqrt{q}} (1 - \frac{\gamma}{\alpha})(\pi_\xi^T \xi - \pi_\chi^T \chi)(\pi_\xi^T \xi - \pi_\chi^T \chi) \Bigg)
\end{aligned}
\tag{5.7}
$$

again with $\beta := \gamma + \frac{1}{\alpha}$. The top line of this expression is the most important one because its derivative terms are dominant in relativistic regimes. In addition to those, we highlight the presence of four-fermion interactions in the second line, which we summarize as

$$
\begin{aligned}
B \;:=\; & \frac{\gamma\kappa\beta}{2(1+\gamma^2)} \left(3 - \frac{\gamma}{\alpha} + 2\gamma^2 \right) (\pi_\xi^T \tau_l \xi + \pi_\chi^T \tau_l \chi)(\pi_\xi^T \tau^l \xi + \pi_\chi^T \tau^l \chi) \\
& + \frac{3\gamma\kappa}{8\alpha} (1 - \frac{\gamma}{\alpha})(\pi_\xi^T \xi - \pi_\chi^T \chi)(\pi_\xi^T \xi - \pi_\chi^T \chi)
\end{aligned}
\tag{5.8}
$$

as it multiplies $q^{-1/2}$. Thus, from (6.33) and (5.7), the energy density is

$$
\begin{aligned}
\rho \;=\; & \frac{2E_i^a}{q} \left(-\frac{\beta}{2} \mathcal{D}_a \left(\pi_\xi^T \tau^i \xi + \pi_\chi^T \tau^i \chi \right) + i \left(-\theta_L \pi_\xi^T \tau^i \mathcal{D}_a \xi + \theta_R \pi_\chi^T \tau^i \mathcal{D}_a \chi - c.c. \right) \right) \\
& + \frac{B}{q}
\end{aligned}
\tag{5.9}
$$

The canonical formula for pressure is

$$
P = -\frac{2}{3N\sqrt{q}} E_i^a \frac{\delta H_{\mathrm{Dirac}}}{\delta E_i^a} \tag{5.10}
$$

as shown by a straightforward adaptation of the calculation done in [100] for metric variables. Now using the functional derivative

$$
\frac{\delta \sqrt{q(x)}}{\delta E_i^a(y)} = \frac{1}{2} e_a^i \delta(x-y) \,, \tag{5.11}
$$

and thus

$$
\frac{\delta}{\delta E_j^b(y)} \left(\frac{2E_i^a(x)}{\sqrt{q(x)}} \right) = \frac{1}{\sqrt{q}} (2\delta_b^a \delta_i^j - e_i^a e_b^j) \delta(x-y) \,, \tag{5.12}
$$

and inserting (5.12) in (5.10), we obtain the pressure

$$P = \frac{2E_i^a}{3q}\left(-\frac{\beta}{2}\mathcal{D}_a\left(\pi_\xi^T\tau^i\xi+\pi_\chi^T\tau^i\chi\right)+i\left(-\theta_L\pi_\xi^T\tau^i\mathcal{D}_a\xi+\theta_R\pi_\chi^T\tau^i\mathcal{D}_a\chi-c.c.\right)\right) + \frac{B}{q} \tag{5.13}$$

This results in an equation of state

$$w_{\mathrm{Dirac}} = \frac{P}{\rho} = \frac{1}{3} - \frac{2B}{3\rho}\,. \tag{5.14}$$

In relativistic regimes, the kinetic term involving partial derivatives ∂_a contained in $\mathcal{D}_a$ is dominant, which leaves us with an equation of state

$$w = \frac{P}{\rho} = \frac{1}{3} + \epsilon \tag{5.15}$$

whose leading term agrees with the parameter for a Maxwell field. But there are clearly correction terms for fermions already in the classical first order theory. They do not arise for the Maxwell field, implying a difference in the coupling to gravity due to torsion, which is present even in relativistic regimes. The order of magnitude of the additional term depends on the fermion current density and is thus not expected to be large unless regimes are very dense. We will not consider this correction further in this article, but highlight its role as a consequence of torsion. Next, we discuss the quantum corrections arising from the discrete geometry for each of these matter fields.

5.3 Quantum Corrections

Being interested in effects from quantum gravity, we have to quantize the gravitational components such as metric q_{ab}, densitized triad P_i^a, and $\sqrt{q}$ in the matter Hamiltonians, not just the matter fields themselves. As noted in Chapter I, an immediate consequence of loop quantization is that fluxes and spatial geometrical operators such as area and volume [16, 17, 18] have discrete spectra containing zero. Hence, their inverses do not exist as densely defined operators. However, a quantization of the matter Hamiltonian such as (3.31) and (5.7) demands the quantization of such inverse expressions since, e.g., $q^{-\frac{1}{2}}$ or the metric q_{ab} which can only be obtained by inverting the densitized triad, appear in the matter Hamiltonian. Therefore, the quantization of the matter Hamiltonians seem, at first, to be seriously problematic. However, as shown in Chapter III, a well-defined quantization is possible after noticing that the Poisson bracket of the volume with connection components,

$$\left\{A_a^i, \int \sqrt{|\det E|}d^3x\right\} = 2\pi\gamma G\epsilon^{ijk}\epsilon_{abc}\frac{E_j^b E_k^c}{\sqrt{|\det E|}} = 4\pi\gamma G e_a^i\,, \tag{5.16}$$

amounts to an inverse of densitized triad components [21]. Thus, the gravitational components in the matter Hamiltonians can be quantized using the techniques of loop quantization. Leading to well-defined operators, this quantization process implies characteristic modifications of the classical expressions such as (3.31) and (5.7) on small scales, where densitized triad components are small. Moreover, since there are many different but classically equivalent ways to rewrite expressions like (5.16) for which the quantization would give different results, there are quantization ambiguities. However, several characteristic effects occur for any quantization choice such that they can be studied reliably with phenomenological applications in mind.

In loop quantum gravity, there are three main effects which imply correction terms in effective matter equations. Despite a proper quantization of the matter Hamiltonians, along the lines of [25], that gives a well-defined operator with the correct semiclassical limit, there are deviations from the classical behavior on small length scales, which are the first source of correction terms.

In addition, there are qualitatively different correction terms. First, loop quantum gravity is spatially discrete, with states supported on spatial graphs. Quantizations of Hamiltonians thus lead to a discrete representation of any spatial derivative term as they also occur for fermions. The classical expression arises in a continuum limit, but for any given state the discrete representation implies corrections to the classical derivatives as the leading terms in an expansion. Secondly, the connection is quantized through holonomies rather than its single components. Thus, the quantum Hamiltonians are formulated in terms of exponentials of line integrals of the connection which also give the leading classical term plus corrections in an expansion. Finally, whenever a Hamiltonian is not quadratic, there are genuine quantum effects as they occur in typical low energy effective actions. They can be computed in a Hamiltonian formulation as well [55, 56], contributing yet another source of corrections.

One certainly needs to know the relative magnitude of all corrections in order to see which ones have to be taken into account. For all of them, the magnitude depends on details of the quantum state describing the regime. Here, properties of states have to be taken into account, and dimensional arguments are no longer sufficient. For instance, discretization and curvature corrections depend on the patch size occurring in the discrete state underlying quantum gravity. This patch size is typically small compared to scales on which the matter field changes, even in relativistic regimes assumed here. Thus, such corrections can be ignored in a first approximation. What remains are corrections from inverse powers. While other corrections shrink in the continuum limit where the patch size becomes small, inverse corrections actually grow when the patch size approaches the Planck length. The regimes where the two classes of corrections are dominant are thus neatly separated, and we can safely focus on inverse triad corrections only. A brief summary of the origin of such corrections is presented below ; see also [57]. A detailed and complete derivation is not yet available since precise properties of a quantum gravity state would be required. Still, many general qualitative insights can be gained in this way.

Perturbative loop quantum cosmology

Hamiltonian operators of a quantum theory can, in semiclassical regimes, be approximated by effective expressions which amend the classical ones by quantum correction terms. The general procedure, detailed in [55, 56], requires one to evaluate expectation values of the Hamiltonian in suitable semiclassical states. A crucial ingredient in loop quantum gravity is the discrete, non-local nature of states written in terms of holonomies as basic objects. Although Hamiltonian operators on such discrete lattice states are quite complicated, expectation values can often be evaluated explicitly in perturbative regimes where one assumes the geometry to be close to a symmetric one. This is certainly allowed in our applications to derive the effective equation of state of radiation in a flat FRW universe. The background symmetry implies the existence of three approximate spatial Killing vector fields X_I^a generating transitive isometries. We will only make use of this translational symmetry, not of the additional rotations in the construction of states. These vector fields can be used as a tangent space basis, thus denoting tensor indices for components in this basis by capital letters $I, J, \ldots$

The background symmetry also has implications for the selection of states of the quantum theory. A general quantization has to consider arbitrary states, but for effective equations one computes expectation values only in states suitable for a semiclassical regime. For perturbative inhomogeneities, one can restrict lattices as they occur in general graphs to regular cubic ones and thus simplify geometrical operators. This has been developed recently in [57] for metric perturbations as well as for a scalar field, and we can directly apply the same techniques to the Maxwell and the Dirac Hamiltonian. We refer the reader to this paper for more details.

Gravitational variables and lattice states

In a perturbative regime around a spatially flat isotropic solution, one can choose the canonical variables to be given by functions $(\tilde{p}^I(x), \tilde{k}_J(x))$ which determine a densitized triad by $E^a_i = \tilde{p}^{(i)}(x)\delta^a_i$ and extrinsic curvature by $K^i_a = \tilde{k}_{(i)}(x)\delta^i_a$. Thus, one can diagonalize the canonical variables compared to the general situation where all matrix elements of E^a_i and K^i_a would be independent. As seen in many symmetric models, this simplifies the calculations considerably: it allows one to replace involved SU(2) calculations by much simpler U(1) calculations [58, 59]. SU(2) matrices arise because loop quantum gravity is based on holonomies $h_e = \mathcal{P}\exp(\int_e \mathrm{d}t\dot{e}^a A^i_a \tau_i)$ of a connection A^i_a related to extrinsic curvature. For unrestricted connections, holonomies can take any SU(2) value, but a diagonalization implies that all quantities can be reduced to a maximal Abelian subgroup U(1). Matrix elements of Hamiltonians and other operators can then be computed in explicit form.

Using properties of the general loop representation mentioned before, basic variables of the quantum theory are, for a chosen lattice, U(1) elements $\eta_{v,I}$ attached to a lattice link $e_{v,I}$ starting at a vertex v and pointing in direction X^a_I, and their conjugate fluxes $F_{v,I}$. The U(1) elements $\eta_{v,I}$ appear as matrix elements in SU(2) holonomies $h_{v,I} = \mathrm{Re}\eta_{v,I} + 2\tau_I \mathrm{Im}\eta_{v,I}$ along edges $e_{v,I}$. Following the construction of the Hilbert space using holonomies as "creation" operators by acting on a state which is constant on the space of connections, a general state is a functional $|\ldots,\mu_{v,I},\ldots\rangle = \prod_{v,I} \eta_{v,I}^{\mu_{v,I}}$. Allowing all possible values of assignments of integers $\mu_{v,I} \in \mathbb{Z}$ to the lattice edges $e_{v,I}$, this defines an orthonormal basis of the Hilbert space. Basic operators are represented as holonomies

$$\hat{\eta}_{v,I}|\ldots,\mu_{v',J},\ldots\rangle = |\ldots,\mu_{v,I}+1,\ldots\rangle \tag{5.17}$$

for each pair (v,I) where all labels other than $\mu_{v,I}$ remain unchanged, and fluxes

$$\hat{\mathcal{F}}_{v,I}|\ldots,\mu_{v',J},\ldots\rangle = 2\pi\gamma\ell_{\mathrm{P}}^2(\mu_{v,I}+\mu_{v,-I})|\ldots,\mu_{v',J},\ldots\rangle \tag{5.18}$$

where $\ell_{\rm P} = \sqrt{\hbar G}$ is the Planck length and a subscript $-I$ means that the edge preceding the vertex v in the chosen orientation is taken. These and the following constructions are explained in more detail in [57].

Effective equations are obtained by taking expectation values of the Hamiltonian operator and computing a continuum approximation of the result (similar to a derivative expansion in low energy effective actions). The result is a local field theory which includes quantum corrections. This is done by relating holonomies

$$\eta_{v,I} = \exp(i \int_{e_{v,I}} {\rm d}t \gamma \tilde{k}_I/2) \approx \exp(i\ell_0\gamma\tilde{k}_I(v+I/2)/2) \tag{5.19}$$

to continuum fields $\tilde{k}_I$ through mid-point evaluation on the edges $e_{v,I}$ (denoted by an argument $v+I/2$ of the fields), and similarly for fluxes

$$F_{v,I} = \int_{S_{v,I}} \tilde{p}^I(y){\rm d}^2y \approx \ell_0^2\tilde{p}^I(v+I/2)\,. \tag{5.20}$$

Although the non-local basic objects do not allow us to define continuum fields at all spatial points, in a slowly-varying field approximation the mid-point evaluations are sufficient to define the continuum fields by interpolation. Here, ℓ_0 is the coordinate length of lattice links. It does not appear in the quantum theory which only refers to states and their labels $\mu_{v,I}$. This is independent of coordinates and only makes use of an abstract, labelled graph. The parameter ℓ_0 only enters in the continuum approximation since it is classical fields which are integrated and related to holonomies and fluxes. These continuum fields, or tensor components $\tilde{p}^I$ and $\tilde{k}_I$, must depend on which coordinates are chosen to represent them. For the situation given here, the combinations $p^I := \ell_0^2\tilde{p}^I$ and $k_I := \ell_0\tilde{k}_I$, as they appear in holonomies and fluxes evaluated for slowly-varying fields, are coordinate independent.

A further operator we can immediately define is the volume operator. Using the classical expression $V = \int {\rm d}^3x\sqrt{|\tilde{p}^1\tilde{p}^2\tilde{p}^3|} \approx \sum_v \ell_0^3\sqrt{|\tilde{p}^1\tilde{p}^2\tilde{p}^3|} = \sum_v\sqrt{|p^1p^2p^3|}$, we introduce the volume operator $\hat{V} = \sum_v\prod_{I=1}^3\sqrt{|\hat{\mathcal{F}}_{v,I}|}$ which, using (5.18), has eigenvalues

$$V(\{\mu_{v,I}\}) = \left(2\pi\gamma\ell_{\rm P}^2\right)^{3/2}\sum_v\prod_{I=1}^3\sqrt{|\mu_{v,I}+\mu_{v,-I}|}\,. \tag{5.21}$$

This operator is not only interesting for geometrical purposes, but also for making use of the identity (5.16) or, more generally,

$$\{A_a^i, V_v^r\} = 4\pi\gamma G\, rV_v^{r-1}e_a^i \tag{5.22}$$

which gives inverse powers of the densitized triad for any $0 < r < 2$ often appearing in matter Hamiltonians. When quantizing this expression using holonomies, the volume operator and a commutator for the Poisson bracket, we obtain

$$\begin{aligned}\widehat{V_v^{r-1}e_I^i} &= \frac{-2}{8\pi ir\gamma\ell_{\rm P}^2\ell_0}\sum_{\sigma\in\{\pm1\}}\sigma\,{\rm tr}(\tau^i h_{v,\sigma I}[h_{v,\sigma I}^{-1},\hat{V}_v^{\,r}])\\ &= \frac{1}{2\ell_0}(\hat{B}_{v,I}^{(r)}-\hat{B}_{v,-I}^{(r)})\delta_{(I)}^i =: \frac{1}{\ell_0}\hat{C}_{v,I}^{(r)}\,.\end{aligned} \tag{5.23}$$

For symmetry, we use both edges $e_{v,I}$ and $e_{v,-I}$ touching the vertex v along direction X_I^a. The operator $\hat{B}_{v,I}^{(r)}$ is obtained by taking the trace in (5.23) and using $h_{v,I} = {\rm Re}\eta_{v,I} + 2\tau_I{\rm Im}\eta_{v,I}$,

$$\hat{B}_{v,I}^{(r)} := \frac{1}{4\pi i\gamma G\hbar r}\left(s_{v,I}\hat{V}_v^r c_{v,I} - c_{v,I}\hat{V}_v^r s_{v,I}\right) \tag{5.24}$$

with

$$c_{v,I} = \frac{1}{2}(\eta_{v,I}+\eta_{v,I}^*) \quad \text{and} \quad s_{v,I} = \frac{1}{2i}(\eta_{v,I}-\eta_{v,I}^*)\,.$$

Maxwell Hamiltonian

Such expressions can be used for the electric field part of the Maxwell Hamiltonian (3.31) where the metric factor to be quantized is

$$\frac{q_{ab}}{\ell_0\sqrt{q}} = \frac{e_a^ie_b^i}{\ell_0\sqrt{q}} \approx \frac{\ell_0^2e_a^ie_b^i}{V_v}$$

in terms of the volume $V_v \approx \ell_0^3\sqrt{q(v)}$ of a lattice site. This can then be quantized, using (5.23) with $r = 1/2$, to

$$\widehat{\frac{q_{IJ}}{\ell_0\sqrt{q}}} = (\ell_0\widehat{V_v^{-1/2}e_I^i})(\ell_0\widehat{V_v^{-1/2}e_J^i}) = \hat{C}_{v,I}^{(1/2)}\hat{C}_{v,J}^{(1/2)}\,. \tag{5.25}$$

