Small Cancellation and the Assouad-Nagata Dimension of Finitely Generated Groups

By

Levi Sledd

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Approved:

Denis Osin, Ph.D.

Michael Mihalik, Ph.D.

Constantine Tsinakis, Ph.D.

Spencer Dowdall, Ph.D.

Kelly Holley-Bockelmann, Ph.D.

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

Preliminaries

1.1 Introduction

Much of geometric group theory is concerned with the large-scale geometry of groups considered as metric spaces. One of the fundamental concepts of geometry is that of dimension. Thus, one should have a notion of the dimension of a metric space which only sees its large-scale structure. There are many such notions of dimension, two of which are the subject of this paper. These are asymptotic dimension (asdim) and asymptotic Assouad-Nagata dimension ($\mathrm{asdim}_{\mathrm{AN}}$). In this dissertation, we consider them in the following contexts, from most general to most specific.

- Arbitrary metric spaces. For a metric space X equipped with any metric, $\mathrm{asdim}(X)$ or $\mathrm{asdim}_{\mathrm{AN}}(X)$ can be any natural number, or ∞. When the metric is fixed we always have $\mathrm{asdim}(X) \leq \mathrm{asdim}_{\mathrm{AN}}(X)$.
- Countable groups. If G is a countable group, then we assume G is equipped with a proper left-invariant metric: in particular, G is discrete. While $\mathrm{asdim}_{\mathrm{AN}}(G)$ may depend on the proper left-invariant metric chosen, $\mathrm{asdim}(G)$ is independent of this choice.
- Finitely generated groups. If G is a finitely generated group, then G is assumed to be equipped with the word metric with respect to a finite generating set. In this case both $\mathrm{asdim}(G)$ and $\mathrm{asdim}_{\mathrm{AN}}(G)$ are independent of the choice of finite generating set.

Remark. For discrete groups, *asymptotic* Assouad-Nagata dimension and Assouad-Nagata dimension (usually abbreviated $\dim_{\mathrm{AN}}$) are equivalent. Since all groups mentioned in this paper will be discrete, whenever we speak of groups we will use the shorter "Assouad-Nagata dimension," which we continue to denote by $\mathrm{asdim}_{\mathrm{AN}}$.

These two invariants have turned out to be useful tools in geometric group theory. We refer the reader to [1] or [2] for a good introductory survey on asymptotic dimension in group theory, and to [3] for many of the corresponding results for Assouad-Nagata dimension. In the next section we will give a brief introduction to asymptotic dimension and Assouad-Nagata dimension, providing the precise definitions and stating the results needed in later sections. For now we avoid definitions, instead focusing on the history of asymptotic dimension and Assouad-Nagata dimension, their importance to group theory and to mathematics in general, and their relationship to each other.

Asymptotic dimension was introduced by Gromov in his landmark 1993 paper [4], intended to serve as a large-scale analogue to the Lebesgue covering dimension of a topological space. Since then, the study of asymptotic dimension has yielded some deep results in mathematics, most notably regarding the Novikov Conjecture. This conjecture can be rephrased as a statement about the class of all groups, so the Novikov conjecture is true if every group satisfies it. In 1998, Yu proved the following theorem.

Theorem 1.1.1. [5, Theorem 1.1] *Let G be a finitely generated group whose classifying space BG has the homotopy type of a finite CW-complex. If G has finite asymptotic dimension, then G satisfies the Novikov conjecture.*

The result is all the more powerful since the class of finitely generated groups with finite asymptotic dimension is quite large. It includes hyperbolic groups [6], one-relator groups [7], Coxeter groups [8], mapping class groups, braid groups, and certain types of Artin and Torelli groups [9], and is closed under subgroups, extension [10], amalgamated products [8], HNN extensions [11], and relative hyperbolicity [12].

Asymptotic Assouad-Nagata dimension ($\mathrm{asdim}_{\mathrm{AN}}$) was first defined in 1982 by Assouad and influenced by the work of Nagata [13]. It can be considered a more "restrictive" version of asymptotic dimension, in the sense that $\mathrm{asdim}_{\mathrm{AN}}(X) \leq n$ is by definition a harder condition to satisfy than $\mathrm{asdim}(X) \leq n$, hence $\mathrm{asdim}(X) \leq \mathrm{asdim}_{\mathrm{AN}}(X)$ for every space X with a fixed metric. In particular, groups with finite Assouad-Nagata dimension also have finite asymptotic dimension, and thus satisfy the Novikov conjecture. It is also true that in many cases, theorems regarding asdim have corresponding analogues for $\mathrm{asdim}_{\mathrm{AN}}$: see [3] for examples. However, the study of asymptotic Assouad-Nagata dimension in its own right has produced some interesting results that set it apart from asymptotic dimension theory. For one, spaces with asymptotic Assouad-Nagata dimension at most n admit a quasi-symmetric embedding into a product of $n+1$ metric trees [14]. Another interesting fact is that a Morita-type theorem holds for asymptotic Assouad-Nagata dimension: that is, for any metric space X, we have that $\mathrm{asdim}_{\mathrm{AN}}(X \times \mathbb{R}) = \mathrm{asdim}_{\mathrm{AN}}(X) + 1$ [15]. Neither of these properties are known to hold for asymptotic dimension.

So far we have not ruled out the possibility that asymptotic dimension and Assouad-Nagata dimension are merely the same invariant by different names. However, this is not the case. In 2010, Higes proved the following theorem.

Theorem 1.1.2. [16] *For any $n \in \mathbb{N} \cup \{\infty\}$ there exists a countable, locally finite group G_n and a proper left-invariant metric d_n on G_n such that $\mathrm{asdim}(G_n, d_n) = 0$ but $\mathrm{asdim}_{\mathrm{AN}}(G_n, d_n) = n$.*

Asymptotic dimension and Assouad-Nagata dimension may differ even among finitely generated groups. A result by Brodskiy, Dydak, and Lang on the Assouad-Nagata dimensions of wreath products of groups provides a class of examples such as the following.

Theorem 1.1.3. [17] *We have* $\operatorname{asdim}(\mathbb{Z}_2 \wr \mathbb{Z}^2) = 2$ *but* $\operatorname{asdim}_{\mathrm{AN}}(\mathbb{Z}_2 \wr \mathbb{Z}^2) = \infty$.

In the examples constructed in [17], the Assouad-Nagata dimension is always infinite. In [16], Higes asks whether this is always the case.

Question 1.1.4. [16, Question (2)] *Does there exist a finitely generated group G such that* $\operatorname{asdim}(G) < \operatorname{asdim}_{\mathrm{AN}}(G) < \infty$*?*

Given a way of defining the dimension of an algebraic structure, it is natural to ask whether it is monotonic with respect to substructures: that is, whether $A \leqslant B$ implies that the dimension of A is no greater than the dimension of B. Is our dimension like that of a vector space, where this natural monotonicity holds, or is it like the rank of a free group, where it fails spectacularly? Since asymptotic dimension is well defined for any countable group, it follows that if G is a countable group and $H \leqslant G$, then $\operatorname{asdim}(H) \leq \operatorname{asdim}(G)$. Previously, it was unknown whether the same was true of Assouad-Nagata dimension.

Question 1.1.5. [3, Questions 8.6 and 8.7] *Does there exist a finitely generated group G with a finitely generated subgroup H such that* $\operatorname{asdim}_{\mathrm{AN}}(G) < \operatorname{asdim}_{\mathrm{AN}}(H)$*?*

In this dissertation we answer both of these questions with the following theorem.

Theorem 1. *For every* $k, m, n \in \mathbb{N} \cup \{\infty\}$ *with* $4 \leq k \leq m \leq n$*, there exist finitely generated, recursively presented groups H and G with* $H \leqslant G$*, such that*

$$\operatorname{asdim}(G) = k$$
$$\operatorname{asdim}_{\mathrm{AN}}(G) = m$$
$$\operatorname{asdim}_{\mathrm{AN}}(H) = n\,.$$

Note that if $H \leqslant G$ but $\operatorname{asdim}_{\mathrm{AN}}(H) > \operatorname{asdim}_{\mathrm{AN}}(G)$, it must be that H is distorted in G, and that this distortion collapses H to a space of lesser Assouad-Nagata dimension. However, distortion does not always affect the Assouad-Nagata dimension of the distorted subgroup. For example, in $BS(1,2) = \langle a, b \mid b^{-1}aba^{-2} \rangle$, the subgroup $\langle a \rangle$ is distorted, but still has Assouad-Nagata dimension 1. We call distortion which affects Assouad-Nagata dimension *Assouad-Nagata dimension distortion*. The author hopes that Assouad-Nagata dimension distortion will be an interesting phenomenon to study in its own right, and that more examples can be found in nature.

We construct a group G satisfying the conclusion of Theorem 1 in the following way. First, we adapt Higes' example from Theorem 1.1.2 to construct a countable, locally finite abelian group K with proper left-invariant metrics d_m and d_n such that $\operatorname{asdim}_{\mathrm{AN}}(K, d_m) = m$ and $\operatorname{asdim}_{\mathrm{AN}}(K, d_n) = n$. We then build finitely generated groups A and B as short exact sequences

$$\begin{aligned} 1 \to K_A \to A \to H_A \to 1 \\ 1 \to K_B \to B \to H_B \to 1 \end{aligned} \tag{1.1}$$

where K_A and K_B are isomorphic and bi-Lipschitz equivalent to (K, d_m) and (K, d_n), respectively. The group G from Theorem 1 is then the amalgamated product $A *_\phi B$, where $\phi : K_A \to K_B$ is an amalgamating isomorphism. The idea is that $\operatorname{asdim}_{\mathrm{AN}}(K_B) = n$ when considered as a subspace of B, thus $\operatorname{asdim}_{\mathrm{AN}}(B) \geq n$. But ϕ "crushes" K_B to the size of K_A within G, lowering the asymptotic dimension of G to m. The difficulty is in proving that each group mentioned above has the asymptotic and Assouad-Nagata dimension that we claim it does. For this we use small cancellation theory.

Historically, small cancellation theory has played an important role in geometric group theory. Groups satisfying the small cancellation condition $C'(1/6)$ are often used to create groups satisfying various exotic properties, the most famous example being the Rips construction. In our construction of a group G as in Theorem 1, the groups H_A and H_B from (1.1) are given by carefully-chosen $C'(1/6)$ presentations. We then use techniques of small cancellation theory to prove results guaranteeing that A and B satisfy the desired metric properties. The main result that allows the construction to work is the following theorem, which we believe is of independent interest.

Theorem 2. *If G is a finitely generated $C'(1/6)$ group, then* $\operatorname{asdim}_{\mathrm{AN}}(G) \leq 2$.

We note that in the finitely presented case, this fully classifies the Assouad-Nagata dimension of $C'(1/6)$ groups. Since a finitely presented group has asymptotic dimension 1 if and only if it is virtually free [18, 19], Theorem 2 implies that the Assouad-Nagata dimension of a finitely presented $C'(1/6)$ group is 1 if the group is virtually free, and 2 otherwise. However, the finitely presented case of Theorem 2 was likely already known to experts. Although apparently not in the literature, a MathOverflow post by Agol [20] shows how to obtain that $\operatorname{asdim}(G) \leq 2$ for G a finitely presented $C'(1/6)$ group, using a theorem of Buyalo and Lebedeva that $\operatorname{asdim}(G) = \dim(\partial G) + 1$ when G is hyperbolic [21].

The real importance of Theorem 2 is that it applies to *infinitely* presented $C'(1/6)$ groups, and that with this fact one can readily apply the techniques of small cancellation theory to the study of asymptotic dimension. Indeed, in the construction of G from Theorem 1, the auxiliary $C'(1/6)$ groups H_A and H_B must be infinitely presented for the construction to work. One might then wish to derive Theorem 2 for infinitely presented groups using the same result for finitely presented groups, but this approach cannot work in general. This is because in [22], Osajda constructs a sequence of groups and surjective homomorphisms $G_0 \to G_1 \to G_2 \to \cdots$ such that $\operatorname{asdim}(G_n) = 2$ for all $n \in \mathbb{N}$, but the inductive limit of the sequence has infinite asymptotic dimension.

Instead, the proof of Theorem 2 uses a version of the “tight geodesics” technique pioneered by Bowditch in [23]. As we believe it is of independent interest, we section it off as Proposition 2.1.4. This technique has been used previously to show that certain hyperbolic spaces have finite asymptotic dimension [6, 9, 23]. Although infinitely presented $C'(1/6)$ groups are not hyperbolic, they have enough hyperbolic-like properties to use the tight geodesics strategy. Specifically, the key property is that simple geodesic triangles can take a limited number of forms, which are classified in Strebel's appendix to Ghys and de la Harp's book on hyperbolic groups [24]. The proof of Theorem 2 seems to be the first application of this technique in a non-hyperbolic setting.

The paper is organized as follows. The first four sections are devoted to proving Theorem 2. In Section 1.2 we review the definitions of asymptotic dimension and asymptotic Assouad-Nagata dimension, and collect various results from the literature that are needed in later sections. In Section 2.1, we introduce the notion of an (ε, k)-tight geodesic combing for $\varepsilon > 0$ and $k \in \mathbb{N}$, and show that a geodesic metric space admitting a (ε, k)-tight geodesic combing for some $\varepsilon > 0$ has asymptotic Assouad-Nagata dimension at most k. In Section 2.2 we give some preliminaries on van Kampen diagrams and the classical small cancellation condition $C'(1/6)$. We also review the classification of van Kampen diagrams over simple geodesic triangles in $C'(1/6)$ groups due to Strebel, the essential tool needed in the proof of Theorem 2. In Section 2.3 we use Strebel's classification to prove that $C'(1/6)$ groups admit a $(1/12, 2)$-tight geodesic combing, and thus have Assouad-Nagata dimension at most 2.

Sections 3.1-3.3 are devoted to proving Theorem 1. In Section 3.1, we fix countable group K, constructed as a direct sum of cyclic groups of increasing order. We then show that for each $m, n \in \mathbb{N} \cup \{\infty\}$ with $m < n$, there are two different proper left-invariant metrics on K such that $\operatorname{asdim}_{\mathrm{AN}}(K) = m$ with respect to one, and $\operatorname{asdim}_{\mathrm{AN}}(K) = n$ with respect to the other. In Section 3.2, we use techniques from small cancellation theory to establish a highly technical lemma. This lemma allows us to quasi-isometrically embed K, with respect to each proper left-invariant metric, into a finitely generated group. In Section 3.3, we embed K into finitely generated groups A and B. This is done is such a way that, calling K_A the copy of K in A and K_B the copy of K in B, we have that $\operatorname{asdim}_{\mathrm{AN}}(K_A) = m$ and $\operatorname{asdim}_{\mathrm{AN}}(K_B) = n$. We then identify the two with an isomorphism $\phi : K_A \to K_B$, and let $G = A *_\phi B$. Our technical small cancellation lemma comes back to help us a second time by showing that ϕ “crushes” the image of K_B in G to the size of K_A. With a few calculations using well-known extension theorems for asymptotic and Assouad-Nagata dimension, we are able to prove the following.

Proposition 1. *For any $m, n \in \mathbb{N} \cup \{\infty\}$ with $m < n$, there exists a group $G = A *_\phi B$ where G, A, and B are* *finitely generated and recursively presented, such that*

$$1 \leq \operatorname{asdim}(G) \leq 2$$

$$m+1 \leq \operatorname{asdim}_{\mathrm{AN}}(G) \leq m+2$$

$$n+1 \leq \operatorname{asdim}_{\mathrm{AN}}(B) \leq n+2\,.$$

Using the free product formulas for asymptotic and Assouad-Nagata dimension and the Morita theorem for Assouad-Nagata dimension, it is then easy to derive Theorem 1 from Proposition 1.

There are many technical restrictions placed on the presentations of A and B from Proposition 1. In Section 3.4 we give explicit presentations where these conditions are satisfied. Curiously, although we are able to give an explicit presentation of a group G satisfying the conditions of Proposition 1, we are not quite able to do the same for Theorem 1. However, we can explicitly give presentations of two groups, one of which must be a group satisfying the conclusion of Theorem 1.

1.2 Overview of Asymptotic Dimension

Conventions. When talking about metric spaces, we suppress notation referring to the metric as much as possible. Thus when we say that X is a metric space, we mean that (X,d) is a metric space with metric d. The letter d always stands for a metric, on whatever set makes sense in context. Occasionally we use subscripts, e.g. d_X stands for the metric on X and d_Y stands for the metric on Y, but only when failing to do so would cause confusion. Since we are concerned here with asymptotics, a *linear* function will really mean a function whose growth is linear, i.e. an affine function, and not a function which is linear in the sense of linear algebra. In this paper, $0 \in \mathbb{N}$. The set of positive integers is $\mathbb{Z}^+$. The set of positive real numbers is denoted $\mathbb{R}^+$, and the set of non-negative real numbers is $\mathbb{R}_0^+$.

1.2.1 Metric spaces

The first appearance of asymptotic dimension is in Gromov's [4], where it is introduced as a large-scale analog of the Lebesgue covering dimension, or topological dimension, of a topological space. So before defining asymptotic dimension, we will define topological dimension. Let X be a topological space, $\mathscr{U}$ a cover of X. The *multipilicity* of $\mathscr{U}$ is defined to be the maximum cardinality of a subset of $\mathscr{U}$ whose elements all contain a common point, if this maximum is finite. We say that the multiplicity of $\mathscr{U}$ is ∞ if this maximum is infinite or there is no maximum. We say that a cover $\mathscr{V}$ of X *refines* $\mathscr{U}$ if every element of $\mathscr{V}$ is contained in some element of $\mathscr{U}$.

Definition 1.2.1. Let X be a topological space. Then we write $\dim(X) \leq n$ if for every cover $\mathscr{U}$ of X there exists a cover $\mathscr{V}$ of X of multiplicity at most $n+1$ which refines $\mathscr{U}$. The *topological dimension* of X, denoted $\dim(X)$, is defined to be the least $n \in \mathbb{N}$ such that $\dim(X) \leq n$. If no such n exists, we write $\dim(X) = \infty$.

Before continuing, we fix notation. Let X be a metric space. The open ball of radius r about $x \in X$ is denoted $B(x,r)$. The *diameter* of a set $A \subset X$, denoted $\mathrm{diam}(A)$, is defined to be $\sup\{d(a,a') \mid a,a' \in A\}$. We say that A is *D-bounded* if $\mathrm{diam}(A) \leq D$. If $A,B \subseteq X$, then $d(A,B)$ is defined to be $\inf\{d(a,b) \mid a \in A, b \in B\}$, and we write $d(a,B)$ for $d(\{a\},B)$. If $\mathscr{U}$ is an arbitrary family of subsets of X and $D > 0$, then $\mathscr{U}$ is *uniformly D-bounded* if $\mathrm{diam}(U) \leq D$ for all $U \in \mathscr{U}$. We say that $\mathscr{U}$ is *uniformly bounded* if there exists $D > 0$ such that $\mathscr{U}$ is uniformly D-bounded.

Now we are ready to define asymptotic dimension.

Definition 1.2.2. Let X be a metric space. We write $\mathrm{asdim}(X) \leq n$ if for every uniformly bounded open cover $\mathscr{U}$ of X, there exists a uniformly bounded open cover $\mathscr{V}$ of X of mulitplicity at most $n+1$, such that $\mathscr{U}$ refines $\mathscr{V}$. The *asymptotic dimension* of X, denoted $\mathrm{asdim}(X)$, is defined as the least $n \in \mathbb{N}$ such that $\mathrm{asdim}(X) \leq n$. If no such n exists, we write $\mathrm{asdim}(X) = \infty$.

For metric spaces, Definitions 1.2.1 and 1.2.2 are exactly the same except in the last line, where the answer to the question "which cover refines which?" is reversed. In topological dimension, we start with a cover and lower its multiplicity by passing to a "smaller" cover. In asymptotic dimension, we start with a cover of a space and reduce its multiplicity by passing to a "larger" one. For this reason asymptotic dimension is often considered "dual" to topological dimension. Indeed, there are at least six equivalent definitions of asymptotic dimension, most of which are "dual" to a definition of topological dimension: for an interesting side-by-side comparison, see [25]. We list those six definitions after introducing a bit more notation.

For the following definitions, let X be a metric space, $\mathscr{V}$ a cover of X, $\mathscr{U}$ an arbitrary family of subsets of X, and $r > 0$.

The *r-multiplicity* of $\mathscr{V}$ is the maximum, over all $x \in X$, of the number of elements of $\mathscr{V}$ met by $B(x,r)$. If this maximum is infinite or there is no maximum, we write that the r-multiplicity of $\mathscr{V}$ is ∞. The *Lebesgue number* of $\mathscr{V}$ is the infimum of all positive real numbers λ such that, for any $A \subseteq X$, $\mathrm{diam}(A) < \lambda$ implies that $A \subseteq V$ for some $V \in \mathscr{V}$.

We say $\mathscr{U}$ is *r-disjoint* if $d(U,U') \geq r$ for all $U,U' \in \mathscr{U}$.

Given $A \subseteq X$, an *r-path* from a to a' in A is a sequence $a = a_0, \ldots, a_n = a'$ of elements of A, such that $d(a_i, a_{i+1}) < r$ for all $i \in \{0, \ldots, n-1\}$. A subset $B \subseteq A$ is *r-connected* if there is an r-path from b to b' for all $b,b' \in B$. An *r-component* of A is a maximal r-connected subset of A.

Let K be a countable simplicial complex. Then the *uniform metric* on K is defined by mapping the vertices of K to points which form an orthonormal basis for ℓ^2, extending affinely, and giving K the subspace metric that its image inherits from ℓ^2. If K is endowed with the uniform metric, we call K a *uniform* simplicial complex. A map $\varphi : X \to K$ is *D-cobounded* if $\operatorname{diam}(\varphi^{-1}(\sigma)) < D$ for all simplices σ in K. For a constant $a > 0$, a map $\varphi : X \to K$ is *a-Lipschitz* if $d(\varphi(x), \varphi(x')) \le ad(x,x')$ for all $x, x' \in X$.

Definition 1.2.3. $[1,2,15,17]$ Let X be a metric space, $n \in \mathbb{N}$. Then $\operatorname{asdim}(X) \le n$ if for all arbitrarily large $r > 0$ there exists an $D > 0$ such that any one of the following conditions hold:

(a) For every cover $\mathscr{U}$ of X which is uniformly r-bounded, there exists a cover $\mathscr{V}$ of X such that $\mathscr{U}$ refines $\mathscr{V}$, $\mathscr{V}$ has multiplicity at most $n+1$, and $\mathscr{V}$ is uniformly D-bounded.

(b) There exists a cover $\mathscr{V}$ of X which has r-multiplicity at most $n+1$ and is uniformly D-bounded.

(c) There exist $n+1$ families of sets $\mathscr{U}_0, \dots, \mathscr{U}_n$ whose union covers X, such that each $\mathscr{U}_i$ is r-disjoint and uniformly D-bounded.

(d) There exists a cover $\{X_0, \dots, X_n\}$ of X such that the r-components of each X_i are uniformly D-bounded.

(e) There exists a cover $\mathscr{V}$ of X with Lebesgue number at least r and multiplicity at most $n+1$, which is uniformly D-bounded.

(f) There exists a $\frac{1}{r}$-Lipschitz, D-cobounded map $\varphi : X \to K$, where K is a uniform simplicial complex of dimension n.

The equivalence of all of the above definitions except for (d) is proved in [2], and the equivalence of (d) with, say, (c), is an easy exercise. Note that in Definition 1.2.3, D depends on r. We say that a function $D : \mathbb{R}^+ \to \mathbb{R}^+$ is an *n-dimensional control function for X* if for all $r > 0$, X satisfies $\operatorname{asdim}(X) \le n$ with $D = D(r)$. Notice that replacing r by some $r' < r$ weakens any of the predicates (a)-(f). Therefore we can assume without loss of generality that any n-dimensional control function for X is non-decreasing.

Asymptotic Assouad-Nagata dimension is a version of asymptotic dimension in which the control function is required to be linear.

Definition 1.2.4. $[3,15]$ Let X be a metric space. Then we write $\operatorname{asdim}_{\mathrm{AN}}(X) \le n$ if there exist $a, b > 0$ such that $D(r) = ar + b$ is an n-dimensional control function for X. The *asymptotic Assouad-Nagata dimension of* X, denoted $\operatorname{asdim}_{\mathrm{AN}}(X)$, is defined as the least $n \in \mathbb{N}$ such that $\operatorname{asdim}_{\mathrm{AN}}(X) \le n$, if such an n exists; otherwise we write $\operatorname{asdim}_{\mathrm{AN}}(X) = \infty$.

Given a fixed $r > 0$, it is quite possible that the n-dimensional control function $D(r)$ arising from the equivalent definitions of asymptotic dimension depends on which definition you choose. However, through tedious calculation it is possible to prove that if the control function according to one definition is linear, then all of them are.

Some (but not all) properties of asymptotic dimension also hold for asymptotic Assouad-Nagata dimension. In order to avoid having to say many things twice, if $\operatorname{asdim}_{(\mathrm{AN})}$ appears in a sentence, then the sentence holds true for asdim, and also when asdim is replaced with $\operatorname{asdim}_{\mathrm{AN}}$ consistently throughout the sentence.

Exercise 1.2.5. The following facts about asymptotic dimension and asymptotic Assouad-Nagata dimension are quite useful, but also easy to prove.

- For any metric space (X,d) we have $\operatorname{asdim}(X,d) \leq \operatorname{asdim}_{\mathrm{AN}}(X,d)$.
- For any metric space (X,d) and $A \subseteq X$, we have $\operatorname{asdim}_{(\mathrm{AN})}(A, d|_{A\times A}) \leq \operatorname{asdim}_{(\mathrm{AN})}(X,d)$.
- If X is a bounded metric space, then $\operatorname{asdim}_{(\mathrm{AN})}(X) = 0$.
- If T is a tree, then $\operatorname{asdim}_{(\mathrm{AN})}(T) \leq 1$.

Here we define types of maps between metric spaces that we will refer to frequently. A set $B \subseteq Y$ is called *cobounded* if there exists a constant $c > 0$ such that $d(y,B) \leq c$ for all $y \in Y$.

Definition 1.2.6. Let X and Y be metric spaces, and let $f : X \to Y$ be a function. Suppose that there exist nondecreasing, unbounded functions $\rho_0, \rho_1 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, such that for all $x, x' \in X$,

$$\rho_0(d(x,x')) \leq d(f(x), f(x')) \leq \rho_1(d(x,x')). \tag{1.2}$$

- Then f is a *coarse embedding*.
- If $f(X)$ is cobounded in Y, then f is a *coarse equivalence*, and X and Y are *coarsely equivalent*.
- If there exist constants $a_1, b_1 > 0$ such that (1.2) is satisfied with $\rho_1(d) = a_1 d + b_1$, then f is *asymptotically Lipschitz*.
- If f is asymptotically Lipschitz and there exist constants a_0 and b_0 with $a_0 > 0$ such that (1.2) is satisfied with $\rho_0(d) = \max\{a_0 d + b_0, 0\}$, then f is a *quasi-isometric embedding*.
- If f is a quasi-isometric embedding and $f(X)$ is cobounded in Y, then f is a *quasi-isometry*, and X and Y are *quasi-isometric*.
- Suppose $a_1 > 0$ is a constant such that (1.2) is satisfied with $\rho_1(d) = a_1 d$. Then f is called a_1-*Lipschitz*. We say that f is *Lipschitz* if there exists an $a_1 > 0$ such that f is a_1-Lipschitz.

- If f is Lipschitz and there exists a constant $a_0 > 0$ such that (1.2) is satisfied with $\rho_0(d) = a_0 d$, then f is *bi-Lipschitz*. Note that a bi-Lipschitz map must be injective.

