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SIMPLICIAL COMPLEXES OBTAINED FROM QUALITATIVE PROBABILITY ORDERS*

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Abstract. The goal of this paper is to introduce a new class of simplicial complexes that naturally generalize the threshold complexes. These will be derived from qualitative probability orders on subsets of a finite set that generalize subset orders induced by probability measures. We show that this new class strictly contains the threshold complexes and is strictly contained in the shifted complexes. We conjecture that this class of complexes is exactly the set of strongly acyclic complexes, a class that has previously appeared in the context of cooperative games. Beyond the results themselves, this new class of complexes allows us to refine our understanding of one-point extensions of a particular oriented matroid.

Key words. simplicial complexes, threshold complex, qualitative probability orders

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1. Introduction.

A *qualitative probability order* in this paper is an order[1] on the set of subsets of $[n] = \{1, 2, \dots, n\}$ satisfying the axioms

$$\emptyset \preceq A$$

for all nonempty subsets $A \subseteq [n]$ and

$$A \preceq B \iff A \cup C \preceq B \cup C$$

for all subsets A, B, and C of $[n]$ such that $(A \cup B) \cap C = \emptyset$. A qualitative probability order $\preceq$ is *linear* if this relation is antisymmetric. These axioms date back to de Finetti [6], who introduced them in his foundational work on probability. The axiomatic perspective of qualitative probability was developed because of its epistemological advantages over traditional probability measures. Qualitative probability orders have been studied continuously since then in a diverse range of fields, appearing not only in studies of probability [23], but also in fair division [2] and in the theory of Gröbner bases [15]. See [17] for a recent survey.

Qualitative probability orders are motivated by the following geometric construction. Let $p = (p_1, \dots, p_n)$ be a probability measure on $[n]$, where p_i is the probability of i, and let $p(A) = \sum_{i \in A} p_i$. The relation $A \preceq B \Leftrightarrow p(A) \le p(B)$ satisfies the axioms of a qualitative probability order. Such orders are called *representable*. Alternatively, we can think of a hyperplane $p_1x_1 + \cdots + p_nx_n = t$ moving from its position at $t = 0$ to the position at $t = 1$ that orders the vertices of the unit n-cube as it goes, so that $x \preceq y$ for two vertices x and y if vertex x is met by the hyperplane no later than

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[1] An order in this paper is any reflexive, complete (every two elements are comparable), and transitive binary relation.

vertex y. We may then identify the vertices of the n-cube with the subsets of $[n]$ in the natural way and the resulting order is a qualitative probability order. Fine and Gill [7] showed that representable qualitative probability orders correspond to regions that the hyperplane arrangement $\mathcal{H}_n$ with normals drawn from $\{-1, 0, 1\}^n$ induce on the simplex S_n of probability distributions on $[n]$. Maclagan [15] showed it to be the discriminantal arrangement of the root system B_n.

In many instances in which combinatorial axioms are used to capture geometric constructions (matroids, oriented matroids, etc.) there appear exceptional objects that satisfy the axioms but cannot be obtained from the original construction. The qualitative probability orders are no exception: those that do not arise from a probability measure are called *nonrepresentable*. The first nonrepresentable order arises when $n = 5$. Representable orders were characterized by Kraft, Pratt, and Seidenberg [14] in terms of an infinite collection of *cancellation conditions*.

Now let us turn to abstract simplicial complexes. Every such complex can be represented by a set of vertices of the unit cube. Returning to our moving hyperplane example we note that at any moment in time the set of vertices that the hyperplane has already met corresponds to a simplicial complex and this complex is a *threshold* complex. In a similar vein we can take an arbitrary qualitative probability order and consider its initial segment, that is, all sets that come earlier than some fixed subset; it too will be a simplicial complex. This complex is easily seen to be a *shifted* complex but whether it is threshold is not so easy to answer. We devote this paper to answering this question in the negative. We can show that for all $n \geq 26$ there exists an initial segment of a qualitative probability order which is not threshold (Theorem 5.1). This is the main result of this paper.

It is clear, of course, that in order to obtain a nonthreshold initial segment we have to start with a nonrepresentable qualitative probability order. However, nonrepresentability of the order alone, as was shown by Maclagan [15], does not ensure that the resulting initial segment is not threshold. Christian, Conder, and Slinko [3] investigated initial segments of qualitative probability orders that contain exactly half of all 2^n subsets of $[n]$. (These correspond to constant-sum games.) Using the MAGMA computing package they showed that for $n = 5$ and $n = 6$ all such initial segments are threshold. This indicates that the required nonrepresentable qualitative probability order must be on a reasonably large set of atoms and cleverly constructed.

How might we construct such a nonrepresentable order? One way to try is the following: start with a hyperplane p as above and use it to begin sweeping through the n-cube. Make sure that p is chosen in such a way that the resulting order is nonlinear, that is, at some point the hyperplane meets a number of vertices simultaneously. Break ties so as to obtain a linear order. If the ties are broken carefully, the resulting order will be a nonrepresentable qualitative probability. Can one construct a nonrepresentable qualitative probability order in this way which possesses a nonthreshold initial segment? We show in this paper that the answer is "Yes" for all $n \geq 26$. This example motivates the further study of the complexes that are initial segments of qualitative probability orders.

Our result, while interesting in its own right, has close connections to simple games, a special type of cooperative game. A simple game is a partition of the collection of all subsets of a set (which are called coalitions in the game theory context) into two parts, the losing coalitions and the winning ones, so that the losing coalitions form an abstract simplicial complex. Winder [24] defined a natural desirability relation on coalitions: roughly speaking coalition A is more desirable than coalition B if $A \cup C$ is a winning coalition whenever $B \cup C$ is, for any C. Winder's desirability relation is *strongly acyclic* for weighted threshold simple games, simple games whose losing coalitions form threshold complexes. Winder [24] constructed a game which was strongly acyclic but not weighted threshold. Taylor and Zwicker [20, 21], answering a question of Peleg, constructed a strongly acyclic constant-sum game which is not weighted threshold. The construction is very complicated and the cardinality of the finite set in this example is huge. None of these examples are known to result from a qualitative probability order. Our example gives the first game arising from a qualitative probability order, which is strongly acyclic and not weighted threshold. Moreover, we conjecture that the initial segments are exactly those simplicial complexes whose Winder relation is strongly acyclic.

Our result also sheds some light on the one-element extensions of the oriented matroid $M(B_n)$ associated with the root system B_n consisting of vectors

$$\{\mathbf{e}_i : 1 \le i \le n\} \cup \{\pm(\mathbf{e}_i \pm \mathbf{e}_j) : 1 \le i < j \le n\}$$

(see, e.g., [1, Ch. 7] for more information). Maclagan [15] has shown that the one-element extensions of $M(B_n)$ are canonically the same as the linear qualitative probability orders on $[n]$. Moreover, the representable one-element extensions are exactly the representable linear qualitative probability orders. Our result shows that the nonrepresentable extensions are of two different types, those that have all initial segments threshold and those that do not. This represents a new way of classifying nonrepresentable one-element extensions of $M(B_n)$ that may be helpful in analyzing extensions of other types of oriented matroids.

The structure of the paper is as follows. In section 2 we introduce the basics of qualitative probability orders. In section 3 we provide the necessary background on abstract simplicial complexes and introduce the initial segments. It is here that we show that the initial segments contain the threshold complexes and are strictly contained in the shifted complexes [13]. We also introduce the collection of conditions that characterize threshold complexes and show that, for small n, initial segments satisfy them. Section 4 contains a construction technique that provides us with examples of nonrepresentable qualitative probability orders. This technique is employed in section 5 to produce the example of an initial segment that is not threshold. Section 6, the last substantive section, introduces ideas from cooperative game theory in order to state our conjecture characterizing the initial segments. We also provide some preliminary evidence supporting our conjecture.

Before proceeding, we wish to note one additional goal of this paper. While well known in other disciplines, much of the work on qualitative probability and cooperative games has escaped notice in the combinatorics community. We think this is unfortunate since the ideas are not only attractive but powerful as well. We hope that this paper will encourage others to investigate these areas where there is a lot of interesting combinatorics to be done.

2. Qualitative probability orders and discrete cones. In this paper all our objects are defined on the set $[n] = \{1, 2, \ldots, n\}$. We denote the set of all subsets of $[n]$ by $2^{[n]}$. An order $\preceq$ on $2^{[n]}$ is called a *qualitative probability order* on $[n]$ if

$$\emptyset \preceq A \tag{2.1}$$

for every nonempty subset A of $[n]$, and $\preceq$ satisfies *de Finetti's axiom*, namely, for each $A, B, C \in 2^{[n]}$

$$A \preceq B \iff A \cup C \preceq B \cup C \text{ whenever } (A \cup B) \cap C = \emptyset\,. \tag{2.2}$$

These orders also appear in the theory of Gröbner bases in exterior algebras [15], where they are called *Boolean term orders*. We write $A \prec B$ if $A \preceq B$ but not $B \preceq A$ and $A \sim B$ if $A \preceq B$ and $B \preceq A$. We say that sets A and B are *tied* if $A \sim B$.

If $\preceq$ is a qualitative probability order, then $A \subseteq B$ implies $A \preceq B$ (we can add A to both sides of $\emptyset \preceq B \setminus A$); hence it is a refinement of the partial semiorder by inclusion. (Notice that it is fairly possible to have a partial semiorder by inclusion which is not a qualitative probability order. For example, $\emptyset \prec 1 \prec 2 \prec 3 \prec 23 \prec 13 \prec 12 \prec 123$ is not a qualitative probability order as $1 \prec 2$ but $23 \prec 13$.)

Given a vector of nonnegative weights $w = (w_1, \ldots, w_n)$ we can define a qualitative probability order on $[n]$ by setting

$$A \preceq B \text{ if and only if } w(A) = \sum_{a \in A} w_a \leq w(B) = \sum_{b \in B} w_b.$$

It is easy to see that $\preceq$ is a qualitative probability order. Any order arising in this way is called *representable*, e.g., [8, 17]. Those not obtainable in this way are called *nonrepresentable*. Kraft, Pratt, and Seidenberg [14] showed that nonrepresentable orders exist for all $n \geq 5$.

We begin with some standard properties of qualitative probability orders.

Lemma 2.1. *Let $\preceq$ be a qualitative probability order on $[n]$. Suppose $A, B, C, D \in 2^{[n]}$ are such that $A \preceq B$, $C \preceq D$, and $B \cap D = \emptyset$. Then $A \cup C \preceq B \cup D$. Furthermore, if $A \prec B$ or $C \prec D$, then $A \cup C \prec B \cup D$.*

Proof. First, consider the case $A \cap C = \emptyset$. Let $B' = B \setminus C$ and $C' = C \setminus B$ and $I = B \cap C$. Then, by (2.2), we have

$$A \cup C' \preceq B \cup C' = B' \cup C \preceq B' \cup D,$$

where $A \cup C' \prec B' \cup D$ if $A \prec B$ or $C \prec D$. We obtain

$$A \cup C' \preceq B' \cup D \Leftrightarrow A \cup C = (A \cup C') \cup I \preceq (B' \cup D) \cup I = B \cup D.$$

Consider the second case when $A \cap C \neq \emptyset$. Let $A' = A \setminus C$. By (2.1) and (2.2) we have $A' \preceq B$. Since $A' \cap C = \emptyset$ by the previous case

$$A \cup C = A' \cup C \preceq B \cup C \preceq B \cup D.$$

One can check that if either $A \prec B$ or $C \prec D$ we will get a strict inequality $A \cup C \prec B \cup D$ in this case as well. □

A weaker version of Lemma 2.1 can be found in [15, Lemma 2.2].