Noticing that the momentum π^a of the electromagnetic field is quantized, just as the densitized triad, by a flux operator $\Pi_{v,I} := \int_{S_{v,I}} \mathrm{d}^2 y n_a \pi^a \approx \ell_0^2 \pi^I(v)$, the whole electric field term can be written as

$$\begin{aligned} H_\pi &= 2\pi \int \mathrm{d}^3 x N(x) \frac{q_{ab}}{\sqrt{q}} \pi^a \pi^b \approx 2\pi \sum_v N(v) \ell_0^3 \frac{q_{ab}}{\sqrt{q}} \pi^a \pi^b \\ &= 2\pi \sum_{v,I,J} N(v) \frac{q_{IJ}}{\ell_0 \sqrt{q}} \Pi_{v,I} \Pi_{v,J} \end{aligned}$$

which is then quantized to

$$\hat{H}_\pi = 2\pi \sum_v N(v) \hat{C}_{v,I}^{(1/2)} \hat{C}_{v,J}^{(1/2)} \hat{\Pi}_{v,I} \hat{\Pi}_{v,J} \,. \tag{5.26}$$

For the magnetic field term in (3.31), at first sight, a different metric expression arises: $\sqrt{q} q^{ac} q^{bd}$ which also involves inverse components when expressed in terms of the densitized triad. The term appears different from the electric field term and could thus be quantized differently. However, noting

$$\begin{aligned} F_{ab} F_{cd} q^{ac} q^{bd} &= B^e B^f \epsilon_{eab} \epsilon_{fcd} q^{ac} q^{bd} \\ &= \epsilon_{eab} B^e B^f q_{fd} \epsilon^{abd} q^{-1} = 2 q^{-1} q_{ab} B^a B^b \end{aligned}$$

in terms of the magnetic field $B^a = \epsilon^{abc} F_{bc}$ shows that the metric dependence is the same as in the electric part. We thus expect the same metric operator and correspondingly the same quantum gravity corrections in both terms, although different ones are mathematically possible owing to quantization ambiguities. The magnetic contribution to the Maxwell Hamiltonian then is

$$\begin{aligned} H_B &= \frac{1}{8\pi} \int \mathrm{d}^3 x N(x) \frac{q_{ab}}{\sqrt{q}} B^a B^b \approx \frac{1}{8\pi} \sum_v N(v) \ell_0^3 \frac{q_{ab}}{\sqrt{q}} B^a B^b \\ &= \frac{1}{8\pi} \sum_{v,I,J} N(v) \frac{q_{IJ}}{\ell_0 \sqrt{q}} B_{v,I} B_{v,J} \end{aligned}$$

with the magnetic flux $B_{v,I} := \int_{S_{v,I}} \mathrm{d}^2 y n_a B^a \approx \ell_0^2 B^I(v)$. Magnetic flux components $B_{v,I}$ are quantized using U(1) holonomies of the electromagnetic vector potential along closed loops transversal to the direction I:

$$\hat{B}_{v,I} = \frac{1}{4} \sum_{J,K} \sum_{\sigma_J,\sigma_K \in \{\pm 1\}} \sigma_J \sigma_K \epsilon^{IJK} \lambda_{v,\sigma_J J,\sigma_K K} \,.$$

We use the symbol λ to distinguish an electromagnetic holonomy λ from a gravitational one, η. The loop holonomy $\lambda_{v,\pm J,\pm K}$ is then computed around an elementary lattice loop starting in v in direction $\pm X_J^a$ and returning to v along $\pm X_K^a$. Summing over J, K and the two sign factors σ_J and σ_K accounts for all four loops starting in v transversally to $e_{v,I}$. The resulting quantized magnetic part of the Hamiltonian is

$$\hat{H}_\pi = \frac{1}{8\pi} \sum_v N(v) \hat{C}_{v,I}^{(1/2)} \hat{C}_{v,J}^{(1/2)} \hat{B}_{v,I} \hat{B}_{v,J} \tag{5.27}$$

with the same gravitational operator $\hat{C}_{v,I}^{(1/2)} \hat{C}_{v,J}^{(1/2)}$ as in the electric term. It is thus natural to use the same quantum operators and corresponding corrections in both terms, even though mathematically it is possible to quantize them differently. This aspect will be used in the derivations for the effective equation of state.

Dirac Hamiltonian

The Dirac Hamiltonian can be quantized in the similar fashion presented in Chapter III and IV. Since our goal is the derivation of the quantum correction functions arising from the quantization of the gravitational components, we avoid the explicit quantization of the total Dirac Hamiltonian. Therefore, we focus only on the gravitational components in the Dirac Hamiltonian. In the Dirac Hamiltonian (5.7) the factor to be quantized containing inverse powers of the densitized triad is

$$\frac{2E_i^a}{\ell_0 \sqrt{q}} = \epsilon^{abc} \epsilon_{ijk} \frac{e_b^j e_c^k}{\ell_0 \sqrt{q}} \approx \epsilon^{abc} \epsilon_{ijk} \frac{\ell_0^2 e_b^j e_c^k}{V_v}$$

in terms of the volume $V_v \approx \ell_0^3 \sqrt{q(v)}$ of one discrete patch at a point v. We can already notice the close resemblance to the Maxwell Hamiltonian, where the corresponding expression is $q_{ab}/\ell_0 \sqrt{q} = e_a^i e_b^i / \ell_0 \sqrt{q}$ which differs only by the additional

ϵ-tensors. This close relation will, in the end, lead to quite similar quantum corrections for photons and fermions.

We proceed using (5.22) for $r = 1/2$, and write

$$\frac{2E^a_i}{\ell_0\sqrt{q}} = \left(\frac{\ell_0}{2\pi G\gamma}\right)^2 \epsilon^{abc}\epsilon_{ijk}\{A^j_b, V_v^{\frac{1}{2}}\}\{A^k_c, V_v^{\frac{1}{2}}\}, \tag{5.28}$$

which can then be quantized by turning Poisson brackets into commutators of operators. This results in

$$\frac{\widehat{2E^a_i}}{\ell_0\sqrt{q}} = \epsilon^{KIJ}\epsilon_{ijk}(\ell_0\widehat{V_v^{-1/2}e^j_I})(\ell_0\widehat{V_v^{-1/2}e^k_J}) = \epsilon^{KIJ}\epsilon_{ijk}\hat{C}^{(1/2)}_{v,I}\hat{C}^{(1/2)}_{v,J}\delta^j_{(I)}\delta^k_{(J)} \tag{5.29}$$

with $\hat{C}^{1/2}_{v,I}$ defined in (5.23).

Correction functions

As in [57] we can include effects of the quantization of metric coefficients by inserting correction functions in the classical Hamiltonian which follow, e.g., from the eigenvalues [57]

$$\begin{aligned} C^{(1/2)}_{v,I}(\{\mu_{v',I'}\}) &= 2(2\pi\gamma\ell_{\mathrm{P}}^2)^{-1/4}|\mu_{v,J}+\mu_{v,-J}|^{1/4}|\mu_{v,K}+\mu_{v,-K}|^{1/4} \\ &\quad \left(|\mu_{v,K}+\mu_{v,-K}+1|^{1/4} - |\mu_{v,K}+\mu_{v,-K}-1|^{1/4}\right) \end{aligned} \tag{5.30}$$

(where indices J and K are defined such that $\epsilon_{IJK}\neq 0$) of operators $\hat{C}^{(1/2)}_{v,I}$. Although for large $\mu_{v,I}$ these eigenvalues approach the function

$$C^{(1/2)}_{v,I}(\{\mu_{v',I'}\})C^{(1/2)}_{v,J}(\{\mu_{v',I'}\}) \sim (2\pi\gamma\ell_{\mathrm{P}}^2)^{-1/2}\frac{\prod_{K=1}^3\sqrt{|\mu_{v,K}+\mu_{v,-K}|}}{|\mu_{v,I}+\mu_{v,-I}||\mu_{v,J}+\mu_{v,-J}|}$$

expected classically for $q_{IJ}/\sqrt{q} = \sqrt{|p^1p^2p^3|}/p^Ip^J$ with a densitized triad $E^a_i = p^{(i)}\delta^a_i$ and using the relation (5.18) between labels and flux components, they differ for values of $\mu_{v,I}$ closer to one. This deviation can, for an isotropic background, be captured in a single correction function

$$\alpha_{v,K} = \frac{1}{3}\sum_I C^{(1/2)}_{v,I}(\{\mu_{v',I'}\})^2\cdot\frac{\sqrt{2\pi\gamma\ell_{\mathrm{P}}^2}(\mu_{v,I}+\mu_{v,-I})^2}{\prod_{J=1}^3\sqrt{|\mu_{v,J}+\mu_{v,-J}|}} \tag{5.31}$$

which would equal one in the absence of quantum corrections. This is indeed approached in the limit where all $\mu_{v,I} \gg 1$, but for any finite values there are corrections. If all $\mu_{v,I} > 1$ one can directly check that corrections are positive, i.e. $\alpha_{v,K} > 1$ in this regime. Expressing the labels in terms of the densitized triad through fluxes (5.18) results in functionals

$$\alpha[p^I(v)] = \alpha_{v,K}(4\pi\gamma\ell_{\mathrm{P}}^2\mu_{v,I}) \tag{5.32}$$

which enter effective Hamiltonians.

Furthermore, for a *nearly* [2] isotropic background geometry α only depends on the determinant q of the spatial metric and thus $q^{ab}\delta\alpha/\delta q^{ab} = -3q\mathrm{d}\alpha/\mathrm{d}q = -\frac{1}{2}a\mathrm{d}\alpha/\mathrm{d}a$ with the scale factor a related to q by $q = \det(q_{ab}) = a^6$. In this case the quantum gravitational expectation for $\alpha(q)$, as per Eqs. (5.30) and (5.31), simplifies. To use these expressions, we have to relate the scale factor to quantum gravitational excitation levels as they occur in calculations of loop quantum gravity. In the above notation, an elementary discrete patch in a nearly isotropic space-time has, on the one hand, an area of $\ell_0^2 a^2$ if ℓ_0 is the coordinate diameter of the patch. This can be expressed as $\ell_0^2 a^2 = (V_{\mathcal{V}}/\mathcal{N}_{\mathcal{V}})^{2/3}$ where $\mathcal{N}_{\mathcal{V}}$ is the number of patches in a box $\mathcal{V}$ of volume $V_{\mathcal{V}}$. On the other hand, using (5.18) the quantum gravity state assigns a value of $4\pi\gamma\ell_{\mathrm{P}}^2\mu_v$ to this patch via the flux operator, where μ_v is the quantum number of the geometrical excitation of this patch. Thus, we obtain

$$\mu_v = \frac{V_{\mathcal{V}}^{2/3}}{4\pi\gamma\ell_{\mathrm{P}}^2\mathcal{N}_{\mathcal{V}}^{2/3}} =: \frac{a^2}{a_{\mathrm{disc}}^2}$$

where

$$a_{\mathrm{disc}} = 2\sqrt{\pi\gamma}\ell_{\mathrm{P}}\left(\frac{\mathcal{N}_{\mathcal{V}}}{V_0}\right)^{1/3} \tag{5.33}$$

with the coordinate volume V_0 of the box $\mathcal{V}$. The numerical value of a_{disc} depends on coordinates via V_0, or on the normalization of the scale factor. (It does not depend

[2] We are not assuming strict isotropy to compute quantum corrections of inhomogeneous Maxwell and fermion fields. Nevertheless, in leading order corrections one can use the background geometry.

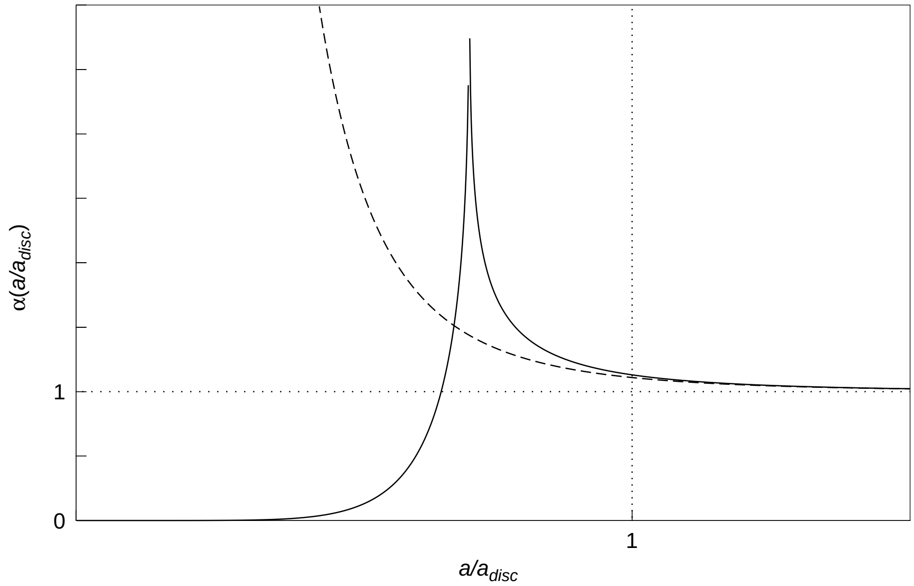

Figure 5.1: The correction function (5.34) as a function of the scale factor (solid line). The asymptotic form (5.35) for large a is shown by the dashed line. (The sharp cusp, a consequence of the absolute value appearing in (5.34), is present only for eigenvalues as plotted here, but would disappear for expectation values of the inverse volume operator in coherent states. This cusp will play no role in the analysis of this paper.)

on the choice of the box $\mathcal{V}$ because a change would multiply $\mathcal{N}_{\mathcal{V}}$ and V_0 by the same factor.) But it is important to note that $a_{\rm disc}$ is not just determined by the Planck length $\ell_{\rm P}$, which appears for dimensional reasons, but also depends on the large number $\mathcal{N}_{\mathcal{V}}$ of discrete patches per volume as given by the quantum gravity state. This is exactly a parameter as expected in the discussion of Sec. 5.1. Replacing μ_v in the equations of the appendix, we obtain

$$\alpha(a) = 8\sqrt{2}(a/a_{\rm disc})^2 \left((2(a/a_{\rm disc})^2+1)^{1/4} - |2(a/a_{\rm disc})^2-1|^{1/4}\right)^2 \tag{5.34}$$

where $a_{\rm disc}$ appears, influencing the size of quantum gravity corrections.

The function is plotted in Fig. 5.1. One can easily see that $\alpha(a)$ approaches the classical value $\alpha = 1$ for $a \gg a_{\rm disc}/\sqrt{2}$, while it differs from one for small a. For $a > a_{\rm disc}/\sqrt{2}$, the corrections are perturbative in a^{-1},

$$\alpha(a) \sim 1 + \frac{7}{64}\left(\frac{a_{\rm disc}}{a}\right)^4 + \cdots . \tag{5.35}$$

This is the first correction in an asymptotic expansion for eigenvalues. If semiclassical states rather than volume eigenstates are used, powers of a^{-1} in the leading corrections can be smaller. Moreover, via $\mathcal{N}_{\mathcal{V}}$ the discreteness scale $a_{\rm disc}$ is expected to be not precisely constant but a function of a itself because the underlying spatial discreteness of quantum gravity can be refined dynamically during cosmological evolution [78, 87]. (Indeed, dynamical refinement is also required for several other phenomenological reasons [117, 118, 119, 120, 121].) In our following analysis we will thus assume a functional form

$$\alpha(a) = 1 + c(a/a_0)^{-n} \tag{5.36}$$

where we traded the fundamental normalization by $a_{\rm disc}$ for normalization with respect to the present-day value of the scale factor a_0. From the derivation, n is likely to be a small, even integer and c is known to be positive. The constant c depends on $a_{\rm disc}$ and inherits the $\mathcal{N}_{\mathcal{V}}$-factor. It can thus be larger than of order one. We will treat this parameter as phenomenological and in the end formulate bounds on c as bounds for $\mathcal{N}_{\mathcal{V}}$.

Effective Hamiltonians and the equation of state

The above correction functions will appear in a Hamiltonian operator and thus also correct expressions for energy density and pressure or the equation of state, which is presented below.

Maxwell Hamiltonian

The general expression for the effective Maxwell Hamiltonian one can expect is thus

$$H_{\rm eff} = \int_\Sigma {\rm d}^3x N \left[\alpha[q_{cd}]\frac{2\pi}{\sqrt{q}}\pi^a\pi^b q_{ab} + \beta[q_{cd}]\frac{\sqrt{q}}{16\pi}F_{ab}F_{cd}q^{ac}q^{bd}\right] \tag{5.37}$$

with two possibly different correction functions α and β depending on the lattice values $\mu_{v,I}$. As shown before, the case $\alpha = \beta$ is preferred, and we will see soon that this has implications for the effective equation of state. (In [25] a Hamiltonian operator was introduced which did not use the same quantizations for metric coefficients in the electric and magnetic parts, thus giving $\alpha \neq \beta$. A quantization as described here, using the same quantization in both parts, was formulated in [60]. Phenomenological implications of a quantization of the latter type, concerning Lorentz invariance, are discussed in [61].) There are other possible sources for corrections, such as higher order powers and higher derivatives of the electric and magnetic fields. But these terms would not be metric dependent and are thus not crucial for the following arguments.