- If f is bi-Lipschitz and surjective, then f is a *bi-Lipschitz equivalence*, and X and Y are *bi-Lipschitz equivalent*.

It is not hard to show that asymptotic dimension is a *coarse invariant*: that is, if $\operatorname{asdim}(X) = n$ and X and Y are coarsely equivalent, then $\operatorname{asdim}(Y) = n$ as well. Similarly, asymptotic Assouad-Nagata dimension is a quasi-isometry invariant.

One very useful result in asymptotic dimension theory is the Hurewicz-type mapping theorem for asymptotic dimension, so called because its statement parallels a theorem of Hurewicz about topological dimension [1]. In order to state it, we need to define the notion of a control function for maps between metric spaces. For Definitions 1.7-1.9, let X and Y be metric spaces, and $f : X \to Y$ a function.

Definition 1.2.7. [3] Let $D_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a function of two variables r and K. For each fixed $K > 0$, we denote by $D_{f,K} : \mathbb{R}^+ \to \mathbb{R}^+$ the function defined by $D_{f,K}(r) = D_f(r,K)$. Then D_f is an *n-dimensional control function for f* if, for every $A \subseteq X$ such that $f(A)$ is K-bounded, $D_{f,K}$ is an n-dimensional control function for A with the metric inherited from X.

Definition 1.2.8. We write $\operatorname{asdim}(f) \leq n$ if f has an n-dimensional control function. The *asymptotic dimension* of f, denoted $\operatorname{asdim}(f)$, is the defined to be the least $n \in \mathbb{N}$ such that $\operatorname{asdim}(f) \leq n$, if such an n exists; otherwise we write $\operatorname{asdim}(f) = \infty$.

Definition 1.2.9. We write $\operatorname{asdim}_{\mathrm{AN}}(f) \leq n$ if there exist constants a, b, c such that $D_f(r,K) = ar + bK + c$ is an n-dimensional control function for f. The *asymptotic Assouad-Nagata dimension* of f, denoted $\operatorname{asdim}_{\mathrm{AN}}(f)$, is defined to be the least $n \in \mathbb{N}$ such that $\operatorname{asdim}_{\mathrm{AN}}(f) \leq n$, if such an n exists; otherwise, we write $\operatorname{asdim}_{\mathrm{AN}}(f) = \infty$.

The Hurewicz-type mapping theorem is the following. The asdim version was proved by Bell and Dranishnikov [10], and the version for $\operatorname{asdim}_{\mathrm{AN}}$ is due to Brodksiy, Dydak, Levin and Mitra [3].

Theorem 1.2.10 (Hurewicz-type mapping theorems)**.** [3, 10] *Let X and Y be metric spaces, and let $f : X \to Y$ be asymptotically Lipschitz. Then*

$$\operatorname{asdim}_{(\mathrm{AN})}(X) \leq \operatorname{asdim}_{(\mathrm{AN})}(f) + \operatorname{asdim}_{(\mathrm{AN})}(Y).$$

The Hurewicz-type mapping theorems are powerful tools in the study of asymptotic dimension and asymptotic Assouad-Nagata dimension. They are used in the proof of the subadditivity of asymptotic and asymptotic Assouad-Nagata dimension with respect to Cartesian products (Lemma 1.2.11) as well as the group-theoretic extension theorems for asymptotic and Assouad-Nagata dimension (Lemma 1.2.13). In Section 2.1, we use Theorem 1.2.10 to provide the best possible upper bound on the asymptotic Assouad-Nagata dimension of a metric space provided by the existence of an (ε,k)-tight geodesic combing (definition to come).

We adopt the convention that the Cartesian product of two metric spaces is always endowed with the ℓ^1 product metric. That is, if X and Y are metric spaces, then $X\times Y$ is equipped with the metric defined by

$$d_{X\times Y}((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$$

for all $x,x'\in X$ and $y,y'\in Y$. With this convention in mind, if $\sim$ stands for either "is coarsely equivalent to," "is quasi-isometric to," or "is bi-Lipschitz equivalent to," then we have that $X\sim X'$ and $Y\sim Y'$ implies $X\times Y\sim X'\times Y'$. In addition, asdim and $\operatorname{asdim}_{\mathrm{AN}}$ are subadditive with respect to taking Cartesian products, in a sense that is made precise by the following pair of theorems. We will use them often throughout this paper.

Lemma 1.2.11. $[1,3]$ *Let X,Y be metric spaces. Then*

$$\operatorname{asdim}_{(\mathrm{AN})}(X\times Y) \le \operatorname{asdim}_{(\mathrm{AN})}(X) + \operatorname{asdim}_{(\mathrm{AN})}(Y)$$

Since $\mathbb{R}$ is a tree, Exercise 1.2.5 and Lemma 1.2.11 readily imply that $\operatorname{asdim}_{\mathrm{AN}}(\mathbb{R}^n)\le n$. Establishing a lower bound for the asymptotic dimension of a space seems to be harder, in general, than establishing an upper bound. However, to prove that $\operatorname{asdim}(\mathbb{R}^n)\ge n$, one can use a lemma of Brodskiy, Dydak and Lang, which states, roughly, that $\operatorname{asdim}(X)\ge n$ if arbitrarily large r-discrete n-dimensional cubes embed into X without much distortion [17]. A lemma of Higes (see Lemma 3.1.5), an essential tool in this dissertation, is the corresponding lemma for asymptotic Assouad-Nagata dimension. Thus we have $\operatorname{asdim}(\mathbb{R}^n) = \operatorname{asdim}_{\mathrm{AN}}(\mathbb{R}^n) = n$, justifying the use of the word "dimension."

1.2.2 Countable groups with proper norms

We denote the identity element of an arbitrary group by 1, and of an abelian group by 0. Let G be a group. A *norm* on G is a function $\|\cdot\| : G\to\mathbb{R}_0^+$ such that, for all $g,h\in G$,

- $\|g\| = 0$ if and only if $g=1$.
- $\|g\| = \|g^{-1}\|$.

- $\|gh\| \leq \|g\| + \|h\|$.

Some authors call this a *length function* or *weight function* on G.

A norm is *proper* if $\{g \in G \mid \|g\| \leq D\}$ is finite for all $D \geq 0$. There is a natural one-to-one correspondence between norms and left-invariant metrics, given by $d(g,h) = \|g^{-1}h\|$ and $\|g\| = d(1,g)$, and a left-invariant metric on a group is proper if and only if the corresponding norm is proper.

Every countable group admits a proper norm; futhermore, if $\|\cdot\|_0$ and $\|\cdot\|_1$ are two norms on the same countable group G, then the identity map from $(G, \|\cdot\|_0)$ to $(G, \|\cdot\|_1)$ is a coarse equivalence [26, Proposition 1.1]. Therefore for any countable group G we define $\operatorname{asdim}(G)$ to be the asymptotic dimension of G with respect to any proper norm. Since asymptotic dimension is a coarse invariant, this is independent of the choice of proper norm, depending only on the group's inherent algebraic structure. In this way asdim is an invariant of countable groups. Since any subgroup of a countable group is countable, if G is a countable group and $H \leqslant G$, then $\operatorname{asdim}(H) \leq \operatorname{asdim}(G)$.

A group is called *locally finite* if all of its finitely generated subgroups are finite. We will use the following fact many times throughout this paper.

Exercise 1.2.12. Let G be a countable group. Then $\operatorname{asdim}(G) = 0$ if and only if G is locally finite.

Asymptotic Assouad-Nagata dimension is decidedly *not* a coarse invariant. While this fact is not new, we demonstrate it in Section 3.1 when we construct, for each $m, n \in \mathbb{N}$, a countable group K and two proper norms $\|\cdot\|_m$ and $\|\cdot\|_n$ on K such that $\operatorname{asdim}_{\mathrm{AN}}(K, \|\cdot\|_m) = m$ but $\operatorname{asdim}_{\mathrm{AN}}(K, \|\cdot\|_n) = n$.

1.2.3 Finitely generated groups with the word norm

Let G be a group, and suppose that S is a generating set of G. Then by definition each $g \in G$ can be expressed as a product of elements of S or their inverses. The word norm on G with respect to S is defined by declaring that $\|g\|_S$ is the minimum number of terms in such a product (see Section 2.2.1 for a more precise definition). It is easy to check that this is in fact a norm on G in the sense described in the last subsection. Note that $\|\cdot\|_S$ is proper if and only if S is finite. Therefore if G is a finitely generated group, $\operatorname{asdim}(G)$ is exactly $\operatorname{asdim}(G, \|\cdot\|_S)$ where S is any finite generating set.

If G is a finitely generated group and S and T are two finite generating sets of G, then the identity map from $(G, \|\cdot\|_S)$ to $(G, \|\cdot\|_T)$ is a quasi-isometry. Therefore for any finitely generated group G we define $\operatorname{asdim}_{\mathrm{AN}}(G)$ to be $\operatorname{asdim}_{\mathrm{AN}}(G, \|\cdot\|_S)$ for any finite generating set of G. Since $\operatorname{asdim}_{\mathrm{AN}}$ is invariant under quasi-isometry, $\operatorname{asdim}_{\mathrm{AN}}(G)$ is independent of the choice of finite generating set, depending only on the group itself. In what follows, we will not need to compare word norms arising from different generating sets of the same group, so we refer to the word norm on a finitely generated group G by $\|\cdot\|_G$, or $\|\cdot\|$ if the group is understood.

Suppose that G is a finitely generated group and $H \leqslant G$. Let $\|\cdot\|_G$ denote the restriction to H of the word norm on G. Then certainly $\|\cdot\|_G$ is a proper norm on H. It may happen that H is not finitely generated, which is our principle motivation for finding ways to calculate the Assouad-Nagata dimension of countable groups with respect to different proper norms. If H is finitely generated, let $\|\cdot\|_H$ denote the word norm on H. Then it may happen that $(H, \|\cdot\|_H)$ and $(H, \|\cdot\|_G)$ are not quasi-isometric, in which case we say that H is *distorted* in G.

The following theorems are known as the extension theorems for asymptotic and Assouad-Nagata dimension. The theorem for asdim is due to Bell and Dranishnikov, while Brodskiy, Dydak, Levin and Mitra later proved the asdim_{AN} version. Both proofs use the appropriate Hurewicz-type mapping theorem in an essential way.

Lemma 1.2.13 (Extension Theorems)**.** [3, 10] *Let*

$$1 \to K \to G \to H \to 1$$

be a short exact sequence, where G and H are finitely generated groups equipped with the word norm with respect to some finite generating set, and the norm on K is the restriction to K of the word norm on G. Then

$$\text{asdim}_{(\text{AN})}(G) \leq \text{asdim}_{(\text{AN})}(K) + \text{asdim}_{(\text{AN})}(H).$$

These results are quite powerful tools in the field. For example, the extension theorem for asymptotic dimension implies that the class of finitely generated groups of finite asymptotic dimension is closed under extension and thus, by Theorem 1.1.1, greatly enlarges the class of finitely generated groups known to satisfy the Novikov conjecture. The extension theorems are also an essential tool in the proof of Theorem 1.

Another tool we will use is the free product formulas for asymptotic and Assouad-Nagata dimension. The theorem for asdim is due to Dranishnikov, and its counterpart for asdim_{AN} is due to Brodskiy and Higes.

Theorem 1.2.14. [8, 27] *Let A and B be finitely generated groups. Then*

$$\text{asdim}_{(\text{AN})}(A * B) = \max\{\text{asdim}_{(\text{AN})}(A), \text{asdim}_{(\text{AN})}(B), 1\}$$

In fact, the result for asymptotic dimension is even stronger. Namely by [8], we have that if A and B are finitely generated groups with a common subgroup C, then $\text{asdim}(A *_C B) \leq \max\{\text{asdim}(A), \text{asdim}(B), \text{asdim}(C) +$

$1\}$. In particular, the class of finitely generated groups of finite asymptotic dimension is closed under taking amalgamated products. Currently it is not known whether the same holds for Assouad-Nagata dimension.

Like the result on amalgamated products, the following theorem distinguishes the state of the art of asymptotic Assouad-Nagata dimension theory from that of asymptotic dimension theory. It is known as the Morita-type theorem for asymptotic Assouad-Nagata dimension.

Theorem 1.2.15 (Morita-type theorem for $\mathrm{asdim}_{\mathrm{AN}}$)**.** [15] *Let X be a metric space. Then*

$$\mathrm{asdim}_{\mathrm{AN}}(X \times \mathbb{R}) = \mathrm{asdim}_{\mathrm{AN}}(X) + 1\,.$$

It is currently unknown whether the same result holds for asymptotic dimension. Note that since $\mathbb{R}$ is quasi-isometric to $\mathbb{Z}$, we have the following group-theoretic corollary.

Corollary 1.2.16. *Let G be a countable group equipped with a proper norm $\|\cdot\|_G$. Then*

$$\mathrm{asdim}_{\mathrm{AN}}(G \times \mathbb{Z}, \|\cdot\|_{G\times\mathbb{Z}}) = \mathrm{asdim}_{\mathrm{AN}}(G, \|\cdot\|_G) + 1\,.$$

In particular, if G is a finitely generated group, then

$$\mathrm{asdim}_{\mathrm{AN}}(G \times \mathbb{Z}) = \mathrm{asdim}_{\mathrm{AN}}(G) + 1\,.$$

1.2.4 Aside: finitely generated groups of finite asymptotic dimension

At this point we have collected all the tools from asymptotic dimension theory that we will need in the proof of Theorem 1. However, in order for the reader to get a better sense of how asymptotic dimension behaves within the class of finitely generated groups, it seems appropriate to mention some of the major results in the field. If the reader is unfamiliar with any terminology used here, they need not worry: any concepts needed for the remainder of the paper will be explained in full later.

In 1993, Gromov posed the question of whether every finitely generated group admits a coarse embedding into a Hilbert space [4]. In 2000, Yu improved on his 1998 result, showing that groups with finite asymptotic dimension satisfy a condition called Property A, which is equivalent to C^*-exactness. Yu then proved that groups with Property A admit a coarse embdedding into a Hilbert space, and that any group which coarsely embeds into a Hilbert space satisfies the Novikov Conjecture [28]. While Gromov constructed examples of groups, now called Gromov monsters, which do not coarsely embed into Hilbert space [29], at the moment it is unknown whether or not all groups satisfy the Novikov Conjecture.

In the last section we reviewed many closure properties of the class of finitely generated groups of finite asymptotic dimension. One closure property which we did not review, but is nevertheless a significant result, is that finitely generated groups of finite asymptotic dimension are closed under taking HNN extensions. That is, if G is a finitely generated group, A and B are subgroups of G, and $\phi : A \to B$ is an isomorphism, then $\operatorname{asdim}(G*_\phi) \leq \max\{\operatorname{asdim}(G), \operatorname{asdim}(A)+1\}$ [11]. Finitely generated groups of finite asymptotic dimension are also closed under relative hyperbolicity. That is, if G is a finitely generated group which is hyperbolic relative to a set of finitely generated peripheral subgroups $\{H_0, \ldots, H_n\}$, and each H_i has finite asymptotic dimension, then $\operatorname{asdim}(G)$ is finite [12].

Since $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$, we have that, as one would expect, $\operatorname{asdim}_{(\mathrm{AN})}(\mathbb{Z}^n) = n$. Other groups which are known to have finite asymptotic dimension include hyperbolic groups [6], one-relator groups [7], Coxeter groups [8], mapping class groups, braid groups, and certain types of Artin groups and Torelli groups [9], and $C'(1/6)$ groups (Theorem 2). A finitely generated group constructed from any of the aforementioned types of groups using extensions, amalgamated products, HNN extensions or relative hyperbolicity constructions, has finite asymptotic dimension. This means that the class of groups which coarsely embed into a Hilbert space, and thus satisfy the Novikov conjecture, is quite large.

Some examples of groups with infinite asymptotic dimension are Gromov monsters, $\mathbb{Z} \wr \mathbb{Z}$, and Thompson's group F. Gromov monsters have infinite asymptotic dimension because they do not coarsely embed into Hilbert space, while $\mathbb{Z} \wr \mathbb{Z}$ and F have infinite asymptotic dimension because each contain $\mathbb{Z}^n$ subgroups for all $n \in \mathbb{N}$. In particular there is no relationship between finite asymptotic dimension and finite presentability, since F can be given by the finite presentation $\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle$.

Since F_n is a tree, $\operatorname{asdim}(F_n) \leq 1$, so we cannot expect any general relationship between the asymptotic dimension of a group and its quotients. Nor is finite asymptotic dimension preserved under inductive limits, as there exists a sequence of finitely generated groups and surjective homomorphisms $G_0 \to G_1 \to G_2 \to \ldots$ such that $\operatorname{asdim}(G_n) \leq 2$ for all $n \in \mathbb{N}$, but the limit group G_∞, defined to be the common quotient of all G_n, has infinite asymptotic dimension [22].

The asymptotic dimension of most groups is not known, although in some cases an exact formula has been found. The asymptotic dimension of $\operatorname{Mod}(S_{2,p})$, the mapping class group of a surface of genus 2 with p punctures, is computed inductively in [9]. By a result of Buyalo and Lebedeva, $\operatorname{asdim}(G) = \dim(\partial G) + 1$ if G is hyperbolic [21]. A group has asymptotic dimension zero if and only if it is locally finite [17], and a finitely presented group has asymptotic dimension 1 if and only if it is virtually free [19]. Currently there is no known classification of finitely presented groups with asymptotic dimension 2.

There is obviously no straightforward relation between the asymptotic dimension and topological dimension of a given metric space, since one can easily construct examples of spaces with (topological, asymptotic) dimension equal to (m, n) for any $m, n \in \mathbb{N} \cup \{\infty\}$. However, by the result of Buyalo and Lebedeva, there is a relation between the topological dimension of a compact hyperbolic manifold and the asymptotic dimension of its universal cover. Namely, if M is a compact hyperbolic manifold and $\tilde{M}$ is its universal cover, we have that $\operatorname{asdim}(\pi_1(M)) = \dim(\partial\tilde{M}) + 1 = \dim(M)$. This relationship fails immediately for non-hyperbolic groups, since every finitely presented group is isomorphic to the fundamental group of a four-dimensional compact manifold, but Thompson's group F is finitely presented and has infinite asymptotic dimension.

One other relation between topological dimension and asymptotic dimension comes through asymptotic cones. Asymptotic cones were introduced by Gromov in [4], and can be thought of as a method of looking at a metric space from infinitely far away. Without going into detail, an asymptotic cone of a metric space X is defined with respect to a (usually non-principle) ultrafilter ω on $\mathbb{N}$ and a sequence of positive real scaling constants $k = (k_i)_{i\in\mathbb{N}}$, and so is often denoted $\mathrm{Cone}_\omega(X,k)$. In geometric group theory, it is often useful to examine the asymptotic cones of finitely generated groups equipped with the word norm. In 2008, Dydak and Higes proved that for any countable group equipped with a proper norm we have $\dim(\mathrm{Cone}_\omega(G,k)) \leq \operatorname{asdim}_{\mathrm{AN}}(G)$ [30]. This inequality can be strict; for each $n \in \mathbb{N}\cup\{\infty\}$ there exists a countable locally finite group G with a proper norm, with respect to which $\operatorname{asdim}_{\mathrm{AN}}(G) = n$ but $\dim(\mathrm{Cone}_\omega(G,k)) = 0$ for any scaling sequence $k = (k_n)$ and non-principal ultrafilter ω [16]. At present, no other relations between $\dim(\mathrm{Cone}_\omega(G,k))$ and either $\operatorname{asdim}(G)$ or $\operatorname{asdim}_{\mathrm{AN}}(G)$ are known.

CHAPTER 2

Assouad-Nagata dimension of $C'(1/6)$ groups

2.1 Tight geodesic combings

In this section, we introduce the notion of an (ε,k)-tight geodesic combing of a geodesic metric space, defined for some $\varepsilon > 0$ and $k \in \mathbb{N}$. For a geodesic metric space, the property of having a (ε,k)-tight geodesic combing is similar to that of having tight geodesics, a property defined by Bowditch in [23]. We then prove that if a geodesic metric space X admits a (ε,k)-tight geodesic combing for some $\varepsilon > 0$, then $\operatorname{asdim}_{\mathrm{AN}}(X) \leq k$ (see Proposition 2.1.4). This is eventually how we provide an upper bound for the Assouad-Nagata dimension of a finitely generated $C'(1/6)$ group in Section 2.3.

Definition 2.1.1. Let X be a geodesic metric space with base point $x \in X$. Then a *geodesic combing* of the pointed metric space (X,x) is a set $T = \{T_y \mid y \in Y\}$, where Y is a cobounded subset of X and T_y is a geodesic from x to y for each $y \in Y$.

Whenever Γ is a connected graph, directed or otherwise, we assume that any edge of Γ may be traversed contrary to its orientation, and that Γ is equipped with the combinatorial metric, so that Γ is naturally a geodesic metric space.

Example 2.1.2. Suppose that Γ is a connected graph equipped with the combinatorial metric, and let $x \in V(\Gamma)$ be a base point. A geodesic tree rooted at x is a subgraph T of Γ such that T is a tree, and for all $y \in V(\Gamma)$, the unique path from x to y in T is geodesic in Γ. If T is a geodesic tree rooted at x and $V(T) = V(\Gamma)$, then we call T a geodesic spanning tree rooted at x. If T is a geodesic spanning tree rooted at x and $y \in V(\Gamma)$, let $[x,y]$ be the path from x to y in T. Then $\{[x,y] \mid y \in V(\Gamma)\}$ is a geodesic combing of (Γ,x).

Suppose that $\{T_y \mid y \in Y\}$ is a geodesic combing of a pointed geodesic metric space (X,x). For each $y \in Y$ and $s > 0$, let

$$T(y,s) = \bigcup\{T_{y'} \mid y' \in Y \cap B(y,s)\}$$

and for each $t \geq 0$, let

$$S(t) = \{x' \in X \mid d(x,x') = t\}$$

be the sphere of radius t centered at x in X.

Definition 2.1.3. Let (X,x) be a pointed geodesic metric space, Y a cobounded subset of X, and $T = \{T_y \mid y \in Y\}$ a geodesic combing of (X,x). Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Then we say that T is (ε,k)-*tight* if for all $r > 0$,

$y \in Y$, and $t \leq d(x,y) - r$, we have $|T(y, \varepsilon r) \cap S(t)| \leq k$.

Figure 2.1 illustrates this definition.

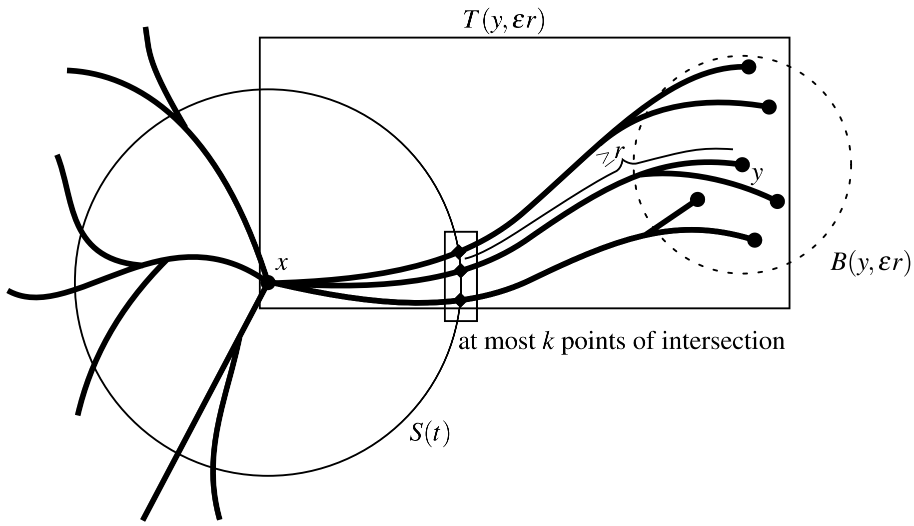

Figure 2.1: An (ε, k)-tight geodesic combing.

Proposition 2.1.4. *Let (X,x) be a pointed geodesic metric space. If X admits an (ε,k)-tight geodesic combing for some $\varepsilon > 0$, then* $\operatorname{asdim}_{\mathrm{AN}}(X) \leq k$.

Proof. Suppose that Y is a cobounded subset of X and $T = \{T_y \mid y \in Y\}$ is an (ε,k)-tight geodesic combing of (X,x). Let $d_x : Y \to \mathbb{R}_0^+$ be defined by $d_x(y) = d(x,y)$. For any $n \in \mathbb{N}$ and $r > 0$, let

$$A(n,r) = \{y \in Y \mid nr \leq d(x,y) \leq (n+2)r\} = d_x^{-1}([nr,(n+2)r])$$

be the n^{th} annulus of width $2r$ in Y.

We claim that for each $n \in \mathbb{N}$ and $r > 0$, there exists a cover $\mathscr{V}(n,r)$ of $A(n,r)$ which has εr-multiplicity at most k and is uniformly bounded by $6r$. To see this, define an equivalence relation $\sim$ on $A(n,r)$ by declaring that $y \sim y'$ if T_y and $T_{y'}$ pass through the same element of $S((n-1)r)$. Let $\mathscr{V}(n,r)$ be the set of $\sim$ equivalence classes. Clearly $y \sim y'$ implies that there is a path in $T_y \cup T_{y'}$ from y to y' of length at most $6r$, hence $\mathscr{V}(n,r)$ is uniformly $6r$-bounded. Furthermore, since T is (ε,k)-tight, for each $y \in A(n,r)$ we have that $|T(y,\varepsilon r) \cap S((n-1)r)| \leq k$, hence any open ball of radius εr in $A(n,r)$ can meet at most k equivalence classes.

Now we claim that $\operatorname{asdim}_{\mathrm{AN}}(d_x) \leq k-1$. Let $s, K > 0$ be given. Now fix $r = \max(\frac{1}{\varepsilon}s, K)$. Let $A \subseteq Y$ be such that $d_x(A)$ is K-bounded. Then $A \subseteq A(n,K) \subseteq A(n,r)$. By the previous argument, there exists a cover $\mathscr{V}(n,r)$ of $A(n,r)$ (and thus of A) with εr-multiplicity at most k, which is uniformly bounded by $6r$.

Therefore $\mathscr{V}(n,r)$ has s-multiplicity at most k and is uniformly bounded by $6r = 6\max(\frac{1}{\varepsilon}s, K) \leq \frac{6}{\varepsilon}s + 6K$. Thus $D_{d_x}(s,K) := \frac{6}{\varepsilon}s + 6K$ is a $(k-1)$-dimensional control function for d_x that is linear in both s and K, and we have $\operatorname{asdim}_{\mathrm{AN}}(d_x) \leq k-1$.

It is easy to check that $\operatorname{asdim}_{\mathrm{AN}}(\mathbb{R}_0^+) \leq 1$, and that d_x is 1-Lipschitz and therefore asymptotically Lipschitz. Therefore by Theorem 1.2.10,

$$\operatorname{asdim}_{\mathrm{AN}}(Y) \leq \operatorname{asdim}_{\mathrm{AN}}(d_x) + \operatorname{asdim}_{\mathrm{AN}}(\mathbb{R}_0^+) = (k-1)+1 = k\,.$$

Since Y is quasi-isometric to X, $\operatorname{asdim}_{\mathrm{AN}}(X) \leq k$. □

A straightforward application of Zorn's Lemma shows that if Γ is a connected graph and $x \in V(\Gamma)$, then Γ has a geodesic spanning tree rooted at x. Hence Example 2.1.2 shows that every connected graph has a geodesic combing, which may or may not be (ε,k)-tight for some $\varepsilon > 0$ and $k \in \mathbb{N}$. Since every quasigeodesic metric space is quasi-isometric to a connected graph, Example 2.1.2 is more general than it appears at first glance.