Definition 2.2. *A sequence of subsets $(A_1, \ldots, A_j; B_1, \ldots, B_j)$ of $[n]$ of even length $2j$ is said to be a* trading transform *of length j if for every $i \in [n]$*

$$|\{k \mid i \in A_k\}| = |\{k \mid i \in B_k\}|. \tag{2.3}$$

In other words, sets $A_1, \ldots, A_j$ can be converted into $B_1, \ldots, B_j$ by rearranging their atoms.

It is worthwhile to note that for a trading transform $(A_1, \ldots, A_j; B_1, \ldots, B_j)$ it is perfectly possible that the sequences $A_1, \ldots, A_j$ or $B_1, \ldots, B_j$ have repetitions. Scott [19] gave a useful reformulation of trading transform in terms of characteristic vectors of subsets: the condition (2.3) can be rewritten as

$$\sum_{i=1}^{j} \chi(A_i) = \sum_{i=1}^{j} \chi(B_i).$$

DEFINITION 2.3. *We say that an order $\preceq$ on $2^{[n]}$ satisfies the kth cancellation condition CC_k if there does not exist a trading transform $(A_1, \ldots, A_k; B_1, \ldots, B_k)$ such that $A_i \preceq B_i$ for all $i \in [k]$ and $A_i \prec B_i$ for at least one $i \in [k]$.*

The key result (Lemma 0) of [14] can now be reformulated as follows.

THEOREM 2.4 (see [14]). *A qualitative probability order $\preceq$ is representable if and only if it satisfies CC_k for all $k = 1, 2, \ldots$.*

Using the de Finetti axiom and theory of linear orders, Fishburn [8, section 2] showed that CC_2 and CC_3 hold for linear qualitative probability orders. With minor modifications that proof also shows that every qualitative probability order satisfies CC_2 and CC_3. Hence, CC_4 is the first nontrivial cancellation condition. Kraft, Pratt, and Seidenberg [14] noticed that for $n < 5$ all qualitative probability orders are representable, but for $n = 5$ there are nonrepresentable ones.

Fishburn [8] observed that every qualitative probability order corresponds to a unique *discrete cone* $C(\preceq)$ in T^n, where $T = \{-1, 0, 1\}$, which is defined as follows.

DEFINITION 2.5. *A subset $C \subseteq T^n$ is said to be a discrete cone if the following properties hold:*

D1. $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subseteq C$, *where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$.*

D2. $\{-\mathbf{x}, \mathbf{x}\} \cap C \neq \emptyset$ *for every $\mathbf{x} \in T^n$.*

D3. $\mathbf{x} + \mathbf{y} \in C$ *whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} + \mathbf{y} \in T^n$.*

We note that Fishburn [8] requires $\mathbf{0} \notin C$ because his orders are antireflexive. In our case, condition D2 implies $\mathbf{0} \in C$.

Given two subsets $A, B \in 2^{[n]}$ we construct the characteristic vector of this pair $\chi(A, B) = \chi(A) - \chi(B) \in T^n$.

LEMMA 2.6. *Qualitative probability orders are in a one-to-one correspondence with discrete cones.*

Proof. Given a qualitative probability order $\preceq$ on $[n]$, we define

$$C(\preceq) = \{\chi(A, B) \in T^n \mid B \preceq A\}.$$

The two axioms of qualitative probability guarantee that $C(\preceq)$ is a discrete cone; see [8, Lemma 2.1]. Every discrete cone $C \subset T^n$ gives rise to a qualitative probability order by the following inverse construction. Every $\mathbf{v} \in C$ uniquely defines two disjoint sets $A, B \subseteq [n]$ such that $A = \{i \colon v_i = 1\}$, $B = \{i \colon v_i = -1\}$, and $\mathbf{v} = \chi(A, B)$, in which case we set $A \preceq B$. By doing this we define comparisons between all disjoint sets. If two sets $A, B \subseteq [n]$ are not disjoint we define

$$A \preceq B \iff A \setminus B \preceq B \setminus A.$$

The order $\preceq$ will then automatically satisfy the de Finetti axiom (2.2). ∎

Following Fishburn [8], the cancellation conditions can be reformulated as follows.

PROPOSITION 2.7. *A qualitative probability order $\preceq$ satisfies the kth cancellation condition CC_k if and only if for every set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\}$ of nonzero vectors in $C(\preceq)$*

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k = \mathbf{0}$$

implies that $-\mathbf{x}_i \in C(\preceq)$ for every $i = 1, \ldots, k$.

Given a system of nonnegative weights $\mathbf{w}$ and two subsets $A, B \subseteq [n]$ we can reformulate the fact that $w(A) \geq w(B)$ as $\mathbf{x} \cdot \mathbf{w} \geq 0$, where $\mathbf{x} = \chi(A, B)$. Hence, geometrically, a qualitative probability order $\preceq$ is representable if and only if there exists a nonnegative vector $\mathbf{w} \in \mathbb{R}^n$ such that

$$\mathbf{x} \in C(\preceq) \iff \mathbf{w} \cdot \mathbf{x} \geq 0 \quad \text{for all } \mathbf{x} \in T^n,$$

where $\mathbf{v} \cdot \mathbf{u}$ is the standard inner (dot) product of vectors $\mathbf{v}$ and $\mathbf{u}$. That is, $\preceq$ is representable if and only if every nonzero vector in the cone $C(\preceq)$ lies in the closed half-space $H^+_{\mathbf{w}} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w} \cdot \mathbf{x} \geq 0\}$ of the corresponding hyperplane $H_{\mathbf{w}} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w} \cdot \mathbf{x} = 0\}$.

3. Simplicial complexes and their cancellation conditions. In this section we will introduce the objects of our study: simplicial complexes that arise as initial segments of a qualitative probability order. Using cancellation conditions for simplicial complexes, we will show that this class contains the threshold complexes and is contained in the shifted complexes. Proving the strict containment of the threshold complexes in the initial segments will require an elaborate construction, which will be developed in the rest of the paper.

DEFINITION 3.1. *A subset* $\Delta \subseteq 2^{[n]}$ *is an* abstract simplicial complex *if it satisfies the following condition:*

$$\textit{if } B \in \Delta \textit{ and } A \subseteq B, \textit{ then } A \in \Delta.$$

Subsets that are in Δ *are called* faces.

Abstract simplicial complexes are in one-to-one correspondence with *simple games* as defined by von Neumann and Morgenstern [22]. A simple game is a pair $G = ([n], W)$, where W is a subset of the power set $2^{[n]}$ which satisfies the following monotonicity condition:

$$\text{if } X \in W \text{ and } X \subseteq Y \subseteq [n], \text{ then } Y \in W.$$

The subsets from W are called *winning coalitions* and the subsets from $L = 2^{[n]} \setminus W$ are called *losing coalitions*. The set of losing coalitions L is a simplicial complex. The reverse is also true: if Δ is a simplicial complex, then the set $2^{[n]} \setminus \Delta$ is a set of winning coalitions of a certain simple game.

A well-studied class of simplicial complexes is the class of *threshold* complexes—mostly as an equivalent concept to the concept of a weighted majority game but also as threshold hypergraphs [5, 18].

DEFINITION 3.2. *A simplicial complex* Δ *is a* threshold *complex if there exist nonnegative reals* $w_1, \ldots, w_n$ *and a positive constant* q *such that*

$$A \in \Delta \Longleftrightarrow w(A) = \sum_{i \in A} w_i < q.$$

The same parameters define a *weighted majority game* by setting

$$A \in W \Longleftrightarrow w(A) = \sum_{i \in A} w_i \geq q.$$

This game has the standard notation $[q; w_1, \ldots, w_n]$.

A much larger but still well-understood class of simplicial complexes is *shifted* simplicial complexes [11, 12].

DEFINITION 3.3. *A simplicial complex is* shifted *if there exists an order* $\trianglelefteq$ *on the set of vertices* $[n]$ *such that for any face* F*, replacing any of its vertices* $x \in F$ *with a vertex* y *such that* $y \trianglelefteq x$ *results in a subset* $(F \setminus \{x\}) \cup \{y\}$ *which is also a face.*

Shifted complexes correspond to complete[2] games [9]. A complete game has an order $\trianglelefteq$ on players such that if a coalition W is winning, then replacing any player

[2] Sometimes also called linear or directed.

$x \in W$ with a player z such that $x \trianglelefteq z$ results in a coalition $(W \setminus \{x\}) \cup \{z\}$ which is also winning.

Definition 3.4. *Let $\preceq$ be a qualitative probability order on $[n]$ and $K \in 2^{[n]}$. Then the corresponding* initial segment *is the set*

$$\Delta(\preceq, K) = \{X \subseteq [n] \mid X \prec K\}.$$

Lemma 3.5. *Each initial segment of a qualitative probability order is an abstract simplicial complex.*

Proof. As we commented above, every qualitative probability order is a refinement of the set-theoretic inclusion. Thus, $\Delta(\preceq, K)$ is an abstract simplicial complex. ∎

From now on, an *initial segment* is a simplicial complex which is an initial segment of a qualitative probability order.

Cancellation conditions, similar to those defined for qualitative probability orders, will play the key role in our analysis of simplicial complexes.

Definition 3.6. *A simplicial complex Δ is said to satisfy cancellation condition CC_k^* if there does not exist a trading transform $(A_1, \ldots, A_k; B_1, \ldots, B_k)$ such that $A_1, \ldots, A_k \in \Delta$ and $B_1, \ldots, B_k \notin \Delta$.*

The following lemma is an immediate consequence of the definitions.

Lemma 3.7. *Suppose $\preceq$ is a qualitative probability order on $[n]$ and $\Delta(\preceq, K)$ is its initial segment. If $\preceq$ satisfies CC_k, then $\Delta(\preceq, K)$ satisfies CC_k^*.*

Since every qualitative probability order satisfies the conditions CC_k with $k = 2, 3$ [8], we obtain the following corollary.

Corollary 3.8. *Every initial segment satisfies CC_k^* for $k \leq 3$.*

Lemma 3.9 (see [13]). *Every initial segment is a shifted complex. Moreover, there are shifted complexes that are not initial segments.*

Proof. Let Δ be a nonshifted simplicial complex. Then it is known to contain an obstruction of the following form: there are $i, j \in [n]$, and $A, B \in \Delta$, neither containing i or j, so that $A \cup \{i\}$ and $B \cup \{j\}$ are in Δ but neither $B \cup \{i\}$ nor $A \cup \{j\}$ are in Δ [11]. But then $(A \cup \{i\}, B \cup \{j\}; B \cup \{i\}, A \cup \{j\})$ is a trading transform that violates CC_2^*. Since all initial segments satisfy CC_2^* they must all be shifted.