Now using (5.37), we get the modified expression

$$\begin{aligned} \frac{1}{N} q^{ab} \frac{\delta H_M}{\delta q^{ab}} &= -\frac{q_{ab}}{N} \frac{\delta H_M}{\delta q_{ab}} = \frac{\pi}{\sqrt{q}} \pi^c \pi^d q_{cd} \left(\alpha + 2q^{ab} \delta\alpha / \delta q^{ab}\right) \\ &\quad + \frac{\sqrt{q}}{32\pi} F_{cd} F^{cd} \left(\beta + 2q^{ab} \delta\beta / \delta q^{ab}\right) , \end{aligned} \tag{5.38}$$

depending on α and β. For a nearly isotropic background geometry, for instance, α only depends on the determinant q of the spatial metric and, from Appendix B, $q^{ab} \delta\alpha / \delta q^{ab} = -3q \mathrm{d}\alpha / \mathrm{d}q$, which we assume in what follows.

The modified energy density and pressure then are

$$\rho_{\text{eff}} = \frac{2\pi}{q} \pi^a \pi^b q_{ab} \alpha + \frac{1}{16\pi} F_{ab} F_{cd} q^{ac} q^{bd} \beta \tag{5.39}$$

$$\begin{aligned} 3P_{\text{eff}} &= \frac{2\pi}{q} \pi^a \pi^b q_{ab} \left(\alpha - 6q \mathrm{d}\alpha / \mathrm{d}q\right) \\ &\quad + \frac{1}{16\pi} F_{ab} F^{ab} \left(\beta - 6q \mathrm{d}\beta / \mathrm{d}q\right) \\ &= \frac{2\pi}{q} \pi^a \pi^b q_{ab} \alpha \left(1 - 6 \frac{\mathrm{d}\log\alpha}{\mathrm{d}\log q}\right) \\ &\quad + \frac{1}{16\pi} F_{ab} F^{ab} \beta \left(1 - 6 \frac{\mathrm{d}\log\beta}{\mathrm{d}\log q}\right) . \end{aligned} \tag{5.40}$$

It follows easily from (5.38), (5.39) and (5.40) that the classical behavior is reproduced for $\alpha = \beta = 1$. Interestingly, for $\alpha = \beta$, the equation of state w can easily be computed and is modified as

$$w_{\text{eff}} = \frac{1}{3} - 2 \frac{\mathrm{d}\log\alpha}{\mathrm{d}\log q} . \tag{5.41}$$

This modification is independent of the specific matter dynamics as in the classical case, and it results in an equation of state which is linear in ρ, but depends on the geometrical scales (and the Planck length) through α.

Dirac Hamiltonian

Thus the general expression one can expect for a phenomenological Dirac Hamiltonian including corrections from inverse powers of the triad is

$$\begin{aligned} H_{\text{phen}} &= \int_{\Sigma_t} \mathrm{d}^3x N\Bigg(\frac{E^a_i}{\sqrt{q}}\alpha(E^b_j)\big(-\beta\mathcal{D}_a\left(\pi_\xi^T\tau^i\xi+\pi_\chi^T\tau^i\chi\right) \\ &\quad -2i\left(\theta_L\pi_\xi^T\tau^i\mathcal{D}_a\xi-\theta_R\pi_\chi^T\tau^i\mathcal{D}_a\chi-c.c.\right)\big)+\frac{\theta(E^b_j)}{\sqrt{q}}B\Bigg) \end{aligned} \tag{5.42}$$

with two possibly different correction functions α and θ. This also affects the energy density and pressure terms, derived by the general expressions (5.2) and (5.10). We are mainly interested in the correction to the one-third in the equation of state (5.14), so we focus on the first term in (5.42) in what follows. Energy density and the pressure then are, ignoring the classical interaction term B,

$$\begin{aligned} \rho_{\text{eff}} &= \frac{2E^a_i}{a^6}\alpha(a)\Big(-\beta\mathcal{D}_a\left(\pi_\xi^T\tau^i\xi+\pi_\chi^T\tau^i\chi\right) \\ &\quad +i\left(-\theta_L\pi_\xi^T\tau^i\mathcal{D}_a\xi-\theta_R\pi_\chi^T\tau^i\mathcal{D}_a\chi-c.c.\right)\Big) \end{aligned} \tag{5.43}$$

and

$$\begin{aligned} 3P_{\text{eff}} &= \frac{2E^a_i}{a^6}\alpha(a)\left(1-\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log a}\right)\big(-\beta\mathcal{D}_a\left(\pi_\xi^T\tau^i\xi+\pi_\chi^T\tau^i\chi\right) \\ &\quad +i\left(-\theta_L\pi_\xi^T\tau^i\mathcal{D}_a\xi-\theta_R\pi_\chi^T\tau^i\mathcal{D}_a\chi-c.c.\right)\big)\,. \end{aligned} \tag{5.44}$$

From this, the equation of state w can easily be computed:

$$w_{\text{eff}}=\frac{1}{3}\left(1-\frac{\mathrm{d}\log\alpha}{\mathrm{d}\log a}\right)\,. \tag{5.45}$$

This quantum gravity correction is independent of the specific matter dynamics as in the classical relativistic case. It results in an equation of state which is linear in ρ, but depends on the geometrical scales (and the Planck length) through α. This is the same general formula derived above for radiation, i.e., (5.41). Thus, on an isotropic background radiation and relativistic fermions are not distinguished by the form of quantum corrections they receive.

Finally, let us derive the correction to the evolution of energy density in an isotropic and homogeneous universe, i.e. Friedmann-Robertson-Walker (FWR) universe. It follows from the FRW metric and Einstein's equation that the evolution of the energy density is given by the continuity equation, i.e.,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + P\right) = 0, \tag{5.46}$$

where a is the scale factor and the dot indicates a proper time derivative. Using the definition of the equation of state and eliminating the time derivative, this equation can be cast into the following useful form:

$$\frac{\mathrm{d}\log\rho(a)}{\mathrm{d}\log a} = -3\left(1 + w(a)\right). \tag{5.47}$$

Here we have shown the dependence of the equation of state on the scale factor explicitly. It can easily be shown that the solution to the above equation is

$$\rho(a) = \rho_0 \exp\left[-3\int\left(1 + w(a)\right)\mathrm{d}\log a\right], \tag{5.48}$$

where ρ_0 is the integration constant. Now by inserting the modified equation of state in the radiation era, (5.45) with $q = a^6$, we obtain

$$\rho(a) = \rho_0 \alpha(a) a^{-4}. \tag{5.49}$$

Again, for $\alpha = 1$, we retrieve the classical result $\rho(a) \propto a^{-4}$. Therefore, loop quantum gravity corrections induced by discreteness of the flux operator are reflected even in the evolution of the FRW universe.

5.4 Effect on Big Bang Nucleosynthesis

Following baryosynthesis, i.e. after a suitable condition for the production of stable protons and neutrons is finally created, the relative abundances of these particles depend on the competition between the weak-interaction rate to reach the thermodynamically favored equilibrium values and the expansion rate, $H = \dot{a}/a$, that causes the rate of change in the mean temparature of the universe. Therefore, the production of elements in the early universe is highly sensitive to the expansion rate, given by

$$\frac{\dot{a}}{a} = \left(\frac{8}{3}\pi G\rho\right)^{1/2}, \tag{5.50}$$

where ρ is the total density, thus including radiation and fermions. As we have seen here for fermions and in [100] for radiation, the effect of loop quantum gravity corrections is to multiply the effective $\rho(a)$ by a factor $\alpha(a)$. Most importantly, we find that $\alpha(a)$ is the *same* for both bosons and fermions (up to possible quantization ambiguities), so a separate treatment of the two types of particles in the early universe (as in Ref. [99]) is unnecessary here.

In the standard treatment of the thermal history of the universe, the density of relativistic particles (bosons or fermions) is given by

$$\rho = \frac{\pi^2}{30} g_* T^4, \tag{5.51}$$

where g_* is the number of spin degrees of freedom for bosons, and 7/8 times the number of spin degrees of freedom for fermions, and T is the temperature, which scales as

$$T \propto a^{-1}. \tag{5.52}$$

The equation of state parameter is

$$w = 1/3. \tag{5.53}$$

Clearly, equations (5.51)–(5.53) are inconsistent with equations (5.45) and (5.49). There is some ambiguity in determining the correct way to modify the expressions for $\rho(T)$ and w. We have chosen to assume that the modifications are contained in the gravitational sector, so that the density is given by

$$\rho = \alpha(a)\frac{\pi^2}{30} g_* T^4, \tag{5.54}$$

with the temperature scaling as in equation (5.52), and the equation of state w is given by equation (5.45). This guarantees that the standard continuity equation (5.46) continues to hold. Note that this is not the only way to incorporate equation (5.49) into the calculation, but it seems to us the most reasonable way. This issue requires a consideration of thermodynamics on a quantum space-time, which is a fascinating but not well-studied area. Instead of entering details here, we note that we interpret the α-correction as a consequence of a quantum gravity sink to energy and entropy. Thus, quantum gravity implies a non-equilibrium situation which would otherwise imply that ρ must be proportional to T^4 without any additional dependence on $a \propto T^{-1}$.

With these assumptions, we can simply treat $\alpha(a)$ as an effective multiplicative change in the overall value of G. Note that this simplification is only possible because we explicitly derived by our canonical analysis that, unexpectedly, quantum corrections of radiation and fermions appear in similar forms. This makes possible a comprehensive derivation of implications for BBN, bearing on earlier work. In fact, a great deal of work has been done on the use of BBN to constrain changes in G (see, e.g. Refs. [101, 102, 103, 104, 105]). The calculation is straightforward, if one has a functional form for the time-variation in G. For the loop quantum gravity corrections considered here, the most reasonable functional form is (5.36). Note that this expression is by construction valid only in the limit where $\alpha(a) - 1 << 1$. In terms of the effective gravitational constant, G, one can then write

$$G(a) = G_0[1 + c(a/a_0)^{-n}], \tag{5.55}$$

where G_0 is the present-day value of the gravitational constant.

In order to constrain the values of c and n, we calculate the predicted element abundances with the indicated change in G and compare with observational constraints. Big Bang nucleosynthesis proceeds first through the weak interactions that interconvert protons and neutrons:

$$\begin{aligned} n + \nu_e &\leftrightarrow p + e^-, \\ n + e^+ &\leftrightarrow p + \bar{\nu}_e, \\ n &\leftrightarrow p + e^- + \bar{\nu}_e. \end{aligned} \tag{5.56}$$

When $T \gtrsim 1$ MeV, the weak-interaction rates are faster than the expansion rate, $\dot{a}/a$, and the neutron-to-proton ratio (n/p) tracks its equilibrium value $\exp[-\Delta m/T]$, where Δm is the neutron-proton mass difference. As the universe expands and cools, the expansion rate becomes too fast for weak interactions to maintain weak equilibrium and n/p freezes out. Nearly all the neutrons which survive this freeze-out are converted into ^{4}He as soon as deuterium becomes stable against photodisintegration, but trace amounts of other elements are produced, particularly deuterium and ^{7}Li (see, e.g., Ref. [106] for a review).

In the standard model, the predicted abundances of all of these light elements are a function of the baryon-photon ratio, η, but any change in G alters these predictions. Prior to the era of precision CMB observations (i.e., before WMAP), Big Bang nucleosynthesis provided the most stringent constraints on η, and modifications to the standard model could be ruled out only if no value of η gave predictions for the light element abundances consistent with the observations. However, the CMB observations now provide an independent estimate for η, which can be used as an input parameter for Big-Bang nucleosynthesis calculations.

Copi et al. [104] have recently argued that the most reliable constraints on changes in G can be derived by using the WMAP values for η in conjuction with deuterium observations. The reason is that deuterium can be observed in (presumably unprocessed) high-redshift quasi stellar object (QSO) absorption line systems (see Ref. [107] and references therein), while the estimated primordial ^{4}He abundance, derived from observations of low metallicity HII regions, is more uncertain (see, for example, the discussion in Ref. [108]). While we agree with the argument of Copi et al. in principle, for the particular model under consideration here it makes more sense to use limits on ^{4}He than on deuterium, in conjunction with the WMAP value for η. The reason is that the ^{4}He abundance is most sensitive to changes in the expansion rate at $T \sim 1$ MeV, when the freeze-out of the weak interactions determines the fraction of neutrons that will eventually be incorporated into ^{4}He. Deuterium, in contrast, is produced in Big Bang nucleosynthesis only because the expansion of the universe prevents all of the deuterium from being fused into heavier elements. Thus, the deuterium abundance is most sensitive to the expansion rate at the epoch when this fusion process operates ($T \sim 0.1$ MeV). The importance of this distinction with regard to modifications of the standard model was first noted in Ref. [109], and a very nice quantitative analysis was given recently in Ref. [110]. Note that our estimate for the behavior of $G(a)/G_0 - 1$, equation (5.55), is a steeply decreasing function of a. Thus, the change in the primordial ^{4}He abundance will always be much larger than the change in the deuterium abundance. Therefore, we can obtain better constraints on this model by using extremely conservative limits on ^{4}He, rather than by using the more reliable limits on the deuterium abundance. For the same reason, we can ignore any effect on the CMB, since the latter is generated at a much larger value of a, and any change will be minuscule. Hence, we can confidently use the WMAP value for η.

WMAP gives [111]

$$\eta = 6.116^{+0.197}_{-0.249} \times 10^{-10}. \tag{5.57}$$

Because the estimated errors on η are so small, we simply use the central value for η; the bounds we derive on c in equation (5.36) change only slightly when η is varied within the range given by equation (5.57). Since c in equations (5.36) and (5.55) is thought to be positive, the effect of LQG corrections is to increase the primordial expansion rate, which increases the predicted ^{4}He abundance. We therefore require an observational upper bound on the primordial ^{4}He abundance. As noted earlier, this is a matter of some controversy. We therefore adopt the very conservative upper bound recommended by Olive and Skillman [108]:

$$Y_P \leq 0.258, \tag{5.58}$$

where Y_P is the primordial mass fraction of ^{4}He. For a fixed value of n in equation (5.55), we determine the largest value of c that yields a primordial ^{4}He abundance consistent with this upper limit on Y_P. Since we are essentially bounding the change in G at $a/a_0 \sim 10^{-10}$, it is convenient to rewrite equation (5.36) as

$$\alpha = 1 + \widetilde{c}/a_{10}^n \tag{5.59}$$

where $a_{10} \equiv 10^{10}(a/a_0)$. This upper bound on $\widetilde{c}$ as a function of n is given in Fig. 5.2. For the special case $n = 4$, we can use these results to place a bound on $a_{\rm disc}$ in equation (5.35). We obtain

$$\frac{a_{\rm disc}}{a_0} < 2.4 \times 10^{-10}\,. \tag{5.60}$$

This is not a strong bound for the parameters of quantum gravity, but clearly demonstrates that quantum corrections are consistent with successful big bang nucleosynthesis.

In terms of more tangible quantum gravity parameters, we have

$$\mathcal{N}_{\mathcal{V}}^{1/3} < \frac{1.2 \cdot 10^{-10}}{\sqrt{\pi\gamma}} \frac{a_0 V_0^{1/3}}{\ell_{\rm P}} \tag{5.61}$$

for the number of patches at the time of big bang nucleosynthesis. In terms of the volume $V_{\mathcal{V}} = (10^{-10}a_0)^3 V_0$ at this time, we have $\mathcal{N}_{\mathcal{V}} < 3V_{\mathcal{V}}/\ell_{\rm P}^3$ with the value $\gamma \approx 0.24$ of the Barbero–Immirzi parameter as derived from black hole entropy calculations. More meaningfully, if we view $2\sqrt{\pi\gamma}\ell_{\rm P}$ as the basic length scale as it appears in the spectrum (5.18) of loop quantum gravity, the bound becomes more interesting: This gives $\mathcal{N}_{\mathcal{V}}^{1/3} < 2.4 V_{\mathcal{V}}^{1/3}/(2\sqrt{\pi\gamma}\ell_{\rm P})$. This upper limit is already quite close to

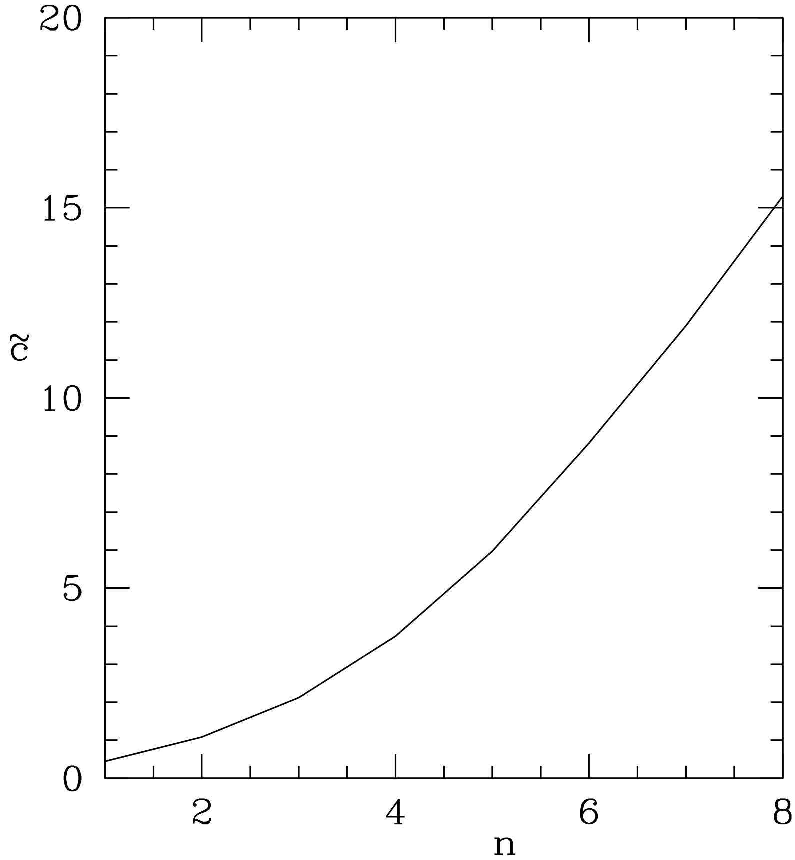

Figure 5.2: The solid curve gives an upper bound on $\widetilde{c}$ as a function of n, for the assumed form for α: $\alpha = 1 + \widetilde{c}/a_{10}^n$, where a_{10} is the value of the scale factor in units for which $a_{10} = 10^{10}$ at present.

what one expects for elementary patch sizes in loop quantum gravity, which would provide $\mathcal{N}_{\mathcal{V}}^{1/3} < V_{\mathcal{V}}^{1/3}/(2\sqrt{\pi\gamma}\ell_{\mathrm{P}})$ as a fundamental upper limit. Given that these values are close to each other, we see a clear potential of improvements by more precise observational inputs. Moreover, other correction terms from quantum gravity could be used to obtain a lower bound for $\mathcal{N}_{\mathcal{V}}$ such that the allowed window would be reduced to a smaller size.