Clearly if a connected graph Γ admits a (ε,k)-tight geodesic combing for some $\varepsilon > 0$ and $k \in \mathbb{N}$, then we may assume without loss of generality that it is given by a geodesic spanning tree. If T is a geodesic spanning tree of Γ, we say that T is (ε,k)-tight if the geodesic combing it induces is (ε,k)-tight. In Section 2.3 we show that if Γ is the Cayley graph of a finitely generated $C'(1/6)$ group with respect to any finite generating set, then any geodesic spanning tree of Γ is $(1/12, 2)$-tight.

2.2 Preliminaries on van Kampen diagrams and small cancellation

We assume that the reader is familiar with van Kampen diagrams and the $C'(\lambda)$ condition. However, in the literature there are myriad definitions of van Kampen diagram, each with subtle differences. In addition, our definition of a "piece" (and thus, of the $C'(\lambda)$ condition) is not stated in the usual way. This is in order to ensure that certain concepts we introduce later (namely signed and unsigned face counts) are well defined. Therefore in Section 2.2.1 and Section 2.2.2 we fix terminology and notation. In summary, our approach is to treat van Kampen diagrams as graphs embedded in the plane, so that the 2-cells are simply the bounded faces enclosed by the graph. In this way we manipulate van Kampen diagrams directly in the plane and keep topological considerations to a minimum. We include inessential edges and faces in our definition. It is assumed that presentations are *not* closed under cyclic shifts and inverses, and a piece is defined, not as a common prefix of two words, but as a common prefix of cyclic shifts of two words or their inverses. If this summary is enough for the reader, they may choose to skip to Section 2.2.3, referring to Sections 2.2.1 and 2.2.2 should the need arise.

In Section 2.2.3, we present a classification of van Kampen diagrams over simple geodesic triangles in $C'(1/6)$ groups. This result, due to Strebel, is the essential tool used in the proof of the main theorem. Then we prove some lemmas regarding the geometry of simple geodesic triangles in $C'(1/6)$ groups that are used repeatedly in Section 2.3.

2.2.1 The $C'(\lambda)$ condition

Let S be a set. Let S^{-1} be the set of formal inverses of S, let 1 be a new symbol not in S, and declare $1^{-1} = 1$. Let

$$\begin{aligned} S_1 &= S \cup \{1\} \\ S_\circ &= S \cup S^{-1} \cup \{1\}. \end{aligned} \tag{2.1}$$

The length of a word w in the free monoid $S_\circ^*$ is denoted $|w|$. There is a unique word of length 0 called the *empty word* and denoted ε. We define w^0 to be ε for any $w \in S_\circ^*$. A word $w \in S_\circ^*$ is *reduced* if w does not contain a subword of the form $1, ss^{-1}$, or $s^{-1}s$ for any $s \in S$, and *cyclically reduced* if every cylcic shift of w (including w itself) is reduced.

Let R be a language over the alphabet $S_\circ$, that is, $R \subseteq S_\circ^*$. Then R_* denotes the closure of R under taking cyclic shifts and formal inverses of its elements. We say that R is *reduced* if every element of R is reduced, and *cyclically reduced* if R_* is reduced. We say that R is *cyclically minimal* if it does not contain two distinct words, one of which is a cyclic shift of the other word or its inverse. That is, R is cyclically minimal if $R \cap \{r\}_* = \{r\}$ for each $r \in R$.

A (group) *presentation* is a pair $\langle S \mid R \rangle$, where S is a set and $R \subseteq S_\circ^*$. The notation $G = \langle S \mid R \rangle$ means that $\langle S \mid R \rangle$ is a presentation and $G \cong F(S)/\langle\langle R \rangle\rangle$, where $F(S)$ is the free group with basis S, and $\langle\langle R \rangle\rangle$ is the normal closure of R as a subset of $F(S)$.

Whenever S is a generating set of a group G, there is a natural monoid epimorphism from $S_\circ^*$ to G that evaluates a word in $S_\circ^*$ as a product of generators and their inverses, and sends 1 to the identity element. If G and S are understood, then for a word $w \in S_\circ^*$, we denote by $\bar{w}$ the image of w under this epimorphism. If we are considering multiple groups with generators S but different relations, it helps to include the group in the notation. Thus if $\bar{w} = g \in G$, then we may write $w =_G g$. If $u, v \in S_\circ^*$ we may write $u =_G v$ to mean $\bar{u} = \bar{v}$ in G.

If both G and S are understood, the *word norm* on G with respect to S is denoted $\|\cdot\|$ and defined by

$$\|g\| = \min\{|w| \mid w \in S_\circ^*, w =_G g\}.$$

We might also denote the word norm on G with respect to S by $\|\cdot\|_G$ or $\|\cdot\|_S$ if the group or generating set is ambiguous. A word $w \in S_\circ^*$ is called *geodesic* in G if $|w| = \|\bar{w}\|$. If $u, w \in S_\circ^*$, $g \in G$, w is geodesic in G, and $w =_G u =_G g$, then w is called a *geodesic representative* of u or of g. If $K, C \geq 0$ are fixed constants, then we say that a word $w \in S_\circ^*$ is (K,C)-*quasigeodesic* if $|w| \leq K\|\bar{w}\| + C$.

Given two words $u, v \in S_\circ^*$, we say that p is a *piece* (of u and of v) if there exists $u' \in \{u\}_*, v' \in \{v\}_*$ such that p is a common prefix of u' and v'.

Definition 2.2.1. Let S be a set, $R \subseteq S_\circ^*$ a language, and λ a real number with $0 < \lambda < 1$. Then R satisfies $C'(\lambda)$ if, whenever $u, v \in R$ and $u' \in \{u\}_*, v' \in \{v\}_*$ witness that p is a piece of u and v, then either $u' = v'$ or $|p| < \lambda \min(|u|, |v|)$.

In this case we say that R is a $C'(\lambda)$ language. If G is a group and $G = \langle S \mid R \rangle$ for some $C'(\lambda)$ language R, then $\langle S \mid R \rangle$ is called a $C'(\lambda)$ presentation and G is called a $C'(\lambda)$ group.

In most treatments of the $C'(\lambda)$ condition, it is assumed that $R = R_*$, and a piece is defined to be a common prefix of two distinct words in R. In our case, however, it is important to assume that R is cyclically minimal (in particular $R \neq R_*$), in order to ensure that the concept of the signed r-face count of a van Kampen diagram (Definition 3.2.1) is well defined. For this reason we give the definition above, which, though not the usual definition of the $C'(\lambda)$ condition, is clearly equivalent.

2.2.2 van Kampen diagrams

Let Γ be a connected graph. By a *path* in Γ we mean a combinatorial path, i.e. an alternating sequence of vertices and edges, as opposed to a continuous map from a closed interval. We allow paths to have repeated edges or vertices: in graph-theoretic terms, our "path" is really a walk. Points in the interiors of edges generally don't matter to us, so we write $x \in \Gamma$ to mean that $x \in V(\Gamma)$. Likewise, if α is a path in Γ, then $x \in \alpha$ means that x is a vertex visited by α.

Let Γ be any directed graph, and suppose that $\mathrm{Lab} : E(\Gamma) \to S_1$ (see (2.1)) is a function which assigns labels from S_1 to the edges of Γ. Then we extend Lab to a map from the set of all paths in Γ to $S_\circ^*$ in the following natural way.

- If $e = (x, y)$ is a directed edge labeled s, then $\mathrm{Lab}(x, e, y) = s$ and $\mathrm{Lab}(y, e, x) = s^{-1}$.
- If $\alpha = (x_0, e_1, x_1, \ldots, x_{n-1}, e_n, x_n)$ is a path, then

$$\mathrm{Lab}(\alpha) = \mathrm{Lab}(x_0, e_1, x_1)\,\mathrm{Lab}(x_1, e_2, x_2) \cdots \mathrm{Lab}(x_{n-1}, e_n, x_n).$$

For a path α we define $\ell(\alpha)$, the length of α, to be the number of edges traversed by α, counting multiplicity. Equivalently, $\ell(\alpha) = |\mathrm{Lab}(\alpha)|$.

A *plane graph* is a graph which is topologically embedded in $\mathbb{R}^2$. A *face* of a plane graph M is the closure of a connected component of $\mathbb{R}^2 \setminus M$. Let F be a face of a finite directed plane graph M with edges labeled by elements of S_1. Choosing a base point $x \in \partial F$ and an orientation counterclockwise $(+)$ or clockwise $(-)$, there is a unique circuit which traverses ∂F exactly once, called the *boundary path* and denoted $(\partial F, x, \pm)$. If all properties of $(\partial F, x, \pm)$ that we care about are preserved after changing its base point and orientation, then we leave these choices out of the notation and write ∂F. We write ∂M instead of ∂F if F is the unbounded face; from now on, "face" will mean "bounded face" unless otherwise stated. The *boundary label* of F is $\mathrm{Lab}(\partial F, x, \pm)$, sometimes denoted by just $\mathrm{Lab}(\partial F)$.

Definition 2.2.2. A *van Kampen diagram* over a presentation $\langle S \mid R \rangle$ is a finite, connected, directed plane graph M with edges labeled by elements of S_1, such that if F is a face of M, then either $\mathrm{Lab}(\partial F) \in R_*$ or $\mathrm{Lab}(\partial F) =_{F(S)} 1$.

A *subdiagram* of a van Kampen diagram M is a simply connected union of faces of M. If M is a van Kampen diagram and D is a subdiagram of M, then we call D *simple* if ∂D is a simple closed curve in the plane. Likewise, a face F of M is called simple of ∂F is a simple closed curve.

Let α and β be two paths in a van Kampen diagram. Then we say that α intersects β *trivially* if $\alpha \cap \beta$ contains at most one vertex, and that α intersects β *nontrivially* otherwise. We say that α and β intersect *simply* if $\alpha \cap \beta$ a single subpath of both α and (β or the reverse path of β). Note that this is *not* the same as saying that $\alpha \cap \beta$ is connected. We apply this terminology to faces as well. For example, if we say that F and α intersect simply, it means that there is a choice of base point $x \in \partial F$ such that $(\partial F, x, +)$ and α intersect simply. If we say that two faces F and F' intersect simply, it means that $(\partial F, x, +)$ and $(\partial F', x, -)$ intersect simply for some $x \in \partial F \cap \partial F'$.

A face F is called *essential* if $\mathrm{Lab}(\partial F) \in R_*$ and *inessential* if $\mathrm{Lab}(\partial F) =_{F(S)} 1$. If R is cyclically reduced then these cases are mutually exclusive. A face with boundary label $r \in R$ is called an r-face. An edge e is called *essential* if $\mathrm{Lab}(e) \in S$, and *inessential* if $\mathrm{Lab}(e) = 1$. We call a van Kampen diagram *bare* if it contains no inessential faces, and *padded* otherwise. Generally speaking, one needs to consider inessential edges and faces in order to make precise arguments with van Kampen diagrams, although in the next section we will only need to consider bare van Kampen diagrams.

Let M be a van Kampen diagram, and suppose F and F' are distinct faces of M. Then we say that F *cancels* with F' if there exists an edge $e = (x, y)$ in $\partial F \cap \partial F'$ such that $\mathrm{Lab}(\partial F, x, +) = \mathrm{Lab}(\partial F', x, -)$. Then we have the following geometric interpretation of the $C'(\lambda)$ condition, which follows immediately from the definition.

Lemma 2.2.3. *Let $\langle S \mid R \rangle$ be a presentation where R satisfies $C'(\lambda)$, and let M be a van Kampen diagram* *over $\langle S \mid R\rangle$. Suppose that F, F' are essential faces of M and α is a common subpath of ∂F and $\partial F'$. Then either F and F' cancel, or $\ell(\alpha) < \lambda \min(\ell(\partial F), \ell(\partial F'))$.*

A van Kampen diagram is called *reduced* if no two of its faces cancel. A van Kampen diagram is *minimal* if, among all van Kampen diagrams with the same boundary label, it minimizes first the number of essential faces, then the number of inessential faces.

Whenever G is a group generated by S, the Cayley graph of G with respect to S is denoted $\Gamma(G,S)$.

Lemma 2.2.4 (van Kampen Lemma)**.** [31, Chapter V, Section 1] *Let $G = \langle S \mid R\rangle$ and $w \in S_\circ^*$. Then $w =_G 1$ if and only if there exists a van Kampen diagram M over $\langle S \mid R\rangle$ and $x \in \partial M$ such that* $\mathrm{Lab}(\partial M, x, +) = w$*. Furthermore, given $g \in G$, there exists a combinatorial map $f : M \to \Gamma(G,S)$ preserving labels and orientations of edges, such that $f(x) = g$. In particular, f does not increase distances, i.e. is 1-Lipschitz.*

2.2.3 Van Kampen diagrams for simple geodesic triangles in $C'(1/6)$ groups

Let a, b, c be distinct elements of $G = \langle S \mid R\rangle$, and let $[a,b], [b,c], [c,a]$ be fixed geodesics between them in $\Gamma(G,S)$. Then $[a,b] \cup [b,c] \cup [c,a]$ is called a *geodesic triangle* and denoted $\Delta(a,b,c)$. We say that $\Delta(a,b,c)$ is a *simple* geodesic triangle if the boundary path $\partial\Delta(a,b,c) := [a,b] * [b,c] * [c,a]$ is a simple closed curve in $\Gamma(G,S)$. If σ is a circuit in $\Gamma(G,S)$ beginning at a group element $g \in G$, we say that M is a van Kampen diagram *for* σ if, for some $x \in \partial M$, $\mathrm{Lab}(\partial M, x, +) = \mathrm{Lab}(\sigma)$ and the combinatorial map $f : M \to \Gamma(G,S)$ sends x to g.

If Γ is a directed graph, the *underlying graph* of Γ is the undirected graph obtained by removing the orientation of every edge of Γ. If Γ is a graph and $e = (x,y)$ is an edge of Γ, then subdividing e means adding a vertex z and edges (x,z) and (z,y) to Γ, and removing e. A *subdivision* of Γ is a graph obtained from Γ by a finite sequence of subdivisions of edges.

Theorem 2.2.5. [24] *Suppose that $G = \langle S \mid R\rangle$, S is finite, R satifies $C'(1/6)$, Δ is a simple geodesic triangle in $\Gamma(G,S)$, and M is a bare, reduced van Kampen diagram over $\langle S \mid R\rangle$ for $\partial\Delta$. Then the underlying graph of M is a subdivision of a member of one of the four infinite families of plane graphs depicted in Figure 2.2.*

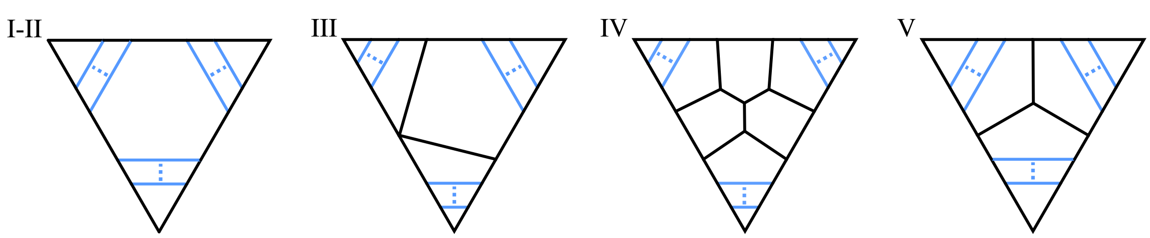

Figure 2.2: Types of van Kampen diagrams for a simple geodesic triangle in $\Gamma(G,S)$

In Figure 2.2, the blue edges and dots signify a sequence of parallel edges which may or may not be present. Vertices are located at the corners and at every juncture of edges. Our notation is slightly different from Strebel's notation in [24]: our I-II encompasses Strebel's I_2, I_3 and II, as well as the van Kampen diagram consisting of a single face, and our III is Strebel's III_1.

For the remainder of this section, suppose that G is a group with presentation $\langle S \mid R \rangle$, S is finite, R satisfies $C'(1/6)$, $\Delta = \Delta(a,b,c)$ is a simple geodesic triangle in $\Gamma(G,S)$, M is a minimal van Kampen diagram for $\partial\Delta$, $f : M \to \Gamma(G,S)$ is the combinatorial map, $\alpha = [b,c], \beta = [c,a]$, and $\gamma = [a,b]$. Note that $f|_{\partial M}^{\Delta} : \partial M \to \Delta$ is bijective, and isometric when restricted to each of the subpaths of ∂M corresponding to α, β, or γ. Thus without harm we blur the distinction between ∂M and $\partial\Delta$, and refer to vertices, edges, paths etc. in ∂M by their images in Δ.

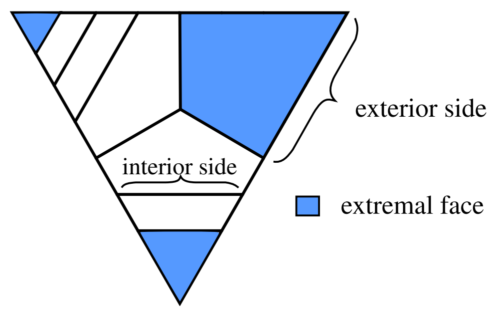

Figure 2.3: Examples of exterior and interior sides, and extremal faces

A face F of M is called *extremal* if F contains a, b, or c. A *side* of F is a maximal subpath of ∂F whose internal vertices all have degree 2 and do not include a, b, or c. A side is called *exterior* if it is contained in ∂M, and *interior* otherwise. An exterior side must be a subpath of α, β, or γ, so all exterior sides are geodesic. We call a face triangular if it has exactly three sides, quadrilateral if it has exactly four sides, etc. Figure 2.3 shows an example of a van Kampen diagram of type V with two triangular faces, four quadrilateral faces, and two pentagonal faces.

For a face F of M, let $i(F)$ denote the number of interior sides of F. The following argument appears so frequently in the proofs that follow that it is worthwhile to section it off as a lemma.

Lemma 2.2.6. *Let F be a face of M and σ an exterior side of F. Then*

$$\sum\{\ell(\tau) \mid \tau \text{ is a side of } F \text{ other than } \sigma\} \geq \tfrac{1}{2}\ell(\partial F).$$

In particular,

$$\sum\{\ell(\tau) \mid \tau \text{ is an exterior side of } F \text{ other than } \sigma\} > \left(\tfrac{1}{2} - \tfrac{i(F)}{6}\right)\ell(\partial F).$$

Proof. If σ is an exterior side, then σ is geodesic, from which the first inequality follows. The second inequality follows from the first inequality and Lemma 2.2.3. □

Definition 2.2.7. Let A be the union of all faces F of M such that ∂F does not share an edge with α, if at least one such F exists: otherwise, set $A = \{a\}$. We call A the *a-corner* of M. Similarly define B and C, the b-corner and c-corner of M. A face which is not included in a corner, i.e. one that shares at least one edge with each of α, β, and γ, is called a *middle face*. This is unique if it exists, and is denoted D. Thus A, B, C, D divide M into three or four (possibly overlapping) regions. Figure 2.4 illustrates where the corners and middle faces are in van Kampen diagrams of various types.

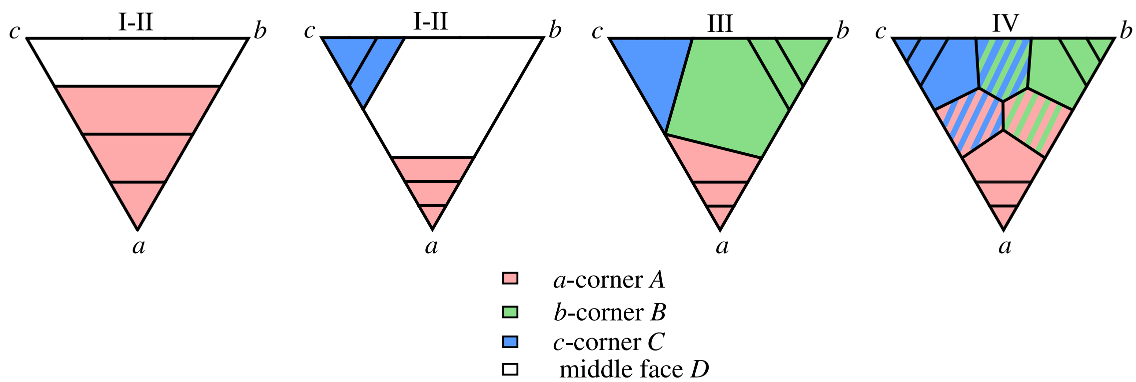

Figure 2.4: Examples of corners and middle faces

A corner may contain no faces if M is of type I-II. A corner containing at least one face contains an extremal face, which is either triangular, or possibly quadrilateral if M is of type IV or V. This may be followed by a sequence of quadrilateral faces; which may be followed by a pentagonal face if M is of type III, IV or V; which may be followed by two pentagonal faces, each with one exterior side, if M is of type IV.

We divide the boundary of the b-corner into three parts

$$\alpha_B = \partial B \cap \alpha \qquad \gamma_B = \partial B \cap \gamma \qquad \iota_B = \partial B \setminus (\alpha \cup \gamma)$$

and assign similar notation for the other two corners. The next proposition shows that α_B, γ_B, and ι_B are of comparable length, and if one is small, then the entire corner is small.

Proposition 2.2.8. *The following inequalities hold, and analogous inequalities hold after switching the roles of a, b, and c.*

(a) $\ell(\iota_B) < 2\min(\ell(\alpha_B), \ell(\gamma_B))$. *If M has a middle face,* $\ell(\iota_B) < \min(\ell(\alpha_B), \ell(\gamma_B))$.

(b) $\max(\ell(\alpha_B), \ell(\gamma_B)) < 3\min(\ell(\alpha_B), \ell(\gamma_B))$. *If M has a middle face,* $\max(\ell(\alpha_B), \ell(\gamma_B)) < 2\min(\ell(\alpha_B), \ell(\gamma_B))$.

(c) *If F is a face of A or D that borders both B and C, then* $\ell(\partial F \cap (\iota_B \cup \alpha_D \cup \iota_C)) < \ell(\alpha)$.

Proof. Assume that M is of type IV, the most complicated case. If M is of a different type the arguments are analogous but shorter. Assume without loss of generality that $\ell(\alpha_B) \leq \ell(\gamma_B)$.

Let $B = \bigcup_{i=0}^{k+3} B_i$, where

- B_0 is the extremal face containing b.
- $B_1, \ldots B_k$ is a (possibly empty) sequence of quadrilateral faces such that B_{j-1} borders B_j for all $j \in \{1, \ldots, k\}$.
- B_{k+1} is the pentagonal face with two exterior sides, if it exists: otherwise, $B_{k+1} = B_0$.
- B_{k+2} and B_{k+3} are the pentagonal faces with one exterior side bordering α and γ, respectively.

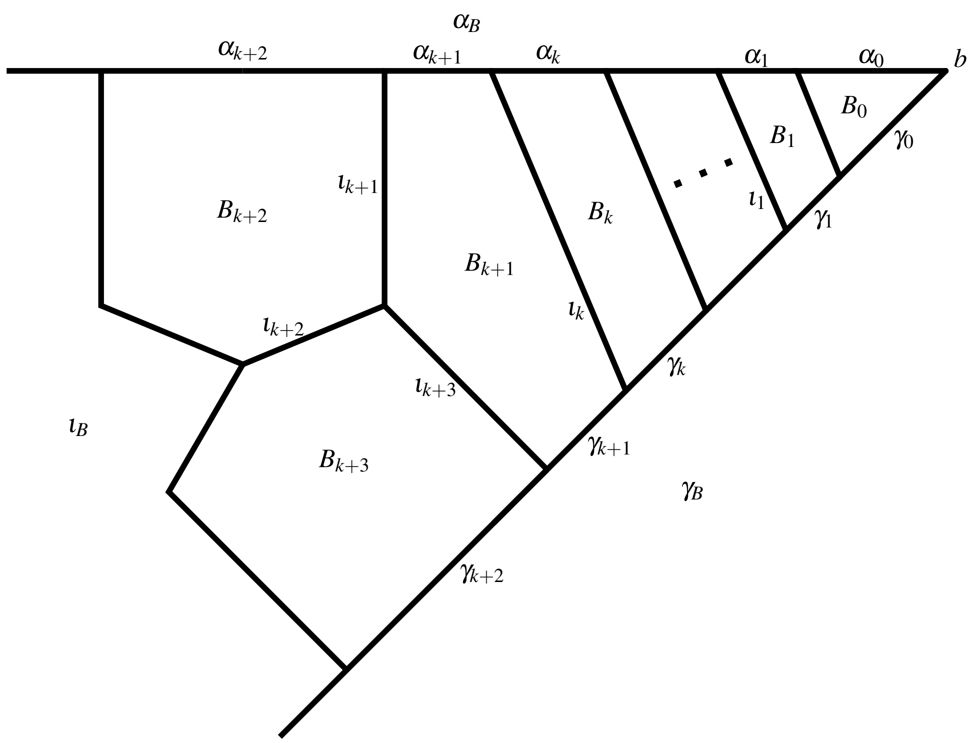

Figure 2.5: The b-corner of M.

We assign the following labels in order to streamline notation: see Figure 2.5.

$$\begin{aligned} \alpha_i &= \partial B_i \cap \alpha && \text{for } i \in \{0, \ldots, k+2\} \\ \gamma_i &= \partial B_i \cap \gamma && \text{for } i \in \{0, \ldots, k+1\}, \text{ and } \gamma_{k+2} = \partial B_{k+3} \cap \gamma \\ \iota_i &= \partial B_i \cap \partial B_{i+1} && \text{for } i \in \{0, \ldots, k+2\}, \text{ and } \iota_{k+3} = \partial B_{k+1} \cap \partial B_{k+3} \end{aligned}$$

Let $i \in \{0, \ldots, k\}$. Then applying Lemma 2.2.6 to γ_i, we obtain $\ell(\alpha_i) > \frac{1}{6}\ell(\partial B_i)$. Since ι_i is an interior side of B_i, $\ell(\iota_i) < \frac{1}{6}\ell(\partial B_i)$ by the $C'(1/6)$ condition. Therefore

$$\ell(\iota_i) < \ell(\alpha_i) \text{ for } i \in \{0, \dots, k\}\,. \tag{2.2}$$

Now consider B_{k+1}. Applying Lemma 2.2.6 to γ_{k+1} yields $\ell(\iota_{k+3}) + \ell(\iota_{k+1}) + \ell(\alpha_{k+1}) + \ell(\iota_k) > \frac{1}{2}\ell(\partial B_{k+1})$, and therefore $\ell(\alpha_{k+1}) + \ell(\iota_k) > \frac{1}{6}\ell(\partial B_{k+1})$ since ι_{k+1} and ι_{k+3} are interior sides. We know by (2.2) that $\ell(\iota_k) < \ell(\alpha_k)$, so $\frac{1}{6}\ell(\partial B_{k+1}) < \ell(\alpha_k) + \ell(\alpha_{k+1})$. Since ι_{k+3} is an interior side of B_{k+1}, we have $\ell(\iota_{k+3}) < \frac{1}{6}\ell(\partial B_{k+1})$. Thus

$$\ell(\iota_{k+3}) < \ell(\alpha_k) + \ell(\alpha_{k+1})\,. \tag{2.3}$$

Notice that ∂B_{k+2} consists of four interior sides and α_{k+2}. Therefore

$$\ell(\alpha_{k+2}) > \tfrac{1}{3}\ell(\partial B_{k+2})\,. \tag{2.4}$$

Since ι_{k+1} and ι_{k+2} are interior sides of B_{k+2}, we have $\ell(\iota_{k+1}) < \frac{1}{6}\ell(\partial B_{k+2})$ and $\ell(\iota_{k+2}) < \frac{1}{6}\ell(\partial B_{k+2})$. Combining this with (2.4) yields

$$\begin{aligned} \ell(\iota_{k+1}) &< \tfrac{1}{2}\ell(\alpha_{k+2}) \\ \ell(\iota_{k+2}) &< \tfrac{1}{2}\ell(\alpha_{k+2})\,. \end{aligned} \tag{2.5}$$

Applying Lemma 2.2.6 to γ_{k+2}, we find that

$$\ell(\iota_{k+2}) + \ell(\iota_{k+3}) > \tfrac{1}{6}\ell(\partial B_{k+3})\,. \tag{2.6}$$

Now ι_B consists of two interior sides of B_{k+2} and two interior sides of B_{k+3}. Thus

$$\ell(\iota_B) < \tfrac{1}{3}\ell(\partial B_{k+2}) + \tfrac{1}{3}\ell(\partial B_{k+3})\,. \tag{2.7}$$

Combining inequalities (2.2-2.7), we have

$$\begin{aligned} \ell(\iota_B) &< \tfrac{1}{3}\ell(\partial B_{k+2}) + \tfrac{1}{3}\ell(\partial B_{k+3}) \\ &< \ell(\alpha_{k+2}) + \tfrac{1}{3}\ell(\partial B_{k+3}) \\ &< \ell(\alpha_{k+2}) + 2\ell(\iota_{k+2}) + 2\ell(\iota_{k+3}) \\ &< 2\ell(\alpha_{k+2}) + 2(\ell(\alpha_{k+1}) + \ell(\alpha_k)) \leq 2\ell(\alpha_B)\,. \end{aligned}$$

If M has a middle face, then $\iota_B = \iota_k$ and inequality (2.2) gives that $\ell(\iota_B) < \ell(\alpha_B)$. This proves part (a). Since γ_B is geodesic,

$$\ell(\gamma_B) \leq \ell(\iota_B) + \ell(\alpha_B) < 2\ell(\alpha_B) + \ell(\alpha_B) = 3\ell(\alpha_B),$$

and if M has a middle face this bound is lowered to $2\ell(\alpha_B)$. This establishes part (b).