On the other hand, there are shifted complexes that fail to satisfy CC_2^* and hence cannot be initial segments. Let Δ be the smallest shifted complex (where shifting is with respect to the usual ordering) that contains $\{1, 5, 7\}$ and $\{2, 3, 4, 6\}$. Then it is easy to check that neither $\{3, 4, 7\}$ nor $\{1, 2, 5, 6\}$ is in Δ but

$$(\{1, 5, 7\}, \{2, 3, 4, 6\}; \{3, 4, 7\}, \{1, 2, 5, 6\})$$

is a trading transform in violation of CC_2^*. ∎

Similarly, the *terminal segment*

$$G(\preceq, K) = \{X \subseteq [n] \mid K \preceq X\}$$

of any qualitative probability order is a complete simple game.

Theorem 2.4.2 of the book [21] can be reformulated to give necessary and sufficient conditions for a simplicial complex to be threshold.

Theorem 3.10. *An abstract simplicial complex $\Delta \subseteq 2^{[n]}$ is a threshold complex if and only if the condition CC_k^* holds for all $k \geq 2$.*

In Lemma 3.9 we showed that the initial segments are strictly contained in the shifted complexes. What is the relationship between the initial segments and threshold complexes?

PROPOSITION 3.11. *The class of threshold complexes and the class of initial segments of representable qualitative probability orders coincide.*

Proof. Let Δ be a threshold complex defined by the weights $w_1, \dots, w_n$ and a positive constant q. If there is no subset in $[n]$ with weight exactly q we may choose a set K whose weight is at least q and the smallest with this property. Then Δ is the initial segment of the representable qualitative probability order defined by the weight vector $(w_1, \dots, w_n)$, and the threshold set K.

Suppose now that $\preceq$ is a qualitative probability order defined by the weights $w_1, \dots, w_n$ and $K \subseteq [n]$. From the definition of a qualitative probability order, for any set A the relations $A \prec K$ and $w(A) < w(K)$ are equivalent. We set $w(K) = q$. Then the initial segment $\Delta(\preceq, K)$ consists of all subsets in $[n]$ whose weight is strictly smaller than q and hence a threshold complex with weights $w_1, \dots, w_n$ and the threshold q. $\square$

We thus have the following containments of classes of simplicial complexes:

$$\text{threshold complexes} \subseteq \text{initial segments} \subsetneq \text{shifted complexes}.$$

This leaves us with the question of whether this first containment is strict, i.e., whether there are initial segments which are not threshold complexes. A natural approach for proving that this containment is strict would be to start with a nonrepresentable qualitative probability order and extract an initial segment which is not threshold. However, the following example, adapted from [15, Examples 2.5 and 3.9], shows that we have to give much thought to the choice of that nonrepresentable order.

Example 3.12. This example gives a nonrepresentable qualitative probability order for which every initial segment is threshold. We start with a representable qualitative probability order on [5] given by the vector of weights $(7, 10, 16, 20, 22)$. This order begins with

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 12 \prec 4 \prec 5 \prec \cdots,$$

where 1 denotes the singleton set $\{1\}$ and by 12 we mean $\{1, 2\}$. Since the qualitative probability order is representable, every initial segment is a threshold complex. Now suppose we interchange the order of 12 and 4. The new order, which begins with

$$\emptyset \prec 1 \prec 2 \prec 3 \prec 4 \prec 12 \prec 5 \prec \cdots,$$

is still a qualitative probability order, but it is no longer representable [15, Example 2.5]. With four exceptions, all the initial segments in this new nonrepresentable qualitative order are initial segments in the original one and, thus, are threshold. One can easily check that the four new initial segments are all threshold with respect to the weight vector $(8, 9, 12, 15, 18)$.

One approach to finding an initial segment that is not threshold is to construct an abstract simplicial complex that violates CC_k^* for some small value of k. As noted above, all initial segments satisfy CC_2^* and CC_3^*, so the smallest condition that could fail is CC_4^*. We will now show that for small values of n, namely, for $n \le 17$, cancellation condition CC_4^* is satisfied for any initial segment. Later, we will use this information in constructing a nonthreshold initial segment.

DEFINITION 3.13. *Two pairs of subsets (A_1, B_1) and (A_2, B_2) are said to be* compatible *if the following two conditions hold:*

$$A_1 \cap A_2 \subseteq B_1 \cup B_2 \textit{ and}$$
$$B_1 \cap B_2 \subseteq A_1 \cup A_2.$$

If one of these conditions fails, the pair of subsets is called incompatible.

We will now show that if a trading transform witnesses a failure of CC_s^* for an initial segment $\Delta = \Delta(\preceq, K)$ and has some compatible pairs, then CC_{s-1} fails for $\preceq$.

LEMMA 3.14. *Let $\preceq$ be a qualitative probability order on $[n]$, $K \subseteq [n]$, and let $\Delta = \Delta(\preceq, K)$ be the corresponding initial segment. Suppose CC_s^* is not satisfied for Δ and, hence, there exists a trading transform $(A_1, \ldots, A_s; B_1, \ldots, B_s)$ such that $A_i \prec K \preceq B_j$ for all $i, j \in [s]$. If pairs (A_i, B_k) and (A_j, B_l) are compatible for some $i, k, j, l \in [s]$, $i \neq j$, and $k \neq l$, then $\preceq$ fails to satisfy CC_{s-1}.*

Proof. Let us remove the intersection from the sets of each pair:

$$\bar{A}_i = A_i \setminus (A_i \cap B_k), \qquad \bar{B}_k = B_k \setminus (A_i \cap B_k),$$
$$\bar{A}_j = A_j \setminus (A_j \cap B_l), \qquad \bar{B}_l = B_l \setminus (A_j \cap B_l).$$

Note that

$$\bar{A}_i \cap \bar{A}_j = \bar{B}_k \cap \bar{B}_l = \emptyset. \tag{3.1}$$

Indeed, suppose, for example, $x \in \bar{A}_i \cap \bar{A}_j$. Then also $x \in A_i \cap A_j$ and, by the compatibility of the pairs, $x \in B_k$ or $x \in B_l$. If $x \in B_k$, then $x \notin A_i \setminus (A_i \cap B_k)$, hence it is impossible for x to be in $\bar{A}_i \cap \bar{A}_j$. The other case is similar. Note also that, by Lemma 2.1, we have

$$\bar{A}_i \cup \bar{A}_j \prec \bar{B}_k \cup \bar{B}_l. \tag{3.2}$$

Observe that

$$(\bar{A}_i, \bar{A}_j, A_{m_1}, \ldots, A_{m_{s-2}}; \bar{B}_k, \bar{B}_l, B_{r_1}, \ldots, B_{r_{s-2}})$$

is a trading transform, where $\{m_1, \ldots, m_{s-2}\} = [s] \setminus \{i, j\}$ and $\{r_1, \ldots, r_{s-2}\} = [s] \setminus \{k, l\}$. Hence, due to (3.1),

$$(\bar{A}_i \cup \bar{A}_j, A_{m_1}, \ldots, A_{m_{s-2}}; \bar{B}_k \cup \bar{B}_l, B_{r_1}, \ldots, B_{r_{s-2}})$$

is also a trading transform. This violates CC_{s-1} since (3.2) holds and $A_{m_t} \prec B_{r_t}$ for all $t = 1, \ldots, s-2$. ∎

Recall that sets in a trading transform might not be distinct. However, we will show that if a trading transform $(A_1, \ldots, A_4, B_1, \ldots, B_4)$ witnesses a failure of CC_4^*, then in fact the A_i (respectively, the B_i) are distinct.

LEMMA 3.15. *Let $\preceq$ be a qualitative probability order on $[n]$, $K \subseteq [n]$, and let $\Delta = \Delta(\preceq, K)$ be the respective initial segment. Suppose $\mathcal{T} = (A_1, \ldots, A_4, B_1, \ldots, B_4)$ is a trading transform that witnesses a failure of CC_4^* for $\preceq$, that is, $A_i \prec K \preceq B_j$ for all $i, j \in [4]$. Then*

$$|\{A_1, \ldots, A_4\}| = |\{B_1, \ldots, B_4\}| = 4.$$

Proof. First, we note that for every $i, j, l, k \in [4]$, if $i \neq k$ and $j \neq \ell$, then the pairs (A_i, B_j) and (A_k, B_ℓ) are not compatible. If they were, by Lemma 3.14, the order $\preceq$ would fail CC_3, which is impossible.

Assume, to the contrary, that we have at least two identical coalitions among $A_1, \ldots, A_4$ or $B_1, \ldots, B_4$. Without loss of generality suppose $A_1 = A_2$. If all sets among $A_1, \ldots, A_4$ are equal, then every atom belongs to each of the four subsets $A_1, \ldots, A_4$ which implies that the same is true for $B_1, \ldots, B_4$ and hence $A_i = B_i$ for all i. Since this is not possible there are at least two distinct sets among $A_1, \ldots, A_4$ and similarly among $B_1, \ldots, B_4$. Suppose that $A_1 \neq A_3$ and $B_1 \neq B_2$.

The pairs $(A_1, B_1), (A_3, B_2)$ are not compatible. It means that either $A_1 \cap A_3 \nsubseteq B_1 \cup B_2$ or $B_1 \cap B_2 \nsubseteq A_1 \cup A_3$. Consider the first case; the other one is similar. Let x be an atom of $[n]$ such that $x \in A_1 \cap A_3$ and $x \notin B_1 \cup B_2$. Then

$$|\{i \in [4] \mid x \in A_i\}| \geq 3 \text{ and } |\{i \in [4] \mid x \in B_i\}| \leq 2.$$

In other words, at least three sets among $A_1, \ldots, A_4$ contain x and, at the same time, at most two sets contain x among $B_1, \ldots, B_4$. This contradicts $\mathcal{T}$ being a trading transform. ∎

The following characterization will be useful.

PROPOSITION 3.16. *Let A_1, A_2, B_1, B_2 be four subsets of $[n]$. Then two pairs (A_1, B_1), (A_2, B_2) are incompatible if and only if there exists $p \in [n]$ such that $|\chi(A_1, B_1)^{(p)} + \chi(A_2, B_2)^{(p)}| = 2$, where $\mathbf{v}^{(p)}$ is the pth coordinate of a vector $\mathbf{v}$.*

Proof. To have $|\chi(A_1, B_1)^{(p)} + \chi(A_2, B_2)^{(p)}| = 2$, we must have either $\chi(A_1, B_1)^{(p)} = \chi(A_2, B_2)^{(p)} = 1$ or $\chi(A_1, B_1)^{(p)} = \chi(A_2, B_2)^{(p)} = -1$. The first case is equivalent to p belonging to $p \in A_1 \cap A_2$ but not to $B_1 \cup B_2$. The second case is equivalent to p belonging to $B_1 \cap B_2$ but not to $A_1 \cup A_2$. This proves the lemma. ∎

Suppose now that eight subsets $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$ of $[n]$ are such that the sequence

$$\mathcal{T} = (A_1, \ldots, A_4; B_1, \ldots, B_4) \tag{3.3}$$

is a trading transform. Let us associate an $8 \times n$ $(0,1)$-matrix $M(\mathcal{T})$ with it. The rows of this matrix will be the characteristic vectors $\chi(A_1)$, $\chi(A_2)$, $\chi(A_3)$, $\chi(A_4)$, $\chi(B_1)$, $\chi(B_2)$, $\chi(B_3)$, $\chi(B_4)$, respectively.