5.5 Conclusion

We have derived here the equation of state of the Maxwell as well as the Dirac field in a canonical form, including corrections expected from loop quantum gravity. In the canonical derivation, the reason for a linear equation of state of Maxwell field, which is trace-freedom in the Lagrangean derivation, is the fact that the same metric dependent factor $q_{ab}/\sqrt{q}$ multiplies both terms in the Hamiltonian. The Maxwell Hamiltonian is thus simply rescaled if the metric is conformally transformed, which explains the conformal invariance of Maxwell's equations. This is special for the Maxwell field and different from, e.g., a scalar field with a non-vanishing potential.

The same fact allows one to quantize the Hamiltonian in a way which affects both the electric and magnetic term in the same way, at least as far as the metric dependence is concerned. One then obtains a single correction function $\alpha = \beta$ which only corrects the metric dependence of the total scale of the Hamiltonian. In this sense, conformal invariance is preserved even after quantization. (But this would not be the case if a quantization is used which results in $\alpha \neq \beta$.)

This preservation of the form of the Hamiltonian explains why we are still able to derive an equation of state independently of the specific field dynamics and that it remains linear. However, the classical value $w = \frac{1}{3}$ is corrected due to quantum effects in the space-time structure. This modification is also understandable from a Lagrangean perspective, together with basic information from the loop quantization. Employing trace freedom of the stress-energy tensor to derive the equation of state, we have to use the inverse metric in $g^{ab}T_{ab}$. But from loop quantum gravity we know that, when quantized, not all components of the inverse metric agree with inverse operators of the quantization. For the scale factor of an isotropic metric, for instance, we have $\widehat{a^{-1}} \neq$ "$\hat{a}^{-1}$" since the right hand side is not even defined [52]. While the left hand side is defined through identities such as (5.16), it satisfies $\widehat{a^{-1}}\hat{a} \neq 1$ and thus shows deviations from the classical expectation $a^{-1}a = 1$ on small scales which were captured here in correction functions. As derived in detail, this implies scale dependent modifications to the equation of state parameter $w_{\rm eff}$.

The result can also be interpreted in more physical terms. The classical behavior $\rho(a) \propto a^{-4}$ can be understood as a combination of a dilution factor a^{-3} and an additional redshift factor a^{-1} for radiation in an expanding universe. As we have seen, this is corrected to $\alpha(a)a^{-4}$ where $\alpha(a)$ corrects the metric factor $q_{ab}/\sqrt{q} \sim a^{-1}\delta_{ab}$. Since this is only a single inverse power of a for an isotropic solution, we can interpret the result as saying that only the redshift receives corrections due to quantum effects on electromagnetic propagation. The dilution factor due to expansion is unmodified, except that the background evolution $a(t)$ itself receives corrections. This agrees with the result for dust, which is only diluted and has an unmodified equation of state even after quantization [3]. Unlike dust, for radiation one has to refer to the inhomogeneous field and its quantum Hamiltonian to derive a reliable equation of state, as presented here.

On the other hand, there are corrections to the simple equation of state $w = \frac{1}{3}$ for fermions even classically. One observation made here is that the interaction term derived in [34] leads to such a correction and might be more constrained by big bang nucleosynthesis than through standard particle experiments [35]. We have not analyzed this further here because more details of the behavior of the fermion current would be required.

A second source of corrections arises for fermions from quantum gravity. Remarkably, while quantum gravity effects on an isotropic background do correct the equations of state, they do so equally for photons and relativistic fermions. Initially, this is not expected for both types of fields due to their very different actions. However, a possible scenario for the same corrections for both fermions and photons is big bang nucleosynthesis since BBN is a highly relativistic regime which, to a good

[3]But it disagrees with [63] both for dust and radiation, where a direct quantization of energy densities exclusively for isotropic fields was attempted.

approximation, implies identical equations of state for both. Thus, quantum gravity effects do not spoil the detailed balance required for the scenario to work and bounds from big bang nucleosynthesis obtained so far are not strong. But there are interesting limits for the primary parameter, the patch size of a quantum gravity state. It is dimensionally expected to be proportional to the Planck length ℓ_{P} but could be larger. In fact, current bounds derived here already rule out a patch size of exactly the elementary allowed value in loop quantum gravity. With more precise estimates, these bounds can be improved further.

We have made use of quantum gravity corrections in a form which does not distinguish fermions from radiation. Although the most natural implementation, quite unexpectedly, provides equal corrections as shown here, there are several possibilities for differences which suggest several further investigations. Small deviations in the equations of state and thus energy densities of fermions and radiation are possible. First, there are always quantization ambiguities, and so far we tacitly assumed that the same basic quantization choice is made for the Maxwell and Dirac Hamiltonians. Such ambiguity parameters can be explicitly included in specific formulas for correction functions; see e.g. [112, 113, 57]. Independent consistency conditions for the quantization may at some point require one to use different quantizations for both types of fields, resulting in different quantum corrections and different energy densities. Such conditions can be derived from an analysis of anomaly-freedom of the Maxwell field and fermions coupled to gravity, which is currently in progress. As shown here, if this is the case it will become testable in scenarios sensitive to the behavior of energy density such as big bang nucleosynthesis. Moreover, assuming the same quantization parameters leads to identical quantum corrections for photons and fermions only on isotropic backgrounds. Small-scale anisotropies have different effects on both types of fields and can thus also be probed through their implications on the equation of state.

For this, it will be important to estimate more precisely the typical size of corrections, which is not easy since it requires details of the quantum state of geometry. The crucial ingredient is again the patch size of underlying lattice states. On the other hand, taking a phenomenological point of view allows one to estimate ranges for patch sizes which would leave one in agreement with big bang nucleosynthesis constraints. Interestingly, corrections studied here provide upper bounds to the patch size, and other corrections from quantum gravity are expected to result in lower bounds. A finite window thus results, which can be shrunk with future improvements in observations.

CHAPTER VI

DARK ENERGY

In the previous chapter, quantum gravity corrections to the equation of state of both fermions and photons have been derived and its effect on BBN have also been explored as an application. It turns out that similar corrections also appear in other approaches to quantum gravity, for example, string theory. Therefore, an example of the effects of such corrections on the evolution of dark energy, k-essence and quintessence, is presented in this chapter.

The universe appears to consist of approximately 30% nonrelativistic matter, including both baryons and dark matter, and 70% dark energy (see Ref. [122] for a recent review, and references therein). The evolution of the dark energy density depends on its equation of state, which is usually parametrized in the form

$$p_{DE} = w\rho_{DE}, \tag{6.1}$$

where p_{DE} and ρ_{DE} are the pressure and density of the dark energy. Then the density of the dark energy scales as

$$\rho_{DE} \propto R^{-3(1+w)}. \tag{6.2}$$

The simplest model for the dark energy is a cosmological constant, for which $w = -1$ and $\rho_{DE} = constant$. More complex models have been proposed, in which the dark energy arises from a scalar field ϕ; these are called quintessence models [123, 124, 125, 126, 127]. These models generally give rise to a time-varying w_ϕ and more complex behavior for ρ_{DE}. One advantage of such models is that certain classes of quintessence potentials lead to tracker behavior, in which the evolution of the scalar field is independent of the initial conditions. The conditions for such tracking behavior have been worked out in detail by Steinhardt, et al. [127].

A second class of models generalizes quintessence to allow for a non-standard kinetic term. These models, dubbed k-essence, have also been explored in great detail [128, 129, 130, 131, 132, 133, 134, 135, 136]. These models can also lead to tracking behavior, and the conditions necessary for such behavior have been discussed by Chiba [133].

Both quintessence and k-essence can be generalized to modified versions of the Friedmann equation. In the standard Friedmann equation, the relation between the scale factor a (or, alternatively, the Hubble parameter H) and the density is

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3}. \tag{6.3}$$

where we set $8\pi G = 1$ throughout. However, various proposals have been put forward to modify this equation at high energy. In type II Randall-Sundrum models, for example, one has [137, 138]

$$H^2 \propto \rho^2, \tag{6.4}$$

in the limit of large ρ, while Gauss-Bonnet models can give [139]

$$H^2 \propto \rho^{2/3}. \tag{6.5}$$

The Cardassian model [140] assumes an expansion law of the form

$$H^2 = \frac{\rho}{3} + B\rho^n \tag{6.6}$$

with $n < 2/3$.

Motivated by these examples, numerous authors have examined the evolution of various dark energy models in the context of non-standard expansion laws [141, 142, 143, 144, 145, 146]. The most general treatments are given in Refs. [144]-[146]. Sami et al. [144] examine quintessence with constant w_ϕ for a power-law modification to the Friedmann equation ($H^2 \propto \rho^q$). Copeland et al. [145] discuss "scaling" quintessence models, i.e., models for which $w_\phi = w_B$, with an arbitrary expansion law, $H^2 \propto f(\rho)$.

Here w_B is the ratio of pressure to density for the dominant, "background" fluid, e.g., $w_B = 0$ for a matter-dominated universe, and $w_B = 1/3$ for a radiation-dominated universe. Tsujikawa and Sami [146] examine arbitrary scalar field models (including both quintessence and k-essence) with scaling behavior ($w_\phi = w_B$) in models with a power-law modification to the Friedmann equation, $H^2 \propto \rho^q$.

Here we generalize this earlier work by examining tracking solutions for both quintessence and k-essence in a general cosmological background characterized by $H^2 \propto f(\rho)$. Although we adopt the approach of Steinhardt et al. [127] for quintessence and Chiba [133] for k-essence, our formalism encompasses tracking solutions not only for a wide range of potentials but also for a wide range of $f(\rho)$. We derive sufficient conditions for both $V(\phi)$ and $f(\rho)$ to obtain tracking solutions with a constant w_ϕ. This formalism provides us with a generic method to study these solutions for a wide variety of scalar field models such as quintessence, tachyon, k-essence, and phantom models.

6.1 Quintessence

Tracking solutions

The equation of motion for the ϕ-field is

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi = 0, \tag{6.7}$$

where

$$V_\phi \equiv dV/d\phi, \tag{6.8}$$

and

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = f(\rho). \tag{6.9}$$

Here a is the Robertson-Walker scale factor, and ρ is the total density, given by

$$\rho = \rho_B + \rho_\phi, \tag{6.10}$$

where ρ_B is the background (radiation + matter) density, and ρ_ϕ is the scalar field energy density. The standard Hubble expansion law corresponds to equation (6.9) with $f(\rho) = \rho$; in this paper we allow $f(\rho)$ to have an arbitrary functional form.

By definition, the tracking solutions are the solutions to which the evolution of the scalar field ϕ converges for a wide range of initial conditions for ϕ and $\dot{\phi}$. We follow the approach prescribed by Steinhardt et al. [127] for quintessence, but now generalize it to the arbitrary expansion law given by equation (6.9). For tracking solutions, w_ϕ is nearly constant [127], where w_ϕ is given by

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}. \tag{6.11}$$

It follows from equation (6.9) that

$$\dot{H} = \frac{3}{2}H^2\eta[(w_\phi - w_B)(1-\Omega_\phi) - (1+w_\phi)], \tag{6.12}$$

where η encodes the information on the generalized expansion law in equation (6.9):

$$\eta = \frac{d\ln f(\rho)}{d\ln(\rho)}. \tag{6.13}$$

For the standard Hubble expansion, $\eta = 1$. In this paper, we will confine our attention to the case $\eta > 0$, and our conclusions will be valid only for this case. However, we note that $\eta < 0$ can lead to interesting types of behavior (e.g., a phantom-like future singularity in a matter-dominated universe). Several specific models of this type are mentioned in Ref. [147].

By combining these relations, it is useful to cast the equation of motion into the following form:

$$\frac{V_\phi}{\sqrt{V}} = \pm 3H\sqrt{\frac{1-w_\phi^2}{2}}(1+\frac{x'}{6}), \tag{6.14}$$

where $x = (1+w_\phi)/(1-w_\phi) = \frac{1}{2}\dot{\phi}^2/V$ is the ratio of the kinetic to potential energy for ϕ, and $x' \equiv d\ln x/d\ln a$. The $\pm$ sign depends on whether $V_\phi > 0$ or $V_\phi < 0$, respectively. It follows from equation (6.14) that the tracker condition ($\dot{w}_\phi \approx 0$)

becomes

$$\frac{V_\phi}{V^{\frac{\eta+1}{2}}} \approx \left(\frac{1}{\Omega_\phi}\right)^{\frac{\eta}{2}}. \tag{6.15}$$

This is the generalization of the Steinhardt et al. [127] tracking condition to an arbitrary expansion law.

As in Ref. [127], we define the function

$$\Gamma_V \equiv V_{\phi\phi}V/(V_\phi)^2, \tag{6.16}$$

whose properties determine whether tracking solutions exist. By taking the time derivative of equation (6.14) and combining with the equation (6.12) and (6.14) itself, we obtain the following equation:

$$\begin{aligned}\Gamma_V - \frac{1+\eta}{2} = & \frac{\eta(w_B - w_\phi)\Omega_B}{2(1+w_\phi)} \\ & - \frac{\eta(w_B - w_\phi)\Omega_B + \eta + (\eta - 2)w_\phi}{2(1+w_\phi)}\frac{x'}{6+x'} \\ & - \frac{2}{(1+w_\phi)}\frac{x''}{(6+x')^2}.\end{aligned} \tag{6.17}$$

where $x'' \equiv d^2 \ln x / d \ln a^2$. As expected, equation (6.17) reduces to the corresponding equation in Ref. [127] for $\eta = 1$. In a universe dominated by a background fluid ($\Omega_B \approx 1$) with $w_\phi \approx$ constant and nearly constant Γ_V, the above equation becomes

$$\begin{aligned}\Gamma_V &\approx \frac{\eta+1}{2} + \frac{\eta(w_B - w_\phi)}{2(1+w_\phi)}, \\ &\approx \frac{1}{2} + \frac{\eta}{2}\left(\frac{1+w_B}{1+w_\phi}\right).\end{aligned} \tag{6.18}$$

In deriving the above equation, the plausibility of the condition that $\Gamma_V \approx$ constant has been discussed in detail in Ref. [127]. The crucial point is that this condition encompasses a wide range of potentials including inverse power law potentials and combinations of inverse power law terms to give rise to tracking solutions.

We must know the appropriate restrictions on η, i.e., on $f(\rho)$ to extract the tracking solutions from equation (6.17). Since the left-hand side of equation (6.18) is nearly constant, it follows that η must be nearly constant during background domination, i.e., the function $f(\rho)$ must satisfy (6.13) for a nearly constant η. Thus we require an extra condition, in addition to the conditions on Γ_V, to derive tracking solutions for both quintessence and k-essence. It is obvious that this extra condition arises from the extra "degree of freedom" in choosing a different cosmological background. The only case for which η is *exactly* constant is $f(\rho) \propto \rho^n$ for a constant n. This power-law behavior includes both the Randall-Sundrum and Gauss-Bonnet models as special cases, and it was studied in detail in Ref. [144]. Of course, more general conditions can produce an expression for $f(\rho)$ that is roughly constant over a wide range in the scale factor. For instance, a sum of power laws, e.g., as in equation (6.6), gives a value for η that is nearly constant over most of the evolution of the universe, i.e., at all times except for the epoch when the two contributions to $f(\rho)$ are roughly equal.

Note that there are a few trivial special cases for which this argument breaks down. In particular, if V is a constant, the right hand side of equation (6.14) must be zero; this can be achieved by taking $w_\phi = \pm 1$. The case $w_\phi = -1$ corresponds to a non-zero constant potential, while $w_\phi = 1$ is the solution for $V = 0$. Both of these results are independent of the value of H on the right-hand side of equation (6.14) and are therefore independent of η.

The validity of equation (6.18) may be checked by comparing with the results obtained by Sami et al. [144]. For scaling solutions with a constant w_ϕ in a background dominated universe, the potential function takes the following form [144]

$$V(\phi) \propto \phi^{-\alpha}, \tag{6.19}$$

where α is constant. Then we obtain from equation (6.18)

$$1 + w_\phi \approx \eta(1 + w_B)\frac{\alpha}{\alpha + 2}. \tag{6.20}$$

This solution agrees with the result obtained in Ref. [144].