Now we prove part (c). If F is the middle face of M, then $\partial F \cap \iota_B = \iota_B$, and $\ell(\iota_B) < \ell(\alpha_B)$. If instead F is a face of A, then $\partial F \cap \iota_B = \iota_{k+3}$ in Fig. 2.5, and $\ell(\iota_{k+3}) < \ell(\alpha_{k+1}) + \ell(\alpha_k) < \ell(\alpha_B)$ by inequality (2.3). In either case we have $\ell(\partial F \cap \iota_B) < \ell(\alpha_B)$; similarly, $\ell(\partial F \cap \iota_C) < \ell(\alpha_C)$. Thus we have $\ell(\partial F \cap (\iota_B \cup \alpha \cup \iota_C)) \leq \ell(\partial F \cap \iota_B) + \ell(\partial F \cap \alpha) + \ell(\partial F \cap \iota_C) < \ell(\alpha_B) + \ell(\alpha_D) + \ell(\alpha_C) = \ell(\alpha)$. □

2.3 Assouad-Nagata dimension of finitely generated $C'(1/6)$ groups

In this section we prove the following proposition.

Proposition 2.3.1. *Let $G = \langle S \mid R \rangle$, where S is finite, and R is a cyclically reduced $C'(1/6)$ language. Then any geodesic spanning tree of $\Gamma(G,S)$ is $(1/12, 2)$-tight.*

We divide this section into two parts. In Section 2.3.1, we fix all notation and assumptions, and give a description of a van Kampen diagram which is obtained by fixing a geodesic spanning tree of $\Gamma(G,S)$ rooted at 1 and assuming it is *not* $(\varepsilon,2)$-tight for some $\varepsilon > 0$. All lemmas in Section 2.3.2 are proved under the assumptions stated in Section 2.3.1. We determine ε along the way, choosing at each stage an ε small enough to make the lemmas work. In the end we reach a contradiction with any $\varepsilon \leq \frac{1}{12}$, meaning that the spanning tree must have been $(1/12, 2)$-tight all along.

2.3.1 Construction of a van Kampen diagram

Let $G = \langle S \mid R \rangle$ be a finitely generated $C'(1/6)$ group. Fix a geodesic spanning tree T of $\Gamma(G,S)$ rooted at 1. Let let $\|\cdot\|$ be the word norm on G with respect to S, and let d be the corresponding metric. For each $g \in G$, let $[1,g]$ be the unique path from 1 to g in T.

Suppose to the contrary that T is not $(\varepsilon,2)$-tight. Let $r \in \mathbb{N}$ witness that T is not $(\varepsilon,2)$-tight. Then there exists an $x \in G$ such that $\|x\| \geq r$ and $B(x,\varepsilon r)$ contains two elements y, y' such that the geodesics $[1,x], [1,y]$, and $[1,y']$ each pass through different elements of the sphere of radius $\|x\| - r$ in $\Gamma(G,S)$. Because T is a tree, for every distinct $g,h \in G$, there is a unique vertex of $\Gamma(G,S)$ where the geodesics $[1,g]$ and $[1,h]$ diverge. Let a be the point at which $[1,x]$ diverges from $[1,y]$, and let a' be the point at which $[1,x]$ diverges from $[1,y']$. Then we have that $d(a,x) \geq r$, $d(a',x) \geq r, d(a,y) > r - \varepsilon r$, and $d(a',y') > r - \varepsilon r$. Without loss of generality suppose that $\|a'\| \geq \|a\|$.

Let $[x,y]$ and $[x,y']$ be arbitrarily chosen geodesics. Let $\Delta(1,x,y)$ be the geodesic triangle in $\Gamma(G,S)$ with sides $[1,x],[1,y]$ and $[x,y]$; similarly define $\Delta(1,x,y')$. Note that $\Delta(1,x,y)$ is not a tripod, since $\ell([x,y]) < \varepsilon r < 2(r-\varepsilon r) \leq \ell([x,a]) + \ell([a,y])$. Therefore $\Delta(1,x,y)$ contains exactly one maximal simple geodesic triangle, and a is the vertex of this triangle which is closest to 1. Let $\Delta = \Delta(a,b,c)$ be the maximal simple geodesic triangle in $\Delta(1,x,y)$, where a,b, and c are the points closest to $1,x$, and y, respectively. Similarly let $\Delta' = \Delta(a',b',c')$ be the maximal simple geodesic triangle of $\Delta(1,x,y')$ where a',b', and c' are the vertices of $\Delta(a',b',c')$ which are closest to $1,x$, and y', respectively. Note that $d(b,x) < \varepsilon r$, $d(b',x) < \varepsilon r$, and $d(a',x) > r$, thus we have that $\|a\| \leq \|a'\| < \|b\| \leq \|b'\|$ or $\|a\| \leq \|a'\| < \|b'\| \leq \|b\|$.

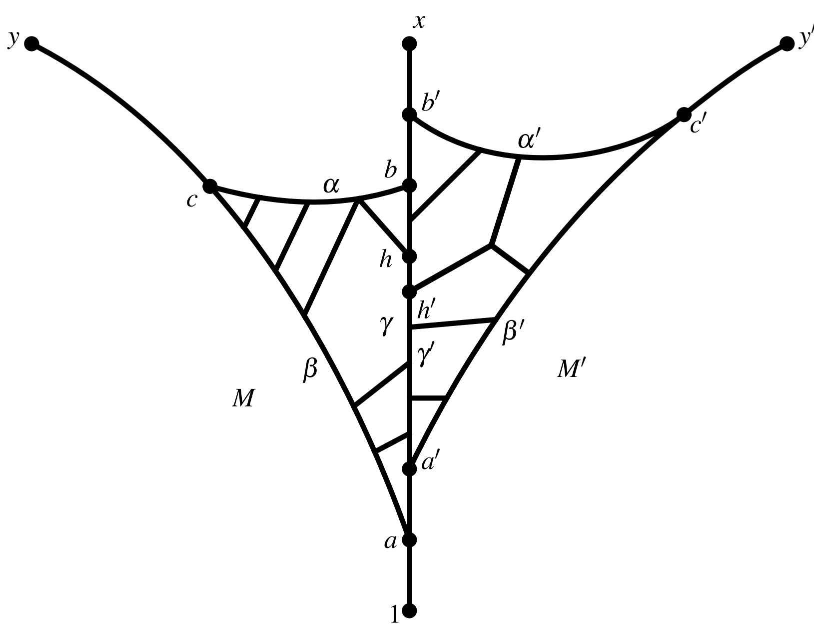

Figure 2.6: $N = M \cup_{[1,x]} M'$

Let M and M' be minimal van Kampen diagrams for Δ and Δ', respectively. Attaching the appropriate geodesic segments and gluing M and M' along $[1,x]$, we obtain a van Kampen diagram, call it N, for the circuit $[1,y]*[y,x]*[x,y']*[y',1]$. Thus N is the diagram shown in Figure 2.6, allowing that b,b' may appear in either order along $[1,x]$, and that M and M' may take any of the forms depicted in Figure 2.2. We retain all notation used in the previous section to describe the geometry of the simple geodesic triangles Δ and Δ' and their van Kampen diagrams M and M', using the symbol $'$ where appropriate. Thus α is the geodesic opposite a, C' is the c'-corner, which is opposite γ', etc. The vertices labeled h and h' in Figure 2.6 are the vertices of γ,γ' at the extremities of the b and b' corner, respectively.

Since M is reduced, no two faces of M cancel: similarly for M'. However, in principle a face of M may cancel with a face of M', so N may or may not be reduced. If F is a face of M, then we refer to the number of sides of F with respect to M, not N. Thus for example if F is a quadrilateral face in M, we will still refer to it as a quadrilateral face, even though a side of ∂F might be split into multiple sides in N if it borders $[1,x]$. Likewise, we call a side of a face F of M exterior if it is exterior in M, even though it may not be a subpath of ∂N.

Note that the combinatorial map f might not be injective when restricted to $\partial M \cup \partial M'$. For example, it may happen that $f(\alpha')$ intersects $f(\alpha)$ or $f(\beta)$ in $\Gamma(G,S)$. However, it is important to note that $[1,x]$, $[1,y]$ and $[1,y']$ do not intersect at any vertex of $\Gamma(G,S)$ farther from the identity than a', so f *is* injective when restricted to $\beta \cup \beta' \cup \gamma \cup \gamma'$.

2.3.2 Proof that a certain geodesic spanning tree is tight

All lemmas in this subsection are proved under the standing assumptions described in Section 2.3.1, which are not restated. The argument is as follows. First, we examine how faces of M and M' may line up along their common boundary, and determine that there is a face of M' that shares more than a third of its boundary with γ and does not cancel with any face of M. Playing around with inequalities provided by the $C'(1/6)$ condition, we find that this situation implies that $\varepsilon > \frac{1}{12}$. Since we were free to choose ε from the start, this is the desired contradiction, proving that T is in fact $(1/12, 2)$-tight.

Lemma 2.3.2. *Let h, h' be the vertices of γ_B, γ'_B, respectively, which are closest to a'. Then* $\min(d(a',h), d(a',h')) > (1-3\varepsilon)r$.

Proof. By Proposition 2.2.8, $\ell(\gamma_B) < 3\ell(\alpha_B)$ and so $d(h,x) = \ell(\gamma_B) + d(b,x) < 3\ell(\alpha_B) + d(b,x) \leq 3(\ell(\alpha) + d(b,x)) = 3d(c,x) \leq 3d(y,x) < 3\varepsilon r$. Since $d(a,x) > r$, we have $d(a,h) > (1-3\varepsilon)r$. Similarly for h'. ☐

Now we examine how faces of M and M' may meet up along $\gamma \cup \gamma'$. We say that a face F of N *cancels* if there is some face F' of M such that F and F' cancel. If F and F' are faces of M and M', respectively, we say that F' *subsumes* F if F does not cancel with F' but $(F \cap \gamma) \subseteq (F' \cap \gamma')$. We say that F is *subsumed* if there is some face F' that subsumes F. We use the same terminology when the roles of F, M, and γ are switched with those of F', M', and γ'.

Lemma 2.3.3. *Let F, F' be faces of M, M'. If F cancels with F', then $\partial F \cap \gamma = \partial F' \cap \gamma'$.*

Proof. Suppose that F cancels with F', but $\partial F \cap \gamma \neq \partial F' \cap \gamma'$. Let $\partial F \cap \gamma = [p,q]$ and $\partial F' \cap \gamma' = [p',q']$. Then either $p \neq p'$ or $q \neq q'$. Suppose that $\|p\| < \|p'\|$: the other cases are similar. Let σ be the side of F *in* N which is incident to p' and is not contained in $\partial F'$. Let τ' be the side of F' incident to p' which is not contained in γ': see Figure 2.7. Then $\sigma * \tau'$ is a subpath of either a face bordering F' or the geodesic $[1,y']$. Since F and F' cancel, if $\mathrm{Lab}(\sigma)$ ends with a letter s, then $\mathrm{Lab}(\tau')$ begins with s^{-1}. Therefore either the boundary label of some face is not freely reduced, or $\mathrm{Lab}([1,y'])$ is not freely reduced. The former contradicts the fact that R is cyclically reduced, and the latter contradicts that $[1,y']$ is geodesic. ☐

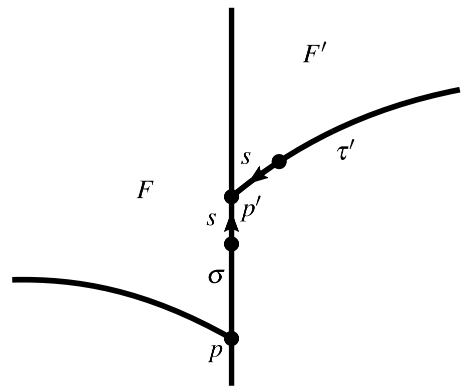

Figure 2.7: Diagram for the proof of Lemma 2.3.3

Lemma 2.3.4. *Let F be a face of M such that F is not the middle face, and such that either F is triangular, F is quadrilateral, or F is pentagonal with only one exterior side. Then F is not subsumed.*

Proof. Let $\sigma = \partial F \cap \gamma$, and let τ be the other exterior side of ∂F if there is one, either $\tau = \partial F \cap \alpha$ or $\tau = \partial F \cap \beta$. If F is triangular or quadrilateral, then applying Lemma 2.2.6 to τ yields that $\ell(\sigma) > \frac{1}{6}\ell(\partial F)$. If F is pentagonal and has only one exterior side, then σ is the only exterior side of F, so $\ell(\sigma) > \ell(\partial F) - \frac{4}{6}\ell(\partial F) = \frac{1}{3}\ell(\partial F)$. In all cases, if σ is also a subpath of the boundary of a face F' which does not cancel with F, then this contradicts the $C'(1/6)$ condition. □

Corollary 2.3.5. *If a face F of N is subsumed, then either F is the middle face, or F is a pentagonal face with two exterior sides. In either case F borders a face in B, so $h \in \partial F$.*

In Lemmas 2.3.6-2.3.11, E' is the extremal face at a', and

$$\rho' = \partial E' \cap \gamma'$$
$$\sigma' = \partial E' \cap \beta'$$
$$\tau' = \partial E' \setminus (\rho' \cup \sigma').$$

Lemma 2.3.6. *If $\varepsilon \le \frac{1}{6}$, then $\ell(\tau') < \frac{1}{6}\ell(\partial E')$.*

Proof. There are three cases to consider: either E' is triangular (Case 1 in Figure 2.8), E' is quadrilateral (Case 2), or A' contains no faces and E' is the middle face (Case 3). In Case 1, τ' is an interior side and the result is immediate. In Cases 2 and 3, we may apply Proposition 2.2.8 (c) to get that $\ell(\tau') < \ell(\alpha') < \varepsilon r$. Also, in these cases E' borders B', so $h \in E'$. Since $a' \in E'$ by definition, $[a', h'] \subseteq \ell(\rho')$ and so $\ell(\partial E') > 2\ell(\rho') \ge 2d(a', h') > 2(1-3\varepsilon)r$. Therefore $\ell(\tau') < \frac{\varepsilon r}{2(1-3\varepsilon)r}\ell(\partial E') = \frac{\varepsilon}{2-6\varepsilon}\ell(\partial E')$. Solving $\frac{\varepsilon}{2-6\varepsilon} \le \frac{1}{6}$ yields $\varepsilon \le \frac{1}{6}$. □

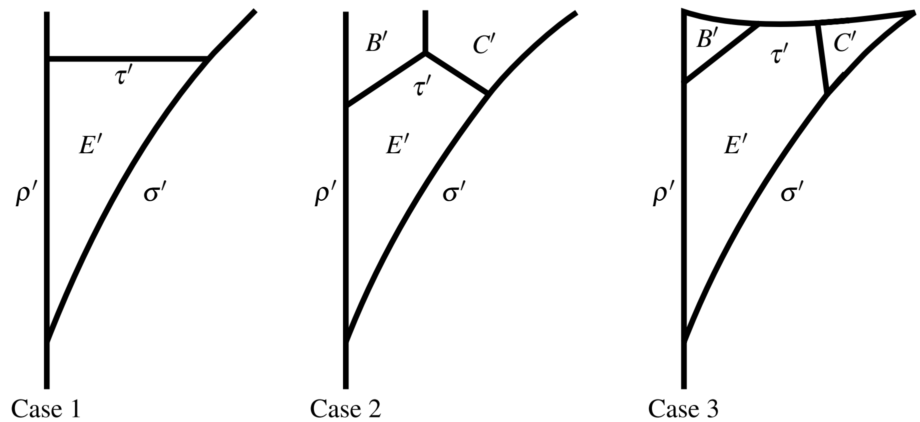

Figure 2.8: Diagram for the proof of Lemma 2.3.6

From now on (that is, Corollary 2.3.7-Corollary 2.3.10), it is assumed that $\varepsilon \leq \frac{1}{6}$, and this will not be restated in the hypotheses. Knowing Lemma 2.3.6, we may then apply Lemma 2.2.6 to ρ' and σ' in turn to obtain the following.

Corollary 2.3.7. *$\ell(\rho') > \frac{1}{3}\ell(\partial E')$ and $\ell(\sigma') > \frac{1}{3}\ell(\partial E')$.*

Lemma 2.3.8. *E' does not cancel.*

Proof. Suppose that E' cancels with some face F of M. Since $F \cap \gamma = E' \cap \gamma'$ by Lemma 2.3.3 and $a' \notin B$, we have that F is not a face of B. Therefore F borders β. Now there are two cases. Either there is an interior side of F which is incident to both γ and β (Case 1 in Figure 2.9), or F is the extremal face at a and $a = a'$ (Case 2). In Case 1, let θ be the side of F incident to a' and β. In Case 2, let θ be the edge of $\partial F \cap \beta$ which is incident to a'. Since F and E' cancel, let θ' be the path starting from a' which is a subpath of $\partial E'$ and has label $\mathrm{Lab}(\theta)$. Let p, p' be the endpoints of θ, θ', respectively.

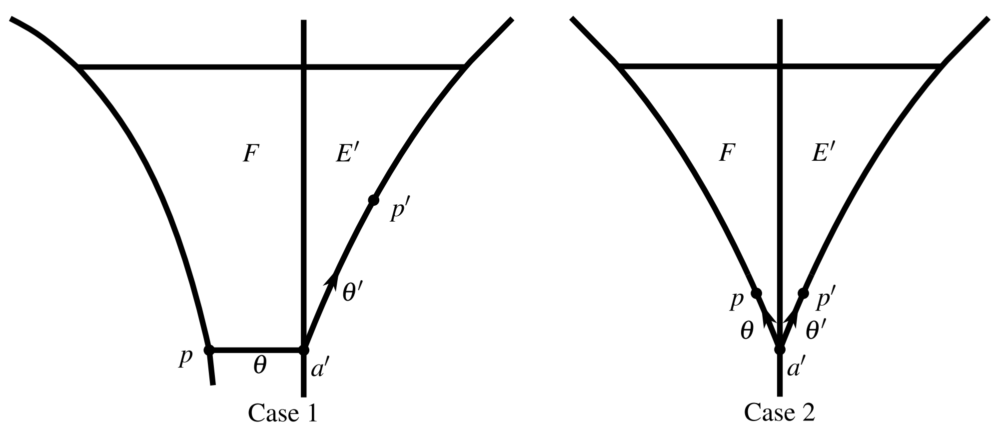

Figure 2.9: Diagram for the proof of Lemma 2.3.8

Now observe that in either case, $\ell(\theta') = \ell(\theta) < \frac{1}{6}\ell(\partial F) = \frac{1}{6}\ell(\partial E')$. By Corollary 2.3.7, $\ell(\sigma') > \frac{1}{3}\ell(\partial E)$, so θ' is a subpath of σ'. Therefore $p' \in \sigma' \subseteq \beta'$. Also, $p \in \beta$ by definition. Since $\mathrm{Lab}(\theta) = \mathrm{Lab}(\theta')$ and the combinatorial map $f : N \to \Gamma(G,S)$ is label-preserving, $f(p) = f(p')$. But f is injective when restricted to $\beta \cup \beta'$ since T is a tree, so this is a contradiction. □

Lemma 2.3.9. $\ell(\rho') > (1-3\varepsilon)r$.

Proof. Since $\|a'\| \geq \|a\|$, either ρ' extends beyond γ or ρ' is a subpath of γ. If ρ' extends beyond γ, then $a' \in \rho'$ and $b \in \rho'$, so $[a,b'] \subset \rho'$ and $\ell(\rho') \geq d(a',b) > (1-\varepsilon)r > (1-3\varepsilon)r$.

Suppose then that ρ' is a subpath of γ. Since E' does not cancel, the $C'(1/6)$ condition implies that each face of M bordering E' must cover less than one sixth of $\partial E'$. Recall that $\ell(\rho') > \frac{1}{3}\ell(\partial E')$ by Corollary 2.3.7. Therefore E' must border at least three faces of M, so E' subsumes some face F. Since F is subsumed, $(\partial F \cap \gamma) \subseteq \rho'$ and $h \in (\partial F \cap \gamma)$ by Corollary 2.3.5. Since ρ' contains a' as well, we have that $[a',h] \subseteq \rho'$ and thus $\ell(\rho') \geq d(a',h) > (1-3\varepsilon)r$. □

Corollary 2.3.10. $\ell(\rho' \cap [a',h]) > \frac{1-6\varepsilon}{3-9\varepsilon}\ell(\partial E')$.

Proof. First, observe that $\ell([h,x]) < 3\varepsilon r$ and $\ell(\rho') > (1-3\varepsilon)r$. Therefore $\ell([h,x]) < \frac{3\varepsilon}{1-3\varepsilon}\ell(\rho')$, so

$$\ell(\rho \cap [a',h]) = \ell(\rho' \smallsetminus [h,x]) \geq \ell(\rho') - \ell([h,x]) > \ell(\rho') - \left(\frac{3\varepsilon}{1-3\varepsilon}\right)\ell(\rho') = \left(\frac{1-6\varepsilon}{1-3\varepsilon}\right)\ell(\rho').$$

By Corollary 2.3.7, $\ell(\rho') > \frac{1}{3}\ell(\partial E')$. Therefore

$$\ell(\rho' \cap [a',h]) > \left(\frac{1-6\varepsilon}{1-3\varepsilon}\right)\ell(\rho') > \left(\frac{1-6\varepsilon}{3-9\varepsilon}\right)\ell(\partial E').$$

□

Lemma 2.3.11. *If $\varepsilon \leq \frac{1}{9}$, there is a face F of M satisfying all of the following conditions.*

(a) *F is subsumed by E'.*

(b) *F is either the middle face of M or the pentagonal face of A with two exterior sides.*

(c) $\ell(\partial F \cap \partial E') > \left(\frac{1-6\varepsilon}{3-9\varepsilon} - \frac{1}{6}\right)\ell(\partial E')$.

(d) $\ell(\partial F) < 6\varepsilon r$.

Proof. From the previous corollary we know that more than $\frac{1-6\varepsilon}{3-9\varepsilon}$ of $\partial E'$ must be covered by faces which are not in B. Since E' does not cancel, if $\frac{1-6\varepsilon}{3-9\varepsilon} \geq \frac{1}{6}$, then E' subsumes some face F of $M \smallsetminus B$. Solving $\frac{1-6\varepsilon}{3-9\varepsilon} \geq \frac{1}{6}$ yields $\varepsilon \leq \frac{1}{9}$, so choose $\varepsilon \leq \frac{1}{9}$ and part (a) follows. Part (b) follows immediately from Corollary 2.3.5. Furthermore, Corollary 2.3.5 implies that no other faces of $M \smallsetminus B$ can be subsumed by E'. Now E' subsumes

F implies that $\ell(\partial F \cap \partial E') < \frac{1}{6}\ell(\partial E')$, so there is still a subpath of ρ' of length more than $\left(\frac{1-6\varepsilon}{3-9\varepsilon} - \frac{1}{6}\right)\ell(\partial E')$, and thus of positive length, to be covered. Therefore E' must border one additional face of $M \setminus B$. Call this face E. Since E cannot be subsumed by E', we have that part of ∂E extends beyond γ', so $a' \in \partial E$. Therefore we have the situation depicted in Figure 2.10.

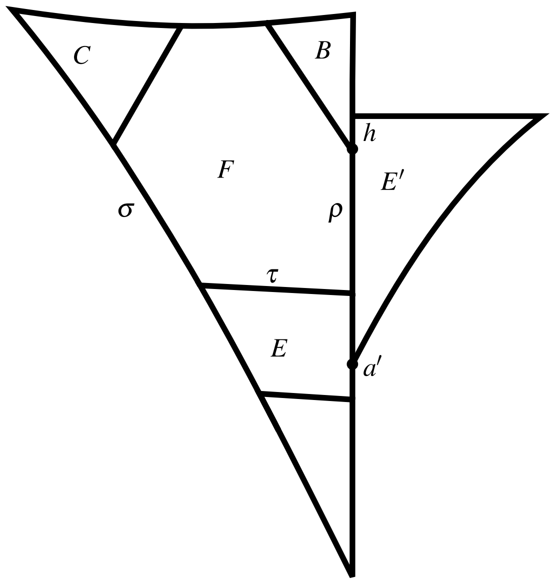

Figure 2.10: Diagram for the proof of Lemma 2.3.11

Notice that $\ell(\partial E \cap \partial E') < \frac{1}{6}\ell(\partial E')$ and $\ell((\partial F \cup \partial E) \cap \partial E') = \ell(\rho' \cap [a', h]) > \frac{1-6\varepsilon}{3-9\varepsilon}\ell(\partial E')$. Therefore $\ell(\partial F \cap \partial E') > \left(\frac{1-6\varepsilon}{3-9\varepsilon} - \frac{1}{6}\right)\ell(\partial E')$. This proves part (c).

Let $\rho = \partial F \cap \gamma$ and $\sigma = \partial F \cap \beta$. Then $\ell(\rho) < \frac{1}{6}\ell(\partial F)$ since E' subsumes F. Since F is either the middle face or the pentagonal face of A with two exterior sides, F has exactly one interior side, call it τ, which does not border either α, B, or C. By Proposition 2.2.8, the sum of the lengths of the other sides of F other than ρ, σ, or τ is less than $\ell(\alpha) < \varepsilon r$. Thus by Lemma 2.2.6 applied to σ, we have that $\ell(\rho) + \ell(\tau) + \varepsilon r \geq \frac{1}{2}\ell(\partial F)$. But $\max(\ell(\rho), \ell(\tau)) < \frac{1}{6}\ell(\partial F)$, so we have $\varepsilon r > \frac{1}{6}\ell(\partial F)$, or $\ell(\partial F) < 6\varepsilon r$. This proves part (d). $\square$

We return to Proposition 2.3.1, which we are now ready to prove.