PROPOSITION 3.17. *Let $\{i, j, k, \ell\} = \{q, r, s, t\} = [4]$. Then the pairs (A_i, B_q) and (A_j, B_r) are incompatible if and only if there exists a column $(a_1, \ldots, a_4, b_1, \ldots, b_4)^T$ of matrix $M(\mathcal{T})$ such that either*

$$a_i = a_j = b_s = b_t = 1 \quad \textit{and} \quad a_k = a_\ell = b_q = b_r = 0$$

or

$$a_i = a_j = b_s = b_t = 0 \quad \textit{and} \quad a_k = a_\ell = b_q = b_r = 1.$$

In such a case the pairs (A_k, B_s) and (A_ℓ, B_t) are also incompatible.

Proof. We may view columns of $M(\mathcal{T})$ as indexed by $[n]$. By Proposition 3.16 we are able to find $p \in [n]$ such that $|\chi(A_i, B_q)^{(p)} + \chi(A_j, B_r)^{(p)}| = 2$ which is equivalent to having one of the two columns described in the proposition. ∎

Consider the following set of 8-dimensional column vectors:

$$U = \{\mathbf{v} \in \mathbb{Z}^8 \mid v_i \in \{0, 1\} \text{ and } v_1 + v_2 + v_3 + v_4 = v_5 + v_6 + v_7 + v_8 = 2\}. \tag{3.4}$$

This set has an involution $\mathbf{v} \mapsto \bar{\mathbf{v}}$, where $\bar{\mathbf{v}} = \mathbf{1} - \mathbf{v}$. Thus, if $\mathbf{v} = (1, 1, 0, 0, 0, 0, 1, 1)^T$, then $\bar{\mathbf{v}} = (0, 0, 1, 1, 1, 1, 0, 0)^T$. There are 36 vectors in U and they are split into 18 nonintersecting pairs $\{\mathbf{v}, \bar{\mathbf{v}}\}$.

We note that the two vectors in Proposition 3.17 whose existence in $M(\mathcal{T})$ is equivalent to the incompatibility of (A_i, B_q) and (A_j, B_r) belong to the same pair. It is easily seen that the incompatibility of (A_k, B_s) and (A_ℓ, B_t) leads to the same two conditions.

THEOREM 3.18. *Let $n \leq 17$ and $\preceq$ be a qualitative probability order on $[n]$. Then for every $K \subseteq [n]$ the initial segment $\Delta = \Delta(\preceq, K)$ satisfies CC_4^*.*

Proof. Suppose $\mathcal{T} = (A_1, A_2, A_3, A_4; B_1, B_2, B_3, B_4)$ is a trading transform witnessing a failure of CC_4^*. Since $A_i \prec K \preceq B_j$ for all $i, j \in [4]$ we have $A_i \neq B_j$. Now by Lemma 3.15 we conclude that all eight sets of $\mathcal{T}$ are distinct. Since $\preceq$ satisfies CC_3, Lemma 3.14 implies that for every choice of indices such that $i \neq j$, $q \neq r$ the pairs (A_i, B_q) and (A_j, B_r) are incompatible. There are 36 such incompatible pairs, so by Proposition 3.17 the matrix $M(\mathcal{T})$ must have at least 18 columns. ∎

While no initial segment on fewer than 18 atoms can fail CC_4^*, we will show that there exists such an initial segment on 26 atoms. This will demonstrate that the class of initial segments strictly contains the class of threshold complexes. The next two sections will be devoted to the construction of this example. It will be based on the construction that converts a representable nonlinear qualitative probability order into a linear one by breaking ties.

4. Constructing new qualitative probability orders from nonlinear representable ones by breaking ties. To find an initial segment that is not threshold, we will start with a nonlinear representable qualitative probability order $\preceq$. The idea is to break ties in this nonlinear order judiciously so as to obtain a linear order that possesses an initial segment violating CC_4^*. In other words we will be coarsening $\preceq$ until it becomes linear. This coarsening procedure is better discussed in the language of discrete cones.

Let $\preceq$ be a representable nonlinear qualitative probability order on $[n]$ with underlying weight vector $\mathbf{w}$. Then all vectors of the discrete cone $C(\preceq)$ lie in the closed half-space $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w} \cdot \mathbf{x} \geq 0\}$ of $\mathbb{R}^n$ bounded by the hyperplane $H_\mathbf{w} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{w} \cdot \mathbf{x} = 0\}$.

The set $S(\preceq) = H_\mathbf{w} \cap C(\preceq)$ consists of all characteristic vectors that correspond to the equivalence relation. More specifically, if $\mathbf{x} \in S(\preceq)$, then there are $A, B \subseteq [n]$ such that $A \sim B$ and $\mathbf{x} = \chi(A, B)$. It is exactly this part of the discrete cone that we would like to thin out. Indeed, if $\mathbf{x} \in S(\preceq)$, then $-\mathbf{x}$ is a vector of $S(\preceq)$ as well. Since the discrete cone of a linear qualitative probability order contains exactly one of these vectors we need to remove $\mathbf{x}$ or $-\mathbf{x}$ from $C(\preceq)$. This will be equivalent to breaking the tie $A \sim B$ one way or another. In order to obtain a discrete cone that corresponds to a linear qualitative probability order we may need to remove several vectors of $S(\preceq)$. If the operation is successful, the new linear order $\sqsubseteq$ corresponding to the new discrete cone will preserve all strict comparisons of the old one, namely, if $A \prec B$, then $A \sqsubset B$. The main question is, what are the conditions under which a set of vectors of $S(\preceq)$ can be removed from $C(\preceq)$?

What can prevent us from removing a vector of $S(\preceq)$ from $C(\preceq)$? It is intuitively clear that we cannot remove a vector $\mathbf{x} = \chi(A, B)$ if the set comparison $A \succeq B$ corresponding to it is a consequence of those remaining. We need to consider what a consequence means formally.

There are at least two ways in which one set comparison may imply another one. The first is by means of the de Finetti condition. This, however, is already built into the definition of the discrete cone as $\chi(A, B) = \chi(A \cup C, B \cup C)$. Another way in which a comparison may be implied from two others is transitivity. This has a nice algebraic characterization. If $C \preceq B$ and $B \preceq A$, then by transitivity we can conclude that $C \preceq A$. In terms of the discrete cone this means that if $\chi(A, B)$ and

$\chi(B,C)$ are in a discrete cone, then $\chi(A,C)$ is in the discrete cone too. However, $\chi(A,C) = \chi(A,B) + \chi(B,C)$. This leads us to the following definition. Following Christian, Conder, and Slinko [4], define a *restricted sum* for vectors in a discrete cone C. Let $\mathbf{u}, \mathbf{v} \in C$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{cases} \mathbf{u}+\mathbf{v} & \text{if } \mathbf{u}+\mathbf{v} \in T^n, \\ \text{undefined} & \text{if } \mathbf{u}+\mathbf{v} \notin T^n. \end{cases}$$

Fishburn [8, Lemma 2.1] showed that the transitivity of a qualitative probability order is equivalent to closedness of its corresponding discrete cone with respect to the restricted addition (without formally defining a discrete cone). The axiom D3 of the discrete cone can be rewritten as follows:

D3. $\mathbf{x} \oplus \mathbf{y} \in C$ whenever $\mathbf{x}, \mathbf{y} \in C$ and $\mathbf{x} \oplus \mathbf{y}$ is defined.

Note that the restricted sum is not associative.

THEOREM 4.1 (construction method). *Let $\preceq$ be a representable qualitative probability order on $[n]$ and $\mathbf{w}$ be its weight vector. Suppose $\preceq$ is not linear. Let $S(\preceq)$ be the set of all vectors of $C(\preceq)$ which lie in the hyperplane $H_{\mathbf{w}}$. Suppose $S(\preceq) = X \cup Y$ is a partition of $S(\preceq)$ into a union of two disjoint subsets X and Y such that*

(i) *$Y \cap \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} = \emptyset$,*
(ii) *$X \cap \{\mathbf{s}, -\mathbf{s}\} \neq \emptyset$ for every $\mathbf{s} \in S(\preceq)$,*
(iii) *X is closed under the operation of restricted sum, i.e., $\mathbf{x} \oplus \mathbf{y} \in X$ whenever $\mathbf{x}, \mathbf{y} \in X$ and $\mathbf{x} \oplus \mathbf{y}$ is defined.*

Then $C_Y = C(\preceq) \setminus Y$ is a discrete cone in T^n.

Proof. By (i) we conclude that C_Y satisfies D1 as none of the $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ can be eliminated. By (ii) we conclude that for every $\mathbf{x}$ either $\mathbf{x}$ or $-\mathbf{x}$ remains in C_Y, hence D2 is also satisfied. Suppose now we have $\mathbf{x}, \mathbf{y} \in C_Y$. Suppose that at least one of them, say, $\mathbf{x}$ is not in $S(\preceq)$, i.e., $\mathbf{x} \cdot \mathbf{w} > 0$. Then $(\mathbf{x}+\mathbf{y}) \cdot \mathbf{w} = \mathbf{x} \cdot \mathbf{w} + \mathbf{y} \cdot \mathbf{w} > 0$. As this element does not belong to $S(\preceq)$ it could not be in Y. So D3 is in this case satisfied. Therefore if $\mathbf{x} \oplus \mathbf{y}$ belonged to $C(\preceq)$ it would also remain in C_Y. If both $\mathbf{x}, \mathbf{y}$ belong to C_Y and hence to X, then D3 follows from (iii). ∎

In the examples below we follow the notation of Theorem 4.1.

Example 4.2 (successful coarsening). This example is based on Example 5 of [10]. The vector of weights $(1, 2, 5, 6, 10)$ defines a qualitative probability order $\preceq$ on [5]. Note that this order is not linear. We want to break all ties in $\preceq$ and to construct a linear qualitative probability order $\sqsubseteq$ which retains all strict comparisons of $\preceq$. Let us show the existence of $\sqsubseteq$.

Consider the four equivalences in $\preceq$,

$$13 \sim 4, \quad 14 \sim 23, \quad 34 \sim 15, \quad \text{and} \quad 25 \sim 134.$$

All equivalences of $\preceq$ follow from the four given above by the de Finetti axiom:

$$\begin{aligned} &13 \sim 4 \text{ implies } 123 \sim 24,\ 135 \sim 45,\ 1235 \sim 245; \\ &14 \sim 23 \text{ implies } 145 \sim 235; \\ &34 \sim 15 \text{ implies } 234 \sim 125; \\ &25 \sim 134 \text{ has no consequences.} \end{aligned}$$

Let $\mathbf{u}_1 = \chi(13, 4) = (1, 0, 1, -1, 0)$, $\mathbf{u}_2 = \chi(14, 23) = (1, -1, -1, 1, 0)$, $\mathbf{u}_3 = \chi(34, 15) = (-1, 0, 1, 1-1)$, and $\mathbf{u}_4 = \chi(25, 134) = (-1, 1, -1, -1, 1)$. Then

$$S(\preceq) = \{\pm\mathbf{u}_1, \pm\mathbf{u}_2, \pm\mathbf{u}_3, \pm\mathbf{u}_4\}$$

and $X = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is closed under the restricted addition as $\mathbf{u}_i \oplus \mathbf{u}_j$ is undefined for all $i, j \in [4]$. Hence, by Theorem 4.1, we can remove the set $Y = -X = \{-\mathbf{u}_1, -\mathbf{u}_2, -\mathbf{u}_3, -\mathbf{u}_4\}$ from the cone $C(\preceq)$ and still get a qualitative probability order. The new order $\sqsubseteq$ corresponds to the discrete cone C_Y and is linear.