Stability of the tracking solutions

So far, we have derived solutions with constant w_ϕ in a general cosmological background; now we want to check the stability of these solutions with constant w_ϕ. In order to check the stability, we perturb the tracker value of w_ϕ, which we will call w_0, by an amount δ. Then we expand equation (6.17) to lowest order in δ and its derivatives to obtain

$$\begin{aligned} &2\delta'' + 3[\eta(1+w_B) - 2w_0]\delta' \\ &+9\eta(1+w_B)(1-w_0)\delta = 0, \end{aligned} \tag{6.21}$$

where the prime means $d/d\ln a$ and w_0 is the value of w_ϕ derived from equation (6.18). The solution of this equation is

$$\delta \propto a^\gamma, \tag{6.22}$$

where

$$\begin{aligned} \gamma = &-\frac{3}{4}\left[\eta(1+w_B) - 2w_0\right] \\ &\pm\frac{3i}{4}\sqrt{8\eta(1+w_B)(1-w_0) - [\eta(1+w_B) - 2w_0]^2}. \end{aligned} \tag{6.23}$$

In the derivation of this equation, Γ_V and η are assumed to be constant.

In order to have δ decay, the real part of γ has to be negative. Hence, it follows that

$$w_0 < \frac{\eta(1+w_B)}{2}, \tag{6.24}$$

provided the quantity under the square root is positive. If the quantity under the square root is negative (so that both values are real), then the above equation is also a necessary condition since the first term under the square root is always positive, provided $\eta > 0$ and $w_0 < 1$. Using equation (6.18), the above inequality can be written in terms of Γ_V as

$$\Gamma_V > \frac{3\eta(1+w_B)+2}{2\eta(1+w_B)+4}. \tag{6.25}$$

Therefore, for a nearly constant Γ_V, η and w_ϕ, the tracker condition, i.e., equation (6.15) gives the following possibilities:

a. If $w_\phi < w_B$, then Ω_ϕ increases with time. Then we conclude from equation (6.15) that $|V_\phi/V^{\frac{n+1}{2}}|$ decreases for a tracker solution. However, taking the time derivative of $V_\phi/V^{\frac{\eta+1}{2}}$, we obtain

$$\frac{d}{dt}\left(\frac{V_\phi}{V^{\frac{\eta+1}{2}}}\right) = \frac{V_\phi^2}{V^{\frac{\eta+3}{2}}}\dot{\phi}\left(\Gamma_V - \frac{\eta+1}{2}\right). \tag{6.26}$$

Hence, $|V_\phi/V^{\frac{\eta+1}{2}}|$ decreases if $\Gamma_V > \frac{1+\eta}{2}$. Thus, $w_\phi < w_B$ is observed for

$$\Gamma_V > \frac{1+\eta}{2}. \tag{6.27}$$

Combining this with the condition for stable tracking behavior (equation 6.25), we obtain

$$\Gamma_V > max\left[\frac{3\eta(1+w_B)+2}{2\eta(1+w_B)+4}, \frac{1+\eta}{2}\right]. \tag{6.28}$$

This is the most interesting case, as it gives viable models for an accelerating universe. These conditions encompass more solutions than the ones derived in Refs. [144, 145, 146]. For example, for the exponential potential, we have $\Gamma_V = 1$, and the above conditions are satisfied as long as $\eta < 1$ (including, for example, the Gauss-Bonnet expansion law).

b. If $w_\phi > w_B$, then tracking behavior is observed for

$$\frac{3\eta(1+w_B)+2}{2\eta(1+w_B)+4} < \Gamma_V < \frac{\eta+1}{2}. \tag{6.29}$$

c. If $\Gamma_V = (1+\eta)/2$, then $w_\phi = w_B$. This is one of the main results (using somewhat different notation) derived in Ref. [145].

6.2 k-essence

Tracking solutions

In general, k-essence can be defined as any scalar field with non-canonical kinetic terms, but in practice such models are usually taken to have a Lagrangian of the form:

$$\mathcal{L} = V(\phi)F(X), \tag{6.30}$$

where ϕ is the scalar field, and X is defined by

$$X = \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi. \tag{6.31}$$

The pressure in these models is given by

$$p_\phi = \mathcal{L}, \tag{6.32}$$

where $\mathcal{L}$ is given by equation (6.30), while the energy density is

$$\rho_\phi = V(\phi)[2XF_X - F], \tag{6.33}$$

where $F_X \equiv dF/dX$. Therefore, the equation of state parameter, $w_\phi \equiv p_\phi/\rho_\phi$, is just

$$w_\phi = \frac{F}{2XF_X - F}. \tag{6.34}$$

In defining the sound speed, we follow the convention of Garriga and Mukhanov [129], who argued that the relevant quantity for the growth of density perturbations is

$$c_s^2 = \frac{(\partial p/\partial X)}{(\partial \rho/\partial X)} = \frac{F_X}{F_X + 2XF_{XX}}, \tag{6.35}$$

with $F_{XX} \equiv d^2F/dX^2$.

In a flat Robertson-Walker metric, the equation of motion for the k-essence field takes the form:

$$(F_X + 2XF_{XX})\ddot{\phi} + 3HF_X\dot{\phi} + (2XF_X - F)\frac{V_\phi}{V} = 0. \tag{6.36}$$

We can express the equation of motion for ϕ in an alternative form which will be useful for subsequent analysis:

$$\pm\frac{V_\phi}{V}\sqrt{2X} = H\left(\frac{1+w_\phi}{2}\right)(6 + Ay'), \tag{6.37}$$

where

$$\begin{aligned} A &= \frac{(XF_X - F)(2XF_{XX} + F_X)}{XF_X^2 - FF_X - XFF_{XX}} \\ &= \frac{1-w_\phi}{c_s^2 - w_\phi}, \end{aligned} \tag{6.38}$$

$y = (1+w_\phi)/(1-w_\phi)$ and $y' = d\ln y/d\ln a$, and plus (minus) sign corresponds to $\dot\phi < 0$ ($\dot\phi > 0$), respectively. The tracker condition ($w_\phi \approx$ constant) becomes

$$\pm \frac{V_\phi}{V^{(n+2)/2}} \approx \left(\frac{F}{\Omega_\phi}\right)^{n/2} \frac{1}{\sqrt{2X}}. \tag{6.39}$$

It is not surprising to see that the tracker condition for k-essence has an extra "degree of freedom" in $F(X)$. The functional form of $F(X)$ plays a crucial role in determining the tracking conditions for k-essence and we shall consider it in the next section.

After taking the time derivative of equation (6.37) and using equation (6.12), we obtain

$$\begin{aligned} \Gamma_V - (1+\frac{\eta}{2}) &= \frac{\eta(w_B - w_\phi)\Omega_B}{2(1+w_\phi)} \\ &- \frac{[\eta(w_B - w_\phi)\Omega_B + \eta + (\eta-2)w_\phi]Ay'}{2(1+w_\phi)(6+Ay')} \\ &- \frac{2(1-w_\phi)y''}{(1+w_\phi)(6+Ay')^2(c_s^2 - w_\phi)} \\ &- \frac{2\left(\dot w_\phi(1-c_s^2) - (dc_s^2/dt)(1-w_\phi)\right)y'/H}{(1+w_\phi)(6+Ay')^2(c_s^2-w_\phi)^2}, \end{aligned} \tag{6.40}$$

where $y'' = d^2\ln y/d\ln a^2$. We note that for $\eta = 1$, equation (6.40) reduces to the one derived in Ref. [133].

For a background-dominated universe with a constant w_ϕ and almost constant Γ_V, the tracker equation (6.40) reduces to

$$\begin{aligned} \Gamma_V &\approx \frac{\eta+2}{2} + \frac{\eta(w_B - w_\phi)}{2(1+w_\phi)}, \\ &\approx 1 + \frac{\eta}{2}\left(\frac{1+w_B}{1+w_\phi}\right) \end{aligned} \tag{6.41}$$

Note that equation (6.41) for k-essence closely resembles equation (6.18) for quintessence; the only difference is the constant appearing in the first term. For the standard Hubble expansion law ($\eta = 1$), we obtain

$$\Gamma_V \approx 1 + \frac{1}{2}\left(\frac{1+w_B}{1+w_\phi}\right), \tag{6.42}$$

in agreement with the results of Ref. [133].

Stability of the tracking solutions

To determine the stability of the tracking solution, we repeat the calculation of Sec. II.B. for the case of k-essence. We assume a k-essence field with equation of state parameter w_0 and perturb w_0 by an amount δ. Then we expand equation (6.40) to lowest order in δ and its derivatives to obtain

$$\begin{aligned} &2\delta'' + 3[\eta(1+w_B) - 2w_0]\delta' \\ &+9\eta(1+w_B)(c_s^2 - w_0)\delta = 0, \end{aligned} \tag{6.43}$$

where the prime means $d/d\ln a$. The solution of this equation is

$$\delta \propto a^\gamma, \tag{6.44}$$

where

$$\begin{aligned} \gamma = &-\frac{3}{4}\left[\eta(1+w_B) - 2w_0\right] \\ &\pm\frac{3i}{4}\sqrt{8\eta(1+w_B)(c_s^2 - w_0) - [\eta(1+w_B) - 2w_0]^2}. \end{aligned} \tag{6.45}$$

Again, in order to have δ decay, the real part of γ has to be negative. Hence, it follows that

$$w_0 < \frac{\eta(1+w_B)}{2}, \tag{6.46}$$

and

$$w_0 < c_s^2. \tag{6.47}$$

At this point, the above conditions cannot be translated into relations in terms of Γ_V without considering the functional form of $F(X)$, since w_ϕ and c_s^2 both depend on $F(X)$. Now we discuss the restrictions on the form of $F(X)$ for constant w_ϕ.

A variety of functional forms for $F(X)$ and $V(\phi)$ have been considered in k-essence models (see, e.g., Refs. [131, 132]). However, we will focus on the form of $F(X)$ responsible for stable tracking solutions for a constant equation of state. In order to find the functional form of $F(X)$ for stable tracking solutions with constant w_ϕ, we note that equation (6.34) can be written as

$$\frac{\partial \ln F(X)}{\partial \ln X} = \frac{1 + w_\phi}{2w_\phi}. \tag{6.48}$$

Case 1. The first possibility emerges if we treat equation (6.48) as a differential equation and derive the general solution, which is

$$F(X) = X^\beta, \tag{6.49}$$

where β is a constant, and w_ϕ is then

$$w_\phi = \frac{1}{2\beta - 1}. \tag{6.50}$$

By inserting equation (6.49) into equation (6.35), we obtain

$$c_s^2 = \frac{1}{2\beta - 1}, \tag{6.51}$$

so that

$$c_s^2 = w_\phi. \tag{6.52}$$

These solutions were previously derived in Ref. [148]; we note here that they are independent of η, and therefore of the expansion law. These solutions also do not depend on the form of $V(\phi)$.

It is obvious from equation (6.52) that $c_s^2 < 0$ for any of these models with negative pressure ($w_\phi < 0$). If $c_s^2 < 0$, then the k-essence fluid is unstable against perturbation. Moreover, equation (6.49) describes a phantom field for $0 < \beta < 1/2$.

Case 2. A second class of solutions arises if the field evolves to a state for which $X = X_0$, where X_0 is a constant [134, 130]. In this case, we have [130]

$$\left.\frac{\partial \ln F(X)}{\partial \ln X}\right|_{X=X_0} = \frac{1+w_\phi}{2w_\phi} \tag{6.53}$$

Again, we see that equation (6.53) is independent of η and hence, independent of the expansion law. However, the condition for a stable solution of the form $X = X_0$ does depend on η, as we now show.

From equation (6.41), the tracking conditions, equations (6.46)-(6.47), take the following form in terms of Γ_V:

$$\Gamma_V > \frac{2\eta(1+w_B)+2}{\eta(1+w_B)+2}, \tag{6.54}$$

and

$$\Gamma_V > 1 + \frac{\eta(1+w_B)}{2(1+c_s^2)}. \tag{6.55}$$

Therefore, for a nearly constant Γ_V, η, and w_ϕ, equation (6.39) gives the following possibilities:

a. If $w_\phi < w_B$, then Ω_ϕ increases with time. Then we conclude from equation (6.39) that $|\sqrt{2X}V_\phi/F^{\frac{\eta}{2}}V^{\frac{\eta+2}{2}}|$ decreases for a tracker solution. However, taking the time derivative of $\left(\sqrt{2X}V_\phi/F^{\frac{\eta}{2}}V^{\frac{\eta+2}{2}}\right)$, we obtain

$$\frac{d}{dt}\left(\frac{V_\phi}{V^{\frac{\eta+2}{2}}}\frac{\sqrt{2X}}{F^{\frac{\eta}{2}}}\right) = \frac{2X}{F^{\frac{\eta}{2}}}\frac{V_\phi^2}{V^{\frac{\eta+4}{2}}}\left(\Gamma_V - \frac{\eta+2}{2}\right). \tag{6.56}$$

In the derivation of this equation, we have used the condition that $X = X_0$. Hence, $|\sqrt{2X}V_\phi/F^{\frac{\eta}{2}}V^{\frac{\eta+2}{2}}|$ decreases if $\Gamma_V > (\eta+2)/2$. Thus, $w_\phi < w_B$ for

$$\Gamma_V > \frac{\eta+2}{2}. \tag{6.57}$$

Combining this with the conditions for stable tracking behavior (equations 6.54-6.55), we obtain

$$\Gamma_V > max[\frac{\eta+2}{2}, \frac{2\eta(1+w_B)+2}{\eta(1+w_B)+2}, 1+\frac{\eta(1+w_B)}{2(1+c_s^2)}]. \tag{6.58}$$

b. If $w_\phi > w_B$, then tracking behavior is observed for

$$max[\frac{2\eta(1+w_B)+2}{\eta(1+w_B)+2}, 1+\frac{\eta(1+w_B)}{2(1+c_s^2)}] < \Gamma_V < \frac{\eta+2}{2}. \tag{6.59}$$

c. If $\Gamma_V = (\eta+2)/2$, then $w_\phi = w_B$. This case encompasses the solutions presented in Ref. [146].

6.3 Conclusion

We have extended the formalism in Refs. [127] and [133] to derive the tracker conditions for quintessence and k-essence, respectively, for an arbitrary cosmological expansion law, $H^2 = f(\rho)$, when the universe is dominated by a background fluid. Our main new result is that, with the exception of the special cases discussed above, tracking solutions for either quintessence or k-essence are possible only for $\eta = d\ln f/d\ln\rho \approx constant$, which is the case only when $f(\rho)$ is well-approximated as a power-law. In fact, such power-law behavior corresponds to most of the models previously considered for non-standard expansion laws.

We note further that the expressions for w_ϕ for both quintessence and k-essence, and the conditions for stable tracking behavior, can be derived by replacing $1+w_B$ with $\eta(1+w_B)$ in all of the corresponding equations for the standard expansion law. This is not surprising, since a given value of w_B corresponds to a background density scaling as $\rho_B \propto a^{-3(1+w_B)}$. Taking a constant value of η in equation (6.9) then gives $H^2 \propto a^{-3\eta(1+w_B)}$, so $1+w_B$ is replaced by $\eta(1+w_B)$ in the expression for H^2 (see also the discussion in Ref. [149]).

6.4 APPENDIX

0.1 Signature And Weyl Representation

It is a well-known fact that the Dirac matrices, γ_μ, in any represtation must satisfy the following relation to form a Clifford algebra, that is,

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}, \tag{60}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. In QFT, the representations of the Dirac matrics are most commonly expressed in the signature $(+---)$. In this signature, the above relation can be decomposed as:

$$\gamma_0^2 = \widehat{I}, \quad \gamma_i^2 = -\widehat{I}, \quad \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \quad \forall \ \ \mu \neq \nu, \tag{61}$$

where $\widehat{I}$ is the 4×4 unit matrix. Note that $\gamma_0^2 = \widehat{I}$ and $\gamma_i^2 = -\widehat{I}$ implies that γ_0 is Hermitean and so unitary, but γ_μ is anti-Hermitean and unitary. Now, changing the signature from $(+---)$ to $(-+++)$, the above relations (61) take the following form:

$$\gamma_0^2 = -\widehat{I}, \quad \gamma_i^2 = \widehat{I}, \quad \gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu \quad \forall \ \ \mu \neq \nu. \tag{62}$$

Notice that the Hermiticity of the Dirac matrices has changed under the above signature-transformation, though the unitarity is preserved. In the new signature $(-+++)$, γ_0 is unitary and anti-Hermitean while γ_μ is unitary and Hermitean; in other words, $\gamma_0^\dagger = -\gamma_0$ and $\gamma_\mu^\dagger = \gamma_\mu$. Now, let us make this observation that the relations in (62) can easily be obtained from (61) just by multiplying each of the Dirac matrices by an i (imaginary). We also need to check the effect the above changes in the Dirac matrices have on the Lorentz transformation and the Dirac bilinears.