Proof of Proposition 2.3.1. Suppose that T is a geodesic spanning tree of $\Gamma(G,S)$ rooted at 1, and T is not $(\varepsilon, 2)$-tight. If $\varepsilon \leq \frac{1}{12}$, then all of the previous lemmas hold. But then, in the notation of Lemma 2.3.11, we have

$$6\varepsilon r > \ell(\partial F) > 6\ell(\partial F \cap \partial E') > 6\left(\frac{1-6\varepsilon}{3-9\varepsilon} - \frac{1}{6}\right)\ell(\partial E') > \frac{1-9\varepsilon}{1-3\varepsilon}2\ell(\rho') = \frac{2-18\varepsilon}{1-3\varepsilon}(1-3\varepsilon)r = (2-18\varepsilon)r.$$

Thus $6\varepsilon > 2 - 18\varepsilon$ or $\varepsilon > \frac{1}{12}$, a contradiction. Therefore T is $(1/12, 2)$-tight. $\square$

Combining this with Proposition 2.1.4, we have the following theorem.

Theorem 2.3.12. *If G is a finitely generated $C'(1/6)$ group, then* $\operatorname{asdim}_{\mathrm{AN}}(G) \leq 2$.

CHAPTER 3

Asymptotic and Assouad-Nagata dimension of finitely generated groups and their subgroups

3.1 Adapting a construction of Higes

In order to prove Theorem 1, we will need to construct a countable group which can attain any positive Assouad-Nagata dimension. To be specific, in this section we construct a countable, locally finite group K and proper norms $\{\|\cdot\|_n | n \in \mathbb{Z}^+ \cup \{\infty\}\}$ such that $\operatorname{asdim}_{\mathrm{AN}}(K, \|\cdot\|_n) = n$ for every $n \in \mathbb{Z}^+ \cup \{\infty\}$. The idea is to take a direct sum of cyclic groups, block every n of them together, and scale the blocks appropriately. Note that since K is locally finite, $\operatorname{asdim}(K) = 0$ no matter the choice of proper norm.

This idea is already present in Higes' work. Namely, in [16] Higes constructs, for any $n \in \mathbb{Z}^+ \cup \{\infty\}$, a group G_n and a proper norm $\|\cdot\|_n$ on G such that $\operatorname{asdim}(G_n, \|\cdot\|_n) = 0$ but $\operatorname{asdim}_{\mathrm{AN}}(G_n, \|\cdot\|_n)$. However, in Higes' examples, if $m \neq n$, then G_m and G_n are not isomorphic. For our purposes, it is necessary that the group remain fixed, with only the norm varying. The rest of this section is devoted to working out the details of this construction.

3.1.1 Scaled normed groups and direct sums

Formally, a normed group should be an ordered pair $(G, \|\cdot\|_G)$. But from now on, whenever we say that G is a *normed* group, it is understood that G is equipped with a norm, which is always called $\|\cdot\|_G$. With this convention in mind we eliminate the norm from the notation wherever possible.

If G is a normed group and s is a positive real number, then the function $s\|\cdot\|_G : G \to \mathbb{R}_0^+, g \mapsto s\|g\|_G$ is also a norm on G. We call the normed group $(G, s\|\cdot\|_G)$ a *scaled normed group*, and denote it briefly by sG.

Given two normed groups G_0 and G_1 and scaling constants s_0, s_1, we define their *scaled direct product* $s_0G_0 \times s_1G_1$ to be the group $G_0 \times G_1$ endowed with the norm $\|\cdot\|_{(s_0,s_1)}$ defined by

$$\|(g_0, g_1)\|_{(s_0,s_1)} = s_0\|g_0\|_{G_0} + s_1\|g_1\|_{G_1}$$

for all $g_0 \in G_0$ and $g_1 \in G_1$. This is just the ℓ^1 product norm on $s_0G_0 \times s_1G_1$. For any $k \in \mathbb{N}$, we define the scaled direct product of finitely many scaled normed groups $\prod_{i=0}^k s_iG_i$ by iterating this construction. Note that for finite direct products we have that $\prod_{i=0}^k s_iG_i$ is bi-Lipschitz equivalent to $\prod_{i=0}^k G_i$ without scaling.

To avoid frequently having to state that certain sets are nonempty, we declare $\prod_{i\in\emptyset} G_i$ to be the trivial group. Let I be a set and $(G_i)_{i\in I}$ an I-tuple of groups. For $g = (g_i)_{i\in I} \in \prod_{i\in I} G_i$, we denote the *support* of g by $\operatorname{supp}(g)$; that is, $\operatorname{supp}(g) = \{i \in I \mid g_i \neq 1\}$. By definition $\bigoplus_{i\in I} G_i$ is the subgroup of $\prod_{i\in I} G_i$ consisting of all $g \in \prod_{i\in I} G_i$ such that $\mathrm{supp}(g)$ is finite. The notion of scaled direct products can then be extended to general direct sums in the following natural way.

Definition 3.1.1. Let I be a set, let $(G_i)_{i\in I}$ be an I-tuple of normed groups, and let $s = (s_i)_{i\in I}$ an I-tuple of scaling constants. Let $G = \bigoplus_{i\in I} G_i$. Then the *scaled direct sum* $\bigoplus_{i\in I} s_i G_i$ is defined to be the normed group $(G, \|\cdot\|_s)$, where $\|\cdot\|_s$ is given by

$$\|g\|_s = \sum_{i\in I} s_i \|g_i\|_{G_i}$$

for all $g \in G$. We call $\|\cdot\|_s$ the norm *induced* by s.

The following lemma just states that, for our purposes, we may assume all scaling constants are positive integers.

Lemma 3.1.2. *Let I be a set, $s = (s_i)_{i\in I}$ an I-tuple of scaling constants bounded away from zero. Then $\bigoplus_{i\in I} s_i G_i$ is bi-Lipschitz equivalent to $\bigoplus_{i\in I} s'_i G_i$, where s'_i is a positive integer for all $i \in I$.*

Proof. Suppose that $\varepsilon > 0$ is such that $s_i \geq \varepsilon$ for all $i \in I$. Let $s' = (s'_i)_{i\in I} = (\lceil s_i \rceil)_{i\in I}$, and let $g = (g_i)_{i\in I} \in \bigoplus_{i\in I} G_i$. Then clearly $\|g\|_s \leq \|g\|_{s'}$, and

$$\|g\|_{s'} = \sum_{i\in I} \lceil s_i \rceil \|g_i\|_{G_i} \leq \sum_{i\in I} \left(\tfrac{s_i+1}{s_i}\right) s_i \|g_i\|_{G_i} \leq \left(1+\tfrac{1}{\varepsilon}\right) \sum_{i\in I} s_i \|g_i\|_{G_i} = \left(1+\tfrac{1}{\varepsilon}\right) \|g\|_s \,.$$

□

3.1.2 A fixed group with varying norms

The next set of lemmas deal specifically with direct sums of cyclic groups. Here we assume that a cyclic group comes equipped with the natural word norm, that is $\|x\|_{\mathbb{Z}_\ell} = \min(x, \ell - x)$ for all $x \in \mathbb{Z}_\ell$, and $\|x\|_{\mathbb{Z}} = |x|$ for all $x \in \mathbb{Z}$. Unless otherwise noted, tuples are sequences indexed by $\mathbb{N}$, e.g. (s_i) stands for $(s_i)_{i\in\mathbb{N}}$.

Definition 3.1.3. Let $(x_i) \in \bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{\ell_i}$. The *geodesic form* of (x_i) is the unique sequence of integers (y_i) such that for all $i \in \mathbb{N}$,

- $y_i \equiv x_i \mod \ell_i$, and
- $y_i \in \left\{ -\left\lfloor \frac{\ell_i - 1}{2} \right\rfloor, \ldots, -1, 0, 1, \ldots, \left\lfloor \frac{\ell_i}{2} \right\rfloor \right\}$.

Note that if $s = (s_i)$ is a sequence of scaling constants, $x = (x_i) \in \bigoplus_{i\in\mathbb{N}} s_i \mathbb{Z}_{\ell_i}$, and (y_i) is the geodesic form of x, then we have

$$\|x\|_s = \sum_{i\in\mathbb{N}} s_i |y_i| \,. \tag{3.1}$$

Definition 3.1.4. For $s \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$, assume that the set

$$s\{0, \ldots, k\}^n = \{0, s, 2s, \ldots, ks\}^n \subset \mathbb{R}^n$$

is equipped with the ℓ^1 metric. Then an *expanded n-dimensional cube* is a space isometric to $s\{0, \ldots, k\}^n$ for some $s \geq 1$ and $k \in \mathbb{N}$.

In accordance with Definition 3.1.4, whenever s is a scaling constant and $s \geq 1$, we call s an *expansion constant*. Sequences of expanded cubes are useful for establishing lower bounds on the asymptotic Assouad-Nagata dimension of a metric space.

Lemma 3.1.5. [16, Corollary 2.7] *Let X be a metric space, $n \in \mathbb{N}$. If X contains a sequence of expanded n-dimensional cubes $s_j\{0, \ldots, k_j\}^n$ where $\lim_{j\to\infty} k_j = \infty$, then* $\operatorname{asdim}_{\mathrm{AN}}(X) \geq n$.

Suppose that P is a set with $|P| \geq n$, $(\ell_i)_{i\in P}$ is a P-tuple of natural numbers, and s_P is an expansion constant. Let k_P be a natural number with $k_P \leq \min\{\ell_i/2 \mid i \in P\}$. Then by (3.1), $s_P \bigoplus_{i\in P} \mathbb{Z}_{\ell_i}$ contains an expanded n-dimensional cube $s_P\{0, \ldots, k_P\}^n$. This observation along with Lemma 3.1.5 is what allows us to construct a countable group which can achieve any positive Assouad-Nagata dimension. To smooth the process, we introduce the following ad hoc notation.

Definition 3.1.6. For each $m \in \mathbb{Z}^+ \cup \{\infty\}$, let $\mathscr{P}_m = \{P_{(m,j)} \mid j \in \mathbb{N}\}$ be the partition of $\mathbb{N}$ given by

$$P_{(m,j)} = \begin{cases} \{jm, jm+1, \ldots, (j+1)m-1\} & \text{if } m \in \mathbb{Z}^+ \\ \{j^2, j^2+1, \ldots, (j+1)^2-1\} & \text{if } m = \infty. \end{cases}$$

Definition 3.1.7. Let $s = (s_i)$ be a sequence, $m \in \mathbb{Z}^+ \cup \{\infty\}$. Let the *m-inflation* of s, denoted $m \times s$, be the sequence defined by

$$(m \times s)_i = s_j \Leftrightarrow i \in P_{(m,j)}\,.$$

For example, if $s = (1, 2, 3, \ldots)$, then $2 \times s = (1, 1, 2, 2, 3, 3, \ldots)$ and $\infty \times s = (1, 2, 2, 2, 3, 3, 3, 3, \ldots)$. By definition,

$$s_i = \begin{cases} (m \times s)_{im} & \text{if } m \in \mathbb{Z}^+ \\ (m \times s)_{i^2} & \text{if } m = \infty \end{cases} \quad \text{and} \quad (m \times s)_i = \begin{cases} s_{\lfloor i/m \rfloor} & \text{if } m \in \mathbb{Z}^+ \\ s_{\lfloor \sqrt{i} \rfloor} & \text{if } m = \infty. \end{cases} \tag{3.2}$$

Lemma 3.1.8. *Let $d \in \mathbb{N}$, and let $(c_0, \ldots, c_{d-1})$ be a finite sequence of scaling constants. Let $m \in \mathbb{Z}^+ \cup \{\infty\}$ be fixed, let (s_i) be an increasing sequence of expansion constants, and let (ℓ_i) be an increasing sequence of* *positive integers. Let*

$$Z_d = \bigoplus_{i=0}^{d-1} c_i \mathbb{Z}, \qquad\qquad K_m = \bigoplus_{i\in\mathbb{N}} (m \times s)_i \mathbb{Z}_{\ell_i}.$$

Then $\operatorname{asdim}_{\mathrm{AN}}(Z_d \times K_m) \geq d+m$.

Proof. By Lemma 3.1.2, we may assume without loss of generality that all s_i are positive integers. Since finite direct products preserve bi-Lipschitz equivalence, we may also assume that all c_i are equal to 1, so that $Z_d = \mathbb{Z}^d$.

Now note that

$$Z_d \times K_m = \mathbb{Z}^d \times \bigoplus_{j\in\mathbb{N}} \left(s_j \bigoplus_{i\in P_{(m,j)}} \mathbb{Z}_{\ell_i} \right),$$

where $\mathbb{Z}^d \times s_j \bigoplus_{i\in P_{(m,j)}} \mathbb{Z}_{\ell_i}$ is an isometrically embedded subgroup for each $j \in \mathbb{N}$. Let

$$k_j = \min\{\lfloor \ell_i/2 \rfloor \mid i \in P_{(m,j)}\} = \begin{cases} \lfloor \ell_{jm}/2 \rfloor & \text{if } m \in \mathbb{Z}^+ \\ \lfloor \ell_{j^2}/2 \rfloor & \text{if } m = \infty. \end{cases}$$

Then $\lim_{j\to\infty} k_j = \infty$.

If $m \in \mathbb{Z}^+$, then $|P_{(m,j)}| = m$ for all $j \in \mathbb{N}$. Then since s_j is a positive integer, $\mathbb{Z}^d \times s_j \bigoplus_{i\in P_{(m,j)}} \mathbb{Z}_{\ell_i}$ contains an isometrically embedded expanded $(d+m)$-dimensional cube $s_j\{0,\dots,k_j\}^{d+m}$ for all $j \in \mathbb{N}$. Since $\lim_{j\to\infty} k_j = \infty$, by Lemma 3.1.5 we have $\operatorname{asdim}_{\mathrm{AN}}(\mathbb{Z}^d \times K_m) \geq d+m$.

If $m = \infty$, let $n \in \mathbb{Z}^+$. Then $|P_{(m,j)}| = (j+1)^2 - j^2 = 2j+1 \geq n$ for all $j \geq n$. Therefore $s_j \bigoplus_{i\in P_{(m,j)}} \mathbb{Z}_{\ell_i}$ contains the expanded n-dimensional cube $s_j\{0,\dots,k_j\}^n$ for all $j \geq n$. Since $\lim_{j\to\infty} k_j = \infty$, by Lemma 3.1.5 we have $\operatorname{asdim}_{\mathrm{AN}}(K_\infty) \geq n$. Since $n \in \mathbb{Z}^+$ was chosen arbitrarily, $\operatorname{asdim}_{\mathrm{AN}}(K_\infty) = \infty$, thus $\operatorname{asdim}_{\mathrm{AN}}(Z_d \times K_\infty) = \infty$. □

Now, in the notation of Lemma 3.1.8, we wish to impose certain conditions on the sequence (s_i) of expansion constants to guarantee that $\operatorname{asdim}_{\mathrm{AN}}(Z_d \times K_m) = d+m$ exactly. We will use a lemma of Higes; in order to do so we need to introduce a little notation, and consider a different norm on countable direct sums of scaled normed groups.

Definition 3.1.9. Let (G_i) be a sequence of normed groups and $s = (s_i)$ a sequence of scaling constants. Let

$G = \bigoplus_{i\in\mathbb{N}} G_i$. For convenience, let us define the *height* function $h : G \to \mathbb{N}$ by

$$h(g) = \begin{cases} 0 & \text{if } g = 1 \\ \max(\operatorname{supp}(g)) & \text{otherwise.} \end{cases}$$

Now define the *quasi-ultranorm* on G induced by s, denoted $\|\cdot\|_s^{\mathrm{qu}}$, by

$$\|g\|_s^{\mathrm{qu}} = s_h \|g_h\|_{G_h} \tag{3.3}$$

for all $g = (g_i) \in G$, where $h = h(g)$.

In [16], Higes calls the metric associated to this norm the *quasi-ultrametric* generated by the sequence of metrics (d_{G_i}), where d_{G_i} is the metric associated to the scaled norm $s_i\|\cdot\|_{G_i}$ for each $i \in \mathbb{N}$. For this reason we call the norm in (3.3) the quasi-ultranorm on G induced by s, and put “qu” in the superscript. The next lemma says that if all G_i are finite then, under mild assumptions about the growth of the sequence s, the norms $\|\cdot\|_s$ and $\|\cdot\|_s^{\mathrm{qu}}$ are, for our purposes, interchangeable.

Lemma 3.1.10. *Let (G_i) be a sequence of finite normed groups and $s = (s_i)$ a sequence scaling constants. Let $G = \bigoplus_{i\in\mathbb{N}} G_i$. Suppose that $G_i, \|\cdot\|_{G_i}, s_i$ satisfy the following conditions for all $i \in \mathbb{N}$:*

- $\|g_i\|_{G_i} \geq 1$ *for all* $g_i \in G_i \smallsetminus \{1\}$.
- $\operatorname{diam}(G_{i+1}) \geq \operatorname{diam}(G_i)$.
- $s_{i+1} \geq 2 s_i \operatorname{diam}(G_i)$.

Then the norm $\|\cdot\|_s$ and quasi-ultranorm $\|\cdot\|_s^{\mathrm{qu}}$ induced by s are bi-Lipschitz equivalent.

Proof. Clearly $\|g\|_s^{\mathrm{qu}} \leq \|g\|_s$ for all $g \in G$.

We now prove by induction on $h(g)$ that $\|g\|_s \leq 2\|g\|_s^{\mathrm{qu}}$. This is clear when $h(g) = 0$. Now suppose that $h(g) = k \geq 1$. Write g as $g'g''$, where $g'_j = g_j$ exactly when $j = k$ and is equal to 1 otherwise, and $h(g'') = i < k$. Then we have

$$\begin{aligned} \|g\|_s &\leq \|g'\|_s + \|g''\|_s = \|g'\|_s^{\mathrm{qu}} + \|g''\|_s \leq \|g'\|_s^{\mathrm{qu}} + 2\|g''\|_s^{\mathrm{qu}} \\ &\leq \|g'\|_s^{\mathrm{qu}} + 2 s_{k-1} \operatorname{diam}(G_{k-1}) \\ &\leq \|g'\|_s^{\mathrm{qu}} + s_k \leq 2\|g'\|_s^{\mathrm{qu}} = 2\|g\|_s^{\mathrm{qu}}. \end{aligned}$$

∎

Lemma 3.1.11. [16, Proof of Corollary 4.11] *Let (ℓ_i) be an increasing sequence of positive integers with $\ell_0 \geq 2$. Let m be a fixed positive integer. Let $s = (s_i)$ be a sequence of expansion constants such that*

$$s_{i+1} \geq 1 + s_i \operatorname{diam}(\mathbb{Z}^m_{\ell_i}) = 1 + (m\lfloor \ell_i/2 \rfloor)s_i\,.$$

Let $K^{\mathrm{qu}}_m = (\bigoplus_{i\in\mathbb{N}} \mathbb{Z}^m_{\ell_i}, \|\cdot\|^{\mathrm{qu}}_s)$. Then for any $k \in \mathbb{N}$ we have $\operatorname{asdim}_{\mathrm{AN}}(\mathbb{Z}^k \times K^{\mathrm{qu}}_m) = k + m$.

We use this lemma in the case $k = 0, m = 1$ to obtain the slightly generalized lemma that we need.

Lemma 3.1.12. *Let $d \in \mathbb{N}$, and let $(c_0, \ldots, c_{d-1})$ be a finite sequence of scaling constants. Let (ℓ_i) be a sequence of positive integers, and let $m \in \mathbb{Z}^+ \cup \{\infty\}$ be fixed. Let (s_j) be an increasing sequence of expansion constants such that, if $m \in \mathbb{Z}^+$, we have*

$$s_{j+1} \geq (\ell_{(j+1)m})s_j$$

for all $j \in \mathbb{N}$. Now let

$$Z_d = \bigoplus_{i=0}^{d-1} c_i\mathbb{Z} \qquad\qquad K_m = \bigoplus_{i\in\mathbb{N}} (m \times s)_i \mathbb{Z}_{\ell_i}\,.$$

Then $\operatorname{asdim}_{\mathrm{AN}}(Z_d \times K_m) = d + m$.

Proof. The lower bound is established in Lemma 3.1.8. For the upper bound, suppose that $m \in \mathbb{Z}^+$. Then

$$K_m = \bigoplus_{r=0}^{m-1} \left(\bigoplus_{j\in\mathbb{N}} s_j \mathbb{Z}_{\ell_{jm+r}} \right).$$

Since (ℓ_i) is increasing, for all $j \in \mathbb{N}$ and $r \in \{0, \ldots, m-1\}$ we have that

$$s_{j+1} \geq (\ell_{(j+1)m})s_j \geq (\ell_{jm+r})s_j \geq (2\lfloor \ell_{jm+r}/2 \rfloor)s_j = (2\operatorname{diam}(\mathbb{Z}_{\ell_{jm+r}}))s_j \geq 1 + s_j \operatorname{diam}(\mathbb{Z}_{\ell_{jm+r}})\,.$$

Therefore for any fixed $r \in \{0, \ldots, m-1\}$, the sequences $(\ell_{jm+r}), (\mathbb{Z}_{\ell_{jm+r}})$, and (s_j) together satisfy the assumptions of Lemmas 3.1.10 and 3.1.11. Hence for all $r \in \{0, \ldots, m-1\}$,

$$\operatorname{asdim}_{\mathrm{AN}} \left(\bigoplus_{j\in\mathbb{N}} s_j \mathbb{Z}_{jm+r} \right) = \operatorname{asdim}_{\mathrm{AN}} \left(\bigoplus_{j\in\mathbb{N}} \mathbb{Z}_{jm+r}, \|\cdot\|_s \right) = \operatorname{asdim}_{\mathrm{AN}} \left(\bigoplus_{j\in\mathbb{N}} \mathbb{Z}_{jm+r}, \|\cdot\|^{\mathrm{qu}}_s \right) = 1\,.$$

Thus by Lemma 1.2.11,

$$\operatorname{asdim}_{\mathrm{AN}}(K_m) \leq \sum_{r=0}^{m-1} \operatorname{asdim}_{\mathrm{AN}} \left(\bigoplus_{j\in\mathbb{N}} s_j \mathbb{Z}_{jm+r} \right) \leq m\,,$$

and $\operatorname{asdim}_{\mathrm{AN}}(Z_d) = \operatorname{asdim}_{\mathrm{AN}}(\mathbb{Z}^d) = d$. Therefore $\operatorname{asdim}_{\mathrm{AN}}(Z_d \times K_m) \leq d + m$. □

The importance of Lemma 3.1.12 lies in the fact that if (ℓ_i) is fixed and $m, n \in \mathbb{Z}^+ \cup \{\infty\}$ are distinct, then K_m and K_n are merely the same group with different norms. Later, we will construct two finitely generated groups A and B with subgroups that are isomorphic and bi-Lipschitz equivalent to K_m and K_n, respectively. Since K_m and K_n are isomorphic, we construct a finitely generated group G which is the amalgamated product of A and B along an isomorphism between K_m and K_n. The isomorphism "collapses" K_n, so that the Assouad-Nagata dimension of G is not much more than m, while the Assouad-Nagata dimension of B is at least n. To construct A, B, and G such that all of the aforementioned geometric properties hold, we use some small cancellation theory. This is the topic of the next section.

3.2 Operations on van Kampen diagrams and signed r-face counts

The goal of this section is to prove Lemma 3.2.18, which states that words of a certain form are quasigeodesic in certain central extensions of $C'(\lambda)$ groups, where $0 < \lambda < 1/12$. This is a generalization [32, Lemma 5.10], originally used to construct finitely generated groups with circle-tree asymptotic cones. The proof of Lemma 3.2.18 is a technical argument that involves performing surgery on van Kampen diagrams.

In Section 3.2.1, we define signed and unsigned r-face counts, where r is a relation of a presentation. We also introduce various operations on van Kampen diagrams, and examine how each of these operations affects the signed and unsigned r-face counts. In Section 3.2.2, we collect some facts about van Kampen diagrams over $C'(1/6)$ presentations that are used in the proof of Lemma 3.2.18. Finally, in Section 3.2.3, we prove Lemma 3.2.18.

3.2.1 Operations on van Kampen diagrams

Given a van Kampen diagram M over a presentation $\langle S \mid R \rangle$, there are various ways to deform M within the plane to get another van Kampen diagram M'. To check that the resulting graph M' is really a van Kampen diagram, it suffices to show that the operation preserves connectedness and produces a planar embedding of M'. If one also requires that M' is a van Kampen diagram over the *same* presentation, one needs to check that any new faces enclosed by the operation have a boundary label which is either in R_* or equal to the identity in $F(S)$. In this section we list a few operations on van Kampen diagrams that are needed for the proof of Lemma 3.2.18. In our case it will be necessary to keep track of how each operation affects the boundary label $\operatorname{Lab}(\partial M)$, as well as two quantities that we call the signed and unsigned r-face counts.

Definition 3.2.1. Let M be a van Kampen diagram over a presentation $\langle S \mid R \rangle$, where R is cyclically minimal. Let $r \in R$. Then the (unsigned) *r-face count* $\kappa(M, r)$ is the total number of r-faces in M.

Definition 3.2.2. Let M be a van Kampen diagram over a presentation $\langle S \mid R\rangle$ where R is cyclically minimal, and let $r \in R$. Then the *signed r-face count* $\sigma(M,r)$ is defined as follows.

- If F is a face of M, then

$$\sigma(F,r) = \begin{cases} 1 & \text{if } \mathrm{Lab}(\partial F, x, +) = r \text{ for some } x \in \partial F \\ -1 & \text{if } \mathrm{Lab}(\partial F, x, -) = r \text{ for some } x \in \partial F \\ 0 & \text{otherwise.} \end{cases}$$

- $\sigma(M,r) = \sum\{\sigma(F,r) \mid F \text{ is a face of } M\}$.

The assumption that R is cyclically minimal ensures that each face contributes to the signed or unsigned r-face count of at most one $r \in R$. Note that if F and F' are two faces of M that cancel with each other, then $\sigma(F,r) = -\sigma(F',r)$ for all $r \in R$.

Operation 3.2.3 (Removing an inessential edge)**.** Suppose that $e = (x,y)$ is an inessential edge of a van Kampen diagram M over a presentation $\langle S \mid R\rangle$, where R is cyclically reduced and cyclically minimal, and $\mathrm{Lab}(\partial M)$ is cyclically reduced. Then e is on the boundary of exactly two inessential bounded faces. There are two possibilities.

(a) If $x \neq y$, contract e to remove it. This will produce a connected, planar embedding of the new graph. This changes two inessential faces with labels $1u$ and $1v$ to two inessential faces with labels u and v. Since R is cyclically reduced, this does not affect the r-face count for any $r \in R$.

(b) If $x = y$, delete e to remove it. Since e is a loop, this will leave the graph connected. This replaces two inessential faces on either side of e with labels $u1$ and $1v$ with a single inessential face labeled uv. Again since R is cyclically reduced, this operation does not affect $\sigma(M,r)$ for any $r \in R$.

Note that neither (a) nor (b) can introduce new self-intersections in the boundary path of any face of M. Also, since $\mathrm{Lab}(\partial M)$ is cyclically reduced, neither operation affects $\mathrm{Lab}(\partial M)$.

Operation 3.2.4 (Removing a simple subdiagram with trivial boundary label)**.** Let M be a van Kampen diagram over $\langle S \mid R\rangle$, where R and $\mathrm{Lab}(\partial M)$ are both cyclically reduced. Suppose that M contains a simple subdiagram D such that ∂D contains no inessential edges and $\mathrm{Lab}(\partial D) =_{F(S)} 1$. Then $\partial D = \alpha_+\alpha_-$, where $\mathrm{Lab}(\alpha_-) = \mathrm{Lab}(\alpha_+)^{-1}$. We may then remove D by replacing D with a simple inessential face F and deforming α_+ onto α_- through the interior of F. This does not affect the boundary label of M.