Example 4.3 (unsuccessful coarsening). This example is a simplified variant of a Gabelman simple game of order 3 [20, 21]. The vector of weights $(1, 2, \ldots, 9)$ defines the qualitative probability order $\preceq$ on [9]. Note that each atom $i \in [9]$ has the weight i.

Consider a 3×3 square with an atom in every cell:

2	7	6
9	5	1
4	3	8

.

The sum of numbers in every row is equal to 15. The sum of numbers in every column is 15 as well. Hence, in the qualitative probability order $\preceq$ we have the following equivalences:

$$249 \sim 357 \sim 168 \sim 267 \sim 159 \sim 348.$$

Suppose that we want to construct a qualitative probability order $\sqsubseteq$ based on $\preceq$ for which $A \sqsubset B$, where

$$A \in \{249, 357, 168\} \quad \text{and} \quad B \in \{267, 159, 348\}.$$

Let us show that such order $\sqsubseteq$ does not exist.

Consider the vectors

$$\begin{aligned}
\mathbf{x}_1 &= (0, 0, 0, -1, 0, 1, 1, 0, -1) = \chi(267, 249),\\
\mathbf{x}_2 &= (1, 0, -1, 0, 0, 0, -1, 0, 1) = \chi(159, 357),\\
\mathbf{x}_3 &= (-1, 0, 1, 1, 0, -1, 0, 0, 0) = \chi(348, 168).
\end{aligned}$$

One can see that $\{\pm\mathbf{x}_1, \pm\mathbf{x}_2, \pm\mathbf{x}_3\} \subset S(\preceq)$. We want to drop vectors $-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3$ from the cone $C(\preceq)$ while leaving the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ there. So if Y is a set to be removed, then $-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3 \in Y$. By Theorem 4.1 the set $X = S(\preceq) \setminus Y$ should be closed under the restricted sum. However $\mathbf{x}_1, \mathbf{x}_2 \in X$ but $\mathbf{x}_1 \oplus \mathbf{x}_2 = -\mathbf{x}_3 \in Y$. Therefore, there is no qualitative probability order with the desired properties.

5. Threshold simplicial complexes and initial segments. In this section we show that the class of initial segments is not equal to the class of threshold simplicial complexes.

THEOREM 5.1. *The class of threshold simplicial complexes is strictly contained in the class of initial segments.*

Proof. Here we give only an outline of the proof. All missing bits and pieces will be gradually filled. We will start with choosing a representable linear qualitative probability order $\preceq$ on [18]. We will then extend it to a representable nonlinear qualitative probability order $\preceq'$ on [26]. A distinguished feature of $\preceq'$ will be the existence of eight sets $A_1^\dagger, \ldots, A_4^\dagger$, $B_1^\dagger, \ldots, B_4^\dagger$ such that

$$(A_1^\dagger, A_2^\dagger, A_3^\dagger, A_4^\dagger; B_1^\dagger, B_2^\dagger, B_3^\dagger, B_4^\dagger) \tag{5.1}$$

is a trading transform and all eight sets involved in it are tied in $\preceq'$. The linear qualitative probability order $\sqsubseteq$ on [26] will be obtained by breaking those ties in $\preceq'$ so that in the new order $\sqsubseteq$ we will have a sequence of strict inequalities

$$A_1^\dagger \sqsubset A_2^\dagger \sqsubset A_3^\dagger \sqsubset A_4^\dagger \sqsubset B_1^\dagger \sqsubset B_2^\dagger \sqsubset B_3^\dagger \sqsubset B_4^\dagger.$$

Then (5.1) will witness a failure of CC_4^* for $\sqsubseteq$ and the initial segment $\Delta(\sqsubseteq, K)$ with $K = B_1^\dagger$ will not be threshold by Theorem 3.10. ∎

5.1. The construction of $\sqsubseteq$. It will involve several steps.

Step 0. First, we define $\preceq$ on [18]. Let $w_1, \ldots, w_{18}$ be arbitrarily chosen positive real numbers from the interval $[0, 1]$ that are linearly independent over $\mathbb{Z}$. Let $\preceq$ be a representable linear qualitative probability order on [18] defined by the weight vector $\mathbf{w} = (w_1, \ldots, w_{18})$.

Step 1. We will again make use of the set $U \subset \mathbb{R}^8$ defined in (3.4). As we know, it splits into a union of 18 pairs $\{\mathbf{v}, \bar{\mathbf{v}}\}$, $\mathbf{v} \in U$. Let M be an 8×18 matrix with exactly one column from each of those 18 pairs. By $A_1, \ldots, A_4, B_1, \ldots, B_4$ we denote the sets whose characteristic vectors are equal to the rows $M_1, \ldots, M_8$ of M, respectively. We note that

$$\mathcal{T} = (A_1, \ldots, A_4; B_1, \ldots, B_4)$$

is a trading transform since by the construction every $i \in [18]$ will be a member of exactly two sets $A_1, \ldots, A_4$ and exactly two sets $B_1, \ldots, B_4$. By construction $M = M(\mathcal{T})$.

Step 2. We add new atoms $\{19, \ldots, 26\}$ and enlarge the sets $A_1, \ldots, A_4, B_1, \ldots, B_4$ forming supersets $A_1^\dagger, \ldots, A_4^\dagger, B_1^\dagger, \ldots, B_4^\dagger$, respectively, so that their characteristic vectors are the rows of the following augmented matrix:

$$M^\dagger = \begin{bmatrix} \chi(A_1^\dagger) \\ \chi(A_2^\dagger) \\ \chi(A_3^\dagger) \\ \chi(A_4^\dagger) \\ \chi(B_1^\dagger) \\ \chi(B_2^\dagger) \\ \chi(B_3^\dagger) \\ \chi(B_4^\dagger) \end{bmatrix} = \begin{array}{c} \begin{array}{c|c|c} \quad 1\ldots18 \quad & \ 19\ 20\ 21\ 22\ & \ 23\ 24\ 25\ 26\ \end{array} \\ \left[\begin{array}{c|c|c} \chi(A_1) & & \\ \chi(A_2) & I & I \\ \chi(A_3) & & \\ \chi(A_4) & & \\ \hline \chi(B_1) & & \\ \chi(B_2) & J & I \\ \chi(B_3) & & \\ \chi(B_4) & & \end{array}\right] \end{array}, \tag{5.2}$$

where I is the 4×4 identity matrix and

$$J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

We note that $A_i^\dagger$ is the only superset of A_i among $A_1^\dagger, \ldots, A_4^\dagger, B_1^\dagger, \ldots, B_4^\dagger$ and a similar thing can be said about $B_i^\dagger$. So the mapping

$$\dagger : D \mapsto D^\dagger \tag{5.3}$$

on $\{A_1, \ldots, A_4, B_1, \ldots, B_4\}$ is then well defined. This construction secures that the sequence

$$\mathcal{T}^\dagger = (A_1^\dagger, A_2^\dagger, A_3^\dagger, A_4^\dagger; B_1^\dagger, B_2^\dagger, B_3^\dagger, B_4^\dagger) \tag{5.4}$$

continues to be a trading transform. Hence $M^\dagger = M(\mathcal{T}^\dagger)$.

Step 3. We extend now $\preceq$ to a qualitative probability order $\preceq'$ on $[26]$ by giving weights $w_{19}, \dots, w_{26}$ to atoms $\{19, \dots, 26\}$ in such a way that each of the sets $A_1^\dagger, \dots, A_4^\dagger$, $B_1^\dagger, \dots, B_4^\dagger$ has the same weight N relative to the new vector of weights $\mathbf{w}' = (w_1, \dots, w_{26})$, where N is a sufficiently large number (to be justified in Lemma 5.5). Then the new qualitative probability order $\preceq'$ on $[26]$ is not linear since $A_1^\dagger, \dots, A_4^\dagger, B_1^\dagger, \dots, B_4^\dagger$ are tied.

Step 4. Let us define

$$\Omega' = \{\chi(C, D) \mid C, D \in \{A_1^\dagger, \dots, A_4^\dagger, B_1^\dagger, \dots, B_4^\dagger\} \text{ and } D \text{ precedes } C \text{ in (5.4)}\}.$$

We show that $S(\preceq') = \Omega' \cup -\Omega'$ (Lemma 5.5) and that Ω' is closed under the operation of a restricted sum (Lemma 5.4). Moreover since the components of $\mathbf{w}'$ are all nonzero, no vector $\mathbf{e}_i$ can satisfy $\mathbf{e}_i \cdot \mathbf{w}' = 0$. Hence Ω' satisfies conditions (i), (ii), (iii) for the set X of Theorem 4.1. By that theorem we conclude that $C(\preceq') \setminus -\Omega'$ is a discrete cone which gives rise to a linear qualitative probability order $\sqsubseteq$ on $[26]$, for which

$$A_1^\dagger \sqsubset A_2^\dagger \sqsubset A_3^\dagger \sqsubset A_4^\dagger \sqsubset B_1^\dagger \sqsubset B_2^\dagger \sqsubset B_3^\dagger \sqsubset B_4^\dagger,$$

as required.

Note that we have a significant degree of freedom in this construction. The matrix M can be chosen in 2^{18} possible ways and the initial linear qualitative probability order $\preceq$ was almost arbitrary.

5.2. Justification of the construction. We start with a justification of Step 3.

LEMMA 5.2. *Let $K > 1026$ and $N > 8K$. Then the weights*

$$\begin{aligned}
w_{19} &= N - K - (-\chi(B_1, A_4) + \chi(A_1)) \cdot \mathbf{w},\\
w_{20} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) + \chi(A_2)) \cdot \mathbf{w},\\
w_{21} &= N - K - (-\chi(B_1, A_4) - \chi(B_2, A_1) - \chi(B_3, A_2) + \chi(A_3)) \cdot \mathbf{w},\\
w_{22} &= N - K - \chi(A_4) \cdot \mathbf{w},\\
w_{23} &= K - \chi(B_1, A_4) \cdot \mathbf{w},\\
w_{24} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1)) \cdot \mathbf{w},\\
w_{25} &= K - (\chi(B_1, A_4) + \chi(B_2, A_1) + \chi(B_3, A_2)) \cdot \mathbf{w},\\
w_{26} &= K
\end{aligned} \tag{5.5}$$

are positive. Furthermore, the eight sets of the sequence (5.4) *are tied having weight N relative to the vector of weights $\mathbf{w}' = (w_1, \dots, w_{26})$.*

Proof. To find weights $w_{19}, \dots, w_{26}$ such that all sets of the sequence (5.4) have weight N, we need to solve the following system of linear equations:

$$\begin{bmatrix} I & I \\ J & I \end{bmatrix} \begin{bmatrix} w_{19} \\ \vdots \\ w_{26} \end{bmatrix} = N\mathbf{1} - M \cdot \mathbf{w}, \tag{5.6}$$

where $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{Z}^8$. The matrix of this system has rank 7. Indeed the only relation between its rows would be that the sum of 4 first rows is equal to the sum of the 4 remaining rows. The augmented matrix of the system has the same rank since the same relation between rows exist in both matrix M and column vector $N\mathbf{1}$.