Let us first note that $\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3$ is invariant under this signature transformation. Next, a generic Lorentz transformation Λ acting on Ψ can be represented by $S(\Lambda) = \exp(-\frac{1}{4} i\omega^{\mu\nu}\sigma_{\mu\nu})$, where $\omega^{\mu\nu}$ is the antisymmetric tensor with six independent components corresponding to the three rotation and three boost parameters, and $\sigma_{\mu\nu}$ is the generator defined as $\sigma_{\mu\nu} := \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. This representation of the Lorentz transformation, $S(\Lambda)$ can be derived from infinitesimal Lorentz transformations in the standard manner presented in QFT, and hence we will not explore this here. However, we do need to show that $\overline{\Psi}\Psi$ is a Lorentz scalar under the above signature transformation. In order to show this, the transformations of the following expressions are needed to be computed first. Thus, in the new signature $(-+++)$,

$$\left(\sigma^{ij}\right)^\dagger = \sigma^{ij} \quad , \quad \left(\sigma^{0j}\right)^\dagger = -\sigma^{0j} \, , \tag{63}$$

$$[\gamma^0, \sigma^{ij}] = 0 \quad , \quad \left[\gamma^0, \sigma^{0j}\right]_+ = 0 \, , \tag{64}$$

where the bracket and the curly-bracket imply commutator and anti-commutator respectively, the lower case roman letters i, j indicates the internal spatial indices, and 0 implies time coordinate. We want to emphasize that lowering or raising the time coordinate change the sign of the gamma matrices. Now, using the above relations (63), we obtain the following relation after some algebra:

$$\gamma^0 S^\dagger \gamma^0 = -S^{-1}, \tag{65}$$

where

$$S^\dagger = \exp\left(\frac{i}{4}\omega_{ij}\sigma^{ij} - \frac{i}{2}\omega_{01}\sigma^{01}\right) \, . \tag{66}$$

Finally, writing the equation (65) as $S^\dagger\gamma^0 = \gamma^0 S^{-1}$, we obtain the following desired result:

$$\begin{aligned} \overline{\Psi}'(x')\Psi'(x') &= \Psi'^\dagger(x')\gamma^0\Psi'(x') = (S\Psi)^\dagger\gamma^0\Psi'(x') \\ &= \Psi^\dagger(x)S^\dagger\gamma^0 S\Psi(x) = \Psi^\dagger(x)\gamma^0 S^{-1}S\Psi(x) \\ &= \overline{\Psi}(x)\Psi(x) \, , \end{aligned} \tag{67}$$

where the primed indices represent the primed frame of reference. Similarly, using the property, $S^{-1}\gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu$, which arises from the covariance (form invariance) of the Dirac equation under Lorentz transformation, one can show that $\overline{\Psi}\gamma^5\Psi$, $\overline{\Psi}\gamma^\mu\Psi$, $\overline{\Psi}\gamma_5\gamma^\mu\Psi$, and $\overline{\Psi}\sigma^{\mu\nu}\Psi$ transform like a pseudoscalar, a vector, a pseudovector, and a second rank tensor respectively. Hence, the set of 16 matrices $\{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma^5, \gamma^5\}$ forms a complete basis of the space of all 4×4 matrices. Therefore, the above signature-transformation does not change the Clifford algebra; it only changes the basis of the Dirac matrices.

So far, the analysis has been independent of a specific representation of the Dirac matrices. However, the Weyl representation of the Dirac matrices is the most commonly used to describe massless fermions, so we work with the Weyl representation in this paper. In the new signature $(-+++)$, the gamma matrices take the following form in the Weyl representation:

$$\gamma^0 = \begin{pmatrix} 0 & i\hat{I}_2 \\ i\hat{I}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\hat{I}_2 & 0 \\ 0 & \hat{I}_2 \end{pmatrix}.$$

0.2 The su(2) Spin Connection Γ^i_a on Σ

Torsion-free spin connection

In the torsion-free case, an explicit expression for the su(2) valued spin connection $\widetilde{\Gamma}^i_a$ can be derived from the fact that the covariant derivative of a co-triad vanishes: $D_a e^i_b = \partial_a e^j_b - \Gamma^c_{ab} e^j_c + \widetilde{\Gamma}_{ai}{}^j e^i_b = 0$. Thus, $\widetilde{\Gamma}_{ak}{}^j = -e^b_k(\partial_a e^j_b - \Gamma^c_{ab} e^j_c)$ and

$$\widetilde{\Gamma}^i_a = \frac{1}{2}\epsilon^{ij}{}_k \widetilde{\Gamma}_{aj}{}^k = -\frac{1}{2}\epsilon^{ij}{}_k e^b_j(\partial_a e^k_b - \Gamma^c_{ab} e^k_c) \tag{68}$$

where Γ^c_{ab} is the usual torsion-free Levi-Civita connection for $\widetilde{\Gamma}_{aj}{}^k := \widetilde{\Gamma}^l_a \epsilon_{jl}{}^k$ is used. With the definition of the Levi-Civita connection and $q_{ab} := e^k_a e^k_b$ we obtain

$$e^j_c \Gamma^c_{ab} = \frac{1}{2}\left(e^{jd} e^k_b \partial_a e^k_d + 2\partial_{(a} e^j_{b)} + e^{dj} e^k_a \partial_b e^k_d - e^{dj} e^k_a \partial_d e^k_b - e^{dj} e^k_b \partial_d e^k_a\right). \tag{69}$$

Inserting (69) into (68) , we finally obtain the desired expression for the spin connection

$$\widetilde{\Gamma}^i_a = -\frac{1}{2}\epsilon^{ij}{}_k\widetilde{\Gamma}_{aj}{}^k = \frac{1}{2}\epsilon^{ij}{}_k e^b_j(\partial_a e^k_b - \Gamma^c_{ab}e^k_c) = \frac{1}{2}\epsilon^{ijk}e^b_k(2\partial_{[b}e^j_{a]} + e^c_j e^l_a\partial_b e^l_c)\,. \quad (70)$$

The following expressions are useful for computing Γ^i_a with torsion from the variational equations in the presence of fermions:

$$e^a_i\widetilde{\Gamma}^i_a = -\frac{1}{2}\epsilon^{ijk}e^b_k e^a_i\partial_a e^j_b = \frac{1}{2\sqrt{q}}\epsilon^{abc}e^j_c\partial_a e^j_b\,, \quad (71)$$

and

$$\begin{aligned}
& \delta^k_l\epsilon^{bcd}e^n_c\partial_b e_{dn} + 2\epsilon^{bcd}e^k_d\partial_b e^l_c \\
= & \operatorname{sgn}\det(e^i_a)(\frac{\sqrt{q}}{2}\epsilon^{ijk}\epsilon_{ijl}\epsilon^{mnp}e^a_m e^b_n\partial_b e_{ap} + 2\sqrt{q}\epsilon^{ijk}e^a_j e^b_i\partial_b e^l_a) \\
= & \operatorname{sgn}\det(e^i_a)\sqrt{q}\epsilon^{ijk}(2e^a_i e^b_l\partial_{[a}e^b_{b]} + e^a_j e^b_i\partial_b e^l_a)\,. \quad (72)
\end{aligned}$$

Finally, the Gauss constraint $D_bP^{bm} = \partial_bP^{bm} + \epsilon_{ij}{}^m\Gamma^i_bP^{bj} = \frac{1}{2(1+\gamma^2)}\sqrt{q}J^m$ for the densitized triad P^a_i implies

$$\begin{aligned}
\Gamma^k_bP^{bl} - \Gamma^l_bP^{bk} &= -\epsilon_m{}^{kl}\partial_bP^{bm} + \frac{1}{2(1+\gamma^2)}\,\epsilon_m{}^{kl}\sqrt{q}J^m \\
&= \frac{\operatorname{sgn}\det(e^i_a)}{\gamma\kappa}\left(-\epsilon^{bcd}e^l_d\partial_b e^k_c + \epsilon^{bcd}e^k_d\partial_b e^l_c\right) + \frac{1}{2(1+\gamma^2)}\,\epsilon_m{}^{kl}\sqrt{q}J^m\,. \quad (73)
\end{aligned}$$

Connection with torsion

Varying the action by connection components, we obtain

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta({}^-A^l_c)} = & \frac{1+\gamma^2}{2}\epsilon^j{}_{lk}P^c_j\omega_t{}^{k0} + \frac{1+\gamma^2}{2}\epsilon^j{}_{kl}P^{[c}_iN^{a]}({}^+A^k_a + {}^-A^k_a) + \frac{1+\gamma^2}{2\gamma\kappa}\epsilon^{acd}\partial_a(e_{dl}N) \\
& + \frac{\gamma^2(1+\gamma^2)\kappa}{2\sqrt{q}}\frac{N}{\kappa}P^a_{[k}P^c_{l]}({}^+A^k_a - {}^-A^k_a) + \frac{N^c}{4}\sqrt{q}\left(\gamma + \frac{1}{\alpha}\right)J_l \\
& - \frac{\gamma\kappa N}{4}P^c_l\left(\gamma + \frac{1}{\alpha}\right)J^0 - \frac{\gamma\kappa N}{4}\,\epsilon^j{}_{lk}P^c_j\left(1 - \frac{\gamma}{\alpha}\right)J^k = 0\,, \quad (74)
\end{aligned}$$

which in the canonical formulation serves as one of the second class constraints. After expressing (74) in terms of Γ_a^i and K_a^i first and then contracting with e_c^m, we obtain

$$\begin{aligned}
&\frac{1+\gamma^2}{2\gamma\kappa}\epsilon^m{}_{lk}\sqrt{q}\ \omega_t{}^{k0} - \frac{1+\gamma^2}{2\gamma\kappa}\epsilon^m{}_{kl}\sqrt{q}\ N^a K_a^k + \frac{1+\gamma^2}{2\gamma\kappa}\sqrt{q}\ e_i^a e_c^m \epsilon^i{}_{kl} N^c K_a^k \\
&+\mathrm{sgn}\det(e_a^i)\frac{1+\gamma^2}{2\gamma\kappa}\epsilon^{bcd}e_c^m e_{dl}\partial_b N + \mathrm{sgn}\det(e_a^i)\frac{1+\gamma^2}{2\gamma\kappa}\epsilon^{bcd}e_c^m N\partial_b e_{dl} \\
&-\frac{(1+\gamma^2)}{2\gamma\kappa}\sqrt{q}\ N(e_i^a\delta_l^m - e_l^a\delta_i^m)\Gamma_a^i \\
&= -\frac{\beta N^c e_c^m}{4}\sqrt{q}J_l + \frac{N\theta}{4}\epsilon^m{}_{lk}\sqrt{q}J^k + \frac{\beta N}{4}\delta_l^m\sqrt{q}J^0\,. \qquad (75)
\end{aligned}$$

Contracting it with δ_m^l and using the Gauss constraint, this equation simplifies considerably to

$$\mathrm{sgn}\det(e_a^i)\frac{1+\gamma^2}{2\gamma\kappa}N\epsilon^{bcd}e_c^l\partial_b e_{dl} - (1+\gamma^2)NP_i^a\Gamma_a^i = \frac{3}{4}\beta N\sqrt{q}J^0\,. \qquad (76)$$

Symmetrizing the indices m and l in (75) and using (76) for $e_i^a\Gamma_a^i$, we obtain the following symmetric combination of P_l^a and Γ_a^m

$$\begin{aligned}
\gamma\kappa(P_l^a\Gamma_a^m + P_m^a\Gamma_a^l) &= \mathrm{sgn}\det(e_a^i)(\delta_l^m\epsilon^{bcd}e_c^n\partial_b e_{dn} - \epsilon^{bcd}e_c^m\partial_b e_{dl} - \epsilon^{bcd}e_{cl}\partial_b e_d^m) \\
&\quad -\frac{\beta\gamma\kappa}{2(1+\gamma^2)}\delta_l^m\sqrt{q}J^0\,. \qquad (77)
\end{aligned}$$

On the other hand, the second class constraints can be seen to provide an equation $2\partial_b P^{bm} + 2\epsilon_i{}^{jm}P_j^b\Gamma_b^i = \theta\sqrt{q}J^m/(1+\gamma^2)$, or

$$\gamma\kappa(P^{al}\Gamma_a^m - P^{am}\Gamma_a^l) = \mathrm{sgn}\det(e_a^i)(\epsilon^{bcd}e_d^m\partial_b e_c^l + \epsilon^{bcd}e_{cl}\partial_b e_d^m) + \frac{\theta\gamma\kappa}{2(1+\gamma^2)}\epsilon_j{}^{ml}\sqrt{q}J^j\,. \qquad (78)$$

Combining (77) and (78) yields

$$\begin{aligned}
2\gamma\kappa P^{al}\Gamma_a^k &= \mathrm{sgn}\det(e_a^i)(\delta^{kl}\epsilon^{bcd}e_c^n\partial_b e_{dn} + 2\epsilon^{bcd}e_d^k\partial_b e_c^l) \\
&\quad +\frac{\gamma\kappa}{2(1+\gamma^2)}\left(\theta\ \epsilon_j{}^{kl}\sqrt{q}J^j - \beta\delta^{kl}\sqrt{q}J^0\right)\,. \qquad (79)
\end{aligned}$$

Next, inserting (72) into (79), we find

$$e_l^c\Gamma_c^k = \frac{1}{2}\epsilon^{ijk}e_i^a(2e_l^b\partial_{[a}e_{b]}^j + e_j^b\partial_a e_{bl}) + \frac{\gamma\kappa}{4(1+\gamma^2)}\left(\theta\ \epsilon_j{}^{kl}J^j - \beta\delta^{kl}J^0\right)\,, \qquad (80)$$

and finally (2.32).

0.3 Pressure

The general, thermodynamical definition of pressure is the negative change of energy by volume, which we can write as

$$P = -\frac{1}{N}\frac{\delta H}{\delta\sqrt{q}} \tag{81}$$

whenever the Hamiltonian $H = \int \mathrm{d}^3x N(x)\mathcal{H}(x)$ is depends isotropically on the metric. Otherwise, one has to use all components of the stress tensor $\delta H/\delta q^{ab}$ which is not proportional to the identity. The derivative by the determinant of the metric can be expressed in terms of metric components by using a suitable change of variables which includes q as an independent one. We thus introduce $q_{ab} =: q^{1/3}\bar{q}_{ab}$ with $\det\bar{q}_{ab} = 1$ such that $\partial q_{ab}/\partial q = \frac{1}{3}q^{-1}q_{ab}$ where all components of $\bar{q}_{ab}$ are kept fixed in the partial derivative. This is exactly what we need to compute pressure since only the volume but not the shape of the fluid is varied. This change of variables implies

$$\frac{\delta}{\delta\sqrt{q}} = 2\sqrt{q}\frac{\delta}{\delta q} = 2\sqrt{q}\sum_{ab}\frac{\partial q_{ab}}{\partial q}\frac{\delta}{\delta q_{ab}} = \frac{2}{3\sqrt{q}}\sum_{ab} q_{ab}\frac{\delta}{\delta q_{ab}}$$

and thus

$$P = -\frac{2}{3N\sqrt{q}}q_{ab}\frac{\delta H}{\delta q_{ab}}\,. \tag{82}$$

We can also verify this by comparing the dynamical effects of H on the metric with the Raychaudhuri equation expressed in terms of the canonical variables which for simplicity we do for homogeneous metrics. Using the following definitions for the extrinsic curvature tensor $K_{ab} = \nabla_a n_b$ (which turns out to be automatically spatial and symmetric without projection if homogeneity is used), the expansion parameter $\theta = K_{ab}q^{ab}$ and the shear $\sigma_{ab} = K_{(ab)} - \frac{1}{3}\theta q_{ab}$, the canonical momentum conjugate to q_{ab} derived from the gravitational Lagrangian is

$$\pi^{ab} = \frac{\sqrt{q}}{16\pi G}\left(K^{ab} - K^c_c q^{ab}\right) = \frac{\sqrt{q}}{16\pi G}\left(\sigma^{ab} - \frac{2}{3}\theta q^{ab}\right)$$

where G is the gravitational constant. Then the Raychaudhuri equation in terms of the canonical variables takes the following form:

$$\dot{\theta} = -8\pi G \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\pi^{ab} q_{ab}}{\sqrt{q}}\right). \tag{83}$$

The canonical equations of motion, in the presence of a matter Hamiltonian H added to the gravitational Hamiltonian to form H_{Total}, become

$$\dot{q}_{ab} = \frac{\delta H_{\mathrm{Total}}}{\delta \pi^{ab}} = \frac{16\pi G N}{\sqrt{q}}\left(2\pi_{ab} - q_{ab}\pi^{c}_{c}\right) + 2D_{(a}N_{b)} \tag{84}$$

and

$$\begin{aligned}\dot{\pi}^{ab} \;=\; & -\frac{\delta H_{\mathrm{Total}}}{\delta q_{ab}} = -\frac{N\sqrt{q}}{16\pi G}\left({}^{(3)}R^{ab} - \frac{1}{2}{}^{(3)}R q^{ab}\right) + \frac{8\pi G N}{\sqrt{q}} q^{ab}\left(\pi_{cd}\pi^{cd} - \frac{1}{2}\pi^2\right) \\ & -\frac{32\pi G N}{\sqrt{q}} q^{ab}\left(\pi^{ac}\pi^{b}_{c} - \frac{1}{2}\pi\pi^{ab}\right) - \frac{\delta H}{\delta q_{ab}} + \frac{\sqrt{q}}{16\pi G}\left(D^a D^b N - q^{ab} D^c D_c N\right) \\ & +\sqrt{q} D_c\left(\frac{N^c \pi^{ab}}{\sqrt{q}}\right) - 2\pi^{c(a} D_c N^{b)},\end{aligned} \tag{85}$$

where D_a is the derivative operator compatible with q_{ab}. Variation of the total action with respect to the lapse function N yields the Hamiltonian constraint equation

$$-\frac{\sqrt{q}}{16\pi G}{}^{(3)}R + \frac{16\pi G}{\sqrt{q}}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right) + H = 0. \tag{86}$$

Upon inserting equations (84), (85), and (86) into equation (83), the Raychaudhuri equation becomes

$$\begin{aligned}\frac{\dot{\theta}}{N} \;=\; & -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - \frac{4\pi G}{N\sqrt{q}} H + \frac{8\pi G}{N\sqrt{q}} q_{ab}\frac{\delta H}{\delta q_{ab}} \\ & +D^a D_a N - 8\pi G D_c\left(\frac{N^c \pi^{a}_{a}}{\sqrt{q}}\right) + \frac{16\pi G}{\sqrt{q}}\pi^{ca} D_c N_a\,,\end{aligned} \tag{87}$$

which, for a homogeneous universe, reduces to

$$\frac{\dot{\theta}}{N} = -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - \frac{4\pi G}{N\sqrt{q}} H + \frac{8\pi G}{N\sqrt{q}} q_{ab}\frac{\delta H}{\delta q_{ab}}. \tag{88}$$