Note that if F and F' are simple faces that intersect simply, and F cancels with F', then $F \cup F'$ is a simple subdiagram of M with trivial boundary label, which may be removed by applying Operation 3.2.4. Perhaps

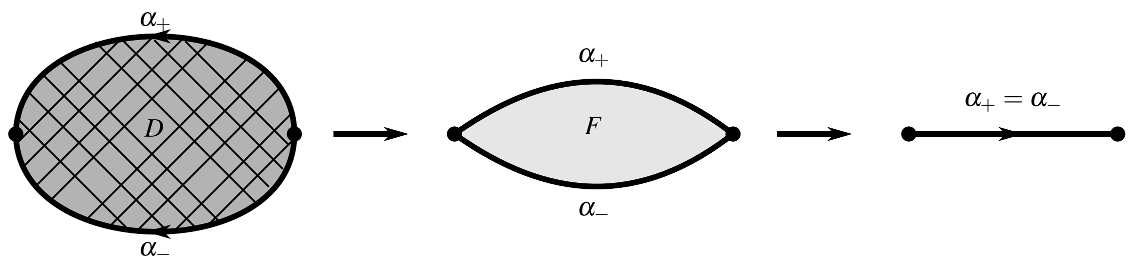

Figure 3.1: Removing a simple subdiagram with trivial boundary label

surprisingly, Operation 3.2.4 does not always preserve the signed r-face count, as the following example shows.

Example 3.2.5. Figure 3.2 depicts a van Kampen diagram M over the presentation $\langle a,b \mid a^2, aba^{-1}b\rangle$ with boundary label bb^{-1}, such that $\sigma(M, aba^{-1}b) = 2$.

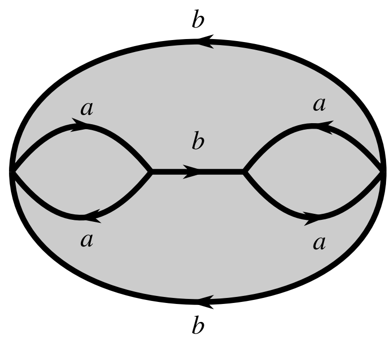

Figure 3.2: Operation 3.2.4 does not always preserve the signed r-face count of a van Kampen diagram

However, Operation 3.2.4 *does* preserve the signed r-face count of van Kampen diagrams over $C'(1/6)$ presentations. This is because $C'(1/6)$ presentations are aspherical. The definition of a spherical van Kampen diagram is the same as that of a van Kampen diagram with $\mathbb{R}^2$ replaced by S^2: in particular, every face is bounded. A presentation $\langle S \mid R\rangle$ is *aspherical* if every bare spherical van Kampen diagram over $\langle S \mid R\rangle$ contains a pair of faces that cancel. The following is a special case of a lemma of Olshanskii.

Lemma 3.2.6. [33, Lemma 31.1 part 2)] *Let $\langle S \mid R\rangle$ be an aspherical presentation, and suppose that M is a van Kampen diagram over $\langle S \mid R\rangle$ with boundary label w, where $w =_{F(S)} 1$. Then $\sigma(M,r) = 0$ for all $r \in R$.*

Operation 3.2.7 (Padding a vertex)**.** Suppose that x is a vertex of M which appears twice in the boundary path of some face (bounded or unbounded) of M. Choose $\varepsilon > 0$ small enough so that $B(x,\varepsilon) \subset \mathbb{R}^2$ contains only the ends of edges incident to x. Now $B(x,\varepsilon) \setminus M$ consists of finitely many connected components: let these be denoted $C_0, C_1, \ldots, C_k$. For each $i \in \{0,\ldots,k\}$, insert a clone x_i of x into C_i, and connect it to x with an inessential edge. Then duplicate the edges on either side of x_i, attaching the endpoint meant for x to x_i

instead: see Figure 3.3. The resulting graph has the same essential faces and boundary path as M, and one fewer vertex that is a point of self-intersection of the boundary path of a face. Each new inessential face has boundary label $1ss^{-1}$ for some $s \in S$.

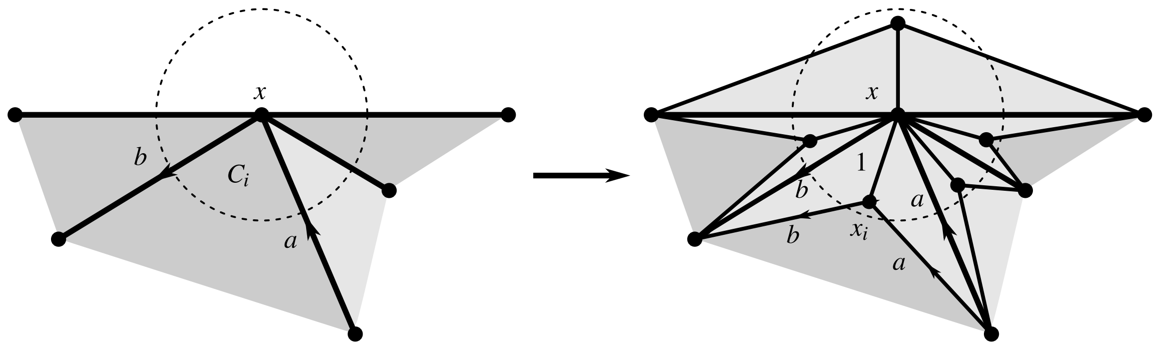

Figure 3.3: Padding a vertex

Operation 3.2.8 (Quotienting simple faces)**.** Suppose that $G = \langle S \mid R_G \rangle$ and $H = \langle S \mid R_H \rangle$ is a quotient of G, so every word in R_G represents the identity element of H. Suppose that M_G is a van Kampen diagram over $\langle S \mid R_G \rangle$. Let F be a simple face of M_G, and let M_F be a chosen van Kampen diagram over $\langle S \mid R_H \rangle$ with boundary label $\mathrm{Lab}(\partial F)$. Then we may quotient F to a copy of M_F without affecting the boundary label of M_G: see Figure 3.4. Applying this operation once produces a van Kampen diagram over $\langle S \mid R_G \cup R_H \rangle$. If F is the last face of M_G with label in $R_G \setminus R_H$, then this results in a van Kampen diagram over $\langle S \mid R_H \rangle$. Thus, if this operation can be applied to every essential face of M_G in sequence, then we obtain a "quotient van Kampen diagram" M_H over $\langle S \mid R_H \rangle$ with the same boundary label as M_G.

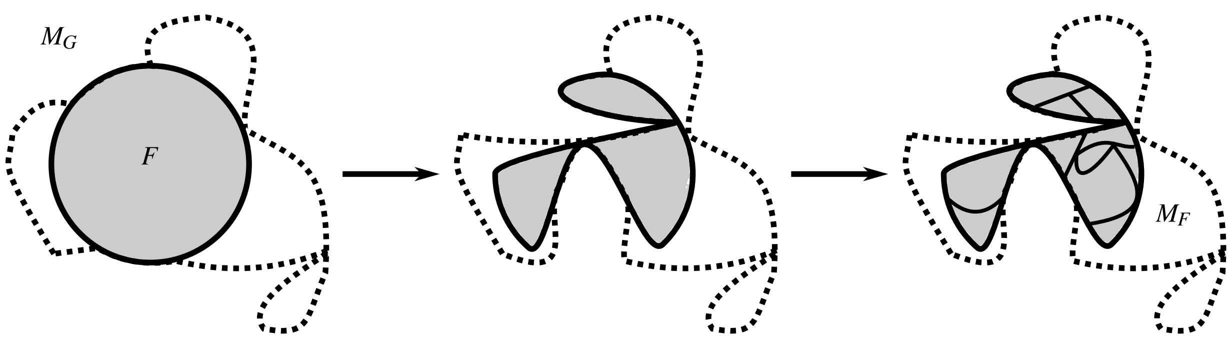

Figure 3.4: Quotienting a simple face

Operation 3.2.9 (Excising a subpath of ∂M)**.** Let M be a van Kampen diagram over a presentation $\langle S \mid R \rangle$, where R is cyclically minimal and cyclically reduced. Let $z \in \partial M$, and suppose we can write $(\partial M, z, +)$ as $\alpha * \beta$, where α and β are paths of positive length. Suppose that $\alpha = \alpha_0 * \rho * \alpha_1$, where $\mathrm{Lab}(\rho)$ is a cyclic shift of $r^{\pm 1}$ for some $r \in R$. Let x be the initial and y the terminal vertex of ρ, and suppose $x \neq y$. Then we may contract x to y through the unbounded face, identifying the two vertices to obtain a new van Kampen diagram M'. Now M' has exactly one new face F', where $(\partial F', x, -) = \rho$, so M' is a van Kampen diagram over the same presentation $\langle S \mid R \rangle$. Also, $(\partial M', z, +) = \alpha' * \beta$, where $\alpha' = \alpha_0 * \alpha_1$: see Figure 3.5. Note that ρ may intersect itself: in that case $\partial F'$ will have self-intersections in M', but this is fine. The only topological feature of M which is essential to this operation is that x and y are distinct.

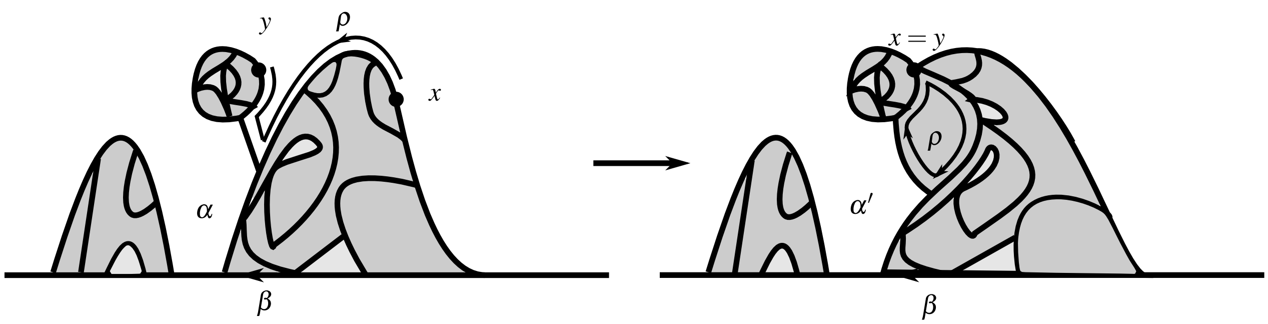

Figure 3.5: Excising a subpath of ∂M

Now $\ell(\alpha') = \ell(\alpha) - |r|$, and the sequence of edges of β is unaffected by the operation. Also, for all $r' \in R$,

$$\kappa(M', r') = \begin{cases} \kappa(M, r') + 1 & \text{if } r' = r \\ \kappa(M, r') & \text{otherwise.} \end{cases}$$

$$\sigma(M', r') = \begin{cases} \sigma(M, r') - 1 & \text{if } r' = r \text{ and } \mathrm{Lab}(\rho) \text{ is a cyclic shift of } r \\ \sigma(M, r') + 1 & \text{if } r' = r \text{ and } \mathrm{Lab}(\rho) \text{ is a cyclic shift of } r^{-1} \\ \sigma(M, r') & \text{otherwise.} \end{cases}$$

Note that, since R is cyclically reduced, these last three cases are all distinct. Indeed, it is an easy exercise to show that if a word $r \in R$ is a cyclic shift of r^{-1}, then r is not reduced.

3.2.2 Reductions that preserve signed r-face counts

Later we will need to use Lemma 3.2.23, a result that applies only to bare, reduced van Kampen diagrams over $C'(1/6)$ presentations. At the same time, we would like to apply this result to van Kampen diagrams with signed r-face counts that are carefully controlled. Thus, we need to establish a method of taking a van Kampen diagram over a $C'(1/6)$ presentation, and making it bare and reduced without affecting the signed r-face counts. In this subsection, we develop such a process, which is encapsulated in Lemma 3.2.16. We then prove Lemma 3.2.17, which allows us to construct certain "quotient" van Kampen diagrams with controlled

r-face counts.

Lemma 3.2.10. *Let M be a van Kampen diagram over $\langle S \mid R\rangle$ such that R is cyclically minimal and cyclically reduced, and $\mathrm{Lab}(\partial M)$ is cyclically reduced. Then there exists a van Kampen diagram M' such that all of the following conditions hold.*

(a) $\mathrm{Lab}(\partial M') = \mathrm{Lab}(\partial M)$.

(b) $\sigma(M', r) = \sigma(M, r)$ *for all* $r \in R$.

(c) *Every inessential face of M' has boundary label ss^{-1} or $1ss^{-1}$ for some $s \in S$.*

(d) *All inessential edges of M' are loops.*

Proof. Let I be the set of all inessential faces in M whose boundary labels are not equal to $1ss^{-1}$ or ss^{-1} for some $s \in S$. Let $F \in I$. If ∂F consists of a single inessential edge loop, we simply contract this loop to remove F; it is easy to see that this preserves the boundary label as well as the signed and unsigned face counts of M, and whether M satisfies (c) or (d). Therefore we assume that ∂F is at least two edges long. Now we repeatedly pad vertices of ∂F (Operation 3.2.7) until F is simple. Since each inessential face added in the process has boundary label $1ss^{-1}$ for some $s \in S$, this does not increase $|I|$.

We claim that, without loss of generality, we may assume that ∂F contains no inessential edges. Suppose that ∂F contains an inessential edge e. Since ∂M and R are both cyclically reduced, e lies on the boundary of exactly two inessential, bounded faces, one of which is F: call the other one F'. Then for some $u, u' \in S_{\circ}^{*}$ we have $\mathrm{Lab}(F) = 1u$ and $\mathrm{Lab}(F') = 1u'$. We know that e must have distinct endpoints since ∂F is a simple closed curve and $\ell(\partial F) \geq 2$. Therefore we may remove e using Operation 3.2.3 (a). This changes the boundary label of F from $1u$ to u, and the boundary label of F' from $1u'$ to u'. Thus it does not change whether or not F or F' is a member of I. Therefore removing e does not change $|I|$, and without loss of generality we may assume that ∂F contains no inessential edges.

Since F is simple, $\mathrm{Lab}(\partial F) =_{F(S)} 1$, and ∂F contains no inessential edges, we may remove F using Operation 3.2.4. This reduces $|I|$ by 1. Since $\mathrm{Lab}(\partial M)$ is cyclically reduced, none of the previous operations affect $\mathrm{Lab}(\partial M)$. Since only inessential faces were removed, and R is cyclically reduced, $\sigma(M, r)$ is also preserved for all $r \in R$. Repeating this process, we obtain a diagram M' for which (a) and (b) hold and $|I| = 0$, i.e. such that (a)-(c) hold. At this point we may repeatedly apply Operation 3.2.3 (a) to remove all inessential edges of M' with distinct endpoints, so that (d) holds in M'. Reasoning as in the previous paragraph, one can see that this does not interfere with conditions (a)-(c). Thus (a)-(d) hold in M', finishing the construction. $\square$

Lemma 3.2.11. *Let G be a group given by presentation $\langle S \mid R\rangle$, where R is cyclically minimal and cyclically reduced, and $s \neq_G 1$ for any $s \in S$. Let M be a van Kampen diagram over $\langle S \mid R\rangle$ such that* $\mathrm{Lab}(\partial M)$ *is cyclically reduced. Then there exists a van Kampen diagram M' over $\langle S \mid R\rangle$ such that all of the following conditions hold.*

(a) $\mathrm{Lab}(\partial M') = \mathrm{Lab}(\partial M)$.

(b) $\sigma(M', r) = \sigma(M, r)$ *for all $r \in R$.*

(c) *Every inessential face of M' is contained in a simple subdiagram whose boundary label is equal to ss^{-1} for some $s \in S$.*

Proof. We may assume that we have a van Kampen diagram M' that satisfies (a)-(d) of Lemma 3.2.10. We prove here that, in the presence of the assumption that $s \neq_G 1$ for all $s \in S$, it follows that M' also satisfies conclusion (c) of the current lemma.

Let F be an inessential face of M'. There are two cases: either $\mathrm{Lab}(\partial F) = ss^{-1}$ or $\mathrm{Lab}(\partial F) = 1ss^{-1}$ for some $s \in S$.

If $\mathrm{Lab}(\partial F) = ss^{-1}$, then since $s \neq_G 1$, we have that F is simple. Thus F itself is a simple subdiagram of M' which contains F and has boundary label ss^{-1}.

Suppose on the other hand that $\mathrm{Lab}(\partial F) = 1aa^{-1}$, where $a \in S$. Let e be the inessential edge of ∂F. Since $\partial M'$ and R are both cyclically reduced, e lies on the boundary paths of exactly two inessential, bounded faces, one of which is F: call the other one F'. Since F' is an inessential face of M' with an inessential edge on its boundary path, we have that $\mathrm{Lab}(F') = 1bb^{-1}$ for some $b \in S$. Now e is a loop since M' satisfies (d) of Lemma 3.2.10. Since $a \neq_G 1$, the endpoints of each of the a-labeled edges of ∂F are distinct: similarly for the b-labeled edges of $\partial F'$. Therefore $F \cup F'$ takes the form depicted in Figure 3.6, allowing that the roles of F and F' may be switched.

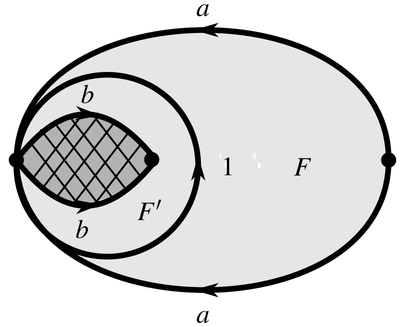

Figure 3.6: Diagram for the proof of Lemma 3.2.11

Notice that in Figure 3.6, $F \cup F'$ is enclosed in a simple subdiagram with boundary label aa^{-1} (or bb^{-1}, if the roles of F and F' are switched). This finishes the second case, thus (c) holds for M' and we are done. □

Corollary 3.2.12. *Let G be a group given by an aspherical presentation $\langle S \mid R \rangle$, where R is cyclically minimal and cyclically reduced, and $s \neq_G 1$ for all $s \in S$. Let M be a van Kampen diagram over $\langle S \mid R \rangle$ such that* $\mathrm{Lab}(\partial M)$ *is cyclically reduced. Then there exists a van Kampen diagram M' such that all of the following conditions hold.*

(a) $\mathrm{Lab}(\partial M') = \mathrm{Lab}(\partial M)$.

(b) $\sigma(M', r) = \sigma(M, r)$ for all $r \in R$.

(c) M' is bare.

Proof. We may assume that M' satisfies (a)-(c) of Lemma 3.2.11. Now all inessential faces of M' are contained in simple subdiagrams of M' with boundary label ss^{-1} for some $s \in S$. Thus we may make M' bare by repeatedly applying Operation 3.2.4. Operation 3.2.4 always preserves the boundary label of a van Kampen diagram, so (a) holds. Since $\langle S \mid R \rangle$ is aspherical, it follows from Lemma 3.2.6 that each application of Operation 3.2.4 preserves $\sigma(M', r)$ for all $r \in R$. Thus (a)-(c) hold for M', and we are done. □

Often one would like to take a van Kampen diagram M which is not reduced, and reduce it using Operation 3.2.4. However, the canceling faces may not be simple, or may not intersect each other simply. A common solution is to pad the van Kampen diagram with inessential faces. However, if M is a van Kampen diagram over a $C'(1/6)$ presentation, then M is topologically well-behaved enough to perform this operation without the use of inessential faces. We make this statement precise in the following two lemmas. The second is a consequence of the first, which is the famous Greendlinger Lemma.

Lemma 3.2.13 (Greendlinger Lemma)**.** [31, Chapter V, Theorem 4.5] *Let M be a bare and reduced van Kampen diagram over a cyclically reduced $C'(\lambda)$ presentation, where $\lambda \leq 1/6$, such that M has at least one bounded face and* $\mathrm{Lab}(\partial M)$ *is cyclically reduced. Then there exists a face F of M such that ∂F and ∂M share a common subpath of length more than $\frac{1}{2}\ell(\partial F)$.*

Lemma 3.2.14. *Let M be a bare van Kampen diagram over a cyclically reduced $C'(1/6)$ presentation. Then*

(a) *If M is reduced, then every face of M is simple.*

(b) *If M is reduced, then every two faces of M that intersect nontrivially also intersect simply.*

(c) *If M is not reduced, then there exists a pair of faces that cancel and intersect simply.*

Proof. We refer the reader to [31, Chapter V, Lemma 4.1] for the proof of part (a). For part (b), suppose that M is a counterexample with the minimum number of faces, and that F and F' are two faces of M that do not intersect simply, i.e. such that ∂F intersects $\partial F'$ in more than one maximal common subpath. Then ∂F and $\partial F'$ together enclose at least one simple subdiagram of M, call it D. Since M is reduced, so is D. By the Greendlinger Lemma, there exists a face E of D such that ∂E intersects ∂D in a subpath of length at least $\frac{1}{2}\ell(\partial E)$. Therefore ∂E intersects one of F or F', say F, in a common subpath of length at least $\frac{1}{4}\ell(\partial E)$. But then E and F cancel, contradicting the assumption that M is reduced.

For part (c), suppose that M is a counterexample with the minimum number of faces. Then M is not reduced, and there are two faces F and F' that cancel but do not intersect simply. Therefore ∂F and $\partial F'$ together enclose a simple subdiagram D. Again D must be reduced, this time by minimality of M. By an argument similar to the one in the preceding paragraph, there is a face E of D that cancels with F. By assumption, E and F cannot intersect simply. But then $D \cup F$ is a subdiagram of M that is a counterexample with strictly fewer faces than M, since it does not include F'. This contradicts minimality of M, finishing the proof. □

One can then use Lemma 3.2.14 to prove the following corollary.

Corollary 3.2.15. *Let G be a group given by presentation $\langle S \mid R \rangle$, where R is cyclically reduced and satisfies $C'(1/6)$. Then for every $r \in R$, if u is a proper subword of an element of $\{r\}_*$, then $u \neq_G 1$. In particular,*

(a) *For all generators $s \in S$, if $s \notin R$ then $s \neq_G 1$.*

(b) *If M is a bare van Kampen diagram over $\langle S \mid R \rangle$, then every face of M is simple.*

(c) *If M is a bare van Kampen diagram over $\langle S \mid R \rangle$ which is not reduced, then there exists a pair of cancelling faces that are simple and intersect simply.*

Proof. Suppose otherwise, and choose $u \in S_\circ^*$ to be a word of minimum length which is a proper subword of $\{r\}_*$ for some $r \in R$. Without loss of generality, suppose that u is a prefix of r, so that $r = uv$ for some $v \in S_\circ^*$. Clearly $v =_G 1$, so by minimality of u we have that $|v| \geq |u|$, hence $|u| \leq \frac{1}{2}|r|$.

Let M be a reduced van Kampen diagram over $\langle S \mid R \rangle$ such that $\mathrm{Lab}(\partial M) = u$. Since R is cyclically reduced, so is u: in particular, M has at least one bounded face. By the Greendlinger Lemma, there exists a face F of M such that F shares a common subpath of length more than $\frac{1}{2}\ell(\partial F)$ with ∂M. Let the label of this common subpath be w. Then w is a piece of r and $\mathrm{Lab}(\partial F)$, of length more than $\frac{1}{2}\ell(\partial F)$. By the $C'(1/6)$ condition, $\mathrm{Lab}(\partial F) = r$. But then $|w| > \frac{1}{2}|r| \geq |u|$. Since w is the label of a subpath of ∂M, this is a contradiction.

Conclusions (a) and (b) follow directly. Part (c) follows from part (b) of the current corollary and Lemma 3.2.14 (c). ☐

Lemma 3.2.16. *Let M be a van Kampen diagram over a $C'(1/6)$ presentation $\langle S \mid R\rangle$, where R is cyclically minimal and cyclically reduced, and $|r| \geq 2$ for all $r \in R$. Then there exists a van Kampen diagram M' over $\langle S \mid R\rangle$ such that*

(a) $\mathrm{Lab}(\partial M') = \mathrm{Lab}(\partial M)$.

(b) $\sigma(M', r) = \sigma(M, r)$ *for all* $r \in R$.

(c) *M' is bare and reduced.*

Proof. We may assume that M' satisfies (a)-(c) of Corollary 3.2.12. Thus we only have to show that it is possible to transform M' so that it is reduced, while preserving the boundary label and signed r-face count for each $r \in R$, and without adding any inessential faces.

Suppose that M' is not reduced. Since M' is bare, by Corollary 3.2.15 there exist two simple faces F and F' that cancel and intersect simply. Thus $F \cup F'$ is a simple subdiagram of M with trivial boundary label. Now remove $F \cup F'$ with Operation 3.2.4. Since $\langle S \mid R\rangle$ is $C'(1/6)$, and therefore aspherical, this operation preserves $\sigma(M', r)$ for all $r \in R$. Repeating, we end up with a reduced van Kampen diagram. ☐

Lemma 3.2.17. *Let G, H be groups given by presentations*

$$G = \langle S \mid R_G\rangle$$
$$H = \langle S \mid R_H\rangle$$

where $\langle S \mid R_H\rangle$ is a cyclically reduced $C'(1/6)$ presentation, and $|r_H| \geq 2$ for all $r_H \in R_H$. Suppose that $r_G =_H 1$ for all $r_G \in R_G$, so H is a quotient of G. Let M_G be a van Kampen diagram over $\langle S \mid R_G\rangle$, and for each face F of M_G, let M_F be a van Kampen diagram over $\langle S \mid R\rangle$ with boundary label $\mathrm{Lab}(\partial F)$. *Then there exists a "quotient van Kampen diagram" M_H over $\langle S \mid R_H\rangle$ such that*

(a) $\mathrm{Lab}(\partial M_G) = \mathrm{Lab}(\partial M_H)$.

(b) $\sigma(M_H, r) = \sum\{\sigma(M_F, r) \mid F$ *is an essential face of* $M_G\}$.

(c) *M_H is bare and reduced.*

Proof. Start with M_G. By repeatedly padding vertices, we may assume that all essential faces of M_G are simple.

Now take an essential simple face F of M_G, and quotient it to M_F. This may introduce self-intersections among essential faces in M_G. Pad vertices again until all essential faces of M_G are simple, and repeat as many times as necessary to quotient all essential faces that were originally in M_G. Since padding vertices and quotienting simple faces preserve the boundary label, we obtain a van Kampen diagram M_H over $\langle S \mid R_H \rangle$, possibly with many inessential faces, such that $\mathrm{Lab}(\partial M_H) = \mathrm{Lab}(\partial M_G)$. Thus (a) holds. In addition, for all $r \in R_H$,

$$\sigma(M_H, r) = \sum \{\sigma(M_F, r) \mid F \text{ is a face of } M_G\},$$

so (b) holds as well.

Note that we do *not* require R_G to by cyclically reduced for any of the previous steps to work. However, R_H *is* cyclically reduced, thus by Lemma 3.2.16 we may ensure that (c) holds, without interfering with conditions (a) or (b). □

3.2.3 A technical lemma

This section is devoted to proving the following lemma. In essence it is similar to [32, Lemma 5.10], but for our purposes we need the more general version stated here. In order to avoid constantly reiterating the assumptions, the notation used in this lemma will be "globally fixed" for this section. Thus until the next section, G will always refer to the group with presentation given in Lemma 3.2.18, etc. Any new notation introduced in the body of this section will also remain fixed until the beginning of the next section.

Lemma 3.2.18. *Let λ be a real number, where $0 < \lambda < 1/12$. Let $\{\ell_i \mid i \in \mathbb{N}\}$ be a set of positive integers, where each $\ell_i \geq 2$. Let S be a finite set. Let*

$$U = \{u_i \mid i \in \mathbb{N}\} \subset S_\circ^* \qquad\qquad V = \{v_i \mid i \in \mathbb{N}\} \subset S_\circ^*$$

be languages, and let $\tilde{u} \in S_\circ^*$ be a word, such that the following conditions are satisfied for all $i, i' \in \mathbb{N}$.