Therefore, the system (5.6) is consistent and there is one free variable in this system. Choose w_{26} as this free variable and let us give it value K. Now we express all other weights $w_{19}, \dots, w_{25}$ (dependent variables) in terms of $w_{26} = K$ and get (5.5). First we note that K is large enough to ensure that $w_{23}, \dots, w_{26}$ are positive. Then we note that N is large enough to ensure that $w_{19}, \dots, w_{22}$ are positive. ∎

We turn to justification of Step 4. Define the following subset of T^{18}:

$$\Omega = \{\chi(C,D) \mid C, D \in \{A_1, \dots, A_4, B_1, \dots, B_4\} \text{ and } D^\dagger \text{ precedes } C^\dagger \text{ in } (5.4)\}.$$

Note that the notation $\chi(Y)$ and $\chi(Y,Z)$ from now on gets ambiguous as it may denote a vector of $\mathbb{Z}^{18}$, $\mathbb{Z}^{26}$, or even $\mathbb{Z}^8$. However, the dimension of the vector will always be clear from the context.

We will need the following.

LEMMA 5.3. *For every partition* $[4] = \{i,j\} \cup \{k,\ell\}$ *there exists* $p \in [18]$ *such that*

$$|\chi(A_i, A_k)^{(p)} + \chi(A_j, A_\ell)^{(p)}| = |\chi(B_i, B_k)^{(p)} + \chi(B_j, B_\ell)^{(p)}| = 2.$$

Proof. Without loss of generality we may assume that our partition is $[4] = \{1,2\} \cup \{3,4\}$. Then either $\mathbf{v} = (0,0,1,1,0,0,1,1)^T$ or $\bar{\mathbf{v}} = (1,1,0,0,1,1,0,0)^T$ is among the columns of M. Choose the atom $p \in [18]$ which corresponds to this column. ∎

LEMMA 5.4. Ω' *is closed under the operation of restricted sum.*

Proof. We will extend the mapping (5.3) to a one-to-one correspondence between Ω and Ω', namely, $\mathbf{v} = \chi(C,D) \in \Omega$ corresponds to the unique $\mathbf{v}^\dagger = \chi(C^\dagger, D^\dagger) \in \Omega'$. We note that if $\mathbf{u} \oplus \mathbf{v}$ is undefined for $\mathbf{u}, \mathbf{v} \in \Omega$, then $\mathbf{u}^\dagger \oplus \mathbf{v}^\dagger$ is undefined as well.

Let $\mathbf{u}$ and $\mathbf{v}$ be any two vectors in Ω. Without loss of generality, we can distinguish five cases.

Case 1. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $j = k$ or $i = m$. Suppose $j = k$ (the other case is similar). Then $\mathbf{u} + \mathbf{v} = \chi(B_i, B_m)$. Since $i > j$ and $j > m$ we have $i > m$. This implies that $\chi(B_i, B_m)$ belongs to Ω. Therefore, $\mathbf{u} \oplus \mathbf{v} \in \Omega$.

Case 2. $\mathbf{u} = \chi(B_i, A_j)$ and $\mathbf{v} = \chi(B_k, A_m)$, where $i \neq k$ and $j \neq m$. Then by Proposition 3.17 the pairs $\chi(B_i, A_j)$ and $\chi(B_k, A_m)$ are not compatible. In this case, by Lemma 3.16, the sum $\mathbf{u} \oplus \mathbf{v}$ is undefined.

Case 3. $\mathbf{u} = \chi(B_i, A_j)$, $\mathbf{v} = \chi(B_i, A_m)$ or $\mathbf{u} = \chi(B_j, A_i)$, $\mathbf{v} = \chi(B_m, A_i)$, where $j \neq m$. Consider the first case; the other one is similar. First choose $k \in [4] \setminus \{i\}$. By Proposition 3.17 pairs $\chi(B_i, A_j)$ and $\chi(B_k, A_m)$ are not compatible. Then, by Lemma 3.16, there is $p \in [18]$ such that $|\chi(B_i, A_j)^{(p)} + \chi(B_k, A_m)^{(p)}| = 2$. However, in this case we must have $\chi(B_i, B_k)^{(p)} = 0$. Hence, the sum $\mathbf{u} \oplus \mathbf{v}$ is undefined as the sum $\chi(B_i, A_j) + \chi(B_k, A_m) + \chi(B_i, B_k)$ has the absolute value of 2 at pth coordinate.

Case 4. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $\{i, j, k, m\} = [4]$. By Lemma 5.3, the sum $\mathbf{u} \oplus \mathbf{v}$ is undefined.

Case 5. $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_k, B_m)$ or $\mathbf{u} = \chi(A_i, A_j)$, $\mathbf{v} = \chi(A_k, A_m)$, where $i = k$ or $j = m$. If $i = k$ and $j = m$, then $\mathbf{u} \oplus \mathbf{v}$ is undefined. Consider the case $i = k$, $j \neq m$. Then $\mathbf{u} = \chi(B_i, B_j)$, $\mathbf{v} = \chi(B_i, B_m)$. Let $s = [4] \setminus \{i, j, m\}$. By Lemma 5.3 applied to $(B_i, B_j), (B_s, B_m)$, there is $p \in [18]$ such that $2 = |\chi(B_i, B_j)^{(p)} + \chi(B_s, B_m)^{(p)}| = |\chi(B_i, B_j)^{(p)} + \chi(B_s, B_m)^{(p)} + \chi(B_i, B_s)^{(p)}| = |\mathbf{u}^{(p)} + \mathbf{v}^{(p)}|$ as $\chi(B_i, B_s)^{(p)} = 0$ in this case. Hence, $\mathbf{u} \oplus \mathbf{v}$ is undefined. ∎

And here comes the last bit of justification of Step 4.

LEMMA 5.5. $S(\preceq') = \Omega' \cup -\Omega'$.

Proof. Assume, to the contrary, that there are two sets $C, D \subseteq [26]$ that have equal weights relative to $\mathbf{w}'$ but $\chi(C,D) \notin \Omega' \cup -\Omega'$. The sets C and D have to contain some of the atoms from $[26] \setminus [18]$, since $w_1, \ldots, w_{18}$ are linearly independent over $\mathbb{Z}$. Let $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$, where $C_1, D_1 \subseteq [18]$ and $C_2, D_2 \subseteq [26] \setminus [18]$ with C_2 and D_2 being nonempty. Since the weights of C and D are equal, we have

$$0 = \chi(C, D) \cdot \mathbf{w}' = \chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+,$$

where $\mathbf{w}^+ = (w_{19}, \ldots, w_{26})^T$. By (5.5), we can express weights $w_{19}, \ldots, w_{26}$ as linear combinations with integer coefficients of N, K, and $w_1, \ldots, w_{18}$ obtaining

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \left(\sum_{i=1}^{4} \gamma_i \chi(A_i) + \sum_{i=1}^{4} \gamma_{4+i} \chi(B_i) \right) \cdot \mathbf{w} + \beta_1 N + \beta_2 K,$$

where $\gamma_i, \beta_j \in \mathbb{Z}$.

The expression in brackets on the right-hand side of this equation is a vector with integer entries. Let us denote this vector by α. Then

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w} + \beta_1 N + \beta_2 K. \tag{5.7}$$

Let us find a bound for coordinates of α. An arbitrary vector $\chi(X, Y)$, by definition, has coordinates in the set $\{-1, 0, 1\}$. Equations (5.5) therefore imply that every γ_i is smaller than or equal to 7. Therefore $\alpha_i \leq 56$ for every $i \in [18]$. We note also that $|\beta_1| \leq 4$ and $|\beta_2| \leq 8$.

Now we may express $\chi(C, D) \cdot \mathbf{w}'$, which is equal to zero, in terms of $\mathbf{w}$, α, K, and N:

$$0 = \chi(C, D) \cdot \mathbf{w}' = (\chi(C_1, D_1) + \alpha) \cdot \mathbf{w} + \beta_1 N + \beta_2 K.$$

Then

$$|(\chi(C_1, D_1) + \alpha) \cdot \mathbf{w}| \leq 57 \sum_{i \in [18]} w_i < 1026.$$

We recap that K was chosen to be greater than 1026 and N is greater than $8K$. So $\beta_1 N + \beta_2 K = 0$ if and only if $\beta_1 = \beta_2 = 0$. In particular, if β_1 or β_2 is nonzero, then $|\beta_1 N + \beta_2 K|$ is a very large number, which cannot be canceled out by $(\chi(C_1, D_1) + \alpha) \cdot \mathbf{w}$. Hence $\beta_1 N + \beta_2 K = 0$ and $\beta_1 = \beta_2 = 0$. Weights $w_1, \ldots, w_{18}$ are linearly independent, so for arbitrary $\mathbf{b} \in \mathbb{Z}^{18}$ the dot product $\mathbf{b} \cdot \mathbf{w}$ can be zero if and only if $\mathbf{b} = \mathbf{0}$. Hence

$$w'(C) = w'(D) \text{ if and only if } \chi(C_1, D_1) = -\alpha \text{ and } \beta_1 = 0, \beta_2 = 0.$$

Taking into account that $\chi(C_1, D_1)$ is a vector from T^{18}, we obtain that $\alpha \in T^{18}$. Hence

$$\alpha \notin T^{18} \Longrightarrow w'(C) \neq w'(D).$$

What is now left is to justify the following two claims.

Claim 5.6. $\chi(C_1, D_1) \in \Omega \cup -\Omega$ *implies* $\chi(C, D) \in \Omega' \cup -\Omega'$.

Claim 5.7. *If* $\alpha \in T^{18}$, *then* $\alpha \in \Omega \cup -\Omega$.

The proofs of Claims 5.6 and 5.7 are given in the appendix. Now let us show how with the help of these two claims the proof of Lemma 5.5 can be completed. The sets C and D have the same weight and this can happen only if α is a vector in T^{18}. By Claim 5.7 $\alpha \in \Omega \cup -\Omega$. The characteristic vector $\chi(C_1, D_1)$ is equal to $-\alpha$, hence $\chi(C_1, D_1) \in \Omega \cup -\Omega$. By Claim 5.6 we get $\chi(C, D) \in \Omega' \cup -\Omega'$. ∎

6. Acyclic games and a conjectured characterization. So far we have shown that the initial segments strictly contain the threshold complexes and are strictly contained within the shifted complexes. In this section we introduce some ideas from the theory of simple games to formulate a conjecture that characterizes initial segments. The idea in this section is to start with an abstract simplicial complex and see if there is a natural relation available on $[n]$ which gives a qualitative probability order and has the original complex as an initial segment. We suggest the *Winder relation* (or Winder desirability relation in [21]) for this role. Threshold logicians considered this relation in the context of switching functions. Apparently the relation is due to Winder; see [24, 16, 21].

Let $\Delta \subseteq 2^{[n]}$ be a simplicial complex. Define the *Winder relation*, $\leq_W$, on $2^{[n]}$ by $A \leq_W B$ if and only if for every $Z \subseteq [n] \setminus ((A \setminus B) \cup (B \setminus A))$ we have that

$$(A \setminus B) \cup Z \notin \Delta \Longrightarrow (B \setminus A) \cup Z \notin \Delta. \tag{6.1}$$

The name Winder *desirability* relation becomes clear if we pass to the related simple game $G = ([n], W)$, where the set of winning coalitions $W = 2^{[n]} \setminus \Delta$. We may now rewrite (6.1) as follows:

$$(A \setminus B) \cup Z \in W \Longrightarrow (B \setminus A) \cup Z \in W.$$

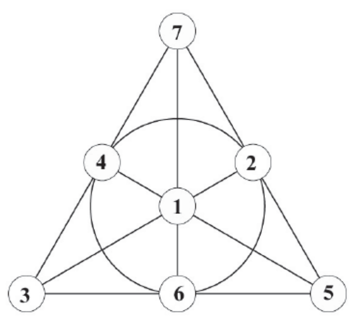
This means that the set of players $A \setminus B$ is a less desirable coalition than $B \setminus A$: if a coalition Z wins merging with $A \setminus B$, then it will also win merging with $B \setminus A$.