On the other hand, for a perfect fluid distribution, the Raychaudhuri equation is found to be

$$\frac{\dot{\theta}}{N} = -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - 4\pi G\left(\rho + 3P\right). \tag{89}$$

Now comparing equation (88) with equation (89), we verify the canonical formula for the average pressure for a perfect fluid distribution in an anisotropic geometry

$$P = -\frac{2}{3N\sqrt{q}} q_{ab} \frac{\delta H}{\delta q_{ab}} = \frac{2}{3N\sqrt{q}} q^{ab} \frac{\delta H}{\delta q^{ab}}. \tag{90}$$

0.4 Bianchi Models

Bianchi models describe most spatially homogeneous anisotropic cosmologies, which is presented in this section for completeness. Bianchi classified 3-D Lie algebra of a symmetry group S for spatial homogeneity to give rise to different inequivalent cosmological models known as Bianchi models. More precisely, spatial homogeneity implies that there is a symmetry group S and a time function t such that S acts transitively[1] on each spatial slice Σ_t. Then, the symmetry group has a Lie algebra of spatial Killing vector fields $(\xi_I^a)_{I=1,\ldots,3}$ such that

$$[\xi_I, \xi_J] = -\tilde{C}^K{}_{IJ} \xi_K \tag{91}$$

with structure constant $\tilde{C}^K{}_{IJ} \in \mathbb{R}$ being antisymmetric in I and J. Given this Lie algebra, a tangent space basis of invariant spatial vector fields X_I^a can be determined since the Lie bracket (Lie derivative) of these fields with all symmetry generators ξ_I^a must vanish, i.e., by expanding $X_I^a = X_I^J \xi_J^a$, the following relations

$$[\xi_I, X_J]^a = \left(\xi_I^b \nabla_b X_J^K\right) \xi_K^a - X_J^K \tilde{C}^L{}_{IK} \xi_L^a = 0 \tag{92}$$

must satisfy for all functions X_I^J. It follows easily from (92) that the invariant vector fields X_I^a form a closed algebra under Lie brackets:

$$[X_I, X_J]^a = C^N{}_{IJ} X_N^a \quad \text{with} \quad C^N{}_{IJ} := X_I^J X_J^K \tilde{C}^M{}_{LK} (X^{-1})_M^N \,. \tag{93}$$

[1]The action of S on Σ_t is *transitive* if any pair of points p, $q \in \Sigma_t$ can be connected by an element of S, i.e., for all p, $q \in \Sigma_t$, there exists $s \in S$ such that $p = sq$. However, if *simply transitive* or *free* action (instead of just *transitive* action) is considered, then Σ_t can be identified with the symmetry group S and the action on Σ_t corresponds to a group automorphism. This is exactly what is needed for LQG since it is formulated in $su(2)$ valued canonical variables and is therefore followed in section 4.2.

The new structure constants $C^N{}_{IJ}$ are equivalent to the original one since X^J_I is invertible and also they are invariant under the action of S, i.e. $\xi^a_L \nabla_a C^K{}_{IJ} = 0$ for all L since $\{X^a_K\}$ is a basis. Now, given a basis vector fields X^a_I, there is a unique dual basis of the cotangent space given by 1-forms ω^I_a satisfying $X^a_J \omega^I_a = \delta^I_J$. Then, it is straightforward to show that the dual basis vectors are invariant under the action S, i.e. $\mathcal{L}_{\xi_J} \omega^I_a = 0$. These invariant 1-forms satisfy the Maurer-Cartan relations

$$D_{[a}\omega^I_{b]} = -\frac{1}{2} C^I{}_{JK} \omega^J_a \omega^K_b \ . \tag{94}$$

With these constructions, one can define invariant tensors which have vanishing Lie derivatives along the ξ_I. For example, a class of homogeneous metrics can be defined as $h_{ab} = h_{IJ}\omega^I_a \omega^J_b$ with a symmetric matrix h_{IJ} with coefficients depending on time only.

Now, the structure constants cannot be chosen arbitrarily since the symmetry generators ξ_I that form a Lie algebra obey antisymmetry of the Lie bracket and the Jacobi identity. This reduces the possible choice of $C^I{}_{JK}$ to nine types as classified by Bianchi. First, using antisymmetry, all the information in the structure constants can equivalently expressed by a matrix

$$\frac{1}{2} C^I{}_{JK} \epsilon^{JKL} =: n^{(IL)} + A^{[IL]} = n^{(IL)} + \epsilon^{ILK} a_K. \tag{95}$$

In the first step, this matric is decomposed into its symmetric and anitsymmetric parts and then its amtisymmetric part is expressed by its three non-trivial components gathered in the vector a_K. Then, it follows that

$$\begin{aligned} a_I &= \frac{1}{2} C^K{}_{JK} \\ \text{and} \quad C^I{}_{JK} &= \frac{1}{2} C^I{}_{LM} \epsilon^{LMN} \epsilon_{NJK} = \epsilon_{NJK} n^{(IN)} + \delta^I_K a_J - \delta^I_J a_K. \end{aligned} \tag{96}$$

Next, the symmetric matix can be diagonalized by a constant change of basis X^a_I as $n^{(IJ)} = n^{(I)} \delta^{IJ}$ (where we are not summing over I on the right hand side as indicated by the brackets).

Now, the vector a_I splits the Bianchi models into two classes: Bianchi class A models with $a_I = 0$ and Bianchi class B models with $a_I \neq 0$. Finally, with the use the Jacobi identity and suitable re-definition of the Lie algebra basis, all components of the structure constants can be further simplified to have values either zero or ± 1 so only relative signs between the parameters are relevant. The details of the complete classification can be found in [76].

For the Bianchi class A models discussed in Chapter IV, the structure constants take the following values: $C^I{}_{JK} = \epsilon^I{}_{JK} n^{(I)}$ with $n^{(I)} = 0$ for Bianchi I models and $n^{(I)} = +1$ for Bianchi IX models.

0.5 Quantization of The Bianchi IX with EM

In this section, we provide a construction of kinematical Hilbert space for anisotropic models with Maxwell's theory. This construction closely mimics that for fermions presented in section 4.3.

Quantum Kinematics

We start with basic variables according to the Poisson structure of homogeneous models and Maxwell's theory. Since the geometrical sector has already been discussed in details in the literature, we mainly focus on Maxwell's theory. It follows from (4.49) that the Poisson structure of reduced Einstein-Maxwell's theory is given by

$$\{\phi_I^{\mathrm{EM}}, p^J_{\mathrm{EM}}\} = \delta_I^J \ . \qquad (97)$$

Now, as in any loop quantization, states in the connection representation are constructed by taking exponentials

$$
\begin{aligned}
&\exp(\mu_1 c_1 \Lambda^i_1 \tau_i) \in \mathrm{SU}(2) \quad , \quad \exp(\mu_2 c_2 \Lambda^i_2 \tau_i) \in \mathrm{SU}(2) \quad , \quad \exp(\mu_3 c_3 \Lambda^i_2 \tau_i) \in \mathrm{SU}(2), \\
&\exp(in_1 \phi^{\mathrm{EM}}_1) \in \mathrm{U}(1) \quad , \exp(in_2 \phi^{\mathrm{EM}}_2) \in \mathrm{U}(1) \quad , \exp(in_3 \phi^{\mathrm{EM}}_3) \in \mathrm{U}(1) \\
&\qquad\qquad \text{for all} \quad \mu_I,\ n_I \in \mathbb{R},, \Lambda^i_I \in SO(3)
\end{aligned}
\tag{98}
$$

as they arise in holonomies. Using holonomies in the general setting is important for a background independent basic algebra of variables. This crucial feature is then reflected also in symmetric models based on exponentials of connection components. The parameters μ_I and n_I can take any real value, corresponding to evaluating holonomies along straight edges (tangential to X^a_I) of arbitrary length.

Matrix elements of the exponentials in (98) form a C^*-algebra of (almost) periodic functions. Any function generated by this set can be written as

$$
g_{\mathrm{grav}}(c_1, c_2, c_1) = \sum_{\mu_1,\mu_2,\mu_3} \xi_{\mu_1,\mu_2,\mu_3} \exp\left(\tfrac{1}{2} i\mu_1 c_1 + \tfrac{1}{2} i\mu_2 c_2 + \tfrac{1}{2} i\mu_2 c_3\right) \ , \tag{99}
$$

and

$$
g^{\mathrm{EM}}(\phi^{\mathrm{EM}}_1, \phi^{\mathrm{EM}}_2, \phi^{\mathrm{EM}}_3) = \sum_{n_1,n_2,n_3} \chi_{n_1,n_2,n_3} \exp\left(\tfrac{1}{2} in_1 \phi^{\mathrm{EM}}_1 + \tfrac{1}{2} in_2 \phi^{\mathrm{EM}}_2 + \tfrac{1}{2} in_3 \phi^{\mathrm{EM}}_3\right) \ , \tag{100}
$$

with coefficients $\xi_{\mu_1,\mu_2,\mu_3},\ \chi_{n_1,n_2,n_3} \in \mathbb{C}$, and $\mu_1, \mu_2, \mu_3, n_1, n_2, n_3 \in \mathbb{R}$. This provides a complete set of continuous functions on $\overline{\mathbb{R}}_{\mathrm{Bohr}} \times \overline{\mathbb{R}}_{\mathrm{Bohr}} \times \overline{\mathbb{R}}_{\mathrm{Bohr}}$, where $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ is the Bohr compactification of the real line. (By definition, $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ is the compactification of $\mathbb{R}$ such that the set of all continuous functions on it is just the set of almost periodic functions.) Again, all spaces in the product are compact Abelian groups and carry a unique normalized Haar measure $\mathrm{d}\mu(c)$ in the case of $\overline{\mathbb{R}}_{\mathrm{Bohr}}$ as described in the fermion case.

By Cauchy completion, we obtain the total Hilbert space as a tensor product $\mathcal{H} = \mathcal{H}_{\mathrm{grav}} \otimes \mathcal{H}_{\mathrm{EM}} = \mathcal{H}^{\otimes 3}_{\mathrm{Bohr}} \otimes \mathcal{H}^{\otimes 3}_{\mathrm{Bohr}}$ with the Hilbert spaces $\mathcal{H}_{\mathrm{Bohr}} = L^2(\overline{\mathbb{R}}_{\mathrm{Bohr}}, \mathrm{d}\mu(c))$ of square integrable functions on the Bohr compactification of the real line. Orthonormal bases for these spaces are given by $\langle c|\mu\rangle = \exp(i\mu c/2)$, $\mu \in \mathbb{R}$, and $\langle\phi^{\mathrm{EM}}|n\rangle = \exp(in\phi^{\mathrm{EM}}/2)$, $n \in \mathbb{R}$, respectively with

$$\langle\mu|\mu'\rangle = \delta_{\mu,\mu'} \quad , \quad \langle n|n'\rangle = \delta_{n,n'} \,. \tag{101}$$

The configuration variables act in the obvious manner: For all g_1 and g_2 of the form (99) and (100), we have

$$\left(\widehat{g}_1 g_2\right)(c_1, c_2, c_3) = g_1(c_1, c_2, c_3) g_2(c_1, c_2, c_3) \tag{102}$$

and

$$\left(\widehat{g}_1^{\mathrm{EM}} g_2^{\mathrm{EM}}\right)(\phi_1^{\mathrm{EM}}, \phi_2^{\mathrm{EM}}, \phi_3^{\mathrm{EM}}) = g_1^{\mathrm{EM}}(\phi_1^{\mathrm{EM}}, \phi_2^{\mathrm{EM}}, \phi_3^{\mathrm{EM}}) g_2^{\mathrm{EM}}(\phi_1^{\mathrm{EM}}, \phi_2^{\mathrm{EM}}, \phi_3^{\mathrm{EM}}) \tag{103}$$

respectively and the corresponding momentum operators are represented by

$$\hat{p}^I = -i\gamma\ell_{\mathrm{P}}^2 \frac{\partial}{\partial c_I} \qquad \text{and} \quad \hat{p}^J_{\mathrm{EM}} = -i\hbar \frac{\partial}{\partial \phi_J^{\mathrm{EM}}} \,, \tag{104}$$

where $\ell_{\mathrm{P}}^2 = \kappa\hbar$. (The densitized triad and electric field in general are quantized via fluxes, i.e. 2-dimensional integrations over surfaces. In a homogeneous context, however, this is not required and densitized triad components can directly be promoted to operators.)

Common eigenstates of all triad operators $\hat{p}^I$ and electric field operators $\hat{p}^J_{\mathrm{EM}}$ are

$$|\mu_1, \mu_2, \mu_3\rangle := |\mu_1\rangle \otimes |\mu_2\rangle \otimes |\mu_3\rangle \qquad \text{and} \qquad |n_1, n_2, n_3\rangle := |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \tag{105}$$

The eigenvalues of the triad operators can be read off from

$$\hat{p}^I|\mu_1, \mu_2, \mu_3\rangle = \frac{1}{2}\gamma\ell_{\mathrm{P}}^2\mu_I|\mu_1, \mu_2, \mu_3\rangle =: p_I|\mu_1, \mu_2, \mu_3\rangle \tag{106}$$

$$\text{and} \quad \hat{p}^I_{\mathrm{EM}}|n_1, n_2, n_3\rangle = \frac{1}{2}\hbar n_I|n_1, n_2, n_3\rangle =: p_I^{\mathrm{EM}}|n_1, n_2, n_3\rangle \,. \tag{107}$$

Using the basic operators $\hat{p}^I$ one can define the volume operator $\hat{V} = \sqrt{|\hat{p}^1\hat{p}^2\hat{p}^3|}$ which will be used later. Its eigenstates are also $|\mu_1, \mu_2, \mu_3\rangle$ with eigenvalues

$$V(\mu_1, \mu_2, \mu_3) = (\frac{1}{2}\gamma\ell_{\mathrm{P}}^2)^{\frac{3}{2}} \sqrt{|\mu_1\mu_2\mu_3|} \,. \tag{108}$$

A kinematical state $|s\rangle$ for the full Hilbert space is described in the triad representation by coefficients $s_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3}$ defined via,

$$|s\rangle = \sum_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3} s_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3}|\mu_1,\mu_2,\mu_3\rangle \otimes |n_1,n_2,n_3\rangle \,. \tag{109}$$

For a state to be gauge invariant under the residual gauge transformations, the coefficients $s_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3}$ have to satisfy

$$s_{\mu_1,\mu_2,\mu_3;n_1,n_2,n_3} = s_{-\mu_1,-\mu_2,\mu_3;n_1,n_2,n_3} = s_{\mu_1,-\mu_2,-\mu_3;n_1,n_2,n_3} = s_{-\mu_1,\mu_2,-\mu_3;n_1,n_2,n_3} \,. \tag{110}$$

These states are left invariant by the gauge invariant triad operators $|\hat{p}^I|$ and the orientation operator $\mathrm{sgn}(\hat{\mathrm{p}}^1\hat{\mathrm{p}}^2\hat{\mathrm{p}}^3)$. In calculations it is often easier to work with non-gauge invariant states in intermediate steps and project to gauge invariant ones in the end.

Together with the basic derivative operators $\hat{p}^I$ we need multiplication operators which usually arise from (point) holonomies $h_I = \exp(c_{(I)}\Lambda_I^i\tau_i) = \cos(\frac{1}{2}c_I) + 2\Lambda_I^i\tau_i\sin(\frac{1}{2}c_I)$ with action

$$\begin{aligned}
\cos(\frac{1}{2}c_1\delta_1)|\mu_1,\mu_2,\mu_3\rangle &= \frac{1}{2}(|\mu_1+\delta_1,\mu_2,\mu_3\rangle + |\mu_1-\delta_1,\mu_2,\mu_3\rangle) && (111)\\
\sin(\frac{1}{2}c_1\delta_1)|\mu_1,\mu_2,\mu_3\rangle &= -\frac{1}{2}i(|\mu_1+\delta_1,\mu_2,\mu_3\rangle - |\mu_1-\delta_1,\mu_2,\mu_3\rangle) && (112)\\
\sin(\frac{1}{2}\phi_1^{\mathrm{EM}}\delta_1)|n_1,n_2,n_3\rangle &= -\frac{1}{2}i(|n_1+\delta_1,n_2,n_3\rangle - |n_1-\delta_1,n_2,n_3\rangle) && (113)
\end{aligned}$$

and correspondingly for c_2,c_3,n_2 and n_3.

REFERENCES

[1] A. Ashtekar, Phys. Rev. D **36**, 1587 (1987).

[2] S. Holst, Phys. Rev. D **53**, 5966 (1996).

[3] J. F. Barbero G., Phys. Rev. D **51**, 5507 (1995).

[4] A. Ashtekar and J. Lewandowski, Class. Quantum Grav. **21**, R53 (2004).

[5] T. Thiemann, gr-qc/0110034.

[6] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, UK, 2004).

[7] A. Ashtekar, *Lectures on Non-perturbative canonical gravity* (World Scientific Publishing Co, 1991).

[8] V.W.Guillemin, S. Sternberg, *Symplectic techniques in Physics* (Cambridge University Press, 1984).

[9] P.A.M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, Yeshiva University, New York, 1964).