(a) *$U \cup V$ is cyclically minimal and cyclically reduced, and satisfies $C'(\lambda)$.*

(b) $2 \leq |u_i| \leq |v_i|$.

(c) *If p is a piece of $\tilde{u}$ and u_i, then $|p| < \lambda |u_i|$, and the same statement holds if u_i is replaced with v_i.*

(d) *If $u_i = u_{i'}$, $v_i = v_{i'}$, or $u_i = v_{i'}$, then $i = i'$.*

Now let

$$G = \langle S \mid R_G \rangle := \langle S \mid [s,u_i], u_i^{\ell_i}, u_i v_i^{-1} : s \in S, i \in \mathbb{N} \rangle$$
$$H = \langle S \mid R_H \rangle := \langle S \mid U \cup V \rangle = \langle S \mid u_i, v_i : i \in \mathbb{N} \rangle .$$

Let (k_i) be a sequence of integers where $|k_i| \leq \ell_i/2$ for all $i \in \mathbb{N}$, and $k_i = 0$ for all but finitely many $i \in \mathbb{N}$. Let $u \in S_\circ^*$ be a word of the form

$$u = \tilde{u} \prod_{i=0}^{\infty} u_i^{k_i} .$$

Then u is $\left(\frac{3}{1-12\lambda}, 0\right)$-quasigeodesic in G.

Note that if $U \cup V$ is $C'(\lambda)$, then so is $(U \cup V \cup \{u_i^{-1}\}) \setminus \{u_i\}$. Therefore assume without loss of generality that all k_i are nonnegative.

Let w be a geodesic representative of u in G. Then $uw^{-1} =_G 1$, so by the van Kampen Lemma, there exists a van Kampen diagram M_G with $\mathrm{Lab}(\partial M_G) = uw^{-1}$.

Lemma 3.2.19. *There exists a van Kampen diagram M_H over $\langle S \mid U \cup V \rangle$ such that*

(a) *M_H is bare and reduced.*

(b) *$\partial M_H = \alpha * \beta$, where $\mathrm{Lab}(\alpha) = u$ and $\mathrm{Lab}(\beta) = w^{-1}$.*

(c) *$\sigma(M_H, u_i) + \sigma(M_H, v_i) \equiv 0 \mod \ell_i$ for all $i \in \mathbb{N}$.*

(d) *$\sigma(M_H, u_i) \equiv 0 \mod \ell_i$ for all $i \in \mathbb{N}$ such that $u_i = v_i$.*

Proof. Each face F of M_G has boundary label equal to either $[s,u_i]$, $u_i^{\ell_i}$, or $u_i v_i^{-1}$. Each of these words represents the trivial elment of H. For each face F of M_G, choose a van Kampen diagram M_F over $\langle S \mid R_H \rangle$, of one of forms depicted in Figure 3.7.

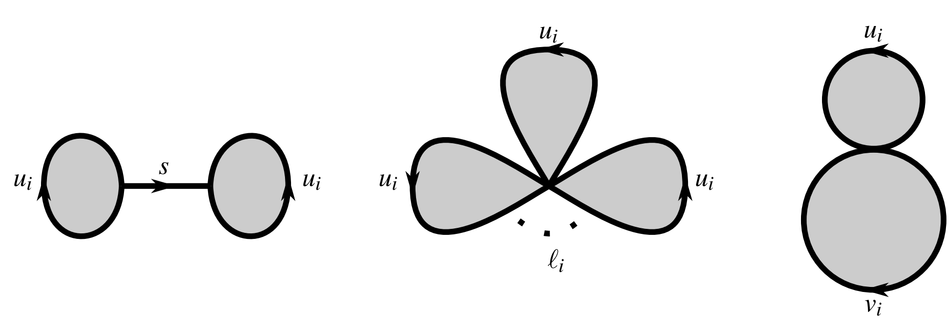

Figure 3.7: Chosen quotient van Kampen diagrams for each face of M_G

Applying Lemma 3.2.17, there exists a bare, reduced van Kampen diagram M_H with $\mathrm{Lab}(M_H) = \mathrm{Lab}(M_G) = uw^{-1}$, such that for all $i \in \mathbb{N}$,

$$\sigma(M, u_i) + \sigma(M, v_i) = \sum \{\sigma(M_F, u_i) + \sigma(M_F, v_i) \mid F \text{ is a face of } M_G\}.$$

Thus (a) and (b) hold. Now notice that for all M_F depicted in Figure 3.7, $\sigma(M_F, u_i) + \sigma(M_F, v_i) \equiv 0 \mod \ell_i$. Also, if $u_i = v_i$, then $\sigma(M_F, u_i) \equiv 0 \mod \ell_i$. Thus (c) and (d) hold as well. ☐

Let M_H be the van Kampen diagram from Lemma 3.2.19. Let $k = \sum_{i \in \mathbb{N}} k_i$. Then we may write

$$\alpha = \tilde{\alpha} * \alpha_0 * \cdots * \alpha_k * \beta$$

where $\mathrm{Lab}(\tilde{\alpha}) = \tilde{u}$, and for all $j \in \{0, \ldots, k\}$ we have $\mathrm{Lab}(\alpha_j) = u_i$ for some $i \in \mathbb{N}$.

Lemma 3.2.20. *There exists a van Kampen diagram M'_H over $\langle S \mid U \cup V \rangle$ and natural numbers $\{h_i \mid i \in \mathbb{N}\}$ satisfying all of the following conditions for all $i \in \mathbb{N}$.*

(a) *M'_H is bare and reduced.*

(b) *$\kappa(M'_H, u_i) = \kappa(M_H, u_i) - h_i$.*

(c) *$\partial M'_H = \alpha' * \beta'$, where $\mathrm{Lab}(\alpha') = \tilde{u} \prod_{i=0}^{\infty} u_i^{k_i - h_i}$ and $\mathrm{Lab}(\beta') = w$.*

(d) *No face F of M'_H intersects α' in a common subpath of length at least $2\lambda \ell(\partial F)$.*

(e) *$0 \leq h_i \leq k_i \leq \ell_i / 2$.*

Proof. If M_H already satisfies (d), then all conditions are satisfied by setting $M'_H = M_H$ and $h_i = 0$ for all $i \in \mathbb{N}$. Therefore suppose that M_H does not satisfy (d), i.e. there exists a face F of M_H such that ∂F intersects α in a common subpath of length at least $2\lambda \ell(\partial F)$. Then there must be a common subpath of ∂F and $\tilde{\alpha}$ or α_j for some $j \in \{0, \ldots, k\}$, of length at least $\lambda \ell(\partial F)$. The former possibility is excluded by condition (c) of Lemma 3.2.18. Thus ∂F intersects α_j in a common subpath of length at least $\lambda \ell(\partial F)$ for some $j \in \{0, \ldots, k\}$. Call this common subpath γ. Since $\mathrm{Lab}(\alpha_j) = u_i$ for some $i \in \mathbb{N}$, by the $C'(\lambda)$ condition we have that $\mathrm{Lab}(\partial F) = u_i$ as well.

Now apply Operation 3.2.9 to excise α_j from α. Let F' be the new u_i-face created by this operation. Then γ is a common subpath of F and F' of length at least $\lambda \ell(\partial F) = \lambda \ell(\partial F')$, so F and F' cancel. Since M_H was reduced, F and F' are the only pair of faces that cancel at this stage. Therefore by Corollary 3.2.15 (c), F and F' are simple and intersect simply. Thus $F \cup F'$ is a simple subdiagram of M with trivial boundary label, which we may remove with Operation 3.2.4.

Let $\hat{M}_H$ be the van Kampen diagram obtained in this way. Then clearly $\hat{M}_H$ satisfies (a). We added one u_i-face and removed two, so $\kappa(\hat{M}_H, u_i) = \kappa(M_H, u_i) - 1$. Since $u_i \neq u_j$ whenever $i \neq j$, no u_j-face counts were affected for any $j \neq i$. Therefore (b) is satisfied with $h_i = 1$. Now after excising α_j the boundary path becomes $\hat{\alpha} * \beta := \tilde{\alpha} * \alpha_0 * \cdots * \alpha_{j-1} * \alpha_{j+1} * \cdots * \alpha_k * \beta$. Removing $F \cup F'$ does not change the boundary label, so (c) is satisfied with $h_i = 1$. Because of this, we may iterate the process. By construction, $\hat{M}_H$ has one fewer face than M_H which fails to satisfy (d). Therefore repeat as many times as there are faces in M_H failing to satisfy (d) to get M'_H. Each such face must be a u_i-face for some $i \in \mathbb{N}$, so for each $i \in \mathbb{N}$, let h_i be the number of u_i-faces in M_H failing to satisfy (d). Since the boundary label becomes shorter at each step, by (c) it follows that $h_i \leq k_i$ for all $i \in \mathbb{N}$. Therefore M'_H satisfies (e), and we are done. □

For the next step in the proof, the following ad hoc lemma is useful.

Lemma 3.2.21. *Let M be a bare, reduced van Kampen diagram over a cyclically reduced $C'(1/6)$ presentation. Let α be a subpath of ∂M such that no face F of M intersects α in a common subpath of length at least $\frac{1}{4}\ell(\partial F)$. Then every face of M that intersects α nontrivially, intersects α simply.*

Proof. Suppose that F is a face of M such that ∂F shares more than one vertex with α, but ∂F does not intersect α simply. Then there exist subpaths of α and ∂F that together enclose a simple subdiagram D of M. Since M, and therefore D, is reduced, by the Greendlinger Lemma there exists a face F' of D such that $\partial F'$ shares a common subpath of length at least $\frac{1}{2}\ell(\partial F')$ with ∂D. Thus $\partial F'$ intersects either ∂F or α in a common subpath of length at least $\frac{1}{4}\ell(\partial F')$. The latter possibility is ruled out by assumption, so F cancels with F' by the $C'(1/6)$ condition. But this contradicts our assumption that M is reduced. □

Definition 3.2.22. Let M be a van Kampen diagram over a presentation $\langle S \mid R \rangle$. Then the *perimeter sum* of M, denoted $\mathrm{PS}(M)$, is defined by

$$\mathrm{PS}(M) = \sum \{\ell(\partial F) \mid F \text{ is a face of } M\}.$$

Note that if M is bare, then

$$\mathrm{PS}(M) = \sum_{r \in R} |r| \kappa(M, r).$$

To obtain bounds on $\ell(\alpha')$ in terms of $\ell(\beta)$, and on $\ell(\alpha)$ in terms of $\ell(\alpha')$, we use the following fact about van Kampen diagrams over $C'(1/6)$ presentations. It is the final puzzle piece in the proof.

Lemma 3.2.23. [32, Lemma 3.8] *Let M be a bare and reduced van Kampen diagram over a cyclically reduced $C'(\lambda)$ presentation, where $\lambda \leq 1/6$. Then $(1-6\lambda)\mathrm{PS}(M) \leq \ell(\partial M)$.*

With this in mind, we resume our proof.

Lemma 3.2.24. $\ell(\alpha') < 2\ell(\beta)$.

Proof. Note that an edge of α' is shared by the boundary path of some face of M'_H if and only if it is not also an edge of β'. We have by Lemma 3.2.20 (d) that no face F intersects α' in a common subpath of length at least $2\lambda\ell(\partial F) < \frac{1}{6}\ell(\partial F) < \frac{1}{4}\ell(\partial F)$. Therefore by Lemma 3.2.21, every face whose boundary path shares an edge with α' intersects α' in a single common subpath. Thus

$$\mathrm{PS}(M_H) > \tfrac{1}{2\lambda}\ell(\alpha' \setminus \beta') \geq \tfrac{1}{2\lambda}\left(\ell(\alpha') - \ell(\beta')\right) = \tfrac{1}{2\lambda}\left(\ell(\alpha') - \ell(\beta')\right)$$

On the other hand, $\ell(\partial M'_H) = \ell(\alpha') + \ell(\beta')$. Thus by Lemma 3.2.23,

$$\begin{aligned}
\tfrac{1}{2\lambda}\left(\ell(\alpha') - \ell(\beta')\right) < \mathrm{PS}(M'_H) \leq \tfrac{1}{1-6\lambda}\ell(\partial M'_H) &= \tfrac{1}{1-6\lambda}\left(\ell(\alpha') + \ell(\beta')\right) \\
(1-6\lambda)(\ell(\alpha') - \ell(\beta')) &< 2\lambda(\ell(\alpha') + \ell(\beta')) \\
(1-8\lambda)\ell(\alpha') &< (1-4\lambda)\ell(\beta') \\
\ell(\alpha') &< \tfrac{1-4\lambda}{1-8\lambda}\ell(\beta') < 2\ell(\beta') = 2\ell(\beta)\,,
\end{aligned}$$

since $0 < \lambda < 1/12$. ☐

Lemma 3.2.25. $\mathrm{PS}(M_H) \geq 2(\ell(\alpha) - \ell(\alpha'))$.

Proof. Let $I = \{i \in \mathbb{N} \mid u_i = v_i\}$. By Lemma 3.2.19, if $i \in I$ then $\sigma(M_H, u_i) \equiv 0 \mod \ell_i$. If $i \notin I$, then $\sigma(M_H, u_i) + \sigma(M_H, v_i) \equiv 0 \mod \ell_i$. Note that $\kappa(M_H, u_i) + \kappa(M_H, v_i) \geq \kappa(M_H, u_i) \geq h_i$ by Lemma 3.2.20 (b). Since $h_i \leq \ell_i/2$, it follows that there are at least $2h_i$ faces in M_H with boundary label either $u_i^{\pm 1}$ or $v_i^{\pm 1}$. If $u_i = v_i$, this says that $\kappa(M_H, u_i) \geq 2h_i$. If $u_i \neq v_i$, this means $\kappa(M_H, u_i) + \kappa(M_H, v_i) \geq 2h_i$. Therefore

$$\begin{aligned}
\mathrm{PS}(M_H) &= \sum_{r \in R_H} |r|\kappa(M_H, r) \\
&= \sum_{i \in I} |u_i|\kappa(M_H, u_i) + \sum_{i \notin I}(|u_i|\kappa(M_H, u_i) + |v_i|\kappa(M_H, v_i)) \\
&\geq \sum_{i \in I} |u_i|\kappa(M_H, u_i) + \sum_{i \notin I} |u_i|(\kappa(M_H, u_i) + \kappa(M_H, v_i)) \\
&\geq \sum_{i \in I} 2h_i|u_i| + \sum_{i \notin I} 2h_i|u_i| = \sum_{i \in \mathbb{N}} 2h_i|u_i| = 2(\ell(\alpha) - \ell(\alpha'))\,,
\end{aligned}$$

where the last equality follows from Lemma 3.2.20 (c). ☐

Now we are ready to prove Lemma 3.2.18.

Proof of Lemma 3.2.18. Continuing to use the terminology and notation built up in this section, since w is a geodesic representative of u, $|u| = \ell(\alpha)$, and $|w| = \ell(\beta)$, it suffices to prove that $\ell(\alpha) < \frac{3}{1-12\lambda}\ell(\beta)$. By Lemmas 3.2.23, 3.2.24, and 3.2.25,

$$\begin{aligned} 2(\ell(\alpha) - \ell(\alpha')) \leq \mathrm{PS}(M_H) \leq \tfrac{1}{1-6\lambda}\ell(\partial M_H) = \tfrac{1}{1-6\lambda}\left(\ell(\alpha)+\ell(\beta)\right) \\ (1-12\lambda)\ell(\alpha) \leq (2-12\lambda)\ell(\alpha') + \ell(\beta) < \ell(\alpha') + \ell(\beta) < 3\ell(\beta) \\ \ell(\alpha) < \tfrac{3}{1-12\lambda}\ell(\beta). \end{aligned}$$

Therefore u is $\left(\frac{3}{1-12\lambda}, 0\right)$-quasigeodesic, as desired. ☐

For this to be a meaningful bound we must have $0 < \lambda < 1/12$, explaining our initial choice of λ.

3.3 Proof of Theorem 1

In this section we prove the following proposition.

Proposition 3.3.1. *Let $m, n \in \mathbb{Z}^+ \cup \{\infty\}$ with $m < n$. Then there exist finitely generated, recursively presented groups G and B such that $B \leqslant G$ and*

$$1 \leq \operatorname{asdim}(G) \leq 2$$

$$m+1 \leq \operatorname{asdim}_{\mathrm{AN}}(G) \leq m+2$$

$$n+1 \leq \operatorname{asdim}_{\mathrm{AN}}(B) \leq n+2\,.$$

Since the proof requires many auxiliary lemmas, we again "globally fix" all notation in this section.

Let m be a fixed positive integer, and let $n \in \mathbb{Z}^+ \cup \{\infty\}$ with $m < n$. Let (ℓ_i) be an increasing sequence of positive integers with $\ell_0 \geq 2$. Let S_A, S_B be disjoint finite sets, and let $0 < \lambda < 1/12$. Suppose we have two languages

$$U_A = \{u_i \mid i \in \mathbb{N}\} \subset (S_A)^*_\circ \qquad\qquad V_B = \{v_i \mid i \in \mathbb{N}\} \subset (S_B)^*_\circ$$

satisfying all of the following conditions for all $i, i', j \in \mathbb{N}$.

(a) U_A, V_B are cyclically minimal and cyclically reduced, and satisfy $C'(\lambda)$.

(b) There exists a nonempty word $y \in (S_B)^*_\circ$ such that, for all $h \in \mathbb{Z}$, if p is a piece of y^h and v_i, then $|p| < \lambda |v_i|$.

(c) $2 \leq |u_i| \leq |v_i|$.

(d) If $u_i = u_{i'}$ or $v_i = v_{i'}$, then $i = i'$.

(e) The sequence of word lengths $(|u_i|)$ is constant on blocks of the partition $\mathscr{P}_m$ and $(|v_i|)$ is constant on blocks of $\mathscr{P}_n$ (see Definition 3.1.6).

(f) $|u_{(j+1)m}| \geq \ell_{(j+1)m}|u_{jm}|$. If $n \in \mathbb{Z}^+$ then $|v_{(j+1)n}| \geq \ell_{(j+1)n}|v_{jn}|$, and if $n = \infty$ then $|v_{(j+1)^2}| \geq \ell_{(j+1)^2}|v_{j^2}|$.

(g) U_A, V_B are recursive.

We construct an example of languages U_A, V_B satisfying (a)-(f) in the next section, and show that they can be recursive in the process. Assuming we already have U_A, V_B satisfying (a)-(g), let H_A, H_B be given by the presentations

$$H_A = \langle S_A \mid U_A \rangle \qquad\qquad H_B = \langle S_B \mid V_B \rangle$$

and let A, B be central extensions of H_A, H_B, respectively, defined by

$$A = \langle S_A \mid R_A \rangle := \langle S_A \mid [a, u_i], u_i^{\ell_i} : a \in S_A, i \in \mathbb{N} \rangle$$
$$B = \langle S_B \mid R_B \rangle := \langle S_B \mid [b, v_i], v_i^{\ell_i} : b \in S_B, i \in \mathbb{N} \rangle .$$

Since all elements in R_A, R_B represent the trivial element in H_A, H_B, respectively, there are natural epimorphisms $\pi_A : A \to H_A$ and $\pi_B : B \to H_B$. Recall that for a word w in $(S_A)^*_\circ$ or $(S_B)^*_\circ$, we denote by $\bar{w}$ the element of A or B, respectively, that w represents. Let

$$K_A = \mathrm{Ker}(\pi_A) = \langle \bar{u}_i : i \in \mathbb{N} \rangle \leqslant Z(A)$$
$$K_B = \mathrm{Ker}(\pi_B) = \langle \bar{v}_i : i \in \mathbb{N} \rangle \leqslant Z(B)$$

where we consider K_A as a normed group, equipped with the restriction to K_A of the word norm on A with respect to the generating set S_A, which we denote $\|\cdot\|_A$: similarly for K_B.

By condition (c), there exist sequences $s = (s_j), t = (t_j)$ such that $|u_i| = (m \times s)_i$ and $|v_i| = (n \times t)_i$ for all $i \in \mathbb{N}$. Define normed groups K_m, K_n similar to the normed group defined in Lemma 3.1.12, as follows:

$$K_m = \bigoplus_{i \in \mathbb{N}} |u_i| \mathbb{Z}_{\ell_i} = \bigoplus_{i \in \mathbb{N}} (m \times s)_i \mathbb{Z}_{\ell_i} \qquad\qquad K_n = \bigoplus_{i \in \mathbb{N}} |v_i| \mathbb{Z}_{\ell_i} = \bigoplus_{i \in \mathbb{N}} (n \times t)_i \mathbb{Z}_{\ell_i} .$$

Suppose that x is a word over S_A satisfying (b) with respect to U_A, except possibly the condition that x is not the empty word. Now condition (d) guarantees that s and t are increasing sequences of positive integers, such that for all $j \in \mathbb{N}$, $s_{j+1} \geq s_j \ell_{(j+1)m}$ and $t_{j+1} \geq t_j \ell_{(j+1)n}$ if $n \in \mathbb{Z}^+$. Condition (e) guarantees that $s_0 \geq 2$ and $t_0 \geq 2$. Therefore all hypotheses of Lemmas 3.1.8 and 3.1.12 are satisfied, and we have

$$\operatorname{asdim}_{\mathrm{AN}}(|x|\mathbb{Z} \times K_m) = \begin{cases} m & \text{if } x = \varepsilon \\ m+1 & \text{otherwise} \end{cases} \qquad \operatorname{asdim}_{\mathrm{AN}}(|y|\mathbb{Z} \times K_n) = n+1\,.$$

Now K_A is abelian, K_A satisfies $\bar{u}_i^{\ell_i} = 1$ for all $i \in \mathbb{N}$, and, since K_A is central in A, we have $\langle \bar{x}, K_A \rangle \cong \langle \bar{x} \rangle \times K_A$. All the corresponding statements hold for y and K_B. Therefore there exist natural epimorphisms ϕ_A and ϕ_B defined by

$$\begin{aligned} \phi_A : |x|\mathbb{Z} \times K_m &\to \langle \bar{x}, K_A \rangle & \qquad \phi_B : |y|\mathbb{Z} \times K_n &\to \langle \bar{y}, K_B \rangle \\ (h,z) &\mapsto \bar{x}^h \prod_{i \in \mathbb{N}} \bar{u}_i^{z_i} & (h,z) &\mapsto \bar{y}^h \prod_{i \in \mathbb{N}} \bar{v}_i^{z_i} \end{aligned}$$

for all $h \in \mathbb{Z}$ and $z = (z_i) \in K_m$ or K_n. In the case that $x = \varepsilon$ we have that $|x| = 0$ and $0\mathbb{Z} = \{0\}$, so $\phi_A : K_m \to K_A$.

Lemma 3.3.2. *Each of the epimorphisms ϕ_A, ϕ_B is bi-Lipschitz, hence each is a quasi-isometry and an isomorphism.*

Proof. We prove the statement for ϕ_A. Let $\|\cdot\|$ be the norm on K_m. Let $h \in \mathbb{Z}$ and $z = (z_i) \in K_m$. Let (k_i) be the geodesic form of z (see Definition 3.1.3). Then

$$\|\phi_A(h,z)\|_A = \left\| \bar{x}^h \prod_{i \in \mathbb{N}} \bar{u}_i^{k_i} \right\|_A \leq \left| x^h \prod_{i \in \mathbb{N}} u_i^{k_i} \right| = h|x| + \sum_{i \in \mathbb{N}} |k_i||u_i| = \|(h,z)\|.$$

Now $k_i \leq \ell_i/2$ for all $i \in \mathbb{N}$, and x^h satisfies condition (c) of Lemma 3.2.18. Furthermore,

$$A = \langle S_A \mid [a,u_i], u_i^{\ell_i} : a \in S_A, i \in \mathbb{N} \rangle = \langle S_A \mid [a,u_i], u_i^{\ell_i}, u_i(u_i)^{-1} : a \in S_A, i \in \mathbb{N} \rangle$$

and $U_A \cup U_A = U_A = \{u_i \mid i \in \mathbb{N}\}$ is a cyclically reduced, cyclically minimal $C'(\lambda)$ language, where $2 \leq |u_i| \leq |u_i|$ and $u_i = u_{i'}$ implies that $i = i'$ for all $i, i' \in \mathbb{N}$. Thus we may apply Lemma 3.2.18 with $G = A, U = U_A, V = U_A$, and $\tilde{u} = x^h$. This yields

$$\|(h,z)\| = h|x| + \sum_{i \in \mathbb{N}} |u_i||k_i| = \left| x^h \prod_{i \in \mathbb{N}} u_i^{k_i} \right| \leq \left(\frac{3}{1-12\lambda} \right) \left\| \bar{x}^h \prod_{i \in \mathbb{N}} \bar{u}_i^{k_i} \right\|_A = \left(\tfrac{3}{1-12\lambda} \right) \|\phi_A(h,z)\|_A\,,$$

hence $\left(\frac{1-12\lambda}{3} \right) \|(h,z)\| \leq \|\phi_A(k,z)\|_A \leq \|(h,z)\|$ and ϕ_A is bi-Lipschitz. $\square$

By replacing x or y with ε, we obtain the following.

Corollary 3.3.3. *Both* $\phi_A|_{K_m} : K_m \to K_A$ *and* $\phi_B|_{K_n} : K_n \to K_B$ *are bi-Lipschitz maps. Therefore* $\operatorname{asdim}_{\mathrm{AN}}(K_A) = \operatorname{asdim}_{\mathrm{AN}}(K_m) = m$ *and* $\operatorname{asdim}_{\mathrm{AN}}(K_B) = \operatorname{asdim}_{\mathrm{AN}}(K_n) = n$.

Corollary 3.3.4. *We have*

$$\begin{aligned} 1 &\leq \operatorname{asdim}(A) \leq 2 & 1 &\leq \operatorname{asdim}(B) \leq 2 \\ m &\leq \operatorname{asdim}_{\mathrm{AN}}(A) \leq m+2 & n+1 &\leq \operatorname{asdim}_{\mathrm{AN}}(B) \leq n+2\,. \end{aligned}$$

Also, if $x \neq \varepsilon$, *then* $\operatorname{asdim}_{\mathrm{AN}}(A) \geq m+1$.