We also define the *Winder existential relation*, $\prec_W$, on $2^{[n]}$ to be

$$A \prec_W B \Longleftrightarrow \text{It is not the case that } B \leq_W A.$$

Both $\leq_W$ and $\prec_W$ are far from being transitive (see, e.g., [21, Proposition 4.7.1]) and may even have cycles, as the following example demonstrates.

Example 6.1. Let us consider the Fano plane with 7 points (shown below) and define the simplicial complex $\Delta \subset [7]$ by $\Delta = \{X \subset [7] \mid X$ does not contain any lines$\}$.



The Winder existential relation is cyclic:

$$12 \prec_W 35 \prec_W 46 \prec_W 12.$$

On the other hand a large class of weighted majority games [24, 21] has an acyclic Winder existential relation. Hence, by duality, there are plenty of abstract simplicial complexes with acyclic Winder existential relation.

DEFINITION 6.2. *A simplicial complex Δ is called* strongly acyclic *if there are no k-cycles*

$$A_1 \prec_W A_2 \prec_W \cdots A_k \prec_W A_1$$

for any k in the Winder existential relation.

The question of finding necessary and sufficient conditions for a simple game to have an acyclic desirability relation has attracted significant attention. Winder [24] constructed the first example of nonweighted switching function for which $\prec_W$ is acyclic. Peleg asked if any constant-sum simple game (i.e., a game in which half of all coalitions were winning and half of them were losing) with complete desirability relation and strongly acyclic desirability relation is a weighted majority game. This question was answered negatively in [20] (see also [21, section 4.10]), but the cardinality of the game in that example is very large (and not even specified).

Note that Taylor and Zwicker [21] prove that the strongly acyclic complexes satisfy the cancellation conditions CC_2^* and CC_3^* but can fail to satisfy CC_4^* [21, Corollary 4.9.9 and Corollary 4.9.10] This is exactly how the initial segments behave. At the moment the theory of simple games does not have a conjecture about the structure of strongly acyclic games. We provide such a conjecture.

CONJECTURE 1. *A simplicial complex Δ is an initial segment if and only if it is strongly acyclic.*

An equivalent companion conjecture relates to games.

CONJECTURE 2. *A simple game G is a terminal segment of a qualitative probability order if and only if this game is strongly acyclic.*

Our evidence in favor of Conjecture 1 is based on the fact that every initial segment is strongly acyclic.

THEOREM 6.3. *Suppose $\preceq$ is a qualitative probability order on $[n]$ and $K \in 2^{[n]}$. Then the initial segment $\Delta(\preceq, K)$ is strongly acyclic.*

Proof. Let $\Delta = \Delta(\preceq, K)$. By definition $A \prec_W B$ if and only if there exists a $Z \in [n] \setminus ((A \setminus B) \cup (B \setminus A))$ such that $(A \setminus B) \cup Z \in \Delta$ and $(B \setminus A) \cup Z \notin \Delta$. Since Δ is an initial segment it follows that

$$(A \setminus B) \cup Z \prec (B \setminus A) \cup Z,$$

which, by the de Finetti's axiom (2.2), implies

$$A \setminus B \prec B \setminus A$$

and hence, again by the de Finetti's axiom (2.2),

$$A \prec B.$$

Thus an m-cycle

$$A_1 \prec_W \cdots \prec_W A_m \prec_W A_1$$

in Δ would imply an m-cycle

$$A_1 \prec \cdots \prec A_m \prec A_1,$$

which contradicts $\prec$ being a total order. $\square$

Conjecture 1 leads us to a slightly stronger version.

CONJECTURE 3. *If Δ is strongly acyclic, then there exists an extension of $\prec_W$ to a qualitative probability order.*

Below we give two lemmas that provide us some support for this belief. We show that the Winder existential relation on a simplicial complex Δ satisfies de Finetti's axiom together with the property $A \prec_W B$ if $A \in \Delta$ and $B \notin \Delta$.

LEMMA 6.4. *For any Δ, the Winder existential relation $\prec_W$ satisfies the property*

$$A \prec_W B \Longleftrightarrow A \cup D \prec_W B \cup D$$

for all D disjoint from $A \cup B$.

Proof. See [21, Proposition 4.7.8]. ∎

LEMMA 6.5. *Suppose $\Delta \subseteq 2^{[n]}$ is a simplicial complex. For every $A, B \in 2^{[n]}$ if $A \in \Delta$ and $B \notin \Delta$, then $A \prec_W B$.*

Proof. Let $Z = A \cap B$. Then

$$(A \setminus B) \cup Z = A \in \Delta \text{ and } (B \setminus A) \cup Z = B \notin \Delta,$$

so $A \prec_W B$. ∎

What are the barriers to proving Conjecture 3? The Winder relation need not be transitive. In fact there are examples of threshold complexes for which $\prec_W$ is not transitive [21, Proposition 4.7.3]. Thus one would have to work with the transitive closure of $\prec_W$, which does not seem to have a tractable description. In particular we do not know if the analogue of Lemma 6.4 holds for the transitive closure of $\prec_W$.

7. Appendix. Here the reader may find the proofs of Claims 5.6 and 5.7. Let us fix some notation first. Suppose $\mathbf{b} \in \mathbb{Z}^k$ and $\mathbf{x}_i \in \mathbb{Z}^n$ for $i \in [k]$. Then we define the product

$$\mathbf{b} * (\mathbf{x}_1, \ldots, \mathbf{x}_k) = \sum_{i \in [k]} b_i \mathbf{x}_i.$$

It resembles the dot product. The difference is that the second argument is not a vector but a sequence of vectors. We need the following technical facts and their corollaries.

FACT 7.1. *Suppose $\mathbf{a} \in \mathbb{Z}^5$ and $\{s, t, u, v\} = \{i, j, k, \ell\} = [4]$. Let*

$$\mathbf{v} = \mathbf{a} * (\chi(B_s, A_i), \chi(B_t, A_j), \chi(B_u, A_k), \chi(A_i, A_\ell), \chi(A_j, A_k)).$$

If $\mathbf{v} \in T^{18}$, then $\mathbf{a}$ belongs to the following set:

$$\begin{aligned} Q = \{&(0,0,0,0,0),\ (\pm1,0,0,0,0),\ (0,\pm1,0,0,0),\ (0,0,\pm1,0,0),\ (0,0,0,\pm1,0),\\ &(0,0,0,0,\pm1),\ (\pm1,0,0,\pm1,0),\ (0,\pm1,0,0,\pm1),\ (0,0,\mp1,0,\pm1),\\ &(0,\pm1,\mp1,0,\pm1),\ (\pm1,\pm1,\pm1,0,0),\ (\pm1,\pm1,\pm1,\pm1,0),\ (\pm2,\pm1,\pm1,\pm1,0)\}. \end{aligned}$$

Proof. In the construction of matrix M the first four rows and the last four rows can be arbitrarily permuted. Hence without loss of generality we may consider $(s, t, u, v) = (i, j, k, l) = (1, 2, 3, 4)$. Then

$$\mathbf{v} = \mathbf{a} * (\chi(B_1, A_1), \chi(B_2, A_2), \chi(B_3, A_3), \chi(A_1, A_4), \chi(A_2, A_3)).$$

By Proposition 3.17 the pairs (B_1, A_3) and (B_2, A_2) are not compatible. Then, there exists an atom p that belongs to $B_1 \cap B_2$ (or $A_2 \cap A_3$), but $p \notin A_2 \cup A_3$ ($p \notin B_1 \cup B_2$, respectively). By construction, we have exactly two copies of every atom among $A_1, \dots, A_4$ and $B_1, \dots, B_4$. Thus, the atom p belongs to $A_1 \cap A_4$ ($B_3 \cap B_4$) and doesn't belong to $B_3 \cup B_4$ ($A_1 \cup A_4$). The table below illustrates both possible cases; the first row corresponds to the case when $p \in B_1 \cap B_2$ and the second corresponds to the case when $p \in A_2 \cap A_3$:

	$\chi(A_1)$	$\chi(A_2)$	$\chi(A_3)$	$\chi(A_4)$	$\chi(B_1)$	$\chi(B_2)$	$\chi(B_3)$	$\chi(B_4)$
pth	1	0	0	1	1	1	0	0
coordinate	0	1	1	0	0	0	1	1

Then at pth position we have $v^{(p)} = \pm a_2$, and hence $a_2 \in T$. By Proposition 3.17 the pairs

$$((B_1, A_3), (B_2, A_2)),\ ((B_1, A_2), (B_3, A_3)),\ ((B_2, A_2), (B_4, A_3)),\ ((B_3, A_3), (B_4, A_2)),$$
$$((B_1, A_3), (B_2, A_4)),\ ((B_1, A_2), (B_3, A_4)),\ ((B_1, A_1), (B_4, A_4)),\ ((B_1, A_1), (B_3, A_2)),$$
$$((B_2, A_1), (B_4, A_2)),\ ((B_1, A_2), (B_2, A_4)),\ ((B_1, A_1), (B_2, A_3)),\ ((B_1, A_1), (B_3, A_3)),$$
$$((B_1, A_1), (B_3, A_3))$$

are incompatible. These allow us to get more equations relating $a_1, \dots, a_5$:

$$\begin{gathered} a_2, a_3, (a_1 - a_2), (a_1 - a_3), (a_4 + a_5), (a_4 - a_5), (a_1 - a_2 - a_3) \in T; \\ (a_1 - a_4 - a_5), (a_2 - a_3 - a_4 - a_5), (-a_2 + a_3 - a_4 + a_5) \in T; \\ (a_1 - a_4 + a_5), (a_1 - a_2 + a_3 - a_4 + a_5), (a_1 + a_2 - a_3 - a_4 - a_5) \in T. \end{gathered}$$

To check that the solution set is, in fact, Q one can argue as follows. Since $a_4 + a_5 \in T$ and $a_4 - a_5 \in T$ we have either $a_4 = 0$ or $a_5 = 0$ (or both). Suppose $a_4 = 0$ and $a_5 \neq 0$. Then

$$\begin{gathered} a_2, a_3, a_5, (a_1 - a_2), (a_1 - a_3), (a_1 - a_2 - a_3) \in T; \\ (a_1 - a_5), (a_2 - a_3 - a_5) \in T; \\ (a_1 + a_5), (a_1 - a_2 + a_3 + a_5), (a_1 + a_2 - a_3 - a_5) \in T. \end{gathered}$$

Since $a_1 + a_5 \in T$ and $a_1 - a_5 \in T$ and $a_5 \neq 0$, we have $a_1 = 0$. Then

$$a_2, a_3, a_5, (a_2 + a_3), (a_2 - a_3 - a_5) \in T.$$

This gives us solutions

$$(0, \pm 1, 0, 0, \mp 1), (0, \pm 1, \mp 1, 0, \pm 1), (0, 0, \pm 1, 0, \mp 1).$$

The cases when $a_4 \neq 0$, $a_5 = 0$, and $a_4 = a_5 = 0$ are considered similarly. In all cases we have $\mathbf{a} \in Q$. ∎

Corollary 7.2. *Let* $\mathbf{v}$ *be the same as in Fact* 7.1. *Then* $\mathbf{v} \in T^{[18]}$ *if and only if* $\mathbf{v}$ *belongs to* $\Omega \cup -\Omega$.