[10] A. Wipf, *Hamilton's Formalism for Systems with Constraints* in Proceedings of the 117th WE Heraeus Seminar (Springer-Verlag, 1993).

[11] G. Immirzi, Class. Quantum Grav. **14**, L177 (1997).

[12] C. Rovelli and L. Smolin, Nucl. Phys. B **331**, 80 (1990).

[13] A. Ashtekar *et al.*, J. Math. Phys. **36**, 6456 (1995).

[14] C. Rovelli and L. Smolin, Phys. Rev. D **52**, 5743 (1995).

[15] J. C. Baez, in *The Interface of Knots and Physics*, edited by L. Kauffman (American Mathematical Society, Providence, 1996), pp. 167–203.

[16] C. Rovelli and L. Smolin, Nucl. Phys. B **442**, 593 (1995), erratum: Nucl. Phys. B **456**, 753 (1995).

[17] A. Ashtekar and J. Lewandowski, Class. Quantum Grav. **14**, A55 (1997).

[18] A. Ashtekar and J. Lewandowski, Adv. Theor. Math. Phys. **1**, 388 (1997).

[19] Marolf D, *Preprint* gr-qc/9508015

[20] Ashtekar A, Lewandowski J, Marolf D, Mourão J and Thiemann T, Jour. Math. Phys. **36** 6456–6493

[21] T. Thiemann, Class. Quantum Grav. **15**, 839 (1998).

[22] T. Thiemann, Phys. Lett. **B380** 257–264

[23] T. Thiemann, Class. Quantum Grav. **15**, 839–873

[24] T. Thiemann, Class. Quantum Grav. **15**, 1207–1247

[25] T. Thiemann, Class. Quantum Grav. **15**, 1281 (1998).

[26] T. Thiemann, Class. Quantum Grav. **18**, 3293–3338

[27] Gambini R, Lewandowski J, Marolf D and Pullin J, Int. J. Mod. Phys. **D7** 97–109

[28] Lewandowski J and Marolf D, Int. J. Mod. Phys. **D7** 299–330

[29] Gaul G and Rovelli C, Class. Quantum Grav. **18** 1593–1624

[30] S. Mercuri, Phys. Rev. D **73**, 084016 (2006).

[31] M. Tsuda, Phys. Rev. D **61**, 024025 (2000).

[32] Alexandrov S, *Preprint* arXiv:0802.1221

[33] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).

[34] A. Perez and C. Rovelli, Phys. Rev. D **73**, 044013 (2006).

[35] L. Freidel, D. Minic, and T. Takeuchi, Phys. Rev. D **72**, 104002 (2005).

[36] T. Thiemann, Class. Quantum Grav. **15**, 1487 (1998).

[37] H. Nicolai, K. Peeters, and M. Zamaklar, Class. Quantum Grav. **22**, R193 (2005).

[38] T. Thiemann, Class. Quantum Grav. **13**, 1383 (1996).

[39] M. Bojowald, B. Sandhöfer, A. Skirzewski and A. Tsobanjan, arXiv:0804.3365.

[40] Bojowald M. and Das R. 2007 Canonical gravity with fermions *Preprint* arXiv:0710.5722

[41] Morales-Técotl H. A. and Rovelli C. 1994 Fermions in quantum gravity *Phys. Rev. Lett.* **72** 3642–3645

[42] Morales-Técotl H. A. and Esposito G. 1994 Selfdual action for fermionic fields and gravitation *Nuovo Cim. B* **109** 973–982

[43] Morales-Técotl H. A. and Rovelli C. 1995 Loop space representation of quantum fermions and gravity *Nucl. Phys. B* **451** 325–361

[44] A. Ashtekar, A. Corichi, and J. Zapata, Class. Quantum Grav. **15**, 2955 (1998).

[45] M. Bojowald, R. Das, and R. Scherrer, arXiv:0710.5734.

[46] S. Alexander and D. Vaid, hep-th/0702064.

[47] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).

[48] N. Barros e Sa, Int. J. Mod. Phys. D **10**, 261 (2001).

[49] D'Eath P. D. and Halliwell J. J. 1987 Fermions in quantum cosmology *Phys. Rev. D* **35** 1100–1123

[50] Alexander S. and Vaid D. 2006 Gravity Induced Chiral Condensate Formation and the Cosmological Constant *Preprint* hep-th/0609066

[51] M. Bojowald, Living Rev. Relativity **8**, 11 (2005). , `http://relativity.livingreviews.org/Articles/lrr-2005-11/`.

[52] M. Bojowald, Phys. Rev. D **64**, 084018 (2001).

[53] M. Bojowald, Phys. Rev. Lett. **89**, 261301 (2002).

[54] M. Bojowald *et al.*, Phys. Rev. D **74**, 123512 (2006).

[55] M. Bojowald and A. Skirzewski, Rev. Math. Phys. **18**, 713 (2006).

[56] M. Bojowald and A. Skirzewski, in *Current Mathematical Topics in Gravitation and Cosmology (42nd Karpacz Winter School of Theoretical Physics)*, Int. J. Geom. Meth. Mod. Phys. **4**, 25–52 (2007), hep-th/0606232.

[57] M. Bojowald, H. Hernández, M. Kagan, and A. Skirzewski, Phys. Rev. D **75**, 064022 (2007).

[58] M. Bojowald, Class. Quantum Grav. **17**, 1489 (2000).

[59] M. Bojowald, Class. Quantum Grav. **21**, 3733 (2004).

[60] H. Sahlmann and T. Thiemann, Class. Quantum Grav. **23**, 867 (2006).

[61] C. N. Kozameh and M. F. Parisi, Class. Quantum Grav. **21**, 2617 (2004).

[62] M. Bojowald, Gen. Rel. Grav. **38**, 1771 (2006).

[63] P. Singh, Class. Quantum Grav. **22**, 4203 (2005).

[64] Bojowald M. 2001 Absence of a Singularity in Loop Quantum Cosmology *Phys. Rev. Lett.* **86** 5227–5230

[65] Bojowald M. 2007 Singularities and Quantum Gravity In *AIP Conf. Proc.* **910** 294–333 *Preprint* gr-qc/0702144 In: Proceedings of the XIIth Brazilian School on Cosmology and Gravitation

[66] Bojowald M. 2001 Loop Quantum Cosmology IV: Discrete Time Evolution *Class. Quantum Grav.* **18** 1071–1088

[67] Bojowald M. 2002 Isotropic Loop Quantum Cosmology *Class. Quantum Grav.* **19** 2717–2741

[68] Ashtekar A. Pawlowski T. and Singh P. 2006 Quantum Nature of the Big Bang *Phys. Rev. Lett.* **96** 141301

[69] Bojowald M. 2007 Large scale effective theory for cosmological bounces *Phys. Rev. D* **75** 081301(R)

[70] Bojowald M. 2001 Dynamical Initial Conditions in Quantum Cosmology *Phys. Rev. Lett.* **87** 121301

[71] Bojowald M. 2003 Initial Conditions for a Universe *Gen. Rel. Grav.* **35** 1877–1883

[72] Bojowald M. 2003 Homogeneous loop quantum cosmology *Class. Quantum Grav.* **20** 2595–2615

[73] Bojowald M. 2000 Loop Quantum Cosmology: I. Kinematics *Class. Quantum Grav.* **17** 1489–1508

[74] Bojowald M. Date G. and Vandersloot K. 2004 Homogeneous loop quantum cosmology: The role of the spin connection *Class. Quantum Grav.* **21** 1253–1278

[75] Ashtekar A. Pawlowski T. and Singh P. 2006 Quantum Nature of the Big Bang: An Analytical and Numerical Investigation *Phys. Rev. D* **73** 124038

[76] MacCallum M. A. H. and Taub A. H. 1972 Variational Principles and Spatially-Homogeneous Universes, Including Rotation *Commun. Math. Phys.* **25** 173–189

[77] Ellis G. F. R. and MacCallum M. A. H. 1969 A Class of Homogeneous Cosmological Models *Commun. Math. Phys.* **12** 108–141

[78] Bojowald M. 2006 Loop quantum cosmology and inhomogeneities *Gen. Rel. Grav.* **38** 1771–1795

[79] Brunnemann J. and Fleischhack C. 2007 On the Configuration Spaces of Homogeneous Loop Quantum Cosmology and Loop Quantum Gravity *Preprint* arXiv:0709.1621

[80] Bojowald M. and Swiderski R. 2006 Spherically Symmetric Quantum Geometry: Hamiltonian Constraint *Class. Quantum Grav.* **23** 2129–2154

[81] Banerjee K. and Date G. 2007 Loop Quantization of Polarized Gowdy Model on T^3: Quantum Theory *Preprint* arXiv:0712.0687

[82] Demianski M. de Ritis R. Platania G. Scudellaro P. and Stornaiolo C. 1987 Inflation in a Bianchi type-I Einstein–Cartan cosmological model *Phys. Rev. D* **35** 1181–1184

[83] Bradas J. C. Fennelly A. J. and Smalley L. L. 1987 Torsion as a source of expansion in a Bianchi type-I universe in the self-consistent Einstein–Cartan theory of a perfect fluid with spin density *Phys. Rev. D* **35** 2302–2308

[84] Fewster C. and Sahlmann H. 2008 Phase space quantization and Loop Quantum Cosmology: A Wigner function for the Bohr-compactified real line *Preprint* arXiv:0804.2541

[85] Ashtekar A. Bojowald M. and Lewandowski J. 2003 Mathematical structure of loop quantum cosmology *Adv. Theor. Math. Phys.* **7** 233–268

[86] Kaminski W. and Lewandowski J. 2007 The flat FRW model in LQC: the self-adjointness *Preprint* arXiv:0709.3120

[87] Bojowald M. 2008 The dark side of a patchwork universe *Gen. Rel. Grav.* **40** 639–660

[88] Bojowald M. Cartin D. and Khanna G. 2007 Lattice refining loop quantum cosmology, anisotropic models and stability *Phys. Rev. D* **76** 064018

[89] Sabharwal S. and Khanna G. 2008 Numerical solutions to lattice-refined models in loop quantum cosmology *Class. Quantum Grav.* **25** 085009

[90] Nelson W. and Sakellariadou M. 2008 Numerical techniques for solving the quantum constraint equation of generic lattice-refined models in loop quantum cosmology *Preprint* arXiv:0803.4483

[91] Böhmer C. G. and Vandersloot K. 2007 Loop Quantum Dynamics of the Schwarzschild Interior *Phys. Rev. D* **76** 104030

[92] Ashtekar A. Pawlowski T. Singh P. and Vandersloot K. 2007 Loop quantum cosmology of k=1 FRW models *Phys. Rev. D* **75** 024035

[93] Martín-Benito M. Garay L. J. and Mena Marugán G. A. 2008 Hybrid Quantum Gowdy Cosmology: Combining Loop and Fock Quantizations *Preprint* arXiv:0804.1098

[94] Bojowald M. 2007 Dynamical coherent states and physical solutions of quantum cosmological bounces *Phys. Rev. D* **75** 123512

[95] Cartin D. and Khanna G. 2006 Wave functions for the Schwarschild black hole interior *Phys. Rev. D* **73** 104009

[96] Moniz P. V. 1996 Supersymmetric Quantum Cosmology — Shaken not stirred *Int. J. Mod. Phys. A* **11** 4321–4382

[97] Kaul R. K. 2007 Holst Actions for Supergravity Theories *Preprint* arXiv:0711.4674

[98] Barvinsky A. Kamenshchik A. and Kiefer C. 1999 Effective action and decoherence by fermions in quantum cosmology *Nucl. Phys. B* **552** 420–444

[99] J. Barrow and R. J. Scherrer, Do Fermions and Bosons Produce the Same Gravitational Field?, *Phys. Rev. D* 70 (2004) 103515, [astro-ph/0406088]

[100] M. Bojowald and R. Das, Radiation equation of state and loop quantum gravity corrections, *Phys. Rev. D* **75**, 123521 (2007).

[101] J.D. Barrow, Mon. Not. R. Astr. Soc. **184**, 677 (1978).

[102] J. Yang, D.N. Schramm, G. Steigman, and R.T. Rood, Astrophys. J. **227**, 697 (1979).

[103] F.S. Accetta, L.M. Krauss, and P. Romanelli, Phys. Lett. B **248**, 146 (1990).

[104] C.J. Copi, A.N. Davis, and L.M. Krauss, Phys. Rev. Lett. **92** (2004) 171301.

[105] T. Clifton, J.D. Barrow, and R.J. Scherrer, Phys. Rev. D **71** (2005) 123526.

[106] K.A. Olive, G. Steigman, and T.P. Walker, Phys. Rep. **333** (2000) 389.

[107] D. Kirkman, D. Tytler, N. Suzuki, J.M. O'Meara, and D. Lubin, Ap.J. Suppl. **149** (2003) 1.

[108] K.A. Olive and E.D. Skillman, Ap.J, **617** (2004) 29.

[109] E.W. Kolb and R.J. Scherrer, Phys. Rev. D **25** (1982) 1481.

[110] C. Bambi, M Giannotti, and F.L. Villante, Phys. Rev. D **71** (2005) 123524.

[111] D.N Spergel, et al., astro-ph/0603449.

[112] M. Bojowald, Quantization ambiguities in isotropic quantum geometry, *Class. Quantum Grav.* 19 (2002) 5113–5130, [gr-qc/0206053]

[113] M. Bojowald, Loop Quantum Cosmology: Recent Progress, In *Proceedings of the International Conference on Gravitation and Cosmology (ICGC 2004), Cochin, India*, *Pramana* 63 (2004) 765–776, [gr-qc/0402053]

[114] M. Bojowald, G. Date and K. Vandersloot, "Homogeneous loop quantum cosmology: The role of the spin connection," Class. Quant. Grav. **21**, 1253 (2004) [arXiv:gr-qc/0311004].

[115] M. Bojowald, H. Hernández, M. Kagan, P. Singh, and A. Skirzewski, Phys. Rev. Lett. **98**, 031301 (2007), [astro-ph/0611685].

[116] M. Bojowald, AIP Conf. Proc. **917**, 130–137 (2007), [gr-qc/0701142].

[117] A. Ashtekar, T. Pawlowski and P. Singh, Quantum Nature of the Big Bang: Improved dynamics, *Phys. Rev. D* 74 (2006) 084003, [gr-qc/0607039]

[118] M. Bojowald, D. Cartin and G. Khanna, Lattice refining loop quantum cosmology, anisotropic models and stability, *Phys. Rev. D* 76 (2007) 064018, [arXiv:0704.1137]

[119] W. Nelson and M. Sakellariadou, Lattice Refining Loop Quantum Cosmology and Inflation, *Phys. Rev. D* 2007, to appear, [arXiv:0706.0179]

[120] W. Nelson and M. Sakellariadou, Lattice Refining LQC and the Matter Hamiltonian, arXiv:0707.0588

[121] M. Bojowald and G. M. Hossain, Quantum gravity corrections to gravitational wave dispersion, arXiv:0709.2365

[122] V. Sahni, astro-ph/0403324.

[123] B. Ratra and P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988).

[124] M.S. Turner and M. White, Phys. Rev. D **56**, 4439 (1997).

[125] R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).

[126] A.R. Liddle and R.J. Scherrer, Phys. Rev. D **59**, 023509 (1999).

[127] P.J. Steinhardt, L. Wang, and I. Zlatev, Phys. Rev. D **59**, 123504 (1999).

[128] C. Armendariz-Picon, T. Damour, and V. Mukhanov, Phys. Lett. B **458**, 209 (1999).

[129] J. Garriga and V.F. Mukhanov, Phys. Lett. B **458**, 219 (1999).

[130] T. Chiba, T. Okabe, M. Yamaguchi, Phys. Rev. D **62**, 023511 (2000).

[131] C. Armendariz-Picon, V. Mukhanov, and P.J. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000).

[132] C. Armendariz-Picon, V. Mukhanov, and P.J. Steinhardt, Phys. Rev. D **63**, 103510 (2001).

[133] T. Chiba, Phys. Rev. D **66**, 063514 (2002).

[134] L.P. Chimento and A. Feinstein, Mod. Phys. Lett. A **19**, 761 (2004).

[135] L.P. Chimento, Phys. Rev. D **69**, 123517 (2004).

[136] R.J. Scherrer, Phys. Rev. Lett. **93**, 011301 (2004).

[137] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999).

[138] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999).

[139] C. Charmousis and J.-F. Dufaux, Class. Quant. Grav. **19**, 4671 (2002).

[140] K. Freese and M. Lewis, Phys. Lett. B **540**, 1 (2002).

[141] G. Huey and R. Tavakol, Phys. Rev. D **65**, 043504 (2002).

[142] N.Yu. Savchenko and A.V. Toporensky, Class. Quant. Grav. **20**, 2553 (2003).

[143] S. Mizuno, S.-J. Lee, and E.J. Copeland, Phys. Rev. D **70**, 043525 (2004).

[144] M. Sami, N. Savchenko and A. Toporensky, Phys. Rev. D **70**, 123528 (2004).

[145] E. J. Copeland, S. J. Lee, J. E. Lidsey and S. Mizuno, Phys. Rev. D **71**, 023526 (2005).

[146] S. Tsujikawa and M. Sami, Phys. Lett. B **603**, 113 (2004).

[147] E. V. Linder, Phys. Rev. D **70**, 023511 (2004).

[148] A. Diez-Tejedor and A. Feinstein, Int. J. Mod. Phys. D **14**, 1561 (2005).

[149] E.J. Copeland, J.E. Lidsey, and S. Mizuno, Phys. Rev. D **73**, 043503 (2006).