Proof. We establish the bounds for A: the argument for B is similar. Since A is finitely generated and infinite, $\operatorname{asdim}_{\mathrm{AN}}(A) \geq 1$. By Corollary 3.3.3, $\operatorname{asdim}_{\mathrm{AN}}(A) \geq \operatorname{asdim}_{\mathrm{AN}}(K_A) = m$. If $x \neq \varepsilon$,

$$\operatorname{asdim}_{\mathrm{AN}}(A) \geq \operatorname{asdim}_{\mathrm{AN}}(\langle \bar{x}, K_A \rangle) = \operatorname{asdim}_{\mathrm{AN}}(|x|\mathbb{Z} \times K_m) = m+1$$

since $|x| > 0$. This gives the lower bounds on the asymptotic and Assouad-Nagata dimension of A. For the upper bounds, note that A is constructed so that there is a short exact sequence

$$1 \to K_A \to A \to H_A \to 1$$

where H_A is a finitely generated $C'(1/6)$ group and hence $\operatorname{asdim}(H_A) \leq \operatorname{asdim}_{\mathrm{AN}}(H_A) \leq 2$ by Theorem 2.3.12. Since K_A is locally finite, $\operatorname{asdim}(K_A) = 0$. Now by Lemma 1.2.13,

$$\begin{aligned} \operatorname{asdim}(A) &\leq \operatorname{asdim}(K_A) + \operatorname{asdim}(H_A) \leq 2 \\ \operatorname{asdim}_{\mathrm{AN}}(A) &\leq \operatorname{asdim}_{\mathrm{AN}}(K_A) + \operatorname{asdim}_{\mathrm{AN}}(H_A) \leq m+2\,. \end{aligned}$$

□

By Corollary 3.3.3, the maps $\phi_A|_{K_m} : K_m \to K_A$ and $\phi_B|_{K_n} : K_n \to K_B$ are isomorphisms. Therefore the map defined by $\bar{u}_i \mapsto \bar{v}_i$ for all $i \in \mathbb{N}$ extends to an isomorphism from K_A to K_B. Let $\phi : K_A \to K_B$ be this isomorphism. Let

$$G = A *_\phi B := \langle A \sqcup B \mid a\phi(a)^{-1} : a \in A \rangle\,.$$

Let $S = S_A \sqcup S_B$. Then G admits the presentation

$$G = \langle S \mid R_G \rangle := \langle S \mid [s, u_i], u_i^{\ell_i}, u_i v_i^{-1} : s \in S, i \in \mathbb{N} \rangle ,$$

which is recursive if U_A and V_B are recursive. Let

$$H = \langle S \mid R_H \rangle := \langle S_A \sqcup S_B \mid U_A \sqcup V_B \rangle = \langle S_A \sqcup S_B \mid u_i, v_i : i \in \mathbb{N} \rangle .$$

Since U_A and V_B are $C'(\lambda)$ languages over disjoint alphabets, H is a $C'(\lambda)$ group. Furthermore, all words in R_G represent the trivial element of H, so there is a natural epimorphism $\pi : G \to H$. Let $K = \operatorname{Ker}(\pi)$. Then

$$K = \langle \bar{u}_i : i \in \mathbb{N} \rangle \leqslant Z(G) .$$

We consider K as a normed group, where the norm on K is the restriction to K of the word norm on G with respect to S. Thus we have a short exact sequence

$$1 \to K \to G \to H \to 1 .$$

Let $b \in S_B$. Considering the relations of K and the fact that K is central in G, there exists a natural epimorphism $\phi_K : \mathbb{Z} \times K_m \to \langle \bar{b}, K \rangle$ given by

$$\phi_K(h, z) = \bar{b}^h \prod_{i \in \mathbb{N}} \bar{u}_i^{z_i}$$

for all $h \in \mathbb{Z}$ and $z = (z_i) \in K_m$. Now we have the following.

Lemma 3.3.5. *The epimorphism ϕ_K is bi-Lipschitz, in particular ϕ_K is a quasi-isometry and an isomorphism.*

Proof. The proof is similar to that of Lemma 3.3.2. The only difference is that now we apply Lemma 3.2.18 with $U = U_A, V = V_B$, and $\tilde{u} = b^h$. Since b is a word over an alphabet disjoint from S_A, clearly condition (c) of Lemma 3.2.18 is satisfied with $\tilde{u} = b^h$ for any $h \in \mathbb{N}$. Since $2 \leq |u_i| \leq |v_i|$ and $u_i \neq v_{i'}$ for all $i, i' \in \mathbb{N}$, all hypotheses of Lemma 3.2.18 are satisfied. □

We are now ready to prove Proposition 3.3.1.

Proof of Proposition 3.3.1. Let B, G be defined as in this section. The bounds on $\operatorname{asdim}_{\mathrm{AN}}(B)$ are established in Corollary 3.3.4. Since G is finitely generated and infinite, $\operatorname{asdim}(G) \geq 1$. For the lower bound on the

Assouad-Nagata dimension of G, note that

$$\operatorname{asdim}_{\mathrm{AN}}(G) \geq \operatorname{asdim}_{\mathrm{AN}}(\langle \bar{b}, K\rangle) = \operatorname{asdim}_{\mathrm{AN}}(\mathbb{Z} \times K_m) = m+1\,.$$

By Theorem 2.3.12, we have $\operatorname{asdim}(H) \leq \operatorname{asdim}_{\mathrm{AN}}(H) \leq 2$. Applying the extension theorems to the short exact sequence $1 \to K \to G \to H \to 1$ yields that $\operatorname{asdim}(G) \leq 2$ and $\operatorname{asdim}_{\mathrm{AN}}(G) \leq m+2$. □

We give a presentation of a group G satisfying the conditions of Proposition 3.3.1 in the next section. For now, we derive the main result of this paper as a corollary.

Theorem 3.3.6. *For all $k,m,n \in \mathbb{N} \cup \{\infty\}$ with $4 \leq k \leq m \leq n$, there exist finitely generated, recursively presented groups G and H with $H \leqslant G$, such that*

$$\operatorname{asdim}(G) = k$$
$$\operatorname{asdim}_{\mathrm{AN}}(G) = m$$
$$\operatorname{asdim}_{\mathrm{AN}}(H) = n\,.$$

Proof. Applying Proposition 3.3.1 with $m-3$ and $n-2$, there exist finitely generated, recursively presented groups G_0 and B_0 with $B_0 \leqslant G_0$, such that

$$1 \leq \operatorname{asdim}(G_0) \leq 2$$
$$m-2 \leq \operatorname{asdim}_{\mathrm{AN}}(G_0) \leq m-1$$
$$n-1 \leq \operatorname{asdim}_{\mathrm{AN}}(B_0) \leq n\,.$$

Let

$$G_1 = \begin{cases} G_0 \times \mathbb{Z}^2 & \text{if } \operatorname{asdim}_{\mathrm{AN}}(G_0) = m-2 \\ G_0 \times \mathbb{Z} & \text{if } \operatorname{asdim}_{\mathrm{AN}}(G_0) = m-1\,. \end{cases}$$

Then $\operatorname{asdim}_{\mathrm{AN}}(G_1) = m$ by the Morita-type theorem for Assouad-Nagata dimension (Corollary 1.2.16). By the extension theorem for asymptotic dimension (Lemma 1.2.13), we have that $\operatorname{asdim}(G_1) \leq \operatorname{asdim}(G_0)+2 \leq 4$. Now let $G = G_1 * \mathbb{Z}^k$. Then since $4 \leq k \leq m$, by the free product formulas for asymptotic and Assouad-Nagata dimension (Theorem 1.2.14) it follows that $\operatorname{asdim}(G) = k$ and $\operatorname{asdim}_{\mathrm{AN}}(G) = m$. Note that B_0 and

$B_0 \times \mathbb{Z}$ are both subgroups of G. Therefore, let

$$H = \begin{cases} B_0 \times \mathbb{Z} & \text{if } \operatorname{asdim}_{\mathrm{AN}}(B_0) = n-1 \\ B_0 & \text{if } \operatorname{asdim}_{\mathrm{AN}}(B_0) = n\,. \end{cases}$$

Again by the Morita-type theorem for Assouad-Nagata dimension, we have that $\operatorname{asdim}_{\mathrm{AN}}(H) = n$. This completes the proof. □

3.4 A concrete example

In this section we construct an example of a group of the sort described in Proposition 3.3.1. In doing so, we show that such a group can be given by an explicit presentation, i.e. is recursively presented. The following lemma shows one way of constructing $C'(\lambda)$ languages, which was used by Bowditch in [34] to construct $2^{\aleph_0}$ small cancellation groups in distinct quasi-isometry classes.

Lemma 3.4.1. *Let $U = \{u_i \mid i \in \mathbb{N}\} \subset \{a,x\}^*_\circ$ be a language where we define*

$$u_i = (a^{m_i} x^{m_i})^{n_i}$$

for some positive integers m_i, n_i, for each $i \in \mathbb{N}$. Let $k \geq 2$ be an integer, and suppose that all of the following conditions hold.

(a) *$n_i \geq k$ for all $i \in \mathbb{N}$.*

(b) *$m_i \neq m_{i'}$ for all distinct $i, i' \in \mathbb{N}$.*

Then all of the following conclusions hold for all $i \in \mathbb{N}$.

(i) *U is cyclically minimal and cyclically reduced, and satisfies $C'\left(\frac{1}{k-1}\right)$.*

(ii) *For all $h \in \mathbb{Z}$, if p is a piece of x^h and u_i, then $|p| < \frac{1}{k-1}|u_i|$.*

(iii) *$2 \leq |u_i|$.*

(iv) *If $u_i = u_{i'}$ then $i = i'$.*

Proof. If $u_i \in U$, then no cyclic shift of u_i^{-1} is in U: if $\tilde{u}_i$ is a cyclic shift of u_i that belongs to U, then $|\tilde{u}_i| = |u_i|$ and $\tilde{u}_i$ must begin with a and end with x, in which case $\tilde{u}_i = u_i$. Therefore U is cyclically minimal. Since all u_i are positive words (that is, do not contain letters a^{-1} or x^{-1}), it is clear that U is cyclically reduced. For the same reason, when talking about pieces of some u_i and another positive word w, it suffices to consider only cyclic shifts of u_i and w, and we may ignore cyclic shifts of u_i^{-1} or w^{-1}. To show that U satisfies $C'(\frac{1}{k-1})$, suppose $i,i' \in \mathbb{N}$ are distinct. Let p be a maximal piece of u_i and $u_{i'}$. Since $m_i \neq m_{i'}$, suppose without loss of generality that $m_i < m_{i'}$. Then p must have the form $a^{m_i}x^{m_i}$. But then $n_i|p| \leq |u_i|$ and $n_{i'}|p| \leq |u_{i'}|$. Since $n_i, n_{i'} \geq k$, we have $|p| \leq \frac{1}{k}\min(|u_i|,|u_{i'}|) < \frac{1}{k-1}\min(|u_i|,|u_{i'}|)$. Therefore U satisfies $C'\left(\frac{1}{k-1}\right)$. Conclusion (ii) says only that any power of x makes up less than $\frac{1}{k-1}$ of a cyclic shift of some u_i. But a maximal subword of a cyclic shift of u_i of the form x^h must be x^{m_i}, which has length at most $\frac{1}{2n_i}|u_i| \leq \frac{1}{2k}|u_i| < \frac{1}{k-1}|u_i|$, so this is clear. Parts (iii) and (iv) are obvious. □

Lemma 3.4.2. *Let $m \in \mathbb{Z}^+ \cup \{\infty\}$, and let $\mathscr{P}_m = \{P_{(m,j)} \mid j \in \mathbb{N}\}$ be the partition of $\mathbb{N}$ given in Definition 3.1.6. Let $k \geq 2$ be an integer. For each $i \in \mathbb{N}$, let $r_i = i - \min(P_{(m,j)})$ whenever $i \in P_{(m,j)}$. Let (p_j) be an increasing sequence of positive integers. Let $U = \{u_i \mid i \in \mathbb{N}\} \subset \{a,x\}^*_\circ$ be given by*

$$u_i = \left(a^{k^{(p_j - r_i)}} x^{k^{(p_j - r_i)}}\right)^{k^{(r_i+1)}}$$

whenever $i \in P_{(m,j)}$. Let (ℓ_i) be an increasing sequence of positive integers. Suppose that the sequence (p_j) satisfies

$$\begin{aligned} p_{j+1} &\geq p_j + \log_k(\ell_{(j+1)m}) + |P_{(m,j+1)}| \text{ if } m \in \mathbb{Z}^+ \\ p_{j+1} &\geq p_j + \log_k(\ell_{(j+1)^2}) + |P_{(m,j+1)}| \text{ if } m = \infty. \end{aligned} \tag{3.4}$$

Then all of the following conclusions hold for all $i \in \mathbb{N}$.

(i) *U is cyclically minimal and cyclically reduced, and satisfies $C'\left(\frac{1}{k-1}\right)$.*

(ii) *For all $h \in \mathbb{N}$, if $p \in \{a,x\}^*_\circ$ is a piece of x^h and w_i, then $|p| < \frac{1}{k-1}|u_i|$.*

(iii) $2 \leq |u_i|$.

(iv) *If $u_i = u_{i'}$, then $i = i'$.*

(v) *The sequence of word lengths $(|u_i|)$ is constant on blocks of $\mathscr{P}_m$.*

(vi) *If $m \in \mathbb{Z}^+$ then $|u_{(j+1)m}| \geq \ell_{(j+1)m}|u_{jm}|$, and if $m = \infty$ then $|u_{(j+1)^2}| \geq \ell_{(j+1)^2}|u_{j^2}|$.*

Proof. Note that, if $i \in P_{(m,j)}$, then $|u_i| = 2k^{p_j - r_i}k^{r_i+1} = 2k^{p_j+1} \geq 2$, which depends only on j. This establishes (iv) and (v). Define the sequence $s = (s_j)$ by

$$s_j = 2k^{p_j+1}$$

for all $j \in \mathbb{N}$. Then $|u_i| = (m \times s)_i$.

For (vi), note that $\log_k(s_{(j+1)m}) = \log_k(2) + p_{j+1} + 1$. If $m \in \mathbb{Z}^+$, then we have $p_{j+1} \geq p_j + \log_k(\ell_{(j+1)m})$, implying that $s_{j+1} \geq \ell_{(j+1)m} s_j$ for all $j \in \mathbb{N}$. If $m = \infty$, then $\log_k(s_{j+1}) \geq p_{j+1} \geq p_j + \log_k(\ell_{(j+1)^2})$, so $s_{j+1} \geq \ell_{(j+1)^2} s_j$ for all $j \in \mathbb{N}$. This establishes (vi).

For parts (i)-(iv), we use Lemma 3.4.1. Obviously part (a) of Lemma 3.4.1 is satisfied, so we only need to check part (b). For this is suffices to show that if $i \in P_{(m,j)}, i' \in P_{(m,j')}$, and $i \neq i'$, then $p_j - r_i \neq p_{j'} - r_{i'}$. If $j' = j$ then this is immediate. If $j' = j+1$ then we have

$$p_{j'} - r_{i'} = p_{j+1} - r_{i'} \geq p_{j+1} - |P_{j+1}| + 1 > p_j \geq p_j - r_i\,.$$

This shows that $p_j - r_i$ increases with j no matter the choice of $i \in P_{(m,j)}$, so we are done. $\square$

We are ready to construct our example. Let $m,n \in \mathbb{Z}^+ \cup \{\infty\}$ with $m < n$. Let $S_A = \{a,x\}$, $S_B = \{b,y\}$ be disjoint two-element alphabets. Let $k = 14$ and let $\ell_i = 14^i$ for all $i \in \mathbb{N}$. Let $(p_j),(q_j)$ be increasing sequences of positive integers. Let $U_A = \{u_i \mid i \in \mathbb{N}\} \subset (S_A)^*_\circ$ be the language constructed with respect to $m,k,(\ell_i)$ and (p_j) as in Lemma 3.4.2. Similarly define $V_B = \{v_i \mid i \in \mathbb{N}\} \subset (S_B)^*_\circ$ with respect to $n,k,(\ell_i)$, and (q_j).

Lemma 3.4.3. *Suppose that for all $i,j \in \mathbb{N}$ we have*

(a) $p_{j+1} \geq p_j + (j+2)m$.

(b) $q_{j+1} \geq q_j + (j+2)n$ *if* $n \in \mathbb{Z}^+$, *and* $q_{j+1} \geq q_j + (j+2)^2$ *if* $n = \infty$.

(c) $p_{\lfloor i/m \rfloor} \leq q_{\lfloor i/n \rfloor}$ *if* $n \in \mathbb{Z}^+$, *and* $p_{\lfloor i/m \rfloor} \leq q_{\lfloor \sqrt{i} \rfloor}$ *if* $n = \infty$.

Then U_A, V_B satisfy conditions (a)-(f) *listed in the proof of Proposition 3.3.1.*

Proof. Note that

$$\log_k(\ell_{(j+1)n}) + |P_{(n,j+1)}| = \log_{14}(14^{(j+1)n}) + n = (j+2)n \qquad \text{if } n \in \mathbb{Z}^+$$
$$\log_k(\ell_{(j+1)^2}) + |P_{(n,j+1)}| = \log_{14}(14^{(j+1)^2}) + (2j+1) \leq (j+2)^2 \qquad \text{if } n = \infty\,.$$

Therefore assumptions (a) and (b) guarantee that (p_j) and (q_j) satisfy (3.4) with respect to (ℓ_i) and m,n, respectively, and so U_A, V_B satisfy all conditions listed in the proof of Proposition 3.3.1, except possibly that $|u_i| \leq |v_i|$ for all $i \in \mathbb{N}$. Now, if $i \in P_{(n,j)} \in \mathscr{P}_n$, then $j = \lfloor i/n \rfloor$ if $n \in \mathbb{Z}^+$, and $j = \lfloor \sqrt{i} \rfloor$ if $n = \infty$. It follows that assumption (c) is necessary and sufficient to guarantee that $|u_i| \leq |v_i|$ for all $i \in \mathbb{N}$. $\square$

Example 3.4.4. Let

$$p_j = m(j+2)^2 \qquad\qquad q_j = \begin{cases} n^2(j+3)^2 & \text{if } n \in \mathbb{Z}^+ \\ m(j+3)^4 & \text{if } n = \infty. \end{cases}$$

Then $(p_j),(q_j)$ satisfy the hypotheses of Lemma 3.4.3. The verification of this is no more than a tedious calculation, so we omit it. Note that, in the notation of Lemma 3.4.2,

$$r_i = \begin{cases} i \mod n & \text{if } n \in \mathbb{Z}^+ \\ i^2 - \lfloor \sqrt{i} \rfloor^2 & \text{if } n = \infty. \end{cases}$$

Also, if $i \in P_{(n,j)}$, then $j = \lfloor i/n \rfloor$ if $n \in \mathbb{Z}^+$, and $j = \lfloor \sqrt{i} \rfloor$ if $n = \infty$. So, expanding the forms of u_i and v_i according to Lemma 3.4.2 with respect to the sequences (p_j) and (q_j) given above yields

$$u_i = \left(a^{14^{m(\lfloor i/m \rfloor + 2)^2 - (i \bmod m)}} x^{14^{m(\lfloor i/m \rfloor + 2)^2 - (i \bmod m)}} \right)^{14^{(i \bmod m)+1}}$$

$$v_i = \begin{cases} \left(b^{14^{n^2(\lfloor i/n \rfloor + 3)^2 - (i \bmod n)}} y^{14^{n^2(\lfloor i/n \rfloor + 3)^2 - (i \bmod n)}} \right)^{14^{(i \bmod n)+1}} & \text{if } n \in \mathbb{Z}^+ \\ \left(b^{14^{m(\lfloor \sqrt{i} \rfloor + 3)^4 - (i - \lfloor \sqrt{i} \rfloor^2)}} y^{14^{m(\lfloor \sqrt{i} \rfloor + 3)^4 - (i - \lfloor \sqrt{i} \rfloor^2)}} \right)^{14^{(i - \lfloor \sqrt{i} \rfloor^2)+1}} & \text{if } n = \infty. \end{cases}$$

Then the languages $\{u_i \mid i \in \mathbb{N}\}$ and $\{v_i \mid i \in \mathbb{N}\}$ satisfy conditions (a)-(f) listed in the proof of Proposition 3.3.1, and are clearly recursive. Thus the group G with presentation

$$G = \langle a,b,x,y \mid [a,u_i],[x,u_i],[b,u_i],[y,u_i],u_i^{14^i},u_i v_i^{-1} : i \in \mathbb{N} \rangle$$

is a finitely generated, recursively presented group of Assouad-Nagata dimension at most $m+2$, containing a finitely generated subgroup of Assouad-Nagata dimension at least $n+1$.

CHAPTER 4

Directions for Further Research

The author hopes that the work done in this dissertation will serve as a springboard for future mathematical research. Thus, in this section we give an overview of various unresolved questions related to the subject of this paper, proceeding from the most general to the most specific.

The first question is more of a broad area of research than a particular open problem. There are many properties that are related to asymptotic dimension: bi-exactness, Assouad-Nagata dimension, Property A, and asymptotic dimension growth, to name a few. At the same time, there are many different types of small cancellation conditions and related properties: conditions $C(n)$ and $T(n)$, graded small cancellation, lacunar hyperbolicity, and so on. Theorem 2 establishes one connection between these two types of properties, so naturally one could ask the following.

Question 4.0.1. *What other connections are there between small cancellation conditions and properties related to asymptotic dimension?*

Next, note that Theorem 1 essentially establishes a "non-relation" between two types of dimension in finitely generated groups. In other words, the relation $\operatorname{asdim}(G) \leq \operatorname{asdim}_{\mathrm{AN}}(G)$ for all finitely generated groups G is immediate, and Theorem 1 shows that this is the only general relation between $\operatorname{asdim}(G)$ and $\operatorname{asdim}_{\mathrm{AN}}(G)$, assuming $\operatorname{asdim}(G) \geq 4$. However, there are other ways of measuring the dimension of a finitely generated group. One such way is to find the topological dimension of an asymptotic cone of the group. In [30], Dydak and Higes show that for any finitely generated group G, non-principle ultrafilter ω, and scaling sequence k, we have the relation

$$\dim(\mathrm{Cone}_{\omega}(G,k)) \leq \operatorname{asdim}_{\mathrm{AN}}(G)\,.$$

Question 4.0.2. *Are there any other relations between* $\operatorname{asdim}(G)$*,* $\operatorname{asdim}_{\mathrm{AN}}(G)$*, and* $\dim(\mathrm{Cone}_{\omega}(G,k))$ *that hold for all finitely generated groups* G*?*

We conjecture that the answer is no. To put it another way, we conjecture that, given positive integers (ℓ,m,n) with $\ell \leq n$ and $m \leq n$, there exists a finitely generated group G, a non-principle ultrafilter ω, and a scaling sequence k, such that $\operatorname{asdim}(G) = \ell$, $\dim(\mathrm{Cone}_{\omega}(G,k)) = m$, and $\operatorname{asdim}_{\mathrm{AN}}(G) = n$.

One can view Theorem 1 as an improvement of Higes' result [16], pushing "countable" to "finitely generated." Can we push "finitely generated" to "finitely presented?"

Question 4.0.3. *Does there exist a finitely presented group G such that* $\operatorname{asdim}(G) < \operatorname{asdim}_{\mathrm{AN}}(G)$*? If so, is there such an example where* $\operatorname{asdim}_{\mathrm{AN}}(G)$ *is finite?*

Question 4.0.4. *Does there exist a finitely presented group G with a finitely generated subgroup H such that* $\operatorname{asdim}_{\mathrm{AN}}(G) < \operatorname{asdim}_{\mathrm{AN}}(H)$*? If so, can H be taken to be finitely presented?*

We conjecture that the answer to all of these questions is yes. Given a finitely generated, recursively presented group G, the Higmann embedding theorem shows how to construct a finitely presented group into which G embeds. The construction uses a finite sequence of HNN-extensions. We also have the following recent result of Tselekidis.

Theorem 4.0.5. [11] *Let H be a finitely generated group, let* $\phi : A \to B$ *be an isomorphism between two subgroups of H, and let* $G = H*_{\phi}$ *be the HNN-extension of H over this isomorphism. Then* $\operatorname{asdim}(G) \leq \max\{\operatorname{asdim}(H), \operatorname{asdim}(A) + 1\}$.

This suggests that it may be possible to take a group of the sort described in Theorem 1 and embed it in a finitely presented group whose asymptotic dimension is not much greater. The problem is controlling the Assouad-Nagata dimension of the larger group. Thus the first step towards a construction seems to be proving the appropriate analogue of Theorem 4.0.5 for Assouad-Nagata dimension.

Looking at the proof of Theorem 1 from Proposition 1, one may get the impression that the condition $k \geq 4$ is more decorative than essential. It gives us the ability to control $\operatorname{asdim}(G)$, $\operatorname{asdim}_{\mathrm{AN}}(G)$, and $\operatorname{asdim}_{\mathrm{AN}}(H)$ precisely, but if one doesn't mind a little uncertainty about $\operatorname{asdim}_{\mathrm{AN}}(G)$ and $\operatorname{asdim}_{\mathrm{AN}}(H)$, one can instead use Proposition 1 and lower $\operatorname{asdim}(G)$ to 2. But as written, Proposition 1 does not rule out the (intuitively unlikely) possibility that, for example, all finitely generated groups G satisfying the conclusion of Proposition 1 have an Assouad-Nagata dimension which is even. Can we have our cake and eat it, too? That is, can we have precise control over $\operatorname{asdim}_{\mathrm{AN}}(G)$ and $\operatorname{asdim}_{\mathrm{AN}}(H)$ as in Theorem 1, while still lowering $\operatorname{asdim}(G)$ as much as possible?

Question 4.0.6. *Let* $k, m, n \in \mathbb{N} \cup \{\infty\}$ *with* $k \leq m \leq n$*. Does there exist a finitely generated group G containing a finitely generated subgroup H such that*

$$\operatorname{asdim}(G) = k$$

$$\operatorname{asdim}_{\mathrm{AN}}(G) = m$$

$$\operatorname{asdim}_{\mathrm{AN}}(H) = n$$

assuming $k \geq 2$*? What about when* $k = 1$*?*

We conjecture that there are such groups for any $k \geq 2$, but when $k = 1$ it is much less clear what the answer should be.

In Proposition 1, the slight uncertainty about $\operatorname{asdim}(G), \operatorname{asdim}_{\mathrm{AN}}(G)$ and $\operatorname{asdim}_{\mathrm{AN}}(H)$ follows from the fact that $G = A *_\phi B$ where the bounds on the asymptotic and Assouad-Nagata dimension of G, A and B are obtained using the extension theorems. These only provide an upper bound for the asymptotic and Assouad-Nagata dimension of an extension, rather than an exact formula, and in general it is unclear when the inequality provided by each extension theorem is strict. In order to relax the condition $k \geq 4$ in Theorem 1, it seems that a more detailed study of the asymptotic and Assouad-Nagata dimensions of extensions of $C'(1/6)$ groups is needed, at least if one wishes to use the results or techniques of this dissertation. In particular, it would be useful to know the following.

Question 4.0.7. *Suppose that $1 \to K \to G \to H \to 1$ is a short exact sequence, where H is a finitely generated $C'(1/6)$ group and $K \leqslant Z(G)$. How can we characterize* $\operatorname{asdim}(G)$ *and* $\operatorname{asdim}_{\mathrm{AN}}(G)$ *in terms of* $\operatorname{asdim}(K)$*,* $\operatorname{asdim}_{\mathrm{AN}}(K)$*,* $\operatorname{asdim}(H)$*, and* $\operatorname{asdim}_{\mathrm{AN}}(H)$*? How does this change if K is not central in G?*

To answer Question 4.0.7, one might begin by simply classifying the asymptotic and Assouad-Nagata dimensions of $C'(1/6)$ groups. This is a more specific question and thus easier to settle, at least in theory, but even the answer to this is unclear. To review what we know so far, suppose that G is a finitely generated $C'(1/6)$ group. If $\operatorname{asdim}(G) = 0$, then G is locally finite; since G is finitely generated by assumption, this means that G is finite and $\operatorname{asdim}_{\mathrm{AN}}(G) = 0$. If G is infinite and finitely presented, then by results of Fujiwara and Whyte [18] and Gentimis [19], we have that $\operatorname{asdim}(G) = 1$ if and only if G is virtually free, in which case $\operatorname{asdim}_{\mathrm{AN}}(G) = 1$ as well. Thus if G is infinite, finitely presented, and not virtually free, then we have $\operatorname{asdim}(G) = \operatorname{asdim}_{\mathrm{AN}}(G) = 2$. This leaves open the case when G is infinitely presented.

Question 4.0.8. *Let G be a finitely generated, infinitely presented $C'(1/6)$ group. When does* $\operatorname{asdim}(G) = 1$ *and when does* $\operatorname{asdim}(G) = 2$*? When does* $\operatorname{asdim}_{\mathrm{AN}}(G) = 1$ *and when does* $\operatorname{asdim}_{\mathrm{AN}}(G) = 2$*? Is it possible that* $\operatorname{asdim}(G) = 1$ *but* $\operatorname{asdim}_{\mathrm{AN}}(G) = 2$*?*

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