Proof. We only need to show that if $\mathbf{v} \in T^{[18]}$, then $\mathbf{v} \in \Omega \cup -\Omega$. By Fact 7.1 $\mathbf{v} \in T^{[18]}$ implies $\mathbf{a} \in Q$. Notice that for the first six elements of Q vector $\mathbf{v}$ belongs to $\Omega \cup -\Omega$. Consider the next four elements of Q. For these elements we use the following strategy: write all $\chi(X, Y)$ in $\mathbf{v}$ as $\chi(X) - \chi(Y)$, simplify the expression, and return to the form $\chi(X, Y)$. In fact, for all eight vectors we will end up with $\chi(X, Y) \in \Omega \cup -\Omega$. For example, if $a = (1, 0, 0, 1, 0)$, then

$$\mathbf{v} = \chi(B_s, A_i) + \chi(A_i, A_l) = \chi(B_s) - \chi(A_i) + \chi(A_i) - \chi(A_l) = \chi(B_s) - \chi(A_l) = \chi(B_s, A_l).$$

Consider the remaining three elements of Q. To deal with these elements we need an additional equation. By the construction the sequence $(A_1,\dots,A_4;B_1,\dots,B_4)$ is a trading transform. So for every $\{i_1,\dots,i_4\}=\{j_1,\dots,j_4\}=[4]$ the equation

$$\chi(B_{i_1},A_{j_1})+\chi(B_{i_2},A_{j_2})+\chi(B_{i_3},A_{j_3})+\chi(B_{i_4},A_{j_4})=\mathbf{0} \tag{7.1}$$

holds. Now we use the following strategy: first, by (7.1) write $-\chi(B_{i_4},A_{j_4})$ instead of $\chi(B_{i_1},A_{j_1})+\chi(B_{i_2},A_{j_2})+\chi(B_{i_3},A_{j_3})$; second, write all $\chi(X,Y)$ as $\chi(X)-\chi(Y)$ and simplify the expression; third, return to the form $\chi(X,Y)$. For all six vectors $a\in\{(\pm1,\pm1,\pm1,0,0),(\pm1,\pm1,\pm1,\pm1,0),(\pm2,\pm1,\pm1,\pm1,0)\}$ we, yet again, will end up with $\chi(X,Y)\in\Omega\cup-\Omega$. For example, if $\mathbf{a}=(2,1,1,1,0)$, then

$$\begin{aligned}\mathbf{v}&=2\chi(B_s,A_i)+\chi(B_t,A_j)+\chi(B_u,A_k)+\chi(A_i,A_l)\\&=\chi(B_s,A_i)+(\chi(B_s,A_i)+\chi(B_t,A_j)+\chi(B_u,A_k))+\chi(A_i,A_l)\\&=\chi(B_s,A_i)-\chi(B_v,A_l)+\chi(A_i,A_l)=\chi(B_s)-\chi(A_i)-\chi(B_v)\\&\quad+\chi(A_l)+\chi(A_i)-\chi(A_l)=\chi(B_s)-\chi(B_v)=\chi(B_s,B_v).\end{aligned}$$

This completes the proof. ∎

COROLLARY 7.3. *Let* $\mathbf{b}\in\mathbb{Z}^6$. *Then*

$$\mathbf{b}*(\chi(B_1,A_4),\chi(B_2,A_1),\chi(B_3,A_2),\chi(A_2,A_1),\chi(A_3,A_1),\chi(A_4,A_1))=\mathbf{0}$$

if and only if $\mathbf{b}=\mathbf{0}$.

Proof. It is sufficient to show that there does not exist a nonzero vector $\mathbf{b}\in\mathbb{Z}^6$ such that

$$\begin{aligned}(-b_5-b_6)\chi(A_4,A_1)&=\mathbf{b}'*\mathbf{x}\\&=(b_1,\dots,b_5)^T*(\chi(B_1,A_4),\chi(B_2,A_1),\chi(B_3,A_2),\chi(A_2,A_1),\chi(A_3,A_4)).\end{aligned}$$

Suppose $b_5+b_6=0$. Then, by Fact 7.1, we have $\mathbf{b}'*\mathbf{x}=\mathbf{0}$ if and only if $\mathbf{b}'=\mathbf{0}$.

On the other hand, if $b_5+b_6\neq0$, then, without loss of generality, we can assume $b_5+b_6=1$. Whence, $\mathbf{b}'*\mathbf{x}$ should be a vector in $T^{[18]}$. By Fact 7.1, the vector $\mathbf{b}'*\mathbf{x}\in T^{[18]}$ if $\mathbf{b}'\in Q$. By construction $\chi(A_1,A_4)\neq\chi(X,Y)$ for every $X,Y\in\{A_1,\dots,A_4,B_1,\dots,B_4\}$ such that $(X,Y)\neq(A_1,A_4)$. One can show that $\mathbf{b}'*\mathbf{x}\neq\chi(A_1,A_4)$ for every $\mathbf{b}'\in Q$. ∎

FACT 7.4. *Let* $\mathbf{a}=(a_1,\dots,a_8)$ *be a vector in* $\mathbb{Z}^8$ *whose every coordinate* a_i *has the absolute value of at most* 100. *Then* $\mathbf{a}\cdot\mathbf{w}^+=0$ *if and only if* $\mathbf{a}=\mathbf{0}$.

Proof. We calculate the dot product $\mathbf{a}\cdot\mathbf{w}^+$ substituting the values of $w_{19},\dots,w_{26}$ from (5.5):

$$\begin{aligned}0=\mathbf{a}\cdot\mathbf{w}^+&=N\sum_{i\in[4]}a_i-K\left(\sum_{i\in[4]}a_i-\sum_{i\in[4]}a_{4+i}\right)\\&\quad-\left[\chi(B_1,A_4)\left(\sum_{i=5}^{7}a_i-\sum_{i=1}^{3}a_i\right)+\chi(B_2,A_1)\left(\sum_{i=6}^{7}a_i-\sum_{i=2}^{3}a_i\right)\right.\\&\quad\left.+\chi(B_3,A_2)(-a_3+a_7)+\sum_{i\in[4]}a_i\chi(A_i)\right]\cdot\mathbf{w}.\end{aligned} \tag{7.2}$$

The number N is much larger than K, and K is much larger than $\sum_{i\in[18]}w_i$. As $|a_i|\le100$, the three summands cannot cancel each other. Therefore, $\sum_{i\in[4]}a_i=0$ and

$\sum_{i\in[4]} a_{4+i} = 0$. Since the coordinates of $\mathbf{w}$ are linearly independent, the expression in the square brackets should be zero.

We know that $a_1 = -a_2 - a_3 - a_4$, so the expression in the square brackets in (7.2) can be rewritten in the following form:

$$\begin{aligned} &b_1\chi(B_1, A_4) + b_2\chi(B_2, A_1) + b_3\chi(B_3, A_2) \\ &\quad + a_2\chi(A_2, A_1) + a_3\chi(A_3, A_1) + a_4\chi(A_4, A_1), \end{aligned} \tag{7.3}$$

where $b_1 = \sum_{i=5}^{7} a_i - \sum_{i=1}^{3} a_i$, $b_2 = \sum_{i=6}^{7} a_i - \sum_{i=2}^{3} a_i$, and $b_3 = a_7 - a_3$.

By Corollary 7.3, the expression (7.3) is zero if and only if $b_1 = 0, b_2 = 0, b_3 = 0$, and $a_2 = 0, a_3 = 0, a_4 = 0$, and this happens if and only if $\mathbf{a} = \mathbf{0}$. ◻

Proof of Claim 5.6. Assume, to the contrary, that $\chi(C_1, D_1) \in \Omega \cup -\Omega$ and $\chi(C, D)$ does not belong to $\Omega' \cup -\Omega'$. Consider $\chi(C_1^\dagger, D_1^\dagger) \in \Omega' \cup -\Omega'$. Sets C and D have equal weights. One can see that the weights of $C_1^\dagger$ and $D_1^\dagger$ are also equal. Equivalently,

$$\begin{aligned} &\chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_2, D_2) \cdot \mathbf{w}^+ = 0, \\ &\chi(C_1, D_1) \cdot \mathbf{w} + \chi(C_1^\dagger \setminus C_1, D_1^\dagger \setminus D_1) \cdot \mathbf{w}^+ = 0. \end{aligned}$$

Adding the two equations, we get

$$(\chi(C_1^\dagger \setminus C_1, D_1^\dagger \setminus D_1) - \chi(C_2, D_2)) \cdot \mathbf{w}^+ = 0.$$

Due to Fact 7.4, this is possible if and only if $\chi(C_1^\dagger \setminus C_1, D_1^\dagger \setminus D_1) - \chi(C_2, D_2) = \mathbf{0}$. The latter is equivalent to $\chi(C, D) = \chi(C_1^\dagger, D_1^\dagger) \in \Omega' \cup -\Omega'$, a contradiction. ◻

Proof of Claim 5.7. We remind the reader that α was defined in (5.7). Sets C and D have the same weights and we established that

$$\chi(C_2, D_2) \cdot \mathbf{w}^+ = \alpha \cdot \mathbf{w}.$$

Without loss of generality we can assume that $C_2 \cap D_2$ is empty. Similarly to the argument in the proof of Fact 7.4, the vector α can be expressed as

$$\alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + \sum_{i\in[4]} b_i\chi(A_i) \tag{7.4}$$

for some $a_i, b_j \in \mathbb{Z}$. If we look at the representation of the last eight weights in (5.5), we note that the first four weights are much larger than the last four. We call the weights $w_{19}, \ldots, w_{22}$ "super-heavy" and weights $w_{23}, \ldots, w_{26}$ "heavy". Since $w(C) = w(D)$, if there are $k \le 4$ atoms with super-heavy (or heavy) weights in C, then there are exactly k atoms with super-heavy (or heavy, respectively) weights in D and visa versa. The characteristic vectors $\chi(A_1), \ldots, \chi(A_4)$ participate in the representations of the super-heavy atoms only. Thus if $18+i \in C_2$, then there is $j \ne i$ such that $18+j \in D_2$, where $i, j \in [4]$. Hence

$$b_{18+i}\chi(A_i) + b_{18+j}\chi(A_j) = \chi(A_i, A_j).$$

Therefore, (7.4) can be rewritten as follows:

$$\alpha = a_1\chi(B_1, A_4) + a_2\chi(B_2, A_1) + a_3\chi(B_3, A_2) + a_4\chi(A_i, A_j) + a_5\chi(A_k, A_l), \tag{7.5}$$

where $a_1, a_2, a_3 \in \mathbb{Z}$; $a_4, a_5 \in \{0, 1\}$; and $\{i, j, k, l\} = [4]$.

One can notice that $\mathbf{v}$ from Corollary 7.2 is the general form of α. Hence $\alpha \in T^{18}$ if and only if $\alpha \in \Omega \cup -\Omega$, which is Claim 5.7. ◻